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Alessandro Giuliani[1], Vieri Mastropietro[2] and Fabio Toninelli[3]

[1] *Dipartimento di Matematica e Fisica, Università di Roma Tre - L.go S. L. Murialdo 1, 00146 Roma, Italy*
[2] *Dipartimento di Matematica, Università degli Studi di Milano - Via Saldini, 50, 20133 Milano, Italy*
[3] *CNRS and Université de Lyon, Université Lyon 1, Institut Camille Jordan - 43 bd du 11 novembre 1918, 69622 Villeurbanne, France*

Abstract – We rigorously establish the asymptotic equivalence between the height function of interacting dimers on the square lattice and the massless Gaussian free field. Our theorem explains the microscopic origin of the sine-Gordon field theory description away from the free fermion point, which has previously been elusive. We use a novel technique, based on the combination of discrete holomorphicity with exact, constructive, renormalization group methods, which has the potential of being applicable to a variety of other non-integrable models at or close to criticality.

High-temperature superconductivity and the physics of Resonance Valence Bonds (RVB) [1] was the original motivation for studying two-dimensional (2D) quantum *dimers*, which later became an important model for frustrated magnetism, cold bosons, and many other systems with hard constraints [2]. In these contexts also classical dimers are of interest, not only because they capture the high-temperature physics of their quantum counterpart, but also because for special values of the parameters the quantum static correlations can be expressed in terms of the classical ones [3]. The properties of a wide class of classical dimer models can be understood by using a celebrated result of half a century ago, the Kasteleyn theorem [4], ensuring exact solvability and explicit expressions of the correlations, which can be written in terms of *Pfaffians*. By using this result and the above-mentioned equivalence, the correlations of certain quantum dimer models at special values of the parameters on the square [3] and triangular lattice [5] were computed, finding a power law (critical), and an exponential (massive) large distance decay, respectively.

However, exact solvability is limited to a special class of systems, and further progress in our understanding of the physics of dimers requires the analysis of what happens away from integrability. We consider a prototypical non-solvable dimer model obtained by assuming a local interaction between parallel dimers: given a periodic box

$\Lambda \subset \mathbb{Z}^2$ of side L (with L even), the partition function is

$$Z_\Lambda(\lambda, m) = \sum_{M \in \mathcal{M}_\Lambda} \Big[\prod_{b \in M} t_b^{(m)} \Big] e^{\lambda \sum_{P \subset \Lambda} N_P(M)}, \quad (1)$$

where $\mathcal{M}_\Lambda$ is the set of dimer coverings of Λ, $\lambda = v/T$ with T the temperature, P is a plaquette (face of $\mathbb{Z}^2$) and $N_P(M) = 1$ if the plaquette P is occupied by two parallel dimers in M, and $N_P(M) = 0$ otherwise; the m-dependence in the reference weight $t_{(\mathbf{x}, \mathbf{x}+\hat{e}_j)}^{(m)} = 1 + \delta_{j,1} m(-1)^{x_1}$ tunes the distance from criticality; λ tunes the distance from solvability, with $\lambda > 0$ corresponding to a local attractive interaction. This model, in the $m = 0$ case, describes polar crystals [6] and it was recently reconsidered in [7–12] where its connection with quantum dimer models, RVB physics and large spin quantum anti-ferromagnets was worked out in detail and used to infer information on the RVB spin-liquid order parameters. Monte Carlo simulations show the presence of *non-universal* anomalous exponents in the dimer correlations decay. This confirms the general picture that the asymptotic properties can be captured by a *Quantum Field Theory* (QFT) of the sine-Gordon type, the fundamental field being a coarse-grained version of the *height* function. Using this effective QFT description, much information was derived about the phase diagram, including the Kosterlitz-Thouless universality of the phase
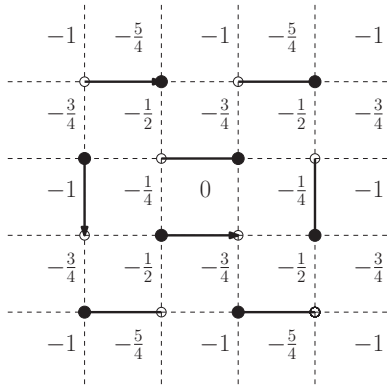
Fig. 1: A dimer configuration for $L = 4$ and the associated height function. The height of the central plaquette is conventionally set to 0.

transition from a liquid to a crystalline phase. The same effective description is believed to be applicable to a variety of dimer and interface models, and it is at the basis of our current understanding of their physics. However, while the validity of the QFT description is supported *a posteriori* by the agreement of its prediction with simulations, a purely deductive and rigorous microscopic argument establishing its correctness is currently not available [2], with the only exception of the integrable, non-interacting, case. Even then, the derivation is very non-trivial, and it has been provided only recently [13] using Discrete Holomorphicity (DH) methods.

In this letter we present the first mathematical justification of the quantum field theory description of *non-integrable* dimer models. We prove a theorem establishing the convergence, in the scaling limit, of the height function of model (1) to the massless Gaussian Free Field (GFF), in a suitable range of parameters. This is done by a new method, based on the combination of DH methods with Constructive Renormalization Group (CRG) techniques [14], which can be applied in a much wider context, including interacting dimers on different lattices (either bipartite or not) and non-integrable deformations of Ising models.

Given a dimer covering M, two faces of Λ centered at $\mathbf{x}$ and $\mathbf{y}$ and a path $C_{\mathbf{x} \to \mathbf{y}}$ from $\mathbf{x}$ to $\mathbf{y}$ with trivial winding around the torus Λ, we define the *height* difference between $\mathbf{x}$ and $\mathbf{y}$ as (see fig. 1)

$$h_{\mathbf{x}} - h_{\mathbf{y}} = \sum_{b \in C_{\mathbf{x} \to \mathbf{y}}} \left(\mathbb{1}_b(M) - \frac{1}{4} \right) \sigma_b, \quad (2)$$

where $\sigma_b = +1/-1$ depending on whether $C_{\mathbf{x} \to \mathbf{y}}$ crosses b with the white site on the right/left. Moreover, $\mathbb{1}_b(M)$ is equal to 1 if b is occupied by a dimer in M, and 0 otherwise. A crucial property of the height function is that $h_{\mathbf{x}} - h_{\mathbf{y}}$ is *independent* of the choice of $C_{\mathbf{x} \to \mathbf{y}}$. The *dimer correlation* is given by $\langle \mathbb{1}_b; \mathbb{1}_{b'} \rangle$, where $\langle \ldots; \ldots \rangle$ is the truncated expectation with the weight in (1), and the

two point *height correlation* is

$$\langle (h_{\mathbf{x}} - h_{\mathbf{y}})^2 \rangle = \sum_{b_1 \in C_{\mathbf{x} \to \mathbf{y}}} \sum_{b_2 \in C_{\mathbf{x} \to \mathbf{y}}} \sigma_{b_1} \sigma_{b_2} \langle \mathbb{1}_{b_1}; \mathbb{1}_{b_2} \rangle. \quad (3)$$

Our main result is the following.

Theorem 1. *For $\lambda \neq 0$ sufficiently small, $L \to \infty$, and $m \to 0$, the height correlation for $\mathbf{x} \neq \mathbf{y}$ verifies:*

$$\langle (h_{\mathbf{x}} - h_{\mathbf{y}})^2 \rangle = \frac{K(\lambda)}{\pi^2} \log |\mathbf{x} - \mathbf{y}| + R(\mathbf{x} - \mathbf{y}), \quad (4)$$

with $K(\lambda)$ an analytic function such that $K(0) = 1$, and $R(\mathbf{x})$ a bounded remainder. The higher-order truncated correlations are bounded uniformly in $|\mathbf{x} - \mathbf{y}|$. At large distances, the coarse graining of $h_{\mathbf{x}}$ converges to the GFF, in the sense that, if $\alpha \in \mathbb{R}$ and f is a smooth, compactly supported function on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} f(u) \mathrm{d}u = 0$, one has

$$\langle e^{i\alpha\varepsilon^2 \sum_{\mathbf{x}} h_{\mathbf{x}} f(\varepsilon\mathbf{x})} \rangle \overset{\varepsilon \to 0}{\to} e^{\frac{K\alpha^2}{4\pi^2} \int f(u)f(v) \log|u-v| \mathrm{d}u\mathrm{d}v}, \quad (5)$$

where ε^{-1} represents the coarse-grain scale, to be sent to infinity after the thermodynamic limit.

Let us mention that the logarithmic growth of the height variance (without sharp control of the constant in front of the log) for some discrete $(2 + 1)$-dimensional interface models (Solid-on-Solid and discrete Gaussian model) was obtained in [15].

The choice of the specific interaction in (1) is just for illustrative purposes: the same result remains valid for generic finite-range interactions, translationally and rotationally invariant.

Equation (5) can be re-read in a more evocative form: if χ_0 is a smooth, compactly supported, probability distribution centered at the origin, $\chi_\xi(u) = \chi_0(u - \xi)$ is its translate, and $\tilde{h}_\varepsilon(\xi) = \varepsilon^2 \sum_{\mathbf{x}} h_{\mathbf{x}} \chi_\xi(\varepsilon\mathbf{x})$, then choosing $f = \chi_\xi - \chi_\eta$ in (5) we find

$$\lim_{\varepsilon \to 0} \langle e^{i\alpha(\tilde{h}_\varepsilon(\xi) - \tilde{h}_\varepsilon(\eta))} \rangle \simeq (\mathrm{const})|\xi - \eta|^{-K\alpha^2/(2\pi^2)},$$

asymptotically as $|\xi - \eta| \to \infty$. The left side is the coarse-grained "electric correlator" ($e^{i\alpha h_{\mathbf{x}}}$ being the lattice analogue of the electric vertex operator in the Coulomb gas picture): our theorem proves its anomalous power law decay at large distances. In the $\lambda = 0$ case, the asymptotic computation of the electric correlator $\langle e^{i\alpha(h_{\mathbf{x}} - h_{\mathbf{y}})} \rangle$, without any coarse graining, has been obtained in [16] for all $\alpha \in (-\pi, \pi)$.

An important step in proving the above result is the computation of the asymptotic behavior of the dimer correlation by CRG methods [14,17]. In the limit $L \to \infty$, if $m \neq 0$, it decays exponentially at large distances with rate $O(m^{1+\eta_m(\lambda)})$ (gaseous phase), with $\eta_m(\lambda)$ an analytic function such that $\eta_m(0) = 0$. If $m \to 0$, it decays as a power law (liquid phase): *e.g.*, if b, b' are both horizontal with $b' - b = (x_1, x_2)$ and $z = x_1 + ix_2$, then it decays

polynomially, with critical exponent $\min\{2, 2+\eta(\lambda)\}$, and $\eta(\lambda) = -(32/\pi)\lambda + O(\lambda^2)$ an analytic function of λ,

$$\langle \mathbb{1}_b; \mathbb{1}_{b'} \rangle = \frac{(-1)^{x_1+x_2}}{2\pi^2} K(\lambda) \mathrm{Re} \frac{1}{z^2} + (-1)^{x_1} \frac{\bar{K}(\lambda)}{|z|^{2+\eta(\lambda)}} + \mathrm{h.o.}$$

$$(6)$$

Here $K(\lambda)$ is the same as in (4), $\bar{K}(\lambda)$ is another analytic function such that $\bar{K}(0) = 1$, and h.o. indicates faster decaying terms at infinity. The above formula reduces as $\lambda \to 0$ to the one known by Kasteleyn's exact solution. The main effect of the interaction is to produce an anomalous exponent in the second term, in agreement with the numerical simulations of [7]. Remarkably, there are no radiative corrections to the exponent of the first term. The model belongs to the same universality class as the XXZ chain, vertex models and Luttinger liquids.

While the dimer characteristic function is a local observable, the height differences are non-local "string" observables, as apparent from (3). Even at $\lambda = 0$, the computation of the height correlation is very subtle. Indeed, by inserting the $\lambda = 0$ version of (6) into (3), one gets an apparently very singular expression: take, *e.g.*, $\mathbf{x}$ and $\mathbf{y}$ on the same horizontal line. In the large separation limit, the object of interest is formally proportional to $\int_\xi^\eta \frac{dudv}{(u-v)^2}$, where ξ, η are the (suitably rescaled) horizontal coordinates of $\mathbf{x}$ and $\mathbf{y}$. Such an integral requires a proper interpretation, because of its singularity at $u = v$, and the result depends on the specific ultraviolet regularization. Of course, an "*ad hoc*" regularization can be chosen [18] in order to reproduce the expected result, but the problem remains of a general derivation, which can unambiguously return the correct exponents without any external bias. The problem was finally solved in [13], and the $(1/\pi^2)$ factor in front of the logarithm in (4) at $\lambda = 0$ was rigorously computed, by taking advantage of DH (lattice) methods. In the interacting case, the problem is much more puzzling. In fact, in addition to the problem of the ultraviolet divergences affecting the computation of the $(1/\pi^2)$ prefactor, the anomalous decay in (6), once inserted into (3), may change the logarithmic growth into an anomalous growth. Our theorem proves that this is not the case: logarithmic fluctuations are robust, stability being guaranteed by sophisticated cancellations arising from emerging chiral symmetry. Spurious ultraviolet divergences are avoided by using the irrelevant terms coming from the lattice: in this respect, the use of exact CRG methods (which, in contrast to the field-theoretic RG, takes the irrelevant terms into full account) is essential. A detailed proof of our main theorem is rather technical and is given elsewhere [17]; below we explain its main ideas.

Sketch of the proof. The first step consists in an exact rewriting of the finite volume/finite lattice *generating function* of dimer correlations, $\mathcal{Z}(A)$ (defined so that $\langle \mathbb{1}_{b_1}; \cdots ; \mathbb{1}_{b_k} \rangle = \frac{\partial^k}{\partial A_{b_1} \cdots \partial A_{b_k}} \log \mathcal{Z}(A)|_{A=0}$, b_i labeling the nearest-neighbor bonds) as a finite Grassmann integral

([17], sect. 2):

$$\mathcal{Z}(A) = \frac{1}{2} \sum_{\theta,\tau} C_{\theta,\tau} \int_{\theta,\tau} P_{\theta,\tau}(\mathrm{d}\psi) e^{V(\psi)+B(\psi,A)}. \quad (7)$$

Here $\psi_\mathbf{x}$ are *Grassmann variables*, V is the sum of monomials in $\psi_\mathbf{x}$ of order 4 or higher, $B(\psi, A)$ is a source term, sum of monomials in ψ and in A, $\theta/\tau \in \{0, 1\}$ label the boundary conditions for the Grassmann variables in the horizontal/vertical directions (0/1 corresponding to periodic/antiperiodic conditions), and $C_{0,0} = -1$, while $C_{\theta,\tau} = +1$ otherwise. By cluster expansion methods, we prove that V and B are analytic in λ. $P_{\theta,\tau}(\mathrm{d}\psi)$ is a Gaussian Grassmann integration with propagator

$$g(\mathbf{x}, \mathbf{y}) = \frac{1}{L^2} \sum_\mathbf{k} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{i \sin k_1 + \sin k_2 + m(-1)^{y_1} \cos k_1}{2D(\mathbf{k}, m)},$$

$$(8)$$

where $D(\mathbf{k}, m) = m^2 + (1 - m^2)(\sin k_1)^2 + (\sin k_2)^2$, and k_1, k_2 are in $(2\pi/L)\mathbb{Z}$ or $(2\pi/L)(\mathbb{Z} + 1/2)$, depending on boundary conditions.

If $\lambda = 0$, then $V = 0$, in which case the integral is Gaussian and can be computed exactly. When $\lambda \neq 0$ the integral is not Gaussian, and it can be evaluated by a multiscale analysis using CRG methods [14]. We are interested in the case of m small or vanishing. As $L \to \infty$ and $m \to 0$, the propagator in (8) becomes singular in correspondence of four momenta, namely $\mathbf{p}_1 = (0, 0)$, $\mathbf{p}_2 = (\pi, 0)$, $\mathbf{p}_3 = (\pi, \pi)$, $\mathbf{p}_4 = (0, \pi)$. Therefore, $g(\mathbf{x}, \mathbf{y})$ can be naturally written as the superposition of four terms, each of which is concentrated in momentum space around one of the singularities. Correspondingly, we decompose the Grassmann field as

$$\psi_\mathbf{x} = e^{i\mathbf{p}_1\mathbf{x}}\psi_{\mathbf{x},1} - ie^{i\mathbf{p}_2\mathbf{x}}\psi_{\mathbf{x},2} + ie^{i\mathbf{p}_3\mathbf{x}}\psi_{\mathbf{x},3} + e^{i\mathbf{p}_4\mathbf{x}}\psi_{\mathbf{x},4}, \quad (9)$$

where $\psi_{\mathbf{x},\gamma}$ are Grassmann variables, often referred to as Majorana variables, since their effective action is a lattice regularization of the standard 2D Majorana action. Their propagator is block-diagonal, the fields with $\gamma = 1, 2$ being independent of $\gamma = 3, 4$; the propagator $G(\mathbf{x} - \mathbf{y})$ of the $\gamma = 1, 2$ fields is the same as that of the $\gamma = 3, 4$ fields, and reads (using the symbol $\int d\mathbf{k}/(2\pi)^2$ as a shorthand for the discrete sum in (8))

$$G(\mathbf{x}) = \frac{1}{Z} \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{\chi(\mathbf{k})e^{-i\mathbf{k}\mathbf{x}}}{2D(\mathbf{k}, m)}$$
$$\times \begin{pmatrix} i\sin k_1 + \sin k_2 & im\cos k_1 \\ -im\cos k_1 & i\sin k_1 - \sin k_2 \end{pmatrix},$$

where $\chi(\mathbf{k})$ is a smoothed characteristic function of the set $\max\{|k_1|, |k_2|\} \leq \pi/2$, and $Z = 1$. To evaluate the Grassmann integral (7) we use (9) and write the propagator $G(\mathbf{x})$ as sum of propagators living on momentum scales 2^h, $h \leq 0$. After integrating the scales $0, \ldots, h+1$, the (θ, τ) contribution to $\mathcal{Z}(A)$ is rewritten as

$$e^{S_h(A)} \int_{\theta,\tau} P_{Z_h, m_h}(\mathrm{d}\psi^{(\leq h)}) e^{V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) + B^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, A)},$$

where P_{Z_h,m_h} has propagator $G^{(h)}(\mathbf{x})$, defined in the same way as $G(\mathbf{x})$, with Z replaced by Z_h, m by m_h and $\chi(\mathbf{k})$ by $\chi_h(\mathbf{k})$, a (smoothed) characteristic function of the set $|\mathbf{k}| \leq (\pi/2)2^h$. The effective potential $V^{(h)}$ is

$$V^{(h)}(\psi) = \lambda_h \sum_{\mathbf{x}} \psi_{\mathbf{x},1}\psi_{\mathbf{x},2}\psi_{\mathbf{x},3}\psi_{\mathbf{x},4} + \text{ir.},$$

where ir. indicates the irrelevant terms (non-local quartic terms, and terms of order 6 or higher in ψ). Remarkably, the kernels of the irrelevant terms in $V^{(h)}$ are *analytic* in λ provided that $|Z_{h+1}/Z_h - 1|, |\lambda_h|$ are sufficiently small, as long as $|m_h| < 2^h$: the proof of this fact uses fermionic cluster expansion methods, including the use of Gram-Hadamard determinant bounds. Similarly, under the same assumptions, the effective source $B^{(h)}$ is analytic in λ. Its structure is expressed most easily by using Dirac rather than Majorana fields: the former are defined as $\psi^{\pm}_{\mathbf{x},1} := \frac{1}{\sqrt{2}}(\psi_{\mathbf{x},1} \mp i\psi_{\mathbf{x},3})$, $\psi^{\pm}_{\mathbf{x},-1} := \pm\frac{i}{\sqrt{2}}(\psi_{\mathbf{x},2} \mp i\psi_{\mathbf{x},4})$, and they are referred to as Dirac variables, because their action is the lattice analogue of that of 2D Dirac fields. In terms of $\psi^{\pm}_{\omega}$, the effective source reads

$$B^{(h)}(\psi) = \frac{Z^{(1)}_h}{Z_h}F_1(\psi,\mathbf{J}) + \frac{Z^{(2)}_h}{Z_h}F_2(\psi,\mathbf{J}) + \text{ir.},$$

where ir. are the irrelevant terms (non-local, or of higher order in A or ψ as compared to $A\psi\psi$). Moreover, denoting $J_{\mathbf{x},i} = J_{(\mathbf{x},\mathbf{x}+\hat{e}_i)}$ with $J_b = e^{A_b} - 1$:

$$F_1 = 2 \sum_{\mathbf{x},\,\omega=\pm} (-1)^{\mathbf{x}}(J_{\mathbf{x},1} + i\omega J_{\mathbf{x},2})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega},$$

$$F_2 = 2 \sum_{\mathbf{x},\,\omega=\pm} [(-1)^{x_1} J_{\mathbf{x},1} + i\omega(-1)^{x_2} J_{\mathbf{x},2}]\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}.$$

Summarizing, the effective theory on scale h has the same structure as a theory of interacting 2D lattice Dirac fermions with a wave function renormalization Z_h, an effective mass m_h, an effective coupling λ_h, and effective source couplings $Z^{(1)}_h, Z^{(2)}_h$. It is completely analogous to that obtained in the multiscale analysis of the 8-Vertex, Ashkin-Teller, XXZ, or Luttinger liquid models [14]: the only differences have to be found in the oscillating factors appearing in the definition of F_1, F_2 and in the specific structure of the irrelevant terms. The flow equation for the effective couplings of all these models is the same, up to irrelevant contributions, which are exponentially negligible in the infrared limit. Therefore, λ_h approaches exponentially, as $h \to -\infty$, a line of fixed points: $\lambda_{-\infty}(\lambda) = -32\lambda(1 + O(\lambda))$. Moreover, $Z_h \sim 2^{\eta(\lambda)h}$, $Z^{(i)}_h \sim 2^{\eta_i(\lambda)h}$, $m_h \sim m\,2^{\eta_m(\lambda)h}$, where "$\sim$" means that the ratio of the two sides is bounded from above and below by two universal positive constants, uniformly in h. Remarkably, using the emergent chiral gauge symmetry of the theory, we find that $\eta = \eta_1$, which implies the robustness (exact non-renormalization) of the exponent 2 in the first term of (6). The integration goes on until $m_h \simeq 2^h$, at which point the Dirac field is massive and
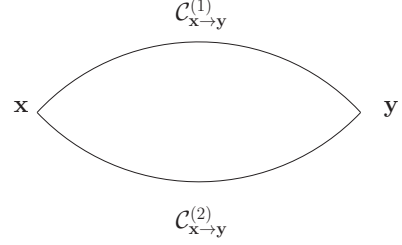

Fig. 2: A schematic view of the paths along which b_1, b_2 are summed over, to be called $\mathcal{C}^{(1)}_{\mathbf{x}\to\mathbf{y}}$ and $\mathcal{C}^{(2)}_{\mathbf{x}\to\mathbf{y}}$.

can be integrated in one step. If $m \to 0$ and $L \to \infty$, the integration has no infrared cutoff.

In order to evaluate the height fluctuations, we use the path independence of the height difference, which is a (weak) instance of DH. We proceed as in the third of refs. [13]. Consider, *e.g.*, the height variance: on the right side of (3) we deform the two paths along which b_1 and b_2 are summed over, in such a way that they are "as much separated as possible", as in fig. 2. In the vicinity of $\mathbf{x}$ and $\mathbf{y}$, the two paths are lattice approximations of straight lines, departing from and arriving at the points $\mathbf{x}, \mathbf{y}$ in *different* directions. After the path deformation, we replace the dimer correlation on the right side of (3) by its asymptotic expression (6) (and its analogues in the cases in which b, b' have different orientations). The h.o. terms contribute a finite constant, uniformly in $|\mathbf{x} - \mathbf{y}|$. The contribution to (3) from the term with decay exponent 2 reads

$$-\frac{K(\lambda)}{2\pi^2} \sum_{b_1 \in \mathcal{C}^{(1)}_{\mathbf{x}\to\mathbf{y}}} \sum_{b_2 \in \mathcal{C}^{(2)}_{\mathbf{x}\to\mathbf{y}}} \text{Re}\frac{\mathcal{D}z_{b_1}\mathcal{D}z_{b_2}}{(z_{b_1} - z_{b_2})^2}, \qquad (10)$$

where z_{b_i} are the representatives in complex coordinates of the centers of the bonds b_i, and $\mathcal{D}z_{b_i}$ are the oriented elementary path elements of $\mathcal{C}^{(i)}_{\mathbf{x}\to\mathbf{y}}$ crossing b_i, expressed in complex coordinates. Note that no oscillatory factor appears in (10): the factors $\sigma_{b_1}\sigma_{b_2}$ in (3) compensate exactly the oscillatory factor of the term under consideration in the dimer correlation. Equation (10) is the Riemann approximation to $-\frac{K(\lambda)}{2\pi^2}\text{Re}\int_{\gamma_1} dz \int_{\gamma_2} dw \frac{1}{(z-w)^2}$, where γ_1 and γ_2 are two *completely disjoint* complex paths (this is what makes the integral non-singular!) going from $z_{\mathbf{x}} = z_{b^{(1)}_{\mathbf{x}}}$ to $z_{\mathbf{y}} = z_{b^{(1)}_{\mathbf{y}}}$, and from $z'_{\mathbf{x}} = z_{b^{(2)}_{\mathbf{x}}}$ to $z'_{\mathbf{y}} = z_{b^{(2)}_{\mathbf{y}}}$, where $b^{(i)}_{\mathbf{x}}$ and $b^{(i)}_{\mathbf{y}}$ are the first and last bonds of $\mathcal{C}^{(i)}_{\mathbf{x}\to\mathbf{y}}$. Its value is $\frac{K}{2\pi^2}\text{Re}\log\frac{(z'_{\mathbf{y}}-z_{\mathbf{x}})(z'_{\mathbf{x}}-z_{\mathbf{y}})}{(z'_{\mathbf{y}}-z_{\mathbf{y}})(z'_{\mathbf{x}}-z_{\mathbf{x}})}$, which is the same as (4) up to a bounded error. Finally, let us consider the contribution to (3) from the term with exponent $2 + \eta$: in this case the factors $\sigma_{b_1}\sigma_{b_2}$ do not compensate exactly with the oscillatory signs in the dimer correlation; the left-over oscillations act, after summation along the paths, as discrete derivative, which effectively makes this term decay faster, thus making its contribution to (3) finite, uniformly in $|\mathbf{x}-\mathbf{y}|$. Similar considerations apply to higher-order cumulants, and (5) follows as a corollary. ∎

In conclusion, we presented a rigorous microscopic derivation of massless Gaussian free field behavior of the height field of a non-integrable interacting dimer model. Our method combines constructive field theory techniques with discrete holomorphicity ideas, which are used for the first time in a unified way to analyze a *non-local* fermionic observable. The method can be applied to several other non-integrable 2D critical or quasi-critical theories and we expect it to be capable, in perspective, of rigorously proving conformal invariance of the scaling limit.

The hypothesis of bipartite lattice, which we used extensively in this work (the very definition of height function requires the lattice to be bipartite), is not necessary for the applicability of our method. Rather, the required ingredient is that the model can be formulated as the perturbation of a Gaussian Grassmann integral, a feature valid much more in general than for dimers on the square lattice: models like the 2D Ising model with next-to-nearest-neighbor interactions on the square, hexagonal or triangular lattices, or interacting dimers on the hexagonal or triangular lattices fall all in the category of models treatable by our method. There are several important open questions still to be investigated in these contexts, for which the methods we introduced may play an important role. Examples include: the study of the crossover from the (anomalous) gapped RVB phase to the liquid phase in interacting dimer models on the triangular lattice as the diagonal hopping is sent to zero [5]; the computation of other non-local fermionic observables such as the spin-spin correlation in non-integrable 2D Ising models, or the monomer and vison [2] correlations in interacting dimer models. For these, the use of DH in a stronger form will be needed, in the spirit of [16,19].

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