

ON TOTALLY GEODESIC SUBMANIFOLDS IN THE JACOBIAN LOCUS

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ABSTRACT. We study submanifolds of A_g that are totally geodesic for the locally symmetric metric and which are contained in the closure of the Jacobian locus but not in its boundary. In the first section we recall a formula for the second fundamental form of the period map $M_g \hookrightarrow A_g$ due to Pirola, Tortora and the first author. We show that this result can be stated quite neatly using a line bundle over the product of the curve with itself. We give an upper bound for the dimension of a germ of a totally geodesic submanifold passing through $[C] \in M_g$ in terms of the gonality of C . This yields an upper bound for the dimension of a germ of a totally geodesic submanifold contained in the Jacobian locus, which only depends on the genus. We also study the submanifolds of A_g obtained from cyclic covers of \mathbb{P}^1 . These have been studied by various authors. Moonen determined which of them are Shimura varieties using deep results in positive characteristic. Using our methods we show that many of the submanifolds which are not Shimura varieties are not even totally geodesic.

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1. INTRODUCTION

1.1. Denote by A_g the moduli space of principally polarized abelian varieties of dimension g , by M_g the moduli space of smooth curves of genus g and by $j : M_g \rightarrow A_g$ the period mapping or Torelli mapping. Both M_g and A_g are complex orbifolds (or smooth stacks) and A_g is endowed with a locally symmetric metric, the so-called Siegel metric. One expects the Jacobian locus, that is the image $j(M_g) \subset A_g$, to be rather curved with respect to the Siegel metric. In particular, it should contain very few totally geodesic submanifolds of A_g . Another reason for this expectation comes from arithmetic geometry. Indeed for a special class of totally geodesic submanifolds (Shimura varieties) it has been conjectured by Coleman and Oort that for large genus no positive dimensional Shimura variety is contained in the closure of the Jacobian locus (in A_g) and meets the Jacobian locus itself. Moonen [20] has proven that an algebraic totally geodesic submanifold is a Shimura subvariety if and only if it contains a complex multiplication point. For results on Shimura subvarieties contained in $\overline{j(M_g)}$ we refer to [13, 20, 22, 25, 21, 1, 19, 15, 18].

Outside the hyperelliptic locus the period map is an orbifold immersion. For $g \geq 4$ the Jacobian locus $j(M_g)$ has dimension strictly smaller than A_g . Therefore it makes sense to compute the second fundamental form of $j(M_g) \subset A_g$ and to study its metric properties by infinitesimal methods. The second fundamental form has been studied by Pirola, Tortora and the first author [6], where an expression for it is given and it is proven that the second fundamental form lifts the second Gaussian map, as stated in an unpublished paper by Green and Griffiths [12]. In particular the computation of the second fundamental form on $\xi_p \odot \xi_p$ (where ξ_p is a Schiffer variation at the point p on the curve) reduces to the evaluation of the second gaussian map at the point p . These results have been used in [5] to compute the curvature of the restriction to M_g of the Siegel metric. In [5] there is an explicit formula for the holomorphic sectional curvature of M_g in the direction ξ_p in terms of the holomorphic sectional curvature of A_g and the second Gaussian map.

It is much harder to use the formula in [6] to compute the second fundamental form on $\xi_p \odot \xi_q$ when $p \neq q$. In fact the formula contains the evaluation at q of a meromorphic 1-form on the curve, called η_p , which has a double pole at p and is defined by Hodge theory. In general it seems rather hard to control the behaviour of η_p , in a way to get constraints on the second fundamental form.

1.2. In this paper we give a global and more intrinsic description of this form. We show that as p varies on the curve the forms η_p glue to give a holomorphic section $\hat{\eta}$ of the line bundle $K_S(2\Delta)$, where $S = C \times C$ and $\Delta \subset S$ is the diagonal. With this interpretation we are able to prove that the second fundamental form coincides with the multiplication by $\hat{\eta}$.

More precisely, fix a genus g , which will always be assumed greater than 3, and fix $[C] \in M_g$ outside the hyperelliptic locus. The conormal bundle

of $j : M_g \subset A_g$ at $[C]$ can be identified with $I_2(K_C)$, which is the kernel of the multiplication map $S^2H^0(C, K_C) \rightarrow H^0(C, 2K_C)$. Hence the second fundamental form can be seen as a map $\rho : I_2(K_C) \rightarrow S^2H^0(C, 2K_C)$. Let $S = C \times C$ and let Δ be the diagonal. By Künneth formula $H^0(S, K_S) = H^0(C, K_C) \otimes H^0(C, K_C)$ and $H^0(S, 2K_S) = H^0(C, 2K_C) \otimes H^0(C, 2K_C)$. In particular $I_2(K_C) \subset H^0(S, K_S(-2\Delta))$.

Theorem A. (See Theorem 3.13). *The second fundamental form ρ is the restriction to I_2 of the multiplication map*

$$H^0(S, K_S(-2\Delta)) \longrightarrow H^0(S, 2K_S) \quad Q \mapsto Q \cdot \hat{\eta}.$$

1.3. Based on these results on the second fundamental form we get some constraints on the existence of totally geodesic submanifolds of A_g contained in M_g . Since our methods are local in nature, the results apply to germs of such submanifolds. We get upper bounds for the dimension of totally geodesic germs passing through $[C] \in M_g$ in terms of the gonality of the curve C .

Theorem B. (See Theorem 4.3). *Assume that C is a k -gonal curve of genus g with $g \geq 4$ and $k \geq 3$. Let Y be a germ of a totally geodesic submanifold of A_g which is contained in the jacobian locus and passes through $j([C]) = [J(C)]$. Then $\dim(Y) \leq 2g + k - 4$.*

This immediately yields a bound which only depends on g .

Theorem C. (See Theorem 4.5). *If $g \geq 4$ and Y is a germ of a totally geodesic submanifold of A_g contained in the jacobian locus, then $\dim Y \leq \frac{5}{2}(g - 1)$.*

1.4. For low genus one can construct examples of totally geodesic submanifolds contained in M_g using cyclic covers of \mathbb{P}^1 , see e.g. [7, 21, 25]. These are in fact Shimura varieties. A complete list of the Shimura varieties that can be obtained in this way has been given in [21] using deep results in positive characteristic. With our methods we check directly that these examples are indeed totally geodesic and we show that a large class of cyclic covers, which are not in the list of Shimura varieties, are not even totally geodesic (see Proposition 5.7 and Corollary 5.8).

1.5. Other works studying totally geodesic submanifolds contained in the Jacobian locus include [26, 13, 8]. In particular Hain [13] proves the following. Let X be an irreducible symmetric domain and consider the locally symmetric variety $\Gamma \backslash X$ (where Γ is a lattice). If there is a totally geodesic immersion $\Gamma \backslash X \rightarrow j(M_g)$, and if some additional conditions are satisfied, then X must be the complex ball. De Jong and Zhang [8] prove a similar result under milder conditions, but still retaining the irreducibility assumption on X . The techniques used in these works are global and are based on group cohomology and on a rigidity theorem for the mapping class group due to Farb and Masur [9]. Our result instead applies to germs of totally

geodesic submanifolds, since it is local in nature and does not require irreducibility assumptions. The same local point of view is present in [17], where the object of study are totally geodesic submanifolds contained in an algebraic subvariety of a complex hyperbolic space form.

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2. NOTATION AND PRELIMINARY RESULTS

2.1. Second fundamental form. Denote by A_g the moduli space of principally polarized abelian varieties of dimension g , by M_g the moduli space of smooth curves of genus g and by $j : M_g \rightarrow A_g$ the period mapping or Torelli mapping. By the Torelli theorem j is injective. To study j one can fix a level structure with $n \geq 3$ and consider $M_g^{(n)} \xrightarrow{j^{(n)}} A_g^{(n)}$ which is a smooth map between manifolds. Since level structures play no role in what we are doing, it is more appropriate to think of M_g and A_g as complex orbifolds or smooth stacks, see e.g. [2, XII, 4]. The period map is smooth in the orbifold sense. Moreover its restriction to the set of non-hyperelliptic curves is an orbifold immersion [23]. By abuse of terminology we will henceforth omit the word "orbifold". The moduli space A_g is endowed with the Siegel metric, which is the metric induced on A_g from the symmetric metric on the Siegel upper halfspace \mathfrak{H}_g . Outside the hyperelliptic locus we have the sequence of tangent bundles:

$$0 \rightarrow TM_g \rightarrow j^*(TA_g) \xrightarrow{\pi} N \rightarrow 0,$$

whose dual, at $[C] \in M_g$ is

$$0 \rightarrow I_2 \rightarrow S^2 H^0(C, K_C) \xrightarrow{m} H^0(C, 2K_C) \rightarrow 0,$$

where $I_2 := I_2(K_C)$ is the set of quadrics containing the canonical curve and m is the multiplication map (see [5] for more details). Denote by

$$\text{II} : S^2 T_{[C]} M_g = S^2 H^1(C, T_C) \rightarrow N_{[C]}$$

the second fundamental form of the period map with respect to the Siegel metric on A_g . Denote by

$$\rho : I_2 \rightarrow S^2 H^0(C, 2K_C)$$

the dual of II. We will refer both to II and to ρ as second fundamental forms.

2.2. Schiffer variations. If C is a curve and $x \in C$, the coboundary of the exact sequence $0 \rightarrow T_C \rightarrow T_C(x) \rightarrow T_C(x)|_x \rightarrow 0$ yields an injection $H^0(T_C(x)|_x) \cong \mathbb{C} \hookrightarrow H^1(C, T_C)$. Elements in the image are called *Schiffer*

variations at x . If (U, z) is a chart centred at x and $b \in C_0^\infty(U)$ is a bump function which is equal to 1 on a neighbourhood of x , then

$$\theta := \frac{\bar{\partial}b}{z} \cdot \frac{\partial}{\partial z}$$

is a Dolbeault representative of a Schiffer variation at x . The map

$$\xi : TC \rightarrow H^1(C, T_C) \quad u = \lambda \frac{\partial}{\partial z}(x) \mapsto \xi_u := \lambda^2[\theta]$$

does not depend on the choice of the coordinates. It is well known that Schiffer variations generate $H^1(C, T_C)$ [2, p.175].

Lemma 2.3. *Let $\beta \in H^0(C, 2K_C)$ and let (U, z) be a chart centred at $x \in C$. If $\beta = f(z)(dz)^2$ on U , then $\beta(\xi_{\frac{\partial}{\partial z}(x)}) = 2\pi i f(0)$.*

Proof.

$$\beta(\xi_{\frac{\partial}{\partial z}(x)}) = \int_C \beta \cup \theta = \int_U f(z) dz \wedge \frac{\bar{\partial}b}{z} = - \int_{U-\{x\}} \bar{\partial} \left(\frac{b(z)f(z)}{z} dz \right).$$

If $\varepsilon > 0$ is small enough, $b \equiv 1$ on $\{|z| \leq \varepsilon\}$. Using Stokes and Cauchy theorems we get

$$\beta(\xi_{\frac{\partial}{\partial z}(x)}) = - \lim_{\varepsilon \rightarrow 0} \int_{U \cap \{|z| > \varepsilon\}} \bar{\partial} \left(\frac{b(z)f(z)}{z} dz \right) = \lim_{\varepsilon \rightarrow 0} \int_{|z|=\varepsilon} \frac{f(z)}{z} dz = 2\pi i f(0).$$

□

2.4. Gaussian maps. We briefly recall the definition of Gaussian maps for curves. Let N and M be line bundles on C . Set $S := C \times C$ and let $\Delta \subset S$ be the diagonal. For a non-negative integer k the k -th *Gaussian* or *Wahl map* associated to these data is the map given by restriction to the diagonal

$$H^0(S, N \boxtimes M(-k\Delta)) \xrightarrow{\mu_{N,M}^k} H^0(S, N \boxtimes M(-k\Delta)|_\Delta) \cong H^0(C, N \otimes M \otimes K_C^k).$$

We are only interested in the case $N = M$. In this case we set $\mu_{k,M} := \mu_{M,M}^k$. With the identification $H^0(S, N \boxtimes M) \cong H^0(C, N) \otimes H^0(C, M)$ the map $\mu_{0,M}$ is the multiplication map of global sections

$$H^0(C, M) \otimes H^0(C, M) \rightarrow H^0(C, M^2),$$

which obviously vanishes identically on $\wedge^2 H^0(C, M)$. Consequently $\ker \mu_{0,M} = H^0(S, M \boxtimes M(-\Delta))$ decomposes as $\wedge^2 H^0(C, M) \oplus I_2(M)$, where $I_2(M)$ is the kernel of $S^2 H^0(C, M) \rightarrow H^0(C, M^2)$. Since $\mu_{1,M}$ vanishes on symmetric tensors, one usually writes

$$\mu_{1,M} : \wedge^2 H^0(C, M) \rightarrow H^0(C, M^2).$$

If σ is a local frame for M and z is a local coordinate, given sections $s_1, s_2 \in H^0(C, M)$ with $s_i = f_i(z)\sigma$, we have

$$(2.1) \quad \mu_{1,M}(s_1 \wedge s_2) = (f_1' f_2 - f_2' f_1) dz \otimes \sigma^2.$$

Consequently, the zero divisor of $\mu_{1,M}(s_1 \wedge s_2)$ is twice the base locus of the pencil $\langle s_1, s_2 \rangle$ plus the ramification divisor of the associated morphism.

Again $H^0(S, M \boxtimes M(-2\Delta))$ decomposes as the sum of $I_2(M)$ and the kernel of $\mu_{1,M}$. Since $\mu_{2,M}$ vanishes identically on skew-symmetric tensors, one usually writes

$$\mu_{2,M} : I_2(M) \rightarrow H^0(C, M^2 \otimes K_C^2).$$

By μ_2 we denote the second gaussian map of the canonical line bundle K_C on C :

$$\mu_2 := \mu_{2,K_C} : I_2(K_C) \rightarrow H^0(K_C^4).$$

2.5. The form η_x and the second fundamental form. We now recall the definition of η_x . Let C be a smooth complex projective curve of genus $g \geq 4$. Fix a point $x \in C$. The space $H^0(C, K_C(2x))$ is contained in the space of closed 1-forms on $C - \{x\}$. The induced map $H^0(C, K_C(2x)) \rightarrow H^1(C - \{x\}, \mathbb{C})$ is injective as soon as $g > 0$. By the Mayer-Vietoris sequence, the inclusion $C - \{x\} \hookrightarrow C$ induces an isomorphism $H^1(C, \mathbb{C}) \cong H^1(C - \{x\}, \mathbb{C})$. Thus we get an injection

$$(2.2) \quad j_x : H^0(C, K_C(2x)) \hookrightarrow H^1(C, \mathbb{C}).$$

$H^{1,0}(C)$ is contained in the image of j_x and $h^0(C, K_C(2x)) = g + 1$, so $j_x^{-1}(H^{0,1}(C))$ is a line. If (U, z) is a chart centred at x , there is a unique element φ in this line such that on $U - \{x\}$

$$\varphi = \left(\frac{1}{z^2} + h(z) \right) dz$$

with $h \in \mathcal{O}_C(U)$. (Applying Stokes theorem on C minus a disc around x shows that the residue at x vanishes, so there is no term in $1/z$.) Define a linear map

$$\eta_x : T_x C \rightarrow H^0(C, K_C(2x))$$

by the rule

$$u = \lambda \frac{\partial}{\partial z}(x) \mapsto \eta_x(u) := \lambda \varphi.$$

We will often drop x and simply write η_u for $\eta_x(u)$. An easy computation shows that η_x does not depend on the choice of the local coordinate.

Theorem 2.6 ([6, Thm. 3.1], [5, Lemma 3.5]). *Let C be a non-hyperelliptic curve of genus $g \geq 4$. Given points $x \neq y$ in C and tangent vectors $u \in T_x C$ and $v \in T_y C$ we have*

$$\begin{aligned} \rho(Q)(\xi_u \odot \xi_v) &= -4\pi i \cdot \eta_x(u)(v)Q(u, v), \\ \rho(Q)(\xi_u \otimes \xi_u) &= -2\pi i \cdot \mu_2(Q)(u^{\otimes 4}). \end{aligned}$$

This theorem is basic to the whole paper. For the reader's convenience we recall its proof.

Proof. First of all we take $w \in H^1(C, T_C)$ and we compute $\rho(Q)(w) \in H^0(C, 2K_C)$ for $Q \in I_2(K_C)$. We can assume that there is a one dimensional deformation $\mathcal{X} \xrightarrow{f} \Delta$ with $C = f^{-1}(0)$ and $w = \kappa(\frac{\partial}{\partial t})$, where $\Delta = \{|t| < 1\}$ and κ is the Kodaira-Spencer map. Take a C^∞ lifting Y of the holomorphic vector field $\frac{\partial}{\partial t}$. So we have a C^∞ trivialization $\tau : \Delta \times C \rightarrow \mathcal{X}$, $\tau(t, x) = \tau_t(x) := \Phi_t(x)$, where $\{\Phi_t\}$ is the flow of the vector field Y . Then $\theta := \bar{\partial}Y|_C$ is a $\bar{\partial}$ -closed form in $A^{0,1}(C, T_C)$ such that $[\theta] = w \in H^1(C, T_C)$. Denote by C_t the fibre of f over t and let $\omega(t)$ be a section of the Hodge bundle, i.e. $\forall t \in \Delta, \omega(t) \in H^0(K_{C_t})$. Since $\omega(t)$ is closed, also $\tau_t^*(\omega(t))$ is closed, so $\tau_t^*(\omega(t)) = \omega + (\alpha + dh)t + o(t)$, where $\omega := \omega(0)$, α is harmonic and h is a C^∞ function. Denote by ∇^{GM} the flat Gauss-Manin connection on $R^1 f_* \mathbb{C}$. So we have $\nabla_{\frac{\partial}{\partial t}}^{GM}[\omega(t)]|_{t=0} = [\alpha]$. By Griffiths' results (see e.g. [29, pp. 234ff]) $\theta \cdot \omega = \alpha^{0,1} + \bar{\partial}h$ and $\kappa(\frac{\partial}{\partial t}) \cdot [\omega] = [\alpha^{0,1}]$, where $\alpha^{0,1}$ is the $(0, 1)$ component of α . Now assume that $\{\omega_i\}_{i=1, \dots, g}$ is a basis of $H^0(K_C)$. Take a quadric $Q = \sum_{i,j} a_{ij} \omega_i \otimes \omega_j \in I_2(K_C)$. Denote by $\nabla^{1,0}$ the Gauss-Manin connection on the Hodge bundle $f_* \omega_{\mathcal{X}/\Delta}$, i.e. $\nabla^{1,0} = \pi \nabla^{GM}$, where π is the projection of $H^1(C_t, \mathbb{C})$ onto $H^0(C_t, K_{C_t})$. Then for all i we have $\nabla_{\frac{\partial}{\partial t}}^{1,0}[\omega_i(t)]|_{t=0} = [\alpha_i^{1,0}]$. Denote by ∇ the induced connection on $S^2 f_* \omega_{\mathcal{X}/\Delta}$. If $\tilde{Q}(t) = \sum_{i,j} a_{ij}(t) \omega_i(t) \otimes \omega_j(t) \in I_2(K_{C_t})$ is a section of the conormal bundle such that $\tilde{Q}(0) = Q$, then

$$\rho(Q)(w) = m(\nabla_{\frac{\partial}{\partial t}} \tilde{Q}|_{t=0}) = \sum_{i,j} a'_{ij}(0) \omega_i \omega_j + 2 \sum_{i,j} a_{ij} \alpha_i^{1,0} \omega_j.$$

Since $\sum_{i,j} a_{ij}(t) \omega_i(t) \omega_j(t) \equiv 0$, also its derivative at $t = 0$ vanishes, i.e. $2 \sum_{i,j} a_{ij}(\alpha_i + dh_i) \omega_j + \sum_{i,j} a'_{ij}(0) \omega_i \omega_j \equiv 0$, and if we take the $(1, 0)$ part we have $2 \sum_{i,j} a_{ij}(\alpha_i^{1,0} + \bar{\partial}h_i) \omega_j + \sum_{i,j} a'_{ij} \omega_i \omega_j \equiv 0$, so

$$(2.3) \quad \rho(Q)(w) = -2 \sum_{i,j} a_{ij} \omega_j \bar{\partial}h_i.$$

This is an instance of a Hodge-Gaussian map, see [6] and also [24, §4].

Now fix a point $x \in C$ and a chart (U, z) centred at x . Let w be the Schiffer variation $\xi_{\frac{\partial}{\partial z}(x)}$ at $x \in C$ with Dolbeault representative $\theta := \frac{\bar{\partial}b}{z} \cdot \frac{\partial}{\partial z}$, where $b \in C_0^\infty(U)$ be a bump function equal to 1 on a neighbourhood of x . Let $\omega_i = f_i(z) dz$ be the local expression of ω_i in U . On $C - \{x\}$ we have $\theta \cdot \omega_i = \frac{f_i}{z} \bar{\partial}b$, so $\alpha_i^{0,1} + \bar{\partial}h_i = \bar{\partial} \left(\frac{b f_i}{z} \right)$. Set $g_i := \frac{b f_i}{z} - h_i$. Then $\alpha_i^{0,1} = \bar{\partial}g_i$. Define $\eta_i := \partial g_i$.

Now we will show that $\sum a_{ij} \omega_i \bar{\partial}h_j = - \sum a_{ij} \omega_i \eta_j$. In the first place, note that η_i is holomorphic in $C - \{x\}$. Indeed, α_i is harmonic, thus $\bar{\partial}\eta_i = \bar{\partial}\bar{\partial}g_i = -\partial\bar{\partial}g_i = -\partial\alpha_i^{0,1} = 0$. Hence, $\sum a_{ij} \omega_i \partial \left(\frac{b f_i}{z} \right) = \sum a_{ij} \omega_i \bar{\partial}h_j + \sum a_{ij} \omega_i \eta_j$ is holomorphic on $C - \{x\}$, because the first term is a holomorphic section of $2K_C$ by (2.3) and the second is holomorphic on $C - \{x\}$ since η_j is holomorphic. In a neighborhood of x where $b \equiv 1$, this expression has the

form

$$\sum a_{ij} f_i \frac{\partial}{\partial z} \left(\frac{f_j}{z} \right) dz^2 = \sum a_{ij} f_i \left(-\frac{f_j}{z^2} + \frac{f_j'}{z} \right) dz^2 = 0,$$

because $\sum a_{ij} f_i f_j = \sum a_{ij} f_i f_j' = 0$. So $\sum a_{ij} \omega_i \partial \left(\frac{bf_j}{z} \right)$ is identically zero. Thus we have

$$\rho(Q)(\xi_{\frac{\partial}{\partial z}(x)}) = -2 \sum a_{ij} \omega_i \partial h_j = 2 \sum a_{ij} \omega_i \eta_j.$$

Now we claim that $\eta_i = -f_i(x) \eta_x \left(\frac{\partial}{\partial z} \right)$. In fact $\eta_i \in H^0(K_C(2x))$, since on U , where $b \equiv 1$, η_i has the form

$$\eta_i = \left(-\frac{f_i(x)}{z^2} + \psi_i(z) \right) dz$$

with $\psi_i(z)$ a holomorphic function, hence η_i is a meromorphic form, with a double pole at x . By definition, $\eta_i + \alpha_i^{0,1} = \partial g_i + \bar{\partial} g_i = dg_i$, so

$$j_x(\eta_i) = -[\alpha_i^{0,1}] \in j_x(H^0(K_C(2x))) \cap H^{0,1}(C).$$

This proves the claim.

We can assume that the chart (U, z) contains also y . Applying Lemma 2.3 the first statement immediately follows for $u = \partial/\partial z(x)$ and $v = \partial/\partial z(y)$. It is clear that this is enough. For the second statement it suffices to use the local expression of μ_2 . \square

Remark 2.7. We remark that Schiffer variations, the forms η_x , Q and $\mu_2(Q)$ are sections of vector bundles, but they become functions as soon as a coordinate chart is fixed. Because of this many statements, like the one above, are usually stated for simplicity, as if these sections were evaluated at points instead of tangent vectors. We will follow this notation when it is convenient. In the next section instead it is better to stick a more formal notation.

3. THE SECOND FUNDAMENTAL FORM AS A MULTIPLICATION MAP

3.1. In this section we show that as x varies on the curve the form η_x varies holomorphically in an appropriate sense and gives rise to a section of a vector bundle on C and to a corresponding section $\hat{\eta}$ of the line bundle $K_S(2\Delta)$ on $S = C \times C$. The two main points are Theorem 3.13 and the invariance of $\hat{\eta}$ with respect to the action of $\text{Aut}(C)$.

3.2. Let (U, z) be a chart centred at $x \in C$. Set $u = \frac{\partial}{\partial z}(x)$. It is a classical result that there is a harmonic function $f_u \in C^\infty(C - \{x\})$ such that $f_u = -1/z + g(z)$ on $U - \{x\}$ for some $g \in C^\infty(U)$. This function is unique up to an additive constant and is called *elementary potential*. Its existence can be proven for example using the (real) Hodge decomposition theorem and the Weyl lemma (see e.g. [10, p. 46-48]) or using the Perron method (see e.g. [16, p. 213ff]).

Lemma 3.3. *If f_u is an elementary potential, then $\partial f_u = \eta_u$, $\bar{\partial} f_u$ is smooth on C and $j_x(\eta_u) = [-\bar{\partial} f_u]$.*

Proof. The $(1, 0)$ -form ∂f_u is holomorphic on $C - \{x\}$ since f_u is harmonic. Moreover $\partial f_u = z^{-2}dz + \partial g$ on $U - \{x\}$. The form ∂g is holomorphic on $U - \{x\}$, but also smooth on U . Hence it is holomorphic on U . This shows that $\partial f_u = \eta_u + \omega$ for some $\omega \in H^0(K_C)$. Set $\alpha := -\bar{\partial} f_u$. Then α is smooth on C by the definition of f_u . It is closed, since f_u is harmonic and of type $(0, 1)$. On $C - \{x\}$ we have $\partial f_u - \alpha = df_u$, so $[\partial f_u] = [\alpha]$ in $H^1(C - \{x\})$. Therefore $j_x(\partial f_u) = [\alpha]$. Since $[\alpha] \in H^{0,1}(C)$, this shows that $j_x(\partial f_u) \in H^{0,1}(C)$. Therefore $\omega = 0$ and $\eta_u = \partial f_u$. \square

Remark 3.4. Notice that one could prove the existence of f_u using η_u and the fact that $C - \{x\}$ is Stein.

Lemma 3.5. *If $H^{0,1}(C)$ is identified with $H^0(C, K_C)^*$ using Serre duality, then $x \mapsto \text{Im } j_x \cap H^{0,1}(C) \in \mathbb{P}(H^{0,1}(C))$ coincides with the canonical map.*

Proof. Let (U, z) be a coordinate centred at x . Set $u = \frac{\partial}{\partial z}(x)$. If $\omega \in H^0(C, K_C)$, let $\omega = \varphi(z)dz$ be its local expression on U . By Lemma 3.3 $j_x(\eta_u) = -[\bar{\partial} f_u]$. Therefore

$$(3.1) \quad \begin{aligned} \int_C \omega \wedge j_x(\eta_u) &= - \int_C \omega \wedge \bar{\partial} f_u = \int_{C-\{x\}} d(f_u \omega) = \\ &= \lim_{\varepsilon \rightarrow 0} \int_{|z|=\varepsilon} \varphi(z) \left(-\frac{1}{z} + g(z) \right) dz = -\varphi(0) = -\omega(u). \end{aligned}$$

\square

3.6. Let $S := C \times C$, let Δ denote the diagonal and let $p, q : S \rightarrow C$ be the projections $p(x, y) = x$, $q(x, y) = y$. Then $K_S = p^*K_C \otimes q^*K_C$. Consider the line bundle $L := K_S(2\Delta)$ on S and set

$$V := p_*(q^*K_C(2\Delta)) \quad E := p_*L.$$

By the projection formula $E = K_C \otimes V$. Since $q^*K_C(2\Delta)|_{\{x\} \times C} = q^*K_C(2x)$, we have $H^0(p^{-1}(x), q^*K_C(2\Delta)) \cong H^0(C, K_C(2x))$. By Grauert semicontinuity theorem V is a holomorphic vector bundle on C with fibre $V_x \cong H^0(C, K_C(2x))$ and the map $x \mapsto \eta_x$ is a section of E . We call this section η .

Proposition 3.7. *η is a holomorphic section of E .*

Proof. Let $W \rightarrow C$ denote the trivial vector bundle with fibre $H^1(C, \mathbb{C})$. We claim that the injection $j : V \hookrightarrow W$ defined in (2.2) is holomorphic. Fix $\alpha \in H^0(C, K_C(2x))$ and a smooth singular 1-cycle c on C . If x does not lie in the support of c , the integral $\int_c \alpha$ is well-defined. It does not change if α is replaced by $\alpha + df$ with $f \in C^\infty(C - \{x\})$. Therefore $\int_c \alpha = \langle [c], j(\alpha) \rangle$. Fix $x_0 \in C$ and choose smooth 1-cycles c_1, \dots, c_{2g} , that do not touch x_0 and whose classes form a basis of $H_1(C, \mathbb{C})$. Let A be an open subset of C , such

that V is trivial over A and \bar{A} does not intersect the supports of the cycles c_i . Fix $s \in H^0(A, V)$. To show that $j(s)$ is a holomorphic section of W over A , it is enough to prove that the functions

$$x \mapsto \langle [c_i], j_x(s(x)) \rangle = \int_{c_i} s(x)$$

are holomorphic on A . Since $V = p_*q^*K_C(2\Delta)$, s corresponds to a section $s \in H^0(A \times C, q^*K_C(2\Delta))$. So $s(x, \cdot)|_{C-A}$ is a holomorphic 1-form on $C - A$ depending holomorphically on the parameter $x \in A$ and its integral over the 1-cycle c_i (which is contained in $C - A$) is a holomorphic function of x . Therefore $j(s)$ is holomorphic. Since s is arbitrary, this proves that j is holomorphic, as claimed. Next fix a chart (U, z) on C . To show that η is holomorphic on U it is enough to prove that $\eta(\frac{\partial}{\partial z})$ is holomorphic or - by the above - that $j(\eta(\frac{\partial}{\partial z}))$ is a holomorphic function $U \rightarrow H^1(C, \mathbb{C})$. Fix a basis $\{\omega_1, \dots, \omega_g\}$ of $H^0(C, K_C)$. The functionals $\int_C(\cdot) \wedge \omega_j$ and $\int_C(\cdot) \wedge \bar{\omega}_j$ form a basis of $H^1(C, \mathbb{C})^*$. Since $j(\eta(\frac{\partial}{\partial z})) \in H^{0,1}(C)$ the latter g functionals vanish on $j(\eta(\frac{\partial}{\partial z}))$. Assume that $\omega_j(z) = f_j(z)dz$ on U . By (3.1)

$$\int_C j\left(\eta\left(\frac{\partial}{\partial z}\right)\right) \wedge \omega_j = f_j(z).$$

This proves that $j(\eta(\frac{\partial}{\partial z}))$ is holomorphic. \square

3.8. Since $E = p_*L$ there is an isomorphism $H^0(C, E) \cong H^0(S, L)$ that associates to $\alpha \in H^0(C, E)$ the section $\hat{\alpha} \in H^0(S, L)$ such that

$$\alpha_x = \hat{\alpha}|_{\{x\} \times C} \in T_x^*C \otimes H^0(C, K_C(2x)) = E_x.$$

It follows that $\alpha_x(u)(v) = \hat{\alpha}((u, 0), (0, v))$ for $x \neq y$, $u \in T_xC$ and $v \in T_yC$.

Thus there is a well-defined section $\hat{\eta} \in H^0(S, L)$ corresponding to η and for $u \in T_xC$ and $v \in T_yC$ with $x \neq y$, we have $\eta_x(u)(v) = \hat{\eta}(u, v)$.

Lemma 3.9. *The form $\hat{\eta}$ is symmetric, i.e. $\hat{\eta}((u, 0), (0, v)) = \hat{\eta}((v, 0), (0, u))$.*

Proof. Fix points $x \neq y$ and tangent vectors $u \in T_xC, y \in T_yC$. Using the identity $d(f_u(\bar{\partial}f_v - \partial f_v) - f_v(\bar{\partial}f_u - \partial f_u)) = 2(f_u\partial\bar{\partial}f_v - f_v\partial\bar{\partial}f_u) = 0$ and applying Stokes theorem on $C - \{|z| < \varepsilon\} \cup \{|w| < \varepsilon\}$ we get

$$\begin{aligned} 0 &= \int_{|z|=\varepsilon} (f_u(\bar{\partial}f_v - \partial f_v) - f_v(\bar{\partial}f_u - \partial f_u)) + \\ &+ \int_{|w|=\varepsilon} (f_u(\bar{\partial}f_v - \partial f_v) - f_v(\bar{\partial}f_u - \partial f_u)) \end{aligned}$$

(This is just Green formula.) Let us denote by A_ε and B_ε these two integrals. Observe that $f_v\partial f_u = \partial(f_u f_v) - f_u\partial f_v$ and $f_u\bar{\partial}f_v = \bar{\partial}(f_u f_v) - f_v\bar{\partial}f_u$. Therefore

$$A_\varepsilon = \int_{|z|=\varepsilon} (d(f_u f_v) - 2f_v\bar{\partial}f_u - 2f_u\partial f_v) = -2 \int_{|z|=\varepsilon} (f_v\bar{\partial}f_u + f_u\partial f_v).$$

Since $f_v \bar{\partial} f_u$ is a smooth form near x , we have $\lim_{\varepsilon \rightarrow 0} \int_{|z|=\varepsilon} f_v \bar{\partial} f_u = 0$. Choose a coordinate (U, z) centred at x such that $u = \partial/\partial z(x)$ and let $g(z)$ be as in 3.2. Near x we have $\eta_y(v) = \partial f_v = h(z)dz$ for some holomorphic function h . So

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} A_\varepsilon &= -2 \lim_{\varepsilon \rightarrow 0} \int_{|z|=\varepsilon} f_u \partial f_v = \\ &= -2 \lim_{\varepsilon \rightarrow 0} \int_{|z|=\varepsilon} \left(-\frac{1}{z} + g(z) \right) h(z) dz = 4\pi i h(0) = 4\pi i \eta_y(v)(u). \end{aligned}$$

By the corresponding computation, $\lim_{\varepsilon \rightarrow 0} B_\varepsilon = -4\pi i \eta_x(u)(v)$, so $\eta_y(v)(u) = \eta_x(u)(v)$ as desired. \square

3.10. $\text{Aut}(C)$ acts diagonally on $C \times C$ and preserves Δ . Therefore the action lifts to K_S and also to $L = K_S(2\Delta)$. This yields a representation of $\text{Aut}(C)$ on $H^0(S, L)$. On the other hand, if $\sigma \in \text{Aut}(C)$, then $(\sigma^{-1})^*$ is a map from $H^0(C, K_C(2x)) = V_x$ to $H^0(C, K_C(2\sigma(x))) = V_{\sigma(x)}$. Tensoring it with $(d\sigma^{-1})^* : T_x^* C \rightarrow T_{\sigma(x)}^* C$ we get a map $T_x^* C \otimes V_x \rightarrow T_{\sigma(x)}^* C \otimes V_{\sigma(x)}$. This yields an action of $\text{Aut}(C)$ on the total space of $E = K_C \otimes V$ which covers the action of $\text{Aut}(C)$ on C . This means that E is an equivariant bundle. In this way we get a representation of $\text{Aut}(C)$ on $H^0(C, E)$. The map $\alpha \mapsto \hat{\alpha}$ considered in 3.8 is an isomorphism of $\text{Aut}(C)$ -representations.

Lemma 3.11. η and $\hat{\eta}$ are invariant with respect to the action of $\text{Aut}(C)$.

Proof. By the above it is enough to show that η is invariant. For $\tau \in \text{Aut}(C)$ we wish to prove that $(d\tau^{-1})^* \otimes (\tau^{-1})^*(\eta_y) = \eta_{\tau(y)}$ for any $y \in C$. For simplicity, set $\sigma := \tau^{-1}$, $x := \tau(y)$. So we wish to prove that $\sigma^*(\eta_y(d\sigma(u))) = \eta_x(u)$ for any $u \in T_x C$. By continuity it is enough to prove this for x such that $\sigma(x) \neq x$. Choose a coordinate patch (U, z) centred at x , such that $U \cap \sigma(U) \neq \emptyset$ and $u = \frac{\partial}{\partial z}|_x$. Then $(\sigma(U), w := z \circ \sigma^{-1})$ is a coordinate system centred at $\sigma(x)$ and $\frac{\partial}{\partial w}(\sigma(x)) = d\sigma(u)$. Assume that

$$\eta_x(u) = \left(\frac{1}{z^2} + h(z) \right) dz$$

on U . Then

$$\tau^* \eta_x(u) = \left(\frac{1}{w^2} + h(w) \right) dw$$

on $\sigma(U)$. Moreover $j_y \tau^* = \tau^* j_x$. So $j_y \tau^*(\eta_x(u)) = \tau^* j_x(\eta_x(u))$. Since $j_x(\eta_x(u)) \in H^{0,1}(C)$, also $j_y \tau^*(\eta_x(u)) \in H^{0,1}(C)$. Hence $\tau^* \eta_x(u) = \eta_y(d\sigma(u))$ as desired. \square

3.12. If we identify $H^0(C, K_C) \otimes H^0(C, K_C)$ with $H^0(S, K_S)$, then I_2 becomes a subspace of $H^0(S, K_S(-\Delta))$. Since elements of I_2 are symmetric, they are in fact in $H^0(S, K_S(-2\Delta))$. So if $Q \in I_2$ the product section $Q \cdot \hat{\eta}$ lies in $H^0(S, 2K_S) \cong H^0(C, 2K_C) \otimes H^0(C, 2K_C)$.

Theorem 3.13. *With the above identifications, if C is non-hyperelliptic and of genus $g \geq 4$, then $\rho : I_2 \rightarrow S^2H^0(C, 2K_C)$ is the restriction to I_2 of the multiplication map*

$$H^0(S, K_S(-2\Delta)) \longrightarrow H^0(S, 2K_S) \quad Q \mapsto Q \cdot \hat{\eta}.$$

Proof. The identification $H^0(C, 2K_C) \otimes H^0(C, 2K_C) \cong H^0(S, 2K_S)$ is compatible with Lemma 2.3, i.e. for $\alpha \in H^0(S, 2K_S)$, $u \in T_x C$ and $v \in T_y C$ we have $\langle \alpha, \xi_u \otimes \xi_v \rangle = \alpha((u, 0), (0, v))$. If $x \neq y$, by Theorem 2.6

$$\begin{aligned} \rho(Q)(\xi_u \odot \xi_v) &= Q(u, v)\eta_x(u)(v) = \\ &= (Q \cdot \hat{\eta})((u, 0), (0, v)) = (Q \cdot \hat{\eta})(\xi_u \odot \xi_v). \end{aligned}$$

(In the last identity we use the fact that both Q and $\hat{\eta}$ are symmetric). So $\rho(Q) - Q \cdot \hat{\eta}$ vanishes on tensors of the form $\xi_u \odot \xi_v$ with $x \neq y$. Clearly we can choose a basis of $S^2H^1(C, T_C)$ formed by such elements. \square

3.14. Since the second fundamental form is symmetric, using this theorem we get another proof of Lemma 3.9.

4. TOTALLY GEODESIC SUBMANIFOLDS AND GONALITY

4.1. In this section we will give an upper bound for the dimension of a germ of a totally geodesic submanifold contained in the Jacobian locus.

Theorem 4.2. *Assume that C is a k -gonal curve of genus g , with $g \geq 4$ and $k \geq 3$. Then there exists a quadric $Q \in I_2(K_C)$ such that $\text{rank } \rho(Q) > 2g - 2k$.*

Proof. Here we follow the simplified notation mentioned in 2.7. So we understand that a local coordinate has been fixed at the relevant points and we write ξ_P for a Schiffer variation at P . Let F be a line bundle on C such that $|F|$ is a g_k^1 and choose a basis $\{x, y\}$ of $H^0(F)$. Set $M = K_C - F$ and denote by B the base locus of $|M|$. By Clifford theorem $\deg(B) < k - 2$. Assume that $\langle t_1, t_2 \rangle$ is a pencil in $H^0(M)$, with base locus B . We can write $t_i = t'_i s$ for a section $s \in H^0(C, \mathcal{O}_C(B))$ with $\text{div}(s) = B$. Then $\langle t'_1, t'_2 \rangle$ is a base point free pencil in $|M - B|$. Let $\psi : C \rightarrow \mathbb{P}^1$ be the morphism induced by this pencil.

Now consider the rank 4 quadric $Q := xt_1 \odot yt_2 - xt_2 \odot yt_1$. Clearly $Q \in I_2(K_C)$. Set $d := \deg(M - B) = 2g - 2 - k - \deg(B)$. We need the following fact: if $\{P_1, \dots, P_d\}$ is a fibre of the morphism ψ over a regular value, then the Schiffer variations $\xi_{P_1}, \dots, \xi_{P_d}$ are linearly independent in $H^1(C, T_C)$. In fact, denote by $W := \langle \xi_{P_1}, \dots, \xi_{P_d} \rangle$. We want to show that the annihilator $\text{Ann}(W)$ of W has dimension $3g - 3 - d$. By lemma 2.3, $\text{Ann}(W) = \{\alpha \in H^0(C, 2K_C) \mid \alpha(P_i) = 0, i = 1, \dots, d\}$, hence by Riemann-Roch $\dim \text{Ann}(W) = h^0(2K_C - P_1 - \dots - P_d) = h^0(2K_C - M + B) = g - 1 + k + \deg(B) = 3g - 3 - d$. The claim is proven. Next denote by φ the morphism induced by the pencil $|F|$ and consider the set $E := \psi(\text{Crit}(\varphi) \cup \text{Crit}(\psi) \cup B)$ where $\text{Crit}(\varphi)$ (resp. $\text{Crit}(\psi)$) denote the set of critical points of φ (resp. ψ).

Let $z \in \mathbb{P}^1 - E$ and let $\{P_1, \dots, P_d\}$ be the fibre of ψ over z . By changing coordinates on \mathbb{P}^1 we can assume $z = [0, 1]$, i.e. $t'_1(P_i) = 0$ for $i = 1, \dots, d$. Then clearly $t_1(P_i) = 0$, so $Q(P_i, P_j) = 0$ for all i, j . Applying Theorem 2.6, one immediately obtains that the restriction of $\rho(Q)$ to the subspace W is represented in the basis $\{\xi_{P_1}, \dots, \xi_{P_d}\}$ by a diagonal matrix with entries $\pi i \cdot \mu_2(Q)(P_i)$ on the diagonal. For a rank 4 quadric the second Gaussian map can be computed as follows: $\mu_2(Q) = \mu_{1,F}(x \wedge y)\mu_{1,M}(t_1 \wedge t_2)$ see [4, Lemma 2.2]. Now $\mu_{1,F}(x \wedge y)(P_i) \neq 0$, because $P_i \notin \text{Crit}(\varphi)$ by the choice of z , see (2.1). Moreover $P_i \notin B$. On $C - B$ the morphism ψ coincides with the map associated to $\langle t_1, t_2 \rangle$. Since $P_i \notin \text{Crit}(\psi)$, it is not a critical point for the latter map. Therefore also $\mu_{1,M}(t_1 \wedge t_2)(P_i) \neq 0$ see (2.1). Thus $\mu_2(Q)(P_i) = \mu_{1,F}(x \wedge y)(P_i)\mu_{1,M}(t_1 \wedge t_2)(P_i) \neq 0$ for every $i = 1, \dots, d$. This shows that in the basis $\{\xi_{P_1}, \dots, \xi_{P_d}\}$ the quadric $\rho(Q)|_W$ is represented by a diagonal matrix with non-zero diagonal entries. So $\rho(Q)$ has rank at least $d = 2g - 2 - k - \deg B > 2g - 2k$. \square

Theorem 4.3. *Assume that C is a k -gonal curve of genus g with $g \geq 4$ and $k \geq 3$. Let Y be a germ of a totally geodesic submanifold of A_g which is contained in the jacobian locus and passes through $[C]$. Then $\dim Y \leq 2g + k - 4$.*

Proof. By Theorem 4.2 we know that there exists a quadric $Q \in I_2$ such that the rank of $\rho(Q)$ is at least $2g - 2k + 1$. By assumption for any $v \in T_{[C]}Y$ we must have that $\rho(Q)(v \odot v) = 0$, so v is isotropic for $\rho(Q)$, hence

$$\dim T_{[C]}Y \leq 3g - 3 - \frac{(2g - 2k + 1)}{2} = 2g + k - \frac{7}{2}.$$

\square

Remark 4.4. In Theorem 4.2 if $|M|$ is base point free – this happens in particular if C is generic in the locus of k -gonal curves – the above proof shows that $\text{rank } \rho(Q) \geq 2g - 2 - k$. So if Y is a germ of a totally geodesic submanifold of A_g contained in the jacobian locus and passing through a generic k -gonal curve, then $\dim Y \leq 2g - 2 + k/2$.

Theorem 4.5. *If $g \geq 4$ and Y is a germ of a totally geodesic submanifold of A_g contained in the jacobian locus, then $\dim Y \leq \frac{5}{2}(g - 1)$.*

Proof. This immediately follows from Theorem 4.3, since the gonality of a genus g curve is at most $\lfloor (g + 3)/2 \rfloor$. \square

Corollary 4.6. *For any $g \geq 4$ and $k \geq 2$ the k -gonal locus is not totally geodesic in A_g .*

Proof. For $k = 2$ this is Proposition 5.1 in [5]. If $k \geq 3$ the dimension of the k -gonal locus is $2g + 2k - 5 > 2g + k - 4$. Hence the statement follows immediately from Theorem 4.3. \square

Remark 4.7. As it is evident from the proof, gonality is used to construct a quadric $Q \in I_2(K_C)$ of rank 4 such that $\rho(Q)$ has large rank. It seems

unlikely that gonality plays any role in this problem. In fact we expect the existence of $Q \in I_2(K_C)$ with image $\rho(Q)$ a nondegenerate quadric on $H^1(C, T_C)$. This would give the upper bound $\frac{3}{2}(g-1)$ for the dimension of a germ of a totally geodesic submanifold. On the other hand, the map ρ is injective. This can be deduced from [5, Cor. 3.4] or from Theorem 3.13, since the form $\hat{\eta}$ is non-zero. Therefore $\rho(I_2(K_C))$ is a linear system of quadrics of dimension $\frac{(g-1)(g-4)}{2}$ on $\mathbb{P}(H^1(C, T_C)) = \mathbb{P}^{3g-4}$. This already gives an upper bound for the dimension of a submanifold Y as in Theorem 4.5. Indeed for any point $[C] \in Y$, the tangent space $T_{[C]}Y \subset H^1(C, T_C)$ is contained in the base locus of $\rho(I_2(K_C))$. This means that $\rho(I_2(K_C))$ is contained in the space of quadrics $q \in S^2H^0(C, 2K_C)$ that vanish on $T_{[C]}Y$. This yields the bound

$$\dim Y \leq \frac{-1 + \sqrt{32g^2 - 40g + 1}}{2}.$$

Nevertheless for any $g \geq 2$ this bound is weaker than the one provided in Theorem 4.5. At any case the study of the base locus of the linear system $\rho(I_2(K_C))$ should clearly improve the understanding of totally geodesic submanifolds. If one could prove that the base locus is empty, one would rule out the existence of totally geodesic submanifolds passing through $[C]$. However to proceed in this direction it is probably necessary to better understand the form $\hat{\eta}$.

Remark 4.8. Observe that for $g \geq 5$, the non existence of germs of totally geodesic hypersurfaces follows directly from Theorem 2.6 and [4, Thm. 6.1]. In fact if Y is a hypersurface in M_g , it passes through a non-trigonal curve $[C]$. Since $\mathbb{P}T_{[C]}Y$ intersects the bicanonical curve in $\mathbb{P}H^1(T_C)$, $T_{[C]}Y$ contains a Schiffer variation ξ_x , for some $x \in C$. By [4, Thm. 6.1], there exists a quadric $Q \in I_2(K_C)$ such that $\mu_2(Q)(x) \neq 0$, so $\rho(Q)(\xi_x \odot \xi_x) \neq 0$ by Theorem 2.6.

5. FAMILIES OF CYCLIC COVERS OF THE PROJECTIVE LINE

5.1. Let C be a curve of genus $g \geq 4$ and let G be a subgroup of the group of automorphisms of C . This yields an inclusion of G in the mapping class group Γ_g [11]. So G acts on the Teichmüller space T_g and we denote by T_g^G the set of fixed points of G on T_g , which is nonempty by the solution of Nielsen realization problem [14, 27, 3], and is a complex submanifold of T_g . The tangent space to T_g^G at a point $[C]$ is given by $H^1(C, T_C)^G$. Moreover T_g^G parametrizes marked curves C endowed with a holomorphic action of G of the given topological type and there is a universal family $\mathcal{C} \rightarrow T_g^G$ containing all such curves. If $t \in T_g^G$ we denote by C_t the corresponding curve.

5.2. Let us now consider the period map at the level of Teichmüller spaces $T_g \rightarrow H_g$. This is an immersion outside the hyperelliptic locus, and we will

consider its restriction to T_g^G . Denote by $Z(G)$ its image in A_g . Given a point $t \in T_g^G$ such that C_t is non-hyperelliptic, the cotangent spaces fit in the exact sequence

$$0 \rightarrow N^* \rightarrow S^2 H^0(K_{C_t}) \xrightarrow{\pi} H^0(2K_{C_t})^G \rightarrow 0,$$

where N^* is the conormal space to $Z(G) \subset A_g$ at the point $J(C_t)$. Since π is G -invariant, N^* is a G -submodule. Let

$$\tilde{\rho} : N^* \rightarrow H^0(2K_{C_t})^G \otimes H^0(2K_{C_t})^G$$

denote the second fundamental form of $Z(G)$ at point $J(C_t)$.

Lemma 5.3. *The second fundamental form $\tilde{\rho}$ is G -equivariant.*

Proof. Recall that by definition, given $a \in N^*$, we have $\tilde{\rho}(a) = \pi(\nabla(a))$, where ∇ is the Gauss-Manin connection on $S^2 f_* \omega_{\mathcal{C}/T_g^G}$ of the family $\mathcal{C} \xrightarrow{f} T_g^G$. Since π is G -invariant, it suffices to prove that ∇ is G -invariant, i.e. $g^{-1} \nabla g = \nabla$ for every $g \in G$. Observe that the flat connection ∇^{GM} on $R^1 f_* \mathbb{C}$ is G -invariant, since the G -action maps flat sections to flat sections. Let $\pi^{1,0} : H^1(C_t, \mathbb{C}) \rightarrow H^{1,0}(C_t)$ be the projection. Then $\pi^{1,0} \circ \nabla^{GM}$ is the connection on the Hodge bundle $f_* \omega_{\mathcal{C}/T_g^G}$. Since the action of G on \mathcal{C} is holomorphic, the projection $\pi^{1,0}$ is G -equivariant, hence $\pi^{1,0} \circ \nabla^{GM}$ is also a G -invariant connection. Finally ∇ is the connection induced by $\pi^{1,0} \circ \nabla^{GM}$ on $S^2 f_* \omega_{\mathcal{C}/T_g^G}$. The result follows. \square

Proposition 5.4. *If there are no nonzero quadrics in $I_2(K_{C_t})$, which are invariant under the action of the group G , then $Z(G)$ is totally geodesic.*

Proof. Consider the cotangent sequence of the period map $j : T_g \rightarrow H_g$ at the point $[C_t]$ and its restriction to T_g^G :

$$(5.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & I_2(K_{C_t}) & \longrightarrow & S^2 H^0(K_{C_t}) & \xrightarrow{m} & H^0(2K_{C_t}) \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow = & & \downarrow \\ 0 & \longrightarrow & N^* & \longrightarrow & S^2 H^0(K_{C_t}) & \longrightarrow & H^0(2K_{C_t})^G \longrightarrow 0. \end{array}$$

(The notation is as in 5.2.) Since the map m is G -equivariant, any G -invariant element in N^* lies in $I_2(K_{C_t})$. Hence by the assumption there are no nontrivial invariant elements in N^* , i.e. the trivial representation does not appear in the decomposition of N^* . On the other hand the representation $H^0(2K_{C_t})^G$ is trivial. By Lemma 5.3 $\tilde{\rho}$ is a morphism of G -representations. Hence Schur lemma implies that $\tilde{\rho}$ is the trivial map. \square

5.5. Now we will consider the case in which $G = \mathbb{Z}/m\mathbb{Z}$, $m \geq 3$ and $C/G \cong \mathbb{P}^1$. These families have been studied by various authors, e.g. [7, 21, 25], since they provide the only known examples of totally geodesic submanifolds contained in the Jacobian locus. These examples are in fact

Shimura varieties. A complete list of the Shimura varieties that can be obtained in this way has been given in [21], which we follow for the notation in the rest of the paper.

We identify G also with the group of m -th roots of unity. Fix an integer $N \geq 4$, together with an N -tuple $a = (a_1, \dots, a_N)$ of positive integers, such that $\gcd(m, a_1, \dots, a_N) = 1$, $a_i \not\equiv 0 \pmod{m}$ and $\sum_{i=1}^N a_i \equiv 0 \pmod{m}$. Given distinct points $t_1, \dots, t_N \in \mathbb{P}^1$ there is a well-defined curve C_t which is a cyclic cover of \mathbb{P}^1 with covering group $\mathbb{Z}/m\mathbb{Z}$, branch points t_i and local monodromy a_i at t_i . It is the normalization of the affine curve

$$(5.2) \quad y^m = \prod_{i=1}^N (x - t_i)^{a_i}.$$

Varying the branch points $t = (t_1, \dots, t_N)$ one obtains a $(N-3)$ -dimensional family of curves $\mathcal{C} \rightarrow B$. There is an action of G on \mathcal{C} given by the rule $\zeta \cdot (x, y, t) := (x, \zeta \cdot y, t)$, $\zeta \in G$. Thus triples (m, N, a) parametrize these families and two triples (m, N, a) and (m', N', a') yield equivalent families if and only if $m = m'$, $N = N'$ and the classes of a and a' in $(\mathbb{Z}/m\mathbb{Z})^N$ are in the same orbit under the action of $(\mathbb{Z}/m\mathbb{Z})^* \times \Sigma_N$, where $(\mathbb{Z}/m\mathbb{Z})^*$ acts diagonally by multiplication and the symmetric group Σ_N acts by permutation of the indices. Notice that to fix the class of the triple (m, N, a) is equivalent to fixing the topological type of the G -action. Thus B and T_g^G have the same image in M_g and we will consider $Z(m, N, a) := Z(G) \subset A_g$ the $(N-3)$ -dimensional subvariety given by the Jacobians of the curves C_t , $t \in B$. By the Hurwitz formula

$$g = 1 + \frac{(N-2)m - \sum_{i=1}^N r_i}{2}$$

with $r_i := \gcd(m, a_i)$. A basis of $H^0(C_t, K_{C_t})$ is given as follows. For $i \in \{1, \dots, N\}$ and $n \in \mathbb{Z}$ set

$$l(i, n) := \left[\frac{-na_i}{m} \right] \quad d_n = -1 + \sum_{i=1}^N \left\langle \frac{-na_i}{m} \right\rangle$$

(Here $[a]$ and $\langle a \rangle$ denote the integral and fractional parts of $a \in \mathbb{R}$.) Since G acts on C_t , there is a decomposition $H^0(C_t, K_{C_t}) = \bigoplus_{n=0}^{m-1} V_n$, where V_n is the subspace of 1-forms ω such that $\zeta \cdot \omega = \zeta^n \omega$. Then $V_0 = \{0\}$, while for $n = 1, \dots, m-1$ a basis of V_n is provided by the forms that have the following expression in the model (5.2):

$$(5.3) \quad \omega_{n,\nu} := y^n \cdot (x - t_1)^\nu \cdot \prod_{i=1}^N (x - t_i)^{l(i,n)} \cdot dx \quad 0 \leq \nu \leq d_n - 1.$$

Remark 5.6. In [21], Moonen proved that there is a finite list of triples (m, N, a) such that the corresponding subvariety $Z = Z(m, N, a) \subset A_g$ is a Shimura variety. By [20] a Shimura variety is a totally geodesic algebraic submanifold of A_g that contains a complex multiplication point. Therefore

the families in Moonen's list are totally geodesic. Using Proposition 5.4 we can immediately verify that all these families are totally geodesic. In fact in all those cases there are no nontrivial quadrics in $I_2(K_{C_t})$ which are invariant under the action of the cyclic group.

Proposition 5.7. *Consider the family $\mathcal{C} \rightarrow B$ associated to the triple (m, N, a) as above and the corresponding subvariety $Z = Z(m, N, a) \subset A_g$ given by the Jacobians of the curves C_t . Assume that not all the curves C_t are hyperelliptic. If there exists an integer $n \in \{1, \dots, m-1\}$ such that $d_n \geq 2$, $d_{m-n} \geq 2$ and $n \neq m-n$, then Z is not totally geodesic.*

Proof. Fix an arbitrary non-hyperelliptic curve $C = C_t$ belonging to the family. We use the representation (5.2). Since $d_n \geq 2$ and $d_{m-n} \geq 2$ and $n \neq m-n$, there are four distinct forms $\omega_{n,0}$, $\omega_{n,1}$, $\omega_{m-n,0}$, $\omega_{m-n,1}$ given in (5.3). We form the quadric

$$Q := \omega_{n,0} \odot \omega_{m-n,1} - \omega_{n,1} \odot \omega_{m-n,0}.$$

One immediately sees from the definition that $Q \in I_2(K_C)$ and that Q is G -invariant. Let D be the divisor of poles of the meromorphic function $x \in \mathcal{M}(C)$. Let $\sigma_0, \sigma_1 \in H^0(C, \mathcal{O}_C(D))$ be the sections corresponding to the meromorphic functions 1 and x . Assume for simplicity that $t_1 = 0$, so we have $\omega_{n,1} = x \cdot \omega_{n,0}$ and $\omega_{m-n,1} = x \cdot \omega_{m-n,0}$. Hence the forms $\omega_{n,0}, \omega_{m-n,0}$ can be seen as elements in $H^0(C, K_C(-D))$. Set $\tau_0 := \omega_{n,0}$, $\tau_1 := \omega_{m-n,0}$. The quadric Q can be written as follows:

$$(5.4) \quad Q = \sigma_0 \tau_0 \odot \sigma_1 \tau_1 - \sigma_0 \tau_1 \odot \sigma_1 \tau_0.$$

Denote by $\varphi : C \rightarrow \mathbb{P}^1$ our covering: it corresponds to the pencil $\langle 1, x \rangle = \langle \sigma_0, \sigma_1 \rangle$. Denote by ψ the map to \mathbb{P}^1 induced by the other pencil $\langle \tau_0, \tau_1 \rangle$. Take a point $p \in C$ that does not belong to the set $\text{Crit}(\varphi) \cup G \cdot \text{Crit}(\psi) \cup B$, where B is the base locus of ψ . Fix a nonzero vector $u \in T_p C$. The vector $v := \sum_{g \in G} \xi_{dg(u)} \in H^1(C, T_C)$ is clearly G -invariant. So it is a tangent vector to T_g^G at the point corresponding to C . Hence by diagram (5.1) we have $\rho(Q)(v \odot v) = \tilde{\rho}(Q)(v \odot v)$. By Lemma (3.11) the map ρ is also G -equivariant. So, using Theorem 2.6 we get

$$\begin{aligned} \tilde{\rho}(Q)(v \odot v) &= \rho(Q)(v \odot v) = \sum_{g_1, g_2 \in G} \rho(Q)(\xi_{dg_1(u)} \odot \xi_{dg_2(u)}) = \\ &= |G| \cdot \left(\sum_{g \neq 1} \rho(Q)(\xi_u \odot \xi_{dg(u)}) - 2\pi i \mu_2(Q)(p) \right) \\ &= -2\pi i m \cdot \left(2 \sum_{g \neq 1} Q(u, dg(u)) \cdot \eta_u(dg(u)) + \mu_2(Q)(p) \right). \end{aligned}$$

By (5.4) Q is the quadric associated to the pencils $\langle \sigma_0, \sigma_1 \rangle$ and $\langle \tau_0, \tau_1 \rangle$, corresponding to the maps φ and ψ respectively. Since the fibre of φ containing

p is the orbit $G \cdot p$, $Q(u, g_*u) = 0$, for all $g \in G - \{1\}$. So finally we get

$$\tilde{\rho}(Q)(v \odot v) = -2\pi i m \cdot \mu_2(Q)(p),$$

and $\mu_2(Q)(p) \neq 0$ by our choice of p . Hence Z is not totally geodesic. \square

Corollary 5.8. *For any m , there is only a finite number of families which can be totally geodesic.*

Proof. By Proposition 5.7, if there exists an integer $n \in \{1, \dots, m-1\}$ such that $d_n, d_{m-n} \geq 2$, $n \neq m-n$, then the family is not totally geodesic. So we can assume that for all $n \in \{1, \dots, m-1\}$ either $d_n \leq 1$, or $d_{m-n} \leq 1$. In particular either $d_1 \leq 1$, or $d_{m-1} \leq 1$. Denote by N_i the cardinality of the set $\{j \mid a_j = i\}$. Then $\sum_{i=1}^{m-1} N_i = N$ and the relation $\sum_{i=1}^N a_i \equiv 0 \pmod{m}$ becomes $\sum_{i=1}^{m-1} i N_i \equiv 0 \pmod{m}$. But

$$d_l = -1 + \sum_{i=1}^N \left\langle \frac{-la_i}{m} \right\rangle = -1 + \sum_{i=1}^{m-1} N_i \left\langle \frac{-li}{m} \right\rangle, \quad \text{for } l = 1, \dots, m-1,$$

$$\text{so } d_1 = -1 + \sum_{i=1}^{m-1} \frac{m-i}{m} N_i, \quad d_{m-1} = -1 + \sum_{i=1}^{m-1} \frac{i}{m} N_i.$$

Hence for $l \in \{1, m-1\}$ we have $d_l \geq -1 + \sum_{i=1}^{m-1} \frac{N_i}{m} = -1 + \frac{N}{m}$. So $d_l \leq 1$ implies $N \leq 2m$. Hence only a finite number of families can be totally geodesic. \square

Now we give some examples of computations for low degree ($m = 3$ and $m = 5$), showing that by the previous results most of the families are not totally geodesic.

Corollary 5.9. *If $m = 3$ and $g \geq 5$, then the varieties $Z(3, N, a)$ are not totally geodesic. If $g = 4$ there is only one totally geodesic family given by the triple $(3, 6, a)$ where $a = (1, 1, 1, 1, 1, 1)$.*

Proof. If $d_1, d_2 \geq 2$, we know by Proposition 5.7 that the family is not totally geodesic. So we can assume that there exists $n \in \{1, 2\}$ such that $d_n = 1$. In this case the space $S^2 H^0(K_{C_t})^{(0)}$ given by the invariant elements equals $\langle \omega_{n,0} \odot V_{3-n} \rangle$. So there are no nonzero invariant elements in $I_2(K_{C_t})$ and by Proposition 5.4 we conclude that the family is totally geodesic. So we have to show that $g = 4$, $N = 6$ and $a = (1, 1, 1, 1, 1, 1)$. Denote as above by N_i the cardinality of the set $\{j \mid a_j = i\}$. Then $N_1 + N_2 = N$ and $N_1 + 2N_2 \equiv 0 \pmod{3}$. So

$$d_1 = -1 + \sum_{i=1}^N \left\langle \frac{-a_i}{3} \right\rangle = -1 + N_1 \left\langle \frac{-1}{3} \right\rangle + N_2 \left\langle \frac{-2}{3} \right\rangle = -1 + \frac{2}{3} N_1 + \frac{1}{3} N_2,$$

$$d_2 = -1 + \sum_{i=1}^N \left\langle \frac{-2a_i}{3} \right\rangle = -1 + N_1 \left\langle \frac{-2}{3} \right\rangle + N_2 \left\langle \frac{-4}{3} \right\rangle = -1 + \frac{1}{3} N_1 + \frac{2}{3} N_2.$$

By Hurwitz formula $g = N - 2$. Since we are assume $g \geq 4$ we get $N_1 + N_2 = N \geq 6$. If $d_1 = 1$,

$$6 \leq N_1 + N_2 \leq 2N_1 + N_2 = 6,$$

so $N = 6$, $g = 4$, $N_1 = 0$, $N_2 = 6$, hence $a = (2, 2, 2, 2, 2, 2)$ which is equivalent to $(1, 1, 1, 1, 1, 1)$. If $d_2 = 1$ we have

$$6 \leq N_1 + N_2 \leq N_1 + 2N_2 = 6,$$

so $N = 6$, $g = 4$, $N_2 = 0$, $N_1 = 6$, $a = (1, 1, 1, 1, 1, 1)$. \square

Remark 5.10. If $m = 5$, the following families are totally geodesic (see the list in [21]):

- (1) $g = 4$, $N = 4$, $a = (1, 3, 3, 3)$,
- (2) $g = 6$, $N = 5$, $a = (2, 2, 2, 2, 2)$.

(1) is the family constructed by de Jong-Noot [7]. Applying the criterion given in Proposition 5.7 we are able to prove that all other families are not totally geodesic except possibly for the following 4 cases

- (3) $g = 4$, $N = 4$, $a = (1, 1, 4, 4)$,
- (4) $g = 4$, $N = 4$, $a = (1, 2, 3, 4)$,
- (5) $g = 6$, $N = 5$, $a = (1, 1, 1, 3, 4)$,
- (6) $g = 6$, $N = 5$, $a = (1, 1, 2, 2, 4)$,
- (7) $g = 8$, $N = 6$, $a = (1, 1, 2, 2, 2, 2)$.

(3) is contained in the hyperelliptic locus, see [21, (5.7)]. (4) has been studied in detail in [28]. Since one can check that it contains a CM point, it is not totally geodesic by Moonen's classification.

It would be interesting to get a complete list of the families of cyclic coverings which are totally geodesic and to compare it with the list in [21].

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