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Endogenous Fluctuations in Macroeconomics: the Role of Heterogeneity

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Chapter 1

The Adaptive Belief System: theoretical applications and empirical validations

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1.1 Introduction

The purpose of this survey is to highlight the fundamental contribution that heterogeneous expectations and the adaptive belief system may bring to the explanation of economic phenomena if systematically used in economic models. Before describing these two approaches in detail, this introduction will illustrate the reasons why they should be preferred to the standard assumption of homogeneous rational expectations.

The psychological human element is probably the key difference between economics and the natural science. Individual beliefs about the future affect the decision of today, so any dynamic economic system can be defined as an expectation feedback system.

For this reason the expectations on aggregate variables are the cornerstone of several economic models. Since the seminal work of Muth (1961) and its application in macroeconomics by Lucas (1972), the Rational Expectations Hypothesis (REH) has become the leading approach on expectations formation in economics. According to REH, expectations are model consistent, agents are optimizers and have perfect knowledge of the market equilibrium equations which are used to derive expectations. REH offers an efficient fixed point solution to the economic expectations feedback system. Hence, in the absence of exogenous shocks, the REH implies that agents have perfect foresight and make no mistakes in their predictions. An important contributions to the popularity of this framework have been the Lucas critique.[1]

This asserts that policy prescriptions are likely to affect expectations. Moreover, policy conclusions based on statistical relationships are potentially misleading as do not take into account the changes in the decision rules that may influence the aggregate structure.

In finance, the REH is related to market efficiency[2]: asset price and returns are the outcomes of a competitive market with rational agents. The efficiency of the market is the result of the arbitrage strategies of agents exploiting every profit opportunity to drive the prices to their correct and fundamental values. Consequently, in an efficient market there cannot be foreseeable structure in asset returns and the value of a risky asset should be set at its fundamental price.

Nevertheless, drawbacks of the rational expectations/optimizing agents paradigm are widely recognized. First of all, it is not realistic to assume perfect and widespread knowledge about the law of motion of the variables of interest. Moreover, agents should have extremely strong computing abilities in order to calculate

[1] See Lucas (1976).
[2] See Fama (1965,1970).

the equilibrium. This aspect becomes crucial if it is applied to heterogeneous agents models, where rational agents are expected to have a full knowledge of the market, including the expectations of all the other agents.

The standard reply to this criticism is based on the principle that this is the asymptotic outcome of a learning process concerning the above mentioned law of motion. In other words, considering a fast learning process, the agents are able to converge to the rational solution. Accordingly, it becomes fundamental to study dynamic models that take explicitly into account this learning process in order to investigate whether a convergence to the rational equilibrium is feasible. These conditions seem to have been proved by Fourgeaud et al. (1986), while Schonhofer (1998) suggests that the forecasting errors never vanish in a standard overlapping generations model with least squares learning finding non-convergence.

Secondly, in a world of rational agents there will be no trade.[3] Assuming that all the agents are rational and this condition is common knowledge, even if a trader has higher private information, e.g. on an asset, she cannot benefit from this because the other rational traders should be able to anticipate it. Therefore, they will not sell the asset to her.[4]

Moreover, the survival argument of the FH has been criticized by many authors[5] proposing formal models where non-optimizing firms can survive under certain hypothesis. Blume and Easley (1992) investigate the "informational efficiency" of the market finding that if some agents have inaccurate beliefs, dynamics may not converge to the rational expectations equilibrium allowing

[3]The Friedman hypothesis (Friedman: 1953) suggests that, in an evolutionary competition, non-rational agents will be driven out of the market by rational agents who are capable to avoid systemic errors. So, the Friedman hypothesis (FH) excludes from the economic modeling the "market psychology" and the "sentiment of the market" that are considered irrational and therefore inconsistent with the rational expectation hypothesis.

[4]For an exhaustive discussion about "no trade theorems" see Milgrom and Stokey (1982) or Fudenberg and Tirole (1991).

[5]See for example Winter (1975), Machina (1989) or Koopmans (1957).

the misinformed traders to survive in the market.

Looking back at the practical relevance of REH in financial markets, Soros (1987) introduces the concepts of fallibility and reflexivity to argue that the standard paradigm of rationality is not the description of the market reality. Fallability means that the agents have a view of the world that never perfectly corresponds to the actual state, whereas reflexivity means that these imperfect views influence but not determine the course of the events. Conversely the course of the events influences but does not determine the participants' views. So, there is a continuous feedback loop between imperfect view and state of the events. These concepts find validation both in laboratory and survey results.

Experimental sessions have shown that often the aggregate behaviour does not converge to the rational expectations equilibrium and bubbles can emerge.[6] Anticipating here some of the results presented in Sections 1.4, Hommes et al. (2005) investigate expectations formation in a standard asset pricing model asking to the subjects to predict the price and leaving the trade decisions to an artificial agent (computer). The human subjects do not have knowledge on the underlying market equilibrium equations but they know their predictions and all past realized prices. In this framework, the authors observe both slow or monotonic convergence to the fundamental price and regular oscillations around it. Moreover, in most of the markets there is excess volatility and the assets are under-valuated. Looking at the individual forecast strategies, Hommes et al. find that in the stable market the agents use naive, adaptive or AR(1) forecasting strategies, whereas in the oscillatory markets the majority of the agents acts according to trend-following strategies. Thus this paper seems to confirm the common hypothesis that in finance agents use simple rules of thumb. In a similar framework which involves human agents only, Hommes et al. (2008) observe the emergence of bubbles in 5 over 6 experiments. These bubbles are triggered by

[6]See for example Smith et al. (1988), the chapter of Duffy (2006) or Hommes (2011).

trend-following behaviours with positive feedback expectations. This feedback structure is based on: high expected prices leading to high level of demand and high market equilibrium prices. So, the agents follow the bubble and are not able to predict the fundamental price, contrary to the rational expectation assumption.

In the 1980's, the strong depreciation experienced by the dollar[7] and the large fluctuations in the S&P500 index[8] have been considered evidences of financial markets excess volatility.[9] In addition, survey data analyses[10] show that the financial agents' behaviour is not fully rational, since they use different trading and forecasting strategies.

Over the last decades many economists have become aware of the unrealistic assumptions behind the rational expectations, developing alternative approaches based on bounded rationality. There are at least three reasons why bounded rationality should be incorporated in economic models, which can be summarized in just three words: evidence, scarcity and success.

Firstly, there is empirical evidence that human beings have critical limits on both cognition and computational capabilities. More precisely, many studies in behavioural economics and psychology display that agents fail in understanding statistical relationships or under/overestimate the data patterns producing reasoning errors that are typically systematic. These systematic errors are often modelled as expectations with bias around the steady state value (SS). Moreover, it is interesting to notice how the bias is not a complete departure from the rational approach, indeed, its magnitude and nature could be related to the economic conditions in the maximization problem, like deliberation cost, incentive or learning process. A vast literature has investigated the

[7] Frankel and Froot (1986).
[8] Cutler, Poterba and Summers (1989).
[9] See Shiller (1981,1989).
[10] See Frankel and Froot (1987 a,b and 1990 a,b)

bias problem suggesting that biases do not easily disappear and can have strong consequences.[11]

Secondly, given that Economics studies choices in a scarce resource framework to satisfy unlimited wants, it is crucial to consider human cognition as a scarce resource and here bounded rationality makes the difference. Models should take seriously into account the cost of the decision process, especially if it implies high calculation efforts. On this point, Conlisk (1996) highlights the infinite regress problem: "How can we formulate an optimization problem which takes full account of the cost of its own solution?". According to Johansen (1977), the answer is: "at some point a decision must be taken on intuitive grounds". In other words, deliberation costs and heuristics challenge the rational optimization as the ultimate logical basis for behavioural modelling.

Thirdly, models with bounded rationality provide interesting results in a wide range of economic problems. The use of heuristic rules (often called rules of thumb) is justified by psychological studies which show that agents compare alternatives avoiding deliberation efforts or excessively complicated computation costs.[12] Many economic models use these heuristics following an evolutionary and dynamic approach, explaining the persistent distance from the rational solutions. For example, according to the Adaptive Belief Systems (ABS) of Brock and Hommes (1997), the agents endogenously update their strategy between rational and naive expectations in compliance to the net profit. Hence, taking into account the population distribution and the learning process, the bounded rational models are also useful to investigate whether in a dynamic system there is convergence to perfectly rational solutions.

All this considered, bounded rationality can be modelled following a great variety of approaches. Indeed, if there is only one way to be rational, there

[11] Some examples are Smith (1991), Smith and Walker (1993) or Slonim (1994).
[12] See Conlisk (1980).

are million ways to make mistakes.[13] For example, Adam (2005) and Guse (2005) develop models with heterogeneous expectations, where agents use one of the possible rules of thumb. The main limitation of their approach is that the fractions of agents using the forecasting rules are determined exogenously and therefore they cannot choose the most performing rule.

Differently, in order to cope with this problem and with the wilderness of bounded rationality, the adaptive learning approach assumes that agents act as econometricians or statisticians using econometric forecasting models. The learning process is translated into updating model parameters over time. Following this approach, agents are aware of the underlying structural model but do not know the value of the parameters.[14] For example, Grandmont (1998) develops a model with adaptive learning related to the "uncertainty principle". According to this model, agents are willing to know the position of the equilibrium, they are able to extrapolate regularities and trends, but are uncertain on the system dynamics. The system converges to temporary unstable equilibrium when the expectations are strong enough.

Another interesting approach to bounded rationality is the behavioural learning based on the Restricted Perception Equilibrium[15] or the Misspecification Equilibrium. As in adaptive learning, agents are considered econometricians but it is recognized that in practice econometricians often misspecify the model. So, agents base their expectations on simple heuristics with parameters pinned down by simple requirements between beliefs and realizations. On the same line Branch and Evans (2005) assume that agents underparameterize the forecasting model neglecting a variable or a lag. Accordingly, the optimal parameters value of each misspecified frame depends on the proportion of agents using the different forecasting rules.

[13] See Sims (1980).
[14] For extensive surveys see Evans and Honkapohja (2001, 2011).
[15] Hommes and Zhu (2014).

Concluding, there is one other approach worth to be mentioned which is based on the assumption that agents do not know the correct model underlying the economic dynamics. In Brock and Hommes (1997) and Branch and Evans (2006) agents have heterogeneous expectations and endogenously switch between rules of thumb according to their relative performance. It is interesting to notice how this leads to complex expectations feedback systems that can produce either the REH equilibrium or, more often, self-fulfilling behavioural learning equilibria manifesting excess volatility and persistent discrepancies from the rational solutions.

This literature review aims to analyse one specific dimension of the bounded rationality with heterogeneous expectations: Adaptive Belief Systems models. These are indeed able to face the "wilderness" problem through the expectations updating mechanism and to generate some important stylized facts in many financial and economical series, such as unpredictable returns and fat tails.

Section 1.2 presents the most popular rules of thumb and the discrete choice model, while Section 1.3 presents their different applications in three domains: in a macroeconomic model, in the asset pricing model and in the exchange rate framework. In section 1.4 are discussed some empirical and experimental validations supporting the use of bounded rationality. Finally, section 1.5 concludes presenting the recent development of ABS in the literature.. The literature on bounded rationality is vast and growing, so I have decided to analyse few canonical examples in each section. Therefore, the inclusion or exclusion of specific papers should not be considered in qualitative terms.

1.2 Bounded rationality and heterogeneous expectations

The Adaptive Believe System (ABS) proposed by Brock and Hommes (1997, 1998, 1999) is based on expectations rules and the discrete choice models. In this framework agents are heterogeneous and switch between different rules according to an evolutionary performance measure.

Agents are bounded rational, in other words, in each period they choose the strategy with higher fitness measure. Brock and Hommes define the consequential coupling between dynamic equilibria and expectations as Adaptive Rational Equilibrium Dynamics (ARED). The heterogeneity of expectations among agents introduces non-linearity into the market dynamics and can be source of potential market instability fluctuations. The discrete choice mechanism generates a link among the market equilibrium dynamics and the evolution of the heterogeneous expectations which co-evolve over time. In other words, the market realization depends on the sentiment of the market and on the heuristics considered, i.e. the distribution of the agents among them.

This heterogeneous approach challenges the traditional rational agent framework because it is closely related to the Keynesian view claiming that "expectations matter". Consequently, the investors' sentiment and the market psychology play important roles in the market dynamics

In most of the heterogeneous expectations models agents can be rational, fundamentalists, biased, chartists and adaptive.

Considering the asset price as reference variable and the simple linear forecasting rule of the type:

$$p_t^e = p^* + f_h(x_{t-1}, x_{t-2}, x_{t-3}, ..., x_{t-L}),$$

where p^* represents the fundamental price and $f_h(\cdot)$ the forecasting rule selected by agent h. The perfect forecasting rule or the perfect rational rule can be written as follows:

$$p_t^e = p^* + (p_t - p^*) . \tag{1.1}$$

It should be noticed that Equation (1) assumes perfect knowledge of the market equilibrium. Particularly, this means that, in each period, in a heterogeneous framework the agent has full knowledge about the belief of all the others. Considering now the linear forecasting rule with one lag:

$$p_t^e = p^* + g\,(p_{t-1} - p^*) + b, \tag{1.2}$$

where g and b represent the trend and the bias parameter respectively. Despite its extreme simplicity, Equation (1.2) allows to study at least four interesting rules of thumb. For example, if g and b are both equal to zero, the linear rule is reduced to the fundamentalist forecast. Fundamentalists base their beliefs about future realization upon market fundamentals, so

$$p_t^e = p^*. \tag{1.3}$$

Other interesting cases covered by the forecasting rule (1.2) are the biased belief and the trend-follower behaviour:

$$p_t^e = p^* + b, \tag{1.4}$$

$$p_t^e = p^* + g\,(p_{t-1} - p^*) . \tag{1.5}$$

The simple biased rule of equation (1.4) represents every possible positive or negative constant price above or below the fundamental value. Hence, the biased rule may describe optimistic and pessimistic agents. In the trend-follower forecasting rule of Equation (1.5), if $g > 0$ it represents trend chaser agents expecting stability in the sign of price changes. Conversely, if $g < 0$ the heuristic describes contrarians expecting a reversal in the trend related to the last price change. Finally, if $g = 1$ it considers the standard naive agents who simply base their expectation on the past observed value.

Extending the lag in the forecasting rule and assuming that agents do not know the fundamental price, it possible to observe the following behaviour:

$$p_t^e = p_{t-1} + g\left(p_{t-1} - p_{t-2}\right). \tag{1.6}$$

Heuristic (1.6) describes technical analysts or chartists, who adopt simple trend extrapolating rule upon observed historical pattern in prices using the last observed price as an anchor. It is interesting to note how this rule is completely time varying, because it does not consider the fixed point of the fundamental value as anchor. This rule finds deep support both in laboratory experiments and in theoretical analyses.[16]

1.2.1 Discrete choice mechanism and adaptive belief system

In their seminal paper, Brock and Hommes (1997) assume that an evolutionary selection drives the agents. Accordingly, traders choose the most successful rule among the different forecasting heuristics.

This selection mechanism represents the evolutionary part of the model describing how the fractions $n_{h,t}$ of agent types evolve over time, i.e. how the agents update their beliefs. The performance measures are available to every-

[16] See for example Shiller (2000) or Vissing-Jorgensen (2003).

one, but subject to noise. The fitness is described by the random utility model:

$$\bar{U}_{h,t} = U_{h,t} + \varepsilon_{i,h,t}, \tag{1.7}$$

where $U_{h,t}$ and $\varepsilon_{i,h,t}$ represent the deterministic part and the individual error term respectively. By assumption, the noise error is independent and identically distributed across agents i and types h and it is drawn from a double exponential distribution. So, if the number of agents goes to infinity, the evolutionary selection will be given by the following discrete choice mechanism with multinomial logit probability:[17]

$$n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}. \tag{1.8}$$

The main insight of the choice mechanism is that the strategies with higher fitness measure will attract more agents than the lower performing rules. The parameter β is the intensity of choice and measures how sensitive are the agents to the optimal strategy, i.e. it represents the degree of rationality. This term is inversely related to the error term $\varepsilon_{i,h,t}$. Lower is the noise term, i.e. more "rational" are the agents, higher is their capability to select the best rule. For example, when $\beta = \infty$ the agents can perfectly observe the deterministic part of the fitness measure because there is no noise, so all the agents will choose the optimal forecasting rule. In the opposite case, when $\beta = 0$, the variance of the noise term is infinite and the agents are not able to observe the difference in the fitness measures among the heuristics. Consequently, the share of agents which choose the different strategies will be fixed over time and equal to $1/H$.

It has to be highlighted that the market equilibrium variables and the fractions of different rules coevolve following an interesting time pattern. Indeed, the market equilibrium in period t is function of the strategies shares selected in

[17]See Manski and McFadden (1981) for an extensive analysis.

the previous period. Consequently, the equilibrium equations will affect the performance measure in the current period which will determine the new fraction for the period $t+1$. Again, these new fractions will establish the equilibrium in period $t+1$ and so on.

Common extensions of the simple versions of equations (1.7) and (1.8) may consider weighted average of the fitness measures or asynchronous updating of the strategies. In models like Hommes (2013), a natural candidate for fitness measure is the weighted average of realized profits. So the performance measure is given by:

$$U_{h,t} = \omega U_{h,t-1} + (1-\omega)\,\pi_{h,t} - C_h, \tag{1.9}$$

where $0 \leq \omega \leq 1$ represents the memory parameter and it measures how fast past realized performance is discounted.[18] According to Conlisk (1980), C_h is the cost per period of the forecasting rule. It is assumed that more sophisticated rules require more effort or higher cost compared to the simple heuristics that are freely available. Equation (1.8) represents the case of synchronous updating of the strategies, i.e. in each period all the agents can switch to better strategies. On the contrary, following the studies of Diks and van der Weide (2005) or the literature about the habits in economics, it could be interesting to extend equation (1.8) to the asynchronous updating case:

$$n_{h,t} = (1-\delta)\,\frac{e^{\beta U_{h,t-1}}}{\sum\limits_{h=1}^{H} e^{\beta U_{h,t-1}}} + \delta n_{h,t-1}. \tag{1.10}$$

It is straightforward that if $\delta = 0$, Equation (1.10) simplifies to the synchronous case. If $\delta > 0$, the upgrade of the strategy is more gradual, indeed in every period only a fraction $(1-\delta)$ of the agents can reconsider the strategy

[18] See Hommes et al. (2012) for a deep analysis of the memory's effects on the choice dynamics.

according to the new performance information.

1.2.2 The cobweb example

Before moving to the description of the ABS applications to more complex scenarios, it is worth to briefly summarize how this approach can generate interesting results in a simple cobweb model considering only two different strategies. The cobweb model describes fluctuations of equilibrium prices in a market of nonstorable goods that takes one time to produce. Assuming a demand $D(p_t)$ linearly increasing, a linear supply $S(p^e_{h,t})$ and two types of producers (naive and rational), it is possible to write:

$$D(p_t) = a - dp_t, \quad d > 0, \tag{1.11}$$

$$S(p^e_{h,t}) = sp^e_{h,t}, \quad s > 0. \tag{1.12}$$

The market equilibrium can be derived through the market clearing condition:

$$D(p_t) = \sum_{h=1}^{H} n_{h,t} S(p^e_{h,t}).$$

It can be proved that this model presents only one non-linearity, which is given by the fractions, i.e. by the updating mechanism. Indeed, substituting equations (1.11) and (1.12) in the market equilibrium formula, this becomes:

$$a - dp_t = \sum_{h=1}^{H} n_{h,t} sp^e_{h,t},$$

and considering explicitly the two types of agent, the previous equation can be written as:

$$a - dp_t = n_{R,t} sp_t + n_{N,t} sp_{t-1}, \tag{1.13}$$

where $n_{R,t}$ and $n_{N,t}$ denote the fractions of rational and naive agents respectively. Equation (1.13) can be explicitly solved for the price:

$$p_t = \frac{a - n_{N,t} s p_{t-1}}{d + n_{R,t} s}. \tag{1.14}$$

These agents update their expectations according to a public performance measure based on the realized net profit:

$$\pi_{h,t} = p_t S\left(p^e_{h,t}\right) - c\left(S(p^e_{h,t})\right) - C_h = \frac{s}{2} p^e_{h,t}(2p_t - p^e_{h,t}) - C_h, \tag{1.15}$$

where C_h is the per period information cost that has to be paid to obtain the forecasts. This cost is positive for the perfect foresight rule, whereas it is equal to zero in the naive case.

The fractions of the two types are updated following the standard discrete choice mechanism without stickiness.[19] Hence, the fraction of agents using the rational forecasting rule equals to:

$$n_{R,t} = \frac{exp\left(\beta\left(\frac{s}{2}p_t^2 - C\right)\right)}{exp\left(\beta\left(\frac{s}{2}p_t^2 - C\right)\right) + exp\left(\beta\left(\frac{s}{2}p_{t-1}(2p_t - p_{t-1})\right)\right)}, \tag{1.16}$$

while the fraction of naive agents is $n_{N,t} = 1 - n_{R,t}$.

In order to investigate the local (in)stability of the steady state and the dynamic evolution, it is convenient to reformulate some of the main equations using the difference of the two fractions, $x_t = n_{R,t} - n_{N,t}$, and the difference in the realized profits, $\pi_{R,t} - \pi_{N,t} = \frac{s}{2}(p_t - p_{t-1})^2$. At this point, it is possible to write the Adaptive Rational Equilibrium Dynamics as:

[19] See Equation (1.8).

$$p_t = \frac{2a - (1 - x_{t-1})\, sp_{t-1}}{2d + (1 + x_{t-1})\, s}, \tag{1.17}$$

$$x_t = tanh\left(\frac{\beta}{2}\left[\frac{s}{2}(p_t - p_{t-1})^2 - C\right]\right). \tag{1.18}$$

It should be noticed that $x_t = -1$ describes the situation where all the agents are naive, on the contrary when $x_t = +1$ all the agents are rational.

Therefore, the model has a unique steady state (SS) $(p^*,\, x^*) = \left(\frac{2}{d+s},\, tanh\left(\frac{-\beta C}{2}\right)\right)$. It is worth to underline that if $C = 0$, i.e. if there is no additional cost for the perfect foresight strategy, the agents will move to the two strategies in equal shares. Since at the SS both strategies must have identical forecasts, if $C > 0$, most of the agents will choose the cheapest strategy between the two, which is the naive rule.

The stability properties of the steady state are determined by the ratio of the derivatives of supply and demand at the SS price. Given that the traditional cobweb stability requirement is $|\frac{S'(p^*)}{D'(p^*)}| = |\frac{s}{d}| < 1$, it can be easily proved that the SS of this model is globally stable. To allow possible unstable steady state and endogenous fluctuations in the heterogeneous model, Brock and Hommes (1997) assume that when all the agents are naive the market is locally unstable: $\frac{S'(p^*)}{D'(p^*)} = -\frac{s}{d} < -1$.

Under this assumption, it is possible to assert that, if $C = 0$ the steady state will be globally stable for all the possible values of β, whereas when the information costs are positive there is a critical value of β_1 over which the system becomes unstable presenting a period-doubling bifurcation at $\beta = \beta_1$.

Brock and Hommes (BH) illustrate how in the neoclassical case, i.e. when $\beta = \infty$, all the time paths converge to the steady state where the price is the fundamental SS, p^*, and all the agents are naive. Indeed, until the profit losses due to error of the naive strategy are lower than the information cost for perfect foresight expectations, all the agents will choose the cheapest forecasting strategy. As long as all the agents are naive, the price dynamic follows a linear unstable oscillation around the fundamental value. This oscillation presents an increasing divergence from the steady state. Therefore, at some point, the naive error will generate lower gain than the net profit of the rational strategy. At this stage, all the agents would pay the information cost in order to obtain higher returns, they use rational expectations and the price jumps to its SS value. In the following periods, all the agents will switch to the naive forecasting rule because they will be able to predict the true fundamental value without bearing the cost of the rational expectations. For this reason, the dynamics will converge to the stable path.

Almost the same line of reasoning applies in the finite intensity of choice cases with the difference that, when all the agents become rational, the price is not driven onto the true fundamental price, but only close to it. Being the price close enough to the steady state, the agents will switch to the costless rule of thumb, hence the price will start to move away from its SS value and the complete process will occur again. The bounded rational model shows irregular price fluctuations. Moreover, for high intensity of choice, BH prove that the ABS-cobweb model exhibits strange attractors for a positive set of β, implying chaotic price fluctuations.

These results have been extended by Branch (2002) who proposes a model including another unsophisticated predictor: the adaptive expectations belief. The author considers this rule type because the difference between adaptive and naive expectations consists in the weight of the adaption parameter, i.e. the value of parameter g mentioned in Section 1.2. In this framework the results suggest that adding a second costless unsophisticated choice, if the memory is sufficiently high, the steady state may be locally stable at the asymptote. On the contrary, if the adaptive rule is costly and the intensity of choice parameter is large enough, the steady state will become an unstable saddle point. However, it is interesting to notice how increasing the range of the expectation rules rises the critical value of β, which turns the path from stable to unstable. In conclusion, the intensity of choice has no effect on the stability conditions as demonstrated by Brock and Hommes model (1997).

1.3 Adaptive Believe System in Economics and Finance: some examples

This section presents some applications of the ABS framework in three different domains: the analysis of the animal spirits in an exchange rate market, the study of the financial market and the stability analysis in a macroeconomic context.

Certainly, the adaptive belief system can be applied to many other fields, indeed, in economics and finance there are a lot of witnesses of paradigm switch from the perfect rational expectation hypothesis to bounded rational approach. Another example might be the overlapping generations monetary economy developed by Brock and De Fontnouvelle (2000). As in Brock and Hommes (1997), the agents adopt a discrete choice mechanism to select between the strategies according to the lowest past squared forecast error. The dynamics of the model are strongly influenced by the value of the intensity of choice. When the parameter β increases, the monetary steady state becomes unstable. Also Chiarella and Khomin (1999) have worked on the adaptive evolving expectation of Brock and Hommes (1997) developing a basic Cagan monetary model with fundamentalist and chartist agents. In this framework, they confirm that greater complexity can arise and the share of agents distributed over the two rules of thumb may widely fluctuate.[20]

1.3.1 Biased agents in an exchange rate model

Considering the empirical evidence,[21] De Grauwe and Kaltwasser (2012) present an exchange rate model in which agents are unable to observe the fundamental value. However, agents have beliefs on it, and use them in order to position themselves in the market. Starting from this assumption, De Grauwe and Kaltwasser (2012) develop three possible versions of the model, which allow to replicate interesting phenomena of the exchange rate market, as the disconnession puzzle or the excess volatility puzzle.

The first version assumes two biased agents: optimist and pessimist. The optimist (pessimist) systematically over (under) estimates the fundamental rate. The model postulates the following linear excess demands:

$$d_{opt,t} = \alpha\left[(e^* + a) - e_t\right], \quad \alpha > 0, \tag{1.19}$$

$$d_{pes,t} = \alpha\left[(e^* - a) - e_t\right], \quad \alpha > 0, \tag{1.20}$$

where e_t is the market exchange rate and $d_{opt,t}$ and $d_{ret,t}$ represent the optimistic and pessimist excess demand respectively. It should be noticed that the agent's beliefs are defined as constant differences over or under the steady state value e^*_t. The law of motion of the market value is given by:

$$e_{t+1} = e_t + \mu \sum_{h}^{H} n_{h,t} d_{h,t},$$

where, as stated in the referred literature, $d_{h,t}$ and $n_{h,t}$ are the excess demand

[20]See Hommes (2006) for further examples.

[21]For example the popular work of Kindleberger and Aliber (2005) on 'Manias, panics and crashes'.

and the number of agents using the h-Th forecasting rule respectively. Following Beja and Goldman (1980), $\mu > 0$ represents the speed according to which the value of the market exchange rate is adjusted. Defining $x_t = n_{opt,t} - n_{pes,t}$, as in Section 1.2.2, and substituting the pessimistic and optimistic demands in the previous equation, it is possible to rewrite the law of motion as follows:

$$e_{t+1} = e_t + \alpha\mu\left(e^* - e_t + ax_t\right). \tag{1.21}$$

Equation (1.21) demonstrates how the exchange rate dynamics is determined by two different factors: a convergence factor and a distribution factor. According to the first one, if the market level is below (above) its fundamental in period t, in the next period the exchange rate will increase (decrease). The second factor is related to the share of optimists in the market. For example, if $x_t > 0$ there will be more optimists than pessimists, therefore the exchange rate will increase between the periods. It is straightforward that the amplitude of this effect depends on the belief bias a. Being this an ABS framework, the agents choose the forecasting rule according to the discrete choice mechanism:

$$n_{h,t} = \frac{exp\left(\beta\pi_{h,t}\right)}{\sum_{h}^{2} exp\left(\beta\pi_{h,t}\right)}, \tag{1.22}$$

where the performance measures are the realized profits: $\pi_{h,t} = d_{h,t-1}\left(e_t - e_{t-1}\right)$.

Combining equations (1.21) and (1.22) it is possible to write the difference between the fractions as:

$$x_t = tanh\left(\frac{1}{2}\beta\pi_{opt,t} - \pi_{pes,t}\right). \tag{1.23}$$

Now, it is reasonable to derive the unique steady state of the 2-dim map: $S = (e^*, 0)$. According to the value of the intensity of choice, the model can present different dynamic behaviours: if $(\alpha\mu - 2)/\alpha^2\mu a^2 < \beta < 1/\alpha^2\mu a^2$, the unique steady state is asymptotically stable, if $\beta > 1/\left(\alpha^2\mu a^2\right)$ the dynamic converges to a stable limit cycle, if $\beta = 1/\left(\alpha^2\mu a^2\right)$ there is the Neimark-Sacker bifurcation, whereas if $\beta < (\alpha\mu - 2)/\alpha^2\mu a^2$ the unique steady state becomes unstable and a two-cycle steady dynamic emerges. In other words, the steady state is reached only when the two opposite shares are equal. Since both pessimist and optimist are fundamentalist agents, when their fractions are equal the two diverging forces will nullify each others.

Looking at the stability conditions, as in the cobweb model example, if the value of β becomes too large it has a destabilizing effect on the steady state. This is because agents are more sensitive to the best forecasting rule and therefore they raise the self-fulfilling nature of the expectations. Hence, increasing the sensibility of the agents decreases the space of the other parameters value under which the system is stable. Nevertheless, the stability condition depends also on other variables. For example, for small values of α, the agents exercise a low convergence force on the exchange rate to the fundamental value. Conversely, if a is low, the beliefs of optimist and pessimist are close and do not imply a high pressure on the stable steady state.

De Grauwe and Kaltwasser (2012) have managed to design an exchange rate model with endogenous fluctuations even if the fundamental value is stable. In addition, it should be noticed that, given the assumption of time invariance of the fundamental, every persistent fluctuation in the market implies the "disconnect" puzzle and the excess of volatility.[22] In other words, the exchange rate is disconnected from its fundamentals most of the time and its volatility exceeds the volatility of the underlying economic variables.

[22] See Obstfekd and Rogoff (2000).

The choice to consider only biased traders in the market could be the reason why the market cannot converge to the steady state solution beyond the stability space. De Grauwe and Kaltwasser (2012) analyze two possible extensions of the model: in the first case, they allow for the existence of an unbiased fundamentalist type, whereas in the second case they add to the first extension also a trend following (chartist) forecasting rule.

The excess of demand functions of the unbiased agents $d_{uf,t}$ and the chartists $d_{ch,t}$ can be written as follows:

$$d_{uf,t} = \gamma \left(e^* - e_t\right), \tag{1.24}$$

$$d_{ch,t} = \delta \left(e_t - e_{t-1}\right). \tag{1.25}$$

It can be seen how equation (1.24) is the simplified version of (1.19) or of equation (1.20) for $a = 0$. In equation (1.25) δ is the extrapolation parameter on the past movements. Adding the first new expectation rule in the previous framework it is possible to rewrite the law of motion of the exchange rate as:

$$e_{t+1} = e_t + \left[\alpha\mu \left(e^* - e_t - ax_t\right)\right]\left(1 - n_{uf,t}\right) + \gamma\mu \left(e^* - e_t\right) n_{uf,t}, \tag{1.26}$$

with $n_{uf,t} + n_{opt,t} + n_{pes,t} = 1$.

The new dynamic of the exchange rate depends on the previous level, on the weighted sum of the non-fundamentalist agents effect, as in equation (21), and on the unbiased agents impact. Indeed, if $n_{uf,t} = 0$ the augmented law of motion is reduced to the original one.

Considering both additional rules of thumb (unbiased and chartist), the law of motion of the exchange rate becomes:

$$e_{t+1} = e_t + \mu \left[\alpha \left(e_{opt,t} - e_t \right) n_{opt,t} + \alpha \left(e_{pes,t} - e_t \right) n_{pes,t} + \gamma \left(e^* - e_t \right) n_{uf,t} + \delta \left(e_t - e_{t-1} \right) n_{ch,t} \right]. \tag{1.27}$$

Analysing separately the two extensions, the model presents some interesting insights. On the one side, the existence of unbiased agents increases the region where the model is locally asymptotically stable. In other words, the unbiased agent enhances the range of the β values under which the system is stable. However, it seems that the presence of the unbiased agents leads to a different type of complex attractor, i.e. when γ is high, the exchange rate dynamic may be driven by strange attractors. This is a counter intuitive result, not only the chaos can be due to biased fundamentalist traders but also to their interaction with unbiased agents. On the other side, the chartist agents may act as destabilizing force, their destabilization power is proportional to the amplitude of the parameter δ.

Concluding, in an exchange rate model with two heterogeneous types, the ABS allows to generate a cyclical movement of biased agents even if the fundamental does not change over time. Moreover, if the model is extended by introducing further belief types, the dynamics will become more unpredictable and will be able to replicate some well-known phenomena as the disconnect and the excess volatility puzzles. These results seem to be consistent with the models elaborated by De Grauwe and Grimaldi (2005, 2006). These authors provide examples of exchange rate frameworks with transaction costs where the ABS generates multitude of fixed-point attractors. Furthermore, the authors show how the effect of a permanent shock on the fundamental exchange rate may be chaotic depending on the exact timing of its occurrence. De Grauwe and Grimaldi suggest that history matters, i.e. the market has a memory. It should be noticed that this statement contrasts with the efficient market assumption.

However, using the ABS framework, it is possible to replicate many market evidences: the disconnect puzzle, the presence of the excess volatility or the non normal distribution of the returns, i.e. the existence of fat-tails.

1.3.2 Macroeconomic stability under heterogeneous expectations

This subsection discusses the frictionless dynamic stochastic general equilibrium (DSGE) model with heterogeneous expectations introduced by Anufriev, Assenza, Hommes and Massaro (2013). It analyses the dynamics of committing to an interest rate feedback rule in a framework with endogenous switching between heterogeneous inflation forecasting rules. This analysis consists of two parts: the first can be linked to the frictionless models of Cochrane (2005, 2011), the second applies the ABS approach.

As in standard macroeconomics models, consumers maximize their utility function:

$$max\ E_t \sum_{j=0}^{\infty} \delta^j u\left(C_{t+j}\right),\ 0<\delta<1,$$

where δ represents the discount factor and $E_t C_{t+j}$ is the expectation of the agents in period t concerning consumption in period $t+j$.

The maximization problem is subject to a budget constraint:

$$P_t C_t + B_t = \left(1+i_{t-1}\right) B_{t-1} + P_t Y,$$

where P_t, B_t, i_t and Y represent the price of the good, the bonds hold by the agent, the nominal interest rate and the constant nonstorable endowment respectively.

From the consumption Euler's equation and the market-clearing condition, it is possible to obtain the linearised Fisher relation of the interest rate:

$$i_t = r + E_t\left(\pi_{t+1}\right), \tag{1.28}$$

where r is the constant real interest rate and π_{t+1} represents the inflation rate. Assuming that the central bank adjusts the interest rate according to the following Taylor rule: $i_t = r + \phi_\pi \pi_t$, it is possible to obtain the equilibrium condition of the model:

$$\pi_t = \frac{1}{\phi_\pi} E_t\left(\pi_{t+1}\right), \tag{1.29}$$

therefore the actual inflation rate depends on the inflation expectations of the agents.

Now, postulating that the agents are heterogeneous and choose their inflation forecasts from a set H of different rules of thumb, equation (1.29) can be rewritten as:

$$\pi_t = \frac{1}{\phi_\pi} \sum_{h=1}^{H} n_{h,t} E_{h,t}\left(\pi_{t+1}\right). \tag{1.30}$$

The agents rank the possible forecasting rules according to the past squared forecast error:

$$U_{h,t} = -\left(\pi_{t-1} - E_{h,t-2}\left(\pi_{t-1}\right)\right)^2 - C_h, \tag{1.31}$$

where C_h is the information cost per period of the h-Th expectation rule.

As De Grauwe and Kaltwasser (2012), Anufriev et al. start their analysis investigating a framework with biased (optimist and pessimist) and fundamentalist agents. The forecasting heuristics are:

$$E_{fun,t}\left(\pi_{t+1}\right)=0, \tag{1.32}$$

$$E_{opt,t}\left(\pi_{t+1}\right)=b, \tag{1.33}$$

$$E_{pes,t}\left(\pi_{t+1}\right)=-b. \tag{1.34}$$

Equation (32) represents the belief that the inflation rate will remain at its fundamental level, whereas equations (1.33) and (1.34) describe expected inflation rates above or below the fundamental value. Supposing a discrete choice mechanism, as in equation (1.8), and substituting these last three equations in (1.30) and (1.31), it is possible to write the new inflation law of motion and the switching mechanism as following:

$$\pi_t=\frac{1}{\phi_\pi}\left(n_{opt,t}b-n_{pes,t}b\right), \tag{1.35}$$

$$n_{fun,t}=\frac{exp\left(-\beta\left(\pi_{t-1}^2+C\right)\right)}{exp\left(-\beta\left(\pi_{t-1}^2+C\right)\right)+exp\left(-\beta\left(\pi_{t-1}-b\right)^2\right)++exp\left(-\beta\left(\pi_{t-1}+b\right)^2\right)}. \tag{1.36}$$

It can be demonstrated that the one-dimensional map described in (1.35) has a SS in $\pi^*=0$. The macro-stability and the dynamics depend on the set of belief parameters and on the aggressiveness degree of the monetary policy. Its local and global stability depends on: the biased parameter (b), the cost of the sophisticated rule (C), the intensity of choice β and the reaction coefficient ϕ_π in the monetary policy. Investigating the stability properties, it is possible to understand how the monetary policy plays a relevant role when the cost of the sophisticated rule is sufficiently low, i.e. when $C<b^2$. On the one hand, under very aggressive financial policy the system converges to the rational

steady state whatever is the value of the intensity of choice. On the other hand, gradually decreasing the reaction coefficient of the monetary policy leads the system to more complex dynamics increasing the β parameter. For example, if the monetary policy is weak and there is a low value of β, the result will be a single, rational and stable SS. Increasing the intensity of choice, the SS looses its stability and this leads to the emergence of two other non fundamental stable steady states. Raising again the intensity of choice will stabilize the RE steady state and will generate two unstable SS, hence, the system will present three stable steady states and two unstable steady states. This is why the compliance with the Taylor principle is not sufficient to reach the stable RE steady state. Instead, when the intensity of choice is high it is required to have a stronger monetary policy, otherwise the system will present three stable and two unstable steady states.

This result seems to confirm the main insights presented by Massaro (2013). Massaro designs a micro-founded DSGE model and examines the monetary policy finding that, in a framework with heterogeneous beliefs, the Taylor principle does not imply the existence of a unique and stable equilibrium. Thus, central Banks should take into account the presence of bounded rational agents in designing their policies, otherwise they risk to destabilize the system.

Concluding, the model of Anufurev et al. (2013) investigates the role of heterogeneous expectations in a frictionless DSGE. According to the stability analysis, when the monetary policy is weak, the main result is that agents receive misleading signals from the market. As a consequence of the learning process, a cumulative process of rising inflation is created. This process is reinforced by self-fulfilling expectations on high inflation. When the central bank reacts aggressively to change in inflation it induces convergence to the stable RE steady state because it sends correct signals to the agents.

1.3.3 An asset price model example

This subsection discusses the asset pricing model presented in Brock and Hommes (1998), henceforward BH98. The framework assumes one risky asset and one risk free asset which is perfectly elastically supplied at fixed rate of return r. Let p_t and y_t be the risky asset price per share and the stochastic dividend process respectively. The dynamic of wealth can be written as:

$$\tilde{W}_{t+1} = (1+r)\,W_t + [\tilde{p}_{t+1} + \tilde{y}_{t+1} - (1+r)\,p_t]\,z_t, \tag{1.37}$$

where the *tilde* denotes random variables and z_t is the number of asset shares purchased. Assuming that the agents have the same belief about the conditional variance of excess returns and that this belief is :

$$V_{h,t}\,[\tilde{p}_{t+1} + \tilde{y}_{t+1} - (1+r_t)\,p_t] \equiv \sigma^2.$$

The agents are myopic mean variance maximizer, so the demand for the risky assets is solved as follows:

$$Max_z \left\{ E_{h,t}\left(\tilde{W}_{t+1}\right) - \frac{a}{2} V_{h,t}\left(\tilde{W}_{t+1}\right) \right\}, \tag{1.38}$$

where a is the risk aversion parameter. Therefore the demand for the risky assets is:

$$z_{h,t} = \frac{E_{h,t}\,[\tilde{p}_{t+1} + \tilde{y}_{t+1} - (1+r)\,p_t]}{aV_{h,t}\,[\tilde{p}_{t+1} + \tilde{y}_{t+1} - (1+r)\,p_t]} = \frac{E_{h,t}\,[\tilde{p}_{t+1} + \tilde{y}_{t+1} - (1+r)\,p_t]}{a\sigma^2}. \tag{1.39}$$

Let the supply of outside risky asset z^s be constant and equal to zero. In a heterogeneous framework the equilibrium of demand and supply implies:

$$(1+r)\,p_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}\left[\tilde{p}_{t+1} + \tilde{y}_{t+1}\right], \tag{1.40}$$

where dividends can be interpreted as risk adjusted. Before investigating the competition among the rules of thumb, some assumptions should be enumerated. Firstly, all the traders have a common and constant conditional variance on the excess return. Secondly, all the agents are able to compute the fundamental price p_t^* that prevails in a full rational world. Thirdly, the agents believe that in a heterogeneous framework the price may deviate from the fundamental price following some functions $f_{h,t}$.

The evolutionary part of the model is given by the standard discrete choice mechanism:

$$n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}},$$

where the fitness measure $U_{h,t}$ represents the realized profit for the forecasting rule type h. The fitness measure in the deterministic dividends dynamic case is defined by:

$$U_{h,t} = \left[p_t + y_t - (1+r)\,p_{t-1}\right] \frac{E_{h,t-1}\left[\tilde{p}_t + \tilde{y}_t - (1+r)\,p_{t-1}\right]}{a\sigma^2}. \tag{1.41}$$

Rewriting the fitness measure in deviation from the steady state $x_t = p_t - p_t^*$ and establishing the gross risk free rate of return $1+r = R$, equation (1.41) can be reformulated as:

$$U_{h,t} = \left[x_t - Rx_{t-1}\right]\left(\frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}\right). \tag{1.42}$$

In equation (1.42), $f_{h,t}$ represents the belief types. The model investigates the emerging dynamics assuming three different frameworks: some examples with two expectation rules, one using three rules of thumb and one with four competing heterogeneous expectations. In general terms, the expectations are described by the simple linear function:

$$f_{h,t} = g_h x_{t-1} + b_h,$$

where g_h and b_h represent the trend and the bias parameter respectively as in Section (1.2).

The first example analysed considers fundamentalists with positive information cost and trend chaser agents, the derived forecasting rules are:

$$f_{fun,t} = 0, \tag{1.43}$$

$$f_{tra,t} = g_h x_{t-1}, \quad g > 0. \tag{1.44}$$

Equation (1.43) describes fundamentalist agents, predicting that the price will be equal to the fundamental value. In order to obtain information about the true fundamental level, agents have to pay a positive cost C. In equation (1.44) the agents believe that prices will rise by a constant rate. Therefore the equilibrium equation (1.40) in deviation term from the steady state is:

$$Rx_t = n_{tra,t-1} g x_{t-1}.$$

Substituting equations (1.43) and (1.44) in (1.42) and in the choice mechanism, it is possible to obtain the share of fundamentalist agents:

$$n_{fun} = \frac{exp\left[\beta\left(\frac{1}{a\sigma^2}Rx_{t-1}\left(Rx_{t-1}-x_t\right)-C\right)\right]}{exp\left[\beta\left(\frac{1}{a\sigma^2}Rx_{t-1}\left(Rx_{t-1}-x_t\right)-C\right)\right]+exp\left[\beta\left(\frac{1}{a\sigma^2}\left(Rx_{t-1}-x_t\right)\left(gx_{t-2}-Rx_{t-1}\right)\right)\right]}.$$

Exploring the stability conditions and the dynamics of the system, BH98 obtain the following result: when the extrapolation parameter g is very low, the system has a unique globally stable steady state whatever is the value of the intensity of choice. On the contrary, when agents extrapolate very strongly, i.e. for high values of g, the dynamic of the system may be unstable according to the value of intensity of choice. For low values of β, the fundamental SS remains stable, whereas increasing the intensity of choice the steady state becomes unstable and a pitchfork bifurcation emerges. For very high values of the intensity of choice the two unstable SS exhibit Hopf bifurcation. In presence of the information cost, when the intensity of choice is high, the fundamentalists can not drive the trend followers out from the market. For this reason, persistent deviation from the fundamental steady state can emerge. It should be noticed that this result can also be found by replacing the fundamentalists with rational agents. Hence, the model analyses the dynamic of the system with fundamentalists and contrarians. As defined in Section 1.2, the contrarian agents believe that prices will reverse their trend in the next period.

Summarizing, BH98 studies an asset pricing model with heterogeneous agents who update their expectations according to the past realized profits. In this heterogeneous framework, the dynamics exhibit persistent deviations from the fundamental steady state with irregular and chaotic price fluctuations when the value of the intensity of choice is high. This case presents results in line with the previous analysis with weaker contrarians. However, when the agents become strong contrarians a two cycle period emerges. This happens also when there is no informative cost for the fundamentalists. When costs are positive, as the intensity of choice increases, the system presents a period doubling bifurcation. By further increasing the value of β, the stable two-cycle loses its stability presenting a Hopf bifurcation. Evidence of these results can be found in Chiarella and He (2001), where wealth dynamics are incorporated into the heterogeneous asset pricing model, allowing also variation of the extrapolation parameter over time. This model is able to generate some well-known phenomena of the financial market, such as volatility clustering, skewness or the asset price oscillations around a geometrically growing trend.

After the analyses of the competition between two different belief types, BH98 extends the dynamic evolution in a framework with three and four different expectations rules. The first case considers fundamentalists and opposite biased agents. Assuming positive and negative biases and no information costs for the fundamental prediction, the adaptive belief system can be written as:

$$Rx_t = n_{opt,t-1}b_{opt} + n_{pes,t-1}b_{pes},$$

$$n_{j,t} = \frac{exp\left[\frac{\beta}{a\sigma^2}\left(b_j - Rx_{t-1}\right)\left(x_t - Rx_{t-1}\right)\right]}{Z_{t-1}},$$

where $Z_{t-1} = \sum_{h=1}^{H} e^{\beta U_{h,t-1}}$ represents a normalization factor. Since the optimists ($b_{opt} > 0$) and the pessimists ($b_{pes} < 0$) are symmetrically opposite, i.e. $b_{opt} = -b_{pes}$, it can be proved that the fundamental steady state is stable for low value of intensity of choice. However, as in the analysis concerning the two forecasting rules, when the intensity of choice increases the system becomes unstable and after a Hopf bifurcation the dynamics can be described by cycles around the unstable steady state. In addition, even if the sophisticated rule is costless, the fundamentalists can not drive out from the market the biased agents. The same result is obtained using four types of rules of thumb. Indeed,BH98 consider fundamentalist traders and three different types of simple linear rules as follows:

$$f_{2,t} = 1.1x_{t-1} + 0.2,$$

$$f_{3,t} = 0.9x_{t-1} - 0.2,$$

$$f_{4,t} = 1.21x_{t-1}.$$

Types 2 and 3 are trend-followers with an upward and downward bias respectively, whereas type 4 follows a strong trend chaser rule with one lag. As before, the rise of the intensity of choice destabilizes the system dynamics and, after a threshold, a strange attractor emerges. Chaos is characterized by switching between two phases, one closer to the stable fundamental steady state and one described by an increasing price trend due to the fact that most of agents are of Type 2.

According to the rules of thumb dominating the market, the model can display manifold bifurcations. For example, when trend chasers dominate, the system presents a stable steady state plus two unstable steady states, one above and one below the fundamental. If the dominant type is the contrarian agent, a period doubling bifurcation arises, whereas when the main strategy is the opposite biased rule, this leads to a Hopf bifurcation with quasi-periodic fluctuations around an unstable steady state.

Concluding, it should be underlined that the irregular fluctuations in asset prices are the consequence of a rational choice, they are the result of the switching choice based upon the realized profits, i.e. they are Rational Animal Spirits.

Starting from this seminal paper, further interesting analyses have been developed in the last years. For example, Brock et al. (2005) develop an extension of this framework considering many different agent types. They introduce the notion of large type limit (LTL). This is a type of ensemble limit that can be obtained replacing sample moments by population moments. Brock et al. (2005) prove that the large type limit describes the dynamics of a market with many expectations rules. This extension confirms one of the main results of the original paper: a bifurcation route to chaos occurs in an asset price with many agent types when the agents become less risk adverse (the parameter a decreases) or the intensity of choice β increases.

Boswijk et al. (2007) reformulate the asset pricing model in term to cash flows with the aim to estimate its performance on yearly S&P500 data. According to their estimations, agents believe that in the long run the dynamic of prices is driven by fundamentals, however they interpret differently the deviations from the steady state in the short run. For example, when deviations from the SS occur, a trend follower agent reacts in a different way respect to a fundamentalist. In other words, they find significant evidence for behavioural heterogeneity. Hence, the model is able to explain the "irrational exuberance" in the stock market in the late '90s. In those years the market was dominated by optimistic and bounded rational agents motivated by short run profitability, so their high cash flows expectations reinforced the rise in the stock market prices inflating the "dot-com bubble".

The evidence that heterogeneous bounded rational agents can reproduce the fluctuations of the asset prices seems also confirmed by Hommes and in't Veld (2015). They extend the analysis around two benchmark models: one with constant risk premium, the dynamic Gordon model, and one with time-varying risk premium, the Campbell-Cochrane model. Moreover, this paper introduces agent's memory of realized excess return with the aim to make the switching mechanism consistent with the frequency of the quarterly data.

Considering chartists and agents believing in mean-reversion of stock price to its fundamental, the model is able to give better predictions than the homogeneous model in periods before both the "dot-com" bubble and the financial crisis. Furthermore, the predictions of "dot-com" crisis period seem to be compatible with the available surveys, indeed agents were aware that the difference between the stock prices and their fundamentals was too high. Hommes and in't Veld suggest two interpretations on the price dynamics of the 2008 financial crisis. According to the Campbell-Cochrane model, it seems that the price dynamic was in line during all the period so the 2008 crash could be explained as an overreaction to the unexpected bankruptcy of Lehman Brothers. On the contrary, in the constant risk premium model the stock prices seem to have overvalued the Gordon fundamental since 1995. Hence, the crash of 2008 could be seen as a temporary correction and after that a new bubble started to inflate.

1.4 Experimental analyses

Usually, in the real world, we observe the agents' aggregate behaviours, while it is harder to obtain information about the individual expectations. Nevertheless, there are at least two ways to collect data on the expectations process creation of the individuals: the first is by survey data, the second by performing laboratory experiments.

As underlined by Duffy (2006) or Hommes (2013), experiments may be useful as source of empirical regularities that can be used to calibrate the models. Moreover, these regularities may be helpful to handle the problem of "wilderness of bounded rationality", i.e. what types of rules of thumb should be considered in a model. In addition, data from human experiments can be used to check the macro and microeconomics external validity of models.

This section analyses the results of experimental investigations in three frameworks: cobweb models, heterogeneous new Keynesian models and asset pricing models. It is important to underlined that expectations in the experiments often do not have a direct and unambiguous relationship with the results. So, in order to avoid jointly hypothesis testing, a literature of experimental analysis designed to test only the expectations hypothesis has been growing in the last years.

Hommes (2011) focuses on the so-called learning-to-forecast experiments (LtFEs). In these types of investigation, the subjects are required to forecast the level of a variable for a number of periods. At the beginning of each period, the subjects receive some qualitative information about the market and then they have to make the forecast. This is the only action required, indeed, in these experiments forecasting decisions are separated from trading operations.

Hommes et al. (2007) investigate the expectation formation in a cobweb model. The research questions investigate: 1) if agents are able to learn the average RE steady state; 2) if expectations matter and may cause excess price volatility; and 3) if price evolution has a foreseeable structure. In the experimental environment, the subject is required to predict the next periods prices for fifty consecutive sessions. The subjects do not receive information about either the underlying model or the distribution of the shocks, they only know the market price bounds. The realized market price is function of the prediction of 6 participants with a small random shock. The whole experiment sample consists of 108 participants. They form 18 markets divided over 3 different treatments: a stable, an unstable and a strongly unstable treatment. As in Section 1.2, the realized price in the experiment is determined by the cobweb market equilibrium that is:

$$D\left(p_t\right) = \sum_{i=1}^{K} S\left(p_t^e\right).$$

The authors assume a fixed linear demand and a nonlinear supply function increasing in the expected price, p^e, according to the profit maximization behaviour. Hence, the demand and the supply functions are respectively:

$$D\left(p_t\right) = a - bp_t + \eta_t,$$

$$S\left(p_t^e\right) = tanh\left[\lambda\left(p_{i,t}^e - 6\right)\right] - 1,$$

where $a, b, \lambda > 0$ and η is a normally distributed demand random shock. The parameter λ sets the non-linearity of the supply function and the stability of the cobweb model. Accordingly, the three different treatments depend on the value of the parameter λ. It also affects the stability of the market, higher is its value more unstable becomes the system. Each λ implies a stationary experimental environment with different fixed and constant RE steady states. Given the previous equations on demand and supply and the individual forecasts by all the subjects, the equilibrium price will be:

$$p_t = \frac{a - \sum_{i=1}^{k} S\left(p_{i,t}^e\right)}{b} + \frac{\eta_t}{b}.$$

The experimental results are compared with the aggregate fluctuations of a benchmark cobweb model under some standard expectation rules: rational, naive, adaptive expectations, learning by average and sample autocorrelation learning.

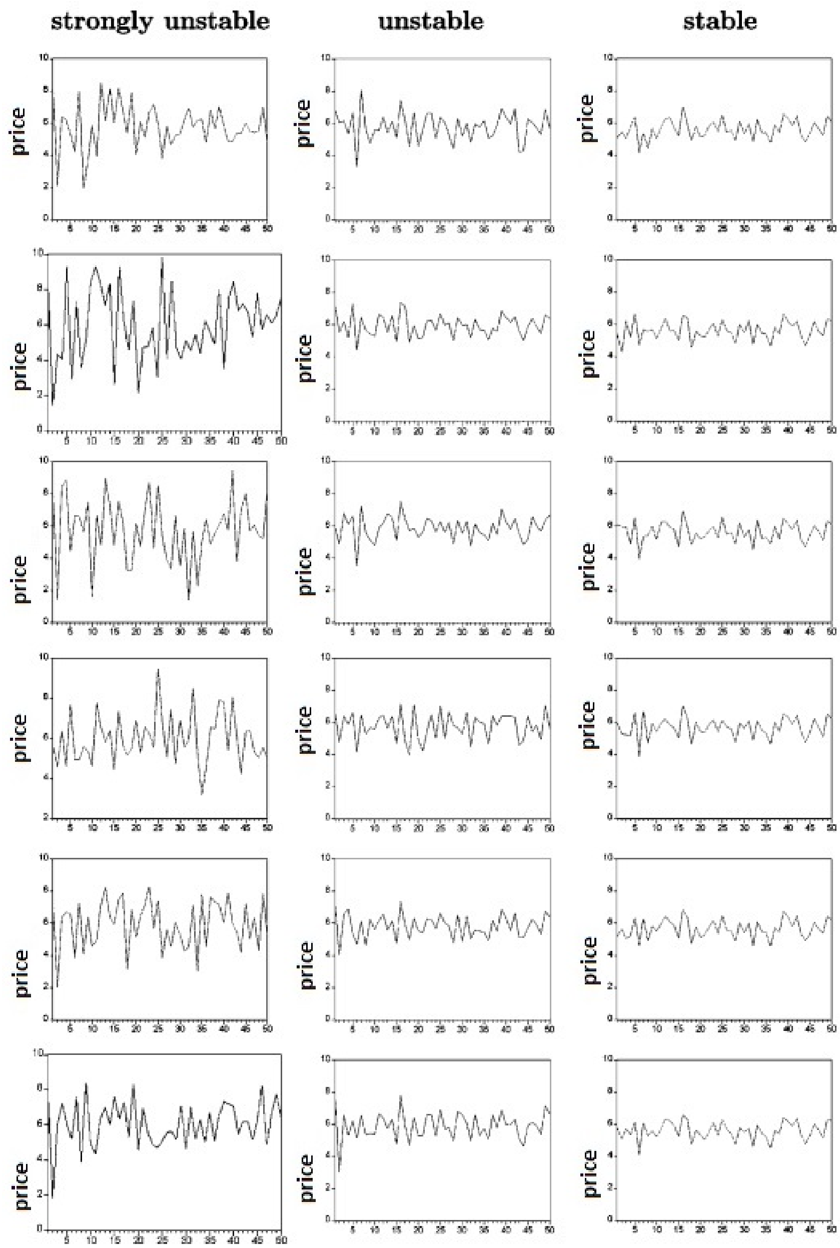

Figure 1: Time series of realized prices of the 18 groups, 6 groups for each treatment: strongly unstable (left panel), unstable (middle panel) and stable treatment (right panel).

As Figures 1 illustrates, the stable and the unstable treatments, right and central panels, present small fluctuations close to the rational expectation benchmark or a decreasing fluctuation amplitude over time. On the contrary, the strongly unstable treatment, represented in the left panel, exhibits sharp fluctuations suggesting excess price volatility. Some interesting results are found by testing the null hypothesis, i.e. the three treatments have the first two moments equal to the results under RE expectations. Firstly, in the stable treatment presents the mean and the variance are in line with those of the RE analysis, whereas the two unstable treatments have only the average of the sample comparable to the rational expectations first moment. Secondly, the unstable and the strongly unstable treatments exhibit statistically significant excess volatility even if the subjects are able to learn the correct price level. Thirdly, the experiments perform non-significant autocorrelation, therefore the prices evolutions are not predictable. Concluding, given that the results of unstable and strongly unstable treatments are different from the rational expectations benchmark but the price structure is not exploitable, the agents are not irrational, they are bounded rational.

Moving to the asset pricing model, Hommes et al. (2005) investigate the formation of expectations with the aim to classify individual forecasting rules. In this environment, subjects are required to give their expectations on the price of a risky asset in the next period. From the submitted forecast, a computer program calculates the associated aggregated demand and consequently the market equilibrium price. It should be noticed that one of the key feature of the controlled environment is the possibility to keep the economic fundamentals constant over time. At the beginning of the experiment, it is explained that agents should act as advisor to a pension fund which can invest in risky or risk free assets. The risk free asset has a rate of return $R = 1 + r$, whereas the risky asset pays uncertain i.i.d. dividends around a given mean. The subjects do not know either the true model underlying the market equilibrium or the fact that the price depends on their own prediction and on the other agents' expectations. However, they know that higher is their forecast, higher will be the demand for stocks. The experiment environment consists of 60 subjects divided in 10 asset markets/groups and a fraction of fundamentalist computerized traders acting as a "stabilizing force" pushing prices towards the fundamental price and excluding the raising of speculative bubbles.

In the experiment, the realized prices are generated using the standard theoretical asset pricing model, hence from equation (40):

$$p_t = \frac{1}{1+r}\left[(1-n_t)\,\bar{p}^e_{t+1} + n_t p^* + \bar{y} + \varepsilon_t\right],$$

where $\bar{p}^e_{t+1} = \frac{1}{6}\sum_{h=1}^{6} p^e_{h,t+1}$ is the mean of predictions by the 6 participants. The risk free return rate is equal to 5% and the fundamental price is $p^* = 60$ (with $\bar{y} = 3$) in 7 groups, whereas for the other 3 groups the fundamental price is $p^* = 40$ (with $\bar{y} = 2$) .

Therefore, the asset price is determined as a weighted sum of the subjects' average forecasts and the fundamental predictions of the "robot" traders with an extra noise term. The weight of these fundamentalist computerized agents is given by:

$$n_t = 1 - exp\left(-\frac{1}{200} \mid p_{t-1} - p^* \mid\right).$$

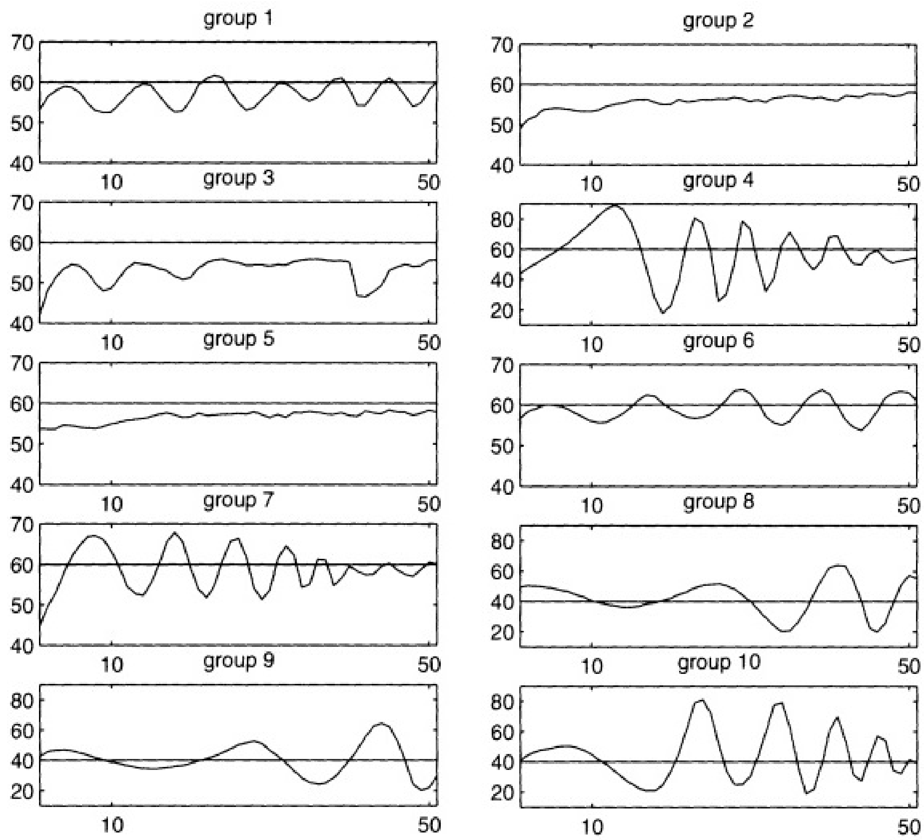

Figure 2: Realized asset prices in the 10 groups of the experiment. The horizontal lines represent the fundamental price levels.

The realized asset prices of the experiments represented in Figure 2 allow to extrapolate three different patterns: 2 groups show a monotonic convergence from below to the fundamental value; 3 groups fluctuate around the steady state but the amplitude decreases over time, generating a convergence; 4 groups present persistent oscillations around the steady state price.

From Figure 3, it is possible to draw an important conclusion comparing the sample mean and the sample variance of the groups with the mean and variance of theoretical examples. The expectations rules which give better descriptions of the experimental results are not the unbounded rational but are the naive expectation and the AR(2) showing both excess volatility with under and over-evaluation of the asset prices.

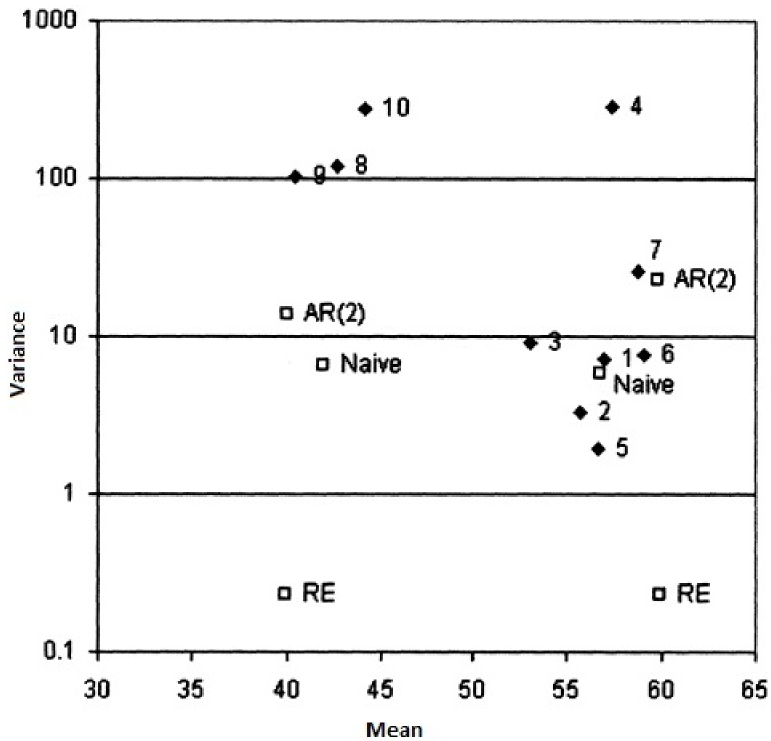

Figure 3: Sample mean and sample variance of 10 different groups (♦) and 3 benchmark rules (□).

Furthermore, in a similar framework, Hommes et al. (2007) investigate the individual expectations in each of the 10 markets. First of all, they find a coordination on a common strategy, as it is represented in Figure 4. It is interesting to notice how the coordination is not on the fundamental value but there is correlation among the subjects' mistakes. For this reason, it seems that the agents' behaviour and the coordination within the market are self-fulfilling. Secondly, even if the participants are not rational, the errors are unbiased and without autocorrelation within two lags, also, the subjects' returns are high.

This result supports the hypothesis that agents use simple rules of thumb when their heuristics are enough successful.

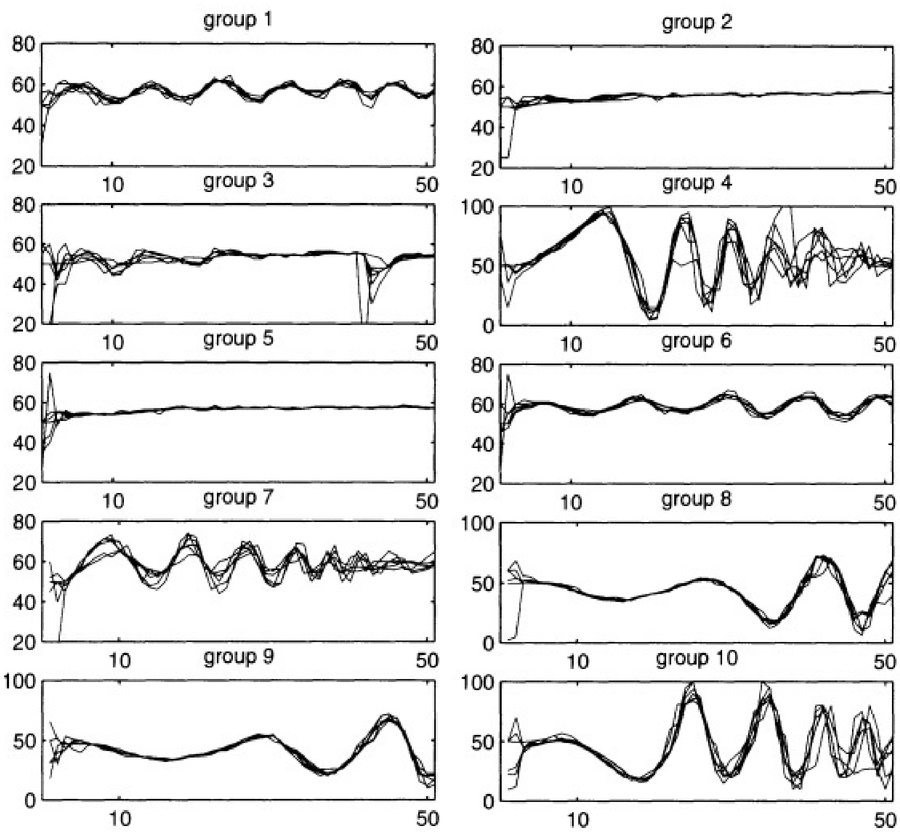

Figure 4: Time series of the 6 individual predictions of the subjects.

In a more recent paper,[23] the same authors analyse the expectations formation in an asset pricing experiment with a sample of 36 participants divided in 6 groups. The framework of the experiment differs from the previous because the model does not inhibit the rise of bubbles, i.e. there are not fundamentalist computer traders which push the forecasts toward the fundamental steady state. As consequence, the laboratory experiments exhibit the endogenous rise of speculative bubbles. The explanation of this price evolution can be found in

[23] Hommes et al. (2008)

the positive feedback of the expectations. Indeed, when the subjects observe a change in price, they try to extrapolate trends affecting their predictions. Due to the selfconfirming nature of the model, these predictions lead to the rise of the actual price and so on. Even if the evolution of prices is far from the rational predictions, it should be noticed that the bubbles do not present a significant autocorrelation structure, i.e. the market seems informationally efficient. Moreover, this analysis seems to confirm the previous results on the coordination among subjects. The participants make structural forecasting errors and deviate from the rational prediction, however it seems that they follow a common path in deviation from the fundamental value.

Heemeijer et al. (2009) ideally extend the previous investigations considering positive and negative expectations feedback. Therefore, the pricing behaviours are derived by an asset pricing model and by a cobweb model respectively. The experiment environment consists of 13 markets, 6 with negative and 7 with positive feedback, with 6 participants each one. The first interesting result is that in the negative feedback model, after an initial phase of high volatility, all the market prices quickly converge to the fundamental level. On the contrary, the positive feedback experiments reproduce fluctuations around the equilibrium price and do not exhibit convergence to the fundamental level. Nevertheless, both models exhibit little dispersion between subjects' forecasts within experimental markets. Hence, in the positive feedback frame there is consensus on the future price that, however, it is not the fundamental price. After the analysis of these aggregate market behaviours, the authors estimate the single individual forecasting rule in the two frameworks. In the asset pricing model, it seems that the trend following strategies are more important. The participants base their forecast on a weighted average of the last price and the last prediction, then extrapolate the trend without considering the fundamental value. They could be labelled as naive and adaptive learning followers. Differently, when the feedback is negative some subjects behave as contrarians and the predictions are a weighted average between the last observed price and the equilibrium price. In this case the expectations of the participants may be described as adaptive-average price expectations.

Moving to the macroeconomic framework, it is difficult to test the expectation hypothesis empirically. Branch (2004) develops a model with agents who forecast one-year ahead inflation rate choosing among a sample of forecasting rules. These rules are: naive, adaptive expectation and VAR forecast. The proportion of agents selecting a rule depends on the mean squared error (MSE) and on a fixed cost for the predictor. Hence, the author infers empirically the heuristics testing the ARED approach using survey data. These data are taken from the Michigan Survey of the Survey Research Center of the University of Michigan and consist in households forecasts on price levels.

When maximum likelihood is used to investigate a new choice mechanism, agents dynamically switch among heuristics and the portion of those using a forecasting rule is inversely related with its MSE. So, two other interesting empirical insights should be pointed out. Firstly, the Michigan data seem to suggest that the expectations formation is not based on a rational choice among the alternatives, but rather there is a positive predisposition to one prediction rule over another. Secondly, it seems that there is a kind of stickiness in the mechanism. More precisely, the forecast error has to cross a threshold to induce the agents to change their forecast rule. This result is in line with the *status quo*-effect analysed in economic psychology.[24]

Considering another empirical application of the ABS frame, Bolt et al. (2014) use an OECD housing data set to search evidence on the heterogeneous expectations in a standard housing market model. The aim of their paper is

[24]See for example Kahneman et al. (1991).

to develop a theoretical model with endogenous switching between two different heuristics. They try to estimate the model using data on rents and house prices from 8 countries (United States, Japan, United Kingdom, the Netherlands, Switzerland, Spain, Sweden and Belgium). These countries have been selected because 1) they have recently experienced a housing bubble, 2) they are quite at the peak of a bubble or 3) they are in a house-price corrective regime.

The switching mechanism between a mean-reverting and a mean-diverting rule inserted in the model displays booms and bursts in the prices evolution around the fundamental level. These dynamics are triggered by stochastic shocks and amplified by the self-fulfilling nature of the expectations. All the countries show a house price bubble driven and intensified by the trend extrapolation. It seems that when the housing bubble burst, the agents amplified the downward price correction switching to the fundamental-reverting forecast strategy. Moreover, the model gives a few policy suggestions. For example, decreasing the mortgage interest rate or rising the housing rents may lead the system closer to instability. On the contrary, stabilizing policies include an increase in the tax rate for home owners or a reduction in the mortgage tax deduction rates. Policy makers should use the market sentiment as a warning indicator on the evolution of the system to prevent bubbles and bursts.

Switching now to laboratory experiments in the macroeconomics framework, Assenza et al. (2013) study the formation of expectations and their interaction with different monetary policies. The theoretical model underlying is the standard New Keynesian model. Hence, the main equations of the model are:

$$y_t = \bar{y}^e_{t+1} - \varphi \left(i_t - \bar{\pi}^e_{t+1} \right) + g_t,$$

$$\pi_t = \lambda y_t + \rho \bar{\pi}^e_{t+1} + u_t,$$

$$i_t = \phi_\pi \left(\pi_t - \bar{\pi}\right) + \bar{\pi},$$

where $\bar{x}^e_{t+1} = \frac{1}{H}\sum\limits_{i=1}^{H} \bar{x}^e_{i,t+1}$ is the average of the participants' predictions.

The authors run experiments using 3 different treatments. In the first and second ones the output gap expectations are fixed and set as fundamentalist and naive respectively, so the agents have to forecast only the inflation level. Conversely, in the third analysis there are 2 groups of subjects and each one forecasts one variable.

At this point, an experimental session is ran twice for each treatment. The aim of these repetitions is to investigate the stabilization properties with two different monetary policies, weak ($\phi_\pi = 1$) and aggressive ($\phi_\pi = 1.5$). The agents entitled to forecast only the inflation level are divided into 6 groups of 6 subjects: 3 groups for each policy. The third treatment consists only of 4 groups: 2 for each policy.

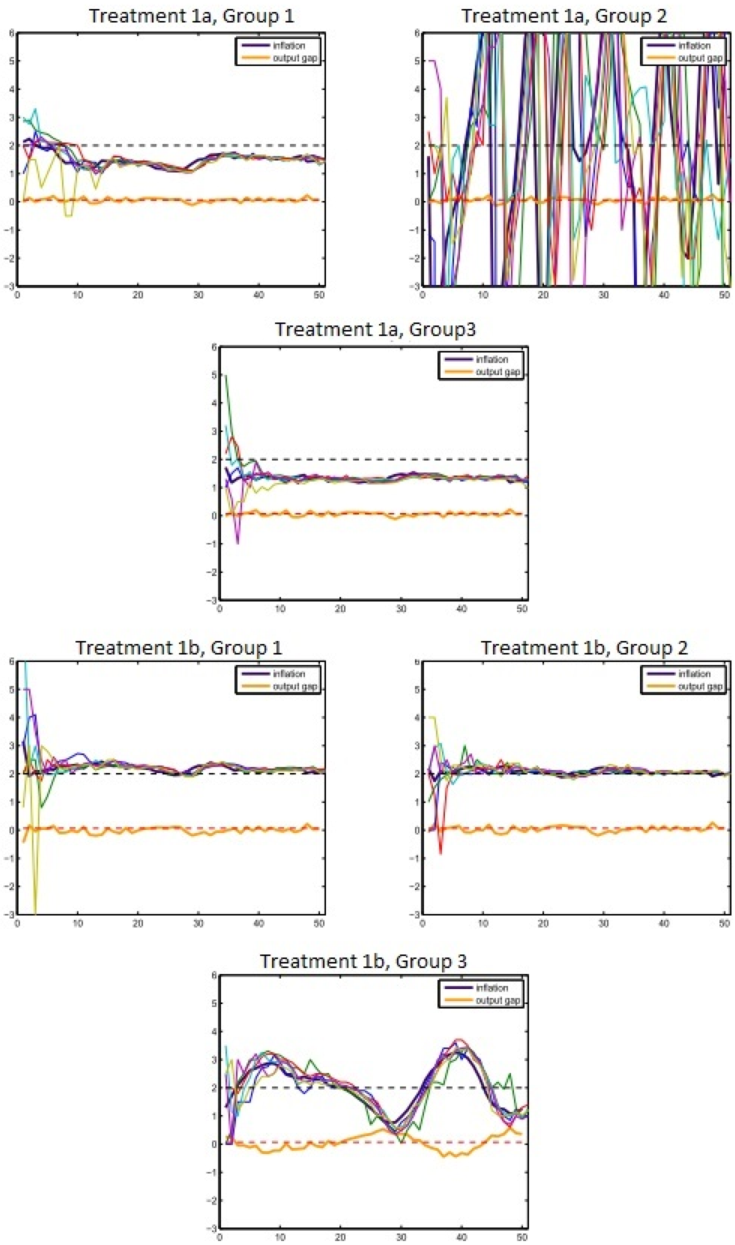

Figure 5: Time series of Treatment 1: Treatment 1a is the weak monetary policy case, Treatment 1b the aggressive monetary policy. The thick lines in the plots represent the realized variables, the thin lines the individual forecasts for the two variables.

Figure 5 illustrates that the first treatment exhibits convergence to a non-fundamental steady state for 2 over 3 groups when the policy is weak, whereas the aggressive monetary policy is able to reach the inflation target in 2 over 3 groups.

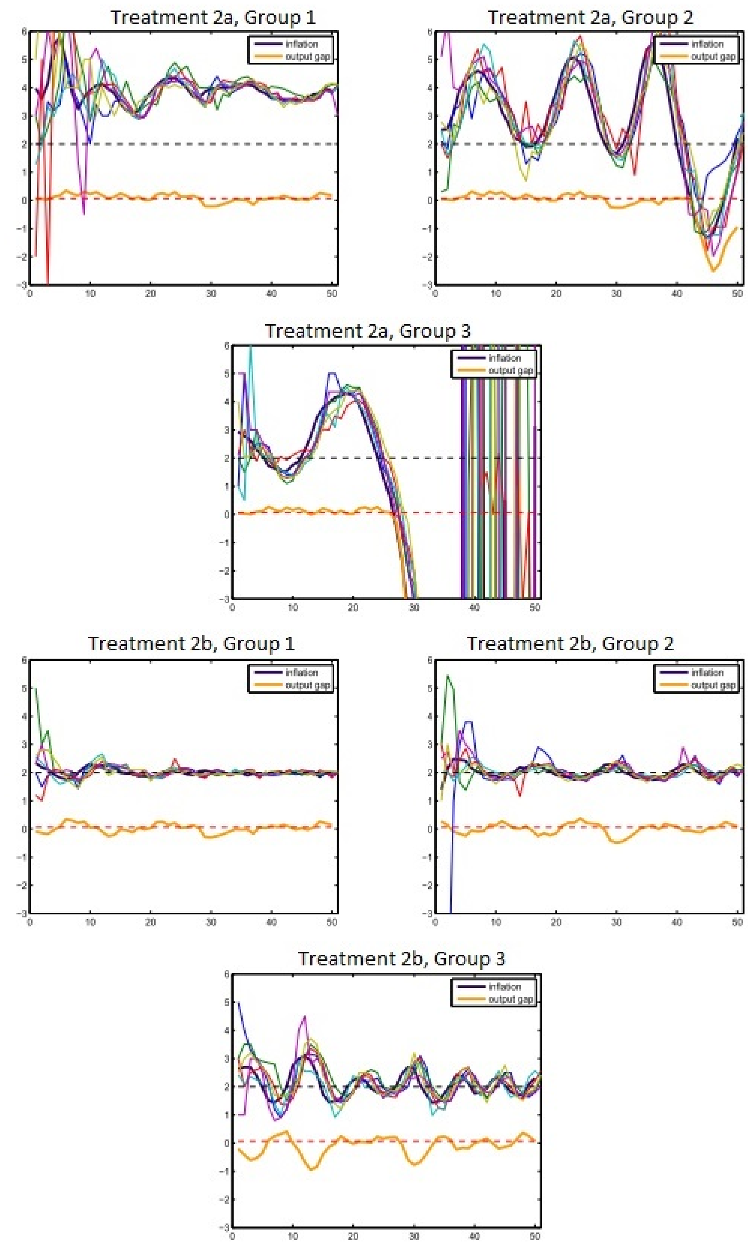

Figure 6: Time series of Treatment 2; Treatment 2a is the weak monetary policy, Treatment 2b is the aggressive monetary policy. The thick lines represent the realized variables and the thin lines the individual predictions.

Figure 6 shows how, in case of weak monetary policy, 3 different patterns emerge under the assumption of naive expectations for output gap. The first group exhibits convergence to a non-fundamental steady state. The second one performs an increasing oscillatory path with high convergence of the forecasts. The third group has a more unstable behaviour with an initial oscillation followed by a drop to the zero lower bound. On the contrary, the evolution of the groups subject to aggressive policy is quite similar to the previous case, in 2 over 3 samples the experiment converges to the steady state level.

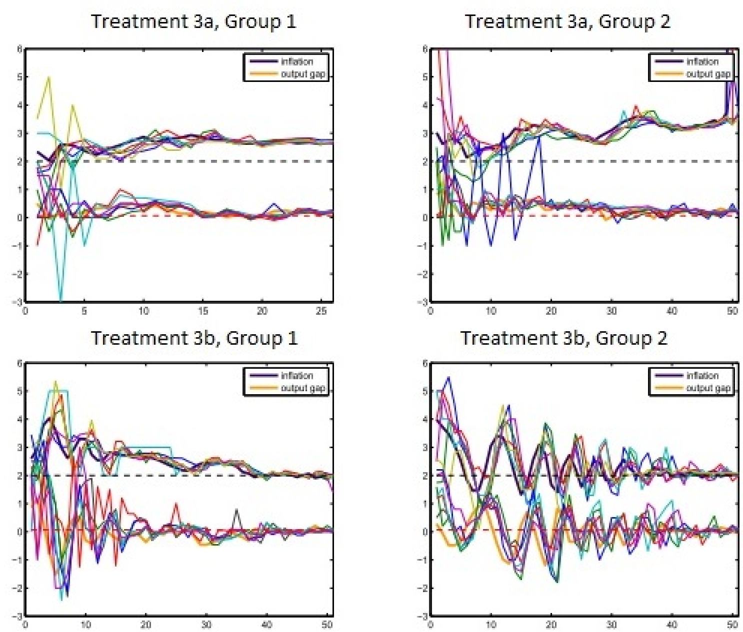

Figure 7: Time series of Treatment 3: Treatment 3a is the weak monetary policy case, Treatment 3b the aggressive monetary policy. The thick lines represent the realized variables and the thin lines the individual predictions.

When the policy is weak and the realized inflation and output gap depend both on individual forecasts (Figure 7), there is convergence to a non-fundamental steady state. On the contrary, when the policy is aggressive, after initial fluctuations the price path converges to the fundamental level in both groups. Summarizing, it seems that a central bank which implements a strong monetary policy can lead the economy to the target results in all the treatments.

After the analysis of the aggregate behaviour, the authors characterize the individual forecasting. First of all, they find a persistent heterogeneity in the forecasting rules used by the participants. In addition, the learning process of the subjects consists in the switch from one heuristic to another. Having found empirical evidence for heterogeneous expectations and switching processes, Assenza et al. (2013) introduce four different rules of thumb and a discrete choice mechanism with asynchronous updating in order to simulate the experimental data.

The four heuristics are:

$$adaptive\,rule:\ x^e_{1,t+1} = 0.65x_{t-1} + 0.35x^e_{1,t},$$

$$weak\,trend-following\,rule:\ x^e_{2,t+1} = x_{t-1} + 0.4\left(x_{t-1} - x_{t-2}\right),$$

$$strong\,trend-following\,rule:\ x^e_{3,t+1} = x_{t-1} + 1.3\left(x_{t-1} - x_{t-2}\right),$$

$$anchoring\,and\,adjustment\,rule:\ x^e_{4,t+1} = 0.5\left(x^{av}_{t-1} - x_{t-1}\right) + \left(x_{t-1} - x_{t-2}\right).$$

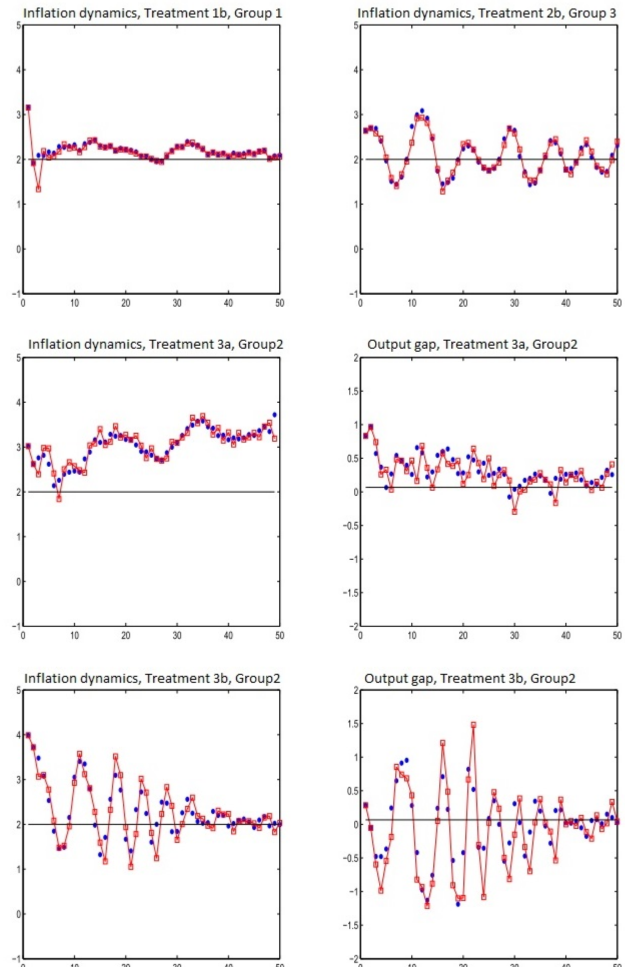

Figure 8: Experimental data (blue points) and one-period ahead simulations (red lines).

The paper proceeds with the empirical validation of the model. As Figure 8 shows, the one-period ahead simulation results fit quite well in the experimental data. So, it seems that subjects tend to base their prediction on past observations by following simple heuristics. The learning process undertaken by participants is simply reduced to switching from one rule to another. It is interesting to notice how in the same economy there can be coordination on different heuristics for both the output and the inflation levels. Agents may behave in a naive way when forecasting inflation but they may follow the adaptive rule in the output prediction. Moreover, the forecasting and the out-of-sample forecasting performances of the model have been evaluated finding that this frame outperforms models with homogeneous expectations, both rational and heuristic. Concluding, this paper evidences empirically the relevance of considering the heterogeneous expectations together with bounded rationality in the macroeconomic modelling.

1.5 Final Remarks

This review presented the general theoretical framework of the Adaptive Belief System showing how it may be used to investigate different research questions. The Hommes's notion of ABS describes a framework where heterogeneous agents switch among expectations rules according to some fitness measures determining the evolution of the actual variables. The switching process introduces in the model non-linear interactions and creates room for sensitive dependence on initial condition. The dynamics generated by this approach are highly non-linear systems and can produce a wide range of behaviours according to the dominant heuristic: from simple convergence to a steady stable to very irregular and unpredictable fluctuations.

Markets are described as complex adaptive systems, so prices, volumes and the expectations population co-evolve over time. Within the ABS approach, the problem of the "wilderness of bounded rationality" is disciplined by parsimony and simplicity of strategies and their relative performance but a problem may persist. Indeed, this approach maintains too many degrees of freedom and too many parameters in the model which can make difficult to assess the main causes of the observed results at the aggregate level. For this reason there is the need for empirical works aimed to reduce the parametrization problem by estimating the economic and financial parameters as well as the rule of thumbs.

Unfortunately, although this literature is still growing, this is mainly computational and theoretically oriented, hence there should be a further, and fundamental, effort to estimate economic and financial data using survey and experimental evidence. Firstly, laboratory analyses give empirical validation to the importance of heterogeneity in the theory of expectations and in the evolution of the economic system. Indeed, heterogeneity allows to explain the path dependence and the complex evolution. It can illustrate different aggregate outcomes across different market settings. Moreover, the empirical results produced by heterogeneous agents models can validate the theoretical frameworks. Conversely, the estimates can be substituted as parameters in the theoretical models in order to have outcomes empirically validated from the bottom.

In the last years, the literature has investigated also some interesting theoretical extensions of the ABS, as the structural heterogeneity in the learning process, i.e. the possibility that different agents use different algorithms. For example, Honkapohja and Mitra (2006) study an overlapping generation model focusing attention on econometric learning with infinite memory. Another interesting extension is proposed by Anufriev, Hommes and Makarewicz (2015) who investigate the generating expectations process using a Genetic Algorithm (GA) optimization procedure derived by biology. Through the genetic algorithm the authors do not have to specify the heuristics because these are the results of a simple optimization process of the agent. Hence, the "wilderness problem" is regulated by the algorithm. Therefore, the results of the GA approach can be used to design and validate simple heuristic switching models.

Concluding, the Adaptive Belief System seems to be useful to describe a wide range of economic problems. Its performance is in many cases even better than the one of the standard homogeneous and fully rational approach. The main persisting problem is the extreme freedom given by heuristics, thus, further research should try to find the simplest behavioural heterogeneous hypotheses able to reproduce the observed stylized facts. The effort to discipline the wilderness of this approach may be done by searching empirical validations in laboratory experiments and/or in survey data series.

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Chapter 2

The complex effect of bankruptcy in a financial accelerator framework

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2.1 Introduction

The Bernanke Gertler and Gilchrist (1999) model is a Dynamic Stochastic new Keynesian model incorporating credit-market imperfections (the "financial accelerator"[1]) which can amplify both real and nominal shocks to the economy. This "financial accelerator" is rooted in the link between the external finance premium (the difference between the cost of external and internal founds) and the net worth of the borrower.

Starting from the analysis of the financial accelerator framework, this paper will focus the investigation on the effects of bankruptcy in the credit market.

[1] Other models of financial accelerator are for example Bernanke and Gertler (1989,1990).

Representative-Agent macroeconomic models with financial frictions[2] take into account the probability of default, however actual bankruptcies never occur in these models otherwise, by assumption, the entire corporate sector would collapse. Indeed, the representative agent assumption makes it impossible to distinguish between financially fragile and financially robust agents. The heterogeneity of financial conditions is a key to introduce actual bankruptcy in macroeconomic models and to explain the effects of default both on aggregate demand and on aggregate supply. For this reason, in this paper, the credit market relationships will be described using the Agent Based (AB henceforward) approach.[3]

More precisely, this paper will investigate the indirect effects of bankruptcies on the financial intermediaries' balance sheet showing how changes in the net worth of the lender can amplify and propagate fluctuations to the whole system. Hence, with the goal to consider these delinquencies' effects on the financial intermediary's behaviour, the paper will explicitly model the share of bankrupt agents who leave the market introducing some real consequences on the lender-borrower relationship. Summarizing, the final purposes of this paper are to investigate: the "real effect" of bankruptcy on the system fluctuations, the corporate sector and the financial intermediary net worth; the propagation and persistence of unexpected shocks in a model with actual bankruptcies.

The paper can be ideally divided in two main parts: the next two sections describe the financial accelerator model and its possible weaknesses, whereas the last two sections introduce an Agent Based financial accelerator and perform some simulations.

In the second section, I will summarize the main non-standard contributions of the Bernanke Gertler and Gilchrist (BGG henceforward) model. In the

[2]For example see Kiyotaki and Moore (1997), Christiano et al. (2010) or Gertler and Karadi (2011).

[3]For a thorough analysis on this approach see Delli Gatti et al. (2011).

third section, I will look at the two main problems which do not seem properly solved by BGG model. The first shortcoming is the exogeneity of the share of entrepreneurs going bankrupt and leaving the market. The second weakness is that banks are always willing to lend to entrepreneurs even if they have gone bankrupt. These two aspects are closely linked to the representative agent assumption. More precisely, given that banks are able to perfectly diversify the risk, they are always willing to lend, also to entrepreneurs who have gone bankrupt in the past. Moreover, the model assumes that these individuals are risk-neutral and have finite horizons. Each entrepreneur must have a constant probability of surviving to the next period for two different reasons, first of all to preclude the possibility that the entrepreneurial sector will ultimately accumulate enough wealth to be fully self-financed, secondly because in this way the authors are able to explain the financial accelerator. In the fourth section, I will present an agent based variant of the BGG model with the aim to find a significant role for bankruptcy clarifying its role on credit network relationships. With the new specification, I investigate the face-to-face relationship between the financial intermediary and each single entrepreneur. The AB approach allows to consider explicitly the bankruptcy adding in the financial contract a premium to the risk-free rate. This premium is equivalent to the difference between the amount received when the default occurs and the expected return on loan. Hence, this paper fits in the reference literature[4] investigating both the direct bank-firm credit relationship and the indirect effect of defaults on the financial intermediary network.

The aim of this part of the paper is to model the finite debt possibilities, i.e. the scenario in which after some episodes of default the financial intermediary shrinks his credit supply. Indeed, consequently to some borrower defaults, the bank suffers some losses due to the default of the borrower, for the same

[4]See for example Delli Gatti et al. (2010), Battiston et al. (2012) or Vitali et al. (2015).

reason the entrepreneur's net worth decreases. Hence, the probability that the entrepreneur will be able to repay a more expensive contract debt, due to the willingness of the bank to cover losses, decreases growing the number of past defaults. This evolution brings the probability of repayment to zero and to a stage in which the financial intermediary is no more willing to lend. Through this mechanism the model should be able to endogenise the share of entrepreneurs that exit from business. Moreover, considering heterogeneous firms, I am able to show the consequences of the defaults of some firms on the balance sheet and on the credit policy of the bank. The fifth section proposes some simulations of the system with three different bounded rational expectation rules with the aim to illustrate how bankruptcy affects the business cycle dynamics of the system and to perform some policy evaluations. The last part (Section 6) concludes suggesting possible future extensions.

2.2 The Bernanke, Gertler and Gilchrist Model

The BGG framework is quite standard according to the related DSGE literature. There are five types of agents (households, entrepreneurs, capital goods producers, financial intermediary and retailers) and the public sector (the central bank and the government) which implements monetary and fiscal policies. Households work, consume and save over an infinite time horizon. Moreover, they can hold money or financial assets which pay interest. Entrepreneurs are the core agent of the model. At the beginning of each period the entrepreneur has a net worth. With this worth entrepreneurs purchase physical capital from capital goods producers financing the difference through loans from the bank. The accumulated net worth plays an important role because it affects the cost of external finance and the agency problem. Indeed, there is a conflict between the interests of the lender and the borrower. The financial contract is set from the bank in order to minimize the expected agency cost. Entrepreneurs combine physical capital with labour in order to produce wholesale output in the following period. The retailers buy wholesale output from entrepreneurs and re-sell the good to households setting in a monopolistic competition. This introduces nominal stickiness in prices in the economy.

2.2.1 The demand for capital and the financial contract

At the end of period t the entrepreneur who manages the *j-th* firm buys capital to be used in *t+1*, K_{t+1}^{j}. The price of capital is denoted by Q_t. By assumption capital is homogeneous. Therefore, the financial constraints apply to the whole capital of the firm and not just to investment.

The entrepreneur purchases capital goods $Q_t K_{t+1}^{j}$using the available net worth N_{t+1}^{j} and bank loans B_{t+1}^{j}:

$$B_{t+1}^{j} = Q_t K_{t+1}^{j} - N_{t+1}^{j}. \tag{2.1}$$

Bank loans are extended by a financial intermediary (henceforward a bank) who faces an opportunity cost equal to the risk free gross rate, R_{t+1}. Entrepreneurs are risk neutral and households are risk averse, so the entrepreneur absorbs any risk. In order to motivate a non-trivial financial structure, BGG assume a "costly state verification" (CSV) framework as Townsend (1979) in which lenders pay an auditing or monitoring cost in order to observe the realization of the entrepreneurial return. This monitoring cost can be interpreted as a cost of bankruptcy.

The return on invested capital is subject to aggregate and idiosyncratic risks. The individual (firm-specific) return to capital is $\omega^j R^k_{t+1}$, where R^k_{t+1} is the average gross return (uniform across firms) and ω^j is the idiosyncratic risk, a stochastic variable, i.i.d. across time and firms, with a continuous and one-differentiable c.d.f. $F(\omega)$ over a non-negative support and $E(\omega^j) = 1$.

Given the choices of the entrepreneur on K^j_{t+1}, B^j_{t+1} and given the aggregate return on capital R^k_{t+1}, the optimal contract is characterized by a non-default firm-specific interest rate, Z^j_{t+1}, and by a threshold value of the idiosyncratic shock, $\bar{\omega}_{t+1}$ such that the borrower is able to fulfil her repayment.

$$\bar{\omega}_{t+1} R^k_{t+1} Q_t K^j_{t+1} = Z^j_{t+1} B^j_{t+1}. \tag{2.2}$$

If $\omega^j_{t+1} \geq \bar{\omega}_{t+1}$ the lender obtains $Z^j_{t+1} B^j_{t+1}$ and the borrower gets $\omega^j_{t+1} R^k_{t+1} Q_t K^j_{t+1} - Z^j_{t+1} B^j_{t+1} \geq 0$, whereas if $\omega^j_{t+1} < \bar{\omega}_{t+1}$ the borrower cannot validate the debt commitment and declares default. In this case the bank pays the auditing cost obtaining $(1-\mu)\omega^j_{t+1} R^k_{t+1} Q_t K^j_{t+1}$, where μ is the fraction of the return on capital spent monitoring the borrower.[5]

The lender should receive an expected return on lending at least equal to the opportunity cost of lending his funds. Therefore, his participation constraint is

$$[1 - F(\bar{\omega}_{t+1})]\, Z^j_{t+1} B^j_{t+1} + (1-\mu) \int_0^{\bar{\omega}_{t+1}} \omega^j_{t+1} f(\omega^j_{t+1}) d\omega^j_{t+1} R^k_{t+1} Q_t K^j_{t+1} \geq R_{t+1}(Q_t K^j_{t+1} - N^j_{t+1}), \tag{2.3}$$

[5] Monitoring cost occurs only when the borrower defaults but with probability one (i.e. with certainty). In Bernanke and Gertler (1989) monitoring was stochastic: the financial intermediaries would audit only when the entrepreneur declared default but with a probability lower than one.

where $F(\bar{\omega}_t) = \int_0^{\bar{\omega}_{t+1}} \omega_{t+1}^j f(\omega_{t+1}^j) d\omega_{t+1}^j$ is the bankruptcy probability and $(1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}^j dF(\omega_{t+1}^j) R_{t+1}^k Q_t K_{t+1}^j$ is the expected return for the lender if the borrower defaults. The LHS of inequality (3) is the total expected return on lending while the RHS is the total opportunity cost of lending B_{t+1}^j.

Combining equations (2.1), (2.2) and (2.3), through simple algebra, the participation constraint can be expressed as follows:

$$\left\{ [1 - F(\bar{\omega}_{t+1})]\, \bar{\omega}_{t+1} + (1-\mu) \int_0^{\bar{\omega}_{t+1}} \omega_{t+1}^j f(\omega_{t+1}^j) d\omega_{t+1}^j \right\} R_{t+1}^k Q_t K_{t+1}^j \geq R_{t+1}(Q_t K_{t+1}^j - N_{t+1}^j). \tag{2.4}$$

The expression in brackets is the expected return on lending per unit of capital, i.e. the fraction of profits that the firm gives to the lender per unit of capital (the fraction of the rate of profit going to the lender). An increase in the cutoff value has three effects on the expected return: (i) the non-default payoff $\bar{\omega}_{t+1} R_{t+1}^k Q_t K_{t+1}^j$ will increase; at he same time (ii) the probability of default rises reducing the expected payoff; (iii) the expected return for the lender if the borrower defaults $(1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}^j f(\omega_{t+1}^j) d\omega_{t+1}^j R_{t+1}^k Q_t K_{t+1}^j$ also increases. Effects (i) and (iii) imply an increase of the expected return while effect (ii) implies a reduction. Defining the expressions $\Gamma(\bar{\omega}_{t+1}) \equiv \int_0^{\bar{\omega}_{t+1}} \omega_{t+1}^j f(\omega_{t+1}^j) d\omega_{t+1}^j + \bar{\omega}_{t+1} \int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}^j) d\omega_{t+1}^j$ the expected gross share of profits going to the lender and $\mu G(\bar{\omega}^j) \equiv \mu \int_0^{\bar{\omega}_t} \omega_t^j f(\omega_t^j) d\omega_t^j$ the expected monitoring costs, thence the inequality (2.4) can be rewritten as:

$$[\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})]\, R_{t+1}^k Q_t K_{t+1}^j = R_{t+1}(Q_t K_{t+1}^j - N_{t+1}^j). \tag{2.5}$$

2.2.2 The optimal choice of capital

The optimal debt contract is defined by the following optimization problem

$max\, [1 - \Gamma(\bar{\omega}_{t+1})]\, R^k_{t+1} Q_t K^j_{t+1},$

subject to $[\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})]\, R^k_{t+1} Q_t K^j_{t+1} \geq R_{t+1}(Q_t K^j_{t+1} - N^j_{t+1}).$

BGG assume: $R^k_{t+1}(1-\mu) < R_{t+1}$ in order to avoid unbounded profits for firms (being $h(\bar{\omega}) = \frac{f(\bar{\omega})}{1-F(\bar{\omega})}$ the hazard rate) and that $\bar{\omega}h(\bar{\omega})$ is increasing in $\bar{\omega}$. All this has two implications: first of all the net payoff to the lender reaches a maximum at a certain level $\bar{\omega}^*$, secondly there is the guarantee of a non-rationing outcome.

Defining the premium on external funds $s_{t+1} = R^k_{t+1}/R_{t+1}$ and rewriting $k_{t+1} = Q_t K_{t+1}/N_{t+1}$ (the capital/wealth ratio) as a choice variable and removing the time pedix and the j suffix for the sake of simplicity, the problem can be rewritten as:

$\underset{\bar{\omega},k}{max}\, [1 - \Gamma(\bar{\omega})]\, sk,$

subject to $[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]\, sk \geq (k-1).$

The first order conditions to this problem can be written as:

$\bar{\omega} :\to \Gamma'(\bar{\omega}) - \lambda\, [\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})] = 0,$

$k :\to [(1 - \Gamma(\bar{\omega})) + \lambda\, (\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))]\, s - \lambda = 0,$

$\lambda :\to [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]\, sk - (k-1) = 0,$ where λ is the Lagrange multiplier.

Assuming an interior solution $\bar{\omega} \leq \bar{\omega}^*$; from the first F.O.C., the Lagrange multiplier can be written as a function of $\bar{\omega}$:

$$\lambda(\bar{\omega}) = \frac{\Gamma'(\bar{\omega})}{[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]}.$$

The participation constraint is binding, so that

$$\lambda'(\bar{\omega}) = \frac{\mu\left[\Gamma'(\bar{\omega})G''(\bar{\omega}) - \Gamma''(\bar{\omega})G'(\bar{\omega}\right]}{\left[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right]^2} > 0,$$

this is due to the assumption that $\bar{\omega}h(\bar{\omega})$ is increasing.

Now defining $\rho(\bar{\omega}) = \frac{\lambda(\bar{\omega})}{[(1-\Gamma(\bar{\omega}))+\lambda(\Gamma(\bar{\omega})-\mu G(\bar{\omega}))]}$, from the F.O.C. we infer that the cutoff $\bar{\omega}$ satisfies

$$s = \rho(\bar{\omega}), \tag{2.6}$$

where $\rho(\bar{\omega})$ is the wedge between the expected rate of return on capital and the safe return demanded by the financial intermediaries. Equation (2.6) shows the monotonically increasing relationship between default probabilities and the premium on external funds.

Inverting (2.6), we obtain the relationship $\overline{\omega} = \overline{\omega}(s)$, where $\bar{\omega}'(s) > 0$. So the cutoff value is increasing with the wedge between expected rate of return on capital and the risk-free interest rate.

Now defining $\Psi(\bar{\omega}) \equiv 1 + \frac{\lambda(\Gamma(\bar{\omega})-\mu G(\bar{\omega}))}{1-\Gamma(\bar{\omega})}$, given the cutoff $\bar{\omega} \in (0, \bar{\omega}^*)$, the F.O.C. imply a capital/wealth ratio:

$$k = \Psi(\bar{\omega}). \tag{2.7}$$

Combining equations (2.6) and (2.7), the authors establish a relationship between capital expenditure and entrepreneur's financial conditions. The capital/wealth ratio may be express as the increasing function of the premium on external funds $k = \Psi(\bar{\omega}(s))$ and given that $k = \frac{QK}{N}$, I can rewrite:

$$Q_t K_{t+1}^j = \psi(s_{t+1}) N_{t+1}^j, \tag{2.8}$$

where $\psi'(\cdot) > 0$ and $\psi(1) = 1$.

Capital expenditure is proportional to the entrepreneur's net worth, with a proportionality factor which is increasing in the expected discounted rate of return. Consequently the higher is the wedge between expected return on capital and risk free rate, the higher is the capability of the firm to borrow from the financial intermediary. The entrepreneur is constrained from raising indefinitely the size of her firm by the fact that increasing the amount of capital borrowed, she also increases the expected default costs. Indeed for non-fully self-financed entrepreneur, the return on capital will be equal to the marginal cost of external finance in equilibrium.

2.2.3 General Equilibrium

At this point, BGG incorporate the contracting problem within a dynamic general equilibrium framework. The capital purchased by the entrepreneur is combined with labour in order to produce wholesale output through the following Cobb-Douglas production function

$$Y_t = A_t K_t^{\alpha} L_t^{(1-\alpha)}, \tag{2.9}$$

where Y_t represents the aggregate production in period t, K_t is the aggregate amount of capital purchased by all the entrepreneurs, L_t is the labour input and A is an exogenous technology parameter. The evolution of capital is

$$K_{t+1} = \Phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta)K_t, \tag{2.10}$$

where δ is the depreciation rate of capital, I_t is the aggregate investment expenditure which yields a gross output of new capital goods $\Phi\left(\frac{I_t}{K_t}\right) K_t$. By assumption $\Phi(\cdot)$ is increasing and concave and $\Phi(0) = 0$. Assuming competitive capital producing firms, the cost of capital Q_t in term of the numeraire good will be

$$Q_t = \left[\phi' \left(\frac{I_t}{K_t}\right)\right]^{-1}. \tag{2.11}$$

Given that entrepreneurs sell their output to retailers who have market power, the relative price of wholesale goods will be $1/X_t$ where $X_t = \frac{P_t}{P_t^w}$ is the gross markup of retail goods over wholesale goods. Consequently, the expected rate of return of capital between two periods can be written as

$$E\left\{R_{t+1}^k\right\} = E\left\{\frac{\frac{1}{X_t}\frac{\alpha Y_{t+1}}{K_{t+1}} + Q_{t+1}(1-\delta)}{Q_t}\right\}, \tag{2.12}$$

where the first term in brackets is the rent paid to one unit of capital and the second is the capital gain due to the fact that entrepreneurs resell underpreciated capital to the capital producing firms. Through simple substitutions of (2.9) and (2.11) into (2.12) the authors find the conventional demand curve for new capital where the return on capital depends inversely on the level of investment:

$$E\left\{R_{t+1}^k\right\} = E\left\{\frac{\frac{P_t^w}{P_t}\frac{\alpha A_{t+1} L_{t+1}^{(1-\alpha)}}{K_{t+1}^{1-\alpha}} + (1-\delta)\left[\Phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\right]}{\left[\Phi'\left(\frac{I_t}{K_t}\right)\right]}\right\}.$$

Taking the mean of both sides of (2.8), which expresses the link between expenditure and net worth of the single entrepreneur, it is possible to obtain the supply curve of investment finance. This shows the dependence of the external cost of funds with the aggregate financial condition of the entrepreneurs of the whole economy. From equation (2.8) it can be written $\frac{Q_t K_{t+1}}{N_{t+1}} = \psi(s_t)$ and $s_t = \psi^{-1}\left(\frac{Q_t K_{t+1}}{N_{t+1}}\right)$, therefore

$$E\left\{R_{t+1}^k\right\} = s\left(\frac{N_{t+1}}{Q_t K_{t+1}}\right) R_{t+1} \quad s'(\cdot) < 0, \tag{2.13}$$

where $s(\cdot)$ is the ratio of the costs of external and internal finance and it is decreasing in $N_{t+1}/Q_t K_{t+1}$, i.e. it depends inversely on the share of the firm's invested capital that is financed by the net worth of the entrepreneur.

Besides purchased capital, technology requires labour as an input. The total labour supply of the economy is composed of households and entrepreneurial labour (H_t and H_t^e respectively).

$$L_t = H_t^{\Omega}(H_t^e)^{1-\Omega}.$$

The demand curves for labour in a competitive market imply that the real wage equates marginal product, therefore it will be

(a) $(1-\alpha)\Omega\frac{Y_t}{H_t} = X_t W_t$,

(b) $(1-\alpha)(1-\Omega)\frac{Y_t}{H_t^e} = X_t W_t^e$, where W_t and W_t^e are the real wage rate for the household and the entrepreneur respectively.

The evolution of the aggregate entrepreneurial net worth N_{t+1} is described by the following law of motion:

$$N_{t+1} = \gamma V_t + W_t^e, \tag{2.14}$$

where V_t represents the equity held by entrepreneurs and γ is the fraction of entrepreneurs which survives in each period. Hence γV_t is the net worth of entrepreneurs still in business in the following period. This equity is the residual part of the return of the investment after the repayment of the loans to the financial intermediaries.

$$V_t = R_t^k Q_{t-1}K_t - \left(R_t + \frac{\mu\int_0^{\bar{\omega}_t}\omega_t^j dF(\omega_t^j)R_t^k Q_{t-1}K_t}{Q_{t-1}K_t - N_t}\right)(Q_{t-1}K_t - N_t), \tag{2.15}$$

where the term $\frac{\mu \int_0^{\bar{\omega}_t} \omega_t^j dF(\omega_t^j) R_{t+1}^k Q_t K_{t+1}}{Q_t K_{t+1} - N_{t+1}}$ reflects the premium for external finance and $\bar{\omega}_t$ the state-contingent value of $\bar{\omega}$ set in period t.

2.2.4 Households, government sector and retailers

In order to close the model, in this section will be briefly described the household optimization problem, the retail sector and the government budget constraint.

Households have an infinite horizon, they work, consume, hold money and invest their savings in financial assets which pay the risk free interest rate. The household maximization problem is therefore:

$$\underset{C_t, \frac{M_t}{P_t}, H_t, D_{t+1}}{max} \quad E_t \sum_{k=0}^{\infty} \beta^k \left[ln\left(C_{t+k}\right) + \zeta ln\left(\frac{M_{t+k}}{P_{t+k}}\right) + \xi ln\left(1 - H_{t+k}\right) \right],$$

$s.t. \quad C_t = W_t H_t - T_t + \Pi_t + R_t D_t - D_{t+1} + \frac{(M_{t-1} - M_t)}{P_t}$, where C_t is the consumption of the households, W_t is the household wage, H_t is the supply of labour, T_t are lump sum taxes, Π_t are the dividends of the retail firms, D_t are the deposits held at banks and $\frac{M_t}{P_t}$ is the real money balances between periods.

From the first order conditions, the following equations can be derived:

$$\frac{1}{C_t} = E_t \left\{ \beta \frac{1}{C_{t+1}} \right\} R_{t+1},$$

$$\frac{W_t}{C_t} = \xi \frac{1}{1 - H_t},$$

$$\frac{M_t}{P_t} = \zeta C_t \left(\frac{R_{t+1}^n}{R_{t+1}^n - 1} \right),$$

where $i_{t+1} = R_{t+1}^n \frac{P_{t+1}}{P_t} - 1$ with R_{t+1}^n the gross nominal interest rate.

Moving to the government budget constraint, the basic assumption is that its expenditures are financed by lump sum taxes and money creation, therefore, the government budget will be $G_t = T_t + \frac{(M_{t-1} - M_t)}{P_t}$.

The retail sector is characterized by monopolist competition and costs adjusting nominal prices à la Calvo (1983). Each *nth* retailer sells the quantity of output $Y_t(n)$ at the nominal price $P_t(n)$. The total final goods and their price are therefore the combination of the individual retailer sales:

$$Y_t^f = \left[\int_0^1 Y_t(n)^{(\epsilon-1)/\epsilon} dn \right]^{\epsilon/(\epsilon-1)}, \tag{2.16}$$

with $\epsilon > 1$ and $P_t = \left[\int_0^1 P_t(n)^{(1-\epsilon)} dn \right]^{1/(1-\epsilon)}$.

To introduce stickiness in the prices, in each period a share of firms faces the probability $(1-\theta)$ of being able to reoptimise its price. In every phase a retailer faces a demand curve:

$$Y_t(n) = \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} Y_t^f, \tag{2.17}$$

therefore denoting with: P_t^* the price set by retailers able to reoptimise and $Y_t^*(z)$ the consequent demand given this price, the *n-Th* retailer sets the price in order to maximize his expected discount profits:

$$\sum_{k=0}^{\infty} \theta^k E_{t-1} \left\{ \Lambda_{t,k} \left(\frac{P_t^*}{P_t} \right)^{-\epsilon} Y_t^*(n) \left[\frac{P_t^*}{P_t} - \left(\frac{\epsilon}{\epsilon - 1} \right) \frac{P_{t+k}^w}{P_{t+k}} \right] \right\}. \tag{2.18}$$

In Equation (2.18) $\Lambda_{t,k} = \beta C_t / C_{t+k}$ represents the discount rate equal to the shareholders intertemporal marginal rate of substitution, $P_t^w = P_t / X_t$ is the nominal price of the wholesale goods and θ^k is the probability that the price is fixed for k periods.

Given that the share θ of retailers is not able to reoptimise in period t, the evolution of the price will be:

$$P_t = \left[\theta P_{t-1}^{1-\epsilon} + (1-\theta)\left(P_t^*\right)^{1-\epsilon}\right]^{1/(1-\epsilon)}. \tag{2.19}$$

Finally, according to the standard literature,[6] the model is closed considering the short-term nominal interest rate as the main instrument of the monetary policy. The central bank adjusts the nominal interest rate according to the following Taylor Rule:

$$r_t^n = \rho r_{t-1}^n + \tau \pi_{t-1} + \varepsilon_t^n,$$

therefore the monetary authority reacts to the lagged inflation and the lagged interest rate.

For the sake of fluency and given that it is not central for the development of the following sections, the complete standard log-linearized model will be presented in Appendix A.

2.3 Shortcomings of the model

The two main ingredients of the financial accelerator model are the equations of the investment funds supply, Equation (2.13), and the law of motion of aggregate entrepreneurial net worth, Equation (2.14).

It may be useful to recall them for the reader's convenience:

$E\left\{R_{t+1}^k\right\} = s\left(\frac{N_{t+1}^j}{Q_t K_{t+1}^j}\right) R_{t+1},$

$N_{t+1}^j = \gamma V_t^j + W_t^e.$

Bankruptcy is an important ingredient of the financial accelerator story as

[6] See for example Clarida et al. (2000) or Galì (2008).

it motivates the financial contract. In the BGG model, however, there is no explicit modelling of the share of entrepreneurs who goes bankrupt and leaves the market. Moreover, delinquency, i.e. the self-fulfilment of debt repayment, has no direct impact on the relationship between lender and borrower. Indeed, in the model, financial intermediaries are always willing to supply loans regardless of the "history" of debt repayment of the borrower. Even if the entrepreneur defaults on her repayment, she does not exit from the market because her net worth will always be positive. In order to show this paradoxical implication of the BGG model, notice that if the *j-th* agent defaults - i.e. $\omega_t^j < \bar{\omega}_t$- then $V_t^j = 0$. In this case the net worth in *t+1* will be

$$N_{t+1}^j = 0 + W_t^e. \tag{2.20}$$

In this example the borrower is not able to repay her debt obligation, the investment yields $\omega_{t+1}^j R_{t+1}^k Q_t K_{t+1}^j$ and is entirely appropriated by the bank. Taking into account monitoring costs, the bank obtains $(1-\mu)\omega_{t+1}^j R_{t+1}^k Q_t K_{t+1}^j$. However, the net worth of the entrepreneur is still positive. This is due to the fact that (i) the entrepreneur is working for the firm and gets a wage W_t^e; (ii) the entrepreneur devotes her wage to increase the net worth of the firm. The net worth of the defaulting firm is equal to the entrepreneur's wage. Since the net worth is positive, the possibility for the entrepreneur to borrow also in the following period cannot be ruled out. Moreover, in the next period the bankrupt entrepreneur will not have any additional penalty for defaulting on her payment.

Suppose now that in the previous period $\omega_{t-1} > \bar{\omega}_{t-1}$, so the net worth at the beginning of period t is:

$$N_t^j = \gamma V_{t-1}^j + W_{t-1}^e,$$

where $V_{t-1}^{j} = (\omega_{t-1}^{j} - \bar{\omega}_{t-1})R_{t-1}^{k}Q_{t-2}K_{t-1}^{j}$ is the net share of profit going to the entrepreneur after having paid her debt in the previous period.

Two important aspects emerge observing the evolution of these variables. First of all, after two consecutive defaults, the net worth becomes stable and equal to the equilibrium wage. Indeed, from equation (20), it can be noticed that even if in every period the *j-th* entrepreneur is not able to fulfil her debt payment, she receives anyhow a wage which maintains her net worth positive. Secondly, even if the entrepreneur is defaulting in each period, from equations (2.1) and (2.20) her net worth remains positive and so does her borrowing capacity, hence she can continue to stay in business: $B_{t+1}^{j} = Q_t K_{t+1}^{j} - W_t^{e}$.

Rewriting here the log-linearized form of the equation of the entrepreneurial wage, the law of motion of capital and the expected gross return to hold a unit of capital respectively:

$$w_t^e = y_t - x_t - c_t, \tag{2.21}$$

$$k_{t+1} = \delta i_t + (1-\delta)\, k_t, \tag{2.22}$$

$$r_{t+1}^{k} = (1-\varpi)\,(y_{t+1} - k_{t+1} - x_{t+1}) + \varpi q_{t+1} - q_t, \tag{2.23}$$

where in equation (2.21) it is assumed inelastic entrepreneurial labour supply, $H_t^e = 1$. Substituting Equation (2.22) and (2.23) in the log-linearized version of the equation (8), it can be obtained the following equation;

$$\Upsilon^{b} b_{t+1} = \psi \left\{ (1-\varpi)\, E_t\,(y_{t+1} - k_{t+1} - x_{t+1}) + \varpi E_t q_{t+1} - q_t - r_t \right\} + \left(1 - \Upsilon^{w^e}\right) w_t^e, \tag{2.24}$$

where $0 < \Upsilon < 1$. In equation (2.24), the terms Υ^b and Υ^{w^e} represent the ratios of debt and entrepreneurial wage with the total purchased capital at their steady state level. From equation (2.24) it can be seen that the optimal level of debt in equilibrium depends positively from the depreciation rate of capital and from the production function, whereas it depends negatively on the market power of the retailers. It has to be noticed that the debt and consequently the capital purchased will be always positive, since in the steady state $w_t^e > 0$. This shows that, even if all the firms default, they always have access to the credit system.

Summarizing, the main goal of the BGG model is to clarify the role of credit market frictions. The framework exhibits a financial accelerator that amplifies and propagates the shocks. This financial accelerator links inversely the "external finance premium" and the net worth of the borrower. In doing this the model assumes that firms are risk-neutral and have finite horizons, each entrepreneur must have a constant and exogenous probability of surviving to the next period. In this way the authors are able to explain the financial accelerator. Hence, even if the bankruptcy has a key role in the financial accelerator story, the model does not explicitly represent the number of bankrupt entrepreneurs who leave the market but defines them exogenously. Moreover, the self-fulfilment of debt repayment has no direct impact on the relationship between lender and borrower.

The remainder of this section will try to overcome this shortcoming incorporating an extra cost in the financial contract of the bankrupt entrepreneur. Suppose that the entrepreneur is not able to fulfil the repayment obligation, hence the bank obtains a lower return than expected and registers losses: $Los_{t+1} = Z_{t+1}B_{t+1} - (1-\mu)\omega_{t+1}^j R_{t+1}^k Q_t K_{t+1}^j$. For this reason, the financial intermediary adds an additional cost to the defaulted entrepreneur interest rate in the following period. The new participation constraint of the financial intermediary becomes:

$$\left[\Gamma(\bar{\omega}_{t+2}) - \mu G(\bar{\omega}_{t+2})\right] R^k_{t+2} Q_{t+1} K^j_{t+2} = R_{t+2}(Q_{t+1}K^j_{t+2} - N^j_{t+2}) + \left[Z_{t+1}B_{t+1} - (1-\mu)\omega^j_{t+1} R^k_{t+1} Q_t K^j_{t+1}\right],$$

where the left hand side of the equation represents the expected return on lending and the right hand side represents the risk-free rate plus a premium equal to the losses. With the new framework, the optimal choice for the entrepreneur will be:

$max\left[1 - \Gamma(\bar{\omega}_{t+2})\right] R^k_{t+2} Q_{t+1} K^j_{t+2},$

s.t. $\left[\Gamma(\bar{\omega}_{t+2}) - \mu G(\bar{\omega}_{t+2})\right] R^k_{t+2} Q_{t+1} K^j_{t+2} = R_{t+2}(Q_{t+1}K^j_{t+2} - N^j_{t+2}) + Los_{t+1}.$

Recalling here that the premium on external funds is defined as $s = R^k/R$ and the capital-wealth ratio is $k = QK/N$. Assuming: uniform distribution for the idiosyncratic shock; constant risk free interest rate, $R_{t+1} = R_t$ and that the wealth of the defaulted entrepreneur is $N_{t+2} = w^e_{t+1} = N_{t+1}$,[7] the first order conditions can be written as:

$\bar{\omega} :\rightarrow \Gamma'(\bar{\omega}) - \lambda\left[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})\right] = 0,$

$k :\rightarrow \left[(1 - \Gamma(\bar{\omega})) + \lambda\left(\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right)\right] s - \lambda = 0,$

$\lambda :\rightarrow \left[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})\right] sk - 2\Delta\left(k - 1 - L\right) = 0,$

where $L = \frac{\bar{\omega}_{t+1} R^k_{t+1} Q_t K^j_{t+1} - (1-\mu) G(\bar{\omega}_{t+1}) R^k_{t+1} Q_t K^j_{t+1}}{R_{t+2} N_{t+2}}$. Solving, the optimal level of capital-wealth ratio is:

$$k_{t+2} = \frac{2\Delta - \left(2\Delta\bar{\omega} - \bar{\omega}^2 - \mu\bar{\omega}^2\right) sk}{2\Delta - s\left[2\Delta\left(1 - \bar{\omega}\right) + \left(1 - \mu\right)\bar{\omega}^2\right]}. \tag{2.25}$$

Assuming a fraction of profit lost in bankruptcy, μ, to 0.2 and an initial risk

[7] The assumptions of constant interest rate and constant net worth across the periods in case of default are taken for the sake of simplicity. According to the model, the defaulted entrepreneur net worth should be lower than the initial level. In this case, the negative term in the nominator of Equation (25) will be higher empowering the argument that this type of model may not be optimal to investigate the real effect of bankruptcy.

spread, $R^k - R$, equal to 2%, it is possible to represent the relationships between the external finance premium (s) and the threshold of the idiosyncratic risk ($\bar{\omega}$) and between external premium and the capital-wealth ratio (k). From the first order conditions of the two specifications, it can be seen how both the threshold of the idiosyncratic risk and the capital-wealth ratio are function of the external finance premium. In the left panel, Figure 1 shows how the existence of the extra premium on the relationship between lender and defaulted borrower affects neither the positive relationship between s and $\bar{\omega}(s)$ nor the shape of the function. On the contrary, as displayed in the right panel, the losses registered by the financial intermediary completely change the relationship between the external finance premium (s) and the capital-wealth ratio ($k(s, \bar{\omega})$). Firstly, it should be noticed that, in the standard model with uniform distribution, increasing the external finance premium decreases the optimal investment-wealth ratio. This shape can be explained by the fact that for high level of external finance premium, the threshold of the idiosyncratic risk is higher. Consequently, this increases the cost of loans (Z) and therefore reduces the amount of optimal debt. Conversely, in the framework with real losses for the bank, the optimal level of capital-wealth ratio is function of $\bar{\omega}$, s and L (the losses-wealth ratio) and, as it is displayed in the figure, its exhibits a reverse U-shaped slope. Finally, the investment-ratio function of the second specification is defined in the negative vertical axis. In other words, the consequence of a financial contract that adds an extra cost for the defaulted entrepreneur is the collapse of the credit market (or the invested capital), given that by definition N is positive.

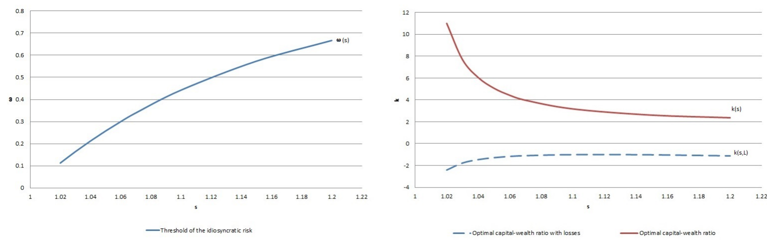

Figure 1: Relationship between external finance premium, threshold of idiosyncratic risk (left panel); relationship between external finance premium and capital wealth ratio (right panel). Thick line: original model; dashed line: framework with losses.

As shown, this approach investigates the credit market frictions but it does not seem optimal to analyse the bankruptcy and to modelling explicitly its effects on the credit market and total investments. Indeed, in the standard model, the default of entrepreneur has no direct impact on her future relationship with the lender, she always has access to the loans. On the contrary, in the specification with losses in the financial contract, the credit market collapses after the first bankruptcy. In order to investigate the effects of the entrepreneurs bankruptcies, from the next section I will introduce an Agent-Based approach.[8] This frame, abandoning the representative agent assumption, will allow to introduce heterogeneity in the firms wealth and on the shocks that hit the productivity. Moreover, analysing the one-to-one relationship in the financial contract, it will also be possible to introduce a premium on the external finance cost for the entrepreneurs defaulted in the past. Despite the change of paradigm, the Agent Based version of the financial accelerator is as close as possible to the original one in its main characteristics, e.g. the relationship between external premium

[8] See for example Delli Gatti et al. (2011), Fagiolo and Roventini (2012) or Assenza and Delli Gatti (2013).

and net worth or the evolution of the entrepreneurial wealth.

2.4 The Agent Based version of the financial accelerator

This section discusses the new agent based specification of the financial accelerator explaining the main assumptions that depart from the original model. Leave the representative agent assumption opens the possibility to analyse the effect of a bankruptcy chain,[9] indeed the AB specification allows to investigate the interactions among the financial intermediaries and heterogeneous firms.

Moreover, following the growing literature which recognizes the drawbacks of the rational agent paradigm, the bounded rational expectations will be introduced in the agent based specification. According to survey data analyses[10] the financial agents are not fully rational: they use different trading and forecasting strategies. Moreover there is empirical evidence that the human being has critical limits on both cognition and computational capabilities and the use of rules of thumb is held by psychological studies which show how agents compare alternative heuristics avoiding deliberation efforts and complicated computational costs.[11]

2.4.1 The heterogeneous financial intermediaries

Conversely, to the original model, the AB specification hypotheses the existence of many lenders and not only a representative financial intermediary. Given the choices of the entrepreneur on K^j_{t+1}, B^j_{t+1} and given the risk free

[9]Interesting examples are Delli Gatti et al. (2003) and Delli Gatti et al. (2005).
[10]See Frankel and Froot (1987 a,b and 1990 a,b).
[11]See Conlisk (1980).

interest rate R_{t+1}, loans are extended by a financial intermediary. The external finance cost for a non-default firm is Z^j_{t+1}, such that:

$$Z^j_{b,t+1} = R_{t+1} + \mu^b \left(\frac{D_t}{N_b}\right)^{\alpha} + \rho \frac{B^j_{t+1}}{N^j_{t+1}}. \tag{2.26}$$

μ^b is a bank specific parameters, $\frac{D_t}{N_b}$ is a financial soundness measure of the bank given by the ratio between deposits, D_t, and net worth, N_b, whereas $\frac{B^j_{t+1}}{N^j_{t+1}}$ is the firm specific leverage ratio. This new mechanism for the interest rate on loans has the same flavour of the original external finance cost but adds some important features. As in BGG model, the firm-specific interest rate on loans is a mark-up on the risk free interest rate and depends positively on the leverage ratio$\left(\frac{B^j_{t+1}}{N_{h,t+1}}\right)$. Besides the firm-specific mark-up, also the financial soundness of the financial intermediary has a role in the credit policy decision for at least two reasons. First, as point out also by Delli Gatti et al. (2010), bank with higher financial strength, i.e. higher $\left(\frac{D_t}{N_b}\right)$, will be able to extend credit at more favourable terms. Second, when a financial intermediary increase its net worth a "too big to fail" problem can arise. In other words, big banks do not consider the extreme hypothesis of their default, as consequence of low cost loans to firms with high debt. This because they expect that, if a negative financial shock should hit them, they will be bail-out from the public authority.

The entrepreneur investment return is affected by idiosyncratic risk, ω^j_t. This risk is a stochastic variable with an uniform distribution. When the idiosyncratic shock is such that

$$\omega^j_{t+1} R^k_{t+1} Q_t K^j_{t+1} \geq Z^j_{t+1} B^j_{t+1},$$

the lender obtains $Z^j_{t+1} B^j_{t+1}$ and the borrower earns the difference between the investment return and the debt cost.

Conversely, if

$$\omega^j_{t+1} R^k_{t+1} Q_t K^j_{t+1} < Z^j_{t+1} B^j_{t+1},$$

the borrower cannot validate her debt contract, therefore the entrepreneur defaults and the bank obtains $(1-\mu)\omega^j_{t+1} R^k_{t+1} Q_t K^j_{t+1}$.

In this case, the balance sheet of the financial intermediary should be negatively affected by the default of the entrepreneurs. Now, suppose that in the period t, the bank's total amount of lending is:

$$B^b_t = \frac{1}{\gamma^b} N^b_t, \tag{2.27}$$

N^b_t represents the total wealth of the bank γ^b represents a capital requirement share.

The evolution of the bank wealth can be described by the following law of motion:

$$N^b_{t+1} = N^b_t + \sum_{j=1}^{J} (\Pi^j_t) - \alpha \sum_{j=1}^{J} Los^j_t - r_t D_t + r_t \mathit{\Delta}_t, \tag{2.28}$$

where $\sum\limits_{j=1}^{J} (\Pi^j_t)$ is the sum of return on entrepreneurs' loans of the previous period, $\sum\limits_{j=1}^{J} Los^j_t = \sum\limits_{j=1}^{J} \left[Z^j_t B^j_t - (1-\mu)\omega^j_t R^k_t Q_{t-1} K^j_t \right]$ is the sum of the total unexpected losses, α is a dummy parameter, it is zero if in the previous period the entrepreneur fulfilled her debt obligation, one otherwise, D_t are the deposits and $\mathit{\Delta}_t$ are non risky assets investment.

Introduce the wealth of the financial intermediary implicitly means to insert in the model a defined amount of available funds. The existence of many financial intermediaries does not ensure that the credit is bound in all the periods, i.e. in some periods there could be accessible funds that are not borrowed. From the balance sheet of the financial intermediary, it should be noticed that $\mathit{\Delta}_t > 0$

only when all the allocable funds are not hold as reserves or required by the entrepreneurs, i.e. $R_t + B_t^b < D_t + N_t^b$ where $R_t = \delta^b D_t$.

Nevertheless, it can happen that after some periods the credit may be constrained. In this case the financial intermediary gives priority in lending to entrepreneur with higher net worth. This rule of thumb can be justified by two reasons: first of all by the assumption that higher net worth is taken by the banks as a proxy of success of the firms in the previous periods; secondly, higher net worth represents higher collateral for the bank in case of defaults.

2.4.2 Entrepreneurial behaviour

Introducing heterogeneity in the heuristic forecasting rules the expectations assume a crucial role in the investment decision of the single entrepreneur. At time t, each entrepreneur forms her one-step-ahead expectation on her own capital return and on inflation level. It should be noticed that the model assumes expectations on the real capital return, i.e. in their decision process the entrepreneurs consider their own return of capital, $\omega^j R^k$, and not the general return of capital of the whole economy R^k.

Given the expectations, in the first period the optimization problem of the *j-th* entrepreneur is:

$$\max_B E_t^j \left\{R_{t+1}^k\right\} Q_t K_{t+1}^j - Z_{b,t+1}^j B_{t+1}^j \quad st \; Z_{b,t+1}^j = R_{t+1} + \mu^b \left(\frac{D_t}{N_b}\right)^\alpha + \rho \frac{B_{t+1}^j}{N_{t+1}^j},$$

and from the fist order condition of the maximization problem the optimal level of debt is:

$$B_{h,t+1}^j = \frac{E_t^j \left\{R_{t+1}^k\right\} - \left[R_{t+1} + \mu^b \left(\frac{D_t}{N_b}\right)^\alpha\right]}{2\rho} N_{t+1}^j. \tag{2.29}$$

In Equation (2.29), the capital expenditure of the entrepreneur is proportional to her financial condition, with a proportionality factor which increases according to the own capital return expectation. The expected return is by definition firm-specific whereas the mark up on the risk free interest rate is bank specific depending both on its financial soundness and on its lending propensity μ^b. Hence, even if some firms have the same net worth, they may borrow different amount of funds according to their expectation and the financial intermediary which supplies the loans.

As, explained in Subsection 2.3.1, the entrepreneur's return at the end of every period is equal to the difference between the investment return and the debt cost, $\omega_{t+1}^j R_{t+1}^k Q_t K_{t+1}^j - Z_{t+1}^j B_{t+1}^j$. Therefore, it is straightforward that this return depends by: the idiosyncratic shock, what lender supplies the credit and how the entrepreneur forms her own expectations.

Supposing h forecasting rules, the *j-th* entrepreneur updates her beliefs according to a performance measure of the investment. The evolutionary performance measure is publicly available but it is subject to noise, it could be expressed as follows:

$$U'_{h,t} = U_{h,t} + \epsilon_{h,t}^j, \tag{2.30}$$

where the performance of the entrepreneur following the h-Th heuristic is defined as the average return on investment (ROI) using the h-Th rule of thumb, i.e. $U_{h,t} = \frac{\sum_{j=1}^{J} ROI_{j,h,t}}{E_{h,t}}$ where $ROI_{j,h,t} = \frac{\omega_t^j R_t^k Q_{t-1} K_t^j - Z_t^j B_t^j}{Q_{t-1} K_t^j}$ and $E_{h,t}$ is the number of entrepreneurs of the h type in period t. The performance measure defined before has the feature to avoid that the choice of switching is exclusively determined by the firm specific shock, hence to be randomly defined. Considering two heuristics, $h = 1, 2$, the entrepreneur of type 1 will switch to the other rule of thumb if her own return on investment will be lower to the return on investment of her own type, given that this one is lower to the average ROI of the other type. More precisely, the *j-th* entrepreneur changes her expectations rule if:

$$U_{1,t} < U_{2,t} \quad and \quad ROI_{j,1,t} < U_{1,t}.$$

This switching mechanism has the flavour of the Adaptive Belief Systems (ABS) of Brock and Hommes (1997, 1998, 1999), indeed the agents endogenously update their strategy between heuristic according to the performance measure. However, the second inequality introduces in the evolution of expectations a positive predisposition to one prediction rule over another. In other words, there is a stickiness in the mechanism in line to the status-quo effect.[12]

2.4.3 The consequences of Bankruptcy

This sub-section analyses in detail the problem of bankruptcy and how this affects the credit and the investment.

The evolution of the entrepreneurial net worth can be written as:

$$N_{t+1}^{j} = \begin{cases} W_t^e + V_t^j & V_t^j = \left(\omega_{t+1}^j R_{t+1}^k Q_t K_{t+1}^j - Z_{t+1}^j B_{t+1}^j\right) \quad (S1) \\ W_t^e & (S2) \end{cases},$$

Equation (S1) describes the case of non-defaulted entrepreneur, the net worth at the beginning of the next period is given by the wage plus the net return of capital. The second equation $(S2)$ describes the defaulted entrepreneur. In this case the entrepreneur looses all the invested capital and her net worth in the next period will be equal to the wage.

Supposing the case of entrepreneur defaults, the bank obtains $(1-\mu)\omega_t^j R_t^k Q_{t-1} K_t^j$,

[12]For example, see the analysis in economic psychology of Kahneman et al. (1991).

this can be rewritten as:

$$(1-\mu)\omega_t^j R_t^k \left(B_{t+1}^j + N_t^j\right) = (1-\mu)\omega_t^j R_t^k (V_{t-1}^j + W_{t-1}^e + B_t^j),$$

therefore the lender registers a net loss in t. Supposing that the bank does not take any strategical action after the first default of the firm, in the following period the defaulted entrepreneur has access to credit and therefore she could not be able to fulfil the debt contract once again. In this case, the financial intermediary records loan return lower than expected in both periods, by definition:

$$(1-\mu)\omega_t^j R_t^k (V_{t-1}^j + W_{t-1}^e + B_t^j) + (1-\mu)\omega_{t+1}^j R_{t+1}^k (W_t^e + B_{t+1}^j) < Z_t B_t^j + Z_{t+1} B_{t+1}^j. \tag{2.31}$$

Hence, it seems quite reasonable to expect that, after a default, the bank changes its credit policy towards the bankrupt entrepreneur. This aspect is completely absent in the BGG model, indeed in that framework the bank diversifies his risk among entrepreneurs and does not take any action against the insolvent entrepreneurs, i.e. the bank has not memory of the past.

Conversely, this ABM version of the financial accelerator introduces in the financial contract an additional cost to external funds for the defaulted entrepreneurs. This premium increases the cost of credit and alters consequently the amount of available funds. As described in equation (2.31), when an entrepreneur defaults the bank suffers some losses. Therefore, in the following period, the available credit to the default entrepreneur will be at higher cost. This because for the defaulted entrepreneur the bank will set an interest rate on loan able to recover the losses of the past period.

The financial contract for the bankrupt entrepreneur becomes:

$$Z_{t+1}^{j} = R_t + \left[\mu^b \left(\frac{D_t}{N_b}\right)^{\alpha} + \rho \frac{B_{t+1}^{j}}{N_{t+1}^{j}}\right] + \left[Z_t^j B_t^j - (1-\mu)\omega_t^j R_t^k Q_{t-1} K_t^j\right]. \quad (2.32)$$

In equation (2.32), the interest rate on loans is a mark-up on the risk free interest rate and the financial soundness of the bank, as in equation (2.26), plus a premium equal to the losses due to non performing loans of the previous period. For the sake of tractability, the financial contract considers a temporal horizon of two periods, i.e. for each defaulted borrower the bank expects to smooth its losses of t in the next period.

The new optimization problem of the entrepreneur is

$$\max_{B} E_t^j \left\{R_{t+1}^k\right\} Q_t K_{t+1}^j - Z_{b,t+1}^j B_{t+1}^j \;\; st \;\; Z_{b,t+1}^j = R_{t+1} + \mu^b \left(\frac{D_t}{N_b}\right)^{\alpha} + \rho \frac{B_{t+1}^j}{N_{t+1}^j} + Loss_t.$$

Solving the maximization problem, the first order conditions may be written as:

$$B_{t+1}^j = \frac{E_t\left\{R_{t+1}^k\right\} - \left[R_{t+1} + \mu^b \left(\frac{D_t}{N_b}\right)^{\alpha} + Loss_t\right]}{2\rho} N_{t+1}^j. \quad (2.33)$$

It can be easily seen that if the entrepreneur is not defaulted, $Loss_t = 0$, hence equation (2.33) becomes equation (2.29). However, the new financial contract is designed with the aim to take into account the possibility of stop lending. This happens when the spread between the expected return and the mark-up on risk free interest rate is lower or equal to zero. This could have two different reasons. The first occurs if the expectation are too low due to bad entrepreneur's past performances whereas the second takes place if the amount of past losses is huge.

At this point, supposing that the financial intermediary stops lending to the entrepreneur, $B_t^j = 0$, and it registers a negative variation in its wealth for the unexpected losses. From equation (S2), the amount of available funds that the bankrupt entrepreneur may invest in the following period consist only in her wage:

$$Q_t K_{t+1}^j = N_{t+1}^j = W_t^e.$$

The new participation constraint of the entrepreneur does not take into account the external finance cost but concerns the ratio between the expected realization of investment and the risk-free rate:

$$E_t \left\{ R_{t+1}^k \right\} W_t^e = R_{t+1} W_t^e. \tag{2.34}$$

The entrepreneur invests her capital only if her expectation on capital return is such that:

$$\frac{E_t \left\{ R_{t+1}^k \right\}}{R_{t+1}} \geq 1. \tag{2.35}$$

Since the risk-free rate is given, the entrepreneur's choice depends on the expected return on capital, therefore the expectation formation mechanism plays a crucial role in the investment decisions. If Equation (2.35) is not verified, the entrepreneur does not invest and uses her wealth for consumption.

Through these variations, the financial accelerator model is able to explain the fraction of entrepreneurs which does not receive loans, i.e. the share of them whose total wealth is given by the wage. Within this share of entrepreneurs there is a portion γ that leaves the market. With the aim to hold off the problem that after few periods there are no more entrepreneurs in the market, different types of mechanisms could be set. Instead of supposing a mechanism à la Gertler and Kiyotaki (2010) which settles the number of the entrepreneurs entering in the market as a parameter maintaining constant the number of firms, this first version of the model does not fix any inactivity period for the bankrupt entrepreneurs before to obtain again access to the credit market. In other words, if an entrepreneur defaults and drops out of the market in period t she can borrow in the following period.

2.4.4 Aggregate Variables

The entrepreneurs purchase capital and combine it with labour following a standard Cobb-Douglas production function:

$$Y_t = A_t K^{\alpha} L_t^{(1-\alpha)},$$

where Y_t represents the aggregate production, whereas A_t,K_t and $L_t$are the technology parameter, the purchased capital and the hired labour respectively. The total amount of capital purchased in the economy is the sum of the capital invested by the non-defaulted entrepreneurs and by the self-financed one. The price Q_t of this capital is

$$Q_t = Q_{t-1} + \phi\left(\frac{\Delta K_{t+1}}{K_t}\right), \tag{2.36}$$

where the term in brackets represents the percentage variation of purchased capital between two periods and $\phi(\cdot)$ is the elasticity coefficient of the capital price respect to this change.

At the end of the period, the entrepreneurs sell their output to retailers who have market power. Assume that the relative price of wholesale goods is $1/X_t$ where $X_t = \frac{P_t}{P_t^w}$ is the gross markup of retail goods over wholesale goods. Consequently, the market gross return to holding a unit of capital between two periods is

$$R^k_{t+1} = \frac{\alpha A_{t+1} K^{\alpha}_{t+1}}{X_t K_{t+1} Q_t} + (1-\delta)\frac{Q_{t+1}}{Q_t}.$$

The return of capital is the sum between the value of the marginal productivity of capital and the capital gain due to the change of its price.

Besides capital, the production function requires labour. Its supply is composed by households and entrepreneurial labour but, conversely to Bernanke et al., the model assumes an inelastic supply with a full employment economy. These assumptions are functional to the aim of the paper: investigate the bankruptcy effects on the credit market leaving aside the feedbacks on the real economy.

The real wages for the two categories are:

$$W_t = (1-\alpha)\Omega\frac{Y_t}{X_t},$$

$$W^e_t = (1-\alpha)(1-\Omega)\frac{Y_t}{X_t},$$

where $(1-\alpha)\,\Omega$ represents the share of households labour share.

In this new framework households have an infinite horizon, they work, consume and invest their savings in financial assets that pay the risk free interest rate. These households follow a simple Keynesian rule in determining their consumption path instead of the standard maximization problem. Each period they consume a fraction of their wealth and deposit to the financial intermediary the rest. More precisely:

$$C_t = a_1(W_t + R_{t-1}D_{t-1}),$$

$$D_t = (1 - a_1)\,(W_t + R_{t-1}D_{t-1}),$$

with $a_1 \leq 1$. Aggregating, the demand curve can be written as:

$$Y_t = C_t + K_t + C_t^e + G_t,$$

where C_t is the households consumption, K_t the total purchased capital andC_t^e the consumption of the defaulted entrepreneurs that leave the market. As in the original model, the public expenditure follows a stationary auto-regressive process,$G_t = \rho_g G_{t-1} + \varepsilon_t^g$.

The aggregate supply can be interpreted as a New Keynesian hybrid Phillips Curve derived from the staggered Calvo price scheme:

$$\pi_t = \kappa_\pi E_t\left\{\pi_{t+1}\right\} + \kappa_y \tilde{y}_t + \varepsilon_t^\pi. \tag{2.37}$$

As in Bernanke Gertler and Gilchrist, the actual inflation is driven by two components: it depends positively on the inflation expectations and on the output gap. In this version the output gap is defined as the deviation of the actual output from a forecast level. This level is based on the previous output level augmented by a constant rate and it can be interpreted as the output potential level in absence of bankruptcies.

Finally, the short risk free interest rate is adjusted by the Central Bank reacting with a strict inflation targeting according the following non-linear Taylor instrumental rule:

$$R_t = \left(1 + \pi^T\right)(1 + R_t^n)\left(\frac{1 + \pi_t}{1 + \pi^T}\right)^{\phi_\pi}, \tag{2.38}$$

where π^T represents the inflation target and $\phi_\pi > 0$ is the parameter concerning the inflation reaction as in Salle et al. (2013).

2.5 Simulations

This section shows the results of quantitative experiments to illustrate how bankruptcy affects the business cycle dynamics of the system. In subsections 2.5.2 and 2.5.3, are considered the results concerning two different scenarios performing simulations including in the economy the idiosyncratic shock on the return of capital and the aggregate government expenditure shocks. These simulation scenarios separately consider competition in the market between entrepreneur types: naive and following trend agents, naive and biased expectation. Moreover, subsection 2.5.3 will investigate a framework considering naive and following trend agents applying different interest rate rules with the aim to find policy suggestions.

According to the reference literature[13] the agents have cognitive limitations or computational limits, hence the model assumes the following different forecasting rules:

$$x_t^e = g x_{t-1}, \tag{2.39}$$

$$x_t^e = x_{t-1} + w\left(x_{t-1} - x_{t-2}\right), \tag{2.40}$$

$$x_t^e = x_{t-1} + bias, \tag{2.41}$$

where x is the reference variable, $g = 1$, $0 \leq w \leq 1$ and $bias > 0$.

These heuristics introduce a backward-looking components in the system dy-

[13] See for example in Hommes et al. (2005).

namics. Indeed, in all these rules of thumb, the expectation on future variable is based on the past realization, even if with different degree of freedom. Equation (2.39) represents the naive expectations. This rule prescribes that the entrepreneurs form their expectations on future variable rank using the last observed level.[14] Indeed, using the words of Keynes (1936), "it is sensible for producers to base their expectations on the assumption that the most recently realised results will continue". Equation (2.40) represents the chartist rule. Following this heuristic, the entrepreneurs base their actions on the past variable movements. This rule find empirical evidence on the analysis on the financial market trading rules and in laboratory experiments, see Frankel and Froot (1990) and Hommes (2011) respectively. Equation (2.41) describes biased expectations, as in Brock and Hommes (1998). These entrepreneurs have a positive and constant disposition on the trend of the variable.

2.5.1 Model parametrization

The proposed parametrization is quite standard and finds ample validation in the literature. The quarterly discount factor β is 0.98, the capital share α is 0.45, the household labour share $(1-\alpha)\,\Omega$ is 0.64 and the depreciation rate for capital δ is 0.025. Besides these parameters, even if there is not consensus in the literature,[15] the value of elasticity of the price of capital respect to the investment capital ratio $\varphi(\cdot)$ is 0.01. Looking at the financial sector, the natural risk free interest rate R_t required of the economy is fixed to 2%, the shock on investment has a homogeneous distribution with $E(\omega)=1$ and it can reduce or increase the firm specific return of capital by 20%. The share of profit lost in case of default μ is constant at 0.6. The last financial parameters are the percentage

[14] See Hommes et al. (2012) for a thorough discussion on how memory affects the dynamics of the system.

[15] See for example King and Wolman (1996).

of deposits required by the regulator as reserves requirement is fixed to 2%, and the capital requirement in extending loans, according to the Basel regulation it is equal to 8%. Let the probability θ that firms are not able to reoptimise their price within a period equal to 0.75 and the mark-up of the retail sector with respect to the wholesale market is of 1.1. The last parameters selected are related to the monetary policy role, the inflation targeting is 2% and the reaction coefficient on inflation is 0.8. At this point it should be highlighted that the steady state variables concerning the percentage of entrepreneurs that leaves the market or the probability of default of the entrepreneurs must not be defined ex-ante as inputs like in the Bernanke Gertler and Gilchrist model but they will be the results of the simulations, indeed this specification allows to endogenise these variables. For each experiment the economy has $j = 30$ entrepreneurs, two financial intermediary and $T = 800$ periods (quarters). As the model is not deterministic, each simulation is repeated 20 times in order to take into account the randomness of the shock. Moreover the graphical analysis deletes the first and the last 25 period in order to handle the initialization problem of the initial conditions.

Parameter	Legend	Parameter	Legend
$T = 800$	Periods	$j = 30$	Number of firms
$n = 20$	Monte Carlo Simulations	$\alpha = 0.35$	Capital share
$b = 2$	Number of banks	$\beta = 0.98$	Quarterly discount factor
$\delta^b = 0.02$	Reserve requirement	$X = 1.1$	Retailers mark up
$\gamma^b = 0.08$	Capital requirement	$\delta = 0.025$	Depreciation rate of capital
$\rho = 0.5$	Mark-up on firm's leverage	$\pi^\pi = 2\%$	Inflation target
$\mu^b = [0.02;\ 0.05]$	Mark-up on the risk free rate	$\phi^\pi = 0.8$	Reaction coefficient on inflation
$\mu = 0.6$	Share of profit lost in default	$Y = 5\%$	Output long run growth rate
$\kappa_\pi = 0.925$	Phillips curve coefficient on π	$\kappa_y = 0.075$	Phillips curve coefficient on $\tilde{y}$

Table 1: Calibration.

2.5.2 Scenario 1: Naive vs Trend following agents

In the model the dynamic endogenously arises from two sources: one is the heterogeneity in the expectations rule, the other is due to the bankruptcy effect on the credit policy. The first simulations performed in order to understand the bankruptcy effects consider only naive and weak trend followers entrepreneurs. Following the empirical evidence,[16] the model hypothesizes that the trend-follower entrepreneurs behave using the same heuristic in forecasting both inflation and return to capital, but they adopt different weights for the trend parameters ($w_\pi = 0.1$ and $w_{Rk} = 0.3$).

Analysing the results of the first scenario, Figure 2 shows the time series of output (upper panel), total investment (middle panel) and inflation (bottom panel). The thick lines in the figures represent the average among the

[16]See Assenza et al. (2013).

simulations of the variable, whereas the shadow area illustrates the standard deviation. For the sake of clarity argumentation, the first differences plot of output and capital are in appendix B. Looking at the long-run dynamic, it is possible to notice a growing trend for the output and the total investment. On the contrary inflation presents a path with low volatility. This stable evolution of the system can be found also in the low average growth of rate of output and capital, respectively 0.25% and 0.21%. Moving the analysis to the short-run evolution, it is possible to notice how all the three series present perturbation in the dynamics. The existence of entrepreneurs which can make mistakes in the expectations formation, and therefore can default, explains the evolution of the invested capital. Consequently, this affects the evolution of the output and the inflation dynamic.

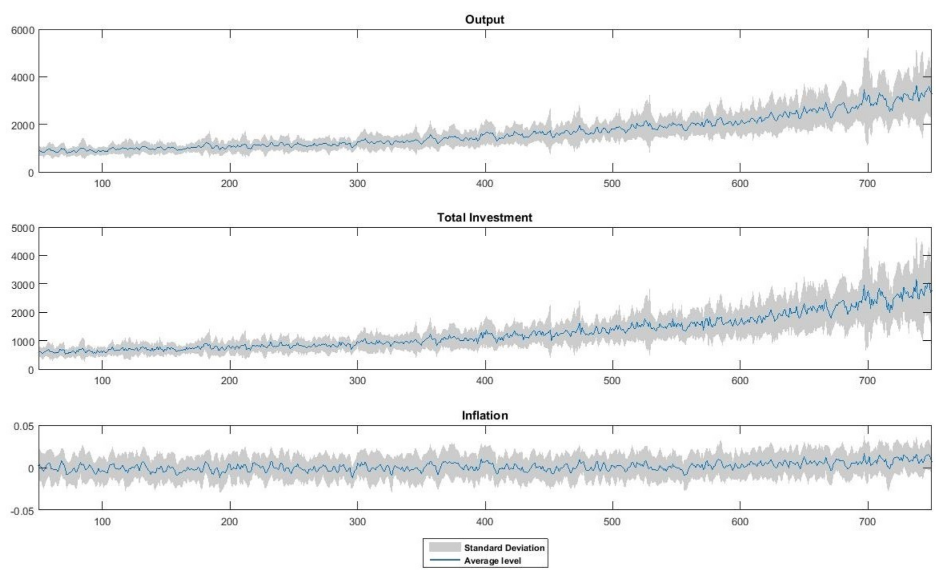

Figure 2: Time series of output (upper panel), total investment (middle panel) and inflation (bottom panel). Blue tick line: average level; grey shadow area: standard deviation.

Moving to the evolution of the core of the agent based specification, Figure

3 illustrates the time series of the difference in the average performance between the rules of thumb[17] (upper panel), the number of naive agents (middle panel) and the number of entrepreneurs which does not invest and leaves the market (bottom panel). The first plot illustrates how none heuristic performs better than the other for all the period but there is as sinuous evolution of the difference between the ROI. When the graph illustrates positive spikes this means that the naive agents perform, on average, better than the type agents, whereas when there are negative spikes trend-follower entrepreneurs register on average higher performance. This evolution partially drives the the switching mechanism. The upper and the middle plots would have had the same dynamics if the share of agents that chooses each strategy would be updated only according to this difference. On the contrary, the model presents a stickiness in choice of heuristics. This explains the non-perfect correspondence between the two dynamics. Investigating the auto-correlation of the difference in performance (Figure 11 in Appendix B), the variable exhibits a negative auto-correlation in the first quarter that may be due by the nature of the two heuristics. For example, extracting the reasoning by the default problem and supposing a positive difference in performance, some agents would change their forecasting rule to the naive behaviour. Given the implicit pro-trend nature of the heuristics,[18] in the next period the following trend behaviour will register higher performance because, in average, exploits better the positive trend. The bottom panel exhibits an oscillating dynamic around the average number of agents leaving the market. It is interesting to notice how: this level is not high, around 7 (6.82), and also the fluctuations are moderate, between 4 and 9 agents. This explains the smoothed trends of the total investment and output of Figure 1 and the

[17] This is defined as the difference between the naive average performance minus the trend-following one, $U_{N,t} - U_{TF,t}$.

[18] Notice that the naive heuristic is equal to the trend-following rule when the weight of the trend is null, $w = 0$.

absence of wide negative spikes. Indeed, leaving the market the defaulted entrepreneurs affect negatively the purchased capital and the total output but, as the panel shows, entrepreneurs stay in the market and, whatever they have access to the credit market or not, they invest sustaining the growing capital trend.

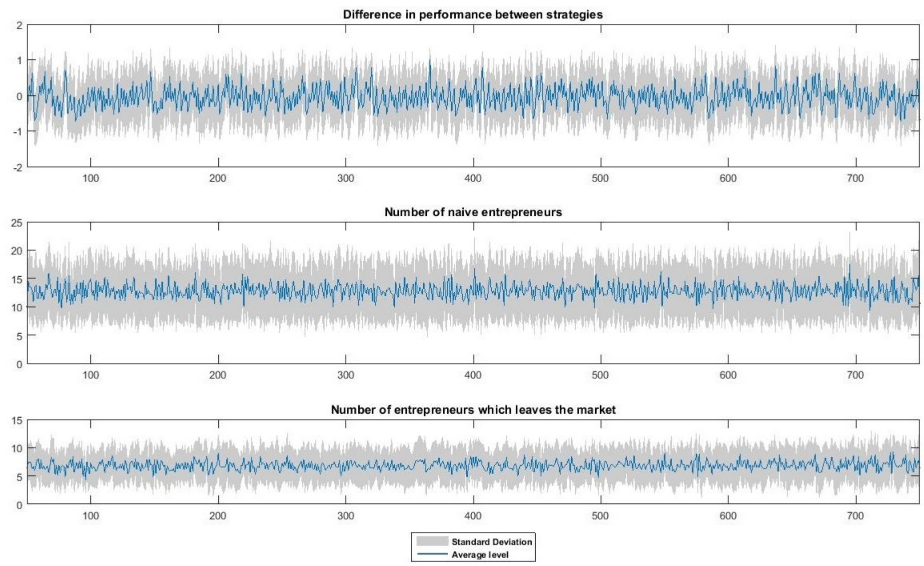

Figure 3: Time series of the difference in performance measure (upper panel), number of naive agents (middle panel) and number of entrepreneurs which leaves the market (bottom panel). Blue tick line: average level; grey shadow area: standard deviation.

Concluding the analysis of scenario 1, Figure 4 presents the variation between periods of the net worth of the financial intermediary. It is interesting to highlight that: the net worth evolution of the two banks is almost identical; this evolution is strictly linked to the bottom panel of Figure 3. Indeed, these entrepreneurs are defaulted agents and, if they leave the market, the financial intermediary has to register their non-collectable credits.

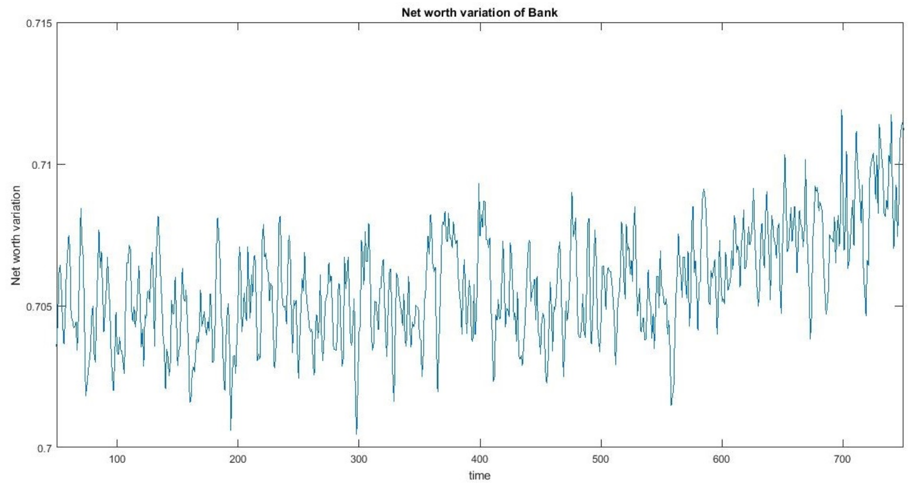

Figure 4: Evolution of the bank net worth (log-scale).

2.5.3 Scenario 2: Naive vs Biased expectations

This subsection describes the results of the simulation concerning naive and biased agents. As in the previous subsection, entrepreneurs behave using the alternative heuristic adopt different values for the bias in forecasting inflation and return to capital, for the first variable the bias is 0.5% whereas it is 5% for the expected return on investment.

Figure 5 depicts the time series of output (upper panel), total capital (middle panel) and inflation (bottom panel) for the scenario 2. The dynamics of output and total investment presents a long-run positive trend with growth rate of 0.13% and 0.12% respectively along the 750 quarters. The inflation evolution exhibits a series fluctuation around an average positive level equal to 3.5%. Looking at the short-run dynamics, all the variables register positive and negative spikes due to the expectations heterogeneity of the model combined with the positive number of bankrupt entrepreneurs which leaves the market generate. Indeed, all these perturbations are the consequences of the real effect of bankruptcy on the economy and the credit market. The default of an investor affects negatively the total amount of invested capital and the output. The effect on capital could be alleviate if in the following period the entrepreneurs invest although they may not have access to the credit market. Besides, if entrepreneurs do not invest, they have a positive weight in the aggregate output through the consumption channel.

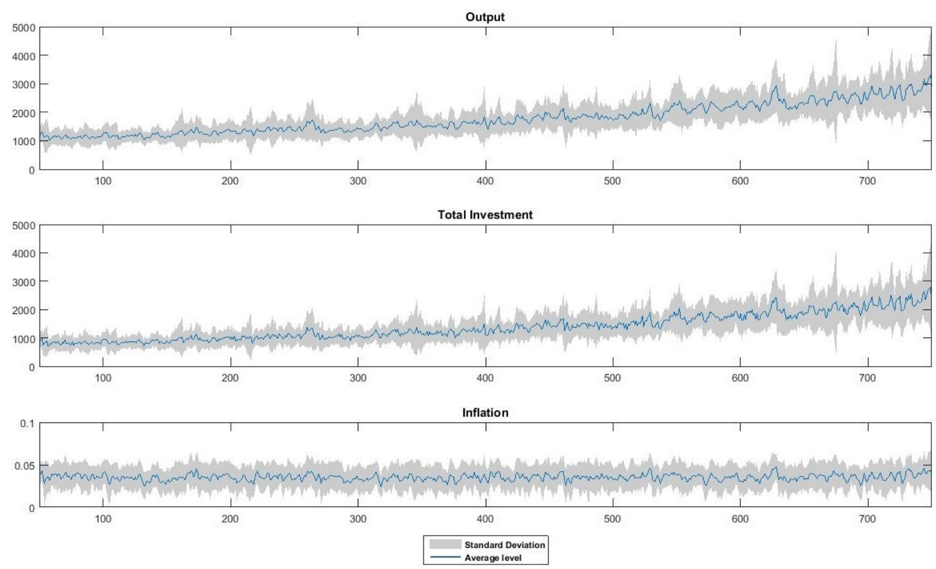

Figure 5: Time series of output (upper panel), total investment (middle panel) and inflation (bottom panel). Blue tick line: average level; grey shadow area: standard deviation.

Figure 6 illustrates the evolution of the difference in the average heuristic performance (upper panel), the number of naive entrepreneurs (middle panel) and the number of entrepreneurs which quits the market (bottom panel). The upper plot shows how in this case the biased heuristic performs, on average, better than the naive rule. All this affects the evolution of the naive agents in the markets, more precisely the average level (12) of agents around which the serie fluctuates. However, the middle plot does not exhibit a stable decreasing trend even if the naive performances on average are worst. This is due by the fact that in the switching mechanism has a crucial role the own return on investment. Indeed, It may happen that, even if the average performance measure of biased type is higher than the naive, the *j-th* entrepreneur registers higher ROI than her average type and therefore she does not switch the forecasting rule. In this scenario, the auto-correlation of the difference in heuristic performance presents a negative and statistically significant auto-correlation for the first two lags. This could be explained by the fact that given a difference in return, on average negative, the number of biased entrepreneurs should increases. This majority of biased agents registers higher levels of purchased capital respect to the naive agent and therefore enhances also the total output and the return on its own investment. Indeed, it should be noticed that the naive and the biased expectations are equal when the bias is null, $b = 0$. Moreover, taking as given all the other variables, when b>0, the expected return of the biased agent is higher than the naive expected return and therefore also the amount of investment. The lower panel represents the evolution of the entrepreneur share which leaves the market. As in the previous scenario, the dynamic displays noisily fluctuation around an average level between 6 and 7 which can partially explain the smoothed trends of capital and output.

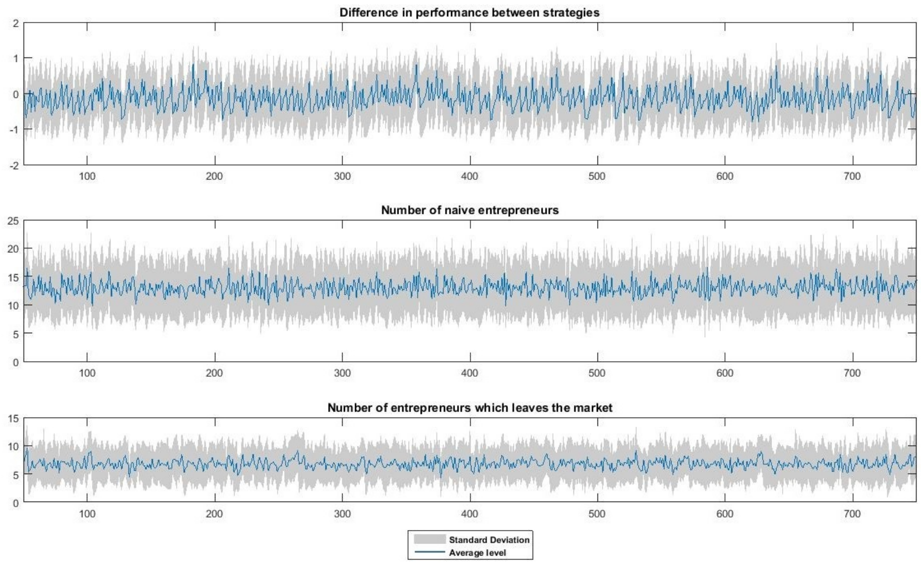

Figure 6: Time series of the difference in performance measure (upper panel), number of naive agents (middle panel) and number of entrepreneurs which leaves the market (bottom panel). Blue tick line: average level; grey shadow area: standard deviation.

Figure 7 reproduces the evolution of the bank net worth. The positive number of the entrepreneurs which defaults and leaves the market in each period affects the wealth of the financial intermediary. It is worthwhile to stress how the positive spikes could have three explanations: high profits, low losses from the non-performing loans or a low level of credit, i.e. an high level of non-risky investment (high level of Δ_t). Indeed, the possibility of default of the entrepreneurs does not ensure to the bank that higher level of extended loans will be follow by higher net worth growth than a scenario with lower level of credit.

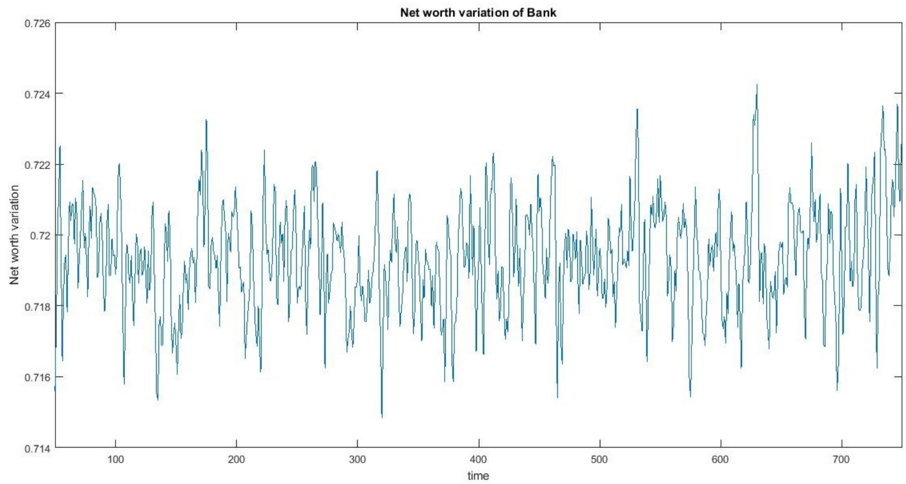

Figure 7: Evolution of the bank net worth (log-scale).

Finally the analysis compares the levels and the volatility between the two scenarios. As displayed in figures 2 and 5, capital and output exhibit comparable levels at the end of the period even if the two scenarios present different level of growth. However, the scenario 2 registers higher initial level for both the variable. These levels could be explained by a lower standard deviation in the number of entrepreneurs which leaves the market (γ) and consequently in a lower volatility in the invested capital. The volatility of γ also affects the level of the bank net worth. As shown by figures 4 and 7, the lower volatility in the scenario 2 allows to reach higher bank net worth. Looking at the average level of inflation (figures 2 and 5), in the first scenario it is around 0% whereas in the second it is 3.5%. This difference is related with the nature of the expectations rules. Indeed, in the naive-bias simulations, the expected inflation is the mean between the forecast levels of naive and biased agent. Therefore, even if in one period the inflation rate is null, it is straightforward that the expectations for the future level will be positive given that the expectations of one heuristic are positively biased.[19] Moreover, the volatility of the number of entrepreneurs which leaves the market and the mechanism of the expectation formation explain the flatter paths in the time series of the second scenario. Comparing the two alternative heuristics (trend-following and bias), by definition, the first alternative rule of thumb, following the evolution of the variables, can generate waves that allows both higher rates of growth and higher volatility. On the contrary, biased agent always forecast a constant path growth reducing the volatility of the invested capital and therefore of the total output. Concluding, a monetary authority should take into account the "sentiment of the market", i.e. the shares of agent types, when designing its policy. Indeed, as shown in this section, applying the same monetary policy may have different consequences on the economic fluctuations when the expectations are non-identical. For this reason, in order to conclude the analysis, in the next subsection will be performed some simulations applying different monetary policies in the Scenario 1.

2.5.4 Monetary policy Evaluation

This subsection presents some simulations using different monetary policies considering weak-trend followers and naive entrepreneurs. The analysis firstly presents the results comparing two empirically founded interest rate rules, then will be investigate a more theoretical problem concerning the effects of policies non-conforming to the Taylor principle. The first two Taylor rules are designed with the aim to represent the Federal Reserve's "reaction function" and the European Central Bank behaviour. Conversely to the previous Taylor rule, equation (2.38), in this subsection the monetary authority will respond to the

[19] Supposing $\pi_t = 0$, the expectation for the next period inflation will be:
$\pi^e_{t+1} = \frac{\left(n_{n,t}\pi^e_{n,t+1} + n_{b,t}\pi^e_{b,t+1}\right)}{H} = \frac{\left[n_{n,t}0 + n_{b,t}(0+b)\right]}{H} = \frac{n_{b,t}b}{H} > 0 \; with \; b > 0,\; n_{b.t} > 0.$

inflation rate and the output oscillations according to the following rule:

$$R_t = \left(1+\pi^T\right)\left(1+R_t^n\right)\left(\frac{1+\pi_t}{1+\pi^T}\right)^{\phi_\pi}\left(1+\widehat{Y}_t\right)^{\phi_Y}, \tag{2.42}$$

where $\widehat{Y}_t = \frac{log(y_t)-log(Y)}{log(Y)}$ represents the logarithmic variation of the output.

Figure 8 shows the time series of output, total investment and inflation. The left row of panels shows the evolution of the variables implementing a monetary policy according to the Fed weight parameters of the Greenspan period ($\phi_\pi = 0.54$ and $\phi_y = 0.99$),[20] whereas, in the right side of the figure, it is applied a Taylor rule in line with the ECB policy ($\phi_\pi = 2.73$ and $\phi_y = 1.44$).[21] Comparing the long run results, the two policies exhibit very similar dynamics in all the variables. However, analysing more in details, it is possible to notice how the Fed scenario displays higher economic growth, 0.22% per quarter compared to 0.15% per quarter of the ECB simulation. Despite of this higher economic growth, the Fed scenario registers lower average level of output with higher volatility. The same result is found looking to the total investment, higher volatility associated with a lower average value (around 63% of the ECB scenario level). This is a foreseeable result, given that the main determinant in the output composition is the invested capital. From this analysis, it seems that the ECB policy, even if it may not be able to reach the same rate of growth of the Fed scenario, being more reactive it allows to reduce the volatility of the output reaching the same inflation level and higher average level of total investment.

[20] For the choice of the parameters in the Fed Taylor rule see Judd and Rudebusch (1998).
[21] See Gerlach-Kristen (2003) for the explanation of the choice of parameter values.

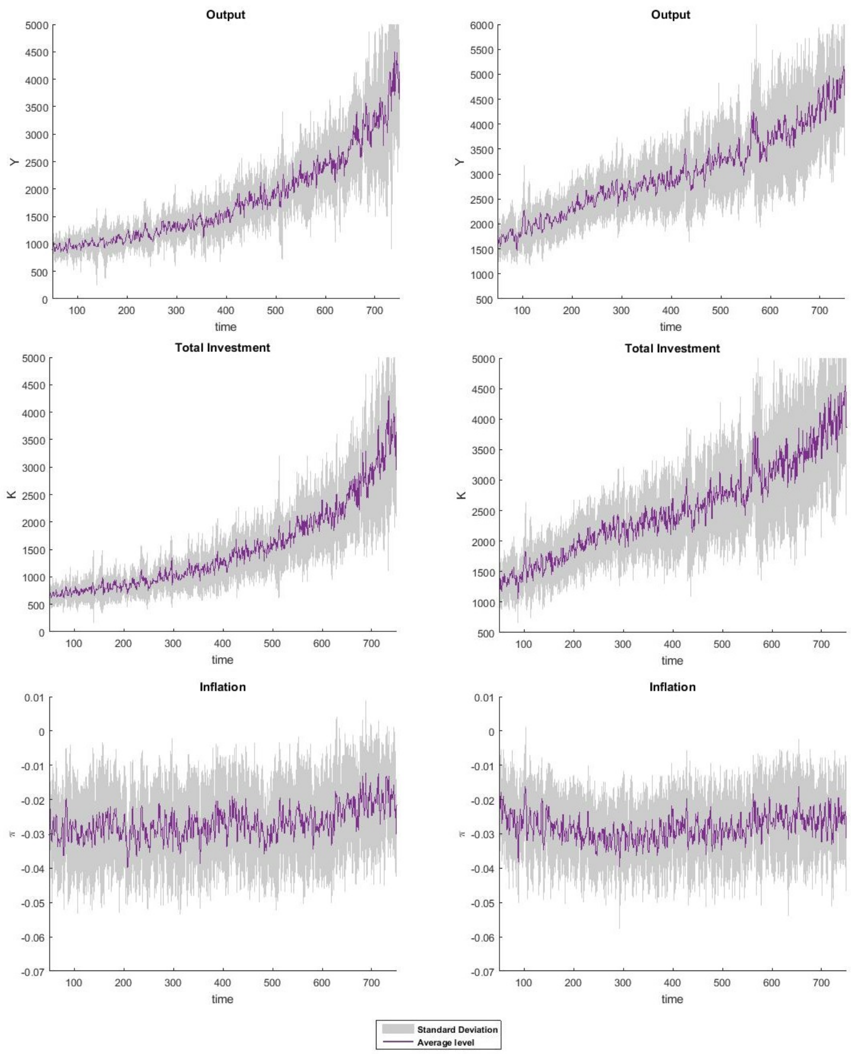

Figure 8: Time series of output (upper panel), total investment (middle panel) and inflation (bottom panel). Left column: Fed Treatment, right column: ECB Treatment. Purple tick line: average level; grey shadow area: standard deviation.

From Figure 9, which illustrates the evolution of the number both of naive agents (upper panels) and of entrepreneurs leaving the market (lower panels), it seems that the different monetary policies do not have any effects on the switching mechanism. Indeed, the dynamics of the first variable present comparable average levels and volatility in both the scenarios. On the contrary, it may be very interesting analyse the number of entrepreneurs leaving the market after some defaults. The lower panels of the figure show how the ECB monetary policy, adopting a more reactive policy and stabilizing the economy, reduces by 40% the number of entrepreneurs which does not invest leaving the market. Indeed, this scenario exhibits a lesser average interest rate likened to the Fed simulation reducing the possibilities for the investment return to be lower than the loans interest rate. This affects the investment choice allowing to the entrepreneurs to purchase capital in the following period avoiding their exit from the market.

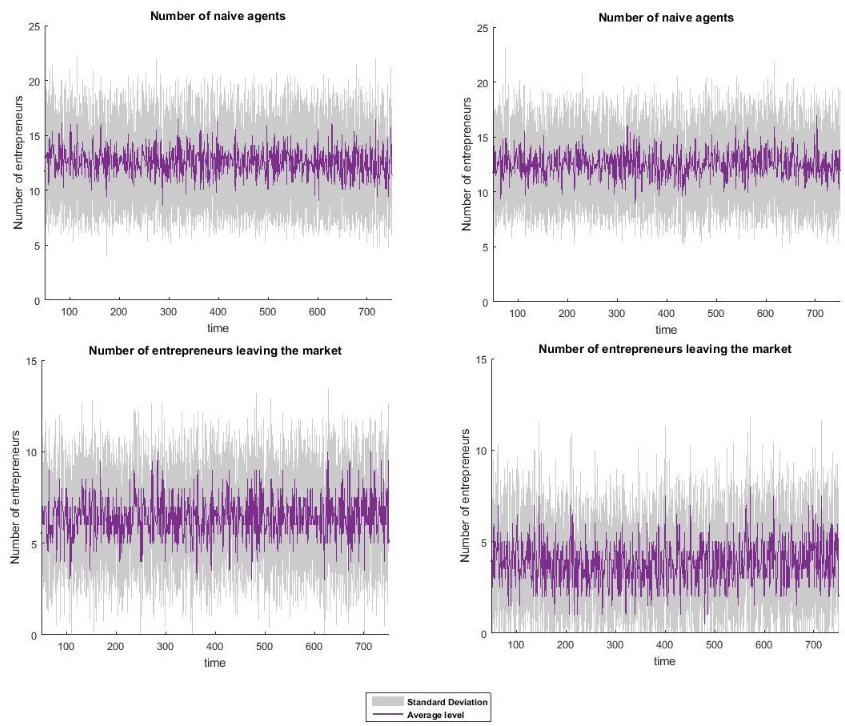

Figure 9: Time series of the number of naive agents (upper panel) and

number of entrepreneurs which leaves the market (lower panel). Left column: Fed Treatment, right column: ECB Treatment. Purple tick line: average level; grey shadow area: standard deviation.

Having described the results of two peculiar and empirically founded interest rate rules, the inflation parameter (ϕ_π) set value will be expanded considering a wider range of values with the aim to find some general insights, from a weaker and non-obeying Taylor principle value to stronger ones. The analysis considers four possible parametrizations: Scenario A ($\phi_\pi = 0.5$, $\phi_y = 1$), Scenario B ($\phi_\pi = 1$, $\phi_y = 1$), Scenario C ($\phi_\pi = 1.5$,$\phi_y = 1$) and Scenario D ($\phi_\pi = 2$,$\phi_y = 1$).

Scenario	A	B	C	D
(ϕ_π, ϕ_y)	(0.5, 1)	(1, 1)	(1.5, 1)	(2, 1)
Y_A	$1.8114e^3$	$1.9411e^3$	$2.1354e^3$	$2.4115e^3$
σ_y	(772.176)	(715.469)	(708.972)	(719.463)
K_A	$1.4284e^3$	$1.5452e^3$	$1.7210e^3$	$1.9731e^3$
σ_k	(694.796)	(645.636)	(644.771)	(657.344)
π	-2.77%	-2.84%	-2.81%	-2.88%
σ_π	(0.46%)	(0.35%)	(0.37%)	(0.32%)
γ	6.5499	6.0673	5.6155	5.1150
σ_γ	(0.8967)	(0.8304)	(0.9135)	(0.8376)

Table 2: Average levels and standard deviations of output, capital, inflation and number of entrepreneurs leaving the market for the four scenarios.

Table 2 depicts the average and the standard deviation of output, total capital, inflation and non-investing entrepreneurs of the four scenarios. From a general overview, it can be noticed how the weaker inflation response policy (Scenario A) exhibits the lowest levels both of output and capital with the highest levels of standard deviation. Moving from this policy to the others respecting the Taylor principle, it seems that greater is the response to the inflation, higher will be the average investment and therefore the average output. This increase in the total capital may be explained by the decreasing value of the number of entrepreneurs leaving the market, γ. Indeed, the entrepreneurs leaving the market are those agents which do not invest. So, lower is their number, higher will be the total investment and therefore the output of the economy. Hereafter in this subsection, will be compared more deeply the policies on the two extremes, Scenario A and Scenario D, leaving the intermediate scenarios time series in Appendix B.

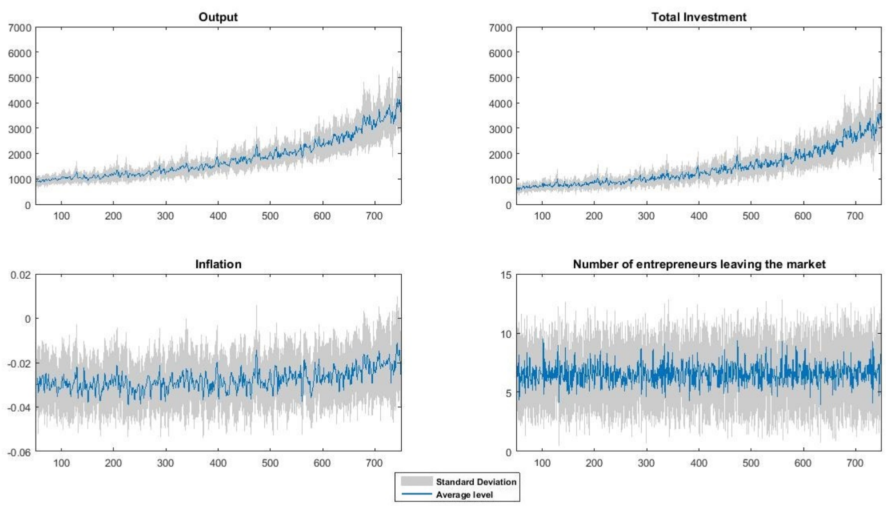

Figure 10: Scenario A: time series of output (upper-left panel), total investment (upper-right panel), inflation (bottom-left panel) and number of entrepreneurs which leaves the market (lower-right panel). Blue tick line: average level; grey shadow area: standard deviation.

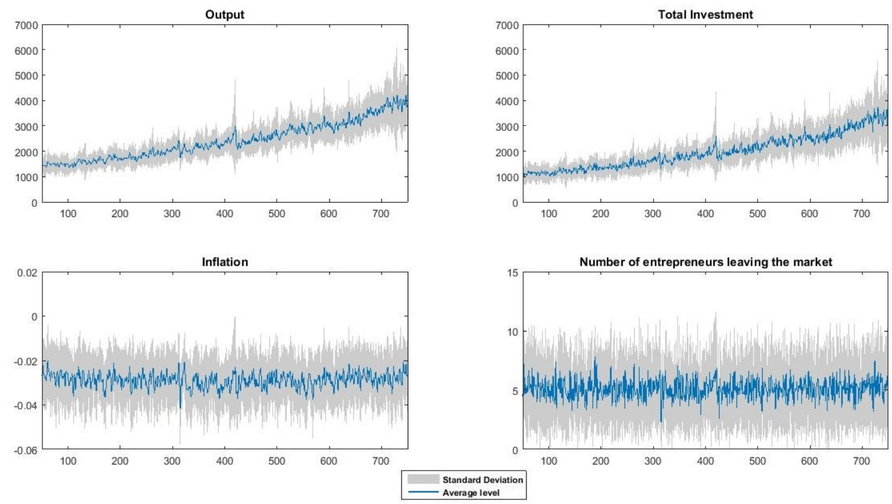

Figure 11: Scenario D: time series of output (upper-left panel), total investment (upper-right panel), inflation (bottom-left panel) and number of entrepreneurs which leaves the market (lower-right panel). Blue tick line: average level; grey shadow area: standard deviation.

Figures 10 and 11 show the time series of the main variables for the two scenarios. Comparing the trends in these figures, it is possible to see how the policies exhibit similar both dynamics and final levels for all the variables. However, Scenario A ($\phi_\pi < 1$) registers: higher economic growth, 0.21% per quarter compared to 0.15% per quarter of the obeying Taylor principle policy, but lower average values of total investment and output. As in the previous Fed-ECB analysis, this finding may be justified by the higher volatility registered by the weaker policy.[22] Indeed, the stronger policy, being more reactive ($\phi_\pi > 1$), is able to stabilize better the oscillations of the variables. Besides to reduce the volatility, the scenario with a strong policy exhibits on average lower interest rates. This combination of outcomes reduces the possibilities for the investment return to be lower than the loans interest rate and therefore entails a lower prob-

[22]This can also be seen in the output first difference plot, Figure 16 in Appendix B, showing how Scenario D registers lower fluctuations than Scenario A.

ability of entrepreneurs to be insolvent. Indeed, as Table 1 illustrates, Scenario D displays a number of bankrupt entrepreneurs leaving the market lower than 22% compared to the weak policy scenario. As a consequence, this lower number of non-investor entrepreneurs positively affects the average levels of the total investment and the output explaining their higher values, respectively 38% and 33% higher than in Scenario A.

Concluding, as found also by Assenza et al. (2013) and Anufriev et al. (2013), a monetary policy responding weakly to inflation may mislead the (heterogeneous and bounded rational) entrepreneurs to take optimal investment decisions. Hence, in a system characterized by backward-looking components in the dynamics, a stronger monetary policy seems give the "possibility for the central bank to reduce the waves of optimism and pessimism by reducing the volatility of output. In doing so, the central bank creates a more stable macroeconomic environment" (De Grauwe: 2012).

2.6 Final remarks

Concluding, the BGG model exhibits a financial accelerator that links the "external finance premium" and the wealth of the entrepreneur. Through the financial accelerator the model is able to amplifies and propagates exogenous shocks in the system. Nevertheless, the bankruptcy of the entrepreneur has not an explicit role in the terms of the financial contract or in the fluctuations of the economy. In the original BGG model, the default of the firm has no impact on the relationship with the bank, this is always willing to extend credit. The version of the financial accelerator model which tries to take into account the losses generates the opposite result. The losses generated by the default bring to the collapse of the credit market and the total investment. Besides, both the specifications are not able to generate endogenously the number of entrepreneurs which goes bankrupt and leaves the market but set it exogenously. The reason for this shortcoming should be found in the representative agent assumption.

Therefore, in section 2.4, the paper develops an agent-based approach to the financial accelerator with heterogeneous agents in order to introduce a proper role of bankruptcy in the credit relationships. In section 2.5, have been performed some simulations evaluations, first comparing two scenarios with different types of agent in the populations, then analysing the effects of distinct monetary rule on the same scenario. The main findings of the first simulations consist in the rise of bankruptcies and fluctuations in the net worth of the financial intermediaries. Indeed, in both the scenarios, the number of defaulted entrepreneurs leaving the market is positive and emerges endogenously. This affects negatively the purchased capital and the total output. Besides, the average number of agents quitting the business is reasonable and also the fluctuations are moderate. These bankruptcies affect the rates of growth of capital, flattening them and avoiding the existence of huge spikes. Moreover, the positive number of the entrepreneurs which defaults influences the wealth of the financial intermediaries and their future credit policy. For every bankrupt entrepreneur, the financial intermediary registers some losses which reduces its net worth. Then the bankruptcy affects the credit channel, firstly because the bank will settle an extra cost to the defaulted entrepreneur financial contract in the following period, secondly because banks with lower financial robustness will fix higher interest rate on loans. From the second group of simulations, comparing interest rate rules with different parametrizations, the model exhibits results in line with the standard literature. Indeed, the simulations suggest that a stronger monetary policy, reducing the volatility, is able to reach both higher final values of output and invested capital and higher average levels. Besides, reducing the fluctuations, the strong monetary policy has a huge impact on the share of agents which defaults and leaves the market, even 40% less than the weaker policy.

Concluding, certainly this is a simple model and in subsequent researches could be consider several extensions to the this work. First, it would be interesting increase the number of the actors, e.g. the entrepreneurs, in order to investigate deeply the constrained borrowing scenarios. Second, this model considers two financial intermediaries but does not present a proper inter-bank lending market. It could be interesting insert in the model a financial system with banks that compete among them or a more sophisticated bank sector, e.g. divided in more branches (wholesale and retails for deposits and loans). Finally, this paper restricts the analysis to a time horizon of two periods in the bank behaviour. It would be interesting to allow to relief debt plans extended for more than two periods. However, even if this paper does not exploit all these possible extensions, it is able to generate endogenously the bankruptcies and takes into account their real effects on the net worth of the bank and on its credit policy. Moreover, it is able to generate some important stylized economic phenomena, such as unpredictable returns and volatility clustering.

Appendix A. Complete log-linearized BGG model

This section presents the complete log-linearization of the standard BGG model. Let lower case variables denote percent deviations from the steady state and the ratios among capital letters without time pedix denote the ratios of the steady state values, the log-linearization of the model is:

$$y_t = \frac{C}{Y}c_t + \frac{C^e}{Y}c_t^e + \frac{I}{Y}i_t + \frac{G}{Y}g_t + ... + \phi_t^y, \qquad (A1)$$

$$c_t = -r_{t+1} + E_t\left\{c_{t+1}\right\}, \qquad (A2)$$

$$c^e_t = n_{t+1} + ... + \phi_t^{c_t^e}, \qquad (A3)$$

$$g_t = \rho_g g_{t-1} + \varepsilon^g_t, \qquad (A4)$$

$$q_t = \varphi\left(i_t - k_t\right), \qquad (A5)$$

$$r^k_{t+1} = (1-\varpi)\left(y_{t+1} - k_{t+1} - x_{t+1}\right) + \varpi q_{t+1} - q_t, \qquad (A6)$$

$$E_t\left\{r^k_{t+1}\right\} - r_{t+1} = -v\left[n_{t+1} - \left(q_t + k_{t+1}\right)\right], \qquad (A7)$$

where

$$\phi^y_t \equiv \frac{DK}{Y}\left[log\left(\mu\int_0^{\bar{\omega}}\omega dF(\omega)R^k_t Q_{t-1}K_t/DK\right)\right],$$

$$D \equiv \mu\int_0^{\bar{\omega}}\omega dF(\omega)R^k_t,$$

$$\phi^{c^e}_t \equiv log\left(\frac{1 - C^e_{t+1}/N_{t+1}}{1 - C^e/N}\right),$$

$$v \equiv \frac{\psi\left(R^k/R\right)}{\psi'\left(R^k/R\right)},$$

$$\varpi \equiv \frac{1-\delta}{(1-\delta) + \alpha Y/\left(XK\right)},$$

$$\varphi \equiv \frac{\left(\Phi\left(I/K\right)^{-1}\right)'}{\left(\Phi\left(I/K\right)^{-1}\right)''}.$$

Equation (A1) represents the log-linearized version of the resource constraint, where the variation in aggregate expenditure y_t is given by changes in consumption of households (c_t) and entrepreneurs (c_t^e), in investment i_t or in government expenditure g_t. The last term represents how variations in monitoring cost (μ) affect the aggregate expenditure but it is a secondary factor. Equations (A5), (A6) and (A7) are the log-linearized versions of equation (2.11) (2.12) and (2.13). They represents the financial sector and the financial accelerator mechanism, in particular, equation (A7) incorporates the capital market frictions, given that the cost of external funds inversely depends on the share of purchased capital financed by the entrepreneur's net worth.

$$y_t = a_t + \alpha k_t + (1-\alpha)\Omega h_t, \qquad (A8)$$

$$y_t - h_t - x_t - c_t = \eta^{-1} h_t, \qquad (A9)$$

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a, \qquad (A10)$$

$$\pi_t = E_{t-1}\left\{\kappa(-x_t) + \beta\pi_{t+1}\right\}, \qquad (A11)$$

$$\kappa \equiv \frac{(1-\delta)}{\delta}(1-\theta\beta)$$

Assuming that the supply of entrepreneurial labour is fixed, the equation (A8) represents the production function whereas equation (A9) describes the labour market equilibrium. In equation (A10) is imposed that the exogenous shock to technology follows a stationary autoregressive process. Equation (A11) characterizes the price adjustment following the stickiness à la Calvo and has the shape of a standard Phillips curve. Indeed the demand changes inversely with the markup x_t, when the demand is high the retail sector purchases more wholesale goods from the entrepreneurs and therefore increases the relative wholesale price and, by definition, reduces its markup. It should be underlined that the slope coefficient κ depends negatively on the degree of price inertia, hence decreases if the probability for an entrepreneur to do not be able to reoptimise her price θ increases.

$$k_{t+1} = \delta i_t + (1-\delta)\, k_t, \qquad (A12)$$

$$n_{t+1} = \frac{\gamma RK}{N}\left(r_t^k - r_t\right) + r_t + n_t + ... + \phi_t^n, \qquad (A13)$$

$$\phi_t^n \equiv \frac{\left(R^k/R - 1\right)K}{N}(r_t^k + q_{t-1} + k_t) + \frac{(1-\alpha)\,(1-\Omega)\,(Y/X)}{N} y_t - x_t,$$

Finally, equation (A12) and (A13) represent the evolution of the two state variables, capital and net worth respectively. It should be notice that the evolution of the net worth depends primarily on the value of lagged net worth and by the net return on the investment weighted by the ratio of gross capital held and entrepreneurial net worth ($\frac{\gamma RK}{N}$).

This description concludes formally the description of the BGG model and allow me to proof one of the shortcomings presented in Section 2.3.

Appendix B.

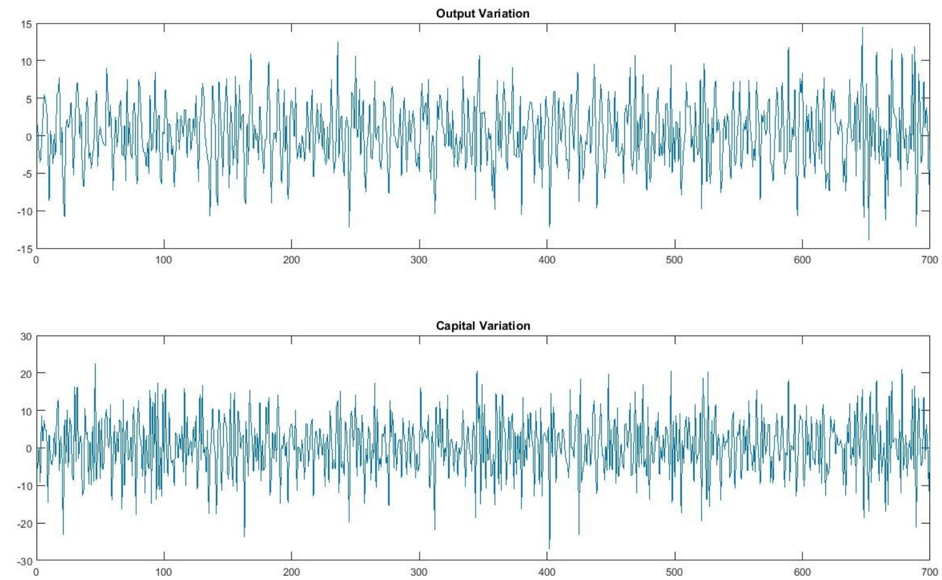

Figure 10: Scenario 1, percentage variations of output (upper panel) and total investment (bottom panel).

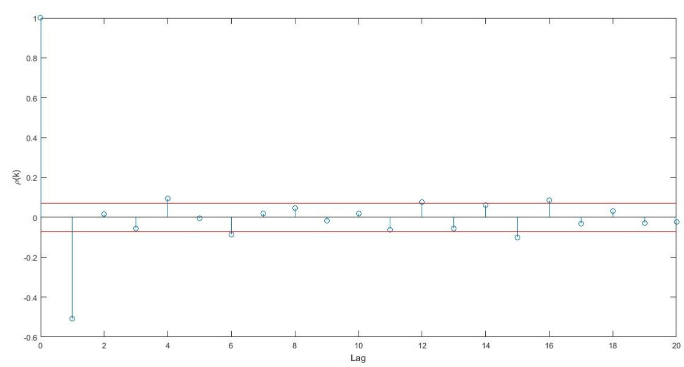

Figure 11: Scenario 1, autocorrelation function for the difference in performance among the strategies.

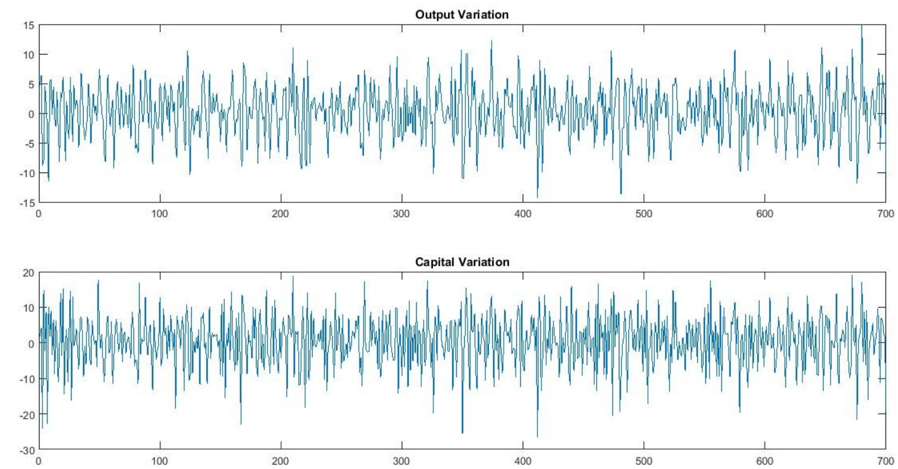

Figure 12: Scenario 2, percentage variations of output (upper panel) and total investment (bottom panel).

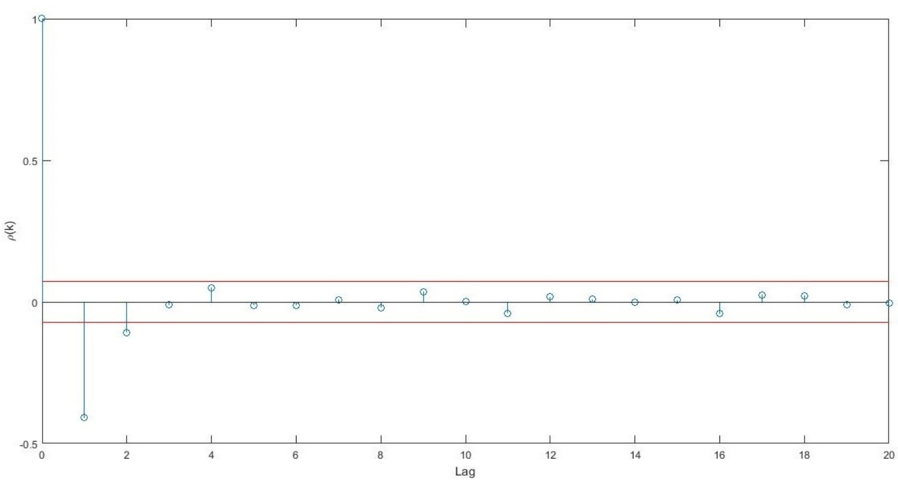

Figure 13: Scenario 2, percentage variations of output (upper panel) and total investment (bottom panel).

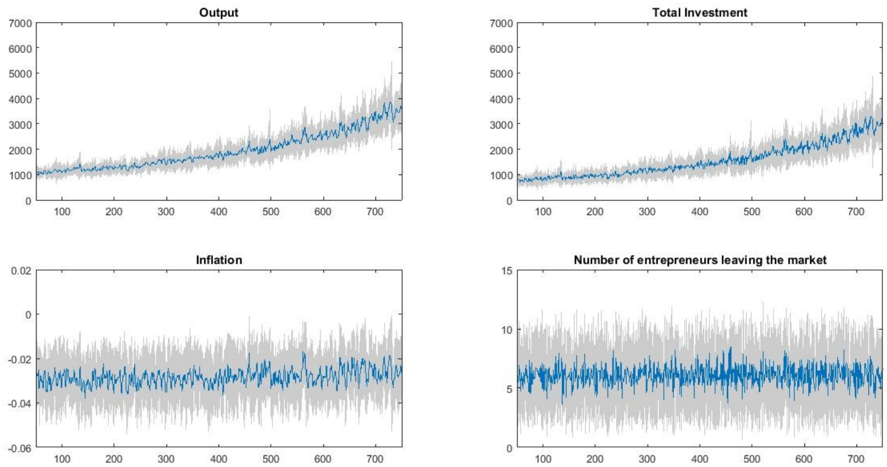

Figure 14: Scenario B: time series of output (upper-left panel), total investment (upper-right panel), inflation (bottom-left panel) and number of entrepreneurs which leaves the market (lower-right panel). Blue tick line: average level; grey shadow area: standard deviation.

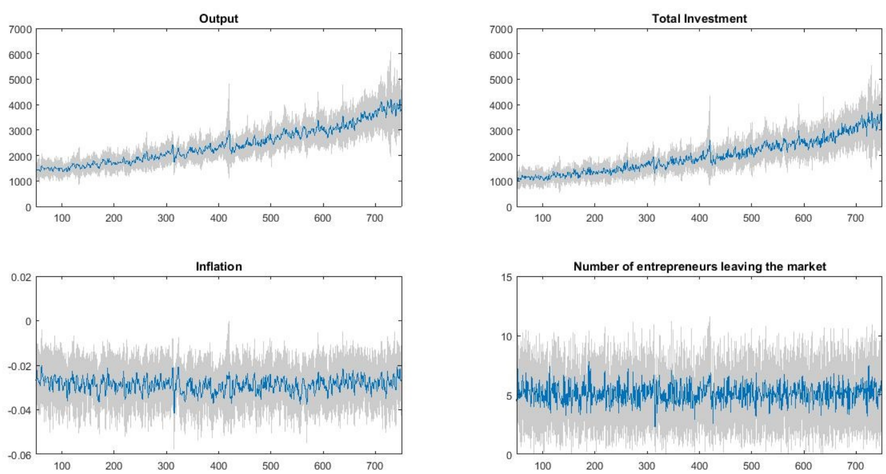

Figure 15: Scenario C: time series of output (upper-left panel), total investment (upper-right panel), inflation (bottom-left panel) and number of entrepreneurs which leaves the market (lower-right panel). Blue tick line: average level; grey shadow area: standard deviation.

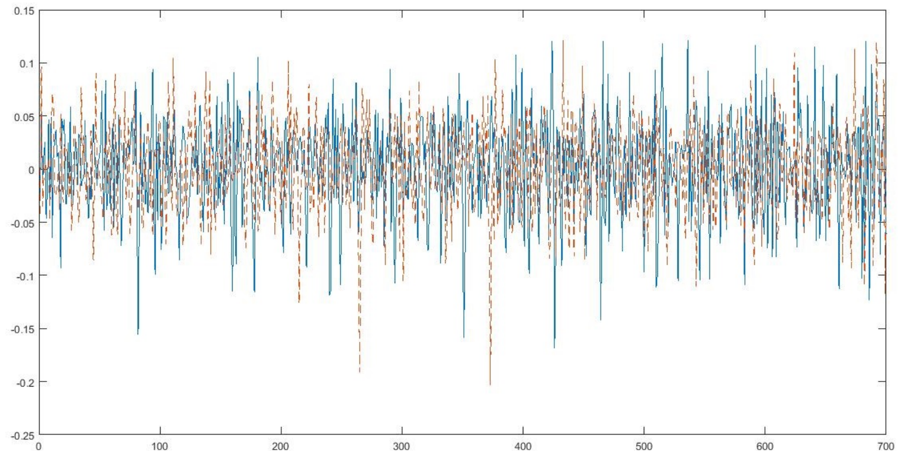

Figure 15: Logarithmic first difference of output. Tick line: weak monetary policy ($\phi_\pi = 0.5$); dashed line: strong monetary policy ($\phi_\pi = 1.5$).

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Chapter 3

Heterogeneous expectations and endogenous fluctuations in the financial accelerator framework

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3.1 Introduction

In last decades, the macroeconomic investigation of monetary policy has been shifting from a Real Business Cycle analysis to the new Keynesian Dynamic Stochastic General Equilibrium modelling approach.[1] This shift of paradigm

[1] See Mankiw (1989).

occurred because, considering a market without imperfections, in the RBC the fluctuations are due to changes in factors productivity; whereas the NK-DSGE assumes nominal and real frictions but maintains the rational and representative agent approach. Both approaches do not recognize the importance of "animal spirits" for the origin and the propagation of the fluctuation in the economy. Indeed, the quoted types of models are not able to endogenize fluctuations and require external shocks to raise variations from the steady state levels of the variables.

During last years and especially from the burst of the Financial Crisis in 2007 behavioural economics has been developing concrete alternatives to the standard rational representative approach using models with bounded rationality and heterogeneous expectations. Keynes already argued that economic fluctuations are not only determined by fundamentals, but investor's animal spirits and the market psychology (e.g. euphoria) influence financial market performance.

The recent macroeconomic literature often investigates business-cycle dynamics by following two parallel paths: financial frictions and heterogeneous expectations. The goal of this paper is to develop a New Keynesian framework which will link these two paths together.

The literature on financial frictions builds a dynamic general equilibrium model with imperfections in the credit market[2] introducing information asymmetry, collateral constraints or costly state verifications (agency cost). These models clarify that the role of frictions is to amplify and propagate shocks to the macroeconomy operating like a "financial accelerator". All of the mentioned papers use the representative agent assumption and the rational expectation hypothesis.

On the other hand, the modelling using heterogeneity is based on the as-

[2]See for example Bernanke et al. (1999), Kiyotaki and Moore (1997) or Iacoviello and Neri (2010).

sumption that agents have different expectations and beliefs about the future and inserts this framework in a standard business cycle model. These models show how heterogeneous beliefs may lead to market instability or to strange attractors using rational expectation with other types of beliefs, e.g based on past performance.[3] Other examples of models with bounded rational agents, naive or biased, can be found in Brock and Hommes (1998) or Branch and McGough (2009).

There are many reasons why they should be used together. First of all, as already mentioned, the assumption of financial frictions implies imperfections in the financial markets, in other words, these models are assuming that borrowers have difficulty to access to credit or that they can default, therefore there are limitations in the credit policy with a credit spread (i.e. a premium in the loan interest rate over the risk free interest rate). This changes according the net worth (or collateral) of the borrower. The imperfections of the financial markets have been well investigated in the literature and the recent crisis has made them even clearer.

Secondly, through heterogeneous expectations it is possible to represent fluctuations abandoning the exogenous shock hypothesis, indeed they allow to endogenous fluctuations to the system. The heterogeneous expectation hypothesis supposes that agents behave in different ways and have different beliefs on the future aggregate outcomes or on the future value of the variables; this assumption has been supported by both empirical and experimental analyses like for example Pfajfar and Zakelj (2011) or Burke and Manz (2011).

Moreover, it seems that the global financial crisis has made clear that not all the agents are driven by rational expectations. As Akerlof and Shiller (2009) explain, agents can be pushed by animal spirits or irrational euphoria in their consumption and saving decisions choosing non-optimal solutions.

[3] See Brock and Hommes (1997).

There are two other reasons why it is important to develop a model with both specifications to better explain the evolution of the business cycle due to shocks. Firstly because the "financial accelerator" can express the propagation and amplification of shocks in the system, like it happened in the last crisis. Furthermore, macroeconomic stability also depends on the set of behaviour strategies of the agents and on how these react to the shocks, like changes in monetary policy, or on how changes in beliefs may produce endogenous fluctuations.

This analysis fits into the same research branch of De Grauwe (2012), Massaro (2013) or Anufriev et al. (2013) which has been investigating the effects of heterogeneous expectations on future inflation and output through the investment channel.

The paper is organized according the following structure: section 3.2 presents the model with the heterogeneous expectations; section 3.3 proposes a simulation of the system with the aim to investigate some policy prescriptions whereas the last part (section 3.4) concludes.

3.2 The Model

The baseline framework of the model is a simplified version of the financial accelerator introduced by Bernanke Gertler and Gilchrist model (1999) with a new behavioural assumption on the expectations formation mechanism. The basic structure of the model considers five types of agents: households, two type of entrepreneurs, retailers, capital producers and the public sector (the fiscal and the monetary authorities). The households' behaviour is quite standard, they live forever and have to take decisions on labour supply, consumption, savings and investment. Indeed, they can choose to hold real money or risk free assets with the aim to maximize their utility.

The two type of entrepreneurs are the core of the model. Conversely to the other agents, they have a finite horizon: each time a portion γ of them survives to the next period and continues to produce. This assumption is made to avoid the possibility of full self-financing by entrepreneurs.

In every period each entrepreneur has a net worth composed by profit and wage of the previous period. With this worth the entrepreneurs purchase physical capital financing the difference through loans from the bank. There are two main variables that drive the borrowing decision: the accumulated net worth and the expectations on the investment return that are heterogeneous among the entrepreneurs. The first variable affects the cost of external finance and the agency problem whereas the second drives the agent behaviour.

Following the extending literature[4] which proofs that private sector behaviours are characterized by different degree of heterogeneity and rationality, the model assumes non perfectly rational agents.

Solved the financial problem, the entrepreneurs hire labour to combine with the physical capital producing output in the following period. This wholesale output is then sell to the retailers that buy and re-sell the goods to the households. By assumption, these retailers compete in a monopolistic market, in this way nominal stickiness is introduced in the economy.

3.2.1 The financial intermediary problem

At the end of time t the *j-th* entrepreneur buys capital to be used in *t+1*. The quantity of purchased capital and its price are denoted by K_{t+1}^{j} and Q_t respectively. Assuming that capital is homogeneous, the financial constraints apply to the whole capital of the firm and not just to investment.

The entrepreneur purchases capital goods $Q_t K_{t+1}^{j}$ using the available net worth N_{t+1}^{j} and bank loans B_{t+1}^{j}:

[4] See for example Carroll (2003), Branch (2004) or Pfajfar and Santoro (2010).

$$B_{t+1}^j = Q_t K_{t+1}^j - N_{t+1}^j. \tag{3.1}$$

Bank loans are extended by a financial intermediary (henceforward a bank) who faces an opportunity cost equal to the risk free gross rate, R_{t+1}. Entrepreneurs are risk neutral and households are risk averse, so the entrepreneur absorbs any risk. Given the choices of the entrepreneur on K_{t+1}^j, B_{t+1}^j and given the risk free interest rate R_{t+1}; the optimal contract is characterized by a non-default firm-specific interest rate, Z_{t+1}^j, such that:

$$Z_{t+1}^j = \left[\chi + \mu \frac{B_{t+1}^j}{N_{t+1}^j}\right] R_{t+1}, \tag{3.2}$$

with $\mu + \chi > 1$ and where $\frac{B_{t+1}^j}{N_{t+1}^j}$ is the leverage ratio. This equation shows the relationship between the external cost of funds and the financial condition of the entrepreneurs. Indeed, the firm-specific interest rate on loans is a mark-up over the risk free interest rate and it is increasing in $\frac{B_{t+1}^j}{N_{t+1}^j}$, or i.e. it depends inversely on the financial soundness.

3.2.2 Heterogeneous expectations and optimal choices of capital

Given the state-contingent debt contract, the expected return of the entrepreneur's investment may be written as:

$$E_t \left\{ R_{t+1}^k Q_t K_{t+1}^j - Z_{t+1}^j B_{t+1}^j \right\}, \tag{3.3}$$

where the expectations are taken upon the return on invested capital, R_{t+1}^k, given that all the other variables are predetermined.

At this point, introducing heterogeneity in the entrepreneurs' behaviour, i.e. in the heuristic forecasting rules, the expectations on the return dynamics assume a crucial role in the investment decision of the entrepreneur.

Abandoning the world of rational expectations and taking the view that agents are not perfectly rational and have cognitive limitations or computational limits,[5] we assume two different forecasting rules:

$$E_{o,t}\left\{R_{t+1}^{k}\right\} = R^{k*} + b, \tag{3.4}$$

$$E_{p,t}\left\{R_{t+1}^{k}\right\} = R^{k*} - b, \tag{3.5}$$

where *"o"* means optimistic expectation and *"p"* represents pessimistic rule. Equations (3.4) and (3.5) describe biased behaviours of bounded rational agents, as in Brock and Hommes (1998), where b represents the bias parameter and R^{k*} is the the fundamental (historical) investment return. More precisely, these entrepreneurs have a commonly shared belief on fundamental investment return plus a type specific bias, i.e. the agents have an approximate knowledge of the correct fundamental value of the investment return but they disagree on the real current level. If the bias reduces the expected return we are considering an agent with "pessimistic" expectation, in the opposite case she is an "optimistic" entrepreneur.

The optimization problem of the *j-th* entrepreneur of *h-Th* type is defined by the following optimization problem:

$$\max_{B} E_{h,t}\left\{R_{t+1}^{k}\right\} Q_t K_{t+1}^{j} - Z_{t+1}^{j} B_{t+1}^{j} \;\; with\, h = o,\, p,$$

where $E_{h,t}\left\{R_{t+1}^{k}\right\}$ represents the expected return on investment of the *j-th* entrepreneur that could be optimist or pessimist.

[5]Some examples in the growing literature on bounded rationality are Duffy (2006) or Conlisk (1980), whereas on the cognitive limitation of the agents see e.g. Hommes and Zhu (2014), Hommes et al. (2005) or Branch and Evans (2005).

From the maximization problem, it is possible to establish a relationship between capital expenditure and entrepreneur's financial expectation measured by the expected discounted spread between the return of capital with the risk free rate and the entrepreneurial net worth. The capital/wealth ratio may indeed be expressed as the increasing function of the premium on external funds and it can be rewritten as:

$$Q_t K_{t+1}^j = \psi(s_{t+1}) N_{t+1}^j, \tag{3.6}$$

where $\psi\left(s_{t+1}\right) = 1 + \frac{E_{h,t}\left\{R_{t+1}^k\right\} - \chi R_{t+1}}{2\mu R_{t+1}}$.

Through simple substitutions equation (3.6) can be rewritten as follow:

$$B_{h,t+1}^j = \frac{E_{h,t}\left\{R_{t+1}^k\right\} - \chi R_{t+1}}{2\mu R_{t+1}} N_{t+1}^j. \tag{3.7}$$

Capital expenditure is proportional to the entrepreneur's net worth, with a proportionality factor increasing in the expected discounted rate of return on capital. Consequently, agents with higher wedge between expected return on capital and risk free rate (optimistic entrepreneurs) will have higher incentive to borrow from the financial intermediary.

The entrepreneurs are constrained from raising the size of their firms by the fact that increasing the amount of capital borrowed, they also increase the leverage ratio and therefore reduce the return on investment. Indeed, increasing debt, the non-fully self-financed entrepreneurs increase the leverage ratio and therefore the external finance costs. In this way they reduce the return on capital that will be equal to the ratio between the investment net return and the total amount of purchased capital as will be explained in subsection 3.2.5.

3.2.3 Households, retailers and public sector

In this section we will describe the features of households, retailer sector, government and central bank. As in the standard literature the households have an infinite horizon, they work, consume, hold money and invest their savings in financial assets which pay the risk free interest rate. The household maximization problem is therefore:

$$\max_{c,h,m} \quad E_t \sum_{k=0}^{\infty} \beta^k \left[ln\left(C_{t+k}\right) + \zeta ln\left(\tfrac{M_{t+k}}{P_{t+k}}\right) + \xi ln\left(1 - H_{t+k}\right)\right],$$

$s.t. \quad C_t = W_t H_t - T_t + \Pi_t + R_t D_t - D_{t+1} + \frac{(M_{t-1} - M_t)}{P_t}$, where C_t is the consumption of the households, W_t is the household wage, H_t is the supply of labour, T_t are lump sum taxes, Π_t are the dividends of the retail firms, D_t are the deposits held at banks and $\frac{M_t}{P_t}$ is the real money balances between periods. The first order conditions of the problem can be written as:

$$C_t :\rightarrow \frac{1}{C_t} = E_t \left\{ \beta \frac{1}{C_{t+1}} \right\} R_{t+1}, \quad (A)$$

$$H_t :\rightarrow \frac{W_t}{C_t} = \xi \frac{1}{1 - H_t}, \quad (B)$$

$$M_t :\rightarrow \frac{M_t}{P_t} = \zeta C_t \left(\frac{R^n_{t+1}}{R^n_{t+1} - 1} \right), \quad (C)$$

where $R_{t+1} = R^n_{t+1} \frac{P_{t+1}}{P_t} - 1$ and therefore R^n_{t+1} is the gross nominal interest rate. It should be remembered that in equilibrium the household deposits are equal to the total amount of loans of the bank, $D_t = B_t$. Households consumption is driven by the consumption Euler equation (Equation A). Assuming the unit coefficient on the intertemporal elasticity of substitution, the log-linearized version of the consumption Euler equation can be written as: $c_t = E_t\left\{c_{t+1}\right\} - r_{t+1}$.

The retail sector is characterized by monopolistic competition and nominal rigidity à la Calvo (1983). The n-Th retailer sells the quantity of output $Y_t(n)$ at the nominal price $P_t(n)$. The total final goods and their price are therefore the combination of the individual retailer sales:

$$Y_t^f = \left[\int_0^1 Y_t(n)^{(\epsilon-1)/\epsilon} dn \right]^{\epsilon/(\epsilon-1)}, \tag{3.8}$$

with $\epsilon > 1$ and $P_t = \left[\int_0^1 P_t(n)^{(1-\epsilon)} dn \right]^{1/(1-\epsilon)}$.

To introduce price stickiness, in each period a share of firms faces the probability $(1-\theta)$ of being able to reoptimize its price. In every phase a retailer faces a demand curve:

$$Y_t(n) = \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} Y_t^f, \tag{3.9}$$

therefore denoting with: P_t^* the price set by retailers able to reoptimize and $Y_t^*(z)$ the consequent demand given this price, the n-Th retailer sets the price in order to maximize his expected discounted profits:

$$\sum_{k=0}^{\infty} \theta^k E_{t-1} \left\{ \Lambda_{t,k} \left(\frac{P_t^*}{P_t} \right)^{-\epsilon} Y_t^*(n) \left[\frac{P_t^*}{P_t} - \left(\frac{\epsilon}{\epsilon-1} \right) \frac{P_{t+k}^w}{P_{t+k}} \right] \right\}. \tag{3.10}$$

In Equation (3.10), $\Lambda_{t,k} = \beta C_t / C_{t+k}$ represents the consumption based discount rate equal to the shareholders intertemporal marginal rate of substitution, θ^k is the probability that the price is fixed for k periods and P_t^w is the nominal price of the wholesale goods.

Given that the share θ of retailers is not able to reoptimize in period t, the evolution of the price will be:

$$P_t = \left[\theta P_{t-1}^{1-\epsilon} + (1-\theta) \left(P_t^* \right)^{1-\epsilon} \right]^{1/(1-\epsilon)}. \tag{3.11}$$

By combining Equations (3.10) and (3.11) and log-linearizing, after some calculations, it is possible to obtain the following Phillips curve:

$$\pi_t = f_1 E_{t-1} \{\pi_t\} + f_2 y_t.$$

Finally, moving to the public sector, government expenditures are financed by lump sum taxes and money creation. Hence the government budget constrain will be $G_t = T_t + \frac{(M_{t-1} - M_t)}{P_t}$. Following the standard literature, e.g. Clarida et al. (2000) or Galì (2008), the main instrument of the monetary policy is the the short-term nominal interest rate. Hence, the central bank adjusts the nominal interest rate as stated by the following Taylor Rule:

$$r_t^n = \rho r_{t-1} + (1-\rho)(\phi_\pi \pi_t + \phi_y y_t) + \varepsilon_t,$$

the monetary authority changes the actual interest rate reacting to: the current inflation, the current output and the lagged interest rate.

3.2.4 Aggregation and General Equilibrium

Thus far we have described how the agents behave solving their own maximization problem. In this subsection these individual choices will be incorporate within a dynamic general equilibrium framework.

The capital purchased by the entrepreneur is combined with labour in order to produce wholesale output through the following Cobb-Douglas production function:

$$Y_{t+1} = A_{t+1} K_{t+1}^{\alpha} L_{t+1}^{(1-\alpha)}, \tag{3.12}$$

where Y_{t+1} represents the aggregate production in period *t+1*, K_{t+1} is the aggregate amount of capital purchased by all the entrepreneurs, L_{t+1} is the labour input and A_{t+1} is an exogenous technology parameter. At this point, it should be noticed that the heterogeneous expectations slightly affect equations (3.12) through the total invested capital, indeed in each time this variable is the result of the different investment choices of agents:

$$K_{t+1} = s_{o,t}\frac{(N_{o,t+1} + B_{o,t+1})}{Q_t} + s_{p,t}\frac{(N_{p,t+1} + B_{p,t+1})}{Q_t}, \tag{3.13}$$

where s_t describes the share of entrepreneurs that uses a specific rule. For the sake of simplicity, and given that the aim of the paper is to investigate the effect of heterogeneous expectations, it will be assumed that $N_{o,t+1} = N_{p,t+1}$, i.e. the different types of entrepreneurs have the same net worth at the beginning of period *t+1*.[6]

Assuming a decentralized capital market in which the perfect competitive capital producing firms act as simple clearing market traders,[7] the price of capital Q_t in term of the numeraire good will be

$$Q_t = \Phi\left(\tfrac{K_t}{K_{t-1}}\right), \tag{3.14}$$

where $\Phi(\cdot)$ is increasing and concave and $\Phi(0) = 0$.

Given that entrepreneurs sell their output to retailers which have market power, the relative price of the wholesale goods will be $1/X_t$ where $X_t = \frac{P_t}{P_t^w}$ is the gross markup of retail goods over wholesale goods. Consequently, the actual rate of return of capital between two periods can be written as:

$$R_{t+1}^k = \frac{\left(\frac{1}{X_t}\frac{\alpha Y_{t+1}}{K_{t+1}}\right) + (1-\delta)Q_{t+1}}{Q_t}, \tag{3.15}$$

[6] This aspect could be modified in the future extensions in order to introduce more heterogeneity in the model.

[7] In other words we are assuming that at the beginning of the period these firms sell capital at price Q_t and at the end of period they purchase the undepreciated invested capital of the entrepreneurs at price Q_{t+1}, clearing the market.

where the first term in brackets is the rent paid to one unit of capital, the second is the capital gain due to the fact that entrepreneurs resell the undepreciated capital.

Besides capital, the technology requires also labour as input. The total labour supply of the economy is composed by households and entrepreneurial labour (H_{t+1} and H^e_{t+1} respectively):

$$L_{t+1} = H^{\Omega}_{t+1}(H^e_{t+1})^{1-\Omega}.$$

In a competitive market the demand curves for labour imply that wage equates marginal product, therefore it will be

$$(1-\alpha)\Omega\frac{Y_{t+1}}{H_{t+1}} = X_{t+1}W_{t+1},$$

$$(1-\alpha)(1-\Omega)\frac{Y_{t+1}}{H^e_{t+1}} = X_{t+1}W^e_{t+1},$$

where W_{t+1} and W^e_{t+1} are the real wage rate to the household and the entrepreneur respectively.

The law of motion of the aggregate entrepreneurial net worth for each type, $N_{h,t+2}$, is described as follows:

$$N_{h,t+2} = \gamma V_{h,t+1} + W^e_{t+1}, \tag{3.16}$$

where $V_{h,t+1}$ represents the equity held by the entrepreneurs of the *h-Th* type. Remembering that γ is the fraction of entrepreneurs which survives in each period, $\gamma V_{h,t+1}$ are the equities held by entrepreneurs still in business in the following period. This equity is the residual part of the return of the investment after the repayment of the loans to the financial intermediaries.

$$V_{h,t+1} = R^k_{t+1} Q_t K_{h,t+1} - \left[\chi R_{t+1} + \mu \frac{R_{t+1} \left(Q_t K_{h,t+1} - N_{t+1} \right)}{N_{t+1}} \right] \left(Q_t K_{h,t+1} - N_{t+1} \right), \tag{3.17}$$

where the term $\left[\chi R_{t+1} + \mu \frac{R_{t+1}(Q_t K_{h,t+1} - N_{t+1})}{N_{t+1}} \right]$ reflects the premium for external finance which has a negative relation with net worth and positive with debt.

3.2.5 Performance measure and dynamic selection mechanism

In order to complete the specification of the model, this subsection will develop how the beliefs are updated over time and how the fractions of the different types change. The evolutionary selection is based upon a performance (or fitness) measure.

The evolutionary performance measure is publicly available but it is subject to noise, it could be expressed as follow:

$$U_{h,t+1} = U^d_{h,t+1} + \epsilon^j_{h,t+1}, \tag{3.18}$$

where $U^d_{h,t+1} = \frac{R^k_{t+1} Q_t K_{h,t+1} - Z_{h,t+1} B_{h,t+1}}{Q_t K_{h,t+1}}$ is the investment performance of the entrepreneur following the *h-Th* forecasting rule. The investment performance defined in equation (3.18) has the flavour of the ROI index, indeed it is the ratio between the net return of the investment and the total amount of purchased capital. The performance measure defined before is not the only possible measure but it has the advantages to be clear and to avoid that the choice of switching is exclusively determined by the firm specific shock, hence to be randomly defined. Alternative performance measures could be the difference between the firm specific investment return and the average of the market or a weighted sum between the average investment performance of the class and the just defined difference, but it adds more complexity to the system and its effect is left to future possible extensions.

As in the standard literature, e.g. Hommes (2013), $\epsilon^j_{h,t+1}$ represents an IID noise across individual *j-th* entrepreneurs and the types $h = 1, 2$ drawn from a double exponential distribution. When the number of entrepreneurs goes to infinity, as showed by Diks and Van der Weide (2005) and Hommes et al. (2005), the probability for an agent to choose the h-Th forecasting rule is given by a discrete choice model with multinomial logit probabilities:

$$s_{h,t+2} = (1-\nu)\frac{e^{\vartheta U^j_{h,t+1}}}{Z_{t+1}} + \nu s_{h,t}, \tag{3.19}$$

where parameter ϑ represents the "intensity of choice", i.e. how the entrepreneurs are sensitive to selecting the optimal forecast strategy, and $Z_{t+1} = \sum_{h=1}^{H} e^{\vartheta U^j_{h,t+1}}$ is a normalizing factor.

There are two main insights related to this dynamic mechanism: the higher is the fitness measure of the h-Th forecasting rule, the larger is the number of entrepreneurs who switch to this strategy; the higher is the intensity of choice, the more "rational" are the agents. When $\vartheta = \infty$ corresponds with the case without noise, so the deterministic part of performance measure can be observed and therefore all the agents will switch to the optimal forecast. On the opposite case, $\vartheta = 0$, the variance of noise term is infinite, thus the differences in the fitness measures cannot be observed and the share of entrepreneurs which choose each forecasting rule will be fixed an equal to $1/H$. It should be noted that the switching model expressed by Equation (3.19) assumes asynchronous updating, indeed in each period only a fraction $(1-\nu)$ of entrepreneur can revise its belief according to the new available information.

The evolution of the optimistic entrepreneur will be therefore:

$$s_{o,t+2} = (1-\nu)\frac{exp\left\{\vartheta\left(\frac{R^k_{t+1}Q_tK_{o,t+1}-Z_{o,t+1}B_{o,t+1}}{Q_tK_{o,t+1}}\right)\right\}}{Z_{t+1}} + \nu s_{o,t+1},$$

with $s_{o,t+2} + s_{p,t+2} = 1$.

3.3 Simulations

This section presents the results of some quantitative analyses aiming to illustrate how the heterogeneous expectations and the financial accelerator are able to endogenously modify the business cycle without the existence of external shocks.

Following Clarida et al. (1999), the choice of the parameter value for the baseline model is quite standard. Looking at the households, their quarterly discount factor (β) is 0.99 and they have a fixed labour supply elasticity (η) at 3. In the production function the capital share (α) is 0.2 with a depreciation rate (δ) of 0.025, the household labour share $(1-\alpha)(1-\Omega)$ is 0.64 whereas the correlation in the technology law of motion (ρ^a) is assumed to be 1.0. The probability θ that firms are not able to reoptimize their price within a period equal to 0.75 and the mark-up of the retail sector with respect to the wholesale market (X) is 1.2. Looking to the financial sector, the rate of the "survival" of the entrepreneurs among the periods (γ) is 0.9728 whereas the weight terms in the financial intermediary interest rate setting are $\mu = 0.7$ and $\chi = 0.6$. Entrepreneurs have a bias of 5% around the fundamental value of the investment return and, according to the parametrization of Anufriev and Hommes (2012), in the asynchronous updating mechanism the fraction of the entrepreneur that can revise its beliefs will be 0.9 with intensity of choice equal to 0.4.

Finally, the simulation are performed for 700 quarters and using 5 Monte Carlo series, each series differs from the others for the initial level of the main variables in $t = 0$, in a range from 0.1 to 1.5. However, for the sake of clarity and without loss of generality, the following figures will show the average of the series dynamic in 25 or 50 quarters.

3.3.1 Homogeneous and fundamentalist vs biased and naives

Before analysing the comparative monetary policies, it is relevant to underline the importance of using both heterogeneity and bounded rationality in the model. In this subsection we perform simulations with homogeneous fundamentalist agents, with and without financial frictions, and heterogeneous naive entrepreneurs applying a flexible inflation targeting monetary policy.[8] Fundamentalist and homogeneous agents have unbiased expectations based on fundamental levels of output gap and current inflation in the NK-IS curve and in the Phillips Curve. In other words, they completely believe to the Central Bank's targets on output gap and inflation, respectively equal to 0 and 2%. Conversely, in the heterogeneous and naive scenario, agents may have optimistic and pessimistic expectations on the investment return and they have the same naive expectation on the current output gap and inflation. The entrepreneurs have biased expectations on their investment return and all the agents in the economy are naive, basing their expectations on output and inflation on the last period level. The simulation assumes a flexible inflation targeting monetary policy satisfying the Taylor Principle. Indeed, in case of weak monetary approach (e.g. $\phi_\pi = 0.5$ and $\phi_y = 0.4$) the model presents an explosive path for every type of expectations rule apart from the fundamentalist one. Hence, the first suggestion is that weak monetary policies seems not able to stabilize the economy. The

[8] $r_t^n = \phi_\pi \pi_t + \phi_y y_t$, with $\phi_\pi = 1.5$ and $\phi_y = 1$.

system shows divergent paths of growth depending on the initial condition. This may be explained by the dominance of positive feedback, i.e. this policy is not able to close the output gap and therefore there is a self-reinforcing mechanism according to which entrepreneurs may become pessimist bringing the capital and output to lower levels.

Figure 1 plots results of three simulations: a heterogeneous framework with naive agents and financial accelerator and two scenarios with homogeneous and fundamentalist agents, with and without financial accelerator. The baseline simulations are based on a model with different initial conditions for output gap and inflation, so the figures represent the average dynamics in the three scenarios. The black line in each picture indicates the evolution in the homogeneous and fundamentalist model without financial accelerator, the magenta line represents the dynamics in the homogeneous and fundamentalist model with the financial accelerator, whereas the blue line is the heterogeneous model evolution with the financial accelerator. The initial values are higher than the steady state levels and have the same effect on fluctuations of a standard technology or demand shocks. As in the original BGG model, the financial accelerator magnifies and propagates the shocks. The explaining mechanism is the raise in the cost of loans associated with higher initial values. This affects the cost of loans bringing to a fall down in investments. In the next periods, the financial accelerator affects investments in a positive way. Looking at the homogeneous scenarios, investments increase with a steeper shape in the simulation with financial accelerator allowing an overshooting reaction in the output gap. However, in both cases, the system converges to the steady state after few periods. Conversely, in presence of heterogeneity and naive expectations more persistent fluctuations emerge. The combination of the financial accelerator with the switching mechanism amplifies the oscillations. As in the homogeneous frame, the initial conditions are higher than the steady state levels affecting the cost of loans and therefore the amount of invested capital. The collapse of investment is further worsen by the increase of pessimist agents. Indeed, by definition, when the share of of pessimists increases the investment in capital decreases. The convergence of the capital to its steady state level may be explained with the reduction of the inflation rate which decreases the risk free interest rate. Lowering this interest rate, it increases the spread between the investment return and the interest rate on loans. All this is translated into a long run positive convergence path to the steady state for capital and output gap and into a reduction of pessimist entrepreneurs in the population.

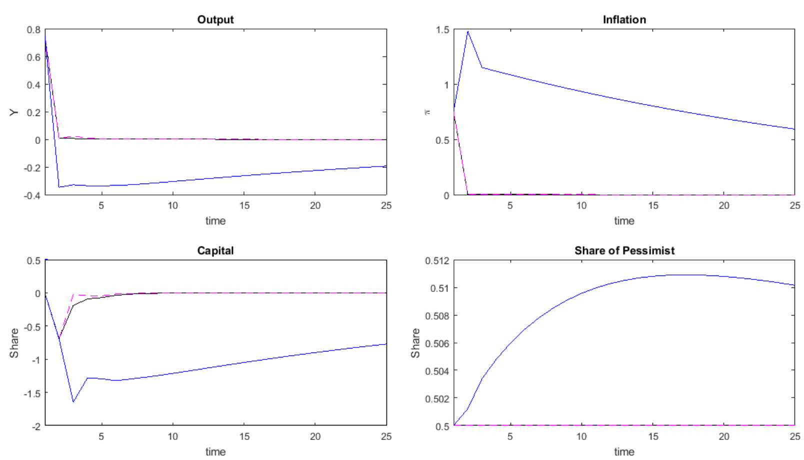

Figure 1: Simulation with flexible inflation targeting Monetary Policy; blue line: heterogeneous entrepreneurs and naive agents; black line: homogeneous and fundamentalist agents without financial accelerator, magenta line: homogeneous and fundamentalist agents with financial accelerator.

3.3.2 Evolution of the system with different Monetary Policies

This subsection expands the analysis performing some comparative statics on different interest rate rules in order to draw some policy recommendations.

This exercise is motivated by the perspective that central banks can pursue different aims.[9] For example, according to EU treaties, the ECB should keep stability of price and financial system, whereas the Fed should also pursue the output stability. In order to perform comparative analyses, the hypothesized monetary policies will be:

$$r_t^n = \phi_\pi \pi_t + \phi_y y_t,$$

where the inflation parameter (ϕ_π) will be equal to 1.5, whereas the coefficient on output gap (ϕ_y) may be null (strict inflation targeting monetary policy) or equal to 1 (flexible inflation targeting monetary policy).

Moreover, the subsection describes how the fluctuations of the economy due to different policies may depend on the composition of the forecasting rules on inflation and output gap. The simulations are performed assuming biased entrepreneurs and four different expectation rules on output gap and inflation: naive, fundamentalist, weak trend-follower and anchoring and adjustment (LAA) rule.

$$x_t^e = x_{t-1}, \tag{3.20}$$

$$x_t^e = x^*, \tag{3.21}$$

$$x_t^e = x_{t-1} + 0.4\left(x_{t-1} - x_{t-2}\right), \tag{3.22}$$

$$x_t^e = 0.5\left(x_{t-1}^{av} + x_{t-1}\right) + \left(x_{t-1} - x_{t-2}\right), \tag{3.23}$$

where x is the hypothetical reference variable, x^* represents its fundamental value and x^{av} is its average level.

As in the previous subsection, Equations (3.20) and (3.21) represent the naive and fundamentalist behaviour respectively. The weak trend-following rule

[9] In Appendix B are performed some simulations comparing two interest rate smoothing rules choosing parametrizations above and below the Taylor Principle boundary ($\phi_\pi = 1$).

(chartist) is given by the extrapolation rule described by Equation (3.22) and it has been deeply analysed (with laboratory experiments) by Hommes (2011) and by Frankel and Froot (1990) in their analysis on the trading rules for the financial market. The most sophisticated heuristics is described by Equation (3.23). Indeed, this strategy exhibits an adaptive learning since the expected value is anchored to the mean between the past average and the last observation plus the short-term variation between the two previous periods. The assumption behind these simulations is that the entrepreneurs have a heterogeneous expectation on the return of capital, biased around its steady state values, but they have the same expectations on the final output and the inflation of the economy. This is in line with the experimental literature, see for example Assenza et al. (2013), according which the agents may coordinate on different heuristic rules for different variables to forecast.

Having described in the previous subsection the differences of the system behaviour among different types of expectations, in this subsection, firstly will be illustrated the differences in dynamics changing the policy but fixing the type of agents in the economy, secondly will be draw some common finding among the scenarios. The considerations of the difference "intra-agents" are left for the last part of the subsection.

Figure 2 considers entrepreneurs and naive expectations. Starting from initial levels which are higher than the inflation and output gap steady states, it is possible to see how in the first periods there is an increase in the number of pessimist entrepreneurs. This can be explained by the fact that these initial levels imply a risk free interest rate higher than its fundamental and therefore higher cost on loans. With high external finance interest rate and relatively low amount of invested capital, the pessimist entrepreneurs register higher performance measure than the optimist ones increasing their share. When the number of pessimists grows, the amount of invested capital in next period will decrease affecting also the total output. Over time, the performance of optimistics increases as a consequence of output levels which are lower than the steady state affecting also the dynamics of the variables which converge to the steady state levels.

Comparing the dynamics of the system with the two monetary policies, it is possible to notice how the strict inflation targeting monetary policy (the red line in the graph) yields deeper fluctuations in both the considered variables (capital and output). These effects may be explained taking in to account that this policy has not in its aims the closure of the output gap and therefore the output and the investment can oscillate affecting the return on capital and the expectations of the entrepreneurs. As a consequence to the fall of the output, the entrepreneurs become pessimist reducing their investment in the following period more than in the flexible inflation targeting scenario. This chain of effects is partially mitigated by the reduction of the inflation allowing the growth in capital return and therefore inverting the trends of capital and output around their steady state values.

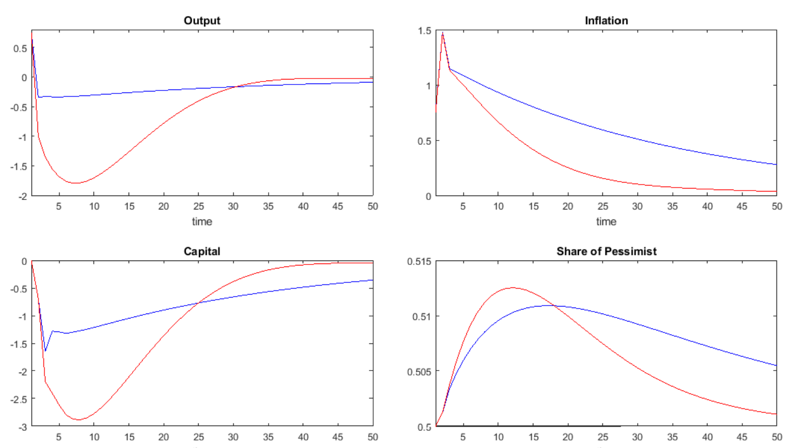

Figure 2: Simulation with biased entrepreneurs and naive agents; red line: strict inflation targeting monetary policy, blue line: flexible inflation targeting

monetary policy.

Considering a population with weak trend following expectations, Figure 3 represents the simulations result with both monetary policies. It is possible to notice how the flexible inflation targeting policy (i.e. the blue line) entails lower fluctuations in the fractions of entrepreneurs. These variations affect the investment choice as it is shown by the negative path of capital in the first periods. As presented by the graph on the top-left hand side, this evolution has a negative effect on the output. Besides, there are two separated effects which influence the evolution of the entrepreneurs' share: the fluctuations in the inflation and the effect on output variations. The first has positive implications on the risk free interest rate which decreases the spread between the investment return and the interest rate on loans. Secondly, the negative variations of the output gap may affect the investment return which will reduce the ROI of the entrepreneur. The strict inflation targeting monetary policy implies higher undershooting phenomena due to the excess of negative feedback. This policy, trying to push the inflation to the steady state level, increases the interest rate and reduces the investment return via the external finance cost. This result may induce to large fluctuations in the expectations on the entrepreneurs' investment return transforming the majority of them in pessimist. The reductions of both investments and production affect negatively the return of capital and accordingly the performance measure.

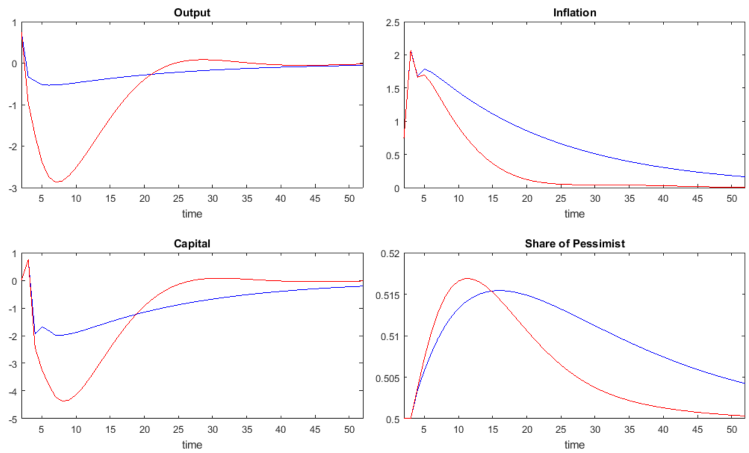

Figure 3: Simulation with biased entrepreneurs and weak trend following expectations; red line: strict inflation targeting monetary policy, blue line: flexible inflation targeting monetary policy.

Figure 4 displays simulations concerning the fundamentalist expectations scenarios maintaining the heterogeneity given by the bias in the expectations on the return on capital. Starting from higher initial values for output gap and inflation than the fundamental levels, we observe that the model converges close to the steady state after few periods in which the share of pessimist entrepreneurs increases. The reason why there is not complete and immediate convergence may be due: firstly to the persistence of the bias in the return of capital expectations which implies fluctuations in the amount of investments, secondly to the stickiness in the discrete choice mechanism. It is interesting to notice that with fundamentalist expectation the system does not present any significant difference in the fluctuations even if there is a change in the monetary policy.

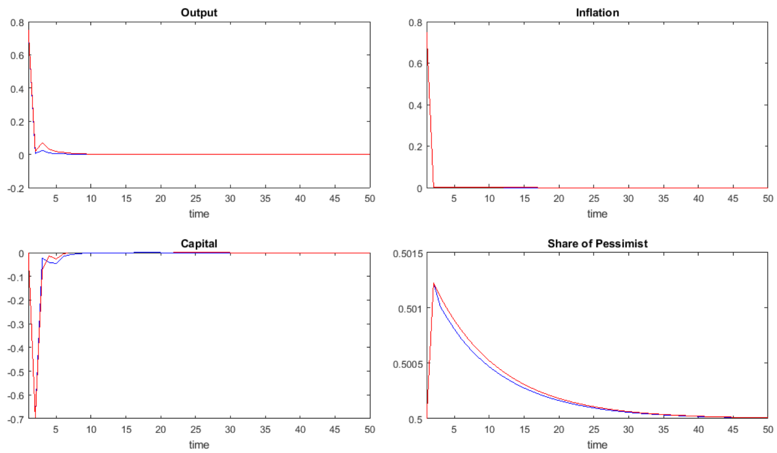

Figure 4: Simulation with biased entrepreneurs and fundamentalist expectations; red line: strict inflation targeting monetary policy, blue line: flexible inflation targeting monetary policy.

The last scenario analyses agents extrapolating the next expected variable level from a reference point, $\frac{x^A_{t-1}+x_{t-1}}{2}$.[10] For both the policies, the model simulations exhibit very peculiar results. Indeed, the environment displays persistent fluctuations with the presence of over and undershooting phenomena which does not allow the convergence to the steady state in the short period. As showed by Figure 5, in both the scenarios the central bank is not able to reach immediately the steady state level. Nevertheless, the fluctuations exhibit a decreasing amplitude converging to the steady state in the long run. This dynamic can be explained by the adaptive learning behaviour in the expectations. In each time the agents update their information on the past realizations of variables, i.e. the averages, and form their expectations adding a trend-component. Therefore, being bounded by the average through this updating process, on the one side the agents reduce the possible range of their predictions affecting also the actual

[10] See Tversky and Kahneman (1974) for a thorough dissertation of this rule.

variables, on the other side the anchor (i.e. the average) acts like a stickiness delaying the convergence process to the steady state.

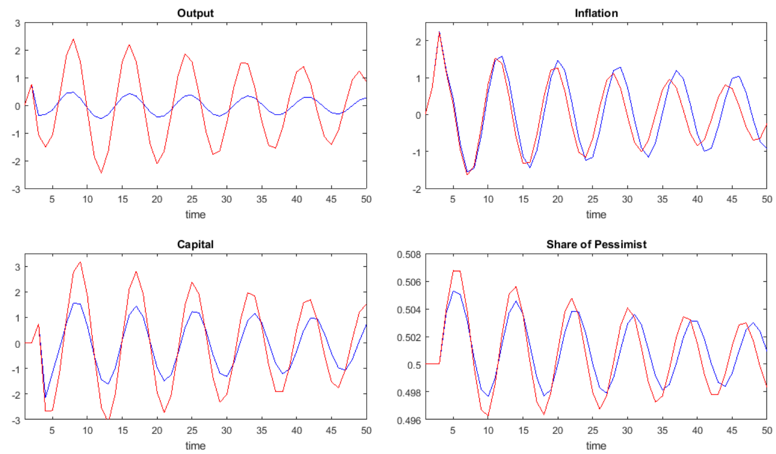

Figure 5: Simulation with biased entrepreneurs and anchoring-adjustment expectation rule; red line: strict inflation targeting monetary policy, blue line: flexible inflation targeting monetary policy.

Summarizing the results of these analyses, on the one hand, it is interesting to notice how the strict inflation targeting policy yields deeper fluctuations in the first periods but, when the timing is considered, it is able to stabilize the economy in fewer quarters. On the one side, these two results may be explained by the aim of the policy (to stabilize the price fluctuations) and by how this is pursued. This policy does not take into account the fluctuations in the output gap, therefore all the related graphs illustrate higher variations for this variable. On the other side, the flexible inflation targeting monetary policy brings the system to lower fluctuations even though these are more persistent. Replying strongly to the fluctuations of the variables, this policy generates continuous and negative feedback on the investment decisions of the entrepreneurs increasing the probability for them to have in each period the “wrong attitude”, i.e. the less profitable expectations. On the other hand, comparing the effects of both the policies on the four scenarios arises an interesting and counterintuitive insight: more sophisticated is the heuristic, higher seem be the fluctuations of the economy.[11] Indeed, looking at the average output gap and the inflation, the anchoring and adjustment expectations rule registers the lower average levels but the higher standard deviations.[12] This can be explained by the mechanism driving the heuristic. Indeed, the anchor parameter, i.e. the mean among the past observations average and the last observation, may be source of a time-varying bias. Besides the anchor, the LAA heuristic extrapolates the last variable change increasing the variance of the predictions. Conversely, naive and weak trend-following scenarios exhibit lower fluctuations. The reason of quite similar statisticsts between these behaviours can be found in their nature: if the trend parameter goes to zero, the trend-following rule simplifies to the naive heuristic. In addition, the trend-following rule exhibits higher (lower) volatility depending on the pro-trend nature of the heuristic which involves waves of pessimism and optimism in the investment decisions and therefore in the output.

Concluding, it should be underlined that the scenarios endogenously generate the economic fluctuations thanks to the heterogeneity in the entrepreneurs' expectations on the capital return. Besides, as described in the related literature on the discrete choice mechanism, it should be highlighted how the results of these simulations are highly dependent from the initial conditions.

Having showed how the heterogeneity and the bounded rationality rise endogenous fluctuation in the system, the next section will present the effects of some macroeconomic shocks with the aim to investigate which monetary policy (between strong and inflation oriented) is able to better stabilize the economy.

[11] Henceforward in the discussion, even if the comments may also apply for the fundamentalist behaviour, this heuristic will be excluded given that it does not set a "proper" computational rule. Indeed, as explained, this rule of thumb assumes that agents know the fundamental and steady state value of the variables but it does not specify how they are able to compute them.

[12] All the statistics are reported in Appendix C.

3.3.3 Stabilization analysis

The simulations described in this section are hit by three types of macroeconomics shocks: monetary, on the supply side and on the production. The treated population has naive expectations and the model has run for 700 quarters. The figures show the impact of unanticipated 1% increase in the variable from its steady state level in scenarios with flexible inflation targeting (FIT) monetary policy and strict inflation targeting (SIT) monetary policy.

Figure 7 represents the dynamics with a demand shock. The shock on prices affects positively the risk free interest rate and it may reduce the spread between the return of capital and the interest rate on loan. As a consequence, more entrepreneurs become pessimist on the profitability of the investment and reduce the purchase of capital causing an output drop down.

Comparing now the magnitude of the fluctuations among the two possible policies, it can be seen how the FIT policy is able to better stabilize the economy. The strict inflation targeting monetary policy, leaving aside the stabilization of the output from its goals, yields five times higher fluctuations than the flexible inflation targeting monetary policy. The huge variations affects also the trends of capital, given that the return of capital is a function of the production and the entrepreneurs are willing to purchase more capital only if it is profitable.

Another channel of propagation is founded in the higher fluctuations of the return of capital bringing to higher variations in the entrepreneur's fractions. The unanticipated variations in the return of capital modify the spread between this and the interest rate on loan and consequently the return on investment, i.e. the performance measure upon which the entrepreneurs base their expectations.

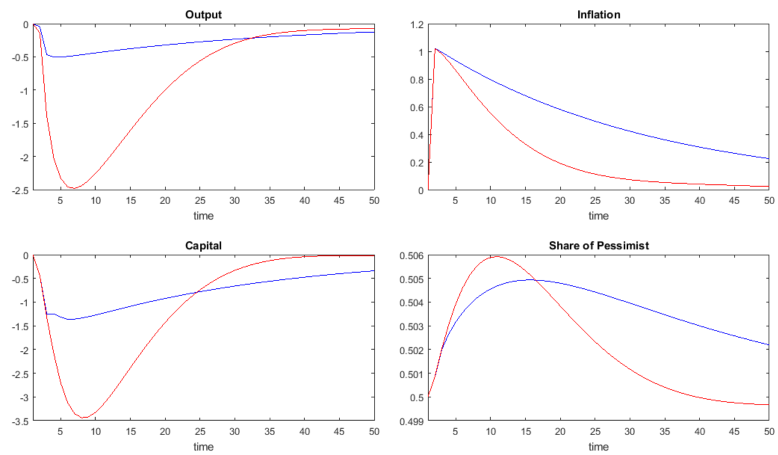

Figure 7: Simulation with price's shock; red line: strict inflation targeting monetary policy, blue line: flexible inflation targeting monetary policy.

Analysing the responses of an unexpected monetary shock - more precisely an unanticipated exogenous movement in the short-term interest rate of 1% - Figure 8 exhibits a drop in output and purchased capital: when the risk free interest rate increases the cost of external finance increases as well reducing the profitability of the investment.

Even if the dynamics in the share types are quite comparable, the increase in the share of pessimist agents has deeper effects on the dynamics in the SIT framework. In this scenario, the monetary authority focuses only on the inflation level without stabilizing the output gap. Thus, it is not able to avoid the propagation in the system of wider fluctuations. The drop in the investment deteriorates the total output and the dynamic of inflation. Consequently, this fall in inflation has positive effects on the risk free interest rate and on the cost of loans in the following periods. This explains why in the framework registering the higher fall in inflation there are also overshooting phenomena in output and capital dynamics.

However, it seems that a policy concerning both the stabilization on price and the output gap (the flexible inflation targeting monetary policy) is able to better stabilize the system. This is true when the entity of the fluctuations is evaluated, but the results are not so clear if the goals are the short run fluctuations. Indeed they seem more persistent, therefore the time required to reach the steady state is wider. According to Figure 8, the strict inflation targeting policy seems to be closer to the steady state after fewer quarters than the FIT monetary policy.

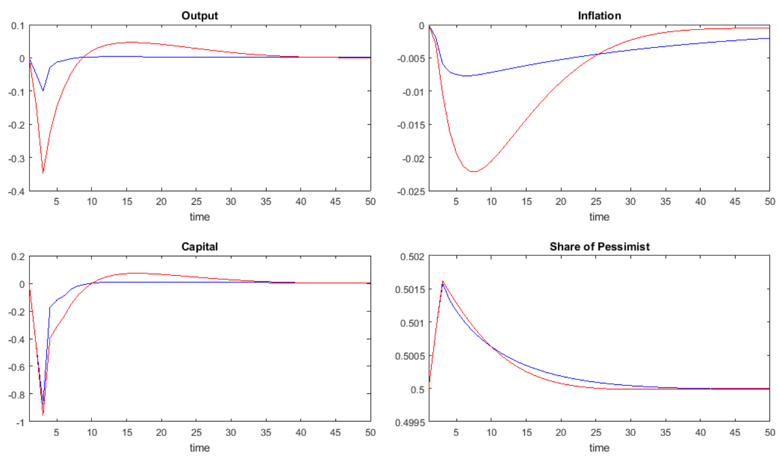

Figure 8: Simulation with monetary shock; red line: strict inflation targeting monetary policy, blue line: flexible inflation targeting monetary policy.

The analysis concludes investigating the impulse responses to a positive shock affecting the total output of the economy. Figure 9 shows how this shock influences positively the inflation in the next period. This dynamic can be explained looking to the nature of the expectations and how these affect the actual inflation. Indeed, the unexpected and temporary growth of production interests the inflation with delay due to naive expectations. Moreover, it is helpful to highlight the substantial impact of the monetary policy goals on the convergence to the steady state. The shock in the strict inflation targeting monetary policy drives the invested capital to an overshooting evolution before it converges to the steady state. This dynamic emerges because this policy does not consider the output gap as a priority. Therefore, in the SIT scenario the increase in the output positively influences the risk free interest rate by means of inflation growth, whereas in the FIT scenario it acts by means of both inflation and output. As a consequence, the external finance premium will be lower in the SIT scenario and the expected net return on investment will be higher as well as the invested capital.

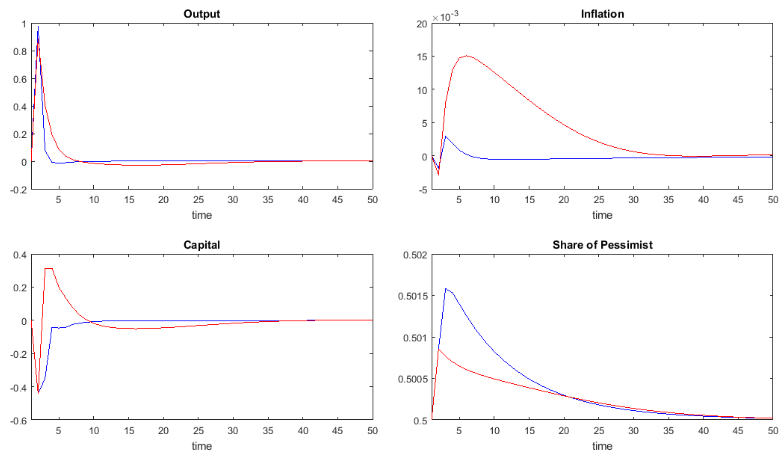

Figure 9: Simulation with production's shock; red line: strict inflation targeting monetary policy, blue line: flexible inflation targeting monetary policy.

3.4 Concluding remarks

This paper presents a financial accelerator framework to study the effects of heterogeneous and bounded rational expectations. It uses the heterogeneous framework of Brock and Hommes (1997) where the decision to switch the expectations arises from an endogenous process aiming to the most profitable strategy. The analysis addresses the investigation of the heterogeneity and financial accelerator effects on the investment decisions, indeed the heterogeneity of the expectations on capital return brings the agents to purchase different amount of capital.

In this framework, the macroeconomic fluctuations emerge endogenously in the model thanks to the expectations updating mechanism. Indeed, this result can be explained by the bounded rationality of the entrepreneurs, i.e. they have not perfect and complete information. First of all, they have only a common belief on fundamental investment return but they do not know the general feeling of the market, in other words, the share of pessimist and optimist in the market. Secondly, the different patterns of investment between the entrepreneur's types and the consequent unknown amount of total capital affect the return on investment of every single entrepreneur. This can misplace the entrepreneurs transforming a former profitable strategy into a non profitable one.

In this model, the financial accelerator and the asset price volatility work to amplify the fluctuations in a significant quantitative way, especially in the investment choices. These channels affect both the single investment decisions and the future fractions, amplifying the future total investment and the fluctuations of the system. Hence, the model is able to generate some important stylized facts in many financial and economical series, e.g. unpredictable returns. After having found endogenous fluctuations analysing the model with different expectation rules, the paper performed some simple monetary policy simulations in order to understand if there are some general intuitions on which policies are better to stabilize the market.

As in the reference literature,[13]in this framework, macroeconomic stability and inflation dynamics depend interestingly on the set of forecasting strategies and the types of the policy rule considered. Summarizing, this analysis seems suggest two policy prescriptions.

On the one side, the core results of these analyses seem to argue that no

[13]See for example Anufriev et al. (2013) or Massaro (2013).

monetary policy is able to quickly stabilize the system completely, actually some fluctuations persist for many quarters. On the one hand, the flexible inflation targeting monetary policies better stabilize the economy yielding smaller fluctuations. The SIT policies allow output gap to register fluctuations higher than the flexible inflation targeting policy. In most of the cases, this output dynamic implies higher fluctuations in the return of capital bringing to higher variations in the entrepreneur's fractions. On the other hand, it is interesting to notice that even if the strict inflation targeting policy yields deeper fluctuations, these are less persistent. As explained, these results may be due to the aim of this specific monetary policy. In other words, the FIT policy replies actively to the fluctuations of the variables and generates continuous and negative perturbations to the investment decisions of the entrepreneurs increasing the probability for them to switch to the other investment strategy.

On the other side, the stabilizing effect of the monetary policy strongly depends on the nature of the forecasting rules. Moreover, this analysis seems suggest a counterintuitive result. Indeed, it seems that in scenarios with more sophisticated heuristics the fluctuations are higher. In other words, increasing the decision-making and computational ability of the agents may not increase both the probability of more performing investment decisions and the system stability.

Concluding, in a framework with heterogeneity and bounded rationality, it seems that the Central Banks should take seriously into account the heterogeneity and the bounded rationality of the agents when designing monetary policy.

Appendix A: The complete log-linearized model

This section presents the complete log-linearization of the model. Let lower case variables denote percent deviations from the steady state and the ratios among capital letters without time pedix denote the ratios of the respective steady state values, the log-linearization of the model is:

$$b_t^h = n_{t-1}^h + \frac{1}{\frac{(RK \pm b - \chi R)}{2\mu R}} \left(\frac{Q\,(RK \pm b)}{2\mu R} \right) (q_{t-1} - r_{t-1}) \qquad (A1),$$

$$z_t^h = \frac{\chi R}{\chi R + \mu \frac{B^h R}{N^h}} r_{t-1} + \frac{\mu B^h R}{N^h \left[\chi R + \mu \frac{B^h R}{N^h} \right]} \left(b_t^h + r_{t-1} - n_{t-1}^h \right) \qquad (A2)\,,$$

$$q_t = \varphi\,(k_t - k_{t-1}) \qquad (A3)\,,$$

$$k_t^h = \frac{B^h}{(B^h + N^h)} b_t^h + \frac{N^h}{(B^h + N^h)} n_{t-1}^h - q_{t-1} \qquad (A4)\,,$$

$$k_t = \frac{s^o K^o}{(s^o K^o + s^p K^p)} \left(k_t^o + s_{t-1}^o \right) + \frac{s^p K^p}{(s^o K^o + s^p K^p)} \left(k_t^p + s_{t-1}^p \right) \qquad (A5)\,,$$

$$r_t^k = \frac{1}{\frac{\alpha Y}{XKQ} + (1-\delta)} \left[\frac{\alpha Y}{XKQ} (y_t - k_t - q_{t-1}) + (1-\delta)(q_t - q_{t-1}) \right] \qquad (A6)\,,$$

$$n_t^h = \frac{\gamma R^k Q K^h}{\gamma \left(R^k Q K^h - Z^h B^h\right) + W^e} \left(r_t^k + q_{t-1} + k_t^h\right) - \frac{\gamma Z^h B^h}{\gamma \left(R^k Q K^h - Z^h B^h\right) + W^e} \left(z_t^h + b_t^h\right) + ...$$

$$... + \frac{W^e}{\gamma \left(R^k Q K^h - Z^h B^h\right) + W^e} w_t^e \qquad (A7),$$

$$U_{h,t}^a = \frac{R^k Q K^h}{R^k Q K^h - Z^h B^h} \left(r_t^k + q_{t-1} + k_t^h\right) - \frac{Z^h B^h}{R^k Q K^h - Z^h B^h} \left(z_t^h + b_t^h\right) - k_t^h - q_{t-1} \qquad (A8)\,,$$

$$y_t = b_1 \left[a_1 E_{t-1} \left\{y_t\right\} + a_2 \left(r_t - E_{t-1} \left\{\pi_t\right\}\right)\right] + b_2 k_t \qquad (A9)\,,$$

$$\pi_t = f_1 E_{t-1} \left\{\pi_t\right\} + f_2 y_t + \varepsilon_t^\pi \qquad (A10)\,,$$

$$r_t^n = \rho r_{t-1}^n + (1+\rho) \left(\tau \pi_t + \iota y_t\right) + \varepsilon_t^r \qquad (A11)\,.$$

Equation (A1) represents the log-linearized version of the solution of the entrepreneur's maximization problem. Equation (A2) is the log-linearized interest rate on loans and represents the positive influence of the leverage ratio on the external finance cost and represents the financial accelerator. Equations (A3), (A4), (A5) and (A6) characterize the investment demand. Equations (A3) and (A6) are the standard relations for marginal product of capital and the link between asset price and investment. Equation (5) represents the total invested capital and is the weighted sum of purchased capital of the two fractions, described by Equation (A5).

The evolution of the entrepreneur's net worth is described by Equation (A7), this affects the investment decision affecting the cost of loans and their required amount to the financial intermediary. Equation (A8) is the log-linearized version of the performance measure, the return on investment. Finally Equations (A9), (A10) and (A11) are conventional for the NK framework.[14] Equation (A10) and (A11) impose exogenous shocks on the inflation and interest rates. The monetary policy rule expressed by Equation (A11) is the more general as possible in order to leave some degrees of freedom for the dynamic analyses.

Appendix B: Interest rate smoothing policies

This appendix shows the different dynamics comparing a strong monetary policy ($\phi_\pi > 1$) with an alternative specifications in which the Taylor principle is violated ($\phi_\pi < 1$) considering the following interest rate smoothing rule:

$$r_t^n = \rho r_{t-1}^n + (1-\rho)\left(\phi_\pi \pi_t + \phi_y y_t\right), \, with \, \phi_y = 1.$$

As Figure 10 shows, if the coefficient of interest rate's autocorrelation is low ($\rho = 0.2$), the convergence paths of long run trajectories are in line with the simulation run with flexible inflation targeting monetary policy (Figure 2). On the contrary, there are fluctuations with higher frequencies in the short run. This may be explained by the fact that the Central Bank is particularly active pursuing the financial stabilization paying less attention on output gap and inflation. Hence, the interest rate is hit in every period, bringing micro-fluctuations in the share of the entrepreneur's types which affects the investment decisions and the output. If the Central Bank increases its effort to stabilize the financial sector (if the autocorrelation of the interest rate grows to $\rho =$

[14]See Romer (2012).

0.8), it will further increase the fluctuations both in the long and short run because this policy does not ensure the convergence to the steady state level. Indeed, according to this parametrization, the interest rate rule does not satisfy the Taylor Principle.[15] In other words, the central bank does not adjust the interest rate with "sufficient strength" misleading the investment decision of the entrepreneurs. More precisely, the weak response to inflation growth, affecting positively both the expected and the net investment returns, entails expanding levels of capital and output avoiding the convergence to the steady state value.

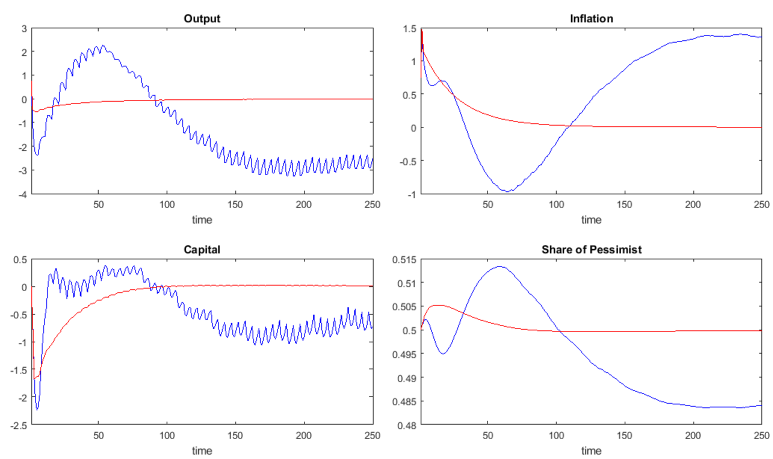

Figure 10: Simulation with biased entrepreneurs and weak trend following expectations; red line: flexible inflation targeting monetary policy obeying to the Taylor Principle, blue line: flexible inflation targeting monetary policy with inflation parameter lower than 1.

[15]It should be noticed that, with an interest rate's autocorrelation of 0.8, the inflation parameter in the interest rate rule becomes equal to 0.3, hence lower than 1.

Appendix C: Statistics

Scenario	Heuristic	Y	π	K	# of pessimist
$\phi_\pi = 1.5,\ \phi_y = 0$	Naive	-0.0451	0.0238	-0.0759	0.5005
		(0.2398)	(0.1258)	(0.3854)	(0.0019)
$\phi_\pi = 1.5,\ \phi_y = 1$	Naive	-0.0172	0.0584	-0.0719	0.5008
		(0.0634)	(0.1810)	(0.221)	(0.0023)
$\phi_\pi = 1.5,\ \phi_y = 0$	Chartist	-0.0449	0.0243	-0.0739	0.5062
		(0.3080)	(0.1671)	(0.4824)	(0.0445)
$\phi_\pi = 1.5,\ \phi_y = 1$	Chartist	-0.0171	0.0590	-0.0691	0.5008
		(0.0784)	(0.2395)	(0.2838)	(0.0028)
$\phi_\pi = 1.5,\ \phi_y = 0$	LAA rule	-0.0441	0.0194	-0.0733	0.5005
		(0.3995)	(0.2568)	(0.5158)	($8.420e^{-04}$)
$\phi_\pi = 1.5,\ \phi_y = 1$	LAA rule	-0.0161	0.0552	-0.0640	0.5008
		(0.0923)	(0.3047)	(0.3065)	($6.912e^{-04}$)
$\phi_\pi = 1.5,\ \phi_y = 0$	Fundamentalist	-0.0262	0.0201	-0.0505	0.5004
		(0.0296)	(0.0276)	(0.0246)	($7.607e^{-05}$)
$\phi_\pi = 1.5,\ \phi_y = 1$	Fundamentalist	-0.0071	0.0205	-0.0249	0.5003
		(0.0286)	(0.0276)	(0.0255)	($7.668e^{-05}$)

Table 1: Averages and standard deviations, in brackets, of the simulations considering different policies and heuristics.

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