First and Second Sound in Cylindrically Trapped Gases

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We investigate the propagation of density and temperature waves in a cylindrically trapped gas with radial harmonic confinement. Starting from two-fluid hydrodynamic theory we derive effective 1D equations for the chemical potential and the temperature which explicitly account for the effects of viscosity and thermal conductivity. Differently from quantum fluids confined by rigid walls, the harmonic confinement allows for the propagation of both first and second sound in the long wavelength limit. We provide quantitative predictions for the two sound velocities of a superfluid Fermi gas at unitarity. For shorter wavelengths we discover a new surprising class of excitations continuously spread over a finite interval of frequencies. This results in a nondissipative damping in the response function which is analytically calculated in the limiting case of a classical ideal gas.

Superfluids are known to exhibit, in addition to usual sound, an additional mode, called second sound, where the superfluid and normal components move with opposite phase [1]. In a weakly compressible fluid, like superfluid helium, second sound reduces to a temperature wave, leaving the total density practically unaffected. In helium its velocity was systematically measured as a function of temperature, providing a high precision determination of the superfluid density. First measurements of second sound were recently reported also in dilute Bose-Einstein condensed gases confined in elongated traps [2]. In this Letter we show that the propagation of sound in a nonuniform trapped gas exhibits new interesting features both in the superfluid and in the normal phase. We will focus on highly elongated configurations where the radial harmonic trapping provides the relevant nonuniformity. These configurations are well suited for the experimental excitation and detection of sound waves [3–5]. We will consider the usual hydrodynamic regime $l \ll \lambda$ where l is the mean free path and λ is the wavelength of the sound wave. We will also consider the condition of strong radial confinement $R_\perp \ll \lambda$ where $R_\perp$ is the radial size of the sample.

The above conditions are compatible with two different regimes, depending on the ratio between the viscous penetration depth $\delta = \sqrt{\eta/\rho_n \omega}$ and the radius $R_\perp$ where η is the shear viscosity coefficient, ρ_n is the normal density and ω is the frequency of the sound wave. Let us first consider the case of a uniform fluid confined by the hard walls of a tube. If $\delta \ll R_\perp$ viscosity plays an unimportant role and sound propagates similarly to the case of bulk matter. If instead $\delta \gg R_\perp$, the viscosity imposes the uniformity of the normal velocity field as a function of the radial coordinate. Since friction further requires that the normal velocity be zero on the walls, the normal part of the fluid cannot move at all along the tube and only the superfluid can. This mode is known as 4th sound [6]. In the case of a harmonically trapped gas the two regimes

exhibit new features whose investigation is the object of the present paper.

Our analysis is based on the two-fluid hydrodynamic equations (HD) [7]

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \qquad m\partial_t \mathbf{v}_s = -\boldsymbol{\nabla}(\mu + U), \quad (1)$$

$$\partial_t j_i + \partial_i P + \rho \partial_i U/m = \partial_k(\eta \Gamma_{ik}), \quad (2)$$

$$\partial_t s + \boldsymbol{\nabla} \cdot (s\mathbf{v}_n) = \nabla \cdot (\kappa \nabla T/T), \quad (3)$$

which reduce, above T_C, to the classical equations of hydrodynamics. In the above equations $\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$ is the current density, ρ_s and ρ_n are the superfluid and normal mass densities for a fluid with total mass density $\rho = \rho_s + \rho_n$, $\mathbf{v}_s$ and $\mathbf{v}_n$ are the corresponding velocity fields, $\Gamma_{ik} = (\partial_k v_{ni} + \partial_i v_{nk} - 2\delta_{ik}\partial_j v_{nj}/3)$, s is the entropy density, while P and μ are, respectively, the local pressure and chemical potential. U is the external trapping potential which will be assumed of axial harmonic form: $U = m(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2)/2$ with $r_\perp^2 = x^2 + y^2$ and $\omega_z \ll \omega_\perp$. Since we are interested only in the linear solutions, in the above equations we have omitted terms quadratic in the velocity. Finally η and κ are, respectively, the shear viscosity and the thermal conductivity (we have not considered bulk viscosities terms which give smaller contributions). In a uniform fluid the effect of viscosity and thermal conductivity is irrelevant in the long wavelength limit, providing only higher order corrections to the dispersion law. In the presence of confinement their effect is instead crucial. This opens new perspectives to explore the behavior of viscosity and thermal conductivity in interacting trapped gases. The two-fluid hydrodynamic equations, in the dissipationless regime ($\eta = \kappa = 0$), have been already the object of numerical investigations [8–10] in the presence of harmonic confinement. These works, so far limited to isotropic trapping, have provided predictions

for the discretized frequencies in the superfluid phase with special emphasis to the hybridization effects between the first and second sound solutions.

In this Letter we are interested in the solutions of the hydrodynamic equations satisfying the condition $\omega \ll \omega_\perp$. As a consequence of the tight radial confinement, Eq. (2) for the current implies the important condition $\nabla_\perp P + \rho \nabla_\perp U = 0$ of mechanical equilibrium along the radial direction. Violation of this condition would in fact result in frequencies of the order of $\omega_\perp$. The tight radial confinement also implies that the radial component of the velocity field must be much smaller than the longitudinal one.

Low frequency regime.—In the presence of harmonic trapping the condition $\delta \gg R_\perp$ of large viscous penetration depth is equivalent to requiring the low frequency condition $\omega \ll \omega_\perp^2 \tau$ where τ is a typical collisional time.

Excluding $\mathbf{v}_s$ from Eqs. (2) and (1) and using the thermodynamic identity $dP = s dT + \rho d\mu/m$ one can write the equation for the relevant z component of the velocity field of the normal component in the form

$$m\rho_n \partial_t v_n^z + \rho_n \partial_z(\delta\mu) + ms\partial_z(\delta T) = m\nabla \cdot (\eta \nabla v_n^z), \quad (4)$$

where we have ignored terms containing the small radial components of the velocity field.

The presence of viscosity in Eq. (4) results in the independence of v_n^z on the radial coordinate $r_\perp$. In fact violation of such a behavior would be incompatible with the low frequency condition $\omega \ll \omega_\perp^2 \tau$. It is worth noticing that, differently from the case of the tube with hard walls discussed in the introduction, for harmonic trapping the uniformity of the velocity field does not stop the motion of the normal component. Analogously, the presence of thermal conductivity in Eq. (3) for the entropy results in the independence of the temperature fluctuations on $r_\perp$ in the same low frequency limit [11]. This in turns implies that also the fluctuations of the chemical potential will be independent of the radial coordinate. This follows from the radial mechanical equilibrium condition and the use of the thermodynamic identity $dP = s dT + \rho d\mu/m$. Thus in the low frequency limit both the fluctuations δT and $\delta\mu$ are independent of the radial coordinates.

It is now convenient to integrate radially both the equations of continuity for the density and for the entropy (1) and (3) as well as Eq. (4) for v_n^z, profiting of the fact that the large terms containing the derivatives with respect to $r_\perp$ vanish due to Gauss theorem, and that those containing the second derivative with respect to z can be ignored since they give rise to higher order corrections in the dispersion law. Integration of Eq. (4) then gives $-m\langle\rho_n\rangle\partial_t v_n^z = m\langle s\rangle\partial_z \delta T + \langle\rho_n\rangle\partial_z\delta\mu$, where the symbol $\langle\ldots\rangle \equiv \int\ldots dx dy$ stands for the integral on the transverse variable $r_\perp$. The above equation for v_n^z and the analog equation $m\partial_t v_s^z = -\partial_z\delta\mu$ for the superfluid component can be used to derive the following coupled equations for the averaged density and entropy fluctuations:

$$-\partial_t^2\delta\langle\rho\rangle + \partial_z(\langle s\rangle\partial_z\delta T) + \partial_z(\langle\rho\rangle\partial_z\delta\mu)/m = 0, \quad (5)$$

$$-\partial_t^2\delta\langle s\rangle + \partial_z(\langle s\rangle^2/\langle\rho_n\rangle\partial_z\delta T) + \partial_z(\langle s\rangle\partial_z\delta\mu)/m = 0. \quad (6)$$

One can further write $\delta\langle\rho\rangle = \partial_\mu\langle\rho\rangle\delta\mu + \partial_T\langle\rho\rangle\delta T$ and $\delta\langle s\rangle = \partial_\mu\langle s\rangle\delta\mu + \partial_T\langle s\rangle\delta T$. The spatial dependence of the thermodynamic quantities entering the above equations is fixed by the trapping potential U through the local density approximation equilibrium condition $\mu(\rho, T) + U = \mu_0$, with μ_0 fixed by the normalization condition for ρ. In the presence of axial trapping ($\omega_z \neq 0$), Eqs. (5) and (6) allow for the calculation of the low frequency discretized states of the system. In the following we will assume $\omega_z = 0$ (cylindrical geometry) in order to calculate the velocity of the sound waves propagating along the axial direction. The solutions for $\delta\mu$ and δT have the t and z dependence $\propto e^{i(qz-\omega t)}$ with $\omega = cq$, yielding the most relevant equation

$$c^4[m\partial_\mu\langle\rho\rangle\partial_T\langle s\rangle - (\partial_T\langle\rho\rangle)^2] + c^2[2\partial_T\langle\rho\rangle\langle s\rangle - \langle\rho\rangle\partial_T\langle s\rangle$$
$$- m\partial_\mu\langle\rho\rangle\langle s\rangle^2/\langle\rho_n\rangle] + \langle\rho_s\rangle\langle s\rangle^2/\langle\rho_n\rangle = 0, \quad (7)$$

which generalizes the well-known Landau equation [1,7] for the first and second sound velocities to the case of a cylindrically trapped gas.

Above T_C, where the superfluid density vanishes, Eq. (7) admits only one solution with velocity different from zero. In this case the value of c can be more conveniently calculated by direct radial integration of the HD Eqs. (1)–(3). In the low frequency limit the equations of continuity for the density and the entropy yield the 1D isoentropic condition $\partial_t(\langle s\rangle/\langle\rho\rangle) = 0$ and one immediately finds the result $c^2 = (\partial\langle P\rangle/\partial\langle\rho\rangle)_{\langle s\rangle/\langle\rho\rangle}$, which generalizes the most famous expression for the adiabatic sound velocity of uniform matter as well as the zero temperature result $mc^2 = (\partial_\mu\langle\rho\rangle/\langle\rho\rangle)^{-1}$ derived in [12]. At unitarity, as well as for the ideal gas, where s/ρ depends on the combination $\rho T^{-3/2}$, the new adiabaticity condition takes the form $\langle\rho\rangle T^{-5/2} = $ const. A nontrivial implication concerns the sound velocity in the ideal classical limit where one finds $c^2 = (7/5)(T/m)$ rather than the value $(5/3) \times (T/m)$ holding in uniform gases. Here and in the following we put the Boltzmann constant $k_B = 1$.

Before discussing the solutions of Eq. (7) in the presence of harmonic trapping, it is useful to calculate the dynamic response function of the system to an external potential of the form $\lambda e^{i(qz-\omega t)}$. The response function is determined by the density fluctuations induced by the external potential according to $\delta\langle\rho\rangle = \lambda\chi(q, \omega)e^{i(qz-\omega t)}$. Its calculation is important because it provides information on the actual possibility of exciting the two sound waves using a density probe. The inclusion of the external perturbation affects both the equation for the current and the equation for $\mathbf{v}_s$. Calculations yield the following result for the imaginary part of the response function

$$\text{Im}\chi(q,\omega) = -\langle\rho\rangle\frac{\pi}{2}\{W_1\omega[\delta(\omega - c_1q) + \delta(\omega + c_1q)]$$
$$+ W_2\omega[\delta(\omega - c_2q) + \delta(\omega + c_2q)]\}, \quad (8)$$

where the weights W_1 and W_2 obey the relationships $mc_1^2W_1 + mc_2^2W_2 = 1$ and $W_1 + W_2 = (\partial_\mu\langle\rho\rangle/\langle\rho\rangle)_T$ determined, respectively, by the f- and the isothermal compressibility sum rules. The knowledge of $\text{Im}\chi$ is relevant for experiments based on the propagation of density pulses and can be measured directly in Bragg scattering experiments [13,14].

We are now ready to calculate the values for the two sound velocities and the relative weights in the density response. We will specialize to the case of the unitary Fermi gas [15] since in this case the collisional regime, needed to apply hydrodynamic theory, is more easily achieved due to the large value of the scattering length. At unitarity the effects of the interactions in the thermodynamic functions can be expressed in a universal way [16] in terms of the dimensionless parameter $u \equiv \mu/T$. Thus the pressure can be written as $P(\mu, T) = T^{5/2}H(u)$ where the function $H(u)$ can be determined through microscopic many-body calculations [17–19] or, in some ranges of temperature, directly extracted from experiment [20]. All the thermodynamic functions can be expressed in terms of the function H. For example the density takes the form $\rho(\mu, T) = mT^{3/2}\nu(u)$ with $\nu(u) = H'(u)$, the entropy density is given by $s = 5P/2T - u\rho/m$, etc. Only the superfluid mass density $\rho_s = mT^{3/2}\nu_s(u)$ requires the knowledge of another independent function for which we use the regularized form introduced in [21].

The radially integrated quantities entering the dispersion relation (7) can also be easily calculated. Since in the presence of radial trapping the chemical potential has the radial dependence $\mu(\mathbf{r}) = \mu_0 - m\omega_\perp^2 r_\perp^2/2$ the integration over $dxdy$ can be usefully transformed into integrals over u. For example, we find $\langle\rho\rangle(\mu_0, T) = (2\pi T^{5/2}/\omega_\perp^2) \times \int_{-\infty}^{\mu_0/T} \nu(u)du$ and analogously for the other quantities. In conclusion, all the coefficients of the dispersion relation (7) can be calculated as a function of T and of the dimensionless variable $u_0 = \mu_0/T$. In practice, in order to determine the function $H(u)$ we have used a fit to the experimental data of [20] above the critical temperature T_C and assumed that, below T_C, the pressure is constant as a function of T, following the analysis in the same reference. The results for the sound velocities c_1 and c_2 and for the relative weights W_2/W_1 [see (8)] are reported in Fig. 1 for the unitary Fermi gas as a function of T/T_C in the relevant interval $0.5T_C < T < T_C$, where $T_C \approx 0.19T_F$, having defined the Fermi temperature for our cylindrical geometry as $T_F = \left(\frac{15\pi}{4}\left(\frac{\hbar^2}{2m}\right)^{3/2}\omega_\perp^2\langle\rho\rangle\right)^{2/5}$ [22]. In Fig. 2 we show the ratio between the relative density ($\delta\langle\rho\rangle/\langle\rho\rangle$) and temperature ($\delta T/T$) fluctuations in the two modes. The ratio provides a physical insight on the nature of the modes. In the first sound solution (high velocity mode) the relative variations of density are larger than the ones of temperature
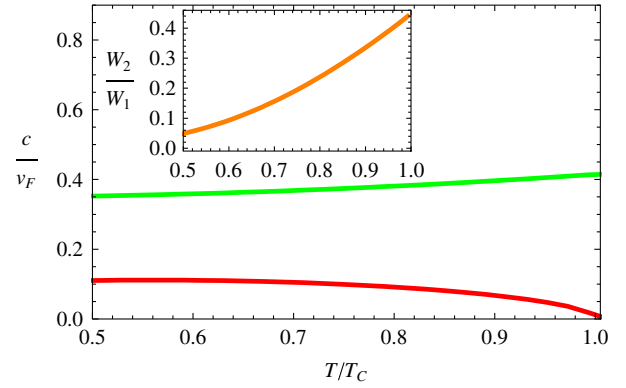

FIG. 1 (color online). First (green) and second (red) sound calculated using Eq. (7), as a function of T/T_C, normalized to the Fermi velocity $v_F = \sqrt{2T_F/m}$. The inset reports the ratio W_2/W_1 between the weights of the two sounds modes in $\text{Im}\chi$ [see Eq. (8)].

and have the same sign. The opposite happens for second sound. The value of the ratio W_2/W_1 is smaller than the value found for a uniform fluid [13] and this is due to the reduction of the superfluid component as one moves from the symmetry axis of the trap. We have also checked that our predictions are not very sensitive to the fitting procedure used to calculate the thermodynamic functions.

High frequency regime.—We have so far discussed the solutions of the hydrodynamic equations in the low frequency limit. In the opposite limit $\omega \gg \omega_\perp^2\tau$ of large frequencies viscosity and thermal conductivity can be ignored [23]. In this limit, still compatible with the condition $\omega \ll \omega_\perp$, the solutions are also affected by the presence of the radial confinement, but in a different way. In particular the fluctuations of the temperature and of the chemical potential are no longer independent of the radial coordinate, except in the superfluid region [24]. This limit is still compatible with the usual hydrodynamical condition $\omega\tau \ll 1$ only if $\omega_\perp\tau \ll 1$, a condition which is rather severe from the experimental point of view.

An explicit and instructive solution in the dissipationless regime is available above T_C where the superfluid component is absent. We will consider the extreme regime of the ideal gas where all the thermodynamic quantities are
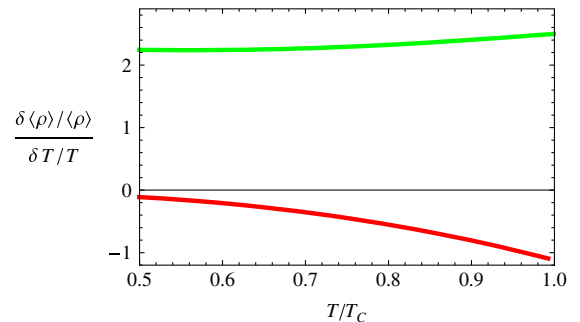

FIG. 2 (color online). Ratio between the relative temperature ($\delta T/T$) and density ($\delta\langle\rho\rangle/\langle\rho\rangle$) fluctuations in the two sounds as a function of T/T_C (see Fig. 1 for the notation).

known explicitly. In this limit the classical equations of hydrodynamics can be recast in the form of an equation for the velocity field [25]:

$$- m\omega^2 \mathbf{v} = \tfrac{5}{3} T \nabla[\nabla \cdot \mathbf{v}] - \nabla[\mathbf{v} \cdot \nabla U] - \tfrac{2}{3}[\nabla \cdot \mathbf{v}]\nabla U. \quad (9)$$

Solutions of this equations for 3D harmonic traps were considered in [26,27]. The advantage of using 1D cylindrical configurations is due to the availability of plane wave solutions along the z direction, providing new physical insight on the effects of trapping on sound propagation. It was found in [28] that an exact solution of this equation in the presence of cylindrical radial trapping ($U = m\omega_\perp^2 r_\perp^2/2$) is given by $v_z \propto e^{\xi^2/5} e^{iqz}$, $v_\perp = 0$, where $\xi^2 = m\omega_\perp^2 r_\perp^2/T$, and is characterized by the adiabatic velocity of sound $mc_a^2 = 5T/3$. This solution is quite different from the one holding in the low frequency regime discussed in the first part of the Letter where v_z does not depend on the radial coordinate. Also the temperature and the chemical potential fluctuations exhibit a strong radial dependence as a consequence of the radial trapping and the absence of thermal conductivity and viscosity. From the explicit knowledge of the velocity field and the orthogonality condition $\int \rho \mathbf{v}_{(i)}^* \cdot \mathbf{v}_{(j)} dx dy = 0$ if $\omega_i \neq \omega_j$ for the solutions of the HD equations [9], it is possible to calculate the contribution of the adiabatic sound to the dynamic response function. Using the expression $\rho \propto e^{-\xi^2/2}$ for the density distribution at equilibrium one obtains, after straightforward algebra, the result

$$\chi_a(q, \omega) = \frac{5}{9} \frac{\langle \rho \rangle}{m} \frac{q^2}{q^2 c_a^2 - \omega^2}, \quad (10)$$

showing that the adiabatic mode does not exhaust the f-sum rule $\chi(\omega \to \infty) = -q^2 \langle \rho \rangle/m\omega^2$, differently from the case of a uniform gas, and hence revealing that additional solutions of the hydrodynamic Eq. (9) should exist. This equation actually admits an additional class of low frequency solutions lying in the continuum and consequently exhibiting damping. On the $\omega - q$ plane these solutions occupy the region $\omega/q < \sqrt{24/25} c_a \equiv c_0$. The velocity field is given by the expressions $v_z = A e^{\xi^2/4}[\sigma \cos(\sigma \xi^2) - \sin(\sigma \xi^2)/20] e^{iqz}$, where $\sigma = \sqrt{c_0^2 q^2/\omega^2 - 1}/4$ is a continuous parameter, and $v_\perp = B e^{\xi^2/4} \sin(\sigma \xi^2) e^{iqz}/\xi$, with $B = -iA\sqrt{3/20}[(c_a^2 q^2 - \omega^2)/q c_a \omega_\perp]$. We present here the result for the imaginary part of the dynamic response function in the region of the continuum

$$\text{Im} \chi_c(q, \omega) = \frac{4}{3} \frac{\langle \rho \rangle}{mc_0^2} \frac{\omega\sqrt{c_0^2 q^2 - \omega^2}}{c_a^2 q^2 - \omega^2}. \quad (11)$$

It is easy to show that inclusion of the continuum allows for a complete fulfillment of both the compressibility and the f-sum rules. The function $\text{Im}\chi_c$ exhibits a sharp maximum at $\omega/q \approx 0.98 c_0$. Thus the new excitation can be considered as the analog of second sound above T_C. It is worth emphasizing that the damping associated with the continuum is not a consequence of dissipative effects as happens in uniform

fluids above T_C, but a result of the energy leakage from the gas due to the increase of the velocity field on large distances. The existence of a continuous spectrum of excitations and the value of the limiting velocity c_0 are universal properties holding also at lower temperatures, being fixed by the asymptotic behavior at large $r_\perp$ where the gas is classical and ideal.

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