Structure-based SVAR identification

Emanuele Bacchiocchi* Riccardo (Jack) Lucchetti†

November 29, 2016

Abstract

It may be desirable, for various reasons, to establish criteria for SVAR identification that do not depend on unknown parameters, but only on the set of restrictions that are imposed on the system a priori on theoretical grounds.

In the context of linear systems, this was accomplished in Johansen (1995). This paper extends and amends the approach proposed by Lucchetti (2006); we introduce a set of criteria which ensure identification independently of unknown parameters for a reasonably general class of models and discuss its possible generalization.

Keywords: SVAR, identification, Rado condition.
JEL codes: C01, C30, C32.

1 Introduction

As is well known, Structural VARs have been an essential tool for macroeconometrics for over 30 years, since they were introduced by Sims (1980). Identification issues have been a thorny issue ever since, and were first analyzed in a systematic way a decade later in the 1992 edition of Amisano and Giannini (1997).

It may be desirable, for various reasons, to establish criteria for SVAR identification that do not depend on unknown parameters, but only on the set of restrictions that are imposed on the system a priori on theoretical grounds. In other words, our focus will be on what we consider a very important issue: is it possible to assess identification of a SVAR on the basis of its constraints alone? Because if it were so, identification would cease to be a property of

*Dipartimento di Economia, Management e Metodi Quantitativi (DEMM), Università degli Studi di Milano.
†Dipartimento di Scienze Economiche e Sociali (DISES), Facoltà di Economia "Giorgio Fuà", Università Politecnica delle Marche

the DGP. Instead, it would be possible to consider identification an inherent characteristic of the *structure* that we superimpose onto the data, independent of the observable evidence. In this case, we would call the model *structurally identified*.

In the context of linear systems, this was accomplished in Johansen (1995), who mostly focused on cointegrated systems, but introduced into the economic literature a few results from matroid theory which enabled him to draw necessary and sufficient conditions for identification in linear systems in a general and elegant way.

A similar approach was used in Lucchetti (2006) to analyze identification of models, such as Structural VARs, in which linear constraints are imposed on a covariance system. As we will show, the "structure condition" defined in that article is not sufficient, as claimed. In this paper, we extend and amend the same approach and introduce a set of criteria which ensure identification independently of unknown parameters for the most widely used class of models (the C-model, in Amisano and Giannini's taxonomy).

In doing so, we also achieve the goal of generalising the approach by Rubio-Ramirez, Waggoner, and Zha (2010), which has gained a notable popularity in recent years.

The structure of the article is as follows: in the next section, we examine the issue of identification and spell out our main result; then, in Section 3, we give a few examples of application of our criterion. Section 4 extends the results to the more general AB-model, with some examples discussed in Section 5. Section 6 concludes and outlines desirable extensions and generalizations of our present work.

2 Identification in the C-model

2.1 Basic concepts and notation

In this section, we review the identification issue in the C-model variety of Structural VARs, mainly to establish notation. Following Amisano and Giannini (1997), we call a C model a SVAR of the kind

$$\begin{aligned} \Phi(L)y_t &= \mu_t + \varepsilon_t && (1)\\ \varepsilon_t &= Cu_t && (2) \end{aligned}$$

in which the n-vector of one-step-ahead prediction errors ε_t is assumed to be a linear function of orthogonal structural shocks u_t, so that $V(\varepsilon_t) = \Sigma = CC'$. In Amisano and Giannini's original classification, this can be seen as a special case of the so-called AB model $A\varepsilon_t = Bu_t$ which is, however, much more seldom used in its full generality. The vast majority of empirical applications relies on a structure such as the one displayed in equation (2).

Under normality, the average log-likelihood can be written as

$$\mathscr{L} = \text{const} - \ln|C| - 0.5 \cdot \text{tr}\left[\hat{\Sigma}(CC')^{-1}\right]$$

and first-order conditions for maximization are equivalent to

$$\hat{\Sigma} = CC' \qquad (3)$$

where $\hat{\Sigma}$ is the sample covariance matrix of the VAR residuals; greater generality can easily be achieved by considering equation (3) simply as a set of orthogonality relationships of the kind

$$E\left[\varepsilon_{it}\varepsilon_{jt} - \sum_{k=1}^{n} c_{ik}c_{jk}\middle|\mathscr{F}_{t-1}\right] = 0,$$

which makes it possible to extend the present setup to QMLE or GMM estimation if necessary.

As is well known, this problem is underidentified, since if (3) holds, then also $\Sigma = C_* C_*'$ holds, where $C_* = CP$ and P is an arbitrary orthogonal matrix.

In order to achieve identification, it is necessary to make more qualifications to the above setup: the traditional solution has been to impose a system of p linear constraints on C of the kind

$$R\operatorname{vec} C = d. \tag{4}$$

A special case of linear restrictions occurs when C is assumed to be lower-triangular. This was Sims's (1980) original proposal, and is sometimes called a "recursive" identification scheme. It has a number of interesting properties, among which the fact that the ML estimator of C is just the Cholesky decomposition of $\hat{\Sigma}$, the sample covariance matrix of VAR residuals. This has been the most frequently used variant of a SVAR model, partly for its ease of interpretation, partly for its ease of estimation.

The general case (4), first analyzed in Amisano and Giannini (1997), has been recently re-assessed and somewhat extended in a recent contribution by Rubio-Ramirez, Waggoner, and Zha (2010), who provide conditions for global identification based on linear and some nonlinear restrictions of the parameters. However, their general results are all obtained by considering equation-by-equation restrictions (which are less general than those expressible via equation 4); moreover, the rank condition they obtain rely on the availability of the unknown parameters, which is precisely what we want to avoid.

An alternative identification strategy was introduced by Rigobon (2003), who suggested to exploit heteroskedasticity to identify simultaneous equation models without having to impose restrictions on the parametric space.[1] This approach has been extended to SVAR models by Lanne and Lütkepohl (2008) and generalized in Bacchiocchi (2014), Bacchiocchi (2016) and Bacchiocchi and Fanelli (2015), who combine heteroskedasticity and structural breaks with the traditional approach of imposing restrictions on the parameters of the model.

In this paper, we focus on the traditional identification strategy and we leave generalizations for future work; as a consequence of equation (4), the vector $c = \operatorname{vec} C$ can be written as

$$c = S\theta + s \tag{5}$$

where θ is the vector containing the $q = n^2 - p$ free parameters (assumed variation-free). This setup is more general than it looks, since it can also handle long-run constraints of the kind proposed by Blanchard and Quah (1989)

[1] The idea that heteroskedasticity can be helpful in identifying econometric models has been originally proposed by Sentana and Fiorentini (2001) in a factor model context, that could be extended to SVAR models too.

or King, Plosser, Stock, and Watson (1991). However, it cannot be used with more exotic identification schemes such as inequality-based ones like in Uhlig (2005).

As argued in Amisano and Giannini (1997) and Lucchetti (2006), local identification is tied to the rank of the matrix

$$\mathcal{H} = R(I \otimes C)\widetilde{D}_n \tag{6}$$

being full; in this case, $\widetilde{D}_n$ is a $n^2 \times \frac{n(n-1)}{2}$ matrix whose columns span the null space of the duplication matrix D_n (see Magnus, 1988).[2] Note that, in the case of exact identification, this matrix is square; introducing extra restrictions adds extra rows, but the number of columns, which only depends on n, remains unchanged.

This introduces a necessary and rather obvious order condition by which the number of linearly independent restrictions p must be at least $\frac{n(n-1)}{2}$. However, as is well known, the order condition is necessary but not sufficient. Evidently, the elements of $\mathcal{H}$ depend on the unknown parameters, since the C matrix is a function of θ, so it would seem that its rank cannot be established in general, but only on a case-by-case basis for each point in the parameter space $\mathbb{R}^q$. Lucchetti (2006) introduces an additional condition, called the *structure condition*, which is independent of the matrix C and should ensure (together with the order condition) sufficiency for identification apart from a zero-Lebesgue-measure set in the parameter space. Unfortunately, there are cases when this does not work. In the next section, we explain why and provide a counter-example.

2.2 An example when the structure condition fails

As an illustrative example, consider a case with 4 variables, in which the constraints are

$$c_{41} = c_{32} = c_{23} = c_{12} = c_{13} = c_{14} = 0$$

Since $n = 4$, $\frac{n(n-1)}{2} = 6$ so the order condition is satisfied. In this particular case the $\mathcal{H}$ matrix in equation 6 is square but singular for any choice of C. As an example, consider this particular choice for C (non-zero entries are random uniforms, just for the sake of the example):

$$C = \begin{bmatrix} 0.74 & 0 & 0 & 0 \\ 0.40 & 0.40 & 0 & 0.13 \\ 0.31 & 0 & 0.23 & 0.86 \\ 0 & 0.65 & 0.25 & 0.82 \end{bmatrix}$$

in which we take R to be

$$R = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

[2] The matrix $\widetilde{D}_n$ can also be defined as a basis for the space of the vectorization of $n \times n$ hemisymmetric matrices, so that any hemisymmetric matrix H has a vectorized form which satisfies $h = \widetilde{D}_n \varphi$ for some $n(n-1)/2 \times i$ vector φ.

and $d = 0$. Clearly, the ordering of the rows of R is immaterial for the purposes of identification. The matrix $\mathcal{H}$, whose rank ought to be full if the model were identified, is

$$\mathcal{H} = \begin{bmatrix} 0.65 & 0.25 & 0.82 & 0 & 0 & 0 \\ -0.31 & 0 & 0 & 0.23 & 0.86 & 0 \\ 0 & -0.40 & 0 & -0.40 & 0 & 0.13 \\ -0.74 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.74 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.74 & 0 & 0 & 0 \end{bmatrix}$$

which is, instead, singular: the set of its three rightmost columns has evidently rank 2. Following the arguments in Lucchetti (2006), it can be shown that this implies the existence of an infinitesimal rotation matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & -a & 0 & c \\ 0 & -b & -c & 0 \end{bmatrix}$$

which, in turn, implies the existence of an orthogonal matrix which rotates, in a neighborhood of C, the space of structural shocks in an observationally equivalent way, so to make the system under-identified.

Furthermore, it is also instructive to explore the rank deficiency of $\mathcal{H}$ by considering its rows as well as its columns. First, it should be noted that, under the above set of restrictions, $\Sigma_{1,4} = \Sigma_{4,1} = 0$ identically, since the first and the fourth rows of C are orthogonal by construction. We conjecture that this leads to underidentification since the corresponding moment condition on $E(\varepsilon_{1,t} \cdot \varepsilon_{4,t})$ contains no information on any of the elements in C.

This is mirrored by considering that the top row in $\mathcal{H}$ is a linear combination of the bottom three rows. In terms of identification, the meaning of this fact is that, given $\Sigma_{1,4} = 0$, the four restrictions on c_{41}, c_{12}, c_{13} and c_{14} could be combined into three linear restrictions on C. Note that this happens despite none of the linear constraints being linearly dependent on the other ones, as the constraints matrix R has full row rank.

In Lucchetti (2006), it was argued that checking for the rank of $\mathcal{H}$ is equivalent to considering the existence of solutions to the system

$$\varphi' \widetilde{D}_n' (I \otimes R_i') [S\theta + s] = \varphi' T_i \theta + \varphi' t_i = 0 \quad \text{for } i = 1 \ldots p \tag{7}$$

with

$$T_i \equiv \widetilde{D}_n' (I \otimes R_i') S \tag{8}$$

$$t_i \equiv \widetilde{D}_n' (I \otimes R_i') s \tag{9}$$

and R_i is a $n \times n$ matrix whose vectorization is the ith row of R. In other words, the system is unidentified at θ if some $\varphi \neq 0$ exists which satisfies the above equations for every i. While this is true, the check proposed in Lucchetti (2006) is flawed, and it is instructive to see why; construct the matrix

$$T = \begin{bmatrix} R(I \otimes S_1)\widetilde{D}_n \\ R(I \otimes S_2)\widetilde{D}_n \\ \vdots \\ R(I \otimes S_q)\widetilde{D}_n \end{bmatrix}$$

where S_i is an $n \times n$ matrix whose vectorisation is the i-th column of S; solution of the system of equations (7) amounts to finding θ and φ such that

$$(\theta' \otimes I_p) T \varphi = 0;$$

clearly, this is always possible if the column rank of T is not full, so the structure condition in Lucchetti (2006) amounts to checking for the rank of T; however, the equation above may also have solutions if a nonzero φ exists such that the product $T\varphi$ is non-null, but lies in the kernel space of $(I \otimes \theta')$, that is $\text{Sp}(\theta_\perp \otimes I_p)$.

Since the structure condition fails to capture this particular feature of the model, Lucchetti (2006)'s setup erroneously labels the above model as identified. In the next sections, we propose some new results that overcome this deficiency, by defining a set of necessary and sufficient conditions for structural identification.

2.3 The measure of under-identified points in the parameter space

The next result, already presented in Johansen (1995) for linear systems of equations and Rubio-Ramirez, Waggoner, and Zha (2010) for global identification in SVAR models, helps in practically checking whether a model is effectively identified or not.

Theorem 1. *Consider the SVAR model in Eq. (2), with restrictions on the parameters as described in Eqs. (4)-(5). If the order condition is met, then either $\mathcal{H}$ is singular for all values of $\theta \in \mathbb{R}^q$ or the set of vectors θ that makes it singular has zero Lebesgue measure.*

Proof. $\mathcal{H}$ has p rows and $\frac{n(n-1)}{2}$ columns, so if the order condition $p \geq \frac{n(n-1)}{2}$ is met, then the rank of $\mathcal{H}'\mathcal{H}$ equals that of $\mathcal{H}$. Since C in equation (6) is an affine function of θ, it follows that the determinant $|\mathcal{H}'\mathcal{H}|$ is a polynomial of finite order in θ. As is well known (see Traynor and Caron (2005)), a finite-order polynomial function $\mathbb{R}^q \mapsto \mathbb{R}$ is either identically 0, or non-zero almost everywhere. Call Θ_0 the set of θ satisfying $|\mathcal{H}'\mathcal{H}| = 0$; then either $\Theta_0 = \mathbb{R}^q$, or $\Theta_0 \subset \mathbb{R}^q$, where $\lambda(\Theta_0) = 0$, and the claim follows.[3] □

The most powerful consequence of the above theorem is that *if one θ exists such that the model is locally identified at it, then it is locally identified almost everywhere*. This result provides theoretical justification to the standard practice of checking identification by simply calculating $\text{rk}(\mathcal{H})$ for randomly selected values for the free parameters θ.

Moreover, it should be noted that, by the continuity of the differential function, if $\Theta_0 \subset \mathbb{R}^q$, there exists some $\theta^* \notin \Theta_0$ in every neighborhood of $\theta \in \Theta_0$, so the subset of parameter points for which identification is attained is either empty, or dense in $\mathbb{R}^q$.

[3] We use $\lambda(\cdot)$ for Lebesgue measure, as per standard notation.

2.4 A new necessary condition for structural identification: the Rado condition

In Johansen (1995) the main result in terms of the identification of the model as a function of the constraints only, is proved by recurring to two interesting theorems in classical mathematics and algebra by Hall (1935) and Rado (1942). In the next theorem we use the same idea and provide a necessary condition for the structural condition to hold that, in part, overcomes the previous deficiency in Lucchetti (2006).

Before providing the details, however, it is worth looking at the problem from a different point of view. Starting from the Amisano and Giannini's (1997) $\mathcal{H}$ matrix reported in Eq. (6), and working on each single column, we obtain

$$R(I_n \otimes C)\widetilde{D}_n e_i = R\text{vec}\,(CH_i) = -R(H_i \otimes I_n)[S\theta + s], \qquad i = 1, \ldots, m \tag{10}$$

where e_i is the i-th column of the identity matrix, H_i is a $n \times n$ matrix whose vectorization corresponds to the i-th column of $\widetilde{D}_n$ and, for simplifying the notation, we fix $m = \frac{n(n-1)}{2}$. If we define

$$V_i \equiv R(H_i \otimes I_n)S \tag{11}$$

$$v_i \equiv R(H_i \otimes I_n)s, \tag{12}$$

the Amisano and Giannini's (1997) $\mathcal{H}$ matrix can also be written as

$$\mathcal{H} = \mathbf{V}(\theta) = \left[\ V_1\theta + v_1 \mid V_2\theta + v_2 \mid \cdots \mid V_m\theta + v_m\ \right] \tag{13}$$

where $\mathbf{V}(\theta)$ emphasizes the dependence of $\mathcal{H}$ from the unknown parameters θ. Note that, in the case of $d = 0$, all the u_i-s are zero.[4]

Now, define $\mathbf{V}_i = (V_i \mid v_i)$, $i = 1, \ldots, m$, and consider

$$\mathbf{V} = \left[\ \mathbf{V}_1 \mid \mathbf{V}_2 \mid \cdots \mid \mathbf{V}_m\ \right], \tag{14}$$

whose size is $p \times m(q+1)$ with $m(q+1) > p$. This matrix collects all the vectors composing V_i and v_i, $i = 1 \ldots m$, as defined in Eq.s (11)-(12). The next theorem restates the *structure condition* introduced by Lucchetti (2006) simply as a necessary condition for the identification of the SVAR. Combined with Theorem 1, if the following condition, that does not depend on the unknown parameters, is not satisfied, the SVAR model cannot be identified.

Theorem 2. *Consider the specification in equations (2)-(3) and the set of restrictions given by equations (4)-(5), with $p \geq m$, where $m = \frac{n(n-1)}{2}$. A necessary condition for the structure condition to hold is that, for all $k = 1, \ldots, m$, and all set of indices: $1 \leq i_1 < i_2 < \ldots i_k \leq m$, then*

$$\text{rk}\left(\mathbf{V}_{i_1} | \ldots | \mathbf{V}_{i_k}\right) \geq k. \tag{15}$$

We call the Rado condition *the condition by which the inequalities (15) are simultaneously satisfied. Therefore, the SVAR model is structurally identified only if the Rado condition is met, except for a zero Lebesgue measure set of values for θ.*

[4] This, for example, is the case of recursive identification and is by far the most common setup in the C model.

Proof. The theorem directly derives from Theorem 1 and Theorem 6, discussed in the Appendix. In fact, if the Rado condition holds, then there exists a set of $q \times 1$ vectors $\theta_1, \ldots, \theta_m$ such that $\mathbf{V}(\theta)$ is of full column rank. On the contrary, if it does not hold, there are no θs, distinct or equal as in Eq. (13), making the $\mathbf{V}(\theta)$ matrix of full column rank. The Rado condition is thus necessary for the identification of the SVAR model. □

It is perhaps useful to give an explicit example of the condition on Theorem 2: Consider a trivariate system with 4 restrictions so that $n = 3$, $m = \frac{n(n-1)}{2} = 3$ and $p = 4$. Theorem 2 states that we first have to check the rank of each single matrix $\mathbf{V}_i$ to be different from zero, then all the matrices of the type $\left(\mathbf{V}_{i_1}|\mathbf{V}_{i_2}\right)$, that is for (i_1, i_2) equal to

$$(1,2), (1,3), (2,3);$$

the rank of all the above combinations should be at least 2. Finally, we consider the matrix of the kind $\left(\mathbf{V}_{i_1}|\mathbf{V}_{i_2}|\mathbf{V}_{i_3}\right)$ that must have rank 3.

If $p \leq \frac{n(n-1)}{2}$ the order condition fails, and the model cannot be identifiable as there are not enough empirical moments in the covariance matrix Σ to estimate the free parameters of the model. If, instead, $p \geq \frac{n(n-1)}{2}$, the number of imposed restrictions is large enough compared to the number of distinct elements in Σ, and the necessary condition is met. At this stage, the condition in Eq. (15) becomes a more stringent necessary condition for the identifiability of the SVAR model than simply counting the number of restrictions, as in the standard order condition.

As will be discussed in one of the examples in Section 3, the Rado condition plays an interesting role as it indicates that the model in Section 2.2 cannot be identified, as instead suggested by the structure condition in Lucchetti (2006).

Remark 1. Differently from Theorem 6, the condition in Theorem 2 cannot be also sufficient for the $(V_1\theta + v_1\,,\, V_2\theta + v_2\,,\, \cdots\,,\, V_m\theta + v_m)$ to be linearly independent. As shown in the next counter-example, the problem is that we constraint all the θs to be equal, while in the original Rado's theorem nothing is said about these vectors $\theta_1, \ldots, \theta_m$. In fact, suppose that the set of constraints in a trivariate SVAR generates the following matrices

$$V_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

with $v_1 = v_2 = v_3 = 0$. The Rado condition gives the following results

$$\begin{array}{lll} \text{rk}\,(V_1) = 1 \geq 1, & \text{rk}\,(V_2) = 3 \geq 1, & \text{rk}\,(V_3) = 2 \geq 1 \\ \text{rk}\,(V_1|V_2) = 3 \geq 2 & \text{rk}\,(V_1|V_3) = 3 \geq 2 & \text{rk}\,(V_2|V_3) = 3 \geq 2 \\ \text{rk}\left(\begin{array}{c|c|c} V_1 & V_2 & V_3 \end{array} \right) = 3 \geq 3 & & \end{array}$$

and it is clearly satisfied. There exist, hence, three possible vectors θ_1 ,θ_2, θ_3 such that $(V_1\theta_1\,,\ V_2\theta_2\,,\ V_3\theta_3)$ are linearly independent. However, when the constraint

$$\theta_1 = \theta_2 = \theta_3 = \theta = \left(\begin{array}{cccccc} \theta^{(1)}, & \theta^{(2)}, & \theta^{(3)}, & \theta^{(4)}, & \theta^{(5)}, & \theta^{(6)} \end{array} \right)'$$

is imposed (for some scalars $\theta^{(1)}, \dots, \theta^{(6)}$), then

$$\mathbf{V}(\theta) = (V_1\theta\,,\ V_2\theta\,,\ V_3\theta) = \left(\begin{array}{c|c|c} \theta^{(1)} & \theta^{(2)} & 0 \\ 0 & \theta^{(3)} & \theta^{(3)} \\ 0 & \theta^{(4)} & \theta^{(4)} \end{array} \right)$$

is composed by linearly dependent columns. Although the Rado condition holds, the rank of $\mathbf{V}(\theta)$ cannot be full for whatever choice of the 6×1 vector θ. This simple counter-example clearly proves that the Rado condition is not sufficient.

2.5 A new necessary and sufficient condition

Starting from the $\mathbf{V}$ matrix introduced in Eq. (14), let define the new matrices M_1, $M_2, \dots,$ M_{q+1}, where the generic one, of dimension $(p \times m)$, is

$$M_i = (\mathbf{V}_1 e_i,\ \mathbf{V}_2 e_i, \dots,\ \mathbf{V}_m e_i) \tag{16}$$

and collects, side by side, all the i-th columns of $\mathbf{V}_1$, $\mathbf{V}_2, \dots,$ $\mathbf{V}_m$, being e_i the i-th column of the identity matrix I_{q+1}.[5]

The next theorem provides a new necessary and sufficient condition for $\mathbf{V}(\theta)$ to have full column rank that does not depend on the unknown parameters θ.

Theorem 3. *Consider the specification in equations (2)-(3) and the set of restrictions given by equations (4)-(5), with $p \geq m$, where $m = \frac{n(n-1)}{2}$. A necessary and sufficient condition for the structure condition to hold is that there exists a linear combination of the M_1, $M_2, \dots,$ M_{q+1} defined as in Eq. (16), i.e.*

$$M = \lambda_1 M_1 + \lambda_2 M_2 + \ldots + \lambda_{q+1} M_{q+1} \tag{17}$$

for some scalars λ_1, $\lambda_2, \dots,$ λ_{q+1}, such that M has full column rank equal to m.

Therefore, if $\text{rk}(M) = m$, the SVAR model is structurally identified except for a zero Lebesgue measure set of values for θ.

[5] If $v_1 = v_2 = \ldots = v_m = 0$, then the number of such new matrices will be q instead of $q + 1$, and e_i will be the i-th column of the identity matrix I_q.

Proof. First of all, let indicate with $\theta^{(i)}$ the i-th element of the vector of parameters θ. It is worth remembering that the SVAR model is identified if and only if $\mathbf{V}(\theta)$, defined in Eq. (13) has full column rank. Focusing on the i-th matrix M_i, each of its column is multiplied by the scalar $\theta^{(i)}$ in the original rank problem in Eq. (13). If the matrix M_i has full column rank it is possible to find m linearly independent vectors in $\mathbf{V}(\theta)$, apart for the trivial case $\theta^{(i)} = 0$, which has, however, zero Lebesgue measure. This sufficient condition, that does not depend on θ, however, is not necessary, since some linearly independent vectors can be detected from the i-th columns of $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_m$ (i.e. M_i), but some others from the collection of the other columns. The necessary condition, thus, is that there must exist at least one linear combination of $M_1, M_2, \ldots, M_{q+1}$ that has full column rank.

For simplicity, consider a linear combination of the first two:

$$M = \lambda_1 M_1 + \lambda_2 M_2.$$

If M has full column rank m, it means that from $\mathbf{V}(\theta)$ it is possible to generate m vectors of the form

$$\lambda_1 M_1 \theta^{(1)} + \lambda_2 M_2 \theta^{(2)}$$

that, for whatever non-null values of the two scalars $\theta^{(1)}$ and $\theta^{(2)}$, will be linearly independent. The generalization to a linear combination involving all the $q+1$ matrices $M_1, M_2, \ldots, M_{q+1}$ is straightforward and makes the previous condition both necessary and sufficient. □

The previous theorem provides a necessary and sufficient condition for the model to be identified that is only based on the imposed restrictions, and that can be checked before the estimation process. As in Johansen (1995), it can be said that identification of the SVAR is a characteristic of the model, rather than being tied to the parameter matrix C. The practical implementation of the previous condition will be discussed in the following section, through a set of identified and not identified SVAR models.

3 A few example C-models

The next examples provide some details on the implementation of the identification conditions developed in Theorems 2 and 3 presented in Section 2.

3.1 Unidentified bivariate SVAR

This example is analyzed in detail both in Amisano and Giannini (1997) and in Lucchetti (2006). Consider the following bivariate SVAR model where the matrix of contemporaneous relations is defined as

$$C = \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{pmatrix}.$$

The structure of the C matrix presents a problem similar to the one discussed in Section 2.2, i.e. it implicitly imposes a covariance matrix of the reduced form as follows

$$CC' = \begin{pmatrix} \theta_1^2 + \theta_2^2 & 0 \\ 0 & \theta_1^2 + \theta_2^2 \end{pmatrix}$$

where the covariance of the VAR residuals is restricted to be zero and the two variances are equal. The model thus presents two parameters to be estimated but only one empirical moment. The matrices of restrictions are

$$R = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \widetilde{D}_n = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

As $m = \frac{n(n-1)}{2} = 1$, the unique U_1 becomes

$$\begin{aligned} V_1 &= \tilde{D}_2' \left(I_2 \otimes R_1' \right) S = \\ &= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

The order condition is clearly met ($m = 1$ and $p = 2$). Using Theorem 2, however, gives the following condition

$$\text{rk}\left(V_1\right) = 0 < 1.$$

The model, thus, cannot be identifiable as the necessary Rado condition does not hold. This result is trivially confirmed by considering the necessary and sufficient condition reported in Theorem 3.

3.2 Trivariate Cholesky SVAR

Consider the triangular trivariate SVAR model. Rubio-Ramirez, Waggoner, and Zha (2010) prove that this model is not simply locally identified, but also globally. The matrix of simultaneous relations is defined as

$$C = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$$

where asterisks ('$*$') denote unrestricted coefficients. The matrices of restrictions can be written as

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order condition is clearly satisfied. The V_1, V_2 and V_3 matrices are

$$V_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank condition provided by Theorem 2 gives the following results

$$\text{rk}\,(V_1) = 1 \geq 1, \qquad \text{rk}\,(V_2) = 1 \geq 1, \qquad \text{rk}\,(V_3) = 2 \geq 1$$

$$\text{rk}\,(V_1|V_2) = 3 \geq 2 \qquad \text{rk}\,(V_1|V_3) = 2 \geq 2 \qquad \text{rk}\,(V_2|V_3) = 2 \geq 2$$

$$\text{rk}\left(V_1 \mid V_2 \mid V_3 \right) = 3 \geq 3$$

indicating that the necessary condition holds. Moreover, let consider the necessary and sufficient condition of Theorem 3 and define M_1, $M_2, \ldots M_6$ as follows

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A simple linear combination of M_1 and M_2 provides a matrix

$$M = M_1 + M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of full rank, indicating that the model is structurally identified.

3.3 Trivariate SVAR with cross-equation restrictions

Consider the trivariate SVAR model with the following matrix of simultaneous relations

$$C = \begin{pmatrix} * & 0 & 0 \\ * & * & \circledast \\ * & * & \circledast \end{pmatrix}$$

where asterisks ('$*$') denote unrestricted coefficients while circled asterisks ('$\circledast$') denote unrestricted but *equal* coefficients. This model is extremely interesting for at least two reasons. Firstly, the restriction $c_{23} = c_{33}$ involves parameters of two different equations, the second and the third. These kind of restrictions are not allowed in the well-known contribution by Rubio-Ramirez, Waggoner, and Zha (2010), where only equation-by-equation restrictions can be included. Although in a different framework, cross-equations restrictions are not allowed in the Johansen (1995) set up. Our methodology, thus, represents an improvement with respect to the existing literature allowing for more general restrictions.

Secondly, when verifying the identification of this SVAR model, a largely diffused econometric software like Eviews immediately says that it is not identified. However, based on the results of Theorem 1, is effectively the model not identified, or does the value of θ chosen for checking the identification represent an isolated point such that the model is not identified in this point but instead is identified almost everywhere in the parametric space? In fact, Eviews uses the Amisano and Giannini's approach and calculates by default the rank of $\mathcal{H}$ at $\theta = (1, 1, 1, 1, 1, 1)'$, for which the rank is not full. However, when selecting the *advanced options* of randomly generating θ, the SVAR model results to be identified. Our procedure, that is independent of the selected parameters θ, but is model-based, has this further advantage with respect to the existing literature, and can avoid the unpleasant situation of discarding a model when it is effectively identified almost everywhere in the parametric space.

The matrices of restrictions can be written as

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad , \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order condition is clearly satisfied as $p = 3$ and $m = \frac{n(n-1)}{2} = 3$. The V_1, V_2 and V_3 matrices are

$$V_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The rank condition provided by Theorem 2 gives the following results

$$\text{rk}\,(V_1) = 1 \geq 1, \qquad \text{rk}\,(V_2) = 1 \geq 1, \qquad \text{rk}\,(V_3) = 2 \geq 1$$

$$\text{rk}\,(V_1|V_2) = 3 \geq 2 \qquad \text{rk}\,(V_1|V_3) = 2 \geq 2 \qquad \text{rk}\,(V_2|V_3) = 2 \geq 2$$

$$\text{rk}\,(\;V_1 \;\;|\;\; V_2 \;\;|\;\; V_3\;) = 3 \geq 3$$

indicating that the necessary Rado condition holds. Concerning the necessary and sufficient condition, it can be seen that

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad M_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and a simple linear combination of M_1 and M_4 provides a full rank matrix

$$M = M_1 + M_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

indicating that the model is structurally identified.

3.4 Unidentified four-variable SVAR

In this section we use the new results in Theorem 2 and Theorem 3 to show that the unidentified SVAR model described in Section 2.2 can now be correctly detected. The necessary and sufficient condition for identification in Eq. (15) suggests to calculate the rank of all possible combinations of the $V_1, \ldots, V_6$ matrices, where $p = 6$ indicates the number of restrictions (rows of R). This is rather boring to do by hand, but is also easy to check automatically, via appropriate software.[6] Considering the example described in Section 2.2, the rank condition is verified for singles and pairs of V_i, $i = 1, \ldots, 6$, but fails when we check

$$\text{rk}\,(\,V_4 \,|\, V_5 \,|\, V_6\,) = 2 \leq 3$$

indicating that the model is under-identified, in accordance to the previous analysis reported in Section 2.2.

Furthermore, if we consider the M_i matrices, not reported here to save space, for each of them the last three columns present non-null entries only in the third and fifth rows, preventing thus the construction of three linearly independent vectors. It is thus impossible to find a full-rank linear combination of the matrices, indicating that the model cannot be identifiable.

4 Identification in the AB-model

Following the terminology of Amisano and Giannini (1997) and Lütkepohl (2006), the AB-model considers possible simultaneous relationships among the observable variables and the structural shocks. Classic empirical applications using this specification can be found, among many others, in Bernanke (1986), Blanchard (1989) and Blanchard and Perotti (2002). Concentrating out the dynamic part of the VAR, the simultaneous relations are defined by

$$A\varepsilon_t = Bu_t \tag{18}$$

[6] A Gretl code is available from the author upon request.

where, as before, ε_t is the one-step-ahead prediction errors, with $E\left(\varepsilon_t \varepsilon_t'\right) = \Sigma$, and u_t collects the standardized and uncorrelated structural shocks. The specification in Eq. (18) naturally induces a set of non-linear restrictions on the parameter space given by

$$A\Sigma A' = BB'. \tag{19}$$

The number of parameters, here, is clearly $2n^2$, while that of empirical moments remains 'at most' $n(n+1)/2$, i.e. the number of elements in the estimable covariance matrix of the residuals Σ. As a consequence, at least $p = 2n^2 - n(n+1)/2 = n^2 + n(n-1)/2$ restrictions on the parameters must be included for identification.

The first systematic contribution in terms of the identification has been provided by Amisano and Giannini (1997). Write the p_a and p_b independent restrictions on A and B as

$$R_a vec\, A = r_a \qquad \text{and} \qquad R_b vec\, B = r_b, \tag{20}$$

or equivalently, in explicit form,

$$vec\, A = S_a\theta_a + s_a \qquad \text{and} \qquad vec\, B = S_b\theta_b + s_b; \tag{21}$$

where θ_a and θ_b are of dimension $q_a \times 1$ and $q_b \times 1$, respectively, and contain the $q = q_a + q_b$ free parameters to estimate; Amisano and Giannini (1997) introduce a necessary and sufficient identification condition based on the following $p_b \times [n(n-1)/2 + q_a]$ matrix

$$R_b\left(B \otimes BB'\right)\left[\widetilde{D}_n \,\middle|\, -\left[A^{-1} \otimes \left(BB'\right)^{-1}\right] S_a\right] \tag{22}$$

that must have full column rank for the model to be identified. As for the C-model previously discussed, this matrix depends on the true values of A and B, that are unknown.

In what follows, we use the intuition in Lucchetti (2006) and check for the existence of a pair of matrices

$$\begin{aligned} A_* = A + \mathrm{d}A &= (I+Q)\,A \\ B_* = B + \mathrm{d}B &= (I+Q)\,B\,(I+H) \end{aligned}$$

that are observationally equivalent to A and B in a neighborhood of the two true matrices A and B.

The two infinitesimal transformation matrices $(I+Q)$ and $(I+H)$ must obey different requisites: $(I+H)$ should be orthogonal, while invertibility is sufficient for $(I+Q)$. This implies that, while it is sufficient to consider a simple infinitesimal non-zero matrix $Q \neq 0$, H must be an infinitesimal rotation with $H = -H'$. The two new matrices of parameters A_* and B_* will be admissible if

$$\begin{aligned} R_a \mathrm{d}a &= R_a\,(I \otimes Q)\,(S_a\theta_a + s_a) = 0 \\ R_b \mathrm{d}b &= R_b\left[(I \otimes Q) + \left(H' \otimes I\right)\right](S_b\theta_b + s_b) = 0. \end{aligned}$$

where $a = vec\, A$ and $b = vec\, B$. With a little algebra, the identification condition can be shown to be equivalent to the following system of equations[7]

$$\left[\; q' \mid \phi' \;\right] \mathbf{U}(\theta) = 0 \tag{23}$$

[7] See Lucchetti (2006), pag 247, Eq. (23).

where

$$\mathbf{U}(\theta)=\left[\begin{array}{c|c|c|c|c|c} U_1^a\theta+u_1^a & \cdots & U_{p_a}^a\theta+u_{p_a}^a & U_1^b\theta+u_1^b & \cdots & U_{p_b}^b\theta+u_{p_b}^b \\ \hline 0 & \cdots & 0 & T_1^b\theta+t_1^b & \cdots & T_{p_b}^b\theta+t_{p_b}^b \end{array}\right] \tag{24}$$

with

$$\begin{array}{lll} U_i^a=K_{nn}\left(R_{a,i}\otimes I_n\right)S_a & u_i^a=K_{nn}\left(R_{a,i}\otimes I_n\right)s_a & i=1,\dots,p_a \\ U_i^b=K_{nn}\left(R_{b,i}\otimes I_n\right)S_b & u_i^a=K_{nn}\left(R_{b,i}\otimes I_n\right)s_b & i=1,\dots,p_b \\ T_i^b=\widetilde{D}'_n\left(I_n\otimes R'_{b,i}\right)S_b & u_i^b=\widetilde{D}'_n\left(I_n\otimes R'_{b,i}\right)s_b & i=1,\dots,p_b. \end{array}$$

in which $R_{a,i}$ and $R_{b,i}$, similarly to R_i defined in Section 2.1, are defined such that their vectorization is equal to the i-th row of A and B, respectively. The matrix K_{nn} is the $n^2\times n^2$ matrix defined by the property $K_{nn}\text{vec}\,(A)=\text{vec}\,(A')$.[8]

If the system admits as its unique solution the null vector $[\ q'\mid\phi'\ ]=[\ 0\ ]$, or equivalently, $\mathbf{U}(\theta)$ has full row rank, then the model is identified. Obviously, $\mathbf{U}(\theta)$ is a function of the free parameters θ, and the identification can only be checked conditionally on the unknown parameters.

4.1 A useful transformation

Starting from Eq. (23), let rewrite $\mathbf{U}(\theta)$ as follows

$$\begin{aligned}\mathbf{U}(\theta) &= \left[\ \mathbf{U}_1\ \mid\dots\mid\mathbf{U}_{p_a}\ \mid\ \mathbf{U}_{p_a+1}\ \mid\dots\mid\mathbf{U}_{p_a+p_b}\ \right]\left(I_p\otimes\theta\right)= \\ &= \mathbf{U}\left(I_p\otimes\theta\right)\end{aligned}$$

where $\theta=\left(\theta'_a\mid 1\mid\theta'_b\mid 1\right)'$, and the generic $[(n^2+m)\times(q+2)]$ matrix $\mathbf{U}_i$ can be defined as

$$\mathbf{U}_i=\begin{cases}\left(\begin{array}{c:c} U_i^a\mid u_i^a & 0 \\ \hline 0 & 0\end{array}\right) & \text{for } 1\le i\le p_a \\ \left(\begin{array}{c:c} U_i^b\mid u_i^b & 0 \\ \hline 0 & T_i^b\mid t_i^b\end{array}\right) & \text{for } p_a+1\le i\le p_a+p_b.\end{cases}$$

The $\mathbf{U}(\theta)$ matrix is of dimension $(n^2+m)\times p$ where, as before, $m=\frac{n(n-1)}{2}$, $p=p_a+p_b$ denotes the total number of restrictions and, according to the already mentioned order condition, $p\ge n^2+m$. If the equality holds, then $\mathbf{U}(\theta)$ will be squared and potentially exactly identified. On the contrary, if we have more restrictions than necessary, $\mathbf{U}(\theta)$ will have more columns than rows, making problematic the application of the Rado necessary condition as developed in Section 2.4. In this vein, the next transformation reveals extremely useful for reconciling the problem with the previous analysis developed for the C-model.

Each row of $\mathbf{U}(\theta)$ can be written as

$$e'_i\mathbf{U}(\theta)=[e'_i\mathbf{U}_1\theta\ ,\ \dots\ ,\ e'_i\mathbf{U}_p\theta]$$

[8] It can be shown that K_{nn} is symmetric and orthogonal; see for instance Magnus and Neudecker (1988, p. 46).

where e_i is the i-th column of the identity matrix. Taking the transpose, it is easily obtained that

$$\mathbf{U}(\theta)'e_i = \begin{bmatrix} e_i'\mathbf{U}_1\theta \\ \vdots \\ e_i'\mathbf{U}_p\theta \end{bmatrix} = \left[\begin{pmatrix} e_i'\mathbf{U}_1 \\ \vdots \\ e_i'\mathbf{U}_p \end{pmatrix} \theta \right].$$

Considering all the rows of $\mathbf{U}(\theta)$, or equivalently, all the columns of $\mathbf{U}(\theta)'$, it is possible to define

$$\begin{aligned} \mathbf{V}(\theta) = \mathbf{U}(\theta)' &= \left[\begin{pmatrix} e_1'\mathbf{U}_1 \\ \vdots \\ e_1'\mathbf{U}_p \end{pmatrix} \theta \;\vdots\; \cdots \;\vdots\; \begin{pmatrix} e_{(n^2+m)}'\mathbf{U}_1 \\ \vdots \\ e_{(n^2+m)}'\mathbf{U}_p \end{pmatrix} \theta \right] \\ &= \left[\mathbf{V}_1\theta \mid \quad \ldots \quad \mid \mathbf{V}_{(n^2+m)}\theta \right] \\ &= \left[\mathbf{V}_1 \mid \mathbf{V}_2 \mid \cdots \mid \mathbf{V}_{(n^2+m)} \right] (I \otimes \theta) \\ &= \mathbf{V}(I \otimes \theta), \end{aligned} \tag{25}$$

where the $p \times q$ generic matrix $\mathbf{V}_i$, with $i = 1, \ldots, (n^2 + m)$, is defined as

$$\mathbf{V}_i = \begin{pmatrix} e_1'\mathbf{U}_1 \\ \vdots \\ e_1'\mathbf{U}_p \end{pmatrix}, \tag{26}$$

and does not depend on the unknown parameters θ.

The matrix $\mathbf{V}(\theta)$, that is equal to $\mathbf{U}(\theta)'$, is of dimension $p \times (n^2 + m)$ and, having at least as many rows as column, allows us to use the same strategy followed for the C-model to provide necessary and sufficient conditions for the structural-identification of the AB-model.

4.2 The Rado condition for the AB-model

The following theorem extends the results of Theorem 2 to the case of the AB-model and provides a necessary condition for the identification.

Theorem 4. *Consider the specification in Eqs. (18)-(19) and the set of p restrictions given by Eqs. (20)-(21), with $p = p_a + p_b \geq n^2 + m$, with $m = \frac{n(n-1)}{2}$ and $p \geq n^2 + m$. A necessary condition for the structure condition to hold is that, for all $k = 1, \ldots, n^2 + m$, and all set of indices: $1 \leq i_1 < i_2 < \ldots i_k \leq n^2 + m$, then*

$$\text{rk}\left(\mathbf{V}_{i_1}| \ldots |\mathbf{V}_{i_k}\right) \geq k. \tag{27}$$

We call the Rado condition *the condition by which the inequalities (27) are simultaneously satisfied. Therefore, the AB-model is structurally identified only if the Rado condition is met, except for a zero Lebesgue measure set of values for θ.*

Proof. The theorem directly derives from Theorem 1, Theorem 2 and Theorem 6, discussed in the Appendix. □

The necessary condition provided in Theorem 4, being based on the $\mathbf{V}_i$ matrices that do not depend on θ, can be checked before the estimation process takes places. Moreover, combined with the result in Theorem 1, when it is not satisfied, the model cannot be identified in any point on the parametric space.

4.3 The new necessary and sufficient condition for the identification of the AB-model

The next theorem provides a new necessary and sufficient condition for $\mathbf{V}(\theta)$ defined in Eq. (25) to have full column rank. As the condition does not depend on the unknown parameters θ, the theorem provides a necessary and sufficient condition for identification that can be checked before the estimation process. The idea is the same as in Theorem 3. Starting from the $\mathbf{V} = \left[\mathbf{V}_1 \mid \mathbf{V}_2 \mid \cdots \mid \mathbf{V}_{(n^2+m)} \right]$ matrix introduced in Eq. (25), let define the new matrices $M_1, M_2, \ldots, M_{q+2}$, where the generic one, of dimension $[p \times (n^2 + m)$, is

$$M_i = (\mathbf{V}_1 e_i, \mathbf{V}_2 e_i, \ldots, \mathbf{V}_{(n^2+m)} e_i) \tag{28}$$

and collects, side by side, all the i-th columns of $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_{(n^2+m)}$, being e_i the i-th column of the identity matrix I_{q+2}.[9]

Theorem 5. *Consider the specification in Eqs. (18)-(19) and the set of p restrictions given by Eqs. (20)-(21), with* $p = p_a + p_b \geq n^2 + m$, *with* $m = \frac{n(n-1)}{2}$ *and* $p \geq n^2 + m$. *A necessary and sufficient condition for the structure condition to hold is that there exists a linear combination of the* $M_1, M_2, \ldots, M_{q+1}$ *defined as in Eq. (16), i.e.*

$$M = \lambda_1 M_1 + \lambda_2 M_2 + \ldots + \lambda_{q+2} M_{q+2} \tag{29}$$

for some scalars $\lambda_1, \lambda_2, \ldots, \lambda_{q+2}$, *such that M has full column rank equal to* $(n^2 + m)$.

Therefore, if $\mathrm{rk}(M) = n^2 + m$, *the SVAR model is structurally identified except for a zero Lebesgue measure set of values for* θ.

Proof. The result immediately follows from Theorem 3. □

As for the C-model discussed in Section 2, the previous theorem presents a necessary and sufficient condition for the identification of the parameters of the AB-model that only depends on the number and kind of restrictions imposed. Being independent of the estimated parameters, the condition can be checked before the estimation process and, when satisfied, guarantees that the SVAR model will be locally identified almost everywhere in the parametric space.

[9] As before, if $v_1 = v_2 = \ldots = v_{(n^2+m)} = 0$, then the number of such new matrices will be q instead of $q+2$, and e_i will be the i-th column of the identity matrix I_q.

5 AB-models: two examples

In this section we present two examples on the implementation of the previously developed strategy to study identification in the general AB-model. The former is an artificial bivariate model of some theoretical interest; the latter is a trivariate SVAR model presented by Blanchard and Perotti (2002) in a well-known empirical application.

5.1 Bivariate AB-model

Consider the bivariate AB-SVAR model in Eq. (18) with

$$A = \begin{pmatrix} 1 & \theta_1 \\ -\theta_1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_3 \end{pmatrix}. \tag{30}$$

The non-linear restrictions implicitly imposed are given by

$$\begin{aligned} \Sigma &= A^{-1}BB'(A')^{-1} \\ &= \frac{1}{\left(1+\theta_1^2\right)^2} \begin{pmatrix} \theta_2^2 + \theta_1^2\theta_3^2 & -\theta_1\theta_2^2 + \theta_1\theta_3^2 \\ -\theta_1\theta_2^2 + \theta_1\theta_3^2 & \theta_3^2 + \theta_1^2\theta_2^2 \end{pmatrix}. \end{aligned}$$

Global identification dictates that a unique solution exists to the non-linear system of three equations in three unknowns connecting the empirical moments in Σ to the parameters θ_1, θ_2 and θ_3. It is interesting to note that the Rubio-Ramirez, Waggoner, and Zha (2010) approach for checking for global identification cannot be applied to AB-SVAR models. It is true that this model could be turned into a C-model, but then identification would rely on a non-linear, cross-equation constraint, which is also impossible to handle in Rubio-Ramirez, Waggoner, and Zha's approach.

Local identification, instead, can be checked by using the result in Theorem 5, when considering the set of restrictions given by

$$R_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad r_a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad R_b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad r_a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$S_a = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad s_a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad S_b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad s_a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Checking for the rank of all combinations of indices as discussed in Theorem 5 shows that the model is locally identified. However, if one imposes the further restriction that $\theta_2 = \theta_3$, then the relation between the empirical moments and the parameters becomes

$$\Sigma = \begin{pmatrix} \frac{\theta_2^2}{1+\theta_1^2} & 0 \\ 0 & \frac{\theta_2^2}{1+\theta_1^2} \end{pmatrix}$$

that is similar to the C-model investigated in Section 3.1. As expected, although the number of restrictions is larger than required, the Rado condition fails and the model is not identified.

5.2 Blanchard-Perotti AB-model

The following model is inspired by the seminal contribution by Blanchard and Perotti (2002) on the effects of fiscal shocks to the real economy. They consider a three-dimensional VAR for taxes, spending and GDP, all in real per capita terms. Once concentrating out the reduced-form parameters, the model can be written as

$$\begin{pmatrix} 1 & 0 & -a_1 \\ 0 & 1 & -b_1 \\ -c_1 & -c_2 & 1 \end{pmatrix} \begin{pmatrix} t_t \\ g_t \\ x_t \end{pmatrix} = \begin{pmatrix} a_3 & a_2 & 0 \\ b_2 & b_3 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} \varepsilon_t^t \\ \varepsilon_t^g \\ \varepsilon_t^x \end{pmatrix} \tag{31}$$

where $\varepsilon_t = \left(\varepsilon_t^t\ ,\ \varepsilon_t^g\ ,\ \varepsilon_t^x\right)'$ is the vector of reduced-form residuals while $u_t = \left(u_t^t\ ,\ u_t^g\ ,\ u_t^x\right)'$ is the vector of uncorrelated unit-variance structural shocks. The model is clearly not identified as the number of parameters to estimate exceeds the number of distinct elements in the reduced-form covariance matrix ($9 > \frac{n(n-1)}{2} = 6$). For the order condition to hold, then, at least three more restrictions must be imposed.

Starting from the specification in Eq. (31), here below we consider alternative combinations of further restrictions (more or less backed by economic intuition) and, for each of these, check for the identification of the model using the necessary and sufficient condition developed in Theorem 5. All the results are confirmed by the traditional Amisano and Giannini's approach by calculating the rank of the matrix in Eq. (22) for randomly generated A and B. A horizontal bar over a parameter, as in $\bar{a}_1$, means that the parameter is calibrated to some non-zero value, usually on the basis of previous studies.

5.2.1 Further restrictions I

$$b_1 = 0 \quad a_1 = \bar{a}_1 \quad b_2 = 0$$

or alternatively

$$b_1 = 0 \quad a_1 = \bar{a}_1 \quad a_2 = 0$$

This is the specification used by the authors in their empirical application. In particular, they justify the choice of $b_1 = 0$ by saying that it is not possible to identify any automatic feedback from economic activity to government spending. Moreover, the elasticity to output of net taxes (tax minus transfer) can be calibrated by using external information and thus considered as fixed within the SVAR framework. Concerning the last restriction, the authors state that there are no convincing ways to establish *a priori*, after a simultaneous and unexpected change in taxes and expenditure, which of the two variables, taxes and public spending, reacts first. For this reason they present both the results with $b_2 = 0$ or alternatively $a_2 = 0$.

These three further restrictions satisfy the necessary order condition of identification, being $p = p_a + p_b = 7 + 5 = 12$ equal to $n^2 + \frac{n(n-1)}{2} = 9 + 3 = 12$. Moreover, using the Rado condition defined in Eq. (27), it is possible to show that, independently of the free parameters, the model is almost everywhere identified. In fact, given the restrictions $b_2 = 0$ and $a_1 = \hat{a}_1$, the *cyclically adjusted* tax and public spending, $\hat{t}_t = t_t - \hat{a}_1 x_t$ and $\hat{g}_t = g_t - b_1 x_t$ respectively, can be used as instruments to estimate c_1 and c_2 in a regression of x_t on t_t and g_t. Furthermore, the consistently estimated $\hat{t}_t$ and $\hat{g}_t$ allow to estimate the remaining two parameters a_2 and b_1, provided that, in turn, one of the two is restricted to zero. Finally, the diagonal elements of the B matrix are the standard deviations of the estimated structural shocks $u_t = \left(u_t^t \ , \ u_t^g \ , \ u_t^x \right)'$.

5.2.2 Further restrictions II

$$b_1 = 0 \quad a_1 = \overline{a}_1 \quad c_1 = 0$$

or alternatively

$$b_1 = 0 \quad a_1 = \overline{a}_1 \quad c_2 = 0$$

This set of restrictions, differently from the previous one, imposes that one of the two elasticities of output to taxes and public spending is null, i. e. that output does not respond, within the quarter, to changes in the level of taxes or spending. Moreover, we consider that both g_t and t_t react to unexpected shocks in taxes and spending u_t^t and u_t^g, respectively.

As before, the number of restrictions satisfies the necessary order condition. Looking at the sufficiency, instead, the Rado condition is now not supported by the set of restrictions, indicating that the model is not identified. In fact, although $b_1 = 0$ and $a_1 = \hat{a}_1$ help consistently estimating c_1 and c_2, this does not allow to consistently estimate a_2 and b_2.

5.2.3 Further restrictions III

$$b_1 = 0 \quad a_1 = \overline{a}_1 \quad a_2 = b_3$$

In this case, the further restrictions, other than, as before $b_1 = 0$ and $a_1 = \hat{a}_1$, impose that g_t and t_t react to u_t^g through coefficients of the same magnitude. This would be a very interesting case to analyse from an economic point of view because it would describe an economy that, for some reason, follows a policy of automatic stabilizing certain budget items. For example, during the recent financial crisis, the Italian budget contained certain tax increases (mostly excises), that were supposed to be applied automatically and conditionally to a certain deficit target not being met. This was mainly conceived as a way to reassure the markets about the commitment by the Italian government to fiscal discipline, and was repeatedly interpreted as such by Italian and EU authorities in public statements.

From a strictly econometric point of view, this restriction is of particular interest because it involves coefficients in two different equations; this type of constraint cannot be handled in the framework proposed by Rubio-Ramirez, Waggoner, and Zha (2010), in which cross-equation restrictions are not considered. And yet, in this scenario, a cross-equation restriction such as $a_2 = b_3$ would be very a natural and direct way to translate an institutional feature into a parametric restriction.

The necessary order condition continues to hold, but differently from the previous case, now the Rado condition holds too, so the model is locally identified. This last cross-equation restriction, apart from being interesting from an economic point of view is in fact indispensable for identifying all the parameters of the model.

6 Conclusions and future directions

In this paper, we have developed a method for checking whether a given SVAR is identified on the pure basis of the restrictions the covariance matrix is subject to, independently of its parameters. Therefore, it is possible to investigate identification prior to estimation. Our analysis is not limited to C-models, but also extends to the more general AB model.

A very interesting research avenue lies in investigating qualitatively the possible under-identification issues that derive from the relationship between observables and parameters; in the example we gave in subsection 2.2, lack of identification was directly linked to the absence of information contained in one of the moment conditions (3). It may be possible to derive a formal criterion for detecting such cases by considering which one, of the set of inequalities (15), fails to hold. The example given in subsection 3.4 looks quite promising in that respect.

A more difficult open question is whether our theorem can be combined with extra conditions for checking global identification, rather than local. This will be the object of future work, as well as the usage of the results presented here for dealing with the heteroskedastic case and make the analysis in Bacchiocchi (2014) more general and rigorous.

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7 Appendix

In order to facilitate the understanding of the proof of Theorem 2 it is useful to restate Rado's (1942) theorem, in the form employed in Johansen (1995):

Theorem 6. *Suppose you have a collection of matrices $M_1, M_2, \ldots, M_n$; then, in order to obtain a set of linearly independent vectors $v_1, v_2, \ldots, v_n$, where $v_i = M_i \lambda_i$, a necessary and sufficient condition is that for all $k = 1, \ldots n$ and all sets of indices $1 \leq i_1 < ... < i_k \leq n$,*

$$\text{rk}\left[M_{i_1}|\ldots|M_{i_k}\right] \geq k.$$

This theorem provides a necessary and sufficient condition for having n linearly independent vectors from the set of matrices $M_1, M_2, \ldots, M_n$. Put in a different way, if the theorem holds, then there exist some λ_1, $\lambda_2, \ldots$, λ_n such that $v_1, v_2, \ldots, v_n$ will be linearly independent. By construction, however, our problem in Section 2.4 fixes all the λs to be equal. This *apparently simple* difference, however, is crucial for the Rado condition to be no more sufficient in our context.