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Mathematical pluralism and some generic absoluteness results in Set theory: a philosophical inquiry

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To my dear family:
Adele, Stefano, and Luca.

Introduction

A possible way to introduce to the phenomenon of independence in Mathematics is to start focusing on the notion of *axiomatic system* trying, first of all, to give an explanation for the centrality of the above notion in the mathematical thinking. As it is stressed in [16], in order to motivate the relevance of the notion of an axiomatic system in Mathematics it can be useful to consider the way in which, *ideally*[1], the process of justification takes place in the mathematical practice.

Generally speaking, when mathematicians reflect on a particular mathematical domain, the process of justification of one sentence is reduced (through *derivability*) to the process of justification of another sentence, and so on, until one reaches some basic principles which themselves cannot be reduced to something else more basic. Such irreducible principles are called *axioms* and the complex body of mathematical theorems that can be produced starting from such principles is then organized in terms of derivability from the axioms. If we consider, for example, the arithmetic of natural numbers, there the process of justification leads one to isolate the axiomatic system PA (*Peano's arithmetic*), or, in the case of Set theory, it leads one to isolate the standard axiomatic system ZFC (*Zermelo-Fraenkel axioms with the axiom of Choice*). As it is stressed in [16], at least two questions naturally emerge concerning the notion of axiomatic system, as we briefly characterized it above.

- The first question concerns the possibility to give a justification to the axioms.
- The second question asks if the *axiomatic basis* that can be isolated through the process of justification described above is rich enough to let us *decide* (prove or refute) all the sentences expressible in the language of our axiomatic system.

As Koellner says in [16], there seem to be initially two general beliefs connected with the notion of axiomatic system. The first conception is that axioms do not need to be further justified because they are already *self evident* (on this side, as it seems to

[1]Historically, things are quite more complicated

us, falls for example, Frege's *logicism*). The second conception conceives axioms as *defining tools* of the objective domain of our specific mathematical inquiry, and as such, because they define the objects we are dealing with, they do not stand in need of justifications (this seems to be, as it seems to us, the proper attitude, for example, of Hilbert's conception of the axiomatic).

Koellner characterises both the above conceptions of the notion of axiom as *optimistic*[2] and he stresses how, according with them, the question of justification in Mathematics reveals itself as a particularly straightforward one. In a slogan,

a sentence ϕ is an axiom, or it is derivable in first order logic starting from the axioms.

This quite optimistic scenario concerning the process of mathematical justification and the notion of axiomatic system has been deeply put in question starting from 1931 by Gödel's *Incompleteness Theorems*.

I will state here below a formulation of Gödel's *second* Incompleteness Theorem.

Theorem 0.0.1. *(Gödel 1931) Assume that PA is consistent. Then PA does not prove $Con(PA)$.*

$Con(PA)$ is an arithmetical sentence that expresses the informal statement that the axiom system PA is consistent. Under further conditions, that in the present context we can skip, it is possible also to show that PA cannot prove $\neg Con(PA)$[3]. In other words, what we have here is an *independence result*, and we say that $Con(PA)$ is independent from PA. So an arithmetical sentence (and not a particularly complex one[4]) cannot be decided on the basis of the Peano's axioms. If we also consider that the Gödel's Theorem is extremely general and that it doesn't apply only to PA but to whatever axiomatic system T that is strong enough to express Robinson's arithmetic, we can see how the emergence of a similar phenomenon leads to some relevant considerations. The first is that Gödel's results definitely exclude the possibility to work in a unique and fixed axiomatic system. Theoretically, as Koellner says in [16], we will always need to introduce new axioms. But then, in an even more relevant way, we have to deal with the question of the *methodology* through which we can introduce new axioms. In fact, as the axioms we need to introduce become stronger, the initial claims that they are self evident statement or defining tools of

[2]See [16], section 1.
[3]See, for example, [15] (section 1, p. 4.) for some more details on this point.
[4]$Con(PA)$ is a Π^0_1 statement.

our mathematical objective domain may become not easy to defend.

Already at the time of his discovery of Incompleteness, as Koellner observes in [16], Gödel indicated a possible strategy for introducing (and *justifying*) new axioms, and so for dealing with the *kind* of independence he discovered. He suggested that if one goes further than the natural numbers through the hierarchy of types (sets of natural numbers, sets of sets of natural numbers, and so on) one reaches new axioms (the axioms of second order arithmetic, the axioms of third order arithmetic, etc..) that are enough strong in solving the undecidable statements discovered by himself. For example, PA_2, the axiomatic system for second order arithmetic, is strong enough to solving the statements left undecided at PA, specifically, $Con(PA)$. Now, anyway, we must deal with $Con(PA_2)$, and according with Gödel's Incompleteness, PA_2 cannot prove $Con(PA_2)$. If we move at the level of third order arithmetic, $Con(PA_2)$ is now provable. This iterative pattern goes on in the same way. Climbing the hierarchy of types one reaches always stronger systems that are able to decide the open problems left undecided by the weaker ones. If we restate the structure of the hierarchy of types into the more uniform iterative structure of sets, we can introduce the *cumulative hierarchy* of sets[5], as follows.

By transfinite induction on the ordinals, we start with the empty set and we take the power set at the successor stage, $\alpha + 1$, and the union at limit stage, λ.

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= P(V_\alpha) \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \ for\ \lambda\ a\ limit. \end{aligned} \tag{1}$$

The *universe* of sets V is the union of V_α for α an ordinal.

As Koellner observes, if the phenomenon of independence in mathematics were restricted exclusively to the cases of undecidability isolated by Gödel through his Theorems, then the sequence of stronger systems internal to the cumulative hierarchy

[5]It seems relevant to recall here that the universe of sets, V, can be seen as the universe of (almost) all of classical Mathematics. Consider, for example, the following passage from [34] (section 1, p. 1), by the set theorist John R. Steel: "In the 19th and the early 20th centuries, it was shown that all mathematical language of the time could be translated into the language of Set theory (LST), and all mathematical Theorems of the time could be proved in ZFC. A century later, mathematicians have yet to develop any mathematics that cannot be expressed in LST, and there are probably few who believe that this will happen any time soon".

of sets (or to the hierarchy of types) would solve every problem. Quoting directly from [16] (section 1),

> "although we would never have a *single* system that gave us a *complete axiomatization of mathematical truth*, we would have a series of systems that *collectively* covered the totality of mathematical truths. (My emphasis)

As one could expect, unfortunately, the question of independence appears to be (extremely) more complicated. The problem, synthetically, is that there exist independent statements that appear to be more *intractable* than the ones displayed by Gödel's through his Theorems. This *new* kind of independent statements appear already at the lower levels of the cumulative hierarchy of types (or at the lower *infinite* levels of the cumulative hierarchy of sets). For example, even at the second level of the cumulative hierarchy (of types) it is possible to formulate (in the language of second order arithmetic) the sentence called PM (asserting that all the projective[6] sets are *Lebesgue* measurable), and at the third level of the hierarchy of types it is possible to formulate CH (the *Continuum Hypothesis*). As Koellner says in [16], these statements were intensively studied during the early era of Set theory, but with modest results. The reason for such a difficulty in holding these statements came finally together with two deep results in mathematical logic, one by Gödel in 1938, and the other by Cohen in 1963, which together changed completely the general perspective towards the phenomenon of independence in mathematics.

Gödel, in 38', discovered the so called *inner model* technique of Set theory and he defined the minimal inner (class) model L (the *constructible* universe). The definition of L is close to that of V, but at successor stages $\alpha + 1$, instead of applying the full power set operation, one takes the *definable*[7] power set of the previous level. We say that , for a set X, the definable power set of X, $Def(X)$, is the set of all the subsets of X that are definable from X with parameters in X.

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= Def(L_\alpha) \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha \ for\ \lambda\ a\ limit. \end{aligned} \tag{2}$$

[6]Projective sets represent a specific family of definable sets of Real numbers.

[7] See, for example, [22], chapter V, for an accurate characterization of the notion of"definability" in Set theory.

L is, then, $\bigcup_{\alpha \in Ord} L_\alpha$.

Gödel showed that L satisfies $ZFC + CH$. As a consequence of Gödel's *Completeness* Theorem for the first order predicative calculus, ZFC cannot prove $\neg CH$. Some years later, in 1963, Paul Cohen discovered the method of *forcing* (or of the *outer* model) and, in this way, he complemented the work of Godel. Given a complete Boolean algebra $\mathbb{B}$, one defines $V^{\mathbb{B}}$ and shows that $V^{\mathbb{B}}$ satisfies $ZFC + \neg CH$. As a consequence, ZFC cannot prove CH[8].

The above results together show that CH is independent from the axiomatic basis ZFC. The same happens also for PM and, starting from 63', and through the sophisticated developments of the forcing technique in the following decades, *many* different other questions in Set theory and other different areas of Mathematics have been proved to be independent.[9]. Similar cases of Independence, anyway, appear to be more problematic than the ones considered above in the context of Gödel's Incompleteness Theorems. For example, the iteration of the operation of *power set* along the cumulative hierarchy of sets doesn't lead automatically to a solution of those statements. Also, unlike the cases isolated by Gödel, *their independence doesn't imply their truth.*

There seems to be, however, an even deeper consideration to add concerning the new cases of Independence generated by the forcing technique. The fact seems to be that, broadly speaking, unlike the cases of Independence isolated by Gödel, the problems shown to be independent by forcing have a stronger *mathematical flavour* compared with the content of the "artificial" or "logical" (or even "pathological") examples of Independence constructed with the Gödel's Theorems. It seems possible to say, then, that with the discovery and the impressive developments of the forcing technique the phenomenon of Independence came much closer to the regions of the "real" (or "core") Mathematics. Such a restriction of the distance between the possible applications of the Independence phenomenon and the *concrete* work of the mathematician generated (consequently) an impact or a "shock"[10] among the community of mathematicians and of philosopher of mathematics more destabilizing than the first effects of the Incompleteness Theorems did. We can say that, in some sense, the new form of Independence seems to disturb in a more intrinsic (or radical)

[8]We are going into (some of) the details of this method (with the so called Boolean valued models) in the first chapter of the present work.

[9]See [15].

[10]See for example [30] , where section 1 of Magidor's paper is called, *The Shock of Independence*

way the notion of set theoretical truth, and- by the *foundational* role played by Set theory towards the rest of all classical Mathematics- that of *mathematical* truth, and as such it seems to represent an unavoidable challenge for everyone working at the Foundations of Mathematics. In the Introduction of his article, *Set theory after Russell: The Journey back to Even*[11], Hugh Woodin accurately summarizes what seems to be the main point of the challenge represented by the emergence of the forcing-Independence phenomenon.

> "Does the phenomenon of formal independence in Set Theory fulfil the prophecy some might claim is the content of Russell's discovery of the now famous Russell Paradox?
> This claim of course is that there can be no meaningful axiomatisation of Set theory because the concept of set is inherently vague, moreover *any choice of axioms is an arbitrary one* and this is the reason that to date essentially all investigations have rather quickly led to inconsistency or to unsolvable problems". (My emphasis)

Schematically, it seems possible to observe two basic different kinds of reactions in front of the emergence of the forcing-Independence phenomenon. On the one side, there is the so-called pluralistic (or skeptical) stance toward the notion of mathematical truth. Generally, the pluralist (or the skeptic) thinks that the forcing-independence phenomenon in Set theory sanctions the end of the story concerning the possibility to find *the correct* answer to the independent questions. The pluralistic stance toward the notion of mathematical truth, actually, covers many different positions, which may be philosophically different one from another. We are not going to offer in the present work an accurate presentation of the different pluralistic positions concerning the notion of mathematical truth, but we stress that, as a common feature, they seem to agree on the idea of a substantial *dispersion* of the notion of mathematical truth after the emergence of the forcing technique. In this sense we believe that the following quotation from Mostowski can well represent the pluralistic point of view. (We refer to [30], section 1, p. 1. for the following quotation.)

> "Such results show that axiomatic Set theory is hopelessly incomplete..
> If there are a *multitude* of Set theories then none of them can claim the central place in Mathematics".

Not all the reactions anyway have been so pessimistic concerning the possibility of reconstructing from the ruins of forcing a (as much as possible) *unified* Set theory.

[11] See [41].

Gödel's reaction and the so called *program for large cardinal axioms* stems out as an important historical and philosophical case. Here is a quotation from Gödel that gives a hint in catching the "spirit" of his program for new axioms.

> "I disagree about the philosophical consequences of Cohen's result. In particular I don't think realists need expect any *permanent ramifications* . . . as long as they are guided, in the choice of axioms, by mathematical intuition and by other criteria of rationality." (My emphasis) (See [15], section 1.5, p. 12)

Gödel's attitude toward the question of undecidability in Set theory determined by forcing techniques has had a remarkable influence during the subsequent decades, and, as a matter of fact, in the Contemporary debate in Set theory and the Foundations of Mathematics, the search and the classification of new axioms has acquired the status of a definite line of research in Set theory. [12] We saw in the recent years[13] the flourishing of different programs for the search of new axioms to add to the standardly accepted axiomatic basis of Set theory, ZFC. Each of these programs, we may say, through the modulation of epistemological criteria that makes the enterprise for the search of new axioms, actually, an enterprise for the search of the *justification* of new axioms, aims at isolating mathematical principles able to reduce as much as possible (and modulo the *unavoidable* Gödel's Incompleteness) the inability of ZFC in settling the solution of the many undecidable questions. We can refer to the kind of completeness searched for by the different programs for the search of new axioms as to a kind of *empirical*[14] completeness.

[12] Pay attention to the *Stanford Encyclopedia of Philosophy* and to its voice "Set theory". You find the following statement: " A *central* theme of Set theory is thus the search and classification of new axioms.These fall currently into two main types:

- The axioms of large cardinals
- The forcing axioms"

[13] This information can appear misleading since, actually, much of the mathematical work on forcing axioms, for example, comes back to the Seventies and to the Eighties. Nevertheless, what we consider to be recent is the philosophical conceptualization of the framework of the search of new axioms within which the work on the forcing axioms can be reframed. Our impression is that, in this sense, the *EFI Project* (2011) represented a crucial moment in the process of elaboration of the *Contemporary* conceptual framework for the search of new axioms. Regarding, in particular, the forcing axioms, we think that Todorcevic's paper [35] for the EFI Project is one very important document for the elaboration of the *Forcing axioms program* for the search of new axioms

[14] See for example [9] (section 2, p. 4), where Dehornoy adds, concerning the notion of being "empirically complete", that it is an 'obviouslyill-defined and imprecise notion".

We would like to try to schematize here below, and without any claiming of exhaustiveness, some paradigmatic aspects of the Contemporary framework for the research of new axioms that, with some degree of plausibility, we think that can be shared by most of the programs[15], and despite of the differences of the *incompatible* mathematical principles proposed by the different programs for the search of new axioms.

- The first point that we want to stress is the Foundational role conferred to Set theory. The universe of sets, V, is the *arena* where all classical Mathematics takes place.
- The goal of Set theory is to *shape* the structure of all mathematical space. This is accomplished, primarily, by deciding *in a reasonable way* the biggest class of mathematical problems shown to be independent by forcing
- The methodology proper to the programs is essentially inherited by Gödel. The key notions are those of *intrinsic* and *extrinsic* evidences. It seems remarkable to note the special relevance attached to the notion of *extrinsic* evidence and the assimilation of the methodology for the search of new axioms with the methodology proper to the empirical sciences.
- Forcing, by itself (and *as a source of undecidability*), is considered as a pathology and it needs to be neutralized[16].
- The programs by themselves don't exclude *dogmatically* the possibility that there will be a ramification in Set theory. But such an analysis ideally has to show *mathematical traction*.

In the present work we primarily deal with some aspects of what we may call the *Forcing axioms* program for the research of new axioms. The methodology proper to this program seems to inherits some main aspects of what in the literature is known

[15] **Procedural note**. Our schematization is based especially on the consideration of the following programs for the research of new axioms in Set theory.

- *Program of Definable determinacy*
- *Forcing axioms program*
- $V = Ultimate L$

There are other programs and different conceptions. We didn't consider them in the present work.

[16] The 'spectre of undecidability'

as *Woodin's program.*

Essentially, and how we understand it, Woodin's program aims at neutralising the power of forcing in destroying the decidability of so a huge variety of mathematical problems and, correspondingly, at finding a satisfactory description of the universe of sets, V, step by step, with an approach that has been called *incremental*[17], giving a sufficiently complete description, through the selection of new axioms, of initial segments of the universe $V = \{x : x = x\}$.
As we stressed before, Gödel's incompleteness makes the axiomatic system PA (*Peano's Axioms*) for the structure $(\mathbb{N}, +, \times)$ incomplete, showing that there are sentences true in $(\mathbb{N}, +, \times)$ that are not provable from PA. Such an *intrinsic* limitation of PA can be at least reduced if one passes, as we saw before, to a stronger axiomatic system such as, for example, PA_2. Woodin's idea is to get into the axiomatic framework of ZFC[18] and to recast the classical mathematical structures, as the structure of the natural number $(\mathbb{N}, +, \times)$, in terms of initial fragments of the universe of sets, $H_{\aleph_0}$, $H_{\aleph_1}$,.., where, more generally, each initial fragment H_k is the set of all the sets whose cardinality is *hereditarily less* than k, for k an arbitrary cardinal number[19]. In particular it is possible to show that the structure $(H_{\aleph_0}, \in)$ is equivalent to $(\mathbb{N}, +, \times)$, that is, the hereditarily finite level of Set theory coincides with the realm of arithmetic. Once noticed that, the benefits of coding the structure $(\mathbb{N}, +, \times)$ in terms of ZFC is that we can analyze the properties of the structure of the natural numbers in terms of the *consistency strength* of ZFC. This means that, instead of asking whether, for a sentence ϕ, ϕ is provable from PA, we study if the *new* sentence "$H_{\aleph_0}$ satisfies ϕ" is provable from ZFC. The new setting of the analysis of $(\mathbb{N}, +, \times)$ in terms of ZFC discloses some interesting facts. First of all, even if Gödel's Incompleteness still applies at the level of ZFC, it seems that most of the cases of sentences ϕ such that the sentence "$(H_{\aleph_0}, \in)$ satisfies ϕ" is not provable neither refutable from ZFC are restricted to cases where ϕ can be seen as a *monstrum* stemming more or less directly from logic. What seems more remarkable is that the analysis of $(H_{\aleph_0}, \in)$ in terms of ZFC shows that the properties of $(H_{\aleph_0}, \in)$ (and hence those of $(\mathbb{N}, +, \times)$) are *invariant under forcing.* With the last expression we only mean that forcing cannot be invoked for *altering* the structure theory of $(H_{\aleph_0}, \in)$. We can then stress the following important idea that seems to emerge as

[17]See, for example, [6] (section 2, p. 3).

[18]We are basically following here the presentation of Woodin's program given in [9], as we understand it.

[19]We refer to chapter 2 of the present work, definition (2.2.3) for a formal definition of this kind of sets.

a kind of *empirical* observation from the analysis just sketched.

> **Empirical observation**: There seems to be a difference in essence between the manifestations of the Incompleteness of ZFC at the level of arithmetic and the manifestations of the Incompleteness of ZFC at higher levels, as for example the level of the Continuum Hypothesis, where forcing effects are strongly active.

We can briefly reformulate the purpose of Woodin's program saying that it aims at recovering for the structure $H_{\aleph_\alpha}$, for $\alpha \geq 1$, the same level of empirical completeness, or of invariance under forcing, that is *naturally* proper to the structure $(H_{\aleph_0}, \in)$.

It has been proved by Woodin in the past years that exploiting some basilar assumptions on the existence of *large cardinals*, it is possible to subtract from the action of forcing a very important fragment of the universe of sets, called $H_{\aleph_1}$. Actually, the result by Woodin holds for a bigger structure than $(H_{\aleph_0}, \in)$, namely it holds for $L(\mathbb{R})$, and it secures the structure theory of all the projective sets of real numbers offering a solution for a vast array of open questions in the branch of Set theory known as Descriptive Set theory.

A natural question, then, is if it is, in some way, possible to *extend* the previous result, that in the present context we could call *Woodin's absoluteness*, to the *next* level of the hierarchy of sets immediately after $H_{\aleph_1}$, namely the level $H_{\aleph_2}$. This would guarantee a solution for the Continuum problem. We can summarize the motivation for extending Woodin's Absoluteness as follows.

- **Goal**: Figure out the picture of the set theoretic universe that would accomodate the *right* structure theory of $P(\omega_1)$ and, so, in particular, solve the Continuum Problem.

Schematically, we may say, this requires offering a solution to the following problem, that, in the present context, we can refer to as the **Fundamental Equation**

$$\frac{L(\mathbb{R})}{AD^{L(\mathbb{R})}} = \frac{P(\omega_1)}{?}$$

In recent years, the methodology of Woodin's program has been originally rephrased by the work of Matteo Viale in terms of the research of increasingly stronger *Completeness Theorems* for the relation of derivability "$\vdash$" of the first order predicative calculus with regard to the relation of forceability "$\Vdash$". The key idea isolated by Viale behind the application of Woodin's program (and, maybe, the way in which Viale understand the concept of *invariance under forcing*) can be stated in the following fundamental *key idea.*

Key Idea: *Transform forcing into a strong tool for proving Theorems over certain (natural) theories which extend ZFC*[20].

It turns out that, actually, there are at least two different strategies for obtaining such a generic absoluteness result concerning the structure $(H_{\aleph_2}, \in)$.

- One obtained by Woodin himself introducing Ω-logic and the axiom $(*)$.
- One obtained by Viale applying the so-called *forcing axioms*

The present work is, first of all, an attempt to understand some of the technical aspects of Viale's and Woodin's strategy of extending the invariance under forcing to the level of the Continuum problem. (This will mainly take place in Chapters 2 and 3). Then, we will try to inquire on their results from a more philosophical point of view. It seems to us, in fact, quite legitimate to *philosophically* compare the above results and to ask to which extent do they show philosophical justification as attempts to uncover *the truths* of the universe of sets, V.

The last chapter of the present dissertation will try to deal with such a question reducing it, in some sense, to the philosophical analysis of a specific *prediction*[21] that has been formulated within the *forcing axioms program* for the search of new axioms, and that (as we will see) it properly concerns the relationship between the axiom called $\mathbf{MM}^{+++}$ and Woodin's axiom $(*)$. Although, naturally, this is only a little contribution toward a better analysis of the philosophical meaning of the previous generic absoluteness results, one possible claim of the final chapter of the present work is that the way in which the prediction we referred to above will be eventually solved may have an impact on the way in which we can *philosophically* appreciate Viale's generic absoluteness and Woodin's generic absoluteness, at the level of $H_{\aleph_2}$. We will contextually present some general features (as we understand them) of the Contemporary framework for the research of new axioms in Set theory, and we will *partially* frame some classical philosophical challenges to it, such as, in particular, the question concerning the philosophical presuppositions that **give sense** to the enterprise itself for the search of new axioms.

[20]See [39], Introduction and [37], Introduction. See also [2] (section 4, p. 6).
[21]See for example [30].

The thesis is organized as follows. In chapter (1) we recall some main aspects of the forcing technique developed following the Boolean valued-models approach introduced by Scott, Solovay, and Vopenka starting from 1965. In chapter (2) we analyze some of the motivations that lie behind Viale's generic absoluteness results, as we understand them, and we sketch a presentation of his first partial generic absoluteness result for the Π_2-theory of the structure $(H_{\aleph_2}, \in)$. Chapter (3) is devoted to the study of Woodin's generic absoluteness for $H_{\aleph_2}$ and to his introduction of Ω-logic as the proper setting for studying generic absoluteness at the level of the CH. In chapter (4) we open the philosophical discussion and we try to contextualize the generic absoluteness results in chapter 2 and 3 into the more general philosophical debate concerning the question of pluralism in Set theory.

We would like to end this Introduction with the following quotation from James Cummings in [8] (section 1, Thesis 1, p. 1), since it has been our ideal source of inspiration all across the present dissertation.

> "A successful philosophy of Set theory should be faithful to the mathematical practices of set theorists. In particular, such a philosophy will require a close reading of the mathematics produced by set theorists, an understanding of the history of Set theory, and an examination of the community of Set theorists and its interactions with other mathematical community. *Note.* I'm not making any claim about the nature of mathematical truth. In particular I'm not claiming (and do not in fact believe) that mathematical truth can be reduced either to the practices of individual mathematicians or to the sociology of the mathematical community. I am also not claiming that the philosophy of mathematics should be purely descriptive, just that it can only get any traction on the main issues by staying closely engaged with actual mathematics."

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Chapter 1

The basic Forcing

1.1 Introduction

In this chapter we will give a formal presentation of **some aspects** of the Boolean-valued models approach to forcing, originally developed by Scott, Solovay, and Vopenka starting from 1965. Forcing was introduced in 1963 by Cohen and, since then, it revealed to be, during the following years, a very powerful tool to provide independence results in Set theory. First of all, as it is well known, the independence of CH, the *Continuum Hypothesis*. Actually, because of the foundational role played by Set theory with regard to the rest of classical mathematics, and because of the possibility to mimic from the standard axiomatic basis ZFC the proof for the existence of almost *any* mathematical object, forcing has been applied to different areas of mathematics, revealing to us the **undecidability** of important questions connected with group theory, functional analysis, operator algebras, general topology and different others subjects[1].

In what follows, we will try to give a general presentation of the basic forcing technique. Nevertheless, it is probably useful to refer for a moment to the original proof of the *relative consistency* of ZFC+ $\neg$ CH in order to get the fundamental idea that is, in some sense, at the very basis of the forcing construction.
The Continuum hypothesis is a specific conjecture regarding the 'size' of the power set of the natural numbers. It asserts that $2^{\aleph_0} = \aleph_1$[2]. As we stressed in the Introduction to the present work, the universe of sets, $V = \{x : x = x\}$, can be seen as the entire universe of all the mathematical entities. All the objects of mathematics

[1] See [39], Introduction.
[2] Assuming the Axiom of Choice.

can be found inside the universe V, and among the other objects we can find in V also the *models* of Set theory. These last, actually, have a kind of strong degree of resemblance with the universe V. In fact, because of the possibility to prove inside ZFC the existence of almost any mathematical object, a model of Set theory M contains an object X^* that is an analogue of the object X contained inside V. It turns out that (quite) often, if one consider a special kind of set theoretic models, called *standard transitive* models, the level of similarity between X and X^* is the identity. Set theorists usually refer to this phenomenon introducing the notion of *absoluteness*[3], and saying that the object X^* in M is absolute. The classification of the absolute set theoretical notions is a very important part of modern Set theory; what it is enough to say here, in the present context, is that along with basic set theoretical notions that are absolute for every standard transitive models M, there are also non absolute notions. Among the non absolute notions, we find the *power set operation* and the notion of *being uncountable*. If we consider a *countable* transitive model of ZFC, M, and we ask for the notion of power set of $\aleph_0$ inside this model, we can see that the object that *plays* the role of $2^{\aleph_0}$ in M is not the same object that plays the role of $2^{\aleph_0}$ in V, otherwise, by transitivity, M could not be countable. The fact that M is countable makes $(2^{\aleph_0})^M$ missing a lot of sets. What we really have is that M "thinks"[4] of $(2^{\aleph_0})^M$ to be the power set of $\aleph_0$, but from the external (*real*) point of view of V we see that $(2^{\aleph_0})^M \neq (2^{\aleph_0})^V$. It is basically this property of the model M to miss some sets that motivates the fundamental construction of the forcing technique. Essentially, the forcing idea is to add to M some of its missing elements in a way that doesn't possibly *perturbate* too much the structure of M, so that we can obtain a new model of ZFC, that we can call $M[G]$, where G stays for the new **main** element added to M. There are different obstacles to surmount. Probably, the first thing to note is that one cannot clearly add only G to M and expects to obtain a model of ZFC. One must add, at least, every set that is *constructible*[5] from G together with elements of M. G itself, also, must be a *generic* object and, as we will see, the property of an object of *being generic* is a crucial aspect of the forcing construction.

It can be usefull to consider more concretely how the object G is added to M for proving that $(2^{\aleph_0})^M = (\aleph_2)^M$. The idea is to exploit the properties of partial orders and to consider the set of all partial functions from the Cartesian product $\aleph_2 \times \aleph_0$ into $\{0, 1\} = 2$. Elaborating on this construction, and modulo some finesses on the

[3]To be distinguished from the notion of *generic* absoluteness we referred to in the Introduction

[4]Or, maybe, it could be better to say "dream", following a beautiful metaphor that I heard once in a Set theory lesson by Peter Koellner

[5]We refer to [22], Chapter VI, for an introduction to the concept of 'constructibility'

property of partial orders to preserve cardinalities, it is possible to generate G as an appropriate unification of all these partial functions such that G lists $(\aleph_2)^M$ functions from $\aleph_0$ into $\{0, 1\}$, where these last functions can be seen as coding of subsets of $\aleph_0$. G, therefore, can be considered as showing us that the power set of $\aleph_0$ in M[G] must be at least of size $(\aleph_2)^{M[G]}$ in M[G].

As we pointed out before, we will give a basic presentation of forcing following the so-called Boolean valued approach, instead of Cohen's account of forcing with partial orders and the **explicit** forcing relation. It will emerge in section (1.5) of the present chapter that the two approaches are, in fact, equivalent and that it is actually possible to recover the Cohen's original approach inside the Boolean valued one. **This is also what justifies the confusion of the two approaches that later on in this work we will assume**.

1.2 The construction of $M^{\mathbb{B}}$

Assume M is a countable transitive model of ZFC.[6] As we saw above, our final goal is to build a new model $M[G]$ (the *generic extension* of M) of ZFC from M, the *ground model*, adding new elements that are missing from M. It turns out that a good starting point for conceiving how to proceed in the construction of the new model is to begin with our set theoretic language and, properly, with the set **S** of all possible formulas that we might want to be true in the new structure $M[G]$. In this sense, first of all we need to formalize the notion of language using the concept of formal language of first order logic. Let **LST** denote the first order language of set theory, i.e. all the formula built out of atomic statements such as "$x = y$" and "$x \in y$" using the Boolean connectives $\vee$, $\wedge$, and $\neg$ and the quantifier $\exists, \forall$. It is possible to show that all the ZFC axioms and all the theory of ZFC can be formulated in **LST**.
Once we have our formal language, it has to be noted that as far as we want to say that some of our formulas are true in the new model while some others are not, we

[6] If we work in ZFC we cannot prove the existence of M. This is because of Goedel's incompleteness theorem which makes not possible for ZFC (unless ZFC is not consistent) to prove the existence of a (countable transitive) model of ZFC. We can bypass the difficulty here following, for example, Kunen's approach in [22] and considering M as a countable transitive model for an **arbitrary finite list** of axioms of ZFC and not as a model of ZFC. In the present chapter we will adopt a similar approach. Nevertheless, just for making our presentation simpler, we will simulate that M is a countable transitive model of ZFC. It is important to remember in what follows that our speaking of c.t.m. M of ZFC should be always rigorously replaced by speaking of c.t.m. M of finite fragments of ZFC.

are, nevertheless, subjected to certain constraints. For example, if we decide that the sentences ϕ and ψ hold, then the sentence "$\phi \wedge \psi$" must hold too.

A good way to track similar constrictions is by means of the very central notion of a Boolean algebra[7].

1.2.1 Boolean algebras

Generally, a Boolean algebra is defined as follows.

Definition 1.2.1. *A Boolean algebra $\mathbb{B}$ is a set B with at least two elements* $\mathbf{0}$*,* $\mathbf{1}$*, endowed with binary operations $\wedge$ and $\vee$ and a unary operation *. The Boolean operations satisfy the following axioms:*

$$\begin{aligned}
u \wedge v &= v \wedge u, \\
u \vee v &= v \vee u \\
u \wedge (v \wedge w) &= (u \wedge v) \wedge w, \\
u \vee (v \vee w) &= (u \vee v) \vee w, \\
u \vee (v \wedge w) &= u \vee v \wedge u \vee w, \\
u \wedge (v \vee w) &= (u \wedge v) \vee (u \wedge w), \\
u \wedge (u^*) &= \mathbf{0}, \\
u \vee (u^*) &= \mathbf{1}
\end{aligned}$$

In a Boolean algebra $\mathbb{B}$ we have that $u \leq v$ iff $u \wedge v = u$ iff $u \vee v = v$.

We introduce contextually some other definitions that will become useful in what follows.

Definition 1.2.2. *A subset A of a Boolean algebra $\mathbb{B}$ is a sub algebra if it contains* $\mathbf{0}$*,* $\mathbf{1}$*, and if it is closed under the Boolean operations:*

(i) $\mathbf{0} \in A, \mathbf{1} \in A$

(ii) if $u, v \in A$, then $u \vee v \in A$, $u \wedge v \in A$, $(u^*) \in A$

Definition 1.2.3. *A complete Boolean algebra is a Boolean algebra in which arbitrary subsets of elements have a least upper bound and a greatest lower bound.*

[7] See [7].

Definition 1.2.4. *A completion of a Boolean algebra* $\mathbb{B}$ *is a complete Boolean algebra* $\mathbb{C}$ *such that* $\mathbb{B}$ *is a dense sub algebra of* $\mathbb{C}$

What seems remarkable here is that there is a natural *correspondence*between the notions **0**, **1**, $\vee$, $\wedge$, and * in a Boolean algebra and the notions of *Falsehood*, *Truth*, $\vee$, $\wedge$ and $\neg$ in classical logic, and that we can exploit such a natural correspondence to record our state of partial knowledge regarding the set **S** of all possible formulas in **LST**. The general idea is that we can map every sentence $\phi \in \mathbf{S}$ to some element of $\mathbb{B}$, that we can denote by $\|\phi\|^{\mathbb{B}}$. If the Boolean algebra we choose is not trivial, its internal structure can be used to reflect our state of *partial* knowledge in the followng way:

- If ϕ is 'definitely true' then we set $\|\phi\|^{\mathbb{B}} = \mathbf{1}$,
- if ϕ is 'definitely false', then we set $\|\phi\|^{\mathbb{B}} = \mathbf{0}$,
- otherwise, $\|\phi\|^{\mathbb{B}}$ takes on some intermediate value between **0** and **1**.

It is quite immediate to see how the mapping, $\phi \to \|\phi\|^{\mathbb{B}}$, should behave in the case of complex formulas built up with propositional connectives . Exploiting the natural correspondence between Boolean algebra and logic we have :

$$\|\phi \vee \psi\|^{\mathbb{B}} = \|\phi\|^{\mathbb{B}} \vee \|\phi\|^{\mathbb{B}} \tag{1.1}$$

$$\|\phi \wedge \psi\|^{\mathbb{B}} = \|\phi\|^{\mathbb{B}} \wedge \|\psi\|^{\mathbb{B}} \tag{1.2}$$

$$\|\neg\phi\|^{\mathbb{B}} = (\|\phi\|^{\mathbb{B}})^{*} \tag{1.3}$$

What is more complicated is to understand how we have to set up the behaviour of the mapping, $\phi \to \|\phi\|^{\mathbb{B}}$, in the cases of atomic expressions such as "$x = y$" and "$x \in y$".

In some sense, dealing with atomic formulas requires us to think directly about the 'core'[8] of the new structure that we want to build. The point here seems to be that if we want some equalities and memberships to hold in the new structure, while postponing our judgements on some other equalities and memberships, this definitely induces us to reflect on the shape, as we may say, of the objects of our new structure and to *demand* that the new objects are *less* determinate than the usual sets we normally deal with. In order to accomplish a similar reduction in the level of determinateness, it may be useful to start observing that we can code an

[8]Not in the technical sense as in Bell's [5].

"ordinary" set as a function whose range is the trivial Boolean algebra with just two elements $\mathbf{0}, \mathbf{1}$, and that sends the members of the set to $\mathbf{1}$, the non-members to $\mathbf{0}$. We can then export this general observation to the case of "fuzzy sets "[9]. What we have to do is to *liberalize* the key notion of Boolean algebra. Let $\mathbb{B}$ be an *arbitrary* Boolean algebra. Then we can conceive a "fuzzy set" as a function x whose domain, $dom(x)$, is a set of potential members, *which should themselves be fuzzy sets*, and that assignes each potential member y a value in $\mathbb{B}$ corresponding to the degree to which $y \in dom(x)$ is a member of x. More precisely, we define a $\mathbb{B}$-*valued set* to be a function from a set of $\mathbb{B}$-valued sets to $\mathbb{B}$. We can now introduce more formally the notion of a Boolean valued structure $M^{\mathbb{B}}$.

Definition 1.2.5. *Given a complete Boolean algebra $\mathbb{B}$ in M (where M is a model of ZFC) such that M models $\mathbb{B}$ to be a Boolean algebra, we can define the structure $M^{\mathbb{B}}$ by recursion on the class of ordinals On in the following way:*

$$M_0^{\mathbb{B}} = \emptyset$$

$$M_\lambda^{\mathbb{B}} = \bigcup_{\beta<\lambda} M_\beta^{\mathbb{B}} \text{ if } \lambda \text{ is a limit}$$

$$M_{\alpha+1}^{\mathbb{B}} = \{f : X \to \mathbb{B} \mid X \subseteq M_\alpha^{\mathbb{B}}\}$$

$$M^{\mathbb{B}} = \bigcup_{\alpha\in On} M_\alpha^{\mathbb{B}}$$ [10]

Our goal now is to complete the description of the general map, $\phi \to \|\phi\|^{\mathbb{B}}$, from $\mathbf{S}$ to our chosen $\mathbb{B}$ in such a way that it sends all the axioms of ZFC to $\mathbf{1}$. What we will obtain then is something very closed to a model of ZFC; for this reason $M^{\mathbb{B}}$ is usually called the Boolean valued *model* of ZFC. However, because $M^{\mathbb{B}}$ will consist of "fuzzy sets" it will not be possible to determine always for an arbitrary $\phi \in \mathbf{S}$ if it is true or false, and instead its truth value will be often some unspecified element of $\mathbb{B}$. To turn $M^{\mathbb{B}}$ into an actual *Tarski model* of ZFC with desired properties, we will have to take a suitable *quotient* of $M^{\mathbb{B}}$ that eliminates the fuzziness.

Note: In what follows we will follow what may be considered a quite standard procedure. We will assume to work in V, the universe[11] of all sets, and we will refer

[9] The expression "fuzzy set" must be intended here not in is technical sense
[10] Observe that the *class* $M^{\mathbb{B}}$ is generated inside M
[11] Explain a little better

to an **ideal extension** of V called $V^{\mathbb{B}}$.[12] A similar assumption makes it possible to *simplify* our presentation without having to relativize and to rephrase many notions and definitions to countable transitive models $M \in V$. It is important to remember however that the next results can be declined and rephrased for any **arbitrary** first order model of ZFC. We will come back to consider transitive countable models of ZFC in Section 1.4 of the present Chapter.

We define $V^{\mathbb{B}}$, the universe of $\mathbb{B}$-valued sets, in the following way by recursion on the ordinals[13].

$$V_{\alpha}^{\mathbb{B}} = \{x : Fun(x) \wedge ran(x) \subseteq \mathbb{B} \wedge \exists \xi < \alpha[dom(x) \subseteq V_{\xi}^{\mathbb{B}}]\} \tag{1.4}$$

We obtain then

$$V^{\mathbb{B}} = \{x : \exists \alpha [x \in V_{\alpha}^{\mathbb{B}}]\} \tag{1.5}$$

The elements of $V^{\mathbb{B}}$ are called $\mathbb{B}$-*names*.

Completing thus the description of our map from $\mathbf{S}$ to $\mathbb{B}$, we have to deal, for example, with expressions involving the quantifiers $\exists$ and $\forall$. Apparently, there doesn't seem to be an immediate counterpart for the logical structure of an existential quantification inside the *structure* of a Boolean algebra. Nevertheless, if we analyze closer the structure of an existential quantification, we see that saying, for example, that there exists an x with a certain property is similar to say that either a has this property, or b has this property, or c etc., where we enumerate all the entities in the universe one by one. This observation leads to the following definition,

$$\|\exists x \phi(x)\|^{\mathbb{B}} = \bigvee_{a \in V^{\mathbb{B}}} \|\phi(a)\|^{\mathbb{B}}$$

There is a potential difficulty with the previous treatment of the existential quantification. A similar difficulty **motivates** an aspect of our previous Definition (1.2.5) of a Boolean valued model. The fact is that an arbitrary B.a. (abbreviation for Boolean algebra) doesn't require necessarily that an infinite subset of its elements has a least upper bound, so, if this is the case, the right-hand side of our last equation may not be always defined. In order to solve this potential difficulty we need to appeal to the notion of completeness of a Boolean algebra (c.B.a) and to **exploit** the conjunction of the following crucial results concerning complete Boolean algebras which asserts the possibility to *switch* from an arbitrary B.a. to its completion.

[12] Appealing to an ideal extension of V is not, in our opinion, extremely intuitive, since V, being the universe of sets, should contain already all possible sets.

[13] See [5].

Lemma 1.2.1. *The completion*[14] *of a Boolean algebra* $\mathbb{B}$ *is unique up to isomorphism.*

Theorem 1.2.2. *Every Boolean algebra has a completion*

The right requirement in order to construct a $\mathbb{B}$-valued extension $V^{\mathbb{B}}$[15] of ZFC is not simply to pick a suitable Boolean algebra, but also to pick one that is complete. Once we satisfy this further requirement we see that our treatment of the previous existential case makes sense as it does the next definition of our mapping $\phi \to \|\phi\|^{\mathbb{B}}$ covering the case of the universal quantification:

$$\|\forall x \phi(x)\|^{\mathbb{B}} = \bigwedge_{a \in V^{\mathbb{B}}} \|\phi(a)\|^{\mathbb{B}}.$$

We have now to deal with the most delicate part of our description of the map $\phi \to \|\phi\|^{\mathbb{B}}$: the case of the Boolean value for the atomic sentences "$x \in y''$ and "$x = y''$. The best strategy here is probably to follow the way in which Bell treats the case of the atomic formulas in [5]. Basically, he lists several equations that one would like to hold for the case of atomic sentences and then he *infers* from that what the *ultimate* definitions must seem like[16].

Following [5], we want, first, that the axiom of extensionality holds in $V^{\mathbb{B}}$. This suggests the equation,

$$\|x = y\|^{\mathbb{B}} = \|(\forall w(w \in x \to w \in y) \ \wedge \ (\forall w(w \in y \to w \in x))\|^{\mathbb{B}}$$

Another plausible equation is,

$$\|x \in y\|^{\mathbb{B}} = \|\exists w((w \in y) \ \wedge \ (w = x))\|^{\mathbb{B}}$$

It is also plausible that the expression $\|\exists w((w \in y) \wedge \phi(w))\|^{\mathbb{B}}$ should depend only on the values of $\|\phi(w)\|^{\mathbb{B}}$ for those w that are actually in the domain of y. Also the value of $\|w \in y\|^{\mathbb{B}}$ should be closely related to the value of $y(w)$. We are thus led to the following equations,

$$\|\exists w(w \in y \ \wedge \ \phi(w)\|^{\mathbb{B}} = \bigvee_{w \in dom(y)} (y(w) \wedge \|\phi(w)\|^{\mathbb{B}}) \tag{1.6}$$

[14]See Definition (1.2.4), Section (1.2.1)
[15]As well as a $\mathbb{B}$-valued model $M^{\mathbb{B}}$ of ZFC.
[16]See [5].

$$\|\forall w(w \in y \to \phi(w))\|^{\mathbb{B}} = \bigwedge_{w \in dom(y)} (y(w) \Rightarrow \|\phi(w)\|^{\mathbb{B}}), \tag{1.7}$$

where $x \Rightarrow y$ his another way of writing $x^* \vee y$. We are thus led to the following definitions.

$$\|x \in y\|^{\mathbb{B}} = \bigvee_{w \in dom(y)} (y(w) \wedge \|x = w\|^{\mathbb{B}}) \tag{1.8}$$

$$\|x = y\|^{\mathbb{B}} = \bigwedge_{w \in dom(x)} (x(w) \Rightarrow \|w \in y\|^{\mathbb{B}} \wedge \bigwedge_{w \in dom(y)} (y(w) \Rightarrow \|w \in x\|^{\mathbb{B}}) \tag{1.9}$$

(1.8) and (1.9) should be understood as a definition of $\|x \in y\|^{\mathbb{B}}$ and $\|x = y\|^{\mathbb{B}}$ **by recursion on a certain well-founded relation**. We make explicit the well-founded relation following [5] (chapter 1, p. 23). Define for $x, y, u, v \in V^{\mathbb{B}}$,

$\langle x, y\rangle < \langle u, v\rangle$ iff either ($x \in dom(u)$ and $y = v$) or ($x = u$ and $y \in dom(v)$).

Then $<$ can be easily seen to be a well-founded relation on the class $V^{\mathbb{B}} \times V^{\mathbb{B}} = \{\langle x, y\rangle : x \in V^{\mathbb{B}}, y \in V^{\mathbb{B}}\}$. It is now possible to apply the recursion (see [5], chapter 1, p. 23).

We can thus sum up all our previous observations in the next definition which **fixes** the general behaviour of the map $\phi \to \|\phi\|^{\mathbb{B}}$.

Definition 1.2.6. *For every formula $\phi(a_1, ..., a_n) \in$ **LST** we define the Boolean value of ϕ*

$$\|\phi(a_1, ..., a_n)\|^{\mathbb{B}} \ (a_1, ..., a_n) \in \mathbb{B}$$

as follows,

1. *If ϕ is a negation, conjunction, or disjunction, we have that*

$$\| \neg \ \psi(a_1, ..., a_n)\|^{\mathbb{B}} = (\|\psi(a_1, ..., a_n)\|^{\mathbb{B}})^*$$

$$\|(\psi \ \wedge \ \chi(a_1, ..., a_n)\|^{\mathbb{B}} = \|\psi(a_1, ..., a_n)\|^{\mathbb{B}} \wedge \|\chi(a_1, ..., a_n)\|^{\mathbb{B}},$$

$$\|(\psi \ \vee \ \chi)(a_1, ..., a_n)\|^{\mathbb{B}} = \|\psi(a_1, ..., a_n)\|^{\mathbb{B}} \vee \|\chi(a_1, ..., a_n)\|^{\mathbb{B}}$$

2. *If ϕ is $\exists x\psi$ or $\forall x\psi$,*

$$\|\exists x\psi(x, a_1, ..., a_n)\|^{\mathbb{B}} = \bigvee_{a \in \mathbb{B}} \|\psi(a, a_1, ..., a_n)\|^{\mathbb{B}}$$

$$\|\forall x\psi(x, a_1, ..., a_n)\|^{\mathbb{B}} = \bigwedge_{a \in \mathbb{B}} \|\psi(a, a_1, ..., a_n)\|^{\mathbb{B}}.$$

3. *For atomic formulas we have,*

$$\|x \in y\|^{\mathbb{B}} = \bigvee_{w \in dom(y)} (y(w) \wedge \|x = y\|^{\mathbb{B}})$$

$$\|x = y\|^{\mathbb{B}} = \bigwedge_{w \in dom(x)} (x(w) \Rightarrow \|w \in y\|^{\mathbb{B}}) \wedge \bigwedge_{w \in dom(y)} (y(w) \Rightarrow \|w \in x\|^{\mathbb{B}}).$$

At this point, we need to perform a long verification that $V^{\mathbb{B}}$ satisfies ZFC, and that the rules of logical inference behave as expected in $V^{\mathbb{B}}$. This is what (partially) we are going to do in Section 3.

As we saw before, we obtain $V^{\mathbb{B}}$ basically by exploiting the idea that we can consider an ordinary set as a given function whose range is the trivial Boolean algebra with just two elements $\mathbf{0}, \mathbf{1}$ and that sends the members of the set to $\mathbf{1}$, the non-members to $\mathbf{0}$. We can **then** generalize the key notion of Boolean algebra obtaining, as we saw earlier, the general notion of $\mathbb{B}$-valued set and building our Boolean valued universe. From this point of view it is important to fix **more formally** the logical connection between our starting model V and his transformation in $V^{\mathbb{B}}$. In order to do this, we will analyze, first, the relationship between V^2 and $V^{\mathbb{B}}$ and we will specify later the relationship between V and V^2. This will give us a **perspective** from which to better appreciate the connection between V and $V^{\mathbb{B}}$.

We need to introduce some new concepts.

1.2.2 V and $V^{\mathbb{B}}$

A complete Boolean algebra $\mathbb{B}'$ is said to be a *complete* sub algebra of $\mathbb{B}$ if $\mathbb{B}'$is a sub algebra of $\mathbb{B}$ and, for any $X \subseteq \mathbb{B}'$, $\bigvee X$, $\bigwedge X$ formed in $\mathbb{B}'$ are the same as $\bigvee X$, $\bigwedge X$ formed in $\mathbb{B}$. The next theorem determines the formal connection between $V^{\mathbb{B}'}$ and $V^{\mathbb{B}}$ whenever $\mathbb{B}'$ is a complete sub algebra of $\mathbb{B}$.

Theorem 1.2.3. *Let $\mathbb{B}'$ be a complete sub algebra of $\mathbb{B}$. Then*

1. $V^{\mathbb{B}'} \subseteq V^{\mathbb{B}}$
   Moreover, for $u, v \in V^{\mathbb{B}'}$,

2. $\|u \in v\|^{\mathbb{B}'} = \|u \in v\|^{\mathbb{B}}$;

3. $\|u = v\|^{\mathbb{B}'} = \|u = v\|^{\mathbb{B}}$.

Proof. 1 is clear, while 2 and 3 are proved simultaneously by induction on the well-founded relation $y \in dom(x)$. Details are left to the reader. (*Hint*: the inductive hypothesis is: for all $y \in dom(v)$ and all $u \in V^{\mathbb{B}}$,

$$\|u \in y\|^{\mathbb{B}'} = \|u \in y\|^{\mathbb{B}}$$
$$\|u = y\|^{\mathbb{B}'} = \|u = y\|^{\mathbb{B}}$$
$$\|y = u\|^{\mathbb{B}'} = \|y = u\|^{\mathbb{B}}.)$$

□

Corollary 1.2.4. *If $\mathbb{B}'$ is a complete sub algebra of $\mathbb{B}$, then, for any* **restricted** *formula $\phi(x_1, ..., x_n)$ and any $u_1, ..., u_n \in M^{\mathbb{B}'}$,*

$$\|\phi(u_1, ..., u_n)\|^{\mathbb{B}'} = \|\phi(u_1, ..., u_n)\|^{\mathbb{B}}$$

Proof. By induction on the complexity of ϕ. For atomic ϕ the result holds by Theorem (1.2.3) The only non trivial induction step arises when ϕ is $\exists x \in u\psi$. And here we argue as follows: if $u, u_1, ..., u_n \in V^{\mathbb{B}}$, then, writing $\bigvee^{\mathbb{B}}$, $\bigvee^{\mathbb{B}'}$ for joins in $\mathbb{B}$, $\mathbb{B}'$ respectively,

$$\|\phi(u, u_1, ..., u_n)\|^{\mathbb{B}'} = \bigvee_{x \in dom(u)}^{\mathbb{B}'} [u(x) \wedge \|\psi(x, u_1, ..., u_n)\|^{\mathbb{B}'}] \tag{1.10}$$

$$= \bigvee_{x \in dom(u)}^{\mathbb{B}} [u(x) \wedge \|\psi(x, u_1, ..., u_n)\|^{\mathbb{B}}] \tag{1.11}$$

$$= \|\phi(u, u_1, ..., u_n)\|^{\mathbb{B}}. \tag{1.12}$$

□

Observing that $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$ is always a complete sub algebra of every complete Boolean algebra $\mathbb{B}$ leads us to our first important acquirement, that is, $V^{\mathbf{2}}$ is a (*kind of*) sub model of $V^{\mathbb{B}}$. What we want to show now is that there exists an *isomorphism* between V and $V^{\mathbf{2}}$. This last point will give us the final understanding on the connection between V and $V^{\mathbb{B}}$. First of all, we define by recursion on the relation $x \in y$ the function $\check{\cdot}$ as follows.

Definition 1.2.7. *For each $x \in V$,*

$$\check{x} = \{\langle \check{y}, 1\rangle : y \in x\}$$

Observe that for all $x \in V$, $\check{x} \in V^{\mathbf{2}} \subseteq V^{\mathbb{B}}$, where $\mathbb{B}$ stays for an arbitrary c.B.a. Also, from Theorem (1.2.3), it follows the next Fact.

Proposition 1.2.5. *For all $x, y \in V$, and for any complete Boolean algebra $\mathbb{B}$,*

$$\|\check{x} \in \check{y}\|^{\mathbb{B}} = \|\check{x} \in \check{y}\|^{\mathbf{2}} \in \mathbf{2}$$

$$\|\check{x} = \check{y}\|^{\mathbb{B}} = \|\check{x} = \check{y}\|^{\mathbf{2}} \in \mathbf{2}$$

By the previous Fact (1.2.5), we can consider the elements $\check{x}$ as the natural representatives in $V^{\mathbb{B}}$ of each $x \in V$, and we can refer to the members of $V^{\mathbb{B}}$ of the form $\check{x}$ as the **standard** members of $V^{\mathbb{B}}$. We can now state the following result.

Theorem 1.2.6. *1. For $x \in V$, $u \in V^{\mathbb{B}}$,*

$$\|u \in \check{x}\|^{\mathbb{B}} = \bigvee_{y \in x} \|u = \check{y}\|^{\mathbb{B}}.$$

2. For $x, y \in V$,

$$x \in y \leftrightarrow V^{\mathbb{B}} \models \check{x} \in \check{y}$$

$$x = y \leftrightarrow V^{\mathbb{B}} \models \check{x} = \check{y}.$$

3. The map $x \mapsto \check{x}$ is one-one from V into $V^{\mathbf{2}}$.

4. For each $u \in V^2$ there is a unique $x \in V$ such that $V^{\mathbb{B}} \models u = \check{x}$.

5. For any formula $\phi(v_1, ..., v_n)$ and any $x_1, ..., x_n \in V$,

$$\phi(x_1, ..., x_n) \leftrightarrow V^{\mathbf{2}} \models \phi(\check{x}_1, ..., \check{x}_n)$$

and if ϕ is restricted then,

$$\phi(x_1, ..., x_n) \leftrightarrow V^{\mathbb{B}} \models \phi(\check{x}_1, ..., \check{x}_n).$$

Proof. We refer the reader to [5] for a proof of Theorem (1.2.6). □

By the previous Theorem (1.2.6), we know that there exists an *isomorphism* between V and $V^{\mathbf{2}}$. In particular, points 3 and 4 of Theorem (1.2.6) tell us that the function $\check{\cdot}$ we defined before is a bijection between V and $V^{\mathbf{2}}$, while the first part of point 5 of Theorem (1.2.6) tells us that V and $V^{\mathbf{2}}$ share their own true sentences. From the final part of point 5 of Theorem (1.2.6), we can also see that V shares with $V^{\mathbb{B}}$, for $\mathbb{B}$ an arbitrary complete Boolean algebra, the **restricted** part of its truths.Thus, as our next step, we shall verify (manually) that the axiomatic standard system, ZFC, which we assume is true in V, is also true in $V^{\mathbb{B}}$. In fact, the axioms of ZFC are **not** restricted. However, before going on with the verification of the axioms of ZFC in $V^{\mathbb{B}}$, let's stop for one moment and try to reflect on the general way we could use a construction such as $V^{\mathbb{B}}$ (actually, maybe, *its transformation into a real Tarski model of* ZFC) in order to produce *relative consistency* proofs in Set Theory. This is what we are going to show with the next theorem which, in some way, defines *formally* how we can exploit $V^{\mathbb{B}}$ in order to generate independence results.

Given a theory T, we write *Consis*(T) for "T is consistent".

Theorem 1.2.7. *Let T, T' be extensions of ZFC such that Consis(ZF) $\rightarrow$ Consis(T'), and suppose that in* **LST** *we can define a constant term $\mathbb{B}$ such that:*

($\star$) $T' \vdash$ "$\mathbb{B}$ *is a complete Boolean algebra" and, for each axiom τ of T, we have* $T' \vdash \|\tau\|^{\mathbb{B}} = 1_{\mathbb{B}}$.

Then Consis(ZF)$\rightarrow$ Consis(T).

Proof. If T is inconsistent, then for some axioms $\tau_1, ..., \tau_n$ of T we would have, for any sentence σ,

$$\vdash \tau_1, ..., \tau_n \rightarrow \sigma \wedge \neg\sigma \tag{1.13}$$

Now let $\mathbb{B}$ be a complete Boolean algebra satisfying ($\star$). Then,

$$T' \vdash \|\tau_1 \wedge, ..., \wedge \tau_n\|^{\mathbb{B}} = 1_{\mathbb{B}}. \tag{1.14}$$

But (1.13) gives

$$T' \vdash \|\tau_1 \wedge ... \wedge \tau_n\|^{\mathbb{B}} \leq \|\sigma \wedge \neg\sigma\|^{\mathbb{B}} = 0_{\mathbb{B}},$$

so that, by (1.14)

$$T' \vdash 1_{\mathbb{B}} \leq 0_{\mathbb{B}},$$

so T', and hence ZF, would be inconsistent. □

1.3 The truth of the axioms of Set theory in $V^{\mathbb{B}}$

Lemma 1.3.1. *All the axioms of the first order predicate calculus with equality are true in $V^{\mathbb{B}}$, and all its rules of inference are valid in $V^{\mathbb{B}}$ for any complete Boolean algebra $\mathbb{B}$.*

Proof. We refer the reader for a proof of Lemma (1.3.1) to [5], Chapter 1. □

Theorem 1.3.2. *All the axioms- and hence all the theorems*[17]*- of ZFC are true in $V^{\mathbb{B}}$ for any complete Boolean algebra $\mathbb{B}$.*

We will sketch the prove of Theorem (1.3.2) by means of a sequence of lemmas. A detailed proof of Theorem (1.3.2) can be found in Bell's [5]. If no confusion can arise, we avoid to put the superscript $\mathbb{B}$.

Lemma 1.3.3. *Extensionality*

Proof. It follows from the definition of $\|x = y\|$. □

Lemma 1.3.4. *Separation*

Proof. The axiom in question is:

$$\forall u \exists v \forall x [x \in v \leftrightarrow x \in u \wedge \psi(x)] \tag{1.15}$$

This is an axiom scheme. To see that *each* of its instance is true in $V^{\mathbb{B}}$, consider an **arbitrary** $u \in V^{\mathbb{B}}$. We need to build an object in $V^{\mathbb{B}}$ correlated to our u such that it *fit with* the biconditional of the axiom scheme of separation. In this regard, we define $v \in V^{\mathbb{B}}$ in the following way:

$$dom(v) = dom(u)$$

and, for each $x \in dom(v)$,

$$v(x) = u(x) \wedge \|\psi(x)\|$$

It is possible to check that v is the object we were looking for. □

Lemma 1.3.5. *Pairing*

Proof. The axiom in question is:

$$\forall u \forall v \exists z \forall x (x \in z \leftrightarrow (x = u \vee x = v))$$

Given $x, y \in V^{\mathbb{B}}$, $z = \{(x, \mathbf{1}), (y, \mathbf{1})\} \in V^{\mathbb{B}}$ is a witness for the pairing axiom. □

Lemma 1.3.6. *Union*

[17] By lemma (1.3.1)

Proof. The axion in question is:

$$\forall u \exists v \forall x [x \in v \leftrightarrow \exists y \in u [x \in y]].$$

We need to build an object correlated with an arbitrary $u \in V^{\mathbb{B}}$, such that our new object satisfies the previous biconditional. We define v in the following way:

$$dom(v) = \bigcup \{dom(y) : y \in dom(u)\}$$

and,

$$v(x) = \|\exists y \in u[x \in u]\|.$$

It is possible to check that v is a witness of the union axiom. □

Lemma 1.3.7. *Power set*

Proof. The axiom in question is :

$$\forall u \exists v \forall x [x \in v \leftrightarrow \forall y \in x [y \in u]].$$

Starting with an arbitrary $u \in V^{\mathbb{B}}$, we build v in the following way:

$$dom(v) = \mathbb{B}^{dom(u)},$$

and for $x \in dom(v)$,

$$v(x) = \|x \subseteq u\| = \|\forall y \in x[y \in u]\|$$

□

Lemma 1.3.8. *Replacement*

Proof. The axiom in question is:

$$\forall u [\forall x \in u \exists y \ \phi(x, y) \rightarrow \exists v \forall x \in u \exists y \in v \ \phi(x, y)].$$

We refer the reader to [5], chapter 1, lemma 1.36 for the definition of the relevant $v \in V^{\mathbb{B}}$. □

Lemma 1.3.9. *Infinity*

Proof. The axiom in question is:

$$\exists u [\emptyset \in u \wedge \forall x \in u \exists y \in u (x \in y)]$$

$\breve{\omega}$ is a witness for the axiom of infinity. □

Lemma 1.3.10. *Foundation*

Proof. The axiom in question is:

$$\forall x(x \neq \emptyset \rightarrow \exists y \in x \forall z \in x(z \notin y))$$

We refer the reader to [5], chapter 1, lemma 1.36 for a proof of (1.3.10). $\square$

Lemma 1.3.11. *Choice*

Proof. The axiom in question is:

$$\forall u \exists f[Fun(f) \wedge dom(f) = u \wedge \forall x \in u[x \neq \emptyset \rightarrow f(x) \in x]]$$

We will give a full presentation of the choice's proof following Bell in [5]. The proof, as we will present it, actually requires to introduce some new technical notions concerning the internal structure of $V^{\mathbb{B}}$. $\square$

1.3.1 Mixtures and the Core

Definition 1.3.1. *(Mixtures) Given a subset $\{a_i : i \in I\} \subseteq \mathbb{B}$ and a subset $\{u_i : i \in I\} \subseteq V^{\mathbb{B}}$, we define the mixture $\sum_{i \in I} a_i \cdot u_i$ of $\{u_i : i \in I\}$ with respect to $\{a_i : i \in I\}$ to be that element $u \in V^{\mathbb{B}}$ such that,*

$$dom(u) = \bigcup_{i \in I} dom(u_i)$$

and, for $z \in dom(u)$,

$$u(z) = \bigvee_{i \in I} [a_i \wedge \|z \in u_i\|].$$

A subset $A \subseteq B$ is called an antichain in B if $a \wedge b = 0$ for any distinct element a, b in A.

The next result justifies the use of the term 'mixture' by showing that under certain mild conditions $\sum_{i \in I} a_i \cdot u_i$ behaves as if it were obtained by mixing the $\mathbb{B}$-valued sets $\{u_i : i \in I\}$ together in (at least) the 'proportions' $\{a_i : i \in I\}$.

Lemma 1.3.12. *(Mixing lemma) Let $\{a_i : i \in I\} \subseteq B$, let $\{u_i : i \in I\} \subseteq V^{\mathbb{B}}$ and put $\sum_{i \in I} a_i \cdot u_i = u$. Suppose that for all $i, j \in I$,*

$$a_i \wedge a_j \leq \|u_i = u_j\| (*)$$

Then, for all $i \in I$,

$$a_i \leq \|u = u_i\|.$$

In particular the result holds if $\{a_i : i \in I\}$ is an antichain.

Proof. We refer the reader to [5] for a proof of lemma (1.3.12). $\square$

Using the previous lemma it is possible to prove the following result.

Lemma 1.3.13. *(The Maximum principle) If $\phi(x)$ is any formula, then there is a $u \in V^{\mathbb{B}}$ such that,*

$$\|\exists x \phi(x)\| = \|\phi(x)\|.$$

In particular, if $V^{\mathbb{B}} \models \exists x \phi(x)$, then $V^{\mathbb{B}} \models \phi(u)$ for some $u \in V^{\mathbb{B}}$.[18]

Proof. We refer the reader to [5] for a proof of lemma (1.3.13). $\square$

Finally, it is useful to introduce the following notion together with a correlated result.

Definition 1.3.2. *(Core) Let $u \in V^{\mathbb{B}}$. A set $v \subseteq V^{\mathbb{B}}$ is called a core for u if the following conditions are satisfied:*

(i) $\|x \in u\| = 1$ for all $x \in v$,

(ii) for each $y \in V^{\mathbb{B}}$ such that $\|y = u\| = 1$, there is a unique $x \in v$ such that $\|x = y\| = 1$.

Lemma 1.3.14. *Any $u \in V^{\mathbb{B}}$ has a core.*

Proof. We refer the reader to [5] for a proof of lemma (1.3.14). $\square$

[18]We refer to the Maximum Principle, also, as to the property of $M^{\mathbb{B}}$ to be a **full** model.

1.3.2 Choice via Zorn's Lemma

In order to verify the axiom of Choice in $V^{\mathbb{B}}$ it suffices to prove the set theoretically **equivalent** principle, *Zorn's lemma.* This last one asserts that any non empty **inductive**[19] partially ordered set has a maximal element.

Lemma 1.3.15. *Zorn's lemma, and hence the axiom of Choice, holds in $V^{\mathbb{B}}$.*

Proof. We need to prove that for any $X, \leq_X \in V^{\mathbb{B}}$, if $V^{\mathbb{B}} \models \langle X, \leq_X \rangle$ *is a non empty inductive partially ordered set*, then $V^{\mathbb{B}} \models \langle X, \leq_X \rangle$ *has a maximal element.* Suppose then that the antecedent of this implication holds. Let Y be a core for X and define the relation $\leq_Y$ on Y by

$$y \leq_Y y' \leftrightarrow \|y \leq_X y'\| = 1$$

for $y, y' \in Y$. It is easy to verify that $\leq_Y$ is a partial ordering on Y. We claim that this partial ordering on Y is inductive. For let C be any chain in Y. It is readily shown that $C' = C \times \{1\} \in V^{\mathbb{B}}$ satisfies

$$V^{\mathbb{B}} \models C' \textit{ is a chain in } X$$

Accordingly, by the Maximum Principle there is $u \in V^{\mathbb{B}}$ for which

$$V^{\mathbb{B}} \models u \textit{ is an upper bound for } C' \textit{ in } X.$$

We can choose now $w \in Y$ such that $\|w = u\| = 1$. Then w is an upper bound for C in Y. For if $x \in C$, then clearly $\|x \in C'\| = 1$, whence $\|x \leq_X u\| = 1$ so that $\|x \leq_X w\| = 1$, and $x \leq_Y w$. Therefore Y is inductive as claimed. By Zorn's Lemma in $V^{\mathbb{B}}$, Y has a maximal element c. Then $\|c \in X\| = 1$. We claim further that

$$V^{\mathbb{B}} \models c \textit{ is a maximal element of } X. \tag{1.16}$$

To prove this, take $x \in V^{\mathbb{B}}$ and fix $y \in Y$ such that $\|x \in X\| = \|x = y\|$. Then,

$$\|c \leq_X x \wedge x \in X\| = \|c \leq_X x \wedge x = y\| \leq \|c \leq_X y\|. \tag{1.17}$$

Now let $v = y \cdot a + c \cdot a^{*}$[20], where $a = \|c \leq_X y\|$. Then $\|v \in X\| = 1$ and so there is $z \in Y$ for which $\|v = z\| = 1$. It is easily shown that $\|c \leq_X v\| = 1$, whence

[19] A partially ordered set is said to be inductive if chains (i.e. linearly ordered subsets) in it have upper bounds.

[20] This is a mixture.

$\|c \leq_X z\| = 1$, and so $c \leq_Y z$. Hence $c = z$ by the maximality of c. Therefore

$$\begin{aligned}\|c \leq_X y\| = a &\leq \|y = v\| \\ &\leq \|y = v\| \wedge \|v = z\| \\ &\leq \|y = z\| \\ &= \|y = c\|,\end{aligned}$$

and so by (1.17)

$$\begin{aligned}\|c \leq_X x \wedge x \in X\| &\leq \|y = c\| \wedge \|x \in X\| \\ &\leq \|y = c\| \wedge \|x = y\| \\ &\leq \|x = c\|.\end{aligned}$$

Thus,

$$V^{\mathbb{B}} \models \forall x \in X[c \leq_X x \rightarrow x = c]$$

that is (1.16). This complete the proof. □

The proof of Theorem (1.3.2) is now complete.

1.4 Modding out by an ultrafilter

We want to come back now to countable transitive models M of ZFC. The key observation that we need for understand how it is possible to relativize the previous **definitions** and **constructions** made in V to a countable transitive M is the following.

> **Observation**. Let M be a transitive model of ZFC, and let $\mathbb{B}$ be a complete Boolean algebra in the sense of M. Then, if $X \in P(\mathbb{B}) \cap M$, then $\bigvee X$ and $\bigwedge X$ exist and are in M (where $P(\mathbb{B}) \cap M$ is the powerset of $\mathbb{B}$ formed in M)[21]

From the previous Sections we know that if we start with a countable transitive model M of ZFC and we pick a $\mathbb{B} \in M$ which M models to be a c.B.a, we can generate in M a new class $M^{\mathbb{B}}$, that we called Boolean valued model, such that the axioms of ZFC are true in $M^{\mathbb{B}}$ ($M \models \ \|\phi\|^{\mathbb{B}}_M = \mathbf{1}_{\mathbb{B}}$ for every axiom ϕ of ZFC). Projecting ourselves from M and its **two valued** logic into the more extensive dimension of the **Boolean valued** logic of $M^{\mathbb{B}}$, where for different first order statements ϕ we haven't

[21]This is crucial if one thinks at the way we defined, for example, $\|x = y\|^{\mathbb{B}}$ and $\|x \in y\|^{\mathbb{B}}$.

yet decided about their truth, put us in the conditions to interfere actively on the truth and the falsehood of the still undecided statements.] This essentially require that we **convert** our Boolean valued model $M^{\mathbb{B}}$ into an **actual** model of ZFC. There is a nice procedure in this sense. We choose a subset U of $\mathbb{B}$ that contains $\|\phi\|^{\mathbb{B}}$ for every formula ϕ that holds in the new model of ZFC. In this sense the set U that we choose is a kind of *truth definition* for our new model of ZFC, and it must have certain properties; for example, since for every ϕ, either ϕ or not ϕ holds in the new model, it follows that for all $x \in \mathbb{B}$, U must contains either x or x^*. Basically, our subset U should have the following properties:

1. $\mathbf{1} \in U$
2. $\mathbf{0} \notin U$
3. if $x \in U$ and $y \in U$ then $x \wedge y \in U$
4. if $x \in U$ and $x \leq y$ (i.e., $x \wedge y = x$) then $y \in U$
5. $\forall x \in \mathbb{B}$ either $x \in U$ or $x^* \in U$.

We call such a subset U of a Boolean algebra an *ultrafilter*.[22]
We can now "mod out" by an ultrafilter our Boolean valued model $M^{\mathbb{B}}$. The idea is to define the quotient $M^{\mathbb{B}}/U$ as follows. First of all, we specify the elements of $M^{\mathbb{B}}/U$ as *equivalence classes* of elements of $M^{\mathbb{B}}$ using the equivalence relation.

$$x \sim_u y \leftrightarrow \|x = y\|^{\mathbb{B}} \in U.$$

We can now define the binary relation on $M^{\mathbb{B}}/U$ that we can denote as $\in_U$. If we write x^U for the equivalence class of x, we have that

$$x^U \in_U y^U \leftrightarrow \|x \in y\|^{\mathbb{B}} \in U.$$

It is now quite straightforward to verify that $M^{\mathbb{B}}/U = \langle \{x^U : x \in M^{\mathbb{B}}\}, \in_U \rangle$ is a *model* of ZFC. We just need the following results.

[22] Actually, it should be pointed here that there is a deeper mathematical characterization of the ultrafilter $U \subseteq \mathbb{B}$ than the one we gave in terms of it being a truth definition for our new Tarski model. A similar characterization is beyond the scope of the present chapter and it can be found, for example, in [38]. It requires to thematise, first, the structural correspondence between Boolean algebras and topological spaces. Then it is possible, given a c.B.a $\mathbb{B}$ to introduce the *space of its ultrafilters* called the **Stone space** of its ultrafilters, $St(\mathbb{B})$, and to see the ultrafilter $U \subseteq \mathbb{B}$ as a point $G \in St(\mathbb{B})$ which *force* us to accept ϕ iff $\|\phi\|^{\mathbb{B}} \in G$.

Theorem 1.4.1. *For any formula $\phi(v_1, ..., v_n)$ and any $x_1, ..., x_n \in M^{\mathbb{B}}$, $M^{\mathbb{B}}/U \models \phi[x_1^U, ..., x_n^U] \leftrightarrow \|\phi(x_1, ..., x_n)\| \in U$*

Proof. We prove the theorem by induction on the complexity of ϕ. If ϕ is atomic, then the result follows by definition.

Suppose $\phi=\neg\psi$. If $M^{\mathbb{B}}/U \models \neg\psi \Rightarrow \neg M^{\mathbb{B}}/U \models \psi \Rightarrow$ by inductive hypothesis, $\|\psi\| \notin U$. Because U is an ultrafilter, then $\|\psi\|^* \in U$. If $\|\psi\|^* \in U \Rightarrow \|\psi\| \notin U$, so by inductive hypothesis $\neg M^{\mathbb{B}}/U \models \psi \Rightarrow M^{\mathbb{B}}/U \models \neg\psi$. The case for disjunction or conjunction similarly follows from the basic properties of ultra filters and truth value.

Suppose now that $\phi = \exists x\psi x$. We need to use the **Maximum Principle**. Let $a \in M^{\mathbb{B}}$ such that $\|\psi(a)\| = \|\exists x\psi(x)\|$, by Maximum Principle. We have that $\|\exists x\psi(x)\| \in U \Rightarrow \exists a\|\psi(a)\| \in U \Rightarrow$ by inductive hypothesis $M^{\mathbb{B}}/U \models \psi(a) \Rightarrow M^{\mathbb{B}}/U \models \exists x\psi(x)$. If $M^{\mathbb{B}}/U \models \exists x\psi(x) \Rightarrow \exists a\|\psi(a)\| \in U \Rightarrow \bigvee_{a\in M^{\mathbb{B}}}\|\psi(a)\| = \|\exists x\psi(x)\| \in U$.

□

> **Remark**. Notice the importance in Theorem (1.4.1) of the property of $M^{\mathbb{B}}$ to be a full model which gives a complete control on how truth in $M^{\mathbb{B}}/U$ is determined by the (topological) properties of the ultrafilter U.

Corollary 1.4.2. *$M^{\mathbb{B}}/U$ is a model of ZFC. More generally, for any sentence σ, if $M^{\mathbb{B}} \models \sigma$, then $M^{\mathbb{B}}/U \models \sigma$.*

Proof. If $M^{\mathbb{B}} \models \sigma \Rightarrow \|\sigma\|^{\mathbb{B}} = 1_{\mathbb{B}} \Rightarrow \|\sigma\|^{\mathbb{B}} \in U \Rightarrow M^{\mathbb{B}}/U \models \sigma$. □

At this point we have a powerful theorem in hand. Quoting from [7] (section 7, p. 12),

> " we can take any model M, any c.B.a. $\mathbb{B}$ in M, and any ultrafilter U of M and form a new model $M^{\mathbb{B}}/U$ of ZFC. We can now experiment with **various choices** of M, $\mathbb{B}$ and U to construct all kinds of models of ZFC with various properties" (My emphasis).

Generally speaking, if we use our newly constructed machinery we soon find that $M^{\mathbb{B}}/U$ need not in general to be **isomorphic to** a standard transitive model of ZFC, even if M is.If we want to transform $M^{\mathbb{B}}/U$ into a well founded countable model $M[U]$ of ZFC and then, accordingly with *Cohen's forcing Theorem*, show how to **relate** the first order properties of the structure $M[U]$ to the **combinatorial** properties that M (the ground model) gives to $\mathbb{B}$ and to the choice of U, we need to impose some extra conditions. Here is precisely where the key notion of *genericity* enters the picture

It is useful to introduce the notion of *genericity* in the context of an arbitrary partially ordered set $\mathbb{P}$. As we will show in Section 6, there exists a **standard method** for switching from a poset (partially ordered set) $\mathbb{P}$ to the complete Boolean algebra $\mathbf{B}(\mathbb{P})$ that is unique up to isomorphism. That method offers essentially to us a way to define, starting from $G \subseteq \mathbb{P}$ an ultrafilter $U \subset \mathbf{B}(\mathbb{P})$. We can define U on $\mathbf{B}(\mathbb{P})$ as the set of elements of $\mathbf{B}(\mathbb{P})$ that are greater than some element of (the image of) G. It is possible to show that starting from a $G \subseteq \mathbb{P}$ M-generic for $\mathbb{P}$ we can obtain a $U \subseteq \mathbf{B}(\mathbb{P})$ such that U is M-generic for $\mathbf{B}(\mathbb{P})$. For similar reasons we can blur in the present context the distinction between G and U.

Definition 1.4.1. *A partial order is a pair $\langle \mathbb{P}, \leq \rangle$ such that $\mathbb{P} \neq \emptyset$ and $\leq$ is a relation on $\mathbb{P}$ which is* transitive *and* reflexive*.*

Definition 1.4.2. *Let $\mathbb{P}$ be a partial order, we say that $D \subseteq \mathbb{P}$ is* dense *in $\mathbb{P}$ if*

$$\forall p \in \mathbb{P} \exists q \leq p (q \in D).$$

Definition 1.4.3. *We call a subset G of $\mathbb{P}$ M-generic if*

1. $\forall p \in G \forall q \geq p (q \in G)$
2. $\forall p, q \in G \exists r \in G (r \leq p, q)$, *and*
3. $\forall D \in M (D \text{ is dense} \rightarrow G \cap D \neq \emptyset)$

Conditions 1) and 2) of Definition (1.4.3) tell us that G is a *filter* in $\mathbb{P}$. Condition 3) tells us that G is M-generic. The crucial condition on the filter G is that it is M-generic.

Proposition 1.4.3. *If U is M-generic, then $\in_U$ is a well founded relation.*

Then, exploiting the previous Proposition, we can use the *Mostowski's Collapsing lemma* and collapse $M^{\mathbb{B}}/U$ to a unique transitive $\in$- structure M[U] via the map h defined recursively on $\in_U$ by

$$h(x^U) = \{h(y^U) : y^U \in_U x^U\} = \{h(y^U) : \|y \in x\| \in U\} \tag{1.18}$$

We have that $h : M^{\mathbb{B}}/U \to M[U]$ is a **bijection** satisfying,

$$x^U \in_U y^U \leftrightarrow h(x^U) \in h(y^U).$$

It is now possible to define the map $i : M^{\mathbb{B}} \to M[U]$ by putting

$$i(x) = h(x^U)$$

for $x \in M^{\mathbb{B}}$. If we consider (1.18) we see that, for x a $\mathbb{B}$-name in $M^{\mathbb{B}}$,

$$i(x) = \{i(y) : \|y \in x\| \in U\} \tag{1.19}$$

The map i is called the **interpretation** map of $M^{\mathbb{B}}$ *onto* $M[U]$. It is important to note that the interpretation map i changes depending by our choice of U. For a similar reason we will refer to it as to the mapping i_U.

Definition 1.4.4. *Let M be a countable transitive model of ZFC and $\mathbb{B}$ be a complete Boolean algebra in M. Let U be M-generic for $\mathbb{B}$. Then*

$$M[U] = \{i_U(x) : x \in M^{\mathbb{B}}\}$$

Lemma 1.4.4. *For any formula $\phi(v_1, ..., v_n)$ and any $x_1, ..., x_n \in M^{\mathbb{B}}$,*

$$M[U] \models \phi[i_U(x_1), ..., i_U(x_n)] \leftrightarrow \|\phi(x_1, ..., x_n)\| \in U$$

Proof. It follows from Theorem (1.4.1) together with the fact that h is an **isomorphism** of $M^{\mathbb{B}}/U$ onto $M[U]$. □

Finally we can define *directly* the relationship between M and M[U] fixing

$$j : M \to M[U]$$

by

$$j(x) = i(\check{x})$$

Theorem 1.4.5. *Let U be an M-generic ultrafilter in $\mathbb{B}$. Then,*

(i) $M[U]$ is a transitive $\in$-model of ZFC

(ii) $M[U]$ is the least transitive $\in$-model of ZF which includes M and contains U,

(iii) M and $M[U]$ have the same ordinals.

Proof. (i) Since $M[U]$ is, by construction, isomorphic to $M^{\mathbb{B}}/U$, (i) is an immediate consequence of Corollary (1.4.2) We refer to [5] for a proof of (ii) and (iii) of Theorem (1.4.5). □

As it emerges from the final Theorem (1.4.5), all our previous construction depends on the assumption that *it does exist* an M-generic U. It is now time to secure that clause. **It is not clear that an M-generic ultrafilter exists in general**[23], anyway if M is countable then there is a theorem by Cohen that guarantees the existence of an M-generic ultrafilter. As we said before, we can essentially **reduce** the question to the existence of M-generic filter G for P a partial order (we will need later to regain our more usual dimension of the complete Boolean algebras). Next theorem (1.4.6) is a fundamental result that, as we will see, lie at the very basis of the kind of **generic absoluteness results** that we try to describe in chapter 2 of the present work.

Theorem 1.4.6. *Let M be a countable transitive model of ZFC and let $\mathbb{P} \in M$ be a partial order. Then there is an M-generic $G \subseteq \mathbb{P}$*

Proof. Let $\langle D_i : i < \omega \rangle$ enumerate the sets in M that are dense in $\mathbb{P}$. Pick $p_0 \in D_0$. Since D_1 is dense in $\mathbb{P}$ there is a $p_1 \leq p_0$ such that $p_1 \in D_1$. Continuing in this way[24] we define a sequence of conditions

$$p_0 \geq p_1 \geq p_2 \geq \ldots$$

such that $p_n \in D_n$ for each $n < \omega$. Thus,

$$G = \{p \in G : \exists n < \omega (p \geq p_n)\}$$

is an M-generic. □

We can thus start with a countable standard transitive model of ZFC and a partial order $\mathbb{P}$. In order to construct our generic extension $M[U]$ we need actually to switch from $\mathbb{P}$ to a Boolean algebra $\mathbb{B}$. There is a standard procedure for completing an arbitrary partially ordered set $\mathbb{P}$ to a complete Boolean algebra $\mathbb{B}$. We will describe the procedure in some details in Section 1.6.

1.5 Reconnection to *Cohen's forcing Theorem*

By the definition of an ultrafilter we see that if $p \leq \|\phi\|^{\mathbb{B}}$, then ϕ must be true, in M[U] if $p \in U$. Forcing is actually implicit in our previous construction and it is

[23] As we will see in Chapter 2, here is precisely where the so called *forcing axioms* will enter our picture.

[24] We are making essential use of the Axiom of Choice.

possible, in principle, to produce a proof of Cohen's result without explicitly using the symbol $\Vdash$ of forcing relation at all, referring only to $\|\cdot\|^{\mathbb{B}}$ and Boolean algebra operations. In what follows we will make substantial use of the assumption, *that we still have to prove*, that for each poset $\mathbb{P}$ there is a complete embedding of $\mathbb{P}$ into $\mathbf{B}(\mathbb{P})$, its boolean completion which is unique up to isomorphism.

Definition 1.5.1. (**Forcing relation**) *Given M a transitive model of ZFC, $\mathbb{P} \in M$ a poset, $p \in \mathbb{P}$, and $j : \mathbb{P} \to \mathbf{B}(\mathbb{P})$ such that M models j to be a complete embedding of $\mathbb{P}$ in its boolean completion $\mathbf{B}(\mathbb{P})$. Then for all formulas $\phi(v_1, ..., v_n)$ in the free variables $v_1, ..., v_n$ and all $x_1, ..., x_n \in M^{\mathbb{B}}$, we say that p forces $\phi(x_1, ..., x_n)$ with respect to M, in formula*

$$p \Vdash^M_{\mathbb{P}} \phi(x_1, ..., x_n)$$

iff for every M-generic filter $G \subseteq \mathbb{B}$ with $j(p) \in G$,

$$M[G] \models \phi(i_G(x_1), ..., i_G(x_n)).$$

Theorem 1.5.1. (**Cohen's forcing theorem**) *Let M be a model of ZFC, $\mathbb{B} \in M$ a Boolean algebra such that $\mathbb{B} = \mathbf{B}(\mathbb{P})$ as in Definition* (1.5.1) *Then for all formulas $\phi(v_1, ..., v_n)$ in the free variables $v_1, ..., v_n$ and all $x_1, ..., x_n \in M^{\mathbb{B}}$,*

$$p \Vdash^M_{\mathbb{P}} \phi(x_1, ..., x_n) \leftrightarrow M \models j(p) \leq \|\phi(x_1, ..., x_n)\|.$$

Proof. ($\Leftarrow$)
From Lemma 4.4 we know that $M[U] \models \phi[i_U(x_1), ..., i_U(x_n)] \leftrightarrow \|\phi(x_1, ..., x_n)\| \in U$. This means that if $j(p) \in U$, then, for the properties of the ultrafilter U, $\|\phi(x_1, ..., x_n)\| \in U$, and so we have $M[U] \models \phi[i_U(x_1), ..., i_U(x_n)]$, hence, by Definition (1.5.1), $p \Vdash^M_{\mathbb{P}} \phi(x_1, ..., x_n)$.
($\Rightarrow$)
For the other direction, let's prove the contrapositive. If $j(p) \nleq \|\phi(x_1, ..., x_n)\|$, it is possible to find a generic U such that U contains $j(p) \wedge (\|\phi(x_1..., x_n)\|)^* \neq 0$, and so $j(p) \in U$ but $\|\phi(x_1, ..., x_n\| \notin U$. Then $M[U] \nvDash \phi[i_U(x_1), ..., i_U(x_n)]$ and so $p \nVdash^M_{\mathbb{P}} \phi(x_1, ..., x_n)$. □

Although Definition (1.5.1) makes sense only in V when M-generic filters exist, the forcing relation can always be defined **inside** M[25] . From the previous Theorem (2.2.1), we can conclude that the two distinct approaches to forcing, the one with boolean valued models and the one with the forcing relation $\Vdash$ are equivalent.

[25] We refer the reader to [22] for such a classical definition of $\Vdash$ inside M.

1.6 Boolean completions of posets

The technical material of the present Section is **mainly taken from** lecture notes that were part of the program of the Set theory class taught by Professor Matteo Viale at the University of Turin, during the Spring 2012/2013.

We remind that a *boolean completion* of a poset $\mathbb{P}$ is a complete Boolean algebra, that we denote by $\mathbf{B}(\mathbb{P})$, such that $\mathbb{P}$ is naturally identified with a dense subset of $\mathbf{B}(\mathbb{P})$. We will prove that each poset $\mathbb{P}$ has a boolean completion $\mathbf{B}(\mathbb{P})$ which is unique up to isomorphism. We will construct the boolean completion $\mathbf{B}(\mathbb{P})$ of $\mathbb{P}$ that will be the regular open algebra $RO(\mathbb{P})$. The approach we deal with is topological. We first work with refined posets and show that they have boolean completions. Then, I will show how to map a poset to a refined poset and thus prove the boolean completion for any poset.

Definition 1.6.1. *Let $\mathbb{P} = \langle \mathbb{P}, \leq \rangle$ be a poset. Two elements $p, q \in \mathbb{P}$ are said to be* compatible*, written $p \| q$, if there is $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq$. Two elements $p, q \in \mathbb{P}$ are said to be* incompatible $p \perp q$ *if they are not compatible.*

Now it is possible to give the following key definition.

Definition 1.6.2. *A poset $\mathbb{P}$ is refined if*

$$\forall p, q \in \mathbb{P}[q \not\leq p \rightarrow \exists p' \leq q : p \perp p']$$

Thus $\mathbb{P}$ is refined if, whenever q is not a refinement of p, q has a refinement which is incompatible with p.

We need to find a topology on $\mathbb{P}$, the *order topology*. Let's, first of all, recall the definition of *topological space*.

Definition 1.6.3 (Topological Space)**.** *A* topology *on a set X is a set $\tau \subseteq P(X)$ such that $\emptyset, X \in \tau, U \cap V \in \tau$ whenever U, $V \in \tau$, and $\bigcup_{\alpha \in I} U_\alpha \in \tau$ whenever $\{U_\alpha\}_{\alpha \in I} \subseteq \tau$. The structure (X, τ) is called a* **topological space** *and we will often refer to X as a topological space. The sets in τ are called* **open** *and a set A is called* **closed** *if $X \setminus A$ is open. Sets which are both open and closed are called* **clopen***. One way to present a topology is by specifying a subset B of $P(X)$ and letting τ be the intersection of all topologies containing B. In such a situation B is referred to as the collection of basic open sets and τ is said to be the topology* **generated** *from B.*

Definition 1.6.4. *For each $p \in \mathbb{P}$, put*

$$\bigcirc_p = \{q \in P : q \leq p\}$$

The $\bigcirc_p$ form a base for a topology on $\mathbb{P}$ called the order topology.

Definition 1.6.5 (Regular open algebra). *Consider $\mathbb{P}$ with the order topology, then the regular open algebra $RO(\mathbb{P})$ is the complete Boolean algebra of regular open sets of $\mathbb{P}$, partially ordered by inclusion. Recall that a set $R \in RO(\mathbb{P})$ is regular open if $\mathring{\bar{R}} = R$*[26]*. The algebraic operations in $RO(\mathbb{P})$ are:*

$$\begin{aligned} 0_{RO(\mathbb{P})} &= \emptyset \\ 1_{RO(\mathbb{P})} &= \mathbb{P}, \\ \forall U, V \in RO(\mathbb{P}) U \vee V &= U \cup V, \\ U \wedge V &= U \cap V, \\ \neg_{RO(\mathbb{P})} V &= \mathbb{P} \setminus V, \\ \{U_i : i \in I\} \subseteq RO(\mathbb{P}) \bigvee_{i \in I} U_i &= \bigcup_{i \in I} U_i, \\ \bigwedge_{i \in I} U_i &= (\bigcap_{i \in I} U_i)^{\circ} \end{aligned}$$

.

Definition 1.6.6. *A subset X of a Boolean algebra $\mathbb{B}$ is dense if $0_{\mathbb{B}} \notin X$ and for each $0_{\mathbb{B}} \neq b \in \mathbb{B}$ there is an $x \in X$ such that $x \leq b$.*

1.6.1 ***Refined*** posets and Boolean Completions

Theorem 1.6.1. *Each poset P has a Boolean completion $\mathbf{B}(\mathbb{P})$ which is unique up to isomorphism.*

We will prove the theorem through a sequence of Lemmas.

Lemma 1.6.2. *(i) $\mathbb{P}$ is refined iff $O_p \in RO(\mathbb{P})$ for all $p \in \mathbb{P}$.*

(ii) If $\mathbb{P}$ is refined, the map $p \mapsto O_p$ is an order isomorphism of $\mathbb{P}$ onto a dense subset of $RO(\mathbb{P})$.

[26] Missing definitions of **closure** and **interior** of A for $A \subseteq X$ and (X, τ) a topological space

Proof. (i) Let's see how the interior of the closure of $X \subseteq \mathbb{P}$ is made. In the order topology the least open containing $p \in \mathbb{P}$ is O_p. The closure of X is constituted by all p in $\mathbb{P}$ such that for all open $O \in RO(\mathbb{P})$ containing p, $O \cap X \neq \emptyset$. Remark that $O_p \subset O$ for all O such that $p \in O$. So we must have,

$$\bar{X} = \{p \in \mathbb{P} : O_p \cap X \neq \emptyset\}$$

The interior of $\bar{X}$ is the set $\mathring{\bar{X}}$ of points $q \in \mathbb{P}$ that have a neighbourhood completely contained in $\bar{X}$, that is

$$\mathring{\bar{X}} = \{q \in \mathbb{P} : O_q \subseteq \bar{X}\} \tag{1.20}$$

Remark

$$\begin{aligned} O_q \subseteq \bar{X} &\text{ iff } \forall p' \in O_q(O_q \cap X \neq \emptyset) \\ &\text{ iff } \forall p' \leq q \exists r \in X(r \leq p') \end{aligned}$$

Thus,

$$\mathring{\bar{X}} = \{q \in \mathbb{P} : \forall p' \leq q \exists r \in X(r \leq p')\} \tag{1.21}$$

If we choose $X = O_p$ in (1.21) we have:

$$\begin{aligned} \mathring{\bar{O}}_p &= \{q \in \mathbb{P} : \forall p' \leq q \exists r \leq p(r \leq p')\} \\ &= \{q \in \mathbb{P} : \forall p' \leq q(p \| p')\}. \end{aligned} \tag{1.22}$$

Since O_p is open we have certainly $O_p \subseteq \mathring{\bar{O}}_p$. Suppose now that $\mathbb{P}$ is refined, if $q \notin O_p$, then $q \nleq p$, so there is $p' \leq q$ such that $p \perp p'$ and by (1.22) $q \notin \mathring{\bar{O}}_p$. Therefore $O_p = \mathring{\bar{O}}_p$, that is, $O_p \in RO(\mathbb{P})$.

Conversely, if $O_p \in RO(\mathbb{P})$, then $O_p = \mathring{\bar{O}}_p$, so by (1.22):

$$q \nleq p \rightarrow q \notin O_p \rightarrow q \notin \mathring{\bar{O}}_p \rightarrow \exists p' \leq q(p \perp p').$$

□

P is then refined.

(ii) Let $p, q \in \mathbb{P}$ such that $p \leq q$, then $O_p \subseteq O_q$ and the map $p \mapsto O_p$ is order preserving. To show that the map is an isomorphism, let $O_p = O_q$. If, by contradiction, $p \neq q$ then suppose $q \nleq p$. As $\mathbb{P}$ is refined, there is some $q' \leq q$ such that $q' \perp p$. Thus there is some $q' \in O_q$ such that $q' \notin O_p$, and hence $O_q \neq O_p$.

The density of $\mathbb{P}$ is easy to prove. We show that for every $\emptyset \neq R \in RO(\mathbb{P})$, there is a $p \in \mathbb{P}$ such that $O_p \subseteq R$. That is immediate by (1): if $R \neq \emptyset$, then $\exists p \in R$ and $O_p \subseteq R$.

Corollary 1.6.3. *$\mathbb{P}$ is refined if and only if it is order isomorphic to a dense subset of a complete Boolean algebra*

Proof. If $\mathbb{P}$ is refined, then by Lemma (1.6.2) (ii), P is order isomorphic to a dense subset of the complete Boolean algebra $RO(\mathbb{P})$. Conversely, suppose that $\mathbb{P}$ is order isomorphic to a dense subset D of a complete Boolean algebra $\mathbb{B}$. We may identify $\mathbb{P}$ with D. If $p, q \in \mathbb{P}$ are such that $q \not\leq p$, then $q \wedge_{\mathbb{B}} \neg p \neq 0_{\mathbb{B}}$. Since $\mathbb{P}$ is dense, there is a $p' \in \mathbb{P}$ such that $p' \leq_{\mathbb{B}} q \wedge_{\mathbb{B}} \neg p$. We have then $p' \leq q$ and $p \perp p'$. Therefore $\mathbb{P}$ is refined. $\square$

1.6.2 Uniqueness up to isomorphism

We now prove that a Boolean completion of a partially ordered set is unique up to isomorphism.

Definition 1.6.7. *We say that a pair $\langle \mathbb{B}, e \rangle$ is a Boolean completion of P if the following conditions are met:*

- *$\mathbb{B}$ is a complete Boolean algebra;*
- *e is an order isomorphism of P onto a dense subset of $\mathbb{B}$.*

We will occasionally use the notation $\mathbf{B}(\mathbb{P})$ to denote the Boolean completion of P.

Lemma 1.6.4. *If $\langle \mathbb{B}, e \rangle$ and $\langle \mathbb{B}', e' \rangle$ are Boolean completions of P, then there is an isomorphism between $\mathbb{B}$ and $\mathbb{B}'$ which interchanges $e[P]$ and $e'[P]$.*

Proof. For each $x \in \mathbb{B}$ put

$$P_x = \{p \in P : e(p) \leq x\}.$$

Then the density of $e[P]$ in $\mathbb{B}$ implies that $\bigvee e[P_x] = x$, for each $x \in \mathbb{B}$. In fact, if $0_{\mathbb{B}} < \bigvee e[P_x] < x$, then $x \setminus \bigvee e[P_x] > 0_{\mathbb{B}}$ and, by the $e[P]$ density, there is some $p \in P$ such that $e(p) \leq x \setminus \bigvee e[P_x]$, thus we get a contradiction from $e(p) \leq x$ and $e(p) > \bigvee e[P_x]$.
Define now the following map:

$$\begin{aligned} f : \mathbb{B} &\to \mathbb{B}' \\ x &\mapsto \bigvee e'[P_x] \end{aligned}$$

We get that f is an isomorphism of complete Boolean algebras.

Join preservation :

$$\begin{aligned} f(x) \vee f(y) &= \bigvee e'[P_x] \vee \bigvee e'[P_y] \\ &= \bigvee \{e'(p) : e(p) \leq x\} \vee \bigvee \{e'(q) \leq y\} \\ &= \bigvee \{e'(r) : e(r) \leq x \ or \ e(r) \leq y\} \\ &= \bigvee \{e'(r) : e(r) \leq x \vee y\} \\ &= \bigvee e'[P_{x \vee y}] \\ &= f(x \vee y). \end{aligned}$$

Complement preservation : We prove $f(\neg x) = \neg f(x)$, showing that:

$$f(\neg x) \vee f(x) = 1_{\mathbb{B}'} \text{ and } f(\neg x) \wedge f(x) = 0_{\mathbb{B}'}$$

$$\begin{aligned} f(\neg x) \vee f(x) &= \bigvee \{e'(p) : e(p) \leq \neg x\} \vee \bigvee \{e'(q) : e(q) \leq x\} \\ &= \bigvee \{e'(p) : e(p) \leq \neg x \vee x\} \\ &= \bigvee \{e'(p) : e(p) \leq 1_{\mathbb{B}}\} \\ &= 1_{\mathbb{B}'} \end{aligned}$$

The last equality comes from the fact that $e'(P)$ and $e(P)$ are dense respectively in $\mathbb{B}'$ and $\mathbb{B}$.

$$\begin{aligned} f(\neg x) \wedge f(x) &= \bigvee \{e'(p) : e(p) \leq \neg x\} \wedge \bigvee \{e'(q) : e(q) \leq x\} \\ &= \bigvee \{e'(p) \wedge e'(q) : e(p) \leq \neg x, e(q) \leq x\} \\ &\leq \bigvee \{e'(p) \wedge e'(q) : p \perp q\} \\ &= \bigvee \{e'(p) \wedge e'(q) = 0_{\mathbb{B}'}\} \\ &= 0_{\mathbb{B}'} \end{aligned}$$

$1_{\mathbb{B}}$**is mapped to** $1_{\mathbb{B}'}$:

$$\begin{aligned} f(1_{\mathbb{B}}) &= \bigvee e'[P_1] \\ &= \bigvee \{e'(p) : e(p) \leq 1_{\mathbb{B}}\} \\ &= \bigvee \{e'(p) : p \in P\} \\ &= 1_{\mathbb{B}'}. \end{aligned}$$

Injectivity :

$$\begin{aligned} kerf &= \{x : f(x) = 0_{\mathbb{B}'}\} \\ &= \{x : \bigvee e'[P_x] = 0_{\mathbb{B}'}\} \\ &= \{x : \bigvee \{e'(p) : e(p) \leq x\} = 0_{\mathbb{B}'}\} \\ &= \{x : \{p : e(p) \leq x\} = \emptyset\} \\ &= \{0_{\mathbb{B}}\}. \end{aligned}$$

Commutativity :

$$\begin{aligned} f \circ e(p) &= \bigvee \{e'(q) : e(q) \leq e(p)\} \\ &= \bigvee \{e'(q) : e(q) = e(p)\} \\ &= e'(p) \end{aligned}$$

The last equality comes from the fact that e and e' are order isomorphisms, thus $e(q) = e(p)$, implies $q = p$ and then $e'(q) = e'(p)$.

Completeness : Let $A \subseteq \mathbb{B}$.

$$\begin{aligned} \bigvee f[A] &= \{f(a) : a \in A\} \\ &= \bigvee_{a \in A} \bigvee \{e'(p) : e(p) \leq a\} \\ &\leq \bigvee \{e'(p) : e(p) \leq \bigvee A\} \\ &= f(\bigvee A). \end{aligned}$$

If $f(\bigvee A) \setminus f[A] \neq 0_{\mathbb{B}'}$ for $e'(P)$ density, there is a $e'(r) \leq f(\bigvee A) \setminus f[A]$.
We affirm that $\exists a \in A(e(r) \wedge a \neq 0_{\mathbb{B}'})$. In fact, if by contradiction, $\forall a \in A(e(r) \wedge a = 0_{\mathbb{B}})$, then $e(r) \wedge \bigvee A = 0_{\mathbb{B}}$ and thus $e'(r) = f(e(r)) \wedge f(\bigvee A) = 0_{\mathbb{B}'}$, which is absurd.

Now, by $e(P)$ density, $\exists q \leq r(e(q) \leq e(r) \wedge a)$. Thus $e(q) \leq \bigvee A$ and $e(q) \leq e(r)$ entail respectively that $e'(q) = f(e(q)) \leq f(\bigvee A)$ and $e'(q) = f(e(q)) \leq f(e(r)) = e'(r)$. Finally, $f(\bigvee A) \wedge e'(r) \neq 0_{\mathbb{B}'}$ which is absurd.

Surjectivity : Let $y \in \mathbb{B}'$, then, by the density of $e'[P]$ in $\mathbb{B}'$, there is a $p \in P$ such that $e'(p) \leq y$. The preimage of y is thus

$$f^{-1}(y) = \bigvee \{e(p) : e'(p) \leq y\}.$$

In fact:

$$\begin{aligned} f(\bigvee\{e(p) : e'(p) \leq y\}) &= \bigvee\{f(e(p)) : e'(p) \leq y\} \\ &= \bigvee\{e'(p) : e'(p) \leq y\} \\ &= y. \end{aligned}$$

□

1.6.3 The connection between refined and non refined posets

We consider now non refined posets and show the connection with refined posets.

Lemma 1.6.5. *Let $\langle P, \leqslant_P \rangle$ a partially ordered set, then there is a unique, up to isomorphism, refined poset $\langle Q, \leqslant_Q \rangle$ and an order preserving map j of P onto Q such that:*

$$\forall p, q \in P(p \| q \leftrightarrow j(p) \| j(q)). \tag{1.23}$$

Proof. **Existence** Define the equivalence relation $\backsim$ on P by

$$p \backsim q \leftrightarrow \forall x \in P(p \| x \leftrightarrow q \| x)$$

and let $Q = P/\backsim$. Elements of Q are denoted by $[p]$. The partial order on Q is defined in the following way:

$$[p] \leqslant_Q [q] \text{ iff } \forall x \in P(x \| p \to x \| q)$$

Let j be the map of P onto Q:

$$j : P \to Q$$

$$p \mapsto [p]$$

We have that j is order preserving. Let $p \leq_P q$, then by definition of the equivalence relation, we get $[p] \leq_Q [q]$, that is $j(p) \leq_Q j(q)$.

Let us verify condition (1.23). If $p \| q$, then $\exists r \in P(r \leq_P p \wedge r \leq_P q)$, thus $j(r) \leq_Q j(p)$ and $j(r) \leqslant_Q j(q)$. That is $j(p) \| j(q)$.
Conversely, if $j(p) \| j(q)$, then

$$\exists j(r) \in Q(j(r) \leq_Q j(p) \wedge j(r) \leq_Q j(q)).$$

Now,

$$j(r) \leq_Q j(p) \leftrightarrow [r] \leq_Q [p] \leftrightarrow \forall x \in P(x \| r \to x \| p)$$

Choosing $x = r$ we get that $r \| p$, thus $\exists r' \in P(r' \leq_P r \wedge r' \leq_P p)$. We also have $j(r) \leq_Q j(q)$. Thus $r' \| r$ implies $r' \| q$. Hence $\exists r'' \in P(r'' \leq_P r' \wedge r'' \leq_P q)$ We have obtained that $r'' \leq_P p \wedge r'' \leqslant_P q$, namely $p \| q$.

We check now that $\langle Q, \leq_Q \rangle$ is refined. To this aim, assume that $[q]$ is not a refinement of $[p]$, that is $[q] \not\leq_Q [p]$. Then certainly $\exists x \in P$ such that $x \| q$, but $x \perp p$. As a consequence, $\exists r \in P(r \leq_P x)$ such that $r \leq_P q$ and $r \perp p$, therefore $[r] \leq_Q [q]$ and $[r] \perp [p]$.

Uniqueness. We prove now that $Q = P/\backsim$ is unique up to isomorphism. Let $\{S, \leq_S\}$ be another refined poset with a surjective order preserving map $k : P \to S$ satisfying (13). Consider the map:

$$f : Q \to S$$

$$[q] \mapsto k(q)$$

The map f is well defined: let $[q] = [p]$, if $k(q) \neq k(p)$ then pick $k(q) \not\leq_S k(p)$. By S refinement $\exists k(r) \in S(k(r) \leq_S k(q) \wedge k(r) \perp k(p))$. By (13) applied to the map k, we get $r \| q$ and $r \perp p$, in contradiction with $[q] = [p]$.

As k is surjectve, f is clearly surjective.

The map f is order preserving: let $[q] \leq_Q [p]$, if $k(q) \not\leq_k (p)$, by S refinement $\exists k(r) \in S(k(r) \leq k(q) \wedge k(r) \perp k(p)$. Thus $r \| q \wedge r \perp p$, in contradiction with $[q] \leq_Q [p]$.

Finally, we check f injectivity: let $k(q) = k(p)$. If, by contradiction, $[q] \neq [p]$, then pick $[q] \not\leq_Q [p]$. Then, by Q refinement, $\exists [r] \in Q([r] \not\leq_Q [q] \wedge [r] \perp [p])$. By f order preserving and by (1.23), we have $k(r) \leq_S k(q) \wedge k(r) \perp k(p)$, which is absurd. $\square$

1.6.4 Summary: The main Theorem

Putting all the Lemmas together we conclude that a poset P can always be carried in a complete Boolean algebra in the following way:

Theorem 1.6.6. *Let P a poset, then there is a complete Boolean algebra $\mathbb{B}$, unique up to isomorphism, and a map $j : P \to \mathbb{B}$ such that*

- *$j[P]$ is dense in $\mathbb{B}$;*

- *j is order preserving;*
- $\forall p, q \in P(p \| q \leftrightarrow j(p) \wedge j(q) \neq 0_{\mathbb{B}})$

Proof. If P is refined, we already see the proof in lemma (1.6.2). If P is non refined, then use lemma (1.6.5) and lemma (1.6.2). The uniqueness comes from lemma (1.6.4). $\square$

Chapter 2

Forcing Axioms

2.1 Introduction

In the present chapter we will try to outline some motivations that are behind Viale's project in [37] and [39] of extending Woodin's absoluteness result for $L(\mathbb{R})$ and $H_{\aleph_1}$ to the level of $H_{\aleph_2}$ and $P(\omega_1)$. In particular, we will try to stress some considerations concerning the *essential* use of forcing axioms that is inherent Viale's generic absoluteness results, in order to compare it with Woodin's proposed solution for freezing the theory of H_{ω_2} and his use of axiom $(*)$.

In sections (2.2) and (2.4), we will try to reconstruct some main aspects of the rationale behind Viale's work in [37] and [39]. The main idea that will emerge, as we already noticed in the Introduction to the present work and as we understand it, is Viale's peculiar way of analyzing the notion of generic absoluteness, conceiving it as a phenomenon given in nature that suggests (or imposes) a radical change of perspective on the forcing technique. From this **new** point of view, forcing is not more (or not only) a source of undecidability in mathematics. In fact when it is possible to relieve generic absoluteness for a certain mathematical structure, a different framework appears, where forcing can be exploited and, so we may say, **integrated into the practice of the mathematician** as a strong tool for proving theorems. Remarkably, as we will try to point out, forcing axioms (in combination with large cardinals) appear as effective devices for diagnosing generic absoluteness. Contextually, we will isolate some key features of what in the Introduction to the present work we called *forcing axioms program* for the search of new axioms. Similar key features determine what can maybe be considered as the **view** of that program concerning the question of how to figure out the picture of the set theoretic universe that would accomodate the right structure theory of $P(\omega_1)$ and how to solve, in particular, the Continuum problem. Both Viale's and Woodin's generic absoluteness results, as we understand the case, share the view of the forcing axioms program, and they can be seen as two distinct theories springing out from a similar program. Insofar as we are interested in spell out Viale's and Woodin's absoluteness results in terms of the search of the **right** axioms for the structure theory of $P(\omega_1)$, the possibility to *unify* the two distinct theories will emerge as one of notable *philosophical* importance]

In section (2.3) we will interpose a quick overview of Woodin's alternative strategy for producing generic absoluteness at the level of the structure $(H_{\omega_2}, \in)$. Section (2.3) will then be extended in some details in chapter 3. We conceive it as an interlude within the present chapter that gives to the interested reader the possibility to notice some immediate differences between Viale's and Woodin's approaches.

The rest of the chapter, starting from section (2.5), will be devoted to a presentation of the result in [37] that under a *natural* strengthening of **MM** (denoted as $\mathbf{MM}^{++}$), and modulo large cardinal axioms, the Π_2-theory of $H_{\aleph_2}$ is invariant with respect to forcings $\mathbb{P}$ belonging to a huge natural class of forcing notions Γ.

2.2 Forcing axioms and Cohen's Absoluteness

For Γ a class of partial orders[1], we will consider the following definition of Γ-consistency.

Definition 2.2.1. *Given a model V of ZFC and a family Γ of partial orders in V, we say that V models that ϕ is Γ-consistent if $V \models 1_{\mathbb{P}} \Vdash \phi$ for some $\mathbb{P} \in \Gamma$.*

We can extend the previous definition in a natural way to the notion of Γ- *validity* and to the notion of Γ-*logical consequence* ($\models_\Gamma$). A clear example of our definition is Woodin's Ω-logic: the Γ-logic obtained by letting Γ be the class of *all* partial orders. The **key idea** connected with Γ -logics is the following:

Γ-logics transform forcing into a tool to prove theorems over certain (natural) theories T which extends ZFC.

Probably, the best way to motivate this key idea is to analyze the following so called *Cohen's absoluteness Lemma*.

It can be useful to introduce, as a preliminary, the following important standard classification of formulas.

[1]See Chapter 1, definition (1.4.1) for a formal definition of partial orders.

Definition 2.2.2. (**Levy's Hierarchy**) *A formula is Σ_0 or Π_0 if its quantifiers are bounded, i.e. a Δ_0 formula. Inductively, a formula is Σ_{n+1} if it is of the form $\exists x \phi$, where ϕ is a Π_n, and Π_{n+1} if it is of the form $\forall \phi$ where ϕ is Σ_n. We say that a property (class, realtion) is Σ_n (or Π_n) if it can be expressed by a Σ_n (or Π_n) formula. A function F is Σ_n (Π_n) if the relation $y = F(x)$ is Σ_n.*

There is an alternative way than the one in (1) given in the Introduction to the present work for stratifying the universe of sets, V. Recall preliminary that $trcl(x)$ is the $\subseteq$-least transitive set such that $x \subseteq y$.

Definition 2.2.3. *For any infinite cardinal k, $H_k = \{x : |trcl(x)| < k\}$.*

We can now state Cohen's absoluteness Lemma.

Lemma 2.2.1. (**Cohen's Absoluteness**) *Let T be any theory extending ZFC ($T \supseteq ZFC$), and $\phi(x,p)$ be a Σ_0 formula in the parameter p such that $T \vdash p \subseteq \omega$. Then, the following are equivalent:*

- $T \vdash H_{\omega_1} \models \exists x \phi(x,p)$
- $T \vdash \exists x \phi(x,p)$ *is Ω-consistent.*

(*Observation*: The idea later will be to extend (or to generalize) Cohen's Absoluteness in two distinct ways:

- Improve the set theoretic type of the parameter p (i.e. $T \vdash p \subseteq \kappa$, for κ an arbitrary cardinal), and
- Improve the syntactic complexity according to the Levy's Hierarchy for the formulas ϕ.).

Let's sketch an analysis of the proof of the lemma as it is given in [37]. By section (1.5) of chapter (1), we will feel free in what follows to blur the distinction between posets and Boolean algebras. We will show the proof of the following formulation of the non trivial direction in the previous equivalence.

Proof. Assume $V \models T$, then $H_{\omega_1} \models \exists x \phi(x,p)$ iff $V \models \exists x \phi(x,p)$ is Ω-consistent. (Further assumption: let V be transitive [2]) The forward implication is trivial (choosing the Boolean algebra $\mathbb{B} = \{0,1\}$). For the reverse implication, assume $\exists x \phi(x,p)$ is Ω-consistent in V with $p \in \mathbb{R}^V$. Let $\mathbb{P}$ be a partial order which witnesses this. Exploiting the downward Lowenheim-Skolem Theorem, pick a model in V, $M \in V$,

[2] The proof can be developed also without this assumption. See [37].

such that $M \prec (H_{|\mathbb{P}|^+})^V$, M is countable, $\mathbb{P}$, $p \in M$. Let $\pi_M : M \to N$ be the transitive collapse and let $\mathbb{Q} = \pi(\mathbb{P})$. $\pi(p) = p$ is not moved. Since π_M is an isomorphism of M with N, we have that $N \models (\Vdash_{\mathbb{Q}} \exists x \phi(x, p))$. Let $G \in V$ be N-generic for $\mathbb{Q}$ (G exists because N is countable), then by **Cohen's fundamental theorem of forcing** applied in V to N we get the generic extension $N[G]$ such that $N[G] \models \exists x \phi(x, p)$. Pick up now $a \in N[G]$ such that $N[G] \models \phi(a, p)$. $N[G] \in H_{\omega_1}$, so a as well belongs to $(H_{\omega_1})^V$. Since $\phi(x, p)$ is a Σ_0 formula, V models that $\phi(x, p)$ is absolute between transitive sets $N[G] \subset H_{\omega_1}$ to which a, p belong. We obtain in this way that a witnesses in V that $H^V_{\omega_1} \models \exists x \phi(x, p)$. The thesis follows by completeness of first order logic. □

As it is stated in [37] (section 1, p. 4), if we carefully analyse the proof of the previous lemma, the key observation seems to be the following.

Key observation. *For any poset $\mathbb{P}$ there is some countable $M \prec (H_{|\mathbb{P}|^+})$ such that $\mathbb{P} \in M$ and $\exists$ G M-generic filter for $\mathbb{P}$.*

The final part of the last sentence concerning the existence of the M-generic filter, as it is known, is an outcome of *Baire's Category Theorem* and it is provable from ZFC (See chapter 1 of the present work, theorem (1.4.6)). What is relevant here is that it is possible to *reformulate* the key observation in Cohen's absoluteness lemma by using the machinery of stationary sets.[3] This will give us a deeper way to analyze [4] the proof of the previous lemma (2.2.1). Next theorem (2.2.2), together with corollary (2.2.3), concretely show how, if we appeal to the notion of stationarity, it is possible to rethink and, actually, to *generalize* Cohen's absoluteness lemma adopting parameters of higher type. In order to do this we need to introduce the general definition of *forcing axiom* together with some conceptual considerations.

As it is well explained in Magidor's [30], one intuitive motivation behind the general notion of Forcing Axiom is connected with the idea that the universe of sets is *as rich as possible*, something that can be summed up by the following slogan[5]:

> A set whose existence is possible and has no clear obstruction to its existence exists. (See [30], section 6.2, p. 15.)

The previous slogan appears to be in some sense vague, but as Magidor observes, it is at least possible to specify it and make its content more transparent. The

[3]For a definition of the notion of *stationarity* see Section 6.

[4]A kind of metric of the proof of *Cohen's Absoluteness* Lemma

[5]See [30]

first observation is that a set is simply specified by a certain property. The second observation concerns the expression " is possible". A good approximation to its meaning seems to be in the actual context "can be *forced* to exist". So, the previous slogan can be restated as follows:

> If one can force the existence of a set satisfying a given property and there is no clear obstruction to its existence, then such a set exists. (see [30], section 6.2, p. 15.)

The main object that we add to the universe when we apply forcing is the *generic* filter with respect to the forcing notion $\mathbb{P}$, and because all the other objects introduced by forcing are defined from the generic filter it is usual to state the general form of a forcing axiom as follows.

Definition 2.2.4. (**General definition of forcing axiom**) *We write $FA_\kappa(\mathbb{P})$ (with κ a cardinal) as an abbreviation for the sentence "for every $\mathscr{D} \subseteq \mathrm{P}(\mathbb{P})$ family of open dense sets of $\mathbb{P}$ with $|\mathscr{D}| \leq \kappa$, there exists a filter $G \subset \mathbb{P}$ such that $G \cap \mathrm{D} \neq \emptyset$ for all $\mathrm{D} \in \mathscr{D}$".*

Towards a *generalization* of Cohen's Absoluteness consider the following Theorem[6].

Theorem 2.2.2. *Let $\mathbb{P}$ be a poset and $\theta \geq 2^{|\mathbb{P}|}$ be a cardinal. Then $FA_\kappa(\mathbb{P})$ holds iff there exists an $\mathrm{M} \prec H(\theta)$, $|M| = \kappa, \mathbb{P} \in M, \kappa \subset M$ and a G filter M-generic for $\mathbb{P}$.*

Proof. First, suppose that $FA_\kappa(\mathbb{P})$ holds and let $M \prec H(\theta)$ be such that $\mathbb{P} \in M$, $\kappa \subset M$, $|M| = \kappa$. There are at most κ dense subsets of $\mathbb{P}$ in M, hence by $FA_\kappa(\mathbb{P})$ there is a filter G meeting all those sets. However, G might not be M-generic since for some $D \in M$, the intersection $G \cap D$ might be disjoint from M. Define

$$N = \{x \in H(\theta) : \exists \tau \in M \cap V^{\mathbb{P}} \exists q \in G (p \Vdash \tau = \check{x})\}.$$

Clearly, N contains M (hence contains κ), and the cardinality $\mid N \mid \leq \mid M \cap V^{\mathbb{P}} \mid = \kappa$ since every τ can be evaluated in a unique way by the elements of the filter G. To prove that $N \prec H(\theta)$, let $\exists x \phi(x, a_1, ..., a_n)$ be any formula with parameters $a_1, ..., a_n \in N$ which holds in $H(\theta)$. Let $\tau_i \in M^{\mathbb{P}}$, $q_i \in G$ be such that $q_i \Vdash \tau_i = \check{a}_i$ for all $i < n$. Define $Q_\phi = \{p \in \mathbb{P} : \exists x \; p \Vdash \phi(\check{x}, \tau_1, ..., \tau_n)\}$. This set is definable in M hence $Q_\phi \in M$. Furthermore, $Q_\phi \cap G$ is not empty since it contains any $q \in G$ below all q_i. By fullness in $H(\theta)$, we have that:

$$H(\theta) \models \forall p \in Q_\phi p \Vdash \exists x \in V \phi(x, \tau_1, ..., \tau_n) \Rightarrow$$

[6] See [2], section 4, p. 7

$$H(\theta) \models \exists\tau\forall p \in Q_\phi p \Vdash \phi(\tau, \tau_1, ..., \tau_n) \wedge \exists x p \Vdash \tau = \check{x} \Rightarrow$$

$$M \models \exists\tau\forall p \in Q_\phi p \Vdash \phi(\tau, \tau_1, ..., \tau_n) \wedge \exists x p \Vdash \tau = \check{x}.$$

Fix such $\tau \in M$, by elementarity the last formula holds also in $H(\theta)$ and in particular for any $q \in Q_\phi$. Since the set $\{p \in \mathbb{P} : \exists x \in H(\theta) p \Vdash \check{x} = \tau\}$ is an open dense (below some $q \in G$) set definable in M, there is a $q' \in G$ below q belonging to this dense set, and an $a \in H(\theta)$ such that $q' \Vdash \tau = \check{a}$. Then q', τ testify that $a \in N$ hence the original formula $\exists x\phi(x, a_1, ..., a_n)$ holds in N.
Finally we need to check that G is N-generic for $\mathbb{P}$. Let $D \in N$ be a dense subset $\mathbb{P}$, and $\dot{D} \in M$ be such that $1_\mathbb{P} \Vdash \dot{D}$ is a dense subset of $\mathbb{P}$ and $\{q : \exists D \in V\ p \Vdash \dot{D} = \check{D}\}$ is dense. Since $1_\mathbb{P} \Vdash \dot{D} \cap \dot{G} \neq \emptyset$, by fullness lemma there exists a $\tau \in H(\theta)$ such that $1_\mathbb{P} \Vdash \tau \in \dot{D} \cap \dot{G}$, and by elementarity there is such a τ also in M. Let $q' \in G$ below q be deciding the value of $\tau, q' \Vdash \tau = \check{p}$. Since q' forces that $\check{p} \in \dot{G}$, it must be $q' \leq p$ so that $p \in G$ hence $p \in G \cap D \cap N$ is not empty.
For the converse implication, let M, G be as in the hypothesis of the theorem, and fix a collection $\mathscr{D} = \langle D_\alpha : \alpha < \kappa \rangle$ of dense subsets of $\mathbb{P}$. Define

$$S = \{N \prec H(|\mathbb{P}|^+) : \kappa \subset N \wedge |N| = \kappa \wedge \exists G \text{ filter } N\text{-generic}\}$$

Note that S is definable in M then $S \in M$. Furthermore, since $\mathbb{P} \in M$ so is $H(|\mathbb{P}|^+)$ hence $M \cap H(|\mathbb{P}|^+) \prec H(|\mathbb{P}|^+)$ and $M \cap H(|\mathbb{P}|^+)$ is in S. Given any $C_f \in M$ club on $H(|\mathbb{P}|^+)$, since $f \in M$ we have that $M \cap H(|P|^+) \in C_f$. Then $V \models S \cap C_f \neq \emptyset$ and by elementarity the same holds for M. Thus, S is stationary in M and again by elementarity S is stationary also in V.
Let $N \in$ S be such that $\mathscr{D} \in N$. Since $\kappa \subset N$ and $\mathscr{D}$ has size κ, $D_\alpha \in N$ for every $\alpha < \kappa$. Thus, the N-generic filter G will meet all dense sets in $\mathscr{D}$, verifying $FA_\kappa(\mathbb{P})$ for this collection. $\square$

Actually, what we proved is something more than the equivalence of our Theorem, as the next corollary shows.

Corollary 2.2.3. *Let $\mathbb{P}$ be a poset with $\mathscr{P}(\mathbb{P}) \in H(\theta)$. Then $FA_\kappa(\mathbb{P})$ holds if and only if there are stationary many* M $\prec H(\theta)$ *such that* $|\mathrm{M}| = \kappa$, $\mathbb{P} \in \mathrm{M}$, $\kappa \subset M$ *and G filter M-generic for $\mathbb{P}$.*

Proof. The forward implication has been proved in the first part of Theorem (2.2.2). The converse implication directly follows from the same Theorem. $\square$

We can now see how to generalize Cohen's absoluteness lemma, and we can appreciate the importance of forcing axioms for this generalization.

Lemma 2.2.4. (**Generalized Cohen's Absoluteness**) *Let T be any theory extending ZFC, κ be a cardinal, ϕ be a Σ_1 formula with a parameter p such that $T \vdash p \subseteq \kappa$.*
Then the followng are equivalent:

- $T \vdash \phi(p)$
- $T \vdash \exists \mathbb{P}(1_{\mathbb{P}} \Vdash \phi(p) \wedge FA_\kappa(\mathbb{P}))$.

Proof. The forward implication is trivial. The converse implication follows the proof of lemma (2.2.1). Given p, $\mathbb{P}$ such that $1_{\mathbb{P}} \Vdash \phi(p)$ and $FA_\kappa(\mathbb{P})$ holds, by Corollary (2.2.3) let $M \prec H(\theta)$ be such that $\mid M \mid = \kappa$, $\mathbb{P} \in M$, $\kappa \subset M$ and there exists a G filter M-generic for $\mathbb{P}$. *Since there are stationary many such* M, we can assume that $p \in M$[7]. Let $\pi : M \to N$ be the transitive collapse map of M, then $H = \pi[G]$ is N-generic for $\mathbb{Q} = \pi[\mathbb{P}]$ and $p \subseteq \kappa \subseteq M$ is not moved[8] by π so that $N[H] \models \phi(p)$. Since ϕ is Σ_1 formula, ϕ is upward absolute for transitive models, hence $V \models \phi(p)$. □

It is useful to rewrite the definition of the stationary sets S above in a way that makes it easier to reintroduce the reference to the classes Γ of partial orders and, more generally, to the notion of Γ-logic. This will make even easier to appreciate the role played by forcing axioms in the context of the generalized Cohen's lemma (2.2.4). Consider, for λ a successor cardinal, the following set:

$$S^\lambda_{\mathbb{P}} = \{M \prec H_{|\mathbb{P}|^+} : M \cap \lambda \in \lambda > |M| \wedge \exists G M\text{-generic for } \mathbb{P}\}.$$

We can then recast the equivalence in corollary (2.2.3) in the following effective way. For $\lambda = \nu^+$[9] a successor cardinal, **the following are equivalent**

- $FA_\nu(\mathbb{P})$ holds
- $S^\lambda_{\mathbb{P}}$ is stationary.

We can consider, for example, the set $S^{\aleph_1}_{\mathbb{P}}$ of models of size $\aleph_0$, or the set $S^{\aleph_2}_{\mathbb{P}}$ of models of size $\aleph_1$.[10] In this way, we can reintroduce classes of posets as follows: for $\mathbb{P}$ a poset, we let

$$\mathbb{P} \in \Gamma_{(\nu^+)} \text{ iff } S^{\nu^+}_{\mathbb{P}} \textit{ is stationary.} \tag{2.1}$$

Exploiting the equivalence in (2.2.3) we can transform (2.1) as follows. For $\lambda = \nu^+$ a successor cardinal,

[7] Club argument.

[8] The transitive structure of κ is not changed by the collapse, so that we can think that everything inside κ is *protected* from collapsing by the structure of κ.

[9] Is this necessary?

[10] Actually, the set $S^{\aleph_2}_{\mathbb{P}}$ will contain models of size $\aleph_0$ as well.

$$\mathbb{P} \in \Gamma_\lambda \text{ iff } FA_\nu(\mathbb{P}).$$

Since $FA_{\aleph_0}(\mathbb{P})$, is true for all posets (provably from ZFC), $\Gamma_{\aleph_1}$ is the class of all posets. In this sense, Cohen's absoluteness lemma tells us that if we find in $\Gamma_{\aleph_1} = \Omega$ a $\mathbb{P}$ poset such that $V \models \exists \mathbb{P}(1_\mathbb{P} \Vdash \phi(p))$, where p is a real parameter and ϕ a Σ_1 formula, then $H_{\omega_1} \models \phi(p)$. Similarly, the generalized version of the lemma tells us that, if we find in Γ_{ν^+} a poset $\mathbb{P}$ such that $V \models \exists \mathbb{P}(1_\mathbb{P} \Vdash \phi(p))$, for $p \in H_{\nu^+}$ and ϕ a Σ_1 formula, then $H_{\nu^+} \models \phi(p)$. As it has emerged from the previous results, the possibility to find posets in the classes Γ's essentially depends on the **internal existential assumption** of the forcing axiom $FA_\nu(\mathbb{P})$, and it should appear more manifest now the way in which forcing axioms (and the correlated Γ-logics) **transform forcing into a strong tool for proving theorems**.

There is even a stronger sense in which forcing axioms transform forcing into a tool for proving theorems. As it is stressed in [37] (Introduction, p. 4), many interesting problems of Set Theory are formulated as Π_2-properties of H_{ν^+} for some cardinal ν (for example, the *Suslin's Hypothesis*[11]) and the generalized version of Cohen's lemma that we showed before gives also a powerful general framework to prove in $V \models ZFC$ whether Π_2-properties, $\forall x \exists y \phi(x, y, z)$, (with ϕ Σ_0) holds for some $H^V_{\nu^+}$ with $p \in H^V_{\nu^+}$ replacing z: it suffices to prove that, for any $a \in H_{\nu^+}$, $V \models$ "$\exists y \phi(a, y, p)$ is Γ_{ν^+} consistent". So, if we are in a model V of ZFC where Γ_{ν^+} contains interesting posets, the Γ_{ν^+}-logic seems a powerful tool for studying the Π_2-properties of H_{ν^+}. As Viale points out in [37] (Introduction, p. 5), this is one of the general reasons for the success of Martin's Maximum, **MM**, (which, as we will see, is an instantiation of $FA_{\aleph_1}(\mathbb{P})$ for certain kinds of $\mathbb{P}$'s) in settling relevant problems that can be formulated as Π_2-properties of $H^V_{\aleph_2}$. It can be useful, at this point, to add the following observation.

Observation 1. It is possible to reduce the proof of a Π_2-property, $\forall x \exists y \phi(x, y, p)$, (with $p \in H_{\nu^+}$), to the proof of **the consistency** of a Σ_1-property, $\exists y \phi(a, y, p)$ (for a an arbitrary object in H_{ν^+}) in the way explained above, by means of some $\mathbb{P}$ in the class Γ_{ν^+} such that $\Vdash_\mathbb{P}$ (the Σ_1 formula) $\exists y \phi(a, y, p)$ (let's call it ϕ). In this way, it is possible to prove Π_2-formulas for H_{ν^+}. Anyway, if $\mathbb{P}$ doesn't force (ϕ), it is possible

[11] A posssible formulation of the Suslin's Hypothesis is as follows : " Suppose $(L, \leq_L)$ is a dense linear ordering without endpoints that is complete and has the Suslin property. Then $(L, \leq_L)$ is separable", where a linear ordering $(L, \leq_L)$ has the *Suslin property* if every collection of disjoint non-empty open intervals is countable. We refer the reader to [12] or to Wikipedia for the definitions of the pertinent notions.

that $\mathbb{P}$ forces $(\neg\phi)$. **But it is generally not clear how to transform the proof the consistency of $\neg\phi$ into the proof of the negation of a Π_2-property (that is, into the proof of a Σ_2-property)**.

This observation leads us directly to the question of the 'completeness' of a first order theory T (that axiomatizes set theory) with respect to its semantics given by the class of boolean valued models[12] of T, and it induces us to ask if it is possible to generalize Cohen's Absoluteness in another sense, i.e. that of the complexity of the formulas involved. It appears in some sense natural to ask whether we can extend the absoluteness results relative to the Σ_1 level to more complex levels of formulas in the Levy's Hierarchy. Here is where the so-called *Woodin's program* (as we understand it) enters into the picture.

As it is observed in [39], if we consider the *first order calculus* and *forcing* as the two main tools at the disposal of the working set theorist, then the question of the 'completeness' of T referred to above appears, in the present context, as the question of reducing the gap that still exists between the syntactic notion of derivability and the semantical notion of forceability. This seems to require, as Viale observes in [39], at least the implementation of the following two criteria[13]:

- T is complete with respect to its intended semantics, i.e. for all statements ϕ at most one among $T + \phi$ and $T + \neg\phi$ is forceable.
- Forceability over T should correspond to a notion of derivability with respect to some proof system, eventually derivability with respect to a standard first order calculus for T.

Woodin's idea is to incrementally close the gap between the two notions of derivability and forceability considering, in progression, the theories: $Th(H_{\aleph_1})$, $Th(H_{\aleph_2})$, and so on. In this direction seems to point the following generic absoluteness result (or, we could say, in Viale's terminology, *completeness theorem* with respect to the notion of forceability) which, with an additional assumption on large cardinals, enhances Cohen's result to *any* formula relativized to $L(\mathbb{R})$[14].

Theorem 2.2.5. [**Woodin's Absoluteness**] *Let T be a theory extending ZFC + there are class many Woodin cardinals. Let ϕ be any formula with a parameter p such that $T \vdash p \subseteq \omega$. Then the following are equivalent:*

---

[12]See Chapter 1 for a characterization of the notion of boolean valued model.

[13]See [39], Introduction.

[14]Where $L(\mathbb{R})$ is the smallest model of Zermelo- Fraenkel Set theory which contains all the reals and all the ordinals. *Remarkably* $L(\mathbb{R})$ doesn't satisfy the Axiom of Choice.

- $T \vdash \phi(p)^{L(\mathbb{R})}$
- $T \vdash \phi(\check{p})^{L(\mathbb{R})})$ *is* $\Gamma_{\aleph_1}$[15] *consistent.*

As it is known, we have that $H_{\aleph_1} \subseteq L(\mathbb{R})$ and that $(H_{\aleph_1})^{L(\mathbb{R})} = H_{\aleph_1}$. If we let ψ be an abbreviation for " There is a proper class of Woodin cardinals", we can say that ψ is a *solution* for $H_{\aleph_1}$ with respect to the relation $\vdash$, where, more generally, we can fix the following Woodin's terminology. (See [36], section 2.1.5, p. 89.)

Definition 2.2.5. *ψ is called a solution*[16] *for H_θ with respect to the relation $\rhd \in \{\models, \models_{WF}, \models_\Omega, \vdash, ..\}$ iff for all $\phi \in Th(H_\theta)$,*

$$ZFC + \psi \rhd \ulcorner H_\theta \models \phi \urcorner \textit{ or } ZFC + \psi \rhd \ulcorner H_\theta \models \neg\phi \urcorner$$

So, modulo the method of forcing, large cardinals are a solution for the structure $H_{(\aleph_1)}$; i.e. they decide the theory of $H_{(\aleph_1)}$ with parameters in $H_{(\aleph_1)}$. It should be carefully stressed, anyway, that interpreting all the results we are presenting in this section as *good* solutions for the corresponding theories presupposes the (strong) assumption that the forcing method is the only effective way to generate independence results for Set theory.

The next attempt[17], then, is to extend Woodin's result at the level of $H_{\aleph_2}$ and $P(\omega_1)$, where CH lives. Clearly, it would be nice to find an analog solution that is *compatible* with the one for $H_{\aleph_1}$ and able to *extend* it. As Giorgio Venturi points out in [36], the first natural attempt would be to look for even stronger large cardinal axioms, since that method has been already very successful. A promising result in this direction has been given by the following Theorem by Woodin.

Theorem 2.2.6. *(Woodin) Given a model M of ZFC, we have that CH is a solution for M, with respect to the class of all Σ^1_2 sentence ϕ (i.e. an existential statement of third order arithmetic the same complexity of CH) written in the language of Set theory. Moreover, if a Σ^2_1 statement ψ is another such a solution for M, then $ZFC + \psi \vdash M \models CH$. Under the assumption of the existence of a class of measurable Woodin cardinals, this fact cannot be changed by forcing*

[15] Where $\Gamma_{\aleph_1}$, as we stressed before, is the class of *all* forcing notions. Actually, this class of posets is referred to by Woodin as the class Ω.

[16] It seems relevant to stress here that the notion of "solution", in this context, is a *technical* notion. In this regard, at least from our point of view, there remains a conceptual distinction between that notion and the more *philosophical* notion of "axiom". This point should emerge more clearly in the broader context of our discussion in the last Chapter of the present work

[17] I will follow here some considerations read in [36] (section 2.1.5, p. 91).

Consider now, in conjunction with the previous Theorem, the following **limitative** result by Asperó, Larson, and Moore[18].

Theorem 2.2.7. *(Asperó, Larson, and Moore) There exists sentences ψ_1 and ψ_2 which are Π_2 over the structure $(H_{\aleph_2}, \in, \omega_1)$ such that*

- *ψ_1 can be forced by a proper forcing not adding ω-sequences of ordinals (i.e. ψ_1 is consistent with CH);*
- *if there exists a strongly inaccessible limit of measurable cardinals, then ψ_2 can be forced by a proper forcing which does not add ω-sequences of ordinals (i.e. ψ_2 is consistent with CH);*
- *the conjunction of ψ_1 and ψ_2 implies that $2^{\aleph_0} \neq \aleph_1$.*

Theorem (2.2.7) says that there are two Π_2 sentences (over the structure $H_{\aleph_2}$) that are mutually compatible with CH, but whose conjunction is incompatible with CH. Hence it seems that any attempts to save CH *together with large cardinals* and find a good solution for $H_{\aleph_2}$ is problematic, **if it aims to include all Π_2 statements provably consistent by means of forcing**.

If we come back to the central problem that we posited at the end of the Introduction to the present work, we can collect together some interesting elements. The question there was the following:

Question. What would it require to solve the fundamental equation

$$\frac{L(\mathbb{R})}{AD^{L(\mathbb{R})}} = \frac{P(\omega_1)}{?} \ ? \tag{2.2}$$

All the previous considerations seems to suggest that we should direct our attention towards different kind of principles that **go beyond** *standard* large cardinals and that can settle all possible problems that can be phrased in the structure $(H_{\aleph_2}, \in, \omega_1)$. Summing up, we can fix some plausible criteria from the previous considerations and try to search for principles such that they conform to the following schematization[19]:

- decide the largest possible fragment of $H_{\aleph_2}$,
- negate CH (because of the Theorem 2.2.7),

[18] See for reference [39]

[19] See [36], (section 2.1.5, p.93).

- not (standard) large cardinal axioms (because they are not sensitive concerning the negation or the confirmation of CH),
- extend Woodin results on $H_{\aleph_1}$ (that is, they should be compatible with large cardinal axioms),
- extend Cohen's Absoluteness

The previous schematization, as it seems to us, sums up some of the main aspects that characterize the *view* -as we could say- proper to the *forcing axioms program* for the research of new axioms concerning a **general strategy** for solving the fundamental equation (2.2).[20] We will come back on this point in Chapter 4. In the next Section, before focalizing on Viale's result in [37] and [39], we will briefly sketch the general aspects of the solution offered by Woodin for the structure $H_{\aleph_2}$. Woodin's solution will be further analyzed in Chapter 3 of the present work which represents the *beginning stage* of our attempt to study the properties of the partial order $\mathbb{P}_{max}$ and its correlation with Ω-logic.

2.3 Woodin's solution and the axiom $(*)$

The main reference for this section is [15].

Woodin characterizes generic absoluteness in terms of a *strong* logic, called Ω-logic ($\models_\Omega$)[21].

Definition 2.3.1. *Suppose there is a proper class of strongly inaccessible cardinals. Suppose T is a theory and ϕ is a sentence, both in the language of Set theory. We*

[20] It seems appropriate, at this point, to remind that the so called *forcing axioms program* for the research of new axioms is one among different possibilities. Other different programs are present. Just to mention another one, alternative to the one we are examining in the present work, recall Woodin's **new** approach and his $V = Ultimate\ L$ program. This last represent a *global* approach toward the problem of completing ZFC^* and it is incompatible with the forcing axioms program, since, for example, $V = Ultimate\ L \Rightarrow CH$. There exists, nevertheless, a very interesting possibility to incorporate the main aspects of forcing axioms *inside* the $V = Ultimate\ L$ program. This attempts is called the forcing axioms *envelope perspective*. I came to know about it during my attendance at the Woodin- Koellner seminar in the Fall 2014. As far as I know, there is not published literature yet concerning this possibility. The only published work that I know where there seems to be a possible reference to this perspective is Todorcevic's EFI paper [35]. We will briefly come back on this point in the last Chapter 4 of the present work.

[21] Here Ω denotes the class of *all* posets.

write

$$T \models_\Omega \phi$$

if whenever $\mathbb{P}$ *is a poset,* α *an ordinal and* $G \subset \mathbb{P}$ *is* V*-generic, then*

$$\textit{if } V[G]_\alpha \models T \textit{ then } V[G]_\alpha \models \phi$$

The Ω relation just defined is invariant under forcing, modulo the existence of a proper class of Woodin cardinals, as the following Theorem shows.

Theorem 2.3.1. *(Woodin) Assume there exists a proper class of Woodin cardinals. Suppose* T *a theory,* ϕ *a sentence,* $\mathbb{P}$ *a poset, and* $G \subset \mathbb{P}$ *is* V*-generic. Then*

$$V \models \text{“}T \models_\Omega \phi''$$

iff

$$V[G] \models \text{“}T \models_\Omega \phi''$$

When $T \models_\Omega \phi$ it is said that ϕ is Ω_T-valid and if $T \not\models_\Omega \neg\phi$ it is said that ϕ is Ω_T satisfiable. It is then possible to reconsider the notion of "completeness" of T discussed in the previous section (2.2) in terms of Ω-logic and say that for Γ a set of sentences, T is said to be Ω-complete for Γ if for all $\phi \in \Gamma$ either $T \models_\Omega \phi$ or $T \models_\Omega \neg\phi$[22]. In particular, there are two cases of interest, that is when Γ is the set of sentences of the form $H_{\aleph_2} \models \phi$ and when it is the set of sentences of the form $L(\mathbb{R}) \models \phi$. If we use the notation[23], respectively, $\Gamma(H_{\aleph_2})$ and $\Gamma(L(\mathbb{R}))$ it is then possible to reformulate Theorem (2.2.5) (Woodin's Absoluteness) above as follows.

Theorem 2.3.2. *Assume there is a proper class of Woodin cardinals. Then ZFC is* Ω*-complete for* $\Gamma(L(\mathbb{R}))$*.*

The main interest of Woodin regarding the set $\Gamma(H_{\aleph_2})$ is to find a solution for CH. This last, in fact, can be formulated as a statement inside the set $\Gamma(H_{\aleph_2})$[24]. Woodin formulated the following conjecture regarding a possible solution for the Continuum Hypothesis.

[22]It should be stressed here that the choice to study the "completeness" of T in terms of Ω-completeness is a crucial aspect of Woodin's strategy for solving the continuum problem, and it is in some way in tension with another possible local approach to the problem. This last one, as we will see in the next Section, appeals to forcing axioms and it *restricts* the class of forcing notions to consider (and, consequently, it reduces the semantics by which to evaluate the completeness of T). This seems to us an important point, and one that asks for a clarification of the philosophical presuppositions of the two approaches. We will come back on this in Chapter 4 of the present work.

[23]I'm strictly following here the presentation in [15]

[24]See, for example, [40], (p. 688). See also [24], (section 6, p. 2151).

(**CH Conjecture**) Assume there is a proper class of Woodin cardinals.

1. Then there is an axiom A such that
   - A is Ω_{ZFC}-satisfiable,
   - $ZFC + A$ is Ω-complete for $\Gamma(H_{\aleph_2})$.
2. Any such axiom A has the feature that

$$ZFC + A \models_\Omega \text{“}H_{\aleph_2} \models \neg CH\text{''}.$$[25]

As Koellner points out in [15], a possible way to rephrase the above conjecture is as follows: Call an axiom A ‘good’ if it satisfies (1) above. The conjecture then is:

1. There is a good axiom,
2. All good axioms Ω-imply $\neg CH$.

Woodin has proved the CH-Conjecture assuming another conjecture that in Koellner's paper is called *Strong Ω-conjecture*. The Strong Ω-conjecture is a conjunction of two other conjectures called, respectively, the Ω-conjecture and the statement that *the AD^+-conjecture is Ω-valid*[26]. What is enough to say here is that the Strong Ω-conjecture requires to refer to a *new* notion of provability that Woodin introduced and that is called Ω-provability ($\vdash_\Omega$)[27]. As the next Woodin's Theorem shows, this notion is invariant under forcing.

Theorem 2.3.3. *Assume a proper class of Woodin cardinals. Suppose T a theory, ϕ a sentence, $\mathbb{P}$ a poset and $G \subseteq \mathbb{P}$ is V-generic. Then,*

$$V \models \text{“}T \vdash_\Omega \phi\text{''} \textit{ iff } V[G] \models \text{“}T \vdash_\Omega \phi\text{''}$$

It is also the case that Ω-logic is sound with respect to $\models_\Omega$.

Theorem 2.3.4. *(Woodin) Suppose T is a set of sentences and ϕ is a sentence.*

$$\textit{Assume } T \vdash_\Omega \phi. \textit{ Then } T \models_\Omega \phi.$$

[25] In particular this point gives to the solution of the CH an *inevitable* aspect.

[26] In the present context we skip on this conjecture, and we refer the reader to [15] for a first introduction to some details.

[27] See [15] for a definition of $\vdash_\Omega$, or Chapter 3 of the present work for a partial characterization of it.

The Ω-conjecture concerns the corresponding Completeness Theorem.

(**Ω-conjecture**). Assume a proper class of Woodin cardinals. Then, for each sentence ϕ,

$$\emptyset \models_\Omega \phi$$

iff

$$\emptyset \vdash_\Omega \phi$$

It is possible now to consider Woodin's solution of the CH-conjecture. First of all we need a candidate as the axiom A. This last has been introduced by Woodin as the axiom $(*)$.

Definition 2.3.2. [28] *Let I_{NS} be the non-stationary ideal on ω_1[29]. $(*)$ is the sentence*

For each projective set A, and for each Π_2-sentence ϕ if

$$\text{“}\langle H_{\aleph_2}, \in, I_{NS}, A\rangle \models \phi\text{''}$$

is Ω_{ZFC} consistent, then

$$\langle H_{\aleph_2}, \in, I_{NS}, A\rangle \models \phi$$

(**Note**: ϕ is said to be Ω_{ZFC}-consistent if its negation is not Ω_{ZFC}-provable, i.e. $ZFC \nvdash_\Omega \neg\phi$)

Woodin proved the following remarkable Theorem.

Theorem 2.3.5. *(Woodin) Assume there is a proper class of Woodin cardinals. Then*

- $(*)$ *is Ω_{ZFC}-consistent,*
- *for every ϕ*[30]

$$ZFC + (*) \vdash_\Omega \text{“}H_{\aleph_2} \models \phi\text{''}$$

or

$$ZFC + (*) \vdash_\Omega \text{“}H_{\aleph_2} \models \neg\phi\text{''}.$$

[28] In Chapter 3 we will give a distinct definition of axiom$(*)$.

[29] See section (2.6), definition (2.6.2) of the present work.

[30] With real parameters.

By soundness (Ω-soundness) $(*)$ freezes the theory of $H_{\aleph_2}$. What it is needed in order for proving the first part of the CH-conjecture is to show that $(*)$ is Ω-satisfiable. That is, we need to prove the Ω-conjecture.[31]

Additionally, we have the following essential (for what we said in section (2.2) concerning the general *direction* where to search for new axioms) information.

Theorem 2.3.6. *(Woodin) Assume there is a proper class of Woodin cardinals and that* $(*)$ *holds. Then*

$$2^{\aleph_0} = \aleph_2.$$

We conclude the section giving the following Theorem that confers to the *negation* of CH an inevitable character.

Theorem 2.3.7. *(Woodin) Assume there is a proper class of Woodin cardinals and that the AD^+-conjecture is Ω-provable in ZFC. Suppose A is an axiom such that*

- *A is Ω_{ZFC}-consistent, and*
- *for every sentence ϕ either*

  $$ZFC + A \vdash_\Omega \text{``}H_{\aleph_2} \models \phi''$$

  or

  $$ZFC + A \vdash_\Omega \text{``}H_{\aleph_2} \models \neg\phi''.$$

  Then

  $$ZFC + A \vdash_\Omega \neg CH.$$

2.4 Generic absoluteness and forcing axioms

Historically, forcing axioms appeared to be very effective principles able to settle many problems in different areas of mathematics that were shown to be independent by the method of forcing[32]. This circumstance together with the *maximality* considerations internal to the general notion of forcing axiom, $FA_\kappa(\mathbb{P})$, that we saw above induced different set theorists with a realistic attitude to think of these principles as the *right* principles to *complete* the inherent incomplete axiom system ZFC. What is remarkable is that, as we disclosed at the end of Section (2.2) of the present Chapter, the *strongest* principles among them fit very well with the criteria that we outlined

[31]**Remark**: The Ω-conjecture could be false and yet $(*)$ could be Ω-satisfiable.

[32]See [39] with particular attention to the Introduction.

before and appear, in this way, to be plausible candidates for being a solution at the level of the structure of $H_{\aleph_2}$.

Viale's preliminary considerations concerning Woodin's strategy for fixing $Th(H_{\aleph_2})$ can be summarized saying that even if Woodin's completeness result (2.3.5) is very powerful since it applies to the **largest possible class of models** produced by forcing, there are at least two points that still require clarification. (See [39], Introduction, p. 8).

- The first point concerns, as we saw before, the question of the Ω_{ZFC}-satisfiability of $(*)$.
- The second point is that: " The correctness and completeness result for $(*)$ are with respect to a natural but non constructive proof system and moreover the completeness theorem is known to hold only under certain assumptions on the set theoretic properties of V[33]."

The aim of Viale in [39] is then to focus on the first order theory with parameters in $P(\omega_1)$ of the structure $H_{\aleph_2}$ (or, more precisely, of the structure $L([Ord]^{\leq\omega_1})$[34]) and to show that the strongest forcing axioms in combination with large cardinals give an axiom system T which extends ZFC and which makes $Th(H_{\aleph_2})$ invariant with respect to forcing notions $\mathbb{P}$ that preserve a suitable fragment of T and *for which we can predicate* $FA_{\aleph_1}(\mathbb{P})$[35]. This last requirement could appear in some sense arbitrary, nevertheless, from Viale's point of view, it comes not without reasons. The fact is that as far as our aim is to generalize Cohen's Absoluteness lemma to the $H_{\aleph_2}$-level *allowing parameters* $p \in H_{\aleph_2}$ we have to face the fact that forcing is able to generate strange phenomena, as for example, to force an uncountable set, say ω_1, to be in 1-1 correspondence mapping with ω. Quoting from [39] (Introduction, p. 5) we can say that:

> " On first glance[...] as we expand the language [...] forcing starts to act randomly on the formulae [...] switching the true values of [...] formulae with parameters in ways which it does not seem simple to describe."

If we want to avoid these forcing distortions and consider only forcings which preserve the *intended meaning* of the parameters, as it seems plausible, appealing to forcing axioms seems to be a good move. For the sake of clearness we will quote a passage

[33]Clarification (??)
[34]A similar structure is called in [39] the Chang model.
[35]See [39], Introduction.

from Magidor that we consider really illuminative of this point. (See [30], section 6.2, p. 16).

> "One can introduce by forcing a set which is an enumeration of all the reals of order type ω_1 but also one can introduce by forcing a list of ω_2 different reals. Of course it is inconsistent to have sets satisfying both properties. The statement is even more problematic if in the property one allow parameters, say a given set A. Because suppose our parameter A is uncountable. By using *Levy's collapse forcing* one can make A countable, so introduce a 1-1 mapping between A and ω. Obviously such a mapping does not exists in our universe. I consider forcing axioms as an attempt to try and get a consistent approximation to the above intuitive principle by restricting the properties we talk about and the forcing extensions we use. The restriction of the forcing notions is usually following the intuition of allowing only forcing notions that *do not make a very dramatic change in the universe*, like making an uncountable set countable. This is somewhat similar to restricting in the interpretations of the modalities " it is possible that..." the set of possible universes which are not too different from the current universe. **So, we restrict the forcing which we consider to "mild" forcing extensions**" (My emphasis).

Clearly, an acceptance of this line of argumentation demands **an analysis of** the class $\Gamma_{\aleph_2}$ (i.e. the class of posets $\mathbb{P}$ such that $FA_{\aleph_1}(\mathbb{P})$), and here is where, as it seems to us, the main reason for adopting **MM** (the forcing axiom *Martin's Maximum*) emerges in the context of Viale's argumentation[36]. (For a finer analysis of the variety of the *Baire Category principles* see [2])

Definition 2.4.1. *A poset* $\mathbb{P}$ *is* stationary set preserving (**SSP**) *iff for every stationary set* $S \subseteq \omega_1$, $1_{\mathbb{P}} \Vdash \forall x \subseteq \check{\omega_1}(x \text{ club} \to x \cap \check{S} \neq \emptyset)$.

Definition 2.4.2. *A poset* $\mathbb{P}$ *is* locally **SSP** *iff there exists a* $p \in \mathbb{P}$ *such that* $\mathbb{P} \restriction p = \{q \in \mathbb{P} : q \leq p\}$ *is an* **SSP** *poset.*

The first thing to note is a result by Shelah, which justifies the adjective "maximum". We will state the relevant theorem referring the reader to [2] (section 4, theorem 4.12, p. 9) for a presentation of its proof.

Theorem 2.4.1. *If* $\mathbb{P}$ *is not locally SSP, then* $FA_{\aleph_1}(\mathbb{P})$ *is false.*

[36]We refer the reader to [37] , and [39] on this point.

Using letters A and B respectively as abbreviation for '$\mathbb{P}$ is $\mathbf{SSP}'$ and '$FA_{\aleph_1}(\mathbb{P})$', we have by the previous Theorem that '$\neg A \to \neg B$', which is '$B \to A$'. Actually, what we don't know is if '$A \to B'$', that is whether not preserving stationarity is a necessary condition for a poset $\mathbb{P}$ in order to generate the failure of $FA_{\aleph_1}(\mathbb{P})$. This last question is not decidable in ZFC[37]. Anyway, in models of **MM** we get the following analysis of the class $\Gamma_{\aleph_2}$:

$$\mathbf{SSP}(\mathbb{P}) \leftrightarrow FA_{\aleph_1}(\mathbb{P}),$$

and also, by Shelah's Theorem, **MM** offers the best analysis of the class $\Gamma_{\aleph_2}$ in the sense that it assert the *maximal* principle that $FA_{\aleph_1}(\mathbb{P})$ holds for any poset $\mathbb{P}$ for which we cannot prove that $FA_{\aleph_1}(\mathbb{P})$ fails.

The main result in [37] is to show that a 'natural'[38] strengthening of **MM** (denoted by $\mathbf{MM}^{++}$) which holds in the standard models of **MM**, in combination with Woodin's cardinals, makes $\Gamma_{\aleph_2}$-logic the correct semantics to *completely* describe the Π_2-theory of $H_{\aleph_2}$ (in models of $\mathbf{MM}^{++}$). Here is the statement of the theorem.

Theorem 2.4.2. *(Viale) Assume* $\mathbf{MM}^{++}$ *holds and there are class many Woodin's cardinals. Then,*

$$H^V_{\aleph_2} \prec_{\Sigma_2} H^{V^{\mathbb{P}}}_{\aleph_2}$$

for all stationary set preserving poset $\mathbb{P}$ *which preserve* **BMM**[39].

Theorem (2.4.2) is a partial result in the direction of a **more general** result. In fact, Viale showed in [39] (which is the natural continuation of the work in [37]) that it is possible to extend the previous result to a *full* solution of $H_{\aleph_2}$ (or more precisely of $L(P(\omega_1))$). Here is the statement of the final Theorem.

Theorem 2.4.3. *(Viale) Assume T extends $ZFC+MM^{+++}+$ There are class many super huge cardinals. Then for every formula $\phi(x)$ in the free variable x and every parameter p such that $T \vdash p \in P(\omega_1)$ the following are equivalent:*

- $T \vdash L(P(\omega_1)) \models \phi(p)$
- $T \vdash$ *There is a stationary set preserving partial order P such that $\Vdash_P \phi^{L(P(\omega_1))}$ and P preserves MM^{+++}.*

[37]See [37].

[38] We will se later in which sense $\mathbf{MM}^{++}$ is considered natural.

[39]$\mathbf{BFA}_\nu(\mathbb{P})$ asserts that $H^V_{\nu^+} \prec_{\Sigma_1} V^{\mathbb{P}}$.

It is worth to note that, as it is stressed in [39], the previous result (Theorem 2.4.3) is *sharp* in the sense that it cannot be obtained by appealing to forcing axioms which are just slightly weaker than MM^{++}[40].

We will conclude this section by listing some general considerations, as they naturally come to us after what we said above.

- First of all, the generalization of Cohen's lemma and of Woodin's Absoluteness by the Viale's construction seems to be conceptually (well) motivated. In particular, the restriction to **SSP** posets seems to be an outcome of restricting the attention to forcing notions that preserve the intended meaning of the parameters that we allow in our language, and this seems to be a plausible requirement.

- What is the relation between Viale's absoluteness result and Woodin's absoluteness result regarding the $H_{\aleph_2}$-level? This leads to reflect on at least two points: the status of Ω-logic and the relation between MM^{+++} and Woodin's axiom (*). Regarding the latter point let me quote from a passage in Magidor's paper.

  > "If this conjecture is true (the conjecture being that MM^{++} implies (*))[41] then it will be strong evidence for adapting MM^{++}. I think that a proof of this conjecture will be a confirmation for both MM^{++} (hence for MM) and for (*) in the same sense that the fact two separate scientific theories with desirable consequences can be merged into one unified theory can be considered to be confirmation for both of them"

- Is it possible to "come up with another "complete" axiom system for the theory of $H_{\aleph_2}$ with parameters in $H_{\aleph_2}$ incompatible with MM^{+++} and which allows us to dispose of a completeness and correctness theorem linking provability and forceability?"[42] In other words, in what sense is Viale's solution *inevitable*?

- To what extent, from a philosophical point of view, is a local approach for uncovering the truths of the universe of sets V interesting? How does this approach deal with the *undefinability* of the universe of sets? Is this approach an attempt in some way to describe the universe from below?

[40] As Viale says in [39] it is an open question if $\mathbf{MM}^{+++}$ is really stronger (in terms of consistency) than $\mathbf{MM}^{++}$ in the presence of large cardinals.

[41] Actually, as we understand things, it is an outcome of Viale's results in [39] that the most appropriate setting for the comparison in between $\mathbf{MM}^{+++}$ and $(*)$.

[42] See [39].

Some of the previous points will be directly faced and discussed in the final Chapter 4, which represents the proper philosophical part of the present dissertation.

In the next Sections, we will concentrate on Viale's main Theorem in [37]. What is the strategy for producing its proof? As we will see, the key point of Viale's argumentation in the paper consists in exploiting the interplay between forcing axioms and the so-called *Stationary Tower Forcing*. The interplay between Forcing axioms and Stationary Tower is actually outlined in a Theorem by Woodin that extends the *first* fundamental equivalence between the stationarity of the sets $S^{\lambda}_{\mathbb{P}}$ and forcing axioms (that we saw in Corollary (2.2.3) of the present Chapter) to a **richer** equivalence between forcing axioms and some kind of *embeddings* within the Stationary Tower forcing. So first of all we need to introduce some new concepts.

For the sake of the philosophical developments exposed in Chapter 4, next sections (2.5), (2.7), and (2.8), are (maybe) not strictly necessary, since they deal with only a *piece*[43] of the generic absoluteness diagnosed by Viale in [39]. We consider important to offer a taste of how these generic absoluteness results are produced. Anyway, the reader not interested in the following technical material can take a quick look to section (2.6) where we introduce the notion of *stationary set* and to section (2.8.1), where we report the important definition of the axiom $\mathbf{MM}^{++}$, and then, skip directly to the final section (2.9) where we stress some final considerations that will set up some of the conditions for the philosophical analysis in Chapter 4.

2.5 Preliminaries I: Models of Set theory

The main reference for this section is [17].

Preliminary note: Our aim in what follows is to present some of the notions and the set theoretical tools that are applied in the context of Viale's argumentation in [37] that we will try to partially present starting from Section (2.8) of the present Chapter. We are not giving all the proofs of the lemmas and of the theorems we refer to, and a complete treatment of all the material that will appear in the following Sections in far beyond the range of the present work. Our goal, more modestly, is to put the interested reader in the condition to "follow" the basic ideas that are behind the construction of the fragment of generic absoluteness presented in [37].

[43]Even though, we may say, a crucial one, that is, the level of the Π_2-theory of the structure $(H_{\aleph_2}, \in)$, where, as we stressed in (2.4), lie many open problems of Set theory.

Let's recall the definition of *filter* and *ultrafilter* but in a set theoretical context.

Definition 2.5.1. *Suppose $A \neq \emptyset$. A filter on A is a set $F \subseteq P(A)$ such that:*

1. $A \in F, \emptyset \notin F$,
2. $\forall XY \in F X \cap Y \in F$,
3. $\forall X \in F \forall Y \in P(A)(X \subseteq Y \to Y \in F)$

If F is a filter on A, we say that F is an ultrafilter if it satisfies the following further condition:

$$\forall X \in P(A) X \in F \vee A \setminus X \in F.$$

We say that F is principal if

$$\exists Y \in P(A)(F = \{X \in P(A) : Y \subseteq X\}).$$

We can now introduce the notion of *ultrapower*. We will restrict our attention to *set* models $\mathscr{M} = (M, E)$, with $E \subseteq M \times M$, in the language of Set theory. This is a simpler context for introducing the main idea behind the notion of Ultrapower, and finally (as we will see in the next sections) we will want to apply this technique to *all* the universe of sets, V. The definition of Ultrapower can be extended, with the appropriate meta-mathematical specifications, also to *class* models of Set theory[44].

Let $\mathscr{M}$ be a set model and let U be an ultrafilter on a set A. (There need to be no special relation between A and $\mathscr{M}$). If we consider the set ${}^A M$ of *all* the functions from A into M, we can generate equivalence classes of the elements of ${}^A M$ as follows. For $f, g \in^A M$ let,

$$f \sim_{U,M} g \leftrightarrow \{a \in A : f(a) = f(g)\} \in U.$$

We can then define, for $f \in^A M$,

$$\|f\|_{U,M} =_{def} \{g \in^A M : g \sim_{U,M} f\},$$

and the *quotient* ${}^A M \setminus U =_{def} \{\|f\|_{U,M} : f \in^A M\}$.

We need now to define a *new* binary relation on ${}^A M \setminus U$, that is we need to define $E_{U,M} \subseteq ({}^A M \setminus U, {}^A M \setminus U)^{45}$. Let

[44]See [17] for the details.

[45]We use, as it is common, the notation (a, b) for ordered pairs.

$$\|f\|E_{U,M}\|g\| \leftrightarrow \{a \in A : f(a)E\ g(a)\} \in U$$

(U is an ultrafilter, so $E_{U,M}$ is well defined)

Definition 2.5.2. *Let $\mathscr{M} = (M, E)$ be a set model with U ultrafilter on a set A. We define*

$$\prod_U \mathscr{M} = ({}^A M \setminus U, E_{U,M})$$

to be the ultrapower of $\mathscr{M}$ with respect to U.

The next fundamental Theorem offers an analysis of the notion of *satisfability* in the sense of the ultrapower $\prod_U \mathscr{M}$.

Theorem 2.5.1. *(Lós) Suppose $\mathscr{M} = (M, E)$ is a set model and U is an ultrafilter on A. Then for each formula $\phi(x_1, ..., x_n)$ and for each $f_1, ..., f_n \in^A M$,*

$$\prod_U \mathscr{M} \models \phi[\|f_1\|, ..., \|f_n\|] \leftrightarrow \{a \in A : \mathscr{M} \models \phi[f_1(a), ..., f_n(a)]\} \in U$$

Proof. We refer to [17] for a proof of Theorem (2.5.1) □

There exists an important "connection" between set models and their ultrapowers. A similar connection appears as a Corollary of the previous Theorem (2.5.1) In order to present it, we need, as a preliminary, to define the notion of *elementary embedding.*

Definition 2.5.3. (**Elementary embedding**) *Suppose $\mathscr{M} = (M, E)$, and $\mathscr{N} = (N, F)$ are set models. An elementary embedding of $\mathscr{M}$ into $\mathscr{N}$ is a function,*

$$j : M \to N,$$

such that for all $\phi(x_1, ..., x_n)$ in the language of Set theory, and for all $a_1, ..., a_n \in M$,

$$\mathscr{M} \models \phi[a_1, ..., a_n] \leftrightarrow \mathscr{N} \models \phi[j(a_1), ..., j(a_n)].$$

Definition 2.5.4. *The critical point of a (non trivial)*[46] *embedding j is the least ordinal γ such that $j(\gamma) \neq \gamma$.*

Corollary 2.5.2. *Suppose $\mathscr{M} = (M, E)$ is a set model and U is an ultrafilter on A. Define,*

$$i_{U,M} : \mathscr{M} \to \prod_U \mathscr{M}^{47}$$

[46] That is, not the identity.

[47] Here what we really mean is $i_{U,M} : M \to M^A$.

$$a \mapsto \|c_a\|_{U,M},$$

where $c_a : A \to M$ is the constant function with value a. Then,

$$i_{U,M} : \mathscr{M} \to \prod_U \mathscr{M}$$

is an elementary embedding.

Proof. We want to show that for each $\phi(x_1, ..., x_n)$ formula in LST and for $a_1, ..., a_n \in M$,

$$\mathscr{M} \models \phi[a_1, ..., a_n] \leftrightarrow \Pi_U \mathscr{M} \models \phi[i_{U,M}(a_1), ..., i_{U,M}(a_n)].$$

Fix $\phi(x_1, ..., x_n) \in LST$ and $a_1, ..., a_n \in M$. Then we can prove the Theorem evaluating the right-hand side of the following:

$$\begin{aligned}
&\Pi_U \mathscr{M} \models \phi[i_{U,M}(a_1), ..., i_{U,M}(a_n)] \\
&\leftrightarrow \Pi_U \mathscr{M} \models \phi[\|c_{a_1}\|, ..., \|c_{a_n}\|] \\
&\leftrightarrow \{a \in A : \mathscr{M} \models \phi[c_{a_1}(a), ..., c_{a_n}(a)]\} \in U \\
&\quad (by\ (2.5.1)) \\
&\leftrightarrow \{a \in A : \mathscr{M} \models \phi[a_1, ..., a_n]\} \in U \\
&\leftrightarrow \mathscr{M} \models \phi[a_1, ..., a_n] \\
&(since\ \emptyset \notin U).
\end{aligned}$$

□

We recall the following important Lemma.

Lemma 2.5.3. (**Mostowski Collapse**) *If $\mathscr{M} = (M, E)$ is a well founded set model that satisfies extensionality, then there is a unique transitive set $\mathscr{N}$ and a unique isomorphism*

$$\pi : (M, E) \cong (N, \in)$$

Proof. We refer the reader to [17] for a proof of Lemma (2.5.3) □

When $\prod_U \mathscr{M}$ is well founded and satisfies extensionality, we can then take its *transitive collapse*:

$$\pi_{U,M} : \prod_U \mathscr{M} \cong (N, \in)$$

and **compose** the ultrapower map $i_{U,M}$ with the collapsing map in the following way

$$\pi_{U,M} \circ i_{U,M} : (M, E) \to (N, \in)$$

obtaining a *canonical* embedding between $\in$- models. If we let $j_{U,M} = \pi_{U,M} \circ i_{U,M}$ and $Ult(M, U)$ be the transitive collapse of the ultrapower $\prod_U \mathscr{M}$ we obtain the following final elementary embedding

$$j_{U,M} : \mathscr{M} \to Ult(M, U)$$

As we will see below, an important construction in Set theory is obtained by taking the *direct limit* of a *directed* system of models. I will report here the main definitions and the statement of the main related Theorem as they are presented in [13]. For further details, and for an actual proof of Theorem (2.5.4), we refer the reader to [13]

A *directed set* is a partially ordered set $(D, \leq)$ such that for every $i, j \in D$ there is a k such that $i \leq k$ and $j \leq k$. Let $\{\mathscr{U}_i : i \in D\}$ be a system of models (of Set theory) indexed by a directed set D such that for all $i, j \in D$ if $i < j$ then $\mathscr{U}_i \prec \mathscr{U}_j$ (where $\prec$ denotes the *elementary substructure* relation). Let $\mathscr{U} = \bigcup_{i \in D} \mathscr{U}_i$. It can be proved by induction on the complexity of the formulas that $\mathscr{U}_i \prec \mathscr{U}$ for all i. In general we consider a directed system of models which consists of models $\{\mathscr{U}_i : i \in D\}$ together with elementary embeddings $e_{i,j} : \mathscr{U}_i \to \mathscr{U}_j$ such that $e_{i,k} = e_{j,k} \circ e_{i,j}$ for all $i < j < k$.

The main Theorem here is the following.

Theorem 2.5.4. *If $\{\mathscr{U}_i, e_{i,j} : i, j \in D\}$ is a directed system of models, there exists a model $\mathscr{U}$, unique up to isomorphism, and elementary embeddings $e_i : \mathscr{U}_i \to \mathscr{U}$ such that $\mathscr{U} = \bigcup_{i \in D} e_i(\mathscr{U}_i)$ and that $e_i = e_j \circ e_{i,j}$ for all $i < j$. The model $\mathscr{U}$ is called the direct limit of $\{\mathscr{U}_i, e_{i,j} : i, j \in D\}$.*

2.6 Preliminaries II: stationary sets

In the present Section we introduce the notion of *stationary set* together with some relevant properties for the definition of the stationary tower forcing. The main reference for the material exposed in the present section is [2] (section 2).

Definition 2.6.1. . *Let X be an uncountable set. A set C is a* club *on* P*(X) iff there is a function $f_c : X^{<\omega} \to X$ such that C is the set of elements of* P*(X) closed under f_c, i.e.*

$$C = \{Y \in \mathrm{P}(X) : f_c[Y]^{<\omega 48} \subseteq Y\}$$

[48] Following [2], we will use $[X]^\kappa$ (resp. $[X]^{<\kappa}$) to denote the set of all subsets of X of size κ (resp. less than κ).

A set is stationary *if and only if it intersects every club on* $\mathrm{P}(X)$*.*

Definition 2.6.2. *The* club filter *on X is $CF_X = \{C \subset \mathrm{P}(X) : C$ contains a club$\}$ Similarly, the* non-stationary ideal *on X is $NF_X = \{A \subset \mathrm{P}(X) : A$ is not stationary$\}$.*

Lemma 2.6.1. *CF_X is a countable complete filter on* $\mathrm{P}(X)$*.*

Proof. We refer the reader to [2] (section 2, lemma 2.7, p. 2) for a proof of lemma (2.6.1). □

Lemma 2.6.2. ($Fodor$) *CF_X is normal, i.e. is closed under diagonal intersection*[49]*. Equivalently, every function $f : P(X) \to X$ that is regressive on CF_X-positive set is constant on a CF_X-positive set.*

Proof. We refer the reader to [2] (section 2, lemma 2.9, p. 3) for a proof of (2.6.2). □

Lemma 2.6.3. (**Lifting and Projection**)

Let $X \subseteq Y$ be uncountable sets. If S is stationary on $\mathrm{P}(Y)$, then $S \downarrow X = \{B \cap X : B \in S\}$ is stationary. If S is stationary on $P(X)$, then $S \uparrow Y = \{B \subseteq Y : B \cap X \in S\}$ is stationary.

Proof. For the first part, given any function $f : [X]^{<\omega} \to X$, extend it in any way to a function $g : [Y]^{<\omega} \to Y$. Since S is stationary, there exists a $B \in S$ closed under g, hence $B \cap X \in S \downarrow X$ is closed under f.
The second part of (2.6.3) is more involved and we refer the reader to [2] (section 2, lemma 2.14, p. 4) for a proof of it. □

We refer the reader to [13] for a presentation of the general template for the formulation of large cardinal hypotheses. In particular, we refer the reader to [23] (chapter 1, section 1.5) for an introduction to some properties of Woodin cardinals.

[49] We give the definition of diagonal union and diagonal intersection:

Definition 2.6.3 (Diagonal intersection and Diagonal union)**.** *Given the family $\{S_a \subseteq \mathrm{P}(X) : a \in X\}$, the diagonal union of the family is $\nabla_{a \in X} S_a = \{z \in P(X) : \exists a \in z\ z \in S_a\}$, and the diagonal intersection of the family is $\triangle_{a \in X} S_a = \{z \in P(X) : \forall a \in z\ z \in S_a\}$.*

2.7 The Stationary Tower

The main reference for this section is [23] [Chapter 2]

We are going to introduce the most relevant notion of the chapter.

Definition 2.7.1 (**Alternative definition of stationarity**). *Say that* $\bigcup a \neq \emptyset$. *Then a is* stationary *if it is stationary in* $\mathscr{P}(\bigcup a)$.

Definition 2.7.2 (The stationary tower). *Suppose that* κ *is a strongly inaccessible cardinal. The (full) stationary tower (up to* κ*) is the following partial order, denoted by* $\mathbb{T}_{<\kappa}$. *Conditions are those* $a \in V_\kappa$ *which are stationary subsets of* $\mathscr{P}(\cup a)$. *Given such* a, b, $a \geq b$ *if and only if* $\cup a \subseteq \cup b$ *and* $Z \cap (\cup a) \in a$ *for each* $Z \in b$[50].

We can now consider how to define a *generic*elementary embedding by taking the ultrapower of the universe V by a **generic object** G defined on the Stationary Tower forcing.

Say that $G \subseteq \mathbb{T}_\infty$ is V-generic if $G \cap D \neq \emptyset$ for each definable dense class $D \subseteq \mathbb{T}_\infty$ allowing arbitrary sets as parameters.

Let κ be a strongly inaccessible cardinal, and suppose that $G \subseteq \mathbb{T}_{<\kappa}$ is V-generic. It is possible to associate to G a generic elementary embedding $j : V \to (M, E)$, where (M, E) and j are defined as a limit of generic ultrapowers, as follows.
For each nonempty $X \in V_k$ the generic G **induces** a V-ultrafilter U_X on $\mathscr{P}(X)$ in the following way. For each $b \in \mathbb{T}_{<\kappa}$ with $X \subseteq \cup b$, let

$$b_X = \{Z \cap X : Z \in b\}.$$

Note that by projection b_X is stationary in $\mathscr{P}(X)$. Let

$$U_X = \{b_X : b \in G \wedge X \subseteq \cup b\}.$$

Proposition 2.7.1. *For each nonempty* $X \in V_\kappa$, U_X *is a V-ultrafilter on* $\mathscr{P}(X)$ *and* U_X *extends the club filter on* $\mathscr{P}(X)$.

[50]In this sense we see how the notions of projection and lifting are the two main operations for **climbing** the stationary tower.

Proof. That U_X extends the club filter follows immediately from projection and the definitions of U_X and b_X. To see that U_X is a V-ultrafilter, it has to be shown (working in V) that for each $b \in \mathbb{T}_{<\kappa}$ with $X \in \cup b$ and each $S \subseteq \mathscr{P}(X)$ there is a $b' \leq b \in \mathbb{T}_{<\ \kappa}$ such that either $b'_X \subseteq S$ or $b'_X \cap S = \emptyset$. By the genericity of G, then, we will have shown that U_X is an ultrafilter.
Fix such b, S. Let

$$b^0 = \{Z \subseteq \cup b : Z \cap X \in S\}$$

and

$$b^1 = \{Z \subseteq \cup b : Z \cap X \notin S\}.$$

Then $b^0_X \subseteq S$ and $b^1_X \subseteq \mathscr{P}(X) \setminus S$. Since b is stationary and $b = b^0 \cup b^1$, at least one of b^0 and b^1 must be stationary. But then we are done, since if b^0 is stationary then $b^0 \leq b$ in $\mathbb{T}_{<\kappa}$ and if b^1 is stationary then $b^1 \leq b$ in $\mathbb{T}_{<\kappa}$. $\square$

Since we know, thanks to Proposition (2.7.1), that for each $\emptyset \neq X \in V_\kappa$ we can take the ultrafilter U_X defined as before, it is possible to apply here the *kind of* ultrapower construction[51] that we sketched in Section (2.5). For each nonempty $X \in V_\kappa$ we can build $\prod_{U_X} V = (M_X, E_X)$, where V is the *class* of all sets, in the following way. As domain of $\prod_{U_X} V$ we take the *space* of function ${}^{P(X)}V$, and we then use the ultrafilter U_X to generate equivalence classes of elements $f \in^{P(X)} V$. First of all, we define the following equivalence relation. For $f, g \in^{P(X)} V$,

$$f \sim_{U_X,V} g \text{ iff } \{A \in P(X) : f(A) = g(A)\} \in U_X,$$

and, as usual, we then take the equivalence class

$$\|f\|_{U_X,V} =_{def} \{g \in^{P(X)} V : g \sim_{U_X,V} f\}.$$

We obtain that

$$\{\|f\| : f \in^{P(X)} V\} =^{P(X)} V \setminus U_X.$$

We define over our quotient ${}^{P(X)}V \setminus U_X$ the relation $E_{U_X,V}$ as

$$\|f\|_{U_X,V} E_{U_X,V} \|g\|_{U_X,V} \text{ iff } \{A \subseteq P(X) : f(A) E g(A)\} \in U_X.$$

Finally, let

$$\textstyle\prod_{U_X} V = ({}^{P(X)}V \setminus U_X, E_{U_X,V}) = (M_X, E_X)$$

[51] Readapted however to the case of class models

If we apply now Corollary (2.5.2), we can define an elementary embedding j from V into (M_X, E_X). For a an element in V, let $c_a : P(X) \to V$ be the constant function with value a. Define then the function,

$$j_X : V \to (M_X, E_X)$$

$$a \mapsto \|c_a\|_{U_X,V}$$

If we reflect on the operations of *projection* and *lifting* as defined before in lemma (2.6.3) , it is possible to see how to isolate a kind of "proximity" between ultrafilters U_X and U_Y such that $X \subseteq Y$. The next Proposition specifies this proximity between U_X and U_Y and gives us the basis for defining a *rich* interrelation between the ultrapowers of V that we can take by U_X.

Proposition 2.7.2. *Suppose X and Y are non empty sets in V_k with $X \subseteq Y$. Then for each $S \subseteq \mathscr{P}(X)$, $S \in U_X$ if and only if $\{Z \subseteq Y : Z \cap X \in S\} \in U_Y$.*

For nonempty sets $X, Y \in V_\kappa$ with $X \subseteq Y$ define the map $j_{XY} : M_X \to M_Y$ by letting

$$j_{XY}(\|f\|_{U_X}) = \|f_Y\|_{U_Y},$$

where $f_Y : \mathscr{P}(Y) \to V$ is defined by letting $f_Y(Z) = f(Z \cap X)$. This defines a *directed system* of embeddings indexed by the nonempty members of V_κ.

Applying Theorem (2.5.4), we can define (M, E) to be the *limit* of the family

$$\langle M_X, j_X, j_{XY} : X, Y \in V_k \setminus \emptyset, X \subseteq Y \rangle,$$

and let j be the corresponding limit of the j_X's. For each $a \in G$ and $f : a \to V$ in V, we let $[f]_G$ denote the member of M represented by f.

2.8 A fragment of Generic Absoluteness

We basically have now some rough elements to analyze in some more details Viale's result in [37] on the Π_2-theory of $H_{\aleph_2}$.

For any regular cardinal λ,

$$R_\lambda = \{X : X \cap \lambda \in \lambda \text{ and } |X| < \lambda\}.$$

Let $\delta > \lambda$ be a Woodin cardinal, and let $\mathbb{T}^{\lambda}_{<\delta}$ be the stationary tower whose elements are stationary sets $S \in V_\delta$ such that $S \subseteq R_\lambda$ with order given by $S \leq T$ if, letting $X = \cup(T) \cup \cup(S)$, S^X is contained in T^X *modulo club* **.

$\mathbb{T}^{\lambda}_{<\delta}/\equiv$ (where $\equiv$ is the equivalence relation induced by its order) can be seen seen as a $<\delta$- complete boolean algebra whose *positive* elements give a forcing which is the *separative* quotient of $\mathbb{T}^{\lambda}_{<\delta}$. It is fine thus to confuse $\mathbb{T}^{\lambda}_{<\delta}/\equiv$with $\mathbb{T}^{\lambda}_{<\delta}$. As we saw before, if we let G be V-generic for $\mathbb{T}^{\lambda}_{<\delta}$, then G induces a direct limit ultrapower embedding

$$j_G : V \to Ult(V, G),$$

where $[f]_G \in Ult(V, G)$ if $f : \mathscr{P}(X_f) \to V$ in V and $[f]_G\ R_G\ [h]_G$ iff for some $\alpha < \delta$ such that X_f, $X_h \in V_\alpha$ we have that

$$\{M \prec V_\alpha : f(M \cap X_f)\ R\ h(M \cap X_h)\} \in G.$$

If *Ult(V, G)* is well founded it is customary to identify *Ult(V, G)* with its transitive collapse. If we assume the existence of Woodin cardinals, then $Ult(V, G)$ is well founded, as the next Theorem states. [52]

Theorem 2.8.1. *(Woodin) Let δ be a Woodin cardinal and let $\mathbb{T}_{<\delta}$ be the stationary tower forcing. Let G be a generic filter on $\mathbb{T}_{<\delta}$, and let $j_G : V \to Ult(V, G)$ be the canonical elementary embedding into the generic ultrapower. Then*

- *(i) Ult (V, G) is well-founded.*
- *(ii) $j_G(\lambda) = \delta$, for λ a successor cardinal.*
- *(iii) In $V[G]$, the model $Ult(V, G)$ is closed under $<\delta$-sequences.*

Given the previous results on stationary tower embeddings, it is possible to see that $Ult(V, G)$ is well founded and, especially, we can obtain the following essential identifications:

$$X = [\{\langle M, \pi_M(X)\rangle : M \prec V_\alpha, X \in M\}]_G$$

and

$$j_G[X] = [\{\langle M, X\rangle : M \prec V_\alpha, X \in M\}]_G.$$

[52] I need some clarifications concerning this important theorem. The following Theorem, we think, contain an interesting example of what it is meant by the expression *Ultrapower analysis*.

In particular these identifications show that[53] :

(a) $(H_{j_G(\lambda)})^{M[G]} = V_\delta[G] = (H_\delta)^{V[G]}$.[54]

(b) $j_G \restriction H^V_\theta \in Ult(V,G)$ and $j_G[H^V_\theta] \prec H^{Ult(V,G)}_{j_G(\theta)}$ for all $\theta < \delta$[55]

(c) $j_G \restriction H^V_\lambda$ is the identity and witness that $H^V_\lambda \prec H^{V[G]}_{j_G(\lambda)}$ [56]

We know from Corollary (2.2.3) that if we let $\lambda = \nu^+$ be a successor cardinal, then the following are equivalent

- $FA_\nu(\mathbb{P})$ holds
- $S^\lambda_{\mathbb{P}}$ is stationary.

It is now possible generalize Corollary (2.2.3) by giving a theorem that essentially **justifies** the introduction of the stationary tower forcing into the present construction. This is the richer equivalence we refer to at the end of Section (2.4). A similar equivalence is also the reason why, as it seems to us, Viale states in [37] that Woodin cardinals are essentially forcing axioms.

Theorem 2.8.2 (Woodin)**.** *Assume V is a model of ZFC+ there are class many Woodin cardinals, and $\lambda = \nu^+$ is a successor cardinal in V.*
Then the following are equivalent for any partial order $P \in V$:

[53]**Very provisional** : it seems the case to stress here an important distinction. The notation $(H_{j_G(\lambda)})^{M[G]}$ as displayed in line (a) means "$H_{j_G(\lambda)}$ computed in $M[G]$". The latter statement, we believe, has to be understood as follows: " consider the object $H_{j_G(\lambda)}$ in V, and see how *this* object in computed in $M[G]$". That is, consider under which form it is possible to find again the same object in $M[G]$ (maybe it will reappear in M[G] as H_η for some η in $M[G]$). When we fix the identity, $(H_{j_G(\lambda)})^{M[G]} = (H_\delta)^{V[G]}$ it means, we think, the following: take (in V) the object $H_{j_G(\lambda)}$ and consider how this same object reappears in $M[G]$. Then, take H_δ in V and consider how it reappears in $V[G]$. The identity means that $H_{j_G(\lambda)}$ reappears in $M[G]$ exactly how H_δ reappears in $V]G]$. Thus, $M[G]$ computes $H_{j_G(\lambda)}$ as $V[G]$ computes H_δ. *But* the H_δ of $V[G]$, that is $H^{V[G]}_\delta$, or the $(H_{j_G(\lambda)}$ of $M[G]$, that is $H^{M[G]}_{j_G(\lambda)}$ are (possibly) different objects from what $V[G]$ computes as H_δ and from what $M[G]$ computes as $H_{j_G(\lambda)}$. We believe that maintaining this conceptual distinction is fundamental for following the argumentation that lies behind lines (a)-(c) of the schematization.

[54]H_δ and V_δ coincides when δ is *strongly inaccessible*

[55]Since $Ult(V,G)$ is definable inside $V[G]$, we can redefine it in $V[G]$ and $V[G]$ has the "power" to see that $Ult(V,G)$ is closed under sequences $< \delta$. This element is essential for stating in line (b) that $j_G \restriction H^V_\theta \in Ult(V,G)$. The fact that $j_G[H^V_\theta] \prec H^{Ult(V,G)}_{j_G(\theta)}$ for all $\theta < \delta$ depends also by the fact that $j_G : V \to Ult(V,G)$ is an elementary embedding.

[56] On the basis of (a), we know that the way in which $V[G]$ computes H_δ is the same as the way in which $M[G]$ computes $H_{j_G(\lambda)}$.

1. S^{λ}_{P} *is stationary.*

2. $FA_{\nu}(P)$ *holds.*

3. *There is a locally complete embedding*[57] *of* $\mathbb{P}$ *into* $\mathbb{T}^{\lambda}_{<\delta} \restriction S$ *for some Woodin cardinal* $\delta >\mid P \mid$ *and some* $S \in \mathbb{T}^{\lambda}_{<\delta}$.

2.8.1 $\mathbf{MM}^{++}$

We can now recall the content of the forcing axiom called Martin's Maximum, **MM**. Let **SSP** denote, as before, the class of posets which preserve stationary subsets of ω_1. **MM** asserts that $FA_{\aleph_1}(\mathbb{P})$ holds for all $\mathbb{P} \in \mathbf{SSP}$. Next Theorem offers an analysis of **MM** in terms of embeddings into the stationary tower forcing and collects together some essential informations.

Theorem 2.8.3. *Assume there are class many Woodin cardinals. Then:*

1. $\Gamma_{\aleph_1}$ *is the class of all posets and for any poset* P *there is a regular embedding into* $\mathbb{T}^{\aleph_1}_{<\delta}$ *for any Woodin cardinal* $\delta >\mid P \mid$.

2. $\mathbb{T}^{\aleph_2}_{<\delta} \in \mathbf{SSP}$ *for any Woodin cardinal* δ.

3. **MM** *holds if and only if* **SSP** *is the class of all posets which regularly embeds into* $\mathbb{T}^{\aleph_2}_{<\delta} \restriction S$ *for some Woodin cardinal* δ *and* $S \in \mathbb{T}^{\aleph_2}_{<\delta}$.*(Foreman, Magidor, Shelah).*

As Viale observes in [37], in the context of the development of the consistency proof for **MM** produced by Magidor, Foreman, and Shelah (exploiting the equivalence in point (3) of Theorem (2.8.3)), it was noticed that the *standard* model offered for **MM** gives some more informations on the nature of the embedding i of $\mathbb{P} \in \mathbf{SSP}$ into $\mathbb{T}^{\aleph_2}_{<\delta} \restriction S$ for some $\delta > |\mathbb{P}|$ Woodin cardinal than the one contained in point 3 of Theorem (2.8.3). The standard model for **MM** provided by Magidor, Foreman, and Shelah shows that the embeding i that we can get has some interesting properties. In fact, what we actually can get is, properly, a complete embedding $i : \mathbb{P} \to \mathbf{B}(\mathbb{T}^{\aleph_2}_{<\delta} \restriction T)$[58] with a *quotient forcing* $(\mathbb{T}^{\aleph_2}_{<\delta} \restriction T) \setminus i[\mathbb{P}]$ such that $\mathbb{P}$ forces it to be stationary set preserving. The general notion of quotient forcing $\mathbb{B} \setminus i[\mathbb{Q}]$ appears

[57] We say that $\mathbb{P}$ *completely embeds* into $\mathbb{Q}$ if there is a map $i : \mathbb{P} \to \mathbb{B}(\mathbb{Q})$ (where $\mathbb{B}(\mathbb{Q})$ is the *boolean completion* of $\mathbb{Q}$) which preserves the order relation and maps maximal antichains of $\mathbb{P}$ into maximal antichains of $\mathbb{B}(\mathbb{Q})$. *Regular embeddings* and *locally complete* embeddings are correlated notions for which I refer to [37]

[58] Boolean completion

in the context of the *iterated* forcing technique and it is basically a forcing notion such that $\mathbb{Q} * (\mathbb{B} \setminus i[\mathbb{Q}])$ is forcing equivalent to $\mathbb{B}$[59]. All this is **deeply connected** with the introduction of $\mathbf{MM}^{++}$. First of all, it seems possible to say that, behind the introduction of $\mathbf{MM}^{++}$ there is an even better approximation of the intuitive concept that, as we saw in Section (2.2) , is at the very basis of the introduction of the *general* notion of forcing axiom. Recall first of all the intuitive concept:

If one can force the existence of a set satisfying a given property and there is no clear obstruction to its existence, then such a set exists.

We can now give the following definition of $\mathbf{MM}^{++}$ as it appears in [37]. Let's recall, first of all, from section (2.8) that for any regular cardinal λ,

$$R_\lambda = \{X : X \cap \lambda \in \lambda \text{ and } |X| < \lambda\}.$$

Definition 2.8.1. $\mathbf{MM}^{++}$ *holds if T_P is stationary for all $P \in$ **SSP** where* $\mathrm{M} \in T_P$ *iff*

- $\mathrm{M} \prec H_{|\mathbb{P}|^+}$ *is in* $R_{\aleph_2}$,
- *There is an M-generic filter H for $\mathbb{P}$ such that, if $G = \pi_M[H]$, $\mathbb{Q} = \pi_M(\mathbb{P})$ and $N = \pi_M[M]$, then $\sigma_G : N^{\mathbb{Q}} \to N[G]$ is an evaluation map such that $\sigma_G(\pi_M(\dot{S})$ is stationary for all $\dot{S} \in \mathrm{M}$ $\mathbb{P}$-name for a stationary subset of ω_1.*

The basic idea, as it seems to us, is that the object G that we can add to our ground model is **correct** (via evaluation map) about subsets of ω_1 being stationary. As Viale points out in [37] "*we shall call correct M-generic filter for $\mathbb{P}$ any M-generic filter H as above*".

Theorem 2.8.4. *(Foreman, Magidor, Shelah)* $\mathbf{MM}^{++}$ *is relatively consistent with respect to the existence of a supercompact cardinal.*

The next Theorem reveals the **strict connection** between the formulation of $\mathbf{MM}^{++}$ and the nature of the embedding $\mathbb{P} \to \mathbb{T}^{\aleph_2}_{<\delta} \restriction S$ (for some Woodin cardinal δ and $S \in \mathbb{T}^{\aleph_2}_{<\delta}$) in the standard model offered by Magidar, Foreman, and Shelah for the consistency proof of $\mathbf{MM}$.

Theorem 2.8.5. *Assume there are class many Woodin cardinals. Then the following are equivalent:*

[59] Clarification

1. $\mathbf{MM}^{++}$ *holds.*

2. *For every Woodin cardinal δ and every stationary set preserving poset $\mathbb{P} \in V_\delta$, there is a complete embedding $i : \mathbb{P} \to \mathbb{B} = \mathbf{B}(T_\delta^{\aleph 2} \restriction T)$ for some stationary set $T \in V_\delta$ such that*

$$\Vdash_{\mathbb{P}} \mathbb{B} \setminus i[\mathbb{P}] \text{ is stationary set preserving.}$$

2.8.2 Π_2-Absoluteness for $\boldsymbol{Th}(H_{\aleph_2})$

We have now all the basic ingredients for introducing the main theorem of the present Section, that is also the main result in [37]. As we said above in Section (2.4), this is a *partial* result in the general context of the project that Viale undertook for generalize Woodin's absoluteness result for $H_{\aleph_1}$ and $L(\mathbb{R})$ to a **full** generic absoluteness result concerning the structure theory of $H_{\aleph_2}$[60], and that Viale presents in [39].

Proof. (Π_2-Absoluteness of the theory of H_{ω_2} in models of $\mathbf{MM}^{++}$). Recall the statement of Theorem (2.4.2). Assume $\mathbf{MM}^{++}$ holds in V and there are class many Woodin cardinals. Then the Π_2 theory of $H_{\aleph 2}$ with parameters **cannot be changed** by stationary set preserving forcings which preserve **BMM**[61].

Let $P \in M$ such that V^P models **BMM**. Let δ be a Woodin cardinal larger than $|P|$. By Theorem there is a complete embedding $i : P \to Q = \mathbb{T}_{<\delta} \restriction T_P$ for some stationary set $T_P \in V_\delta$ such that

$$\Vdash_P Q \setminus i[P] \textit{is stationary set preserving.}$$

Now let G be V-generic for Q and $H = i^{-1}[G]$ be V-generic for P. Then $V \subset V[H] \subset V[G]$ and V[G] is a generic extension of V[H] by a forcing which is stationary set preserving in V[H]. Moreover by Woodin's theorem on stationary tower forcing we have that $H_{\aleph_2}^V \prec H_{\aleph_2}^{V[G]}$.
We show that

$$H_{\aleph_2}^V \prec_{\Sigma_2} H_{\aleph_2}^{V[H]}.$$

This will prove the theorem, modulo standard forcing arguments.
We have to prove the following for any Σ_0-formula $\phi(x, y, z)$:

1. If

$$H_{\aleph_2}^V \models \exists y \forall x \phi(x, y, p)$$

[60] More precisely, the structure theory of the Chang model $L([Ord]^{\leq\omega_1})$.
[61] $\mathbf{BFA}_\nu(\mathbb{P})$ asserts that $H_{\nu^+}^V \prec_{\Sigma_1} V^{\mathbb{P}}$.

for some $p \in H^V_{\aleph_2}$, then also

$$H^{V[H]}_{\aleph_2} \models \exists y \forall x \phi(x, y, p).$$

2. If

$$H^V_{\aleph_2} \models \forall y \exists x \phi(x, y, p)$$

for some $p \in H^V_{\aleph_2}$, then also

$$H^{V[H]}_{\aleph 2} \models \forall y \exists x \phi(x, y, p).$$

To prove 1 we note that for some $q \in H^V_{\aleph 2}$ we have that

$$H^V_{\aleph_2} \models \forall x \phi(x, q, p).$$

Then, since

$$H^V_{\aleph_2} \prec H^{V[G]}_{\aleph_2},$$

we have that

$$H^{V[G]}_{\aleph_2} \models \forall x \phi(x, q, p).$$

In particular, since $q, p \in H^{V[H]}_{\aleph_2}$ and $H^{V[H]}_{\aleph_2}$ is a transitive substructure[62] of $H^{V[G]}_{\aleph_2}$, we get that

$$H^{V[H]}_{\aleph_2} \models \forall x \phi(x, q, p)$$

as well. The conclusion now follows.
To prove 2 we note that, since

$$H^V_{\aleph_2} \prec H^{V[G]}_{\aleph 2},$$

we have that

$$H^{V[G]}_{\aleph_2} \models \forall y \exists x \phi(x, y, p).$$

In particular we have that for any $q \in H^{V[H]}_{\aleph_2}$ we have that

$$H^{V[G]}_{\aleph_2} \models \exists x \phi(x, q, p).$$

[62] Generally, we have the following Lemma.

Lemma 2.8.6. *Let M and N be transitive models such that $M \subseteq N$. Let $\phi(x_1, ..., x_n)$ be a formula and let $\bar{a} \in M^n$ be a parameter sequence. We have,*

- *If ϕ is Δ_0 then $M \models \phi[\bar{a}]$ iff $N \models \phi[\bar{a}]$.*
- *If ϕ is Σ_1 then $M \models \phi[\bar{a}]$ implies $N \models \phi[\bar{a}]$ but not conversely.*
- *If $\phi \in \Pi_1$ then $N \models \phi[\bar{a}]$ implies $M \models \phi[\bar{a}]$ but not conversely.*

now since V[H] models **BMM** and V[G] is an extension of V[H] by a stationary set preserving forcing, we get that

$$H_{\aleph_2}^{V[H]} \prec_{\Sigma_1} H_{\aleph_2}^{V[G]}.$$

In particular we can conclude that

$$H_{\aleph_2}^{V[H]} \models \exists\phi(x, q, p).$$

for all $q \in H_{\aleph_2}^{V[H]}$, from which the desired conclusion follows. The proof of the theorem is completed. □

2.9 Concluding remarks

By Theorem (2.4.2), we know that assuming $\mathbf{MM}^{++}$ one cannot hope, *appealing to stationary set preserving forcings*, to change the Π_2 theory of the initial fragment of the universe of sets $H_{\aleph_2}$. Assuming $\mathbf{MM}^{+++}$ one can do something more, stabilizing the effects of forcing for the full theory of the Chang model $L([Ord]^{\leq\omega_1})$, as Theorem (2.4.3) in Section (2.4) shows. If we introduce the notion of **generic multiverse** as the collection of all the models produced by the method of forcing, it is maybe possible to say that generic absoluteness results, as those expressed by Theorems (2.4.2) and (2.4.3), disclose to us a **pattern of uniformity** among the randomly produced sequences of models in the generic multiverse.

Viale′s solution to the Fundamental equation is

$$\frac{L(\mathbb{R})}{AD^{L(\mathbb{R})}} = \frac{P(\omega_1)}{\mathbf{MM}^{+++}}$$

As we pointed out in section (2.3), there is another strategy and a distinct axiom for producing generic absoluteness result at the level of $H_{\aleph_2}$. That is, Woodin's axiom $(*)$. Crucially, it is not clear (for the moment) which is the mathematical relation between $\mathbf{MM}^{+++}$ and $(*)$. We are going to analyze some aspects of $(*)$ in the next Chapter 3.

In the final Chapter of the present work, we will set up what we believe is a **plausible framework** for a philosophical comparison of $\mathbf{MM}^{+++}$ and Woodin's axiom $(*)$. We would like, however, to stress since now (and it will emerge later) that the philosophical case for $AD^{L(\mathbb{R})}$ is only **partially generalizable** to the case for $\mathbf{MM}^{+++}$ and $(*)$. Last point will be made more precise in Section (4.2). In this sense, we first note that the situation with the structure $P(\omega_1)$ represents plausibly a **transition point** in the kind of justification that one can offer. This in turn opens to the possibility to radically different strategies, than the one proper to the forcing axioms program, for giving a solution to the fundamental equation (2.2)

> "The above case for $\neg CH$ is weaker than the case for $AD^{L(\mathbb{R})}$ in that $\neg CH$ lacks the inevitability had by $AD^{L(\mathbb{R})}$. This, however, is simply an inevitable consequence of the fact that CH is not settled by large cardinal axioms. *With CH one reaches a transition point in the kind of justification that can be given- the case is necessarily going to have to be more subtle.* As a symptom of this consider the following scenario: Suppose that inner model theory reaches "L-like" models $L[E]$ that can accomodate all large cardinals and have much of the rich combinatorial structure of current inner model. An axiom of the form $V = L[E]$ would then be a plausible new axiom". (My emphasis)

Generic absoluteness is the clearest property of $L(\mathbb{R})$ that can be lifted to $P(\omega_1)$. Given the central role of forcing in establishing independence results and the strong desire to find axioms that prove as many statement as possible, this is a very natural constraint to impose. Nevertheless, ultimately, it seems important to clarify a reply to the following question

What is the meaning, in terms of philosophical justification, of lifting generic absoluteness from $L(\mathbb{R})$ to $P(\omega_1)$?

As we will try to point out in Chapter 4, in our opinion a reply to the previous question need to be given *in tandem with* an analysis of the **structure theory** of H_{ω_2} and $P(\omega_1)$ under axiom $\mathbf{MM}^{+++}$ and axiom $(*)$.]

Some basic philosophical questions remain open. One main philosophical question upon the notion of generic absoluteness is *whether* and *in which sense* the existence of a uniformity pattern among the models in the generic multiverse corresponds to something more real regarding the universe of sets. This is we think a crucial question, and one that deserves more investigation. It can be, maybe, recovered in a more general one concerning directly the key notion of extrinsic evidence for the search of new axioms in Set theory.

General question.(*Why, in other words, should regularity properties be understood as criteria of truth in the search for new axioms?*).

The possibility to conceive $\mathbf{MM}^{++}$, or better $\mathbf{MM}^{+++}$ (considering Viale's most general Theorem (2.4.3)), as a **deep truth** regarding the universe of sets V requires, we think, an answer to the previous general question.

Chapter 3

$\mathbb{P}_{max}$ forcing and generic absoluteness

3.1 Introduction

As we briefly sketched in Section (2.3) of Chapter 2, Woodin proposed a strategy for a possible solution of CH. He formulated a conjecture, the CH *conjecture*, concerning the solution of CH, and he proved that conjecture introducing a *new* logic, called Ω logic, and presupposing a further conjecture, the Ω conjecture[1], concerning the relationship between the *semantic* relation of Ω logical consequence $\models_\Omega$ (that we already introduced in Section (2.3) of Chapter 2) and its *proof theoretic* complement $\vdash_\Omega$.

There seems to be an original difficulty with the search for a new axiom (or new axioms) able to offer a satisfactory description of the structure $(H_{\omega_2}, \in)$. Such a difficulty appears to be **directly correlated** with the motivation behind Woodin's idea of introducing a new kind of logic. As we already stressed standard large cardinal axioms are (presumably) not extremely informative regarding a possible solution of the Continuum Hypothesis. It is, in fact, possible to switch the value of the Continuum, $2^{\aleph_0}$, appealing to (small) forcing[2] notions that preserve large cardinals. A similar situation represents a notable **disanalogy** with respect to what happens for the structure $(H_{\omega_1}, \in)$ (and for $L(\mathbb{R})$) and the search for the corresponding *correct* axioms. In that case large cardinals axioms played an essential role in indicating the correct axiom $AD^{L(\mathbb{R})}$. Briefly, the fact that an appealing principle like Projective

[1] Actually, the *Strong* Ω conjecture.

[2] Forcing notions $\mathbb{P}$ such that $|\mathbb{P}| < k$ for k a familiar large cardinal.

Determinacy is deeply intertwined with others *intrinsically plausible*[3] principles such as large cardinal axioms, remarkably reinforces the philosophical case for $AD^{L(\mathbb{R})}$ as a new axiom to add to ZFC. Actually, the case for $AD^{L(\mathbb{R})}$ is much stronger. As Koellner states in [15] (section 3.3, p. 22.), by a result due to Woodin, it is possible to say that

> " as large cardinals are necessary for definable determinacy, definable determinacy **is necessary** for generic absoluteness. [for the inner model $L(\mathbb{R})$]".

Schematically, using Koellner's terminology in [15], the situation for $AD^{L(\mathbb{R})}$ can be summarized in the following way: Let's call a theory 'good' if it stabilizes forcing for the theory of $L(\mathbb{R})$.

1. There is a good theory,
2. All good theories imply $AD^{L(\mathbb{R})}$.

The main motivation of Woodin concerning the structure $(H_{\aleph_2}, \in)$ is to find a solution to the Continuum Problem. Actually, as Theorem (3.6.1) shows, the introduction of Ω-logic makes it possible to **reproduce** at the higher level of CH a situation similar, in some of its abstract features, to that depicted by the previous schematization concerning $AD^{L(\mathbb{R})}$. In fact, *modulo inference in strong logic*, we can summarize the situation for CH as follows:

1. There is a good axiom,
2. All good axioms Ω-imply $\neg CH$.

In Woodin's words the situation can be summed up as follows.

> "Here we have a problem if we regard large cardinal axioms as our sole source of inspiration: even if there is an analog of Projective Determinacy for $(H_{\omega_2}, \in)$, *how can we find it or even recognize it if we find it?* [...] The solution is to take an **abstract approach**. [...] An important possibility arises through strong logics. This is the possibility that augmenting ZFC with a **single axiom** yields a system of axioms powerful enough to resolve, through inference in strong logic, all questions about H_{ω_2}." (My emphasis) (See [40], p. 682 and p. 683.)

[3]See Chapter 4.

The inspection of $(H_{\omega_2}, \in)$ with the **artificial** devise of the consequence relation in Ω-logic corresponds to Woodin's precise choice to formalize the notion of generic absoluteness at the level of CH. The single axiom Woodin refers to in the quoted passage from [40] is Woodin's axiom $(*)$, and we already gave a possible formulation of it in Section (2.3) of Chapter 2. Axiom $(*)$ is the **right place** where to inspect in order to analyze the connection between Ω logic, the *new* logic introduced by Woodin, and the **canonical model** that Woodin isolated and where CH fails. The latter is a generic extension of the inner model $L(\mathbb{R})$ obtained by exploiting the special forcing notion discovered by Woodin and called $\mathbb{P}_{max}$. The strategy here seems to be the following. If $\mathbb{P}$ is a forcing construction definable in $L(\mathbb{R})$ such that $\mathbb{P}$ expresses some **homogeneity** properties, then the theory of the $\mathbb{P}$-extension of $L(\mathbb{R})$ can be computed in the ground model, $L(\mathbb{R})$. Since the theory of $L(\mathbb{R})$ is **invariant under set forcing** (assuming large cardinal axioms), the homogeneity of $\mathbb{P}$ makes the theory of the generic extension of $L(\mathbb{R})$ obtained by $\mathbb{P}$ invariant under set forcing too. As Larson observes in [24] (p. 2122), this suggests that the **absoluteness properties** of $L(\mathbb{R})$ can be lifted to models of the Axiom of Choice,[4] as Choice, despite the fact that $L(\mathbb{R}) \not\models AC$, **can be forced** over $L(\mathbb{R})$. The first property of $\mathbb{P}_{max}$ that we want to mention here is that it expresses an homogeneity property strong enough such that the theory of the generic extension obtained by forcing with $\mathbb{P}_{max}$ over $L(\mathbb{R})$ can be computed in the ground model $L(\mathbb{R})$. $\mathbb{P}_{max}$ has also another remarkable property, that is, it implies the Π_2-maximality of the structure $(H_{\omega_2}, \in)$ *relativized* to the $\mathbb{P}_{max}$ extension.

Theorem 3.1.1. *Suppose that there exists a proper class of Woodin cardinals, $A \subset \mathbb{R}$, $A \in L(\mathbb{R})$, ϕ is a Π_2 in the extended language containing two additional predicates, and in some set forcing extension*

$$\langle H_{\omega_2}, \in, NS_{\omega_1}, A^* \rangle \models \phi$$

(where A^* is the reinterpretation of A in the extension). Then

$$L(\mathbb{R})^{\mathbb{P}_{max}} \models [\langle H_{\omega_2}, \in, NS_{\omega_1}, A \rangle \models \phi]$$

Since axiom $(*)$ (as we will see) is the principle that axiomatizes the structure generated by forcing with $\mathbb{P}_{max}$ over $L(\mathbb{R})$, we can consider the Π_2-maximality property expressed by Theorem (3.1.1) as a property of $(*)$. Let's call this property of

[4] Quotation from [40] (p. 686): " Arguably, the stationary, co-stationary, subsets of ω_1 constitute the **simplest true manifestation** of the Axiom of Choice. (..) These considerations support the claim that the structure $(H_{\omega_2}, \in)$ is indeed the next structure to consider after $(H_{\omega_1}, \in)$, being the simplest structure where the influence of the Axiom of Choice is manifest." (My emphasis)

$(*)$, Π_2-Ω-*maximality.*

In Section (2.2) of Chapter 2, within the context of our discussion of forcing axioms and possible generalizations of Cohen's absoluteness lemma (2.2.1), we saw that a Π_2 sentence in the *Levy Hierarchy* is a formula, $\forall x \exists y \psi(x, y, z)$, where ψ is itself a Δ_0 formula[5]. It has been already noticed how Π_2 sentences play a special role in Set theory since many of its undecidable problems can be formulated as Π_2 sentences. One reason for the interest in this **level of logical complexity** of the formulas has to do, as we understand it, with some *empirical* observations and some experience accumulated so far in different fields of Mathematics. The case referred to by Woodin in [40] concerns the standard structure of Number theory. Here there are different open questions expressible as Π_1 sentences in the language of the structure of Number theory[6]. Among the others, Woodin lists the *Goldbach's conjecture* and the *Riemann hypothesis.* Nevertheless, there are also open problems such as, for example, the *Twin Prime Conjecture*, that are expressible by a Π_2 sentence but are not obviously expressible by a Π_1 formula.

> " This becomes interesting if, say, either of these latter problems were proved to be unsolvable from, for example, the natural axioms for $(H_\omega, \in)$. Unlike the unsolvability of a Π_1 sentence, from which one can infer its "truth", for Π_2 sentences the unsolvability does not immediately yield a resolution " (See [40], p. 686.)

The idea to require a Π_2 maximality property seems to well harmonize with this kind of observation concerning the general level of complexity of the open mathematical problems. One virtue of **MM** (Martin's Maximum), in this sense (as we already noticed in Chapter 2), is that it is attempting to maximize the Π_2 theory of the structure $(H_{\aleph_2}, \in)$. Using the Boolean valued models machinery of forcing we can reformulate the Π_2 maximality property of **MM** as follows.

Lemma 3.1.2. *(Martin's Maximum) Suppose that ϕ is a Π_2 sentence and that there is a* **stationary set preserving** *Boolean algebra $\mathbb{B}$ such that*

$$V^{\mathbb{B}} \models \text{``}(H_{\aleph_2}, \in) \models \phi''$$

Then,

$$(H_{\aleph_2}, \in) \models \phi$$

[5] Σ_0 or Π_0 formula.
[6] See [40].

Thus, when considering a huge natural class of forcing notions such as the stationary set preserving forcings, assuming Martin's Maximum, a Π_2 sentence ϕ for the structure $(H_{\omega_2}, \in)$ which is shown to be forceable by an SSP forcing notion, actually holds in $(H_{\omega_2}, \in)$. Let's call this property of **MM** Π_2-*SSP-maximality*.

The present Chapter represents the **initial stage** of our attempt to understand the poset $\mathbb{P}_{max}$ and its correlation with Ω-logic. It can be considered as our first attempt to deepen some technical aspects of the notions already introduced in Section (2.3) of Chapter 2. Our main sources for the next technical material are [24], and [25].

3.2 *Iterations* and some preliminaries

In Section (2.5) of Chapter 2 we introduced the notion of filter (in a set theoretical context). We now introduce in the present section the **dual** notion of *ideal*.

Definition 3.2.1. *Let X be a nonempty set and $I \subseteq P(X)$. Then I is an ideal over X if and only if the following conditions hold.*

1. $\emptyset \in I$
2. $X \notin I$
3. *For all $A, B \subseteq X$, if $B \in I$ and $A \subseteq B$ then $A \in I$*
4. *For all $A, B \subseteq X$, if $A, B \in I$, then $A \cup B \in I$*

The following lemmas that we state here without proof exemplifies the **duality relation** that exists between filters and ideals.

Lemma 3.2.1. *If F is a filter over X, then*

$$\{X \setminus A : A \in F\}$$

is an ideal over X.

Lemma 3.2.2. *If I is an ideal over X, then*

$$\{X \setminus A : A \in I\}$$

is a filter over X.

Let I be an ideal over X. It is possible to define the following equivalence relation on elements of $P(X)$

$$u \sim v \text{ if and only if } u \triangle v \in I$$

where

$$u \triangle v^7 = (u - v) \cup (v - u)$$

If we consider C as the set of all equivalence classes of elements of X induced by I, that is $C = X/\sim$, we can **transform** C into a Boolean algebra as follows, equipping C with the following operations.

$$\begin{aligned}
[u] \vee [v] &= [u \cup v] \\
u \wedge v &= [u \cap v] \\
&= [X \setminus u] \\
\mathbf{0} &= [\emptyset] \\
\mathbf{1} &= [X]
\end{aligned}$$

We call C the **quotient algebra** and we denote it as X/I. In what follows we will take X to be $P(\omega_1)$ and we will be focused on the Boolean algebra $P(\omega_1)/I$ with I a *normal*[8] ideal on ω_1. Forcing with the Boolean algebra $P(\omega_1)/I$ induces a V-normal ultrafilter G on ω_1 and this is the key ingredient to set up the *ultrapower construction* along the lines sketched in Section (2.5) of Chapter 2. More specifically, we will be concerned with set models M of Set theory. Thus we will consider the elementary embedding

$$j : M \to Ult(M, G) := \{f : \omega_1^M \to M : f \in M\}/ =_G$$

Definition 3.2.2. *I is precipitous if $Ult(M, G)$ thus constructed is well-founded from the point of view of $M[G]$, for all M-generic G.*

We now want to pursue a more elaborated construction starting from the ultrapower construction one. The new construction is called **iterated ultrapower construction** and it is the fundamental idea for generating the $\mathbb{P}_{max}$ forcing notion introduced by Woodin. First of all, following the Larson's presentation in [..], we specify two theories strong enough for making the iterated ultrapower construction possible.

[7] *The symmetric difference* of u and v.

[8] Definition of when an ideal is normal

- T_0: a theory consistent with ZFC and strong enough to make sense of the generic ultrapower construction above and prove that $j : M \to Ult(M, G)$ is elementary.
- T_1: a theory consistent with ZFC and at least as strong as T_0+ "every set lies in some $H_k \models T_0$".

In the present chapter we will take T_0 to be simply ZFC, and we will take T_1 to be $ZFC+$" There is a proper class of strongly inaccessible cardinals".

The main idea for passing from the notion of generic ultrapower to that of iterated ultrapower is to **extend** the generic ultrapower construction. Starting from (M_0, I_0), with $G_0 \subseteq (P(\omega_1)/I)^{M_0}$, and $j_0 : (M_0, I_0) \to Ult(M_0, G_0)$, we let $M_1 = Ult(M_0, I_0)$ and $I_1 = j_0(I_0)$. We then take the ultrapower of M_1 (the ultrapower of the ultrapower) by I_1 and we iterate. When we reach limit stage what we obtain is a *direct system* of elementary embeddings, and we can take the direct limit of the system and keep going with our iteration, until we reach length ω_1. At that point what we get is a model, M_{ω_1}, of size ω_1, that is, an element of H_{ω_2}[9] .

Definition 3.2.3. *An* iteration *of (M, I) (for M countable) of length γ consists of $M_\alpha, I_\alpha(\alpha \leq \gamma), G_\eta(\eta < \gamma)$, and $j_{\alpha,\beta}(\alpha \leq \beta \leq \gamma)$, satisfying*

- $M_0 = M, I_0 = I$
- *G_η is M_η-generic for $(P(\omega_1)/I_\eta)^{M_\eta}$*
- *$j_{\eta,\eta+1}$ is the canonical embedding of M_η into $Ult(M_\eta, G_\eta) = M_{\eta+1}$*
- *$j_{\alpha,\beta} : M_\alpha \to M_\beta$ are a commuting family of elementary embeddings*
- $I_\beta = j_{0,\beta}(I)$
- *For limit β, M_β is the direct limit of $\{M_\alpha : \alpha < \beta\}$ under the embeddings $j_{\alpha,\eta}(\alpha \leq \eta < \beta)$*

Definition 3.2.4. *We will use the following terminology.*

- *M_α's are* **iterates** *of (M, I),*
- *(M, I) is* **iterable** *if all iterates are well-founded,*

[9]So, we are stretching our countable model M we start with making it uncountable.

- *(M, I) is an* **iterable pair** *if M is a countable transitive model of T_0, I a normal ideal on $P(\omega_1)$ in M, and (M, I) is iterable.*

Next lemmas play an essential role in the development of the proof of the main theorem (3.4.1) that we are going to sketch in section (3.4.2). Collectively we can say that their informations improve our power of inspection on the well-foundedness of the iterates of an iteration.

Lemma 3.2.3. *Suppose that $M \in N$ are models of T_0, M is closed under ω_1-sequences from N, and $P(P(\omega_1))^M = P(P(\omega_1))^N$. Let I be an M-normal ideal on ω_1^M. Then the following hold.*

- *For each iteration $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} : \alpha \leq \beta \leq \gamma, \eta < \gamma\rangle$ of (M, I) there is a unique iteration $\langle N_\alpha, I_\alpha, G_\eta, j^*_{\alpha,\beta} : \alpha \leq \beta \leq \gamma, \eta < \gamma\rangle$ of (N, I) such that $(\forall \beta \leq \gamma)$, $j^*_{0,\beta}(M) = M_\beta$, M_β is closed under ω_1 sequences from N_β, $P(P(\omega_1))^{M_\beta} = P(P(\omega_1))^{N_\beta}$, and $j^*_{\alpha,\beta} \restriction M_\alpha = j_{\alpha,\beta}$.*

- *For each iteration $\langle N_\alpha, I_\alpha, G_\eta, j^*_{\alpha,\beta} : \alpha \leq \beta \leq \gamma, \eta < \gamma\rangle$ of (N, I) there is a unique iteration $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} : \alpha \leq \beta \leq \gamma, \eta < \gamma\rangle$ of (M, I) such that $(\forall \beta \leq \gamma, j^*_{0,\beta}(M) = M_\beta$, M_β is closed under ω_1-sequences from N_β, $P(P(\omega_1))^{M_\beta} = P(P(\omega_1))^{N_\beta}$, and $j^*_{\alpha,\beta} \restriction M_\alpha = j_{\alpha,\beta}$.*

Proof. We refer the reader to [24] (section 1, p. 2126) for some observation concerning the proof of lemma (3.2.3). □

Lemma 3.2.4. *Suppose that N is a transitive model of T_1, $\gamma \in Ord^N$, and I is a normal precipitous ideal on ω_1^N in N. Then any iterate of (N, I) by an iteration of lenght γ is wellfounded.*

Proof. We refer the reader to [25] (lemma 6, p. 4) for a proof of lemma (3.2.4). □

3.3 $\mathbb{P}_{max}$

We give now the definition of $\mathbb{P}_{max}$.

Definition 3.3.1. *The partial order $\mathbb{P}_{max}$ is the set of pairs $\langle (M, I), a\rangle$ such that*

1. *M is a countable transitive model of $T_0 + MA_{\aleph_1}$*
2. *(M, I) is an iterable pair*
3. *$a \in P(\omega_1)^M$ and $\exists x \in P(\omega)^M$ such that $\omega_1^{L[x,a]} = \omega_1^M$*

ordered by : $p < q$ *(where* $p = \langle (M, I), a\rangle$*,* $q = \langle (N, J), b\rangle$*) if there is some iteration* $j : (N, J) \to (N^*, J^*)$ *such that*

1. $j \in M$
2. $j(b) = a$
3. $J^* = N^* \cap I$ *(and hence* $j(\omega_1^N) = \omega_1^M$*)*
4. $q \in H(\omega_1)^M$

Definition 3.3.2. *We say that (M, I) is a* $\mathbb{P}_{max}$ *precondition if there exists an a such that* $\langle (M, I), a\rangle \in \mathbb{P}_{max}$*.*

3.3.1 Basic properties of $\mathbb{P}_{max}$

Lemma 3.3.1. *Let* $\langle (M, I), a\rangle$ *be a* $\mathbb{P}_{max}$ *condition and let* A *be a subset of* ω_1*. Then there is at most one iteration of* (M, I) *for which* A *is the image of* a*.*

Proof. We refer the reader to [25] (section 3, lemma 10, p. 7) for a proof of (3.3.1). □

Lemma (3.3.1) **guarantees that the order on each comparable pair of conditions is witnessed by a unique iteration**.

The following lemma states the **homogeneity** property proper of $\mathbb{P}_{max}$.

Lemma 3.3.2. *Suppose for each* $x \in H(\omega_1)$ *there exists a* $\mathbb{P}_{max}$ *precondition* (M, I) *such that* $x \in M$*. Then* $\forall p_0, p_1 \in \mathbb{P}_{max}$*,* $\exists q_0, q_1 \in \mathbb{P}_{max}$ *such that each* $q_i \leq p_i$*, and* $\mathbb{P}_{max} \restriction q_0 \cong \mathbb{P}_{max} \restriction q_1$*.*

Proof. We refer the reader to [25] (section 3, lemma 12, p. 8) for a proof of (3.3.2). □

Proposition 3.3.3. $\mathbb{P}_{max}$ *forcing is* σ*-closed, and so it does not add any reals*[10]

Proof. We refer the reader to [25] (section 3, lemma 13, p. 9) for a proof of (3.3.3). □

[10]This is crucial considering that we are forcing over $L(\mathbb{R})$.

3.4 The existence of the $\mathbb{P}_{max}$ conditions

In order to state the main existence Lemma we need to introduce the following preliminary definition.

Definition 3.4.1. *Given $A \subseteq \mathbb{R}$, and an iterable pair (M, I), we say that (M, I) is A-iterable if $A \cap M \in M$ and for any iteration $j : (M, I) \to (M^*, I^*), j(A \cap M) = A \cap M^*$.*

Lemma 3.4.1. (**Main existence lemma**) *Suppose there are infinitely many Woodin cardinals below some measurable cardinal, and let $A \in P(\mathbb{R}) \cap L(\mathbb{R})$. Then there exists an A-iterable $\mathbb{P}_{max}$ precondition (M, I) such that for every set forcing extension M^+ of M and every precipitous ideal $I^+ \in M^+$ on $\omega_1^{M^+}$, (M^+, I^+) is $A - iterable$[11].*

In order to develope a sketch of the proof of the above existence Lemma, we need to introduce some preliminary notions. This is what we are going to do in the next subsection.

3.4.1 Some preliminaries

It is customary to consider a *real number* as an element of the *Baire Space* $\mathscr{N} = (\omega^\omega, t)$, where t is the *product* topology, with the discrete topology on ω. Such an interpretation of the real numbers will offer us the opportunity to look at *set* of reals as particular *projection* of trees.We briefly recall here below the definition of the Baire Space $\mathscr{N}$.

Definition 3.4.2. *If $n < \omega$ and $s \in \omega^n$, then*

$$N_s = \{x \in \omega^\omega : x \restriction n = s\}$$

These are the *basic open subsets* of ω^ω.

Lemma 3.4.2. *If $s, t \in \omega^{<\omega}$ then,*

- *if $s \subseteq t$, then $N_t \subseteq N_s$,*
- *if $t \subseteq s$, then $N_s \subseteq N_t$, and*
- *otherwise, $N_s \cap N_t = \emptyset$*

[11]This point expresses an invariance under forcing

Definition 3.4.3. *U is an* open *subset of ω^ω iff there is a family F of basic open subsets of ω^ω such that*

$$U = \bigcup F$$

Lemma 3.4.3. *$\{U : U$ is an open subset of $\omega^\omega\}$ is a topology on ω^ω*

Proof. We refer the reader to [33] (section 5.2, lemma 5.8, p. 86) for a proof of the previous Lemma (3.4.3). □

This is the Baire topological space.

There are two fundamental notions in the proof of Lemma (3.4.1) whose **interplay** plays a fundamental role. These are, respectively, the notion of *towers of measures* and the notions of *homogeneous* tree.

We will give the following definitions following [25] (section 4, p. 11. See also [23] sections 1.2 and 1.3).

Definition 3.4.4. *Given $Z \neq \emptyset$, a tower of measures on Z is a sequence $\langle \mu_i : i < \omega \rangle$ such that each $\mu_i \subseteq P(Z^i)$ is an ultrafilter, and for all $k < i < j$ and all $A \in \mu_i$, we have $\{b \in Z^j : b \restriction i \in A\} \in \mu_j$ and $\{b \restriction k : b \in A\} \in \mu_k$.*
Such a tower is countably complete if whenever $\langle A_i : i < \omega \rangle$ is such that each $A_i \in \mu_i$ there is $a \in Z^\omega$ such that $\forall i \ a \restriction i \in A$

As a remark we observe that countable completeness is equivalent to the fact that the *direct limit* of $Ult(V, \mu_i)$ is well-founded.

Definition 3.4.5. *A tree on $\omega \times Z$ is a set $T \subseteq (\omega \times Z)^{<\omega}$ such that for all $i < \omega$, $t \in T$ we have that $t \restriction i \in T$. The projection of T is $p[T] := \{y \in \omega^\omega : \exists c \in Z^\omega \forall i < \omega \ (y \restriction i, c \restriction i) \in T\}$.*
Such a tree is weakly k-homogeneous *(for k a cardinal) if there exists k-complete ultrafilters $\mu_{a,b} \subseteq P(Z^{|a|})$ such that $\forall a, b \in \omega^{<\omega}$ with $|a| = |b|$,*

$$\{c \in Z^{|a|} : (a, c) \in T\} \in \mu_{a,b},$$

and such that for each $x \in p[T]$ there exists a $b \in \omega^\omega$ such that $\langle \mu_{x \restriction i, b \restriction i} : i < \omega \rangle$ is a countably complete tower.

A key fact that we will use in order to sketch a proof of the main existence Lemma is the following due to Woodin.

Theorem 3.4.4. *If δ is a limit of Woodin cardinals and there is a measurable cardinal above δ, then for each $A \in P(\mathbb{R}) \cap L(\mathbb{R})$ and $\gamma < \delta$, there exists a γ-weakly homogeneous tree T such that $p[T] = A$.*

3.4.2 Sketch of a proof of the main existence Lemma

We will list below one lemma and two propositions that are essential for the development of the main existence Lemma. For the respective proofs and references we refer to [24] and [25].

Lemma 3.4.5. *Suppose that $T \subseteq (\omega \times Z)^{<\omega}$ is a γ^+-weakly homogeneous tree, $\theta > (2^{|T|})^+$ is regular, $X \prec H(\theta)$, $T, \gamma \in X, |X| < \gamma$, and $\bar{a} \in p[T]$. Then there exists $Y \prec H(\theta)$ with $X \subseteq Y, X \cap \gamma = Y \cap \gamma, |X| = |Y|$, and $\bar{a} \in p[T \cap Y]$.*

Proposition 3.4.6. *If δ is Woodin, then $Coll(\omega_1, < \delta)$ forces that NS_{ω_1} is pre saturated, and hence precipitous.*

Proposition 3.4.7. *Any c.c.c. forcing preserves that NS_{ω_1} is precipitous.*

We can now try to give a **sketch** of the proof of the main existence Lemma.

Proof. Recalling the hypotheses of the main existence Lemma, we have that δ is a limit of Woodin cardinals, there exists a measurable cardinal greater that δ, and A is in $P(\mathbb{R}) \cap L(\mathbb{R})$. Suppose now that k is the least Woodin cardinal , and γ the least strong inaccessible above k. Fix γ^+-weakly homogeneous trees S, T with $p[S] = A$, $p[T] = \mathbb{R} \setminus A$. Fix a regular $\theta > (2^{|S|})^+, (2^{|T|})^+$. Let X be a countable elementary sub model of $H(\theta)$, with $S, T, \gamma, \kappa \in X$. Apply Lemma [..] repeatedly to obtain $Y \prec H(\theta)$ such that $X \subseteq Y, X \cap \gamma = Y \cap \gamma, A = p[S \cap Y], \mathbb{R} \setminus A = p[T \cap Y]$. Now let N be the transitive collapse of Y, and let $\bar{S}, \bar{T}, \bar{\gamma}, \bar{\kappa}$ be the images of S, T, γ, κ therein. Let h be N-generic for $Coll(\omega_1, < \bar{\kappa})$ followed by a c.c.c. poset of size 2^{ω_1} to make $MA_{\aleph_1}$ hold. Then $N[h] \models MA_{\aleph_1} +$ "NS_{ω_1} *is precipitous*". Then, by Lemma (3.2.4), $N[h]$ is iterable. Let M be $(V_{\bar{\gamma}})^{N[h]}$, and let $j : (M, NS_{\omega_1}^M) \to (M^*, NS_{\omega_1}^*)$ be an iteration. By Lemma (3.2.3), this induces an iteration of $(N[h], NS_{\omega_1}^{N[h]})$ with final model (N^*, I^*) (which we will also call j). Now, N^* is well founded, and $p[\bar{S}] \subseteq p[j(\bar{S})]$ and $p[\bar{T}] \subseteq p[j(\bar{T})]$. But by elementarity, $N^* \models p[j(\bar{S})] \cap p[j(\bar{T})] = \emptyset$, and since N^* is well founded it is correct about this. Then $p[\bar{S}] = p[j(\bar{S})]$ and $p[\bar{T}] = p[j(\bar{T})]$, so $j(A \cap M) = p[j(\bar{S})] \cap M^* = A \cap M^*$. $\square$

It is important to stress, at this point, that instead of $Coll(\omega_1, < \bar{\kappa})$ we could have taken h to be N-generic for any poset in $V_{\bar{\gamma}}^N$ such that $N[h] \models$ "$\exists$ *precipitous I on ω_1*", and the rest of the proof would have still gone through.

3.4.3 The Π_2-maximality

Proof of theorem (3.1.1). (**Π_2 maximality of the $\mathbb{P}_{max}$ extension**) Fix a Π_2 sentence $\phi = \forall x \exists y \psi(x, y)$ (in the extended language with two new unary predicates), and some $A \in P(\mathbb{R}) \cap L(\mathbb{R})$. To show that

$$\langle H(\omega_2), \in, A, NS_{\omega_1} \rangle^{L(\mathbb{R})^{\mathbb{P}_{max}}} \models \phi,$$

it is sufficient to show[12] that for each $\langle (M, I), a \rangle \in \mathbb{P}_{max}$ A-iterable and each $b \in H^M_{\omega_2}$, there exists $\langle (N, NS^N_{\omega_1}), e \rangle \in \mathbb{P}_{max}$ and $j : (M, I) \to (M^*, I^*)$ in N such that $j(a) = e$, $I^* = M^* \cap NS^N_{\omega_1}$ and

$$\langle H^N_{\omega_2}, \in, A \cap N, NS^N_{\omega_1} \rangle \models \exists d \psi(j(b), d).$$

Suppose $\langle (M, I), a \rangle$ is given. Fix $\mathbb{P}$[13] forcing ϕ. Let δ be the least Woodin cardinal with $\mathbb{P} \in V_\delta$. Let κ be the least strong inaccessible above δ. Let S, T be κ^+- weakly homogeneous trees projecting to $A, \mathbb{R} \setminus A$. Let $\theta > (2^{|S|})^+, (2^{|T|})^+$ be regular. Fix $Y \prec H(\theta)$ with $Y \cap \kappa$ countable, $p[S \cap Y] = A, p[T \cap Y] = \mathbb{R} \setminus A$ and $\langle (M, I), a \rangle \in Y$. Let N be the transitive collapse of Y, and let $\bar{P}, \bar{S}, \bar{\delta}, \bar{\kappa}$ be the respective images of P, S, δ, κ, under this collapse. Let h_0 be $\bar{P}$-generic for N. Note that since $P \in V_\delta$, $\bar{\delta}$ remains Woodin in $N[h_0]$. The reinterpretation of A is the projection of $\bar{S}$ in the extension. Thus,

$$\langle H(\omega_2)^{N[h_0]}, \in, (p[\bar{S}])^{N[h_0]}, NS^{N[h_0]}_{\omega_1} \rangle \models \phi.$$

Pick an iteration j of (M, I) in N such that $j(I) = j(M) \cap NS^{N[h_0]}_{\omega_1}$. Then there exists a $d \in H(\omega_2)^{N[h_0]}$ such that

$$\langle H(\omega_2)^{N[h_0]}, \in, (p[\bar{S}])^{N[h_0]}, NS^{N[h_0]}_{\omega_1} \rangle \models \psi(j(b), d)$$

Let h_1 be $N[h_0]$-generic for $Coll(\omega_1, < \bar{\delta})^{N[h_0]}$ followed by some c.c.c. forcing making $MA_{\aleph_1}$ hold. Now $\langle ((V_{\bar{\kappa}})^{N[h_0][h_1]}, NS^{N[h_0][h_1]}_{\omega_1}), j(a) \rangle$ is the desired condition. □

3.5 Ω-logic and $\mathbb{P}_{max}$

We stress now a very important fact about the $\mathbb{P}_{max}$ extension. Every subset of ω_1 in the extension is the image of the first component of a member of the generic filter under the iteration of that member induced by the generic filter. A similar observation **gives context** to the next new definition of $(*)$

[12]But aren't we assuming here Theorem (3.5.1) below, part (a)?

[13]Notice that $\mathbb{P}$ is an **arbitrary** forcing notion

Theorem 3.5.1. *Assume that for every $A \subseteq \mathbb{R}$ there exists a $\mathbb{P}_{\max}$ condition $\langle (M, I), a \rangle$ such that (M, I) is A-iterable and*

$$\langle H_{\omega_1}^M, A \cap M, \in \rangle \prec \langle H_{\omega_1}, A, \in \rangle.$$

Suppose that $G \subseteq \mathbb{P}_{\max}$ is a V-generic filter. Then in $V[G]$ the following hold.

(a) $P(\omega_1) = P(\omega_1)_G$.

(b) $NS_{\omega_1} = I_G$.

Proof. We refer the reader to [24] (section 5, theorem 5.1, p. 2148) for a proof of (3.5.1). □

Woodin **defines** the following axiom.

Definition 3.5.1. (**New definition**) *Axiom $(*)$ is the statement that AD holds in $L(\mathbb{R})$ and $L(P(\omega_1))$ is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.*

In what follows, I will try to fix the relationship between the partial order $\mathbb{P}_{max}$ and Woodin's Ω-logic.
I will try to follow the characterization of Ω-logic presented in [4] and [24].

Let T be a set of sentences and let ϕ be a sentence, both in the language of set theory. Then $T \models_\Omega \phi$ (that is, ϕ is Ω_T-valid) if for every forcing construction P and every ordinal α, if $V_\alpha^P \models T$ then $V_\alpha^P \models \phi$. Given this model-theoretic notion, Woodin defined a proof theoretic relation $(\vdash_\Omega)$ that he conjectured it is the proof theoretic complement of the model theoretic relation $\models_\Omega$

In order for give some hints on[14] explain how the proof-theoretic relation works, we need to introduce some important new concepts.

Definition 3.5.2. *(Universally Baire) Given a cardinal k, a set of reals A is k-universally Baire if there exist trees S and T (contained in $\omega \times Z$ for some set Z) such that $p[S] = A$ and S and T project to complements in all extensions by forcing constructions of cardinality less than or equal to k. The set A is $< \kappa$-universally Baire if it is γ-universally Baire for all $\gamma < \kappa$.*
A set of reals A is universally Baire if it is κ-universally Baire for all cardinals κ.

[14] For a more detailed explanation see [4] (section 2, p. 6).

Definition 3.5.3. *(**A - closed models**) Given a uB set $A \subseteq \mathbb{R}$, a transitive $\in$-model M of (a fragment of) ZFC is A-closed if for all posets $\mathbb{P} \in M$ and all V-generic filters $G \subseteq \mathbb{P}$,*

$$V[G] \models M[G] \cap A_G{}^{15} \in M[G]$$

Proposition 3.5.2. *The following are equivalent:*

- *(i) For all a $A-$closed c.t.m. M of ZFC, all $\alpha \in M \cap On$, and all $\mathbb{B}$ such that $M \models$ "$\mathbb{B}$ is a c.B.a.", if $M_\alpha^{\mathbb{B}} \models T$, then $M_\alpha^{\mathbb{B}} \models \phi$.*
- *(ii) For all A-closed c.t.m. M of ZFC, and for all $\alpha \in M \cap On$, if $M_\alpha \models T$, then $M_\alpha \models \phi$.*

Proof. (ii)$\Rightarrow$ (i): Let M be a A-closed c.t.m. of ZFC, $\alpha \in On \cap M$, and let $\mathbb{B}$ be such that $M \models$ "$\mathbb{B}$ *is ac.B.a.*" Suppose $M_\alpha^{\mathbb{B}} \models T$ and, towards a contradiction suppose that, in M, for some $b \in \mathbb{B}$, $b \Vdash$ "$M[\dot{g}]_\alpha \models \neg\phi$", where $\dot{g}$ is the standard name for the generic filter. It is possible to show that there is a g $\mathbb{B}$-generic over M such that $b \in g$ and $M[g]$ is A-closed. We have $M[g]_\alpha \models T$. Hence, by (ii), $M[g] \models \phi$, in contradiction with the assumption that B forced $M[\dot{g}]_\alpha \models \neg\phi$. $\square$

Let T be a theory containing ZFC and let ϕ be a sentence, both in the language of set theory. Then $T \vdash_\Omega \phi$ (that is, T implies ϕ is Ω-logic) **if there exists** a set of reals A such that

- $L(A, \mathbb{R}) \models DC_{\mathbb{R}} + AD^+$,
- every set of reals in $L(A, \mathbb{R})$ is universally Baire,
- for every countable A-closed model M and every ordinal $\alpha \in M$, if V_α^M satisfies T then V_α^M satisfies ϕ.

A sentence ϕ is Ω_{ZFC}-consistent if $ZFC \nvdash_\Omega \neg\phi$

Lemma 3.5.3. *If $A \subset \mathbb{R}$ is uB and k is such that $V_k \models ZFC$, then A is uB in V_k.*

The first important theorem relative to $\vdash_\Omega$ that we will state shows that statements that can be forced to hold (along with ZFC) in suitable initial segments of the universe are Ω_{ZFC}-consistent.

[15] You have to give a definition of A_G

Theorem 3.5.4. *Suppose that A is a universally Baire set of reals and that k is a strongly inaccessible cardinal. Then any forcing extension (in V) of any transitive collapse of any elementary sub model of V_k containing A is A-closed.*

Theorem 3.5.5. *(***Soundness***) Assume there is a proper class of strongly inaccessible cardinals. For every $T \cup \{\phi\} \in Sent$, $T \vdash_\Omega \phi$ implies $T \models_\Omega \phi$.*

Proof. Let A be a uB set A witnessing $T \vdash_\Omega \phi$. Fix α and $\mathbb{B}$, and suppose $V_\alpha^{\mathbb{B}} \models T$. Let $\lambda > \alpha$ be a strongly inaccessible cardinal such that $A, \mathbb{B}, T \in V_\lambda$ and $V_\lambda \models$ "$\mathbb{B}$ is a c.B.a". Take $X \prec V_\lambda$ countable with $A, \mathbb{B}, T \in X$. Let M be the transitive collapse of X, and let $\bar{\mathbb{B}}$ be the transitive collapse of $\mathbb{B}$. It is possible to show that M is A-closed. Hence $M_\alpha^{\bar{\mathbb{B}}} \models T$, then $M_\alpha^{\bar{\mathbb{B}}} \models \phi$. Since $V_\lambda \models$ "$V_\alpha^{\mathbb{B}} \models T''$, by elementarity, $M \models$ "$M_\alpha^{\bar{\mathbb{B}}} \models T''$. Hence $M \models$ "$M_\alpha^{\bar{\mathbb{B}}} \models \phi''$. So, again, by elementarity, $V_\lambda \models$ "$V_\alpha^{\mathbb{B}} \models \phi''$. Hence, $V_\alpha^{\mathbb{B}} \models \phi$.

□

Actually, Woodin showed that the axiom $(*)$ is Ω_{ZFC}-consistent.

Theorem 3.5.6. *Suppose that there exists a proper class of Woodin cardinals and that there is an inaccessible cardinal which is a limit of Woodin cardinals. Then the theory*

$$ZFC + (*)$$

is Ω_{ZFC}-consistent.

Recall from Section (2.3) the notion of Ω-satisfiability.

Definition 3.5.4. *(Ω-satisfiability) Let T be a theory containing ZFC and ϕ a sentence both in the language of Set theory. Then, if $T \not\models_\Omega \neg\phi$ it is said that ϕ is Ω_T satisfiable.*

Question : Is axiom $(*)$ Ω_{ZFC}-satisfiable?

An answer to the previous question could come from a solution to the following *Conjecture.*

Definition 3.5.5. *Woodin's Ω Conjecture asserts that if there exist proper class many Woodin cardinals then for every sentence ϕ, $\emptyset \models_\Omega \phi$ if and only if $\emptyset \vdash_\Omega \phi$.*

It is possible to show that every Π_2 sentence for $\langle H_{\omega_2}, NS_{\omega_1}, \in\rangle$ which is Ω_{ZFC}-consistent with the existence of a precipitous ideal on ω_1 holds in the $\mathbb{P}_{max}$ extension.

[Here is exactly the **connection point** between Ω-logic and $\mathbb{P}_{max}$ costruction[16]] As Larson says, using the canonical inner models for Woodins cardinals it is possible to do something more, as the following Theorem shows.

Theorem 3.5.7. *If there is a proper class of Woodin cardinals, then for every set of reals A in $L(\mathbb{R})$, every Ω_{ZFC}-consistent Π_2 sentence for $\langle H_{\omega_2}, NS_{\omega_1}, A, \in\rangle$ holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$[17].*

Theorem (3.5.7) plays an essential role when considered in the light of axiom $(*)$. As we saw above in the definition of $(*)$, this latter asserts that $L(P(\omega_1))$ is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$. This together with the fact that $H_{\omega_2} \subseteq L(P(\omega_1))$, and that $H_{\omega_2} = H_{\omega_2}^{L(P(\omega_1))}$ gives us a *maximality* property of the axiom $(*)$, which we could restate in the following way:

Assume ZFC and that there is a proper class of Woodin cardinals. Then the following are equivalent:

- $(*)$
- For each Π_2-sentence ϕ in the language of the structure

$$\langle H_{\omega_2}, \in, NS_{\omega_1}, A : A \in P(\mathbb{R}) \cap L(\mathbb{R})\rangle$$

if

$$ZFC + \text{“}\langle H_{\omega_2}, \in, NS_{\omega_1}, A : A \in P(\mathbb{R}) \cap L(\mathbb{R})\rangle \models \phi''$$

is Ω_{ZFC}-consistent, then

$$\langle H_{\omega_2}, \in, NS_{\omega_1}, A : A \in P(\mathbb{R}) \cap L(\mathbb{R})\rangle \models \phi$$

Using Ω-logic we can restate Woodin's solution for the level H_{ω_1} as follow.

Theorem 3.5.8. *Suppose that there exists a proper class of Woodin cardinals. Then for every sentence ϕ, either $ZFC \vdash_\Omega L(\mathbb{R}) \models \phi$ or $ZFC \vdash_\Omega L(\mathbb{R}) \not\models \phi$.*

[16]See for example [24] (section 9, theorem 9.7, p. 2165) for a first sketch of a similar connection. An accurate analysis of such a connection is beyond the scope of the present work. It requires especially the introduction of some concepts from the branch of Set theory called *Inner model theory*. It suffices to say here that, as it seems to us, it is by exploiting the deep link between the notion of A-closure of a model (which is incapsulated in the $\vdash_\Omega$ relation) and the properties of iterability of a model in the $\mathbb{P}_{max}$ construction that it is possible to offer a proof of the statement in question.

[17] In [40] (p. 687.) Woodin declares:“ The axiom $(*)$ is really a maximality principle somewhat analogous to asserting algebraic closure for a field”

At this point we can appreciate the almost-homogeneity property of the forcing notion $\mathbb{P}_{max}$. Such an homogeneity property imply, in fact, that the theory of the generic extension can be computed in the ground model. Because of the fact that the theory of $L(\mathbb{R})$, as we saw in the theorem (3.5.8), is *generically absolute*, if we add $(*)$ as a new axiom to the standard axiomatic basis ZFC, the theory of the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ (and so, by $(*)$, the theory of $L(P(\omega_1)))$ inherits the generically absoluteness of the theory of $L(\mathbb{R})$. Summarizing we have the following Theorem.

Theorem 3.5.9. *Suppose that there exists a proper class of Woodin cardinals. Then for every sentence ϕ, either*

$$ZFC + (*) \vdash_{\Omega} L(P(\omega_1)) \models \phi$$

or

$$ZFC + (*) \vdash_{\Omega} L(P(\omega_1)) \not\models \phi.$$

Considering that $(*) \Rightarrow \neg CH$ and that, moreover, Woodin showed that if ψ is any sentence for which Theorem (3.5.9) holds (with ψ in place of $(*)$) then $ZFC + \psi \vdash_{\Omega} \neg CH$, we have, modulo the Strong Ω conjecture, a reproduction for the case of CH of **some** of the considerations in favour of $AD^{L(\mathbb{R})}$.

3.6 Concluding remarks

Last considerations in previous Section (3.5) face an important issue. We started, in fact, with a general question concerning essentially the possibility to find a theory T, extending ZFC, and compatible with all (consistent) large cardinals, such that T is Ω-complete for the theory of $(H_{\omega_2}, \in)$ with real parameters. (Let's call it: Question 1). Theorem (3.5.9) gives us a similar theory T. Moreover, in such a theory T the CH fails. Nevertheless, Theorem (3.5.9) leaves open an important possibility. This is the possibility that there exist other **different** theories Ω- complete for $Th(H_{\omega_2}, \in)$. Could be possible, for example, to find another analogue Ω-complete theory for $Th(H_{\omega_2}, \in)$ **but in the context of CH**? (Let's call it: Question 2) We can rephrase Question 2, more generally, in Woodin's words.

> *Under what circumstances can the theory of the structure $(H_{\omega_2}, \in)$ be finitely axiomatized over ZFC in Ω logic?* (My emphasis) (see [40], p. 688.)

Next Theorem (3.6.1), whose content we anticipated at the end of Section (3.5) provides a first important reply to Question 2.

Theorem 3.6.1. *(Woodin) Assume PCWC*[18] *and assume the* Strong Ω*-conjecture holds. Then,*

1. *There is an axiom A such that*

   ZFC+A is Ω satisfiable, and
   ZFC+A is Ω-complete for the structure H_{ω_2}.

2. *Any such axiom has the feature that*

$$ZFC + A \models_\Omega \text{“}H_{\omega_2} \models \neg CH''$$

Following Koellner in [18], it is possible to rephrase the content of theorem (3.6.1) as follows.
For each A satisfying (1) of Theorem (3.6.1), let

$$T_A = \{\phi : ZFC + A \models_\Omega \text{“}H_{\omega_2} \models \phi''\}$$

Then, Theorem (3.6.1) says that in the presence of PCWC and assuming that the Strong Ω-conjecture holds, there are Ω-complete theories T_A of H_{ω_2} and all such theories contains $\neg CH$.

The point now is to understand how much is it possible to extend the level of **agreement** of such Ω- complete theories T_A. We would like to have **only one**[19] such a theory. Unfortunately, the following result due to Woodin and Koellner asserts that if there is one such theory T_A, then there are many such theories T_A.

Theorem 3.6.2. *(Koellner and Woodin) Assume PCW. Suppose that A is an axiom such that:*

- *$ZFC + A$ is Ω-satisfiable, and*
- *$ZFC + A$ is Ω-complete for the structure of H_{ω_2}*

Then there is an axiom B such that

- *$ZFC + B$ is Ω-satisfiable,*
- *$ZFC + B$ is Ω-complete for the structure H_{ω_2}*

[18] Proper class of Woodin cardinals.
[19] This is, we think, the philosophical meaning of Question 2.

and $T_A \neq T_B$

The natural question then is how we can choose between such theories T_A. Here is where the axiom $(*)$, as we saw before, enters the picture and stands out from the others thanks to its **maximality**. The theory $T_{(*)}$, given by $(*)$, maximizes the Π_2 theory of the structure $\langle H_{\omega_2}, \in, I_{NS}, A : A \in \mathbb{P}(\mathbb{R}) \cap L(\mathbb{R})\rangle$, and CH fails in this theory. In fact, $(*)$ implies that the size of the continuum is the second uncountable cardinal, $\aleph_2$.

On the basis of the previous replies to Question 1 and Question 2, we can state Woodin's **solution** to the Fundamental equation (2.2). Actually, given definition (3.5.1) of $(*)$, we could slightly modify fundamental equation (2.2) in the following way.

$$\frac{L(\mathbb{R})}{AD^{L(\mathbb{R})}} = \frac{L(P(\omega_1))}{(*)}$$

We may say that Question 2 aims at measuring the level of philosophical arbitrariness of Woodin's proposal. It would seem important to formulate an analogue question also on the side of Viale's alternative strategy for freezing the theory of $(H_{\aleph_2}, \in)$.

The problem is raised by Viale in [39] (p. 10).

> **Question** : Is it possible to "come up with another 'complete' axiom system for the theory of $H_{\aleph_2}$ with parameters in $H_{\aleph_2}$ incompatible with MM^{+++} and which allows us to dispose of a completeness and correctness theorem linking provability and forceability?"

In other words, in what sense is Viale's solution *inevitable*?

We do not deal with such a problem in the present work and we refer the interested reader to the Introduction of [39] (p. 10) where Viale offers some hints on how one should start to reflect on the problem.

The next question that, as it seems to us, is appropriate to ask concerns the **direct** relationship between axiom $(*)$ and $\mathbf{MM}^{+++}$.

> **Question**: How can we analyze, from a philosophical point of view, the relationship between the theory ZFC^{*}[20] $+ (*)$ and $ZFC^* + \mathbf{MM}^{+++}$? Which one is **more justified** from a philosophical point of view?

This is what we are going to (start to) see in the next and final Chapter of the present work.

[20]Here "ZFC^*" means ZFC supplemented with large cardinal axioms.

Chapter 4

Philosophy of the axioms

4.1 Introduction

Chapters 2 and 3 of the present work offer a description of some main aspects of the distinct techniques employed respectively by Viale and Woodin for producing generic absoluteness at the level of the initial fragment of the universe of sets $H_{\aleph_2}$. Adopting Viale's point of view on the concept of generic absoluteness, we can plausibly say that the above constructions outlined in chapters 2 and 3 aim at substantially change our perspective toward the forcing technique, transforming it in **a generator of mathematical truth instead of a generator of mathematical undecidability.**
In the present chapter we want to **correlate** the pure mathematical phenomenon of generic absoluteness with the philosophical debate on the question of pluralism and the search for new axioms in Set theory. The question that marks our correlation can maybe be expressed paradigmatically as follows:

> to which extent and why the modular generalization of Cohen's absoluteness via forcing axioms, which is, as we understand it, the proper framework of Viale's project to **tame** progressively the **chaotic effects** of the forcing relation, should lead by itself to the recovering of an interesting notion of truth for Set theory?[1]

To which extent, for example, the *restriction* on the class of forcing notions to consider- that is operating in Viale's absoluteness result (as we saw)- is a legitimate restiction and not an arbitrary one when considered under the light of the search for the **correct** theory for the initial fragment $H_{\aleph_2}$?

[1]Why, for example, is it not considered 'only' a remarkable technical result?

It seems that, ultimately, a reply to the previous questions **is not separable from** an analysis of the conceptual validity of the fundamental principles that are inherent Viale's construction of the generic absoluteness result at the level of $H_{\aleph_2}$, that is, of the instantiations of the **forcing axioms schema** $FA_{\aleph_1}(\mathbb{P})$, for $\mathbb{P}$ a poset. The previous questions appear even more urgent if one considers that, in fact, there exists already a distinct completeness result (i.e. Woodin's result inspected in chapter 3) with respect to the notion of forceability for the theory of $H_{\aleph_2}$, and that that result doesn't restrict the assumptions on the class of the forcing notions to consider (in fact Woodin's result, as we saw, strongly depends on the exploitation of Ω-logic). This last result[2], thus, **appears at first sight** to be more general than the one obtained by Viale in [37] and [39].

How can we evaluate the question? That is, how can we evaluate from the point of view of their philosophical justification the above constructions?

A plausible methodology seems to be that of putting under analysis the fundamental assumptions behind the two constructions. These assumptions appears in the shape of two axioms, that is, the forcing axiom Martin's Maximum, actually a strengthening of it indicated as $\mathbf{MM}^{+++}$, and the *close relative* of forcing axioms that is called $(*)$[3]. Which perspectives appears through an analysis of the above axioms? The possible scenarios (as we will see) seems to be those articulated by the possible solutions of a specific conjecture that has been formulated. The conjecture concerns the following question,

$$\textit{Does } \mathbf{MM}^{+++} \textit{ imply } (*)?$$

Plausibly, the most clear case would be if it turned out that $(*)$ is inconsistent with large cardinal axioms. In that case the remaining possibility for someone who is willing to follow the forcing axioms program for the search of new axioms would be to 'go with $\mathbf{MM}^{+++}$' and so to come back positively on Viale's construction. The remaining cases involve the possibility that $(*)$ is consistent (with large cardinals) and, schematically, they can be summarised as

- the case of Compatibility
- the case of Incompatibility

[2] Assuming that $(*)$ is consistent with large cardinal axioms.
[3] Woodin calls $(*)$ the *ultimate* forcing axiom. See [43].

Before facing them directly and speculating on their possible implications for our central question concerning, ultimately, the *truth* of $\mathbf{MM}^{+++}$ and the *truth* of $(*)$, it seems appropriate to schematically present the case of the forcing axioms program for the search of new axioms. This is what we are going to do in Section (4.2). Once presented **some main aspects** of that program, we will introduce a *plausible* epistemological framework from where to evaluate, from a philosophical point of view, the case of the forcing axioms program. We underline that the debate concerning pluralism and the search for new axioms in Set theory is a huge one that goes fairly beyond the scope of the present work. In the next sections we will restrict our attention on some considerations concerning the notion of extrinsic justification and its role for a possible evaluation of the forcing axioms program. Our reasons for a similar restriction should emerge in subsection (4.2.2). We would like to stress how the general notion of extrinsic evidence should be somehow refined as far as we want to gain a more active grasp in our ability to evaluate the case of the forcing axioms program. A similar refinement, as we will try to point out, **comes together with** a significant revision of the way to deal with the question of pluralism in the philosophy of Set theory. The question of truth in mathematics is the horizon of our analysis. However, our analysis **doesn't exhaust** the question of the truth of the forcing axioms. Other different considerations need to be inquired and deeply studied, we think, which fall outside the range of the analysis of the present chapter. It is also important to stress since now that it is not part of the present work to put in question the notion of *extrinsic justification* which, as we understand things, lies essentially at the basis of the philosophical case of the forcing axioms program. The notion of extrinsic evidence has a long and important tradition in the practice of Set theory and that is **the starting point of the present work**. Instead, assuming the eminent role played by the notion of extrinsic evidence in the set theoretical practice, we want to understand which are some of its **possible implications** for an evaluation of the philosophical case of the forcing axioms program. This will take place in section 4.3. We will then concentrate (in section 4.4) on the relationship between $\mathbf{MM}^{+++}$ and $(*)$, and we will **sketch**, reflecting on the philosophical side of the question, the cases of their compatibility and of their incompatibility. An analysis concerning the state of the art of the mathematical relationship between $\mathbf{MM}^{+++}$ and $(*)$ is beyond the scope of the present work. Some useful informations concerning some sensible test cases that are known to be implied by $(*)$ but that is not known whether they are implied by $\mathbf{MM}^{+++}$ can be found in [24] (section 7, p. 2158). Finally, in section 4.5, we will draw some concluding considerations.

4.2 The *forcing axioms program* and the conjecture "$\mathbf{MM}^{+++} \Rightarrow (*)$"

We want to briefly schematize, in the present Section, the **paradigmatic case** for the axiom $AD^{L(\mathbf{R})}$[4], where, as it will emerge later, we can appreciate some concrete applications of the epistemological criteria for the selection and the justification of new axioms in Set theory. What we aim to do is to compose a general schema that characterises the case for the axiom of *Definable Determinacy*, $AD^{L(\mathbf{R})}$. Subsequently, we will try to reproduce the schema concerning the case of $AD^{L(\mathbf{R})}$ to an higher level case for a new axiom. Definable determinacy and standard large cardinal, in fact, leave unmoved the undecidability of important questions like, for example, the Continuum problem. In Chapter 2 we defined a *progressive* strategy for studying the universe of sets, V, in terms of its initial fragments H_k, for k a cardinal. We also identified the *next* structure to be axiomatized, i.e. the fragment $H_{\aleph_2}$. What we would like to see is the extent to which it is possible to reproduce *at the new higher level* our paradigmatic schema for $AD^{L(\mathbb{R})}$. What **we will see** is that, actually, there exist two distinct competing candidates to fill in the general schema, namely the axiom $\mathbf{MM}^{+++}$ and the Woodin's axiom $(*)$, and that a resolution for their evaluation seems to be connected with the possible scenarios for the solution of the Conjecture that we called " $\mathbf{MM}^{+++} \Rightarrow (*)$".

4.2.1 The extrinsic case of forcing axioms

The general schema we referred to in (4.2) for the case of $AD^{L(\mathbf{R})}$ can be found for example in Todorcevic [35] (section 4, p. 19), and it appears as follows.

(a) $AD^{L(\mathbf{R})}$ provides a structure theory for $L(\mathbf{R})$ which is a natural extension of the structure theory that can be established for sets of reals of lower complexity in ZFC.

(b) $AD^{L(\mathbf{R})}$ follows from the structure theory of $L(\mathbf{R})$ that it yields.

(c) $AD^{L(\mathbf{R})}$ follows from Large cardinal axioms.

(d) The Large cardinal axioms give an Ω-complete picture of $L(\mathbf{R})$.

[4]The axiom of determinacy, AD, is the statement that every set of reals is *determined*. We refer the reader to [19] (section 3.1) for a definition of when a set is determined.

(e) $AD^{L(\mathbf{R})}$ is equivalent to the existence of inner models of certain Large cardinal axioms.

(f) $AD^{L(\mathbf{R})}$ is implied by any other statement of sufficiently strong consistency strenght (measured on the scale of Large cardinal axioms).

In Chapter 2 we isolated the basic characterization of what we called the **view** of the *forcing axioms* program for the search of new axioms together with its main goal.

Goal : Figure out the picture of the set theoretic universe that would accomodate the *right* structure theory of $P(\omega_1)$ and, so, in particular, solve the Continuum Problem.

As we already noticed in chapter 2, it has been observed quite early that standard Large cardinal axioms are quite insensitive to the Continuum problem. As a consequence, as Todorcevic notices in [35], the structure theory of $P(\omega_1)$, since it must give an answer to the Continuum problem, cannot be so closely tied with Standard Large cardinal axioms[5]. This means that, in particular, we can't have the analogues of (c), (e), and (f) of our schema.Since PID[6] and OGA[7] (which are both implied by $\mathbf{MM}^{+++}$, see [35]) have substantial "ZFC-shadows", to some extent it is possible to have the analogue of point a) of our schema. What about the analogue of (b)?

Let's work on it a little bit. First of all, recall the following Theorem.

Theorem 4.2.1. *Assume $AD^{L(\mathbf{R})}$. Then,*

1. *Every set of reals in $L(\mathbf{R})$ is Lebesgue measurable*
2. *Every set of reals in $L(\mathbf{R})$ has the property of Baire*
3. *Σ^2_1- uniformization holds in $L(\mathbf{R})$*

[5] See [35]

[6] We give here the formal definition of the set theoretical dichotomy principle PID and we refer the reader to [35] for more informations on it.

Definition 4.2.1. *PID: For every P- ideal $\mathscr{I}$ of countable subsets of some set S either*

1. *there is uncountable $X \subset S$ such that $[X]^{\aleph_0} \subseteq \mathscr{I}$, or else*
2. *there is a decomposition $S = \bigcup_{n<\omega} S_n$ such that $S_n \cap a$ is finite for all $n < \omega$ and $a \in \mathscr{I}$*

[7] OGA is the following graph-theoretic principle: For every open graph $\mathscr{G} = (X, E)$ on a separable metric space X either $\mathscr{G}$ is countably chromatic or else $\mathscr{G}$ has an uncountable clique. See [35] for more informations

Theorem 4.2.1 is a nice Theorem since it asserts that under $AD^{L(\mathbb{R})}$ all sets of reals in $L(\mathbb{R})$ satisfy what are commonly considered *regularity properties*[8] of sets of real numbers. However, there is still space for a possible concern.

Possible Concern. There might be other (**incompatible**) theories sharing these fruitful consequences

Theorem 4.2.2. *(Woodin) Assume*

1. *Every set of reals in* $L(\mathbf{R})$ *is Lebesgue measurable*
2. *Every set of reals in* $L(\mathbf{R})$ *has the property of Baire*
3. Σ_1^2*- uniformization holds in* $L(\mathbf{R})$

Then, $AD^{L(\mathbf{R})}$ *holds.*

Meaning of the Theorem. Perhaps there are other statements which are incompatible with $AD^{L(\mathbf{R})}$ which also have these consequences. But this possibility is closed off by the previous Theorem. One can recover $AD^{L(\mathbf{R})}$ from its consequences, that is $AD^{L(\mathbf{R})}$ is not only a *sufficient* condition for (1), (2), and (3), but it is also a *necessary* condition. This shows that the evidence for the consequences (that is, their intrinsic plausibility) transfers to $AD^{L(\mathbf{R})}$. This is basically the information contained in point b) of our general schema.

Question. *Is it possible to recover point b) at the level of* $P(\omega_1)$ *that we are now inquiring?*

In our new schema for $P(\omega_1)$ we have **two candidates** that satisfies the analogue of condition a) and the analogue of condition d). Namely, the axioms $\mathbf{MM}^{+++}$ and $(*)$. Let's briefly spell out the "evidence equivalence" situation relative to point a) for the axioms under considerations. Our main reference for the following technical results is Woodin's *Forcing axioms and unsolvable problems* [43]. Next results show collectively how forcing axioms produce a structure theory of $H_{\aleph_2}$ where many pathologies are eliminated. We refer the reader also to Todorcevic's [35] (section 2) for a wider presentation of some main aspects of the structure theory of $P(\omega_1)$ under the influence of forcing axioms. In [35] the reader can also find informations for further references for the relevant definitions of the notions involved in the next results.

If we assume *Martin Maximum*, or $\mathbf{MM}^{+++}$, the following holds.

[8]We refer the reader to [19] for the definitions and for a first introduction to the regularity properties listed in theorem (4.2.1).

- Suslin's hypothesis.
- Suppose $X, Y \subseteq \mathbb{R}$ are each dense and locally of cardinality ω_1. Then X and Y are order isomorphic.
- There is a 5 element basis for the uncountable linear orders.
- Every homomorphism $\pi : C([0,1]) \to A$ of $C([0,1])$ into a Banach algebra is automatically continuous.
- Every automorphism of the Calking Algebra is an inner automorphism.

It is possible to show that through the axiom $(*)$ or one **natural** extension of it called[9] $(*)^+$ (such that $(*)^+ \Rightarrow (*)$) is possible to produce the same class of nice consequences for the structure theory of $P(\omega_1)$. Also, as Todorcevic stresses in [35], the situation concerning the **structure theory** of $P(\omega_1)$ under CH is completely different

> "while the Baire Category assumptions like $mm > \omega_1$ reveals a fine structure theory of $P(\omega_1)$ that could also address problems coming from different areas of mathematics, *nothing comparable to this is known if we require CH to be true*" (My emphasis)

By our analysis we are quite naturally faced with the following question

Are the axioms $(*)$ *and* $\mathbf{MM}^{+++}$ *compatible or incompatible?*

- If they are compatible, then we think we have some hints to be, in some sense, quite close to the situation depicted by clausole b) in the general schema for $AD^{L(\mathbf{R})}$
- If they are incompatible we have to face a serious problem. We have two incompatible theories sharing the same **fruitful consequences**.

What does emerge from the previous considerations is the centrality of the Conjecture $\mathbf{MM}^{+++} \Rightarrow (*)$ in order for us to evaluate the case for the axiom $\mathbf{MM}^{+++}$ and the case for the axiom $(*)$. The possible scenarios of the mathematical solution of the Conjecture give rise to the corresponding philosophical possibilities concerning essentially the *truth*[10] of the two axioms.

[9] See, for instance, [43], where the *extension* of $(*)$ to $(*)^+$ is illustrated appealing to the so-called *Sealing Theorem*.

[10] Are we dealing here with a **revisable** notion of truth? Maybe.

- Which one is to be selected if they are *compatible*?
- Which one is to be selected if they are *incompatible*?

4.2.2 The maximality's dilemma

On the basis of the **equivalence situation** determined by the extrinsic evidences accumulated so far at the level of $H_{\aleph_2}$, a possible solution for the choice between the two axioms could maybe come speculating more on the **two distinct notions of maximality** that, as we saw in chapter 3, are behind $\mathbf{MM}^{+++}$ and $(*)$ respectively.

- Π_2-Ω-Maximality
- Π_2-SSP-Maximality

Are there philosophical reasons for preferring one notion of maximality than the other?

Let's recall, first of all, the case for the SSP-Π_2-Maximality as we initially sketched it in chapter 2. The main point, as we understand it, seems to be that, via reduction of the proof of Π_2-formulas $\phi(p)$ (with p in $H_{\aleph_2}$) to the proof of the consistency of Σ_1-formulas, and exploiting for Σ_1-formulas the *generalized Cohen's absoluteness lemma* obtained by introducing forcing axioms of the general form FA_k $(\mathbb{P})$, $\Gamma_{\aleph_2}$-logic appears as a strong tool for studying the Π_2-properties of $H_{\aleph_2}$. We also saw that **MM** offers essentially the best analysis (in terms of maximality) of the class of posets $\Gamma_{\aleph_2}$. It is then asked if similar nice considerations could lead toward a more ambitious *completeness* result (and modulo the unavoidable Gödel's incompleteness) for the Π_2-theory of $H_{\aleph_2}$. Here the requirement of completeness appears in [37] more or less in the following way.

> **Question**. Can we recover inside the scope of our (first order) derivability relation '$\vdash$' not only Π_2-formulas $\phi(p)$ (with p in $H_{\aleph_2}$) via the reduction of their proof to the proof of the consistency of a Σ_1-formula by means of some $\mathbb{P} \in \Gamma_{\aleph_2}$ according with the method exposed in chapter (2), but also the **complement** of Π_2-formulas, that is, their negations expressible as Σ_2-formulas?

That is, given an arbitrary Π_2-formula $\forall x \psi$ and assuming that $\mathbb{P}$ doesn't force the Σ_1-property ψ for $\mathbb{P} \in \Gamma_{\aleph_2}$, so that we cannot recover the proof of $\forall x \psi$ from the proof of the consistency of ψ, we would have that for some $p \in \mathbb{P}$, p forces the negation of ψ, $\neg\psi$. If that is the case it is legitimate and meaningful to ask whether we can move from the forceability of $\neg\psi$ to $T \vdash \neg\forall x\psi$ for $T \supseteq ZFC$. **The requirement of completeness we are interested in is essentially reducible to the above question.** On similar presuppositions the main objection of the Π_2-Ω-Maximality to the SSP-Π_2-Maximality lies in the observation that *enlarging* the class of forcing notions to consider from the class $\Gamma_{\aleph_2}$ to the class Ω makes the completeness result more general (since it considers *all* the possible forcing notions) and this should not be understimated when we are dealing ultimately with questions concerning **mathematical truth**.

Given the previous formulation of the case of the tension between Ω and SSP-Π_2-Maximality, one should be able to reply to the following question.

Is the objection of the Π_2-Ω Maximality against SSP-Π_2 Maximality really tenable?

Another way to formulate the previous question is the following.

Does the abdication to $\Omega \setminus \Gamma_{\aleph_2}$ really represent a limit (from a philosophical point of view) of Viale's construction of the completeness result *compared* with Woodin's construction?

If we analyze closer Viale's approach to the study of the Π_2-theory of $H_{\aleph_2}$ we see that he is allowing parameter in $H_{\aleph_2}$. Apparently, this is the case also for Woodin's approach, but there is an important difference. Such a difference, we believe, emerges if we consider the definition of Ω-consistency[11]. What emerges by that definition is that the parameters in $H_{\aleph_2}$ allowed by Woodin are really *countable* objects in the set $H_{\aleph_2}$ as **computed by** a countable model M. This is how we understand Viale's observation in [37] (Introduction, p. 6) that Woodin is allowing only real parameters. Now, this is not the case in Viale's construction, since the parameters in $H_{\aleph_2}$ that he considers are **really** of size ω_1. Once one allows similar parameters of natural size, one faces a difficulty with the selection of forcing notions he can deal with, as we already stressed in chapter 2 and as the following passage in Bagaria [3] (section 5.1, p. 21) clearly summarized.

> "It is worth noting that it is a theorem of ZFC that all Σ_1 sentences that holds in some Boolean-valued model $V^{\mathbb{B}}$, allowing only sets in H_{ω_1} as parameters, are true. So, the Bounded Forcing Axioms are just natural generalisations of this fact to H_{ω_2}. (...) Moreover, as we pointed out in

[11] See chapter 3 of the present work (section 3.5).

> the last Section, V cannot be a Σ_1-elementary substructure of $V^{\mathbb{B}}$ for any non trivial $\mathbb{B}$. (...) Furthermore, if we want Γ to be the class of all forcing notions, then we cannot even have ω_1 as a parameter, since we can easily collapse ω_1 to ω, and saying that ω_1 is *countable* is Σ_1 in the parameter ω_1. (...) So a natural question is what is the maximal class Γ for which $BFA(\Gamma)$ is consistent with ZFC."

Allowing parameters of size ω_1 requires that one restricts the class of all forcings to some subclasses. **MM** defines the maximal subclass of forcings $\mathbb{P}$ such that $FA_{\aleph_1}\mathbb{P}$. This is Viale's scenario. Woodin's allows only real parameters (countable objects) and he maximizes the class of forcings.

> Which strategy represents the best way to study the right Π_2-theory of $H_{\aleph_2}$?

What (we think) we can say, given the previous considerations, is that at least the apparent advantage given by the more general aspect of Woodin's construction for the level of $H_{\aleph_2}$ doesn't seem so immediate as it could appear at the beginning. On the other hand, allowing parameters of size ω_1, why should one consider forcing notions in $\Omega \setminus \Gamma_{\aleph_2}$ (that is, forcing notions that are potentially able to destroy the intended meaning of the parameters) ? One could claim, in this sense, that the only forcings *pertinent for the inspection of the notion of truth* for the Π_2-theory of $H_{\aleph_2}$ are those in the class $\Gamma_{\aleph_2}$, that is, those that don't betray the meaning of the formula we are working with. So we are left with the following situation concerning the notion of Π_2-Maximality. On the side of Π_2-SSP- Maximality we have *more* formulas (more parameters) but less forcings. On the side of Π_2-Ω-Maximality we have less formulas but more forcings.

Is there a way out from this dilemma?

4.2.3 Summary

The analysis developed in (4.2.1) and (4.2.2) can be summarized in the following schematic way.

- (4.2.1) points to a remarkable similarity from the point of view of their mathematical implications between $\mathbf{MM}^{+++}$ and $(*)$.
- (4.2.2) indicates that behind $\mathbf{MM}^{+++}$ and $(*)$ lie two subtly distinct notions of maximality.

It is not clear to us how it should be framed the philosophical comparison between the two distinct notions of maximality as we characterized them in (4.2.2). For this reason our attempt to give a philosophical analysis for the case of the forcing axioms program in the next sections will be mostly centered on the analysis given in (4.2.1), and in this sense it will be mostly based on the notion of extrinsic evidence. As we already remarked in (4.1) however, other different considerations need to be inquired and, ultimately, for an accurate understanding of the philosophy surrounding forcing axioms, the concept of maximality has to be better examined. In light of a deeper understanding of the notion of maximality, we don't know if our considerations in the next sections would be still plausible or if **the central role that we give to the conjecture for a balanced evaluation of the philosophical case of forcing axioms should be reduced**.

Coming back to the possible cases concerning the relationship between $\mathbf{MM}^{+++}$ and $(*)$, and in light of the next **philosophical** considerations, the following question will emerge.

Question. *What is the impact of the incompatibility-case on the overall philosophical meaning of the forcing axioms program for the search of new axioms? How could we choose at that point between* $\mathbf{MM}^{+++}$ *and* $(*)$*?*

We are going, for the sake of the philosophical clearness, to clarify some presuppositions of a *plausible* epistemological framework that rest on the background of the specific case for new axioms that we are analysing in the present chapter. This should make **more precise** what we think is the role played by our conjecture in the context of the forcing axioms program.

4.3 Some general considerations concerning the program for the search of new axioms

There is a methodological claim that seems to be quite shared by different set theorists[12] engaged in the search of new axioms for Set theory and that can be well summarised by the following quote from [30] (section 4, p. 7).

> "We see the search of new axioms as an ongoing process, not dissimilar to the process in other fields of science, by which a scientific theory is crystallised by a sequence of trials and errors, where at any particular moment there may be several competing options".

[12] Though not by *all* of them.

The main aspect of the methodological approach to the search of new axioms contained in the above quotation is, as it seems to us, the analogy stressed with the methodology of the empirical sciences. A similar analogy **motivates** the relevance attached to the so called *extrinsic* evidences for the selection of new axioms to add to the standard axiomatic basis ZFC. This aspect of the contemporary search of new axioms for Set theory is not new, and the notion of extrinsic evidence, together with the analogy with the methodology of the empirical sciences, come back at least to Gödel. Here is a well known quotation from [11] (section 3, p. 521).

> "Even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its **truth** is possible also in another way, namely, inductively by studying its "success". Success here means fruitfulness in consequences...There might exist axioms so abundant in their *verifiable consequences*[13], shedding so much light upon a whole field, and yielding such powerful methods for solving problems. .. that, no matter whether they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory" (My emphasis)

The **dialectic** between intrinsic and extrinsic justifications seems to be constitutive of the search of new axioms in Set theory almost since its inauguration with Gödel. However, in the present Chapter, we would like to point to a **possible** difference in the way Gödel conceived *the pluralism/ non pluralism* debate in Set theory compared with the way that debate is conceived by (part of) the contemporary framework for the search of new axioms. A similar difference, as it seems to us, is influenced by the way to understand the dialectic between intrinsic and extrinsic justifications. Despite the relevance attached to the notion of extrinsic evidence, in fact, the very notion of extrinsic justification appears to us to be in some way **theoretically disconnected** from the metaphysical framework within which Gödel elaborates the question of pluralism in Set theory. From this point of view, and in light of some possible aspects of the contemporary search of new axioms in Set theory that, on our eyes, deserve attention, it would be meaningful to manifacture a metaphysical framework where to model the question of pluralism in such a way to confer more **theoretical depth** to the notion of extrinsic justification. This is the

[13] There seem to be two main characterizations on the notion 'verifiable':

- The first is when verifiable means 'provable in an accepted theory' (ZFC).
- The second is when 'verifiable' means ' has an high degree of intrinsic plausibility'.

See for example [30] section 4, or see [20].

main point that we want to (start to) make in the present Section. What should emerge from our exposition is a notable difference, if compared with Gödel, in the way **the question of pluralism** in Set theory is conceived in (at least part of) the contemporary debate.

4.3.1 Questions of realism behind the search for new axioms

The problem of pluralism in Set theory is strictly intertwined with the question of realism in the philosophy of mathematics. The question, however, is delicate. In fact, even if it is possible to encourage the practice for the research of new axioms *without* defending a specific metaphysical (or ontological) position, we feel that, ultimately, a genuine philosophical analysis of the overall question should face the ontological aspects of the problem. In its essence, the problem, in the actual context of the search of new axioms, can be reduced to the following one.

> **Problem 1**. *How can we come to know about the existence of an extremely rich realm of objects (the higher infinite) that are not expected to be in any causal relation with our physical world?*

Only if **there is** a mathematical reality to describe it seems theoretically justified the attempt to select the **true** mathematical principles that describe the universe of sets V.

As we already indicated in the Introduction to the present work, the phenomenon of **forcing Independence** motivates a pluralistic conception of Set theory [**]: there are different set theories and different conceptions of the notion of set not only one, **not the true one**.

> " Some have claimed that the early independence results in set theory already suffice to secure such a position. For example, it is claimed that the independence of CH with respect to ZFC shows that the choice between $ZFC + CH$ and $ZFC + \neg CH$ is one of mere expedience. It is maintained that although there may be practical reasons in favour of adopting one axiom over the other (say for a given purpose at hand) there are no theoretical reasons that one can give for one over the other". (See [14], section 5.1, p. 35)

Such a perspective openly annihilates in its own essence the search for the true missing set theoretic axioms that should describe the universe of sets V. There are traditionally at least two kind of reactions to the question presented by Problem 1, that is,

(a) Gödel's metaphysical realism, and

(b) Maddy's mathematical naturalism.

What we would like to stress in the present section is that both the previous philosophical positions don't seem to be very sensible concerning the *problem of selection* imposed by our analysis from section (4.2) between the axioms $\mathbf{MM}^{+++}$ and $(*)$. That problem, in its essence, evokes the possibility that there exist two axioms **remarkably similar but potentially incompatible**. In fact,

- Considering (a), the epistemological framework induced by Gödel's platonism doesn't seem to offer any clear constraint about which one among the two axioms we should select, since for all we know the reasons why one or the other could be true in the hyperuranion-world of mathematical objects lie beyond our possible understanding.

- Regarding (b), consider the following passage from [10] (p. 415).

  > "Justifications in this view come from within couched in simple terms of what means are most effective for meeting the relevant mathematical ends. Philosophy follows afterwards as an attempt to understand the practice not to justify or criticize it".

  According with the previous passage, Maddy's naturalism is unable in principle to face the question raised by our conjecture, since that question springs out directly from the practice that her naturalism identifies as the only source of justification for the selection of a new axiom.

From this point of view, the case of the forcing axioms program, as we presented it, seems to relieve some limitations in the conception both of Gödel and of Maddy, in the sense that the two positions appear for different reasons to be **not reactive** regarding the **essential problem** presented by the conjecture, that is, ultimately, the possible *existence of two very similar axioms potentially incompatible.* For the purpose of a better understanding of the question of pluralism in Set theory, it would seems important to dispose of a more equipped epistemology than the ones expressed by Gödel's and Maddy's fundamental philosophical positions, so that to have more leeway over the question imposed by the relation between $\mathbf{MM}^{+++}$ and $(*)$. Last observation seems to be in line with the following consideration made by John Steel in [10] (p. 433).

> "a solution to the Continuum Problem may need some accompanying analysis of what it is to be a solution to the Continuum Problem, and in this way, Philosophy may have a more active role to play at the foundations of Mathematics".

Given the *reasonable* intention to respect the practice for the research of new axioms, the challenge that the previous considerations place is to formulate a tenable realistic position able to give reason to that practice relatively to Problem 1 that we stated at the beginning of section (4.3.1) and that allows us to deal with the kind of situation presented in section (4.2) of the present chapter.

Instead of facing directly Problem 1, we choose to follow, in the present work, a more indirect way to approach it. Namely, we sketch an **argumentation** that places the conjecture in the middle of the philosophical case for the forcing axioms program. Our argumentation **reflects** some aspects of what we consider an interesting way to **treat** the question of pluralism in the contemporary debate in Set theory and that, on our eyes, seems to have some confirmation in the contemporary literature. We will try then to elaborate a little more on it. What is important to note is that a similar argumentation, in its fundamental aspects, fits quite well with the **methodological attitude** contained in the quotation by Magidor we started with in section (4.3), and that it is *in an essential way* centered on the notion of **extrinsic justification**. In a sense that should be made more precise later, similar aspects shape the search for new axioms in Set theory in a way that is **coherent with** the general methodology of evaluation proper also of the empirical sciences. A similar perspective will give us a possible angle from where to analyze the philosophical case of the forcing axioms program together with its potential difficulty raised at the end of section (4.2), and the reasons should appear why we think that case is to some extents pending on the possible solutions of the conjecture $\mathbf{MM}^{+++} \Rightarrow (*)$.

Once we have presented some main aspects of that promising way to study the question of pluralism in contemporary Set theory, we will come back to Problem 1 and we will restate it in terms of how the notion of mathematical existence **should be thought** in light of the main aspects of the approach to the study of pluralism in Set theory **we want to propose in the present work**. We will not offer a definite proposal for a solution to Problem 1, nevertheless, we will fix some **general characters** that the form of realism we are searching for should calibrate.

4.3.2 The question of pluralism in contemporary Set theory

Regarding the question of realism, it seems possible to distinguish two broadly different ways of thinking in the Philosophy of Mathematics. Under the voice "Large cardinals and Determinacy" of the *Stanford Encyclopedia of Philosophy* we find them characterized in the following way

(a) **Taking reason to be prior to realism**.

(b) **Taking realism to be prior to reason**.

Relatively to point (a) a possible description of it is as follows.

> " People in this category [taking reason to be prior to realism] take objectivity to be the hallmark of realism and they **come to their conclusions concerning realism** about a given domain **only after** one has a good understanding of what kind of theoretical reasons have traction in that domain" (My emphasis) (See [19], section 5)

Now, plausibly, the main unknown in the previous passage concerns the notion of theoretical reason. In this sense, the question that we should ask becomes the following:

Question. What do we exactly take **reason** to be?

The notion of taking reason to be prior to realism is not extremely transparent under this respect by the previous passage. Appealing, though, to the case for $AD^{L(\mathbb{R})}$ that we rapidly sketched in Section (4.2), we can approximate an interpretation for that notion in the present context.

Let's consider, first of all, the following passage from [19] (section 5.).

> " Similarly in the set theoretic case the non-pluralist sees theoretical reason at play at higher level, beyond the theorems and the intrinsically plausible statements. Just as in astronomy the non-instrumentalist finds evidence of a higher level structure in the constellation of connections, likewise in the case of set theory the non-pluralist takes the constellation of connections in the interpretability hierarchy- in particular, the above theorems concerning $AD^{L(\mathbb{R})}$ -as providing **evidence of structure at a higher level** ". (My emphasis)

The general schema that we proposed for the case of $AD^{L(\mathbb{R})}$ in section (4.2.1) captures, as we understand it (in a schematic way) the constellations of connections concerning $AD^{L(\mathbb{R})}$. As it is stressed in [35] (section 1, p. 1), the structure theory of $L(\mathbb{R})$ under the assumption of $AD^{L(\mathbb{R})}$ is "the most widely accepted among all other theories about countable structures ."

That case is essentially based on the notion of extrinsic evidence, as it is, for what we saw in section (4.2), the case for the forcing axioms program. In this sense the **notion of reason** we are dealing with for studying the question of pluralism **embraces** the notion of extrinsic evidence. If we tailor our considerations to the case study of the present work, that is, to the fundamental equation (2.2)

$$\frac{L(\mathbb{R})}{AD^{L(\mathbb{R})}} = \frac{P(\omega_1)}{?}$$

then, the main challenge may appears as follows.

How much evidence of structure can the forcing axioms program provide at the higher level of $P(\omega_1)$?

This was precisely the meaning behind our quest in Section (4.2) relatively to the possibility to reproduce the general schema for $AD^{L(\mathbb{R})}$ to the higher level case of $P(\omega_1)$. At that level we can give to pluralism a quite precise formulation.

Pluralist default position. There is no **correct** axiomatization for the theory of $P(\omega_1)$, but different possible choices that ultimately rely on practical reasons.

By contrast and in light of the previous passages that we considered we can ask the following question.

Which is the vision of Pluralism at the level of $P(\omega_1)$ the set theorist is **committed to** by his taking reason to be prior to realism?

A *first possible methodological difference* is that, from the point of view of the program for the research of new axioms (as we are trying to describe it), a pluralistic scenario relative to $P(\omega_1)$, as the one depicted by the previous pluralistic default position, should not be assumed **in advance** with respect to the enterprise of the research of new axioms.

Toward the formulation of a more precise reply to the pluralist's default position that, on our eyes, could well represent the position of (part of) the contemporary framework for the search of new axioms we could consider the next quotation from [14] (section 5.2, p. 40). A similar quotation puts some light on a further element of the contemporary framework for the search of new axioms that we would like to stress here.

> “ A key virtue of this scenario [a pluralist scenario concerning a certain level of the universe of sets] is that it is *sensitive to future developments in mathematics*. In this way by presenting mathematically precise scenarios that are sensitive to mathematical developments, the pluralist/non pluralist can give the question of pluralism **mathematical traction** and through time tests the robustness of Mathematics” (My emphasis)

The concept of mathematical traction as the property of a (philosophical) scenario to be sensitive of the mathematical developments may appear a little abstract at this point. We are going to propose an interpretation of the concept, as well as, what we consider a possible example of it in what follows. Before doing that and in order to motivate our next reply to the pluralist's default position, let's try to elaborate a little bit more on a point. It can be helpful to consider the next passages.

> “It should be mentioned that the non pluralist about $AD^{L(\mathbb{R})}$ **is open to** the possibility that pluralism holds at higher levels, say at the level of CH.” (My emphasis) (See [19], section 5.))

> (about the $V = Ultimate\ L$ approach) “ while if the answer oscillate one will have evidence that these statements are absolutely undecidable and this will strengthen the case for pluralism. In this way the question of absolute undecidability and pluralism are given mathematical traction”. (See [18], section 7.)

Broadly speaking the non pluralist position is for Gödel a metaphysical assumption (choice) that precedes and lays as the general **unquestioned** premise motivating the practice for the search of new axioms. (This is an instance of taking *realism to be prior to reason*).

> “Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality..” (See [11], section 3, p. 520.)

In light of what we said before regarding the commitment of the Set theorist to the a posteriori mathematical knowledge for evaluating the coherence and hence the existence of a conception**, we think, schematically, that in the contemporary framework for the search of new axioms , as we understand it, non pluralism more than a general metaphysical **premise** is a question that can be inquired with mathematical methods, and, as such, it is to be considered as the result (the outcome) of the activity of research of new axioms rather than as its starting point. The previous considerations can be maybe refined appealing to the following key idea.

Key Idea. Non pluralism in the new context more than a metaphysical option (as for Gödel) is, instead, **a scientific hypothesis**: *as such it is revisable.*

In this sense, we think that, in the contemporary framework for the search of new axioms, **the pluralistic position** too more than being considered as a wrong metaphysical choice, it appears, rather, as a conceivable scenario certified by the pervasive presence of the **forcing independence phenomenon**. This makes Pluralism, in the new context for the search of new axioms, a theoretical possibility. This is a peculiar factor. We call such a peculiar factor of the contemporary framework for the search of new axioms

the theoretical possibility of pluralism.

To give more context to the previous considerations and to the notion of mathematical traction we come back to the case study of the present chapter: the forcing axioms program. We aim at approaching its philosophical case along the lines of the previous considerations. **The overview** from Section (4.2) pointed to two distinct principles, namely $\mathbf{MM}^{+++}$ and $(*)$, which both produce nice structure theory for $P(\omega_1)$. The analysis there drove us quite naturally to the question of the direct relationship between $\mathbf{MM}^{+++}$ and $(*)$. In light of the restatement of the pluralism/non pluralism debate as contained in the previous Key Idea, we will try to center the philosophical case of the program mostly on a specific conjecture that has been formulated within it. The conjecture being, as we already noticed

$$\mathbf{MM}^{+++} \Rightarrow (*)$$

Question. How can we understand the **rational connection** between the formulation of the conjecture $\mathbf{MM}^{+++} \Rightarrow (*)$, and the case for the forcing axioms program within which it is formulated?

The main point here is what we may consider a **corollary** of the previous Key Idea. The corollary, we think, states a *conceptual closeness* between two distinct notions:

- the notion of philosophical tenability of a non pluralist position in the philosophy of Set theory concerning a certain level of the universe of sets (which, as we saw before, requires a positive solution of Problem 1), and
- the notion of philosophical tenability of a scenario for new axioms (which, in our context, is mostly based on the notion of extrinsic evidence).

We are going to spell out the notion of conceptual closeness between the two distinct notions of philosophical tenability by contrast with what we conceive as a *conceptual remoteness* between those two notions both in Gödel's and in Maddy's positions. However we stress in advance that such a conceptual closeness doesn't exhaust by itself the analysis of the question of Pluralism, as we understand it, in the contemporary debate, and we don't want to claim that the question of pluralism can be reduced *tout court* to the question of the philosophical tenability of a program for new axioms. Rather, such a conceptual closeness between the two distinct notions of philosophical tenability licences the possibility to **inquire** on one notion *in terms of* the other. In a schema the situation can be depicted as follows.

Abstract schema :

(a) Philosophical tenability of non pluralism at the level X.
$\Downarrow$ *Shift*
(b) Philosophical tenability of a scenario for a new axiom for the level X.

The previous abstract schema in unconceivable for Gödel, since in his metaphysical framework, (a), (non pluralism at the level of X), doesn't depend on (b), (the existence of an axiom for X). On the contrary, it is (a) that properly justifies (b). In this sense, the shift contained in the abstract schema is not hosted by what we may *euristically* consider the ideal logical implication $(a) \rightarrow (b)$ that is incapsulated in Gödel's metaphysical realism. Note that, along this euristic line, for Gödel

$$\neg((b) \rightarrow (a)),$$

where last implication, in our opinion, expresses the distance of the order of metaphysical questions, (a), from the level of mathematical practice, (b), in Gödel's framework. As regards Maddy's position, her naturalism shapes, we think, the following *ideal* logical implication between (a) and (b): $(b) \rightarrow (a)$. It thus seems available, in principle, to host the shift contained in the abstract schema above. There is, though, an important observation to make. The relation expressed by Maddy's naturalism implies in fact that

$$\neg(a) \rightarrow (b).$$

Last passage expresses, on our view, a substantial disconnection in Maddy's naturalism between the order of mathematical questions, (b), from the order of metaphysical questions, (a). By the way, the case of the conjecture we are analyzing in the present chapter seems to relieve a problem exactly on this point of her naturalism. This seems to suggests that we should **restrict** the relation between (a) and (b). The restatement of the conception of pluralism contained in the Key Idea seems to be the first step in this direction. In fact, it seems to suggest a possible direction for a refinement of the notion of philosophical tenability of a program. Our Key Idea implies, as we saw, what we considered a peculiar factor, and that we characterized as the notion of theoretical possibility of pluralism. A similar notion is absent from Godel's philosophical framework, and it doesn't seem to be really stressed in Maddy's naturalism. It is, instead, the peculiar character of the conception of pluralism we are trying to elaborate in the present chapter. Exploiting a similar notion it seems possible to refine the notion of philosophical tenability of a program and **to track it to a more theoretical dimension**. As we are going to see, the notion of theoretical possibility of pluralism drives us toward the design of a more theoretical property that a program for new axioms should meet. Such a property is not in contradiction with the usual extrinsic criteria of evaluation for a new axiom, but, as we may say, it subsumes them under itself and it confers to them theoretical traction.

What is it, we may ask, that confer **philosophical tenability** *to a program?*

In light of what we said before, in our context of the search of new axioms for set theory *the problem of* the philosophical tenability of a program is evaluated primarily considering the degree of impermeability of the program *with respect to*, what we called, the theoretical possibility of pluralism. (This is a peculiar criterion if compared with the usual criteria for the selection of an axiom. (*A theoretical criterion.*)]

First of all, analyzing the notion of philosophical tenability of a program for new axioms **in terms of** the main character of the conception of pluralism contained in the Key Idea leads, we think, to the following observation.

> **Observation** : It is not enough to introduce a good (both for intrinsic or extrinsic reasons) axiom. It is important to close off or limit the possibility that there exist other good axioms (**from the point of view of a program**) that are incompatible between each others.

We notice how, as it seems to us, the requirement contained in the last sentence of the previous observation (that we are going to call the requirement of uniqueness) plays in the present context a substantially different role than the one it plays in Gödel's metaphysical framework. Non pluralism, from the point of view we are trying to develop in the present chapter, is not secured independently by the level of the mathematical activity, and, in so far as it is an open question studied also in mathematical terms, the requirement of uniqueness becomes a sensible element of evaluation for the possibility of non pluralism relatively to a certain level of the universe of sets. Instead, since for Gödel, as we characterized his position, non pluralism is **the metaphysical datum that justifies the practice for the search of new axioms**, the requirement of uniqueness, though it can be a desirable epistemological requirement, it doesn't seem ultimately to play the same structural role as in our context. On the other side, it doesn't seem theoretically clear how one could ask for such a criterion in the context of Maddy's naturalism, since in that context the mathematical practice is the only source of justification of new axioms. The requirement of uniqueness as a regulative criterion for the evaluation of a new axiom seems to be, in this sense, **an acquisition made legitimate by the restatement of the conception of pluralism contained in the Key Idea.** If we look at the case of the Axiom of Definable Determinacy, $AD^{L(\mathbb{R})}$, and, in particular, at Woodin's recovery theorem (4.2.2) , we can appreciate a nice instantiation of the previous observation. The next quotation stresses the point.

> " Without the **recovery theorems** one could always wonder whether there are incompatible axioms with the same fruitful consequences. In contrast to the case of physics, where one could never hope to show that the data logically implies the theory, in the case of the search for new axioms this is in fact possible. It is the recovery theorem that seal the case". (See [19], section 4.7.)

Referring to the case of $AD^{L(\mathbb{R})}$ as we schematically presented it in Section (4.2), we can appeal to the notion of **convergence** as to the general *theoretical* property of that program for new axioms such that pluralism (**in the form of** distinct incompatible axioms, or **in the form of** the absence of a good consistent candidate axiom) cannot resurrect from within it.

So if we want to **inquire about** non pluralism at the higher level of CH, by the previous considerations, we should follow preliminary the methodological advise contained in the following provisional note.

Provisional note: The extent to which non pluralism is preserved in the philosophy of Set theory at the *higher* level of the Continuum problem **depends** (also) by the extents to which it is possible to obtain convergence for a program for new axioms at that next level.

It is the introduction of the property of convergence what determines the conceptual closeness between the two distinct notions of philosophical tenability. Such a conceptual closeness is the proper setting that makes it possible the application of the concept of mathematical traction. We would like **to propose** the idea the conjecture $\mathbf{MM}^{+++} \Rightarrow (*)$ is aimed at testing the degree of convergence proper of the forcing axioms program for the search of new axioms. It plays a role *similar* to that of the recovery theorem in the case of $AD^{L(\mathbb{R})}$.

The **motivation** for our previous proposal, we think, seems to be the following. If there is no convergence, that is, if the pluralism resurrects from within the forcing axioms program, this, *by the characterisation of the notion of philosophical tenability of a program in terms of the notion of convergence, and by the conceptual closeness between the philosophical tenability of non-pluralism and the philosophical tenability of a program for new axioms*, constitutes **mathematical traction** toward the problematical philosophical nature of the program and toward the pluralist position. If, instead, there is convergence of the Forcing axiom program, this *produce* mathematical traction towards the non-pluralist position and the philosophical tenability of the program.
More generally, the set theorist involved in the search for new axioms doesn't dogmatically exclude the possibility of pluralism, neither he assumes a priori such a pluralistic position. Instead he tries to attach **mathematical substance** to the pluralism/ non pluralism question. From a similar point of view we think that the right distinction is the one between the skeptic and the advocate of the Contemporary framework for the search of new axioms more than the one between the pluralist and the non pluralist. In fact, if the philosophical tenability of a program is put in question by the refutation of a conjecture, this is evidence in favour of the pluralistic stance. This doesn't mean that, for example, the refutation of a conjecture by itself close **necessarily** a program for the search of new axioms, rather it indicates that the way to think at the pluralism/non pluralism question **is changed**.

Idea. *The demarcation line between pluralism and non pluralism is crossable on the basis of mathematical results.*

This, we think, means essentially **treating Set theory as a science** (like biology or physics). Consider the following passage from [28] (section 1.1, p. 17).

> "..one might hold that set theory is about the concept of set, as given in the axioms of ZFC, and that the concept can be extended in various, equally admissible ways. Opposed to these is the view that set theory is a *science*, like physics or biology, that questions undecided by our current theory are nevertheless legitimate, and that what is needed to decide them *is more theory*". (My emphasis)

For similar reasons we formulate the response to the initial pluralist's default position by the contemporary advocate for the search of new axioms as follows.

Ideal response from the contemporary Set theorist: A similar scenario as the endorsement of a pluralistic position relative to the theory of the structure $P(\omega_1)$ shouldn't be assumed *unless it has mathematical traction*. Unless it turns out to be an inevitable mathematical point[14].

What we would like to suggest here, **motivated by** the preceding analysis, is that, somehow, the idea of mathematical traction (as the property of a (philosophical) scenario to be sensitive of the mathematical developments) acquires in the contemporary framework for the search of new axioms the status of a proper method with which the set theorist deals ultimately with the question of pluralism[15]

General remarks. In the previous paragraphs we tried to make an **argument for defending the idea of the centrality of the conjecture MM$^{+++} \Rightarrow (*)$ in order for evaluating the philosophical case of the forcing axioms program**. However, we are aware of the fact that different aspects of the previous argument need to be more inquired and that they rise different issues. We list here below some of the main points that, in our opinion, should be more explored.

- First of all there is the question of the massif use of the notion of extrinsic evidence that is behind the case of the forcing axioms program. A clear criticism to the notion of extrinsic evidence is expressed paradigmatically by the advocates of the so-called *Hyperuniverse Program* for the search of new axioms. Here is an significative passage from [1] (section 3.3, p. 18, footnote 29.)

  > "When declaring the intention of extending ZFC so as to settle independent questions, one also requires that one be as *unbiased* as possible as to the way such questions should be settled and as

[14]But when it is an inevitable mathematical point? How can we come to recognize it?

[15]We are not saying that it is the only method, we are saying that it is the most specific one.

> to which principles and criteria for preferred universes one should formulate. In particular, the latter must not be chosen at the outset so as to be apt for settling questions independent of ZFC, or for meeting the needs of some particular area existing in set-theoretic practice".

By the previous quotation it seems plausible to think that the advocate of the Hyperuniverse Program would conceive forcing axioms ultimately as *ad hoc* hypothesis built up more or less for meeting the needs of some particular area existing in set theoretic practice. A full understanding of the issue would probably require to face the question concerning the philosophical status of the so-called regularity properties as we framed it in (2.9). We limit ourselves to notice here that our characterization of the notion of philosophical tenability of a program in terms of the property of convergence transforms the conjecture in an actual prediction for the forcing axioms program. That prediction exposes the program to a certain **degree of risk**. This seems in line with the general dynamics of evaluation proper of the empirical sciences.

- A note aside should be added concerning the proper method of justification of large cardinal axioms that is coherent with the conception of non pluralism in the philosophy of Set theory that we tried to outline in the previous paragraphs.

- Our considerations don't represent a **conclusive** argument against the skeptic's objection[16] and his eventual choice to reject the search of new axioms in favour of an a priori pluralistic stance toward the notion of truth in Set theory. We notice also that there is an important conceptual framework in the contemporary debate within which to articulate a pluralistic scenario for Set theory, and it is known as the *multiverse* framework. There are different degree of multiverse, and it is beyond the scope of the present work to explore such an interesting notion. We limit ourselves to say that it is on this conception and on the question of the conditions of tenability of such a conception that it is played an important match *between the set theorist and the skeptic*[17].

[16]We wonder however if it is possible to refute absolutely the skeptic's position and **if it does make sense** to ask for something like that. Correspondingly, we refer to the following meaningful quotation by Russell from [20]: " *Universal skepticism, though logically irrefutable, is practically barren: it can only, therefore, give a certain hesitancy to our beliefs, and cannot be used to substitute other beliefs for them".*

[17]See for more informations [18] (section 4).

- The argumentation that we sketched in the previous paragraphs deals primarily with the notion of philosophical tenability of the forcing axioms program, but it doesn't draw any conclusion concerning the truth of the forcing axioms. This is also why we preferred to speak about conceptual closeness between the notions of philosophical tenability of non pluralism and philosophical tenability of a program for new axioms, instead of conceptual equivalence. However, the question of the truth of the forcing axioms remain the ultimate goal of the analysis that we started in the present chapter.

  Can we, from the philosophical tenability of the forcing axioms program, draw some conclusion concerning the truth of forcing axioms?

  Here the question appears remarkably delicate and it goes far beyond the scope of the present work. We limit ourselves to make the following observations.

  As it is emerged from section (4.2), the case for $AD^{L(\mathbb{R})}$ is only partially generalizable to the case of $\mathbf{MM}^{+++}$ and $(*)$. In particular, the disconnection of the CH from the standard large cardinal axioms is an hint that the situation with the structure theory of $P(\omega_1)$ represents plausibly a transition point in the kind of justification that one can offer for new axioms. The kind of justification for the forcing axioms that we schematically summarised in (4.2) is centered in showing the **powerful effects** that forcing axioms have in a vast spectre of different branchs of mathematics. The ramification of theorems produced by forcing axioms offers, in line with the analysis of the previous paragraphs, a hint for relieving existence of structure at the level of $P(\omega_1)$, and our main argument stresses how, on similar presuppositions, the eventual verification of the conjecture would strengthen the case for this point of view. Quoting from [35] (section 1, p. 1),

  > "..many deep part of the theory of $P(\omega_1)$ are of low consistency strength, as measured on the scale of the current large cardinal axioms. This is due to the fact that the theories of $P(\omega_1)$ that we develop are based on quite different set theoretic principles and as a consequence we must develop another way to measure their inevitability. For examples, inevitability, in this context, has to be measured by the relevance of these theories to the rest of mathematics not just to set theory itself."

However, there is space for radically different strategies than the one proposed by the forcing axioms program for giving a solution to the fundamental equation (2.2). In particular, here our main reference is to Woodin's new approach with $V = Ultimate - L$. This last appears to work with the notion of extrinsic evidence as far as the question of his practicability is relied on the verification of a mathematical problem known as the HOD-conjecture. The study of the relation between the $V = Ultimate - L$ approach and the forcing axioms program (and of their respective presuppositions) appears as a crucial topic for better understanding the question of the possibility of the truth of an axiom at the level of CH. We will briefly come back to similar questions in the final part of the present chapter.

We are finally driven back to the following *essential* question:

> **Problem 1 revisited**. How should the notion of mathematical existence be modulated in order to philosophically accomodate the conception of pluralism as the one depicted in the previous paragraphs and expressed by the previous Key Idea? Which are or should be the metaphysical presuppositions of a similar conception of pluralism?

We are not going to offer a solution to the previous question in the present work, but we stress some general aspects towards the formulation of a possible solution to that question:

- The form of realism we are searching for should differ deeply by the kind of strong platonism defended by Gödel. In particular it shouldn't be guaranteed in advance with respect to the practice for the search of new axioms.

- It should recast the dialectic between intrinsic and extrinsic justifications attaching **theoretical legitimacy** to the notion of extrinsic justification. In particular it should give an account of the correlation between the existence of a network of theorems concerning a certain case for a new axiom and the possibility to relieve **existence of structure** at the level of the axiom in question. This seems to point to a realistic conception that is built up carving into the **interdependence** between the epistemological devices that are available to us and the metaphysical issues. The following quotation from Koellner [20] seems to express the same point.

  > "The tension itself points to an interdependence of metaphysics and epistemology and thereby suggest a more nuanced approach, namely,

> the approach of developing the epistemological and metaphysical conception in tandem, working back and forth and implementing a procedure of mutual adjustment until we reach reflective equilibrium, the hope being that we will converge on a view which is both faithful to mathematical practice and **is short of** extravagant metaphysical and epistemological features" (My emphasis)

- Contextually we stress here how the notion of mathematical traction indicates a **direct** engagement of the mathematician in questions standardly considered to belong to the metaphysical side of the activity at the foundations of mathematics. This, we think, is a structural consequence of having substantially shifted (though not reduced) the question of the philosophical tenability of non-pluralism for a certain mathematical structure to the question of the philosophical tenability of a **program** for new axioms concerning that structure. We limit ourselves to say here that in the actual context of the Philosophy of mathematics, where a lot of efforts are put in preventing philosophy from being normative toward mathematics and, substantially, in unfastening the philosophical goals from the mathematical goals at the level of the foundations of mathematics(as the tradition of Naturalism in mathematics, or the invocation on the notion of mathematical practice, from our point of view, seem to indicate) the previous considerations concerning the notion of mathematical traction, assuming they are plausible, seem to express an interesting and different way of thinking to the relationship between philosophy and mathematics. In a slogan

  One does philosophy (also) doing mathematics.

In section 4.4 we will briefly sketch a closer analysis of the conjecture $\mathbf{MM}^{+++} \Rightarrow (*)$ indicating quickly how the respective cases of compatibility and incompatibility should be approached and set up according with our epistemological framework.

4.4 Outline of a philosophical evaluation of *generic absoluteness* through an analysis of "$\mathrm{MM}^{+++} \Rightarrow (*)$"

There seem to be the following possibilities concerning the relationship between $\mathbf{MM}^{+++}$ and the axiom $(*)$.

1. All models of $\mathbf{MM}^{+++}$ are *not* models of $(*)$. That is $\mathbf{MM}^{+++}$ refutes $(*)$.
2. All models of $\mathbf{MM}^{+++}$ are models of $(*)$. That is, $\mathbf{MM}^{+++}$ implies $(*)$.
3. Some model of $\mathbf{MM}^{+++}$ is also a model of $(*)$. That is, $\mathbf{MM}^{+++}$ and axiom $(*)$ are compatible.

As we observed before, if the axiom $(*)$ is inconsistent with Large cardinal axioms, then $\mathbf{MM}^{+++}$ appears, *at least within the scope of the forcing program for the research of new axioms*, as the only plausible possibility. *But what if* $(*)$ *is consistent?* In that case, which are the philosophical scenarios corresponding to the possible mathematical solutions 1), 2), and 3)? In other words, under the fundamental assumption of the consistency of the axiom $(*)$ with Large cardinal axioms, which axiom, among our two, should we choose (that is, should we consider as the *correct* axiom) if the solution of the Conjecture will be in the sense of 1)? And which one if the solution will be in the sense of 2), or in the sense of 3)?
We notice as a preliminary the following remark.

Remark.[18] It is known that **MM** doesn't imply $(*)$. Also $\mathbf{MM}^{+\omega}$[19] does not imply $(*)$.[20]

On the background of the possibilities depicted above, we can better specified the nature of our conjecture saying that it is properly focus on possibility 2). (We may leave open the possibility that it could be verified, to some extent, also in case of 3). Anyway, we don't know exactly how to deal with possibility 3)).[21]Otherwise, the conjecture is falsified, and that can only mean: possibility 1), that is, the *refutation* case. This last possibility seems to raise the most serious problem:

How can we choose among $\mathbf{MM}^{+++}$ *and the axiom* $(*)$?

By itself, possibility 1), since it is a refutation of the conjecture, induces to ask what is the impact of such a possible scenario on the overall *philosophical tenability* of the forcing axioms program for the research of new axioms, and whether it could have a negative effect on both our principles. Although we think that, **at least**

[18]The Remark makes the status of the Conjecture really crucial.

[19]See [25] (section 7, p. 2158) for a definition of these principles. They can be informally described as intermediate strengthening of **MM** weaker than $\mathbf{MM}^{++}$.

[20]Folklore: In all **known** models of **MM**, $(*)^{+}$ fails.

[21]If we adopt, for example, Viale's completeness theorem, then we could reasonably expect that the conjecture consists only of the implication and the refutation case. This follows by the very nature of Viale's result.

on the basis of the analysis sketched in 4.3, an eventual refutation of the conjecture represents a serious problem for the philosophical tenability of the forcing axioms program relative to the question of pluralism, (since it would undermine its property of convergence), we are not suggesting here that the possible refutation of the conjecture would necessarily sanction the philosophical failure of it. It remains still open the possibility that, through the accumulation of further extrinsic evidences it becomes possible to select between $(*)$ and $\mathbf{MM}^{+++}$ the most plausible one. Since $\mathbf{MM}^{+++}$ is a global axiom, contrary to $(*)$ that is in essence an axiom that can be localized to H_{ω_2}, the comparison between the two axioms should take place on two distinct levels

- at the level of H_{ω_2}
- beyond the level of H_{ω_2}

4.4.1 The case of Incompatibility

If we compare MM^{+++} and $(*)$ **at the level of $\mathbf{H}_{\aleph_2}$** we are faced first of all with the following question.

Question: Is it the case that their incompatibility will manifest itself already in the structure theory of $P(\omega_1)$? If this is the case, then, under the hypothesis of their Incompatibility, it seems natural to ask the following questions

- What is the perspective for the structure theory of $P(\omega_1)$ under the axiom $\mathbf{MM}^{+++}$?
- What is the perspective for the structure theory of $P(\omega_1)$ under the axiom $(*)$ (or $(*^+)$)?

In line with the epistemological framework of section 4.3 it is possible that, in case of Incompatibility, through the accumulation of further extrinsic evidences, it becomes possible to select between the two axioms the most plausible one. We notice, anyway, that a similar perspective could also not unfold, since it seem conceivable that the amount of consequences they share concerning the structure theory of $P(\omega_1)$ is more meaningful from a philosophical point of view than the amount of consequences on which they diverge.

Question: What, if we *don't* restrict the comparison at the level of $H_{\aleph_2}$?

$\mathbf{MM}^{+++}$ is, in fact, a **global** axiom that has a lot of consequences[22] which are not consequences of $(*)$. *How much does the situation change in that case?*

4.4.2 The case of Compatibility

If $\mathbf{MM}^{+++}$ implies $(*)$, then this would represent a strengthening, from a philosophical point of view, for the forcing axioms program as it would be a remarkable extrinsic evidence both for $\mathbf{MM}^{+++}$ and for $(*)$. In fact, the only reason for selecting $(*)$ and not $\mathbf{MM}^{+++}$ could come from maximality considerations discussed above. If, as it emerges from the previous analysis of the notion of maximality, there aren't specific philosophical reasons for preferring one notion of maximality with respect to the other, then we have no particular reasons for not selecting both of our principles, and so we should take **their conjunction**. However consider the following potential difficulty raised by Woodin.

Under $(*)$ the following phenomenon happens

$$Th(H_{\omega_2}) \text{ is } \textit{logically reducible} \text{ to } Th(H_{\omega_1})$$

Suppose that $MM^{++} \Rightarrow (*)$. We ask the following question.

Question : Is it the case that the logical reducibility phenomenon concerns also $\mathbf{MM}^{+++}$? If yes, what is the impact of such a phenomenon on $\mathbf{MM}^{+++}$?

Woodin argument. "By the very nature of its conception, the set of all truths of the transfinite universe (the universe of sets) cannot be reduced to the set of truths of some explicit fragment of the universe of sets. *Taking into account the iterative conception of sets, the set of all truths of an explicit fragment of the universe of sets cannot be reduced to the truths of an explicit simpler fragment*"

Possible counter argument. For all we know it could be the case that changes in the structural complexity of the universe happen only sometime and not whenever we pass from H_κ to H_λ for $\lambda > \kappa$.

4.5 Concluding remarks

We tried to relate the philosophical question concerning a possible justification of Viale's and Woodin's generic absoluteness results for the level $H_{\aleph_2}$ with the possi-

[22] Remarkably, among the global implications of $\mathbf{MM}^{+++}$ there is the *Singular Cardinal Hypothesis*. See [13] (chapter 5, p. 58) for a presentation of the Singular Cardinal Hypothesis.

ble scenarios for a solution of the conjecture $\mathbf{MM}^{+++} \Rightarrow (*)$. According with the philosophical framework installed in section 4.3, the conjecture plays the role of a prediction testing the level of philosophical tenability of the forcing axioms program. As we try to argue in section 4.3 this is in our opinion an example of **exercise of mathematical traction**. We already noticed, though, that other threads of investigation are necessary and that ultimately the very notion of maximality incapsulated in the conceptual content of forcing axioms must be examined. It is possible that a better understanding of the notion of maximality could allow for a more intrinsic way of appreciating the philosophical content of the forcing axioms and, hence, of the generic absoluteness results inspected in chapters 2 and 3. The main idea under the surface of the concept of generic absoluteness, as the work of Viale in [37] and [39] clearly states, is to reverse our perspective toward the forcing technique and to conceive forcing as a powerful source of standard mathematical proofs. It is possible that there exists a deeper connection to relieve between that approach to forcing and the idea of maximality incapsulated in forcing axioms. This is anyway material for a different work and it goes beyond the scope of the present dissertation.
We would like to briefly focus our attention on another issue that calls for a clarification. This is the relationship between the forcing axioms program and $V=$ Ultimate-L program. **How should we understand the coexistence between those two different programs for the search of new axioms?** It is maybe under this light that one could better appreciate the sketch-comparison that Todorcevic considers in [35] (section 4.2, p. 20) between the structure theory of $P(\omega_1)$ given by Martin's Maximum and the perspective of Woodin's Ultimate-L approach on the structure theory of $P(\omega_1)$ and, essentially, on the Continuum Problem. As we already stressed, under $V=$Ultimate-L, the CH is true. This last property of Ultimate-L clearly makes it incompatible with the forcing axioms $FA_{\aleph_1}\mathbb{P}$ we discussed in chapter 2, and in particular with $\mathbf{MM}^{+++}$. Yet, one could still speculate on a possible way to *reconcile* the two perspectives. Is it possible to find a *justified way* to unify them? A unified prospective, actually, has been proposed by Woodin and Koellner and it is known as the *Envelope forcing axioms perspective*[23].
Exploiting the analogy with the case of the retreat from AD, the full axiom of determinacy, to $AD^{L(\mathbb{R})}$, its relativisation to the inner model $L(\mathbb{R})$, it is possible to suggest that the right structure theory of $P(\omega_1)$ holds in an inner model of the Universe of sets.[24] So we should have the following shift

$$\frac{L(\mathbb{R})}{AD^{L(\mathbb{R})}} = \frac{P(\omega_1)}{?} \dashrightarrow \frac{L(\mathbb{R})}{AD^{L(\mathbb{R})}} = \frac{L(X)}{?} \tag{4.1}$$

[23]We already briefly referred to such a perspective in chapter (2).
[24]See for example [35] (section 4.2, p. 20).

for X plausibly $P(\omega_1)$.

While, in the case of AD, the main reason for the retreat to an inner model was the preservation of the axiom of Choice (since AD and AC are incompatible), this time, in the case of the structure theory of $P(\omega_1)$, the role of AC would be replaced by the CH. It is beyond the scope of the present work to discuss the details of a similar perspective. It suffices to stress here the following point.

> "At this stage it is difficult to predict the picture of the Set theoretic universe that would accommodate the right structure theory of $P(\omega_1)$ and so, in particular, solve the Continuum problem" (See [35], section 4.2, p. 20).

We would like at this point only to stress the general position assumed by Todorcevic in front of such a perspective, because we feel that it is highly representative of the substantial attitude of patient and careful accumulation of evidences and data from the rest of mathematics that animate the research of new axioms in Set theory.

> " **Question**. What is the true structure theory of $P(\omega_1)$? Is CH or its negation a part of this theory?
>
> **Question**. In order to have the true structure theory of $P(\omega_1)$ do we really need to retreat to an inner model of the universe of sets?
>
> We believe that the tests that will prove crucial are those coming from the rest of mathematics. The combined experience from the rest of mathematics might eventually give us a hint which of the two theories of $P(\omega_1)$ is more useful and should be kept, a CH theory that give us an immense quantity of unrelated mathematical structures, or a fine structure theory of $P(\omega_1)$ that contradicts CH and that resembles the structure theory of $P(\omega)$ under $AD^{L(\mathbb{R})}$".

It should be noticed, however, that the question raised previously is in some sense generic since it concerns only the general possibility of coexistence between distinct incompatible programs for the research of new axioms. We think that we can add here a more specific element in order to consider the previous case from an even different point of view. That is, if we consider the forcing axioms program and the $V{=}UltimateL$ approach, we see that while the first is a *local* approach to the solution of CH, the second one is a *global* approach to the solution of CH. *How do these* *different levels of comprehension of the universe V interact with each other?*. We think that a possible passage in the literature where a similar issue clearly emerges is the following by Woodin [42] (Introduction, p. 19).

> "..The validation of this axiom [$(*)$] requires a synthesis with axioms for V itself for otherwise it simply stands as an *isolated* axiom.(..) I remained convinced that if CH is false then the axiom $(*)$ holds and certainly there are now many results confirming that if the axiom $(*)$ does hold then there is a rich structure theory of H_{ω_2} in which many pathologies are eliminated. But nevertheless for all reasons discussed at length in (...) I think the evidence now favour CH."

Our question is, then, how does the advocate of the forcing axioms program reply to the previous Woodin's worry concerning the problem of, as we may say, the **isolated** axiom?

A definitive decision on the truth of an axiom such as $\mathbf{MM}^{+++}$ or $(*)$, and hence, at least **according with** the epistemological assumptions that we imposed in the present chapter, a final decision on the philosophical justification of Viale's and Woodin's generic absoluteness results should pass through a **better understanding** of the methodological issues raised by the previous considerations.

The following question: " What does it mean to give a solution to the Fundamental equation

$$\frac{L(\mathbb{R})}{AD^{L(\mathbb{R})}} = \frac{P(\omega_1)}{?} \text{ ?"}$$

is still **open** as it is the question of pluralism at the level of the Continuum Hypothesis.

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