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FOURIER-MUKAI TRANSFORMS FOR SINGULAR PROJECTIVE VARIETIES

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Introduction

Fourier-Mukai functors cover almost the entire examples of geometric functors that are known, and this is the main reason why they are so investigated. Unlike the smooth case, in which much has been already said, the singular case still raises many questions. Along this thesis we face two different situations that highlight the difficulties arising when the smoothness hypothesis is taken off.

The first chapter is dedicated to basic definitions of those objects that we will use all along the thesis.

In the second chapter we deal with the first situation. Orlov proved that every exact fully faithful functor $F : X \longrightarrow Y$ between the bounded derived categories of smooth projective schemes is of Fourier-Mukai and the kernel is unique up to isomorphisms ([26]). But what can be said when X is singular? The most relevant result in this sense found in literature is due to Lunts and Orlov:

Theorem 0.0.1. *Let X be a projective variety and Y be a noetherian separated scheme. Denote by $T_0(\mathcal{O}_X)$ the maximal torsion subsheaf of dimension zero of $\mathcal{O}_X$. If $T_0(\mathcal{O}_X) = 0$ then every exact fully faithful functor $F : \mathbf{Perf}(X) \longrightarrow \mathbf{D}^b(Y)$ is such that there exists a unique object $\mathcal{E} \in \mathbf{D}^b(X \times Y)$ with $F \cong \Phi_{\mathcal{E}}$.*

This highly non-trivial result shows immediately that removing the smoothness hypothesis brings complications, because the assumption about the triviality of $T_0(\mathcal{O}_X)$ is all but natural. We decided then to take in consideration an example of a singular projective scheme X in which the maximal torsion subsheaf of dimension zero of $\mathcal{O}_X$ is not trivial: in chapter 2 we show that the results mentioned in Theorem 0.0.1 still hold at least in a case, which is the "double point scheme", that is the spectrum of the ring of dual numbers $A := k[\epsilon]/(\epsilon^2)$. The main result of this thesis, which is a joint work with Riccardo Moschetti ([1]) is the following:

Theorem 0.0.2. *Let:*

$$F : \mathbf{Perf}(A) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(Y)$$

be a fully faithful functor, then there is an object $\mathcal{E} \in \mathbf{D}_{\mathrm{qcoh}}(\operatorname{Spec} A \times Y)$ *such that:*

$$\Phi_{\mathcal{E}}|_{\mathbf{Perf}(A)} \cong F.$$

Furthermore, if Y *is noetherian and* F *sends* $\mathbf{Perf}(A)$ *to* $\mathbf{D}^b(Y)$*, then:*

$$\mathcal{E} \in \mathbf{D}^b(\operatorname{Spec} A \times Y).$$

This could suggest that the assumption $T_0(\mathcal{O}_X) = 0$ may be avoidable.

Using the description of indecomposable objects in the derived category of the double point, we classify all the stability conditions on the category $\mathbf{D}^b(A)$. We prove that $\operatorname{Stab}(\mathbf{D}^b(A))$ is isomorphic to $\mathbb{C}$, the universal covering of $\mathbb{C}^*$.

In the third chapter we study the case of G-equivariant sheaves, where G is a finite group acting on a smooth projective variety X.
If the action of G is free then the geometrical quotient is smooth and everything is easy and well-known: the derived category of G-equivariant sheaves on X is equivalent to the derived category of the geometrical quotient X/G. But what happens when the action of the group is not free, and the quotient is singular? In this situation the category of G-equivariant coherent sheaves on X is isomorphic to the category $\operatorname{Coh}[X/G]$ of coherent sheaves on the smooth stack $[X/G]$, which is different from the quotient variety in general. Kawamata proved Orlov's representability theorem for smooth stacks [19]:

Theorem 0.0.3. *Let* X *and* Y *be normal projective varieties with only quotient singularities and denote by* $\mathcal{X}$ *and* $\mathcal{Y}$ *the smooth stacks naturally associated to them. Let:*

$$\mathbf{D}^b(\mathcal{X}) \longrightarrow \mathbf{D}^b(\mathcal{Y})$$

be an exact functor which is fully faithful and has a left adjoint. Then there exists a unique object $\mathcal{E} \in \mathbf{D}^b(\mathcal{X} \times \mathcal{Y})$ *with* $F \cong \Phi_{\mathcal{E}}$*.*

This suggested us to consider if there is some relationship between $\operatorname{Aut}(\mathbf{Perf}(X/G))$, the group of autoequivalences of the category of perfect complexes on the quotient variety, and $\operatorname{Aut}(\mathbf{D}^b_G(X))$, the group of autoequivalences of the derived category of G-equivariant sheaves on X.

We prove the following:

Theorem 0.0.4. *Let* X *be a smooth projective variety and take* $\mathcal{F}^\bullet \in \mathbf{D}^b_G(X)$*. Then there* *exists an object* $\mathcal{V}^\bullet \in \mathbf{Perf}(X/G)$ *such that* $\mathcal{F}^\bullet$ *is quasi-isomorphic to* $\mathbf{L}\pi^*(\mathcal{V}^\bullet)$ *if and only if the stabilizer* G_x *acts trivially on the* $\mathcal{O}_X$*-modules* $\mathrm{H}^j(\mathcal{F}^\bullet \otimes^{\boldsymbol{L}} k(x))$ *for every point* $x \in X$*.*

The result above implies that, given an autoequivalence $\Phi_{\mathcal{P}}$ of $\mathbf{D}^b_G(X)$, it is possible to obtain an autoequivalence of $\mathbf{Perf}(X/G)$ if the stabilizer G_x acts trivially on the $\mathcal{O}_X$-modules $\mathrm{H}^j(\Phi_{\mathcal{P}}(\mathbf{L}\pi^*(\mathcal{A}^\bullet)) \otimes k(x))$ for every $x \in X$ and $j \in \mathbb{Z}$, where $\mathcal{A}^\bullet \in \mathbf{Perf}(X/G)$. This is a starting point that we believe can be further developed in terms of a deeper criterion which involves only the kernel $\mathcal{P}$.

Chapter 1

Derived Categories

Along this chapter we give basics definitions for those objects that we will use throughout the thesis.

1.1 Categories and functors

Definition 1.1.1. *A category $\mathcal{A}$ is called additive if, for every pair of objects X and Y, the set $Hom(X,Y)$ is equipped with an abelian group structure such that the following conditions hold:*

- *The compositions $Hom(X,Y) \times Hom(Y,Z) \longrightarrow Hom(X,Z)$ are bilinear.*
- *There exists an object $0 \in \mathcal{A}$ such that $Hom(0,0)$ is a trivial group.*
- *$\mathcal{A}$ admits finite coproducts.*

Recall that for an indexed family of object $\{X_j\}_{j\in J}$, a *coproduct* is an object X togheter with a collection of morphisms $i_j : X_j \longrightarrow X$ such that for any other object Y and collection of morphisms $f_j : X_j \longrightarrow Y$ there exists a unique morphism $f : X \longrightarrow Y$ with $f_j = f \circ i_j$.

Usually, the finite coproduct of the family $\{X_j\}_{j=1,\ldots,n}$ is called *direct sum* and indicated with $\oplus_{j=1}^n X_j$.

A functor F between two additive categories is always supposed to be additive too, that is the induced maps $\mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(F(X),F(Y))$ are group homomorphisms.

Definition 1.1.2. *An additive category is called an abelian category if every morphism has kernel and cokernel. Furthermore every monomorphism and every epimorphism is normal: in other words, every monomorphism is a kernel of some morphism, and every epimorphism is a cokernel of some morphism.*

The category $\mathbf{mod}-R$ of modules over a commutative ring is an example of abelian category. Another example is given by $\mathbf{Sh}(X)$, the category of abelian group sheaves over a topological space X. The categories $\mathbf{QCoh}(X)$ and $\mathbf{Coh}(X)$ of quasi-coherent sheaves and coherent sheaves over a projective variety X are abelian.

Definition 1.1.3. *A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called right-exact (left-exact) if for every short exact sequence:*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

the following short sequence:

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0$$

is exact except possibly in $F(Z)$ (respectively in $F(X)$). F is said exact if it is both right-exact and left-exact.

Notice that a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is left-exact if and only if for every short exact sequence of the form:

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$$

the following sequence is exact:

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z).$$

Infact, if F is exact then the conclusion is immediate. Viceversa, the following sequence is exact:

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Im}(g) \longrightarrow 0.$$

The functor F is supposed to be left exact and then the following sequence is exact:

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(\mathrm{im}((g)).$$

But $F(\mathrm{im}((g)) \subset F(Z)$.

In other words, in order to show left-exactness of a functor it is enough to work with short left-exact sequences.
Similarly, a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is right-exact if and only if for every short exact sequence of the form:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

the following sequence is exact:

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0.$$

Definition 1.1.4. *Consider a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$. A functor $H : \mathcal{B} \longrightarrow \mathcal{A}$ is right-adjoint for F if for every $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ there exists a functorial isomorphism:*

$$\mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, H(Y)). \qquad (*)$$

Similarly a functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ is left-adjoint for F if there exists a functorial isomorphism:

$$\mathrm{Hom}(Y, F(X)) \cong \mathrm{Hom}(G(Y), X). \qquad (**)$$

Suppose F admits a right-adjoint. Then for every $X \in \mathcal{A}$ we have:

$$\mathrm{Hom}(F(X), F(X)) \cong \mathrm{Hom}(X, H(F(X))).$$

Therefore the identity morphism $1_{F(X)}$ defines uniquely a morphism $h_X : X \longrightarrow H(F(X))$. Thus we can describe the isomorphism $(*)$. Take $f \in \mathrm{Hom}(F(X), Y)$, then f goes to:

$$X \xrightarrow{h_X} H(F(X)) \xrightarrow{H(f)} H(Y)$$

A similar description holds for $(**)$.

1.2 Triangulated categories

Let $\mathcal{A}$ be an additive category, and consider an equivalence:

$$T : \mathcal{A} \longrightarrow \mathcal{A}.$$

For every object $X \in \mathcal{A}$ we call $X[1] := T(X)$. A *triangle* is given by a collection of objects X, Y and Z and morphisms:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

Let $f \in \operatorname{Hom}(X, Y)$. We write $X[n] := T^n(X)$ and $f[n] := T^n(f)$ for $n \in \mathbb{Z}$.
A morphism between two triangles is given by a commutative diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]. \end{array}$$

An isomorphism of triangles is a morphism such that the first three vertical maps above are isomorphisms.

We now want to define the set of *distinguished triangles*. We need four axioms:

1. - Every triangle of the following form:

   $$X \longrightarrow X \longrightarrow 0 \longrightarrow X[1]$$

   is a distinguished triangle.

   - Every triangle isomorphic to a distinguished triangle is a distinguished triangle.

   - Every morphism $f : X \longrightarrow Y$ defines a distinguished triangle:

   $$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

2. The triangle:

   $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

   is a distinguished triangle if and only if the following:

   $$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

   is a distinguished triangle.

3. Suppose there exists a commutative diagram of distinguished triangles of the form:

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1].
\end{array}
$$

Then the diagram can be completed to become a morphism of distinguished triangles. That is, it is always possible to find a morphism (not necessarily unique) between Z and Z'.

4. Suppose are given the following distinguished triangles:

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow X[1]$$

$$Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow Y[1]$$

$$X \xrightarrow{gf} Z \longrightarrow Y' \longrightarrow X[1]$$

Then there exists a distinguished triangle:

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow Z'[1]$$

such that the following diagram commutes:

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & X[1] \\
\downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow \\
X & \xrightarrow{gf} & Z & \longrightarrow & Y' & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & Y[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z'[1]
\end{array}
$$

Definition 1.2.1. *A triangulated category $\mathcal{A}$ is an additive category together with an equivalence $T : \mathcal{A} \longrightarrow \mathcal{A}$ and a set of distinguished triangles satisfying the previous four axioms.*

Let:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

be a distinguished triangle in a triangulated category $\mathcal{A}$. Then for every object $W \in \mathcal{A}$ the following sequences are exact:

$$\operatorname{Hom}(W, X) \longrightarrow \operatorname{Hom}(W, Y) \longrightarrow \operatorname{Hom}(W, Z)$$

$$\operatorname{Hom}(Z, W) \longrightarrow \operatorname{Hom}(Y, W) \longrightarrow \operatorname{Hom}(X, W).$$

As a corollary, we have the following (five lemma for triangulated categories):

Lemma 1.2.2. *Consider a commutative diagram of distinguished triangles:*

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1].
\end{array}
$$

If two among the first three vertical maps are isomorphism, then so is the third.

1.3 Localization

Given a category $\mathcal{A}$, a *localization* of $\mathcal{A}$ by a class of morphism $\mathcal{S}$ is, basically, another category $\mathcal{S}^{-1}\mathcal{A}$ in which morphisms of $\mathcal{S}$ become isomorphisms. Usually some conditions on the class $\mathcal{S}$ are required:

Definition 1.3.1. *A multiplicative system in a category $\mathcal{C}$ is a class of morphisms $\mathcal{S}$ satisfying the following restrictions:*

- $\mathrm{Id}_X \in \mathcal{S}$ *for any object X.*
- *For any pair of morphisms f and g in $\mathcal{S}$, their composition $g \circ f$ is still in $\mathcal{S}$.*
- *For $X \xrightarrow{f} Y$ and $Z \xrightarrow{g} Y$ there exist $U \xrightarrow{t} X$ and $U \xrightarrow{s} Z$ such that the following diagram commutes:*

$$\begin{array}{ccc} U & \xrightarrow{\quad t \quad} & X \\ \Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{\quad g \quad} & Y. \end{array}$$

- *Suppose $f, g \in \mathrm{Hom}(X, Y)$. Then there exists $t : Y \longrightarrow Z$ such that $t \circ f = t \circ g$ if and only if there exists $s : W \longrightarrow X$ such that $f \circ w = g \circ w$.*

Given a triangulated category $\mathcal{A}$ and a multiplicative system $\mathcal{S}$, we construct a triangulated category $\mathcal{S}^{-1}\mathcal{A}$ and a functor $Q : \mathcal{A} \longrightarrow \mathcal{S}^{-1}\mathcal{A}$ such that:

- If $f \in \mathcal{S}$ then $Q(f)$ is an isomorphism.
- The functor Q is universal. If there exists another category $\mathcal{B}$ and a functor $G : \mathcal{A} \longrightarrow \mathcal{B}$ taking elements of $\mathcal{S}$ to isomorphisms in $\mathcal{B}$, then there exists a unique functor $H : \mathcal{S}^{-1}\mathcal{A} \longrightarrow \mathcal{B}$ such that $Q \circ H$ is isomorphic to G.

We can also start from a class of objects instead of a class of morphisms. Given a triangulated category $\mathcal{A}$, a subclass N of objects of $\mathcal{A}$ is called a *null system* if:

- $0 \in N$.
- $X \in N$ if and only if $X[1] \in N$.

- if X and Y are in N and $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ is a distinguished triangle, then also Z is in N.

Given a triangulated category $\mathcal{A}$ and a null system N, we can define a multiplicative system associated to N:

$\mathcal{S} := \{f : X \longrightarrow Y$ such that there exists a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ with $Z \in N\}$.

The localized category $\mathcal{S}^{-1}\mathcal{A} := \mathcal{A}/N$ is triangulated and it is called the *Verdier quotient* of $\mathcal{A}$ by N. The localization functor Q is a functor of triangulated categories such that $Q(X)$ is isomorphic to the zero object for every $X \in N$. Furthermore Q is universal among those functors which have the above property.

1.4 Derived categories

Given an abelian category $\mathcal{A}$ we consider the category $\mathbf{Kom}(\mathcal{A})$ whose objects are complexes of the form:

$$X^\bullet : \{ \dots \xrightarrow{d^{i-1}} X^{i-1} \xrightarrow{d^i} X^i \xrightarrow{d^{i+1}} X^{i+1} \xrightarrow{d^{i+2}} \dots \}$$

where X^i are objects of $\mathcal{A}$ for every $i \in \mathbb{Z}$ and d^i are morphisms such that $\mathrm{im}(d^{i-1}) \subset \ker(d^i)$ for every i.
A morphism $f : X^\bullet \longrightarrow Y^\bullet$ between two complexes is given by a commutative diagram:

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{d_X^{i-1}} & X^{i-1} & \xrightarrow{d_X^{i}} & X^{i} & \xrightarrow{d_X^{i+1}} & X^{i+1} & \xrightarrow{d_X^{i+2}} & \dots \\
 & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^{i}} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\dots & \xrightarrow{d_Y^{i-1}} & Y^{i-1} & \xrightarrow{d_Y^{i}} & Y^{i} & \xrightarrow{d_Y^{i+1}} & Y^{i+1} & \xrightarrow{d_Y^{i+2}} & \dots
\end{array}$$

The category $\mathbf{Kom}(\mathcal{A})$ is abelian: the zero object is given by the trivial complex and the kernel of a morphism $f : \mathcal{A}^\bullet \longrightarrow \mathcal{B}^\bullet$ is the complex of the kernels of f^i for every $i \in \mathbb{Z}$.

Definition 1.4.1. *Given an object $X^\bullet \in \mathbf{Kom}(\mathcal{A})$, the stupid truncation $\sigma_{\leq i}X^\bullet \in \mathbf{Kom}(\mathcal{A})$ of $X^\bullet$ is the complex defined by $(\sigma_{\leq i}X^\bullet)^j = 0$ if $j > i$ and $(\sigma_{\leq i}X^\bullet)^j = X^j$ if $j \leq i$.*

Given a complex $X^\bullet$, for every $k \in \mathbb{Z}$ we define the complex $X^\bullet[k]$ such that $X^i[k] = X^{k+i}$

and $d^i_{X[k]} = (-1)^k d^{i+k}_X$. The functor:

$$T : \mathbf{Kom}(\mathcal{A}) \longrightarrow \mathbf{Kom}(\mathcal{A}), \qquad X^\bullet \longmapsto X^\bullet[1]$$

defines an equivalence of categories. However, $\mathbf{Kom}(\mathcal{A})$ together with T does not in general determine a triangulated category. Then, in order to obtain a triangulated category we need to introduce a new object:

Definition 1.4.2. *The category $\mathbf{K}(\mathcal{A})$ is the category whose objects are the same of $\mathbf{Kom}(\mathcal{A})$ and morphisms are defined up to homotopy equivalence.*

Remark 1.4.3. Recall that two morphisms $f, g : X^\bullet \longrightarrow Y^\bullet$ are homotopy equivalent if there exist morphisms $h^i : X^i \longrightarrow Y^{i-1}$ such that $f^i - g^i = h^{i+1} \circ d^i_X + d^{i-1}_Y \circ h^i$.

Definition 1.4.4. *Let $f : X^\bullet \longrightarrow Y^\bullet$ be a morphism of complex in $\mathbf{Kom}(\mathcal{A})$. The mapping cone $M(f)$ is the complex defined by:*

$$M(f)^i = X^{i-1} \oplus Y^i \text{ and } d^i_{M(f)} = (-d^{i+1}_X, f^{i+1} + d^i_Y).$$

Notice there exist two natural morphisms: $\tau : Y^\bullet \longrightarrow M(f)$ and $\pi : M(f) \longrightarrow X^\bullet[1]$.

Definition 1.4.5. *A distinguished triangle in $\mathbf{K}(\mathcal{A})$ is a triangle that is isomorphic to a triangle of the following form:*

$$X^\bullet \xrightarrow{\quad f \quad} Y^\bullet \xrightarrow{\quad \tau \quad} M(f) \xrightarrow{\quad \pi \quad} X^\bullet[1].$$

Remark 1.4.6. The category $\mathbf{K}(\mathcal{A})$ together with the functor T and the set of distinguished triangles defined above is a triangulated category.

Let $f : X^\bullet \longrightarrow Y^\bullet$ be a morphism of complexes. It induces naturally a morphism of cohomologies:

$$\mathrm{H}^i(f) : \mathrm{H}^i(X^\bullet) \longrightarrow \mathrm{H}^i(Y^\bullet).$$

We say that f is a *quasi-isomorphism* (often abbreviated with qis) if $\mathrm{H}^i(f)$ is an isomorphism for all i.

The idea which underlies the construction of derived categories is to start from quasi-isomorphic complexes and to make them become isomorphic into the derived category:

Theorem 1.4.7. *Let $\mathcal{A}$ be an abelian category. There exists a category $\mathbf{D}(\mathcal{A})$ and a functor:*

$$Q : \mathbf{Kom}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A})$$

such that:

- *If $f : X^\bullet \longrightarrow Y^\bullet$ is a quasi-isomorphism, then $Q(f)$ is an isomorphism in $\mathbf{D}(\mathcal{A})$.*

- *Every functor $F : \mathbf{Kom}(\mathcal{A}) \longrightarrow \mathcal{D}$ satisfying the previous property factors uniquely over Q. That is, there exists a unique functor (up to isomorphisms) $G : \mathbf{D}(\mathcal{A}) \longrightarrow \mathcal{D}$ such that $F \cong G \circ Q$.*

Proof. See [16], Theorem 2.7 for a reference. □

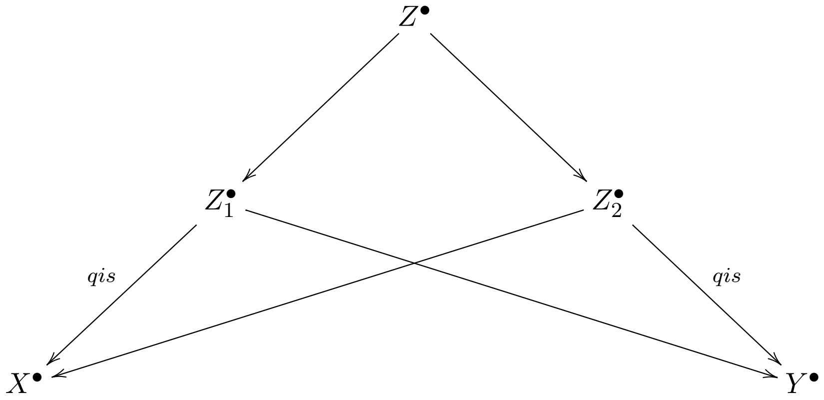
The category $\mathbf{D}(\mathcal{A})$ is called the *derived category* associated to the abelian category $\mathcal{A}$. Notice that $\mathbf{D}(\mathcal{A})$ is the localization of $\mathbf{K}(\mathcal{A})$ by the class of quasi-isomorphisms.

In the following we will show what kind of objects are in this category and how morphisms are made.

By the set theory point of view, objects of $\mathbf{D}(\mathcal{A})$ are the same of $\mathbf{Kom}(\mathcal{A})$. Thus they are complexes made by objects of $\mathcal{A}$.
With regard to morphisms, notice that if $Z^\bullet \longrightarrow X^\bullet$ is a quasi-isomorphism and $Z^\bullet \longrightarrow Y^\bullet$ is any other morphism, then $X^\bullet$ and $Z^\bullet$ are isomorphic in the derived category and therefore there exists a morphism (in $\mathbf{D}(\mathcal{A})$) $X^\bullet \longrightarrow Y^\bullet$.
Thus we define morphisms of the derived category as the set made by diagrams of the form:

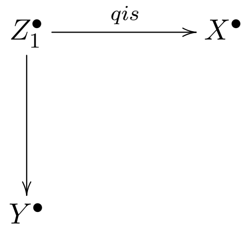
$$\begin{array}{ccc} Z^\bullet & \xrightarrow{qis} & X^\bullet \\ \downarrow & & \\ Y^\bullet & & \end{array}$$

Two such diagrams are equivalent if they are dominated in $\mathbf{K}(\mathcal{A})$ by a third commutative diagram of the same form:

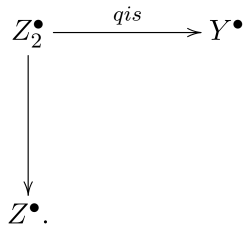


Therefore, morphisms are defined up to homotopy equivalence and in particular $Z^\bullet \longrightarrow Z_1^\bullet \longrightarrow X^\bullet$ and $Z^\bullet \longrightarrow Z_2^\bullet \longrightarrow X^\bullet$ are quasi-isomorphisms.

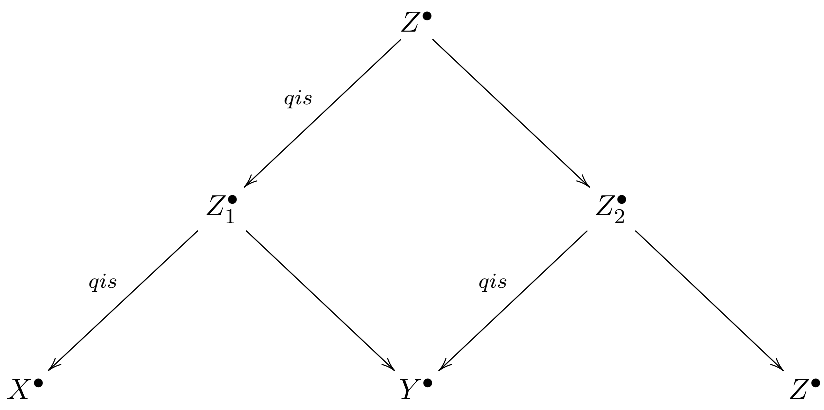
Now take two morphisms:



and



Their composition is a commutative diagram of the form:



Such a diagram always exists and it is unique up to equivalence (see [16], chapter 2 for further explanations).

Last, notice that $\mathbf{D}(\mathcal{A})$ is a triangulated category. The functor $T : \mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A})$ naturally extends to an equivalence of $\mathbf{D}(\mathcal{A})$ and the set of distinguished triangle is made by those triangles:

$$P_1^\bullet \longrightarrow P_2^\bullet \longrightarrow P_3^\bullet \longrightarrow P_1^\bullet[1]$$

which are isomorphic in $\mathbf{D}(\mathcal{A})$ to a triangle of the form:

$$X^\bullet \xrightarrow{\ f\ } Y^\bullet \xrightarrow{\ \tau\ } M(f) \xrightarrow{\ \pi\ } X^\bullet[1].$$

Denote by $\mathbf{Kom}^*(\mathcal{A})$ (with $* = +, -$ and b) the category of complexes $X^\bullet$ with $X^i = 0$ for, respectively, $i << 0$, $i >> 0$ and $|i| >> 0$.
Similarly, we define the categories $\mathbf{K}^*(\mathcal{A})$ and $\mathbf{D}^*(\mathcal{A})$.

We give now some examples.

Example 1.4.8. Consider a commutative noetherian ring R and the abelian category associated $\mathbf{mod}_{\text{fg}} - R$ of finitely generated R-modules. The objects of $\mathbf{D}^b(\mathbf{mod}_{\text{fg}} - R)$, the bounded derived category of $\mathbf{mod}_{\text{fg}} - R$, are chain complexes:

$$P^\bullet := \{\ldots \longrightarrow P^{i-1} \longrightarrow P^i \longrightarrow P^{i+1} \longrightarrow \ldots\}$$

of projective R-modules such that $\mathrm{H}^i(P^\bullet) = 0$ for $|i| >> 0$. Given two objects $P^\bullet$ and $Q^\bullet$ of $\mathbf{D}^b(\mathbf{mod}_{\text{fg}} - R)$, a morphism $P^\bullet \longrightarrow Q^\bullet$ is given by the equivalence class $\tilde{f}$ of morphisms of complexes $f : P^\bullet \longrightarrow Q^\bullet$ modulo the homotopy relation.

Example 1.4.9. Let k be a field. The spectrum $X := \operatorname{Spec} k$ can be viewed as a smooth point. The bounded derived category $\mathbf{D}^b(\mathbf{Coh}(X)) := \mathbf{D}^b(X)$ of coherent sheaves on X is the bounded derived category associated to $\mathbf{mod}_{\mathrm{fg}}-k$. Every object of $\mathbf{D}^b(\mathbf{mod}_{\mathrm{fg}}-k)$ is isomorphic to a direct sum of $M^i[i]$ where M^i is a finitely generated module over k and $i \in Z$. Moreover there are no non trivial morphisms from $M^i[i]$ to $N^j[j]$ unless $i = j$. Therefore we conclude that $\mathbf{D}^b(X)$ is equivalent to the category of k-graded vector spaces with finitely many non-zero components.

Example 1.4.10. Let X be a smooth curve. Then every object in the bounded derived category $\mathbf{D}^b(\mathbf{Coh}(X)) := \mathbf{D}^b(X)$ is isomorphic to a direct sum of $\mathcal{E}^i[i]$ where $\mathcal{E}^i$ are coherent sheaves on X and $i \in \mathbb{Z}$. Notice that the smoothness hypothesis here is crucial; infact, in the singular case, the structure of the derived category is harder to obtain. Moreover, a lot of properties fail, for example: the homological dimension of the category is infinite and Serre duality does not hold anymore in general.

Example 1.4.11. The bounded derived category of $\mathbb{P}^n$ is generated by:

$$\{O(-n), \mathcal{O}(-n+1), \ldots, \mathcal{O}(1), \mathcal{O}\}.$$

Remember that in the derived category we have only two operations: shifts and cones. Therefore, when we say 'generated' we mean that the smallest subcategory of $\mathbf{D}^b(\mathbf{Coh}(\mathbb{P}^n)) := \mathbf{D}^b(\mathbb{P}^n)$ which contains $\{O(-n), \mathcal{O}(-n+1), \ldots, \mathcal{O}(1), \mathcal{O}\}$ is $\mathbf{D}^b(\mathbb{P}^n)$ itself.
This result comes from the Beilinson's resolution of coherent sheaves on $\mathbb{P}^n$. In the singular case, i.e. weighted projective spaces, things are much harder mainly because it is no longer true that any coherent sheaf admits a finite resolution by locally free sheaves of finite type. Anyway, a similar result was given in [7].

1.5 Derived functors and perfect complexes

1.5.1 Derived functors

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ a functor between abelian categories. The natural extension of F to $\mathbf{D}(\mathcal{A})$ does not have any sense (the image of an exact complex, that is the zero complex, could be non-zero if F is not exact).
Thus, if we want to extend F at the level of derived category we must proceed in a different way. Basically, starting from a left-exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ it is possible to construct an associated functor $\mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}^+(\mathcal{B})$. Similarly, for a right-exact functor, one construct a

functor $\mathbf{D}^-(\mathcal{A}) \longrightarrow \mathbf{D}^-(\mathcal{B})$. Both procedures are analogous, thus we will only show the case of a left-exact functor.

Definition 1.5.1. *Given an abelian category $\mathcal{A}$, an object $X \in \mathcal{A}$ is called injective if, for every morphism $g : Y \longrightarrow X$ and every monomorphism $h : Y \longrightarrow Z$, there exists a morphism $f : Z \longrightarrow X$ such that $f \circ h = g$.*

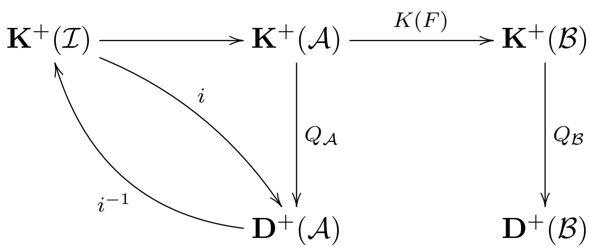
Denote by $\mathcal{I}$ the subcategory of $\mathcal{A}$ made by injective objects. An abelian category $\mathcal{A}$ has *enough injectives* if for every object $Y \in \mathcal{A}$ there exists a monomorphism $Y \longrightarrow X$ where $X \in \mathcal{I}$.

The natural inclusion $\mathcal{I} \subset \mathcal{A}$ and the functor $Q_{\mathcal{A}} : \mathbf{K}^*(\mathcal{A}) \longrightarrow \mathbf{D}^*(\mathcal{A})$ (defined in 1.4.7) induce a natural functor $\mathbf{K}^*(\mathcal{I}) \longrightarrow \mathbf{D}^*(\mathcal{A})$.

Proposition 1.5.2. *Suppose $\mathcal{A}$ has enough injectives. Then the natural functor $i : \mathbf{K}^+(\mathcal{I}) \longrightarrow \mathbf{D}^+(\mathcal{A})$ is an equivalence.*

Proof. [16], Proposition 2.26. □

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ a left-exact functor of abelian categories. Suppose $\mathcal{A}$ has enough injectives. Then we have a commutative diagram:



where $K(F)$ is the natural extension of F at the level of homotopy categories.

The *right derived functor* associated to F is the functor:

$$\mathbf{R}F : Q_{\mathcal{B}} \circ K(F) \circ i^{-1} : \mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}^+(\mathcal{B}).$$

Notice that $\mathbf{R}F$ is an exact functor of triangulated categories because it is obtained by compositions of three exact functors.

The construction of the left derived functor $\mathbf{L}F$ for a right-exact functor F is pretty much similar, what really changes is just the use of projective objects instead of injective objects.

Let now consider some geometrical examples. Recall that for a projective variety X we have the associated abelian categories $\mathbf{Qcoh}(X)$ and $\mathbf{Coh}$(X) of, respectively, quasi-coherent and coherent sheaves on X. We will denote by $\mathbf{D}_{\mathrm{qcoh}}(X)$ and $\mathbf{D}(X)$ the derived categories associated to $\mathbf{Qcoh}(X)$ and $\mathbf{Coh}$(X). Notice that $\mathbf{Qcoh}(X)$ has always enough injectives (this is not in general true also for $\mathbf{Coh}(X)$).
Let $f : X \longrightarrow Y$ be a morphism of projective varieties. We have a left exact functor (the direct image):

$$f_* : \mathbf{Qcoh}(X) \longrightarrow \mathbf{Qcoh}(Y).$$

Hence we can construct a right derived functor:

$$\mathbf{R}f_* : \mathbf{D}^+_{\mathrm{qcoh}}(X) \longrightarrow \mathbf{D}^+_{\mathrm{qcoh}}(Y).$$

For $\mathcal{F}^\bullet \in \mathbf{D}^+_{\mathrm{qcoh}}(X)$, the *higher direct images* $\mathbf{R}^i f_*(F^\bullet)$ are the cohomology sheaves $\mathcal{H}^i(\mathbf{R}f_*(F^\bullet))$. By [13], Proposition 8.1, $\mathbf{R}^i f_*(F^\bullet) = 0$ for $i > \dim(X)$, hence $\mathbf{R}f_*$ may actually be seen as a functor between the bounded derived categories:

$$\mathbf{R}f_* : \mathbf{D}^b_{\mathrm{qcoh}}(X) \longrightarrow \mathbf{D}^b_{\mathrm{qcoh}}(Y).$$

Now, notice that $\mathbf{D}^b(X)$ is equivalent to the full triangulated subcategory of $\mathbf{D}_{\mathrm{qcoh}}(X)$ of bounded complexes of quasi-coherent sheaves with coherent cohomology ([16], Proposition 2.49). Also, by [13], Theorem 8.8 it follows that the higher direct images of a complex $\mathcal{F}^\bullet \in \mathbf{Coh}(X)$ is again a complex of coherent sheaves. Combining these two facts we obtain that the composition:

$$\mathbf{D}^b(X) \longrightarrow \mathbf{D}^b_{\mathrm{qcoh}}(X) \longrightarrow \mathbf{D}^b_{\mathrm{qcoh}}(Y)$$

defines an exact functor of triangulated categories:

$$\mathbf{R}f_* : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(Y).$$

Another example of derived functor is given by tensor products. Take a sheaf $\mathcal{F} \in \mathbf{Coh}(X)$. Then we have a right-exact functor:

$$\mathcal{F} \otimes (-) : \mathbf{Coh}(X) \longrightarrow \mathbf{Coh}(X).$$

The left derived functor associated is:

$$\mathcal{F} \otimes^{\mathbf{L}} (-) : \mathbf{D}^-(X) \longrightarrow \mathbf{D}^-(X).$$

If X is smooth it is possible to show (see [16]) that $\mathcal{F} \otimes^{\mathbf{L}} (-)$ defines a functor at the level of bounded derived categories:

$$\mathcal{F} \otimes^{\mathbf{L}} (-) : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X).$$

As a final example we give the case of the inverse image associated to a morphism $f : X \longrightarrow Y$ of projective varieties. The inverse image is the functor f^* obtained by composition of the exact functor:

$$f^{-1} : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}_{f^{-1}(\mathcal{O}_Y)}(X)$$

with the right exact functor:

$$\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} (-) : \mathbf{Sh}_{f^{-1}(\mathcal{O}_Y)}(X) \longrightarrow \mathbf{Sh}(X).$$

The inverse image is a right-exact functor and then we have a left derived functor:

$$\mathbf{L}f^* = (\mathcal{O}_X \otimes^{\mathbf{L}}_{f^{-1}(\mathcal{O}_Y)} (-)) \circ f^{-1} : \mathbf{D}^-(Y) \longrightarrow \mathbf{D}^-(X).$$

1.5.2 The singular case

Let X be a projective variety not necessarily smooth, and take the derived category $\mathbf{D}_{\mathrm{qoch}}(X)$ of quasi coherent sheaves on X. We define the full subcategory of *perfect complexes*: an object $\mathcal{F}^\bullet \in \mathbf{D}_{\mathrm{qoch}}(X)$ is called *perfect* if it is quasi isomorphic to a bounded complex of locally free sheaves of finite type on X. We denote by $\mathbf{Perf}(X)$ the category of perfect complexes.
Notice that $\mathbf{Perf}(X)$ is a thick subcategory of $\mathbf{D}^b(X)$. Furthermore, if X is smooth then $\mathbf{Perf}(X)$ is equivalent to $\mathbf{D}^b(X)$ (this follows by the fact that on a smooth projective variety X every coherent sheaf admits a finite resolution by locally free sheaves). Actually, X is smooth if and only if $\mathbf{Perf}(X) = \mathbf{D}^b(X)$.

Given a triangulated category $\mathcal{A}$, then an object $C \in \mathcal{A}$ is *compact* if for every family $E^i_{i\in I}$ of objects of $\mathcal{A}$, the canonical map:

$$\bigoplus_{i\in I} \mathrm{Hom}(C, E^i) \longrightarrow \mathrm{Hom}(C, \oplus_i E^i)$$

is an isomorphism.
Denote by $\mathcal{A}^c$ the thick subcategory of $\mathcal{A}$ made by compact objects.

Lemma 1.5.3. *Let $\mathcal{F}^\bullet \in \mathbf{D}_{\mathrm{qoch}}(X)$. Then $\mathcal{F}^\bullet$ is perfect if and only if it is compact.*

See [32], Lemma 3.5.

Let X be a projective variety and Y be a noetherian scheme. Denote by:

$$q : X \times Y \longrightarrow X \text{ and } p : X \times Y \longrightarrow Y$$

the projections.

Let $\mathcal{P}^\bullet$ be an object in the derived category of the product $\mathbf{D}^b(X \times Y)$.
We have derived functors:

$$\mathbf{R}p_* : \mathbf{D}^b(X \times Y) \longrightarrow \mathbf{D}^b(Y) \text{ and } \mathcal{P}^\bullet \otimes^{\mathbf{L}} (-) : \mathbf{D}^b(X \times Y) \longrightarrow \mathbf{D}^b(X \times Y).$$

The projection q is flat, hence we do not need to derive it, since it will sufficient to consider the naive extension.

We say a functor $F : \mathbf{Perf}(X) \longrightarrow \mathbf{D}^b(Y)$ to be of Fourier-Mukai type if there exists an object $\mathcal{P}^\bullet \in \mathbf{D}^b(X \times Y)$ called kernel of the functor, such that $F \cong \Phi_{\mathcal{P}^\bullet}$, with:

$$\Phi_{\mathcal{P}^\bullet} : \mathbf{Perf}(X) \longrightarrow \mathbf{D}^b(Y), \qquad \Phi_{\mathcal{P}^\bullet}(-) := \mathbf{R}(p)_*(\mathcal{P}^\bullet \overset{\mathbf{L}}{\otimes} q^*(-))$$

where $p : X \times Y \longrightarrow Y$ and $q : X \times Y \longrightarrow X$ are the projections.

It is an open question to determine whenever a functor $F : \mathbf{Perf}(X) \longrightarrow \mathbf{D}^b(Y)$ is of Fourier-Mukai type or not. For X projective and Y noetherian separated, every exact fully faithful functor $F : \mathbf{Perf}(X) \longrightarrow \mathbf{D}^b(Y)$ is such that there exists an object $\mathcal{P}^\bullet \in \mathbf{D}^b(X \times Y)$ with $F(\mathcal{A}^\bullet) \cong \Phi_{\mathcal{P}^\bullet}(\mathcal{A}^\bullet)$ for every $\mathcal{A}^\bullet \in \mathbf{Perf}(X)$. In other words, under certain assumptions, it always possible to find a Fourier-Mukai functor which is isomorphic *on the objects* to F. Unfortunately this isomorphism does not turn out, in general, to be an isomorphism of functors. In order to obtain a global isomorphism we have to put some more hypothesis. The following theorem is due to Lunts and Orlov:

Theorem 1.5.4. *Let X be a projective variety and Y be a noetherian separated scheme. Denote by $T_0(\mathcal{O}_X)$ the maximal torsion subsheaf of dimension zero of $\mathcal{O}_X$. If $T_0(\mathcal{O}_X) = 0$ then every exact fully faithful functor $F : \mathbf{Perf}(X) \longrightarrow \mathbf{D}^b(Y)$ is such that there exists a unique object $\mathcal{E} \in \mathbf{D}^b(X \times Y)$ with $F \cong \Phi_{\mathcal{E}}$.*

The fully faithful assumption here is essential: infact, it was proven in a recent paper (see [30]) by Rizzardo and Van den Bergh that the result above is false without the fully faithfulness hypothesis.

1.5.3 The G-equivariant case

A representability theorem was also proved in the case of projective orbifolds by Kawamata in [19]. In other words, every exact fully faithful functor, which has a left adjoint, between the bounded derived categories of smooth stacks naturally associated to normal projective varieties is of Fourier-Mukai type. This suggest us to try to investigate in chapter 3 if there is some relationship between the category of autoequivalences of perfect complexes on the (singular) quotient variety X/G and the bounded derived category of G-equivariant sheaves on X. Infact, those are exactly the kind of sheaves on X which have a good behavior with respect to the action of G.

Let X be a smooth projective variety of dimension n over an algebraic closed field $k = \mathbb{C}$ and G be a finite group with an action on X.

Definition 1.5.5. *A geometric quotient of X is a (singular) projective variety X/G togheter with a map $\pi : X \longrightarrow X/G$ such that:*

- *π is affine and G-equivariant.*
- *π is surjective and $U \in X/G$ is open if and only if $\pi^{-1}(U) \subset X$.*
- *The natural homomorphism $\mathcal{O}_{X/G} \longrightarrow (\pi_*(\mathcal{O}_X))^G$ is an isomorphism.*
- *If W is an invariant closed subset of X, then $\pi(W)$ is a closed subset of X/G. If W_1 and W_2 are disjoint invariant closed subsets of X, then $\pi(W_1) \cap \pi(W_2) = \oslash$.*

Remark 1.5.6. In the case in which the group G is finite, a geometrical quotient always exists.

We allow the action of G to be not free, although we require the points of X associated with non-trivial stabilizers to be isolated.

Take a coherent sheaf $\mathcal{F}$ on X. A *G-linearization* of $\mathcal{F}$ is given by isomorphisms $\lambda_g : \mathcal{F} \longrightarrow g^*\mathcal{F}$ for all $g \in G$ satisfying the cocycle condition: $\lambda_1 = \mathrm{Id}$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$ for all $g, h \in G$. A morphism f between two G-linearized coherent sheaves is *G-invariant* if $f = g^*f$ for all $g \in G$.

Definition 1.5.7. *The category $\mathrm{Coh}^G(X)$ is the category of G-equivariant sheaves whose objects are G-linearized sheaves with G-invariant morphisms.*

Notice that $\mathrm{Coh}^G(X)$ is an abelian category, hence we can consider the bounded derived category $\mathbf{D}^b_G(X) := \mathbf{D}^b(\mathrm{Coh}^G(X))$ associated.

Remark 1.5.8. More generally, given an additive category $\mathcal{A}$ with an arbitrary action of a finite group G we can associate to $\mathcal{A}$ the category $\mathcal{A}^G$ of G-equivariant objects, i.e. the category made by pairs (F, λ_g) where F is an object of $\mathcal{A}$ and λ_g are isomorphisms $F \longrightarrow g^*F$ satisfying the cocycle condition: $\lambda_1 = \mathrm{Id}$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$ for all $g, h \in G$. A morphism between two G-equivariant objects (F^1, λ_g^1) and (F^2, λ_g^2) is a morphism $F^1 \longrightarrow F^2$ compatible with λ_g, which means that all the diagrams:

$$\begin{array}{ccc} F^1 & \xrightarrow{\lambda_g^1} & g^*F \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g^*f} \\ F^2 & \xrightarrow{\lambda_g^2} & g^*F^2 \end{array}$$

are commutative.
If $\mathcal{A}$ is triangulated, then it is natural to ask if $\mathcal{A}^G$ is triangulated as well. It was proven by Elagin in [12] that if $\mathcal{A} := \mathbf{D}(\mathcal{C})$ is a derived category of an abelian category $\mathcal{C}$, then $\mathcal{A}^G$ has a natural structure of triangulated category.
Furthermore, if $\mathcal{C} := \mathbf{Coh}(X)$ is the category of coherent sheaves over an algebraic G-variety, then the category of G-equivariant objects $\mathbf{D}(X)^G$ is equivalent to $\mathbf{D}^b(\mathrm{Coh}^G(X))$, the derived category of G-equivariant sheaves on X. In other words, there is no difference in passing first to the equivariant category and then to the derived category or viceversa.

Let X_1 and X_2 be two smooth projective varieties. Let G_1 and G_2 be two finite groups acting on X_1 and X_2 respectively. Suppose there exists a surjective group homomorphism $\phi : G_1 \longrightarrow G_2$. Let $\alpha : X_1 \longrightarrow X_2$ be a ϕ-map.
We have a canonical pull-back:

$$\alpha^* : \mathrm{Coh}^{G_2}(X_2) \longrightarrow \mathrm{Coh}^{G_1}(X_1).$$

Let K be the kernel of ϕ. Then we also have a push-forward:

$$\alpha_*^K : \mathrm{Coh}^{G_1}(X_1) \longrightarrow \mathrm{Coh}^{G_2}(X_2)$$

defined as follow: notice that G_1 acts naturally on X_2 by ϕ, thus the kernel K acts trivially on X_2; for $\mathcal{F} \in \mathrm{Coh}^{G_1}(X_1)$ the direct image $\phi_*(\mathcal{F})$ is canonically G_1-linearized. Further, if we take K-invariants:

$$\alpha_*^K(\mathcal{F}) := [\alpha_*(\mathcal{F})]^K$$

we get a G_2-linearization because ϕ is surjective. The functors (α^*, α_*^K) form an adjoint pair.

A $G_1 \times G_2$-linearized complex $\mathcal{P}^\bullet \in \mathbf{D}^b_{G_1\times G_2}(X_1 \times X_2)$ gives rise to a *Fourier-Mukai transform*:

$$\Phi_{\mathcal{P}^\bullet} : \mathbf{D}^b_{G_1}(X_1) \longrightarrow \mathbf{D}^b_{G_2}(X_2), \qquad \mathcal{F}^\bullet \longmapsto \mathbf{R}p^K_{2_*}(\mathcal{P}^\bullet \otimes^{\mathbf{L}} p_1^*\mathcal{F}^\bullet)$$

where $p_1 : X_1 \times X_2 \longrightarrow X_1$ and $p_2 : X_1 \times X_2 \longrightarrow X_2$ is compatible with the surjective homomorphism $G_1 \times G_2 \longrightarrow G_2$.

In the case in which $X_1 = X_2 = X$ and $G_1 = G_2 = G$, we define $G \times G := G^2$; then the definition of *Fourier-Mukai transform* becomes as follow:

$$\Phi_{\mathcal{P}^\bullet} : \mathbf{D}^b_G(X) \longrightarrow \mathbf{D}^b_G(X), \qquad \mathcal{F}^\bullet \longmapsto \mathbf{R}p^G_{2_*}(\mathcal{P}^\bullet \otimes^{\mathbf{L}} p_1^*\mathcal{F}^\bullet)$$

where $\mathcal{P}^\bullet \in \mathbf{D}^b_{G^2}(X \times X)$ with p_1 and p_2 being the canonical projections.

Consider the natural direct image functor:

$$\pi_* : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X/G).$$

Let $\{0\}$ be the trivial group. The category $\mathrm{Coh}(X/G)$ can be viewed as the category of $\{0\}$-equivariant sheaves over X/G. The zero map $G \longrightarrow \{0\}$ induces a push-forward as above:

Definition 1.5.9. *The equivariant direct image functor is the exact functor:*

$$\pi_*^G : \mathrm{Coh}^G(X) \longrightarrow \mathrm{Coh}(X/G)$$

defined as:

$$\pi_*^G(\mathcal{F}) := (\pi_*\mathcal{F})^G$$

where $(\pi_*\mathcal{F})^G$ is the G-invariant subsheaf of $(\pi_*\mathcal{F})$.

Notice that the functors (π^*, π_*^G) form an adjoint pair, where π^* is the usual pull-back functor:

$$\pi^* : \mathrm{Coh}(X/G) \longrightarrow \mathrm{Coh}^G(X).$$

Furthermore, $\pi_*^G \circ \pi^* = \mathrm{Id}$.

1.6 t-structures and stability conditions

1.6.1 t-structures

Definition 1.6.1. *Let $\mathcal{T}$ be a triangulated category. A t-structure on $\mathcal{T}$ is given by a full additive subcategory $\mathscr{F}$ such that:*

- $\mathscr{F}[1] \subset \mathscr{F}$
- *For all objects E in $\mathcal{T}$, there exists a distinguished triangle:*

$$F \to E \to G \to F[1]$$

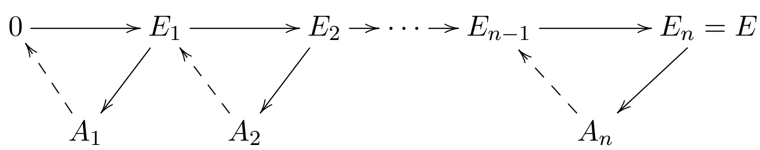
where $F \in \mathscr{F}$ and $G \in \mathscr{F}^{\perp}$.

The heart of a t-structure is the subcategory $\mathcal{A} := \mathscr{F} \cap \mathscr{F}^{\perp}[1]$.

A t-structure is said to be *bounded* if every object $E \in \mathcal{T}$ belongs to $\mathscr{F}[i] \cap \mathscr{F}^{\perp}[j]$ for some i and j. The trivial t-structures on $\mathcal{T}$ are given by $\mathscr{F} = 0$ or $\mathscr{F} = \mathcal{T}$.
We might ask if a specific heart identifies a unique t-structure associated. An answer to this question has been given in [5]:

Proposition 1.6.2. *Let $\mathcal{A}$ be a full additive subcategory of a triangulated category $\mathcal{T}$. $\mathcal{A}$ is the heart of a bounded t-structure if and only if the following properties hold:*

1. *For every objects A and B of $\mathcal{A}$ and for every integer $h_1 > h_2$, then $\mathrm{Hom}(A[h_1], B[h_2]) = 0$.*
2. *For every object E of $\mathcal{T}$, there exist a finite sequence of integers $h_1 > h_2 > \ldots > h_n$ and a collection of distinguished triangles:*



with $A_j \in \mathcal{A}[h_j]$ for all j.

The subcategory $\mathscr{F}$ is then generated by extension of the subcategories $\mathcal{A}[h]$, $h \geq 0$.

The following remark gives some well known properties about t-structures and hearts. See [2], [17] and [5] for more details.

Remark 1.6.3. Given a heart of a bounded t-structure, the filtration provided by the property (2) of Proposition 1.6.2 has the following properties:

1. The heart of a t-structure is an abelian subcategory closed by extensions.

2. The filtration is unique up to isomorphisms. In particular, the shifts k_j are fixed.

3. The filtration of the object $X[h]$ can be deduced from the filtration of X.

4. The filtration of the object $X \oplus Y$ can be deduced from the filtrations of X and of Y.

Example 1.6.4. Let $k = \mathbb{C}$ be a field and consider the ring $A = k[\epsilon]/(\epsilon^2) = \{a + \epsilon b \text{ s.t. } a, b \in k\}$. The spectrum of A consists of a single singular point, which corresponds to the maximal ideal (ϵ). Holm, Jørgensen and Yang proved, in the context of spherical objects ([14]), that there are no non-trivial t-structures on the subcategory $\mathbf{Perf}(\operatorname{Spec} A) \subset \mathbf{D}^b(\operatorname{Spec} A)$. The proof of this fact follows easily by a direct calculation.

Example 1.6.5. A *torsion pair* in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ such that $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$ for all $T \in \mathcal{T}, F \in \mathcal{F}$, and such that every object $E \in \mathcal{A}$ fits into a short exact sequence:

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

for some $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
To a torsion pair $(\mathcal{T}, \mathcal{F})$ is it possible to associate a t-structure on the bounded derived category $\mathbf{D}^b(\mathcal{A})$ by setting:

$$\mathscr{F}^{\leq 0} := \{\mathcal{F}^\bullet \in \mathbf{D}^b(\mathcal{A}) \text{ such that } \mathrm{H}^i(\mathcal{F}^\bullet) = 0 \text{ for } i > 0 \text{ and } \mathrm{H}^0(\mathcal{F}^\bullet) \in \mathcal{T}\}$$

$$\mathscr{F}^{\geq 0} := \{\mathcal{F}^\bullet \in \mathbf{D}^b(\mathcal{A}) \text{ such that } \mathrm{H}^i(\mathcal{F}^\bullet) = 0 \text{ for } i < -1 \text{ and } \mathrm{H}^{-1}(\mathcal{F}^\bullet) \in \mathcal{F}\}.$$

Its heart is called *tilt* and it is the category $\mathscr{F}^{\leq 0} \cap \mathscr{F}^{\geq 0}$.

1.6.2 Stability conditions

Definition 1.6.6. *Let $\mathcal{A}$ be an abelian category. A stability function on $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for every non zero object E of $\mathcal{A}$, the number $Z(E)$ belongs to:*

$$H = \{z \in \mathbb{C} \text{ s.t. } z = \rho \exp(i\pi\phi), \rho \geq 0, 0 < \phi \leq 1\}.$$

The phase of $E \in \mathcal{A}$ is the real number $(1/\pi) \arg(Z(E)) \in (0, 1]$.
A non zero object $E \in \mathcal{A}$ is called semi-stable if every non zero sub-object $S \hookrightarrow E$ has the phase less or equal to the phase of E.

Let $\mathcal{T}$ be a triangulated category. Let $K(\mathcal{T})$ be the Grothendieck group of $\mathcal{T}$. We define a bilinear form on $K(\mathcal{T})$, known as the Euler form, via the formula:

$$\chi(E,F) = \sum_i (-1)^i \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{T}}(E,F[i])).$$

The *numerical Grothendieck* group of $\mathcal{T}$ is the free abelian group $N(\mathcal{T}) = K(\mathcal{T})/K(\mathcal{T})^{\perp}$.

Definition 1.6.7. *Let $\mathcal{T}$ be a triangulated category. A stability condition on $\mathcal{T}$ consists of a pair $(Z, \mathcal{A})$, where:*

1. $Z : N(\mathcal{T}) \longrightarrow \mathbb{C}$.
2. *$\mathcal{A}$ is a heart of a bounded t-structure on $\mathcal{T}$.*

satisfying the following compatibilities:

1. *For every non zero object E of $\mathcal{A}$, the number $Z(E)$ belongs to:*

$$H = \{z \in \mathbb{C}^* \ s.t. \ z = \rho \exp(i\pi\phi), \rho \geq 0, 0 < \phi \leq 1\}.$$

2. *Any non zero object $E \in \mathcal{A}$ admits a Harder-Narasimhan filtration, that is a finite number of inclusion in $\mathcal{A}$:*

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_{n-1} \hookrightarrow E_n = E$$

such that $F_j = \mathrm{Cone}(E_{j-1} \hookrightarrow E_j)$ are semistable objects with phase:

$$\phi(F_1) > \cdots > \phi(F_{n-1}) > \phi(F_n).$$

3. *The support property holds: there exists a constant $C > 0$ such that, for every semistable object E:*

$$||E|| \leq C|Z(E)|.$$

where $||\ ||$ is an arbitrary norm on $N(\mathcal{T}) \otimes \mathbb{C}$.

Remark 1.6.8. $||\ ||$ is arbitrary since all the norms on $N(\mathcal{T}) \otimes \mathbb{C}$ are equivalent.

Notice however that, in those examples that we will consider in the future, the Grothendieck group if finitely generated torsion free, hence the support property is automatically fulfilled. Also, the numerical Grothendieck group coincides with the usual Grothendieck group.

Remark 1.6.9. If $A \longrightarrow B$ is a morphism of semistable objects, calling by N the image we see that $\phi(A) \leq \phi(N) \leq \phi(B)$. This in particular means that the fltration is unique if it exists.

Proposition 1.6.10. *To give a stability condition is the same as to give a stability function on its heart satisfying the Harder-Narasimhan property.*

Proof. See Proposition 5.3 in [5]. ☐

Thus, given a stability function, we would like to have a way to check easily the Harder-Narasimhan property. This is provided by the following:

Proposition 1.6.11. *Suppose an abelian category $\mathcal{A}$ is given with a stability function such that:*

- *Any chain of monomorphisms $\ldots \longrightarrow A_2 \longrightarrow A_1$, with $\phi(A_1) < \phi(A_2) < \ldots$, stabilizes.*
- *Any chain of epimorphisms $A_1 \longrightarrow A_2 \longrightarrow \ldots$, with $\phi(A_1) > \phi(A_2) > \ldots$, stabilizes.*

Then the stability function has the Harder-Narasimhan property.

Proof. See Proposition 5.1.6 in [15]. ☐

Example 1.6.12. Consider X a smooth curve and let $\mathbf{D}^b(X)$ the bounded derived category associated. There are two natural morphisms $K(\mathbf{D}^b(X)) \longrightarrow \mathbb{C}$, the degree and the rank (the alternating sum of the degree and the rank of the complex). We get then the map:

$$Z(E) = -\deg(E) + i\text{rank}(E).$$

which gives a stability function on $\mathbf{Coh}(X)$. It is not hard to see that any chain of subsheaves $\ldots \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, with $\phi(\mathcal{F}_1) < \phi(\mathcal{F}_2) < \ldots$, stabilizes. In the same way, any chain of quotients $\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \ldots$, with $\phi(\mathcal{F}_1) > \phi(\mathcal{F}_2) > \ldots$, stabilizes. Therefore the Harder-Narasimhan property holds and thus we find a stability condition on $\mathbf{D}^b(X)$. Actually, stability conditions on curves are completely classified. In particular, for smooth projective curves of positive genus, the space of stability conditions is isomorphic to $\mathbb{C} \times \mathbb{H}$ where $\mathbb{H}$ denote the complex upper half plane (see [24]).

1.7 DG categories

Here we give some definitions and tools that will be used strongly in chapter 2.

Definition 1.7.1. *A DG category is a k-linear category $\mathcal{A}$ such that:*

- $\text{Hom}(X, Y)$ *is a $\mathbb{Z}$-graded k-module for every* $X, Y \in \text{Ob}(\mathcal{A})$.
- *There is a differential* $d : \text{Hom}(X, Y) \longrightarrow \text{Hom}(X, Y)$ *of degree one, such that for every* $X, Y, Z \in \text{Ob}(\mathcal{A})$ *the composition* $\text{Hom}(X, Y) \otimes \text{Hom}(Y, Z) \longrightarrow \text{Hom}(X, Z)$ *is a morphism of DG k-modules.*

A *DG functor* $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ between two DG categories is given by a map on the objects $\mathcal{F} : \mathrm{Ob}(\mathcal{A}) \longrightarrow \mathrm{Ob}(\mathcal{B})$ and maps on the spaces of morphisms:

$$\mathcal{F}(X,Y) : \mathrm{Hom}_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y))$$

which are morphisms of DG k-modules and are compatible with the compositions and the units. Given a DG category $\mathcal{A}$, we denote by $H^0(\mathcal{A})$ the *homotopy category* associated to $\mathcal{A}$, which has the same objects of the DG category $\mathcal{A}$ and its morphisms are defined by taking the zeroth cohomology $H^0(\mathrm{Hom}_{\mathcal{A}}(X,Y))$.

Definition 1.7.2. *A DG functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is called a quasi-equivalence if $\mathcal{F}(X,Y)$ is a quasi-isomorphism for all objects $X, Y \in \mathcal{A}$ and the induced functor $H^0(\mathcal{F}) : H^0(\mathcal{A}) \longrightarrow H^0(\mathcal{B})$ is an equivalence. We say that two objects $X, Y \in \mathcal{A}$ are homotopy equivalent if they are isomorphic in $H^0(\mathcal{A})$.*

Definition 1.7.3. *Let* dgMod-k *be the DG category of DG k-modules. Given a small DG category $\mathcal{A}$, every DG functor:*

$$\mathcal{M} : \mathcal{A}^{op} \longrightarrow \text{dgMod-}k$$

is called a right DG $\mathcal{A}$-module.

We denote by dgMod-$\mathcal{A}$ the DG category of right DG $\mathcal{A}$-modules. Let $\mathrm{Ac}(\mathcal{A})$ be the subcategory of Mod-$\mathcal{A}$ consisting of all acyclic DG modules.

Definition 1.7.4. *The derived category $\mathbf{D}(\mathcal{A})$ is the Verdier quotient between the homotopy category associated with* Mod-$\mathcal{A}$ *and the subcategory of acyclic DG modules:*

$$\mathbf{D}(\mathcal{A}) := \frac{H^0(\text{Mod-}\mathcal{A})}{H^0(\mathrm{Ac}(\mathcal{A}))}.$$

Every object $X \in \mathcal{A}$ defines a *representable* DG module:

$$h^X(-) := \mathrm{Hom}(-, X).$$

The functor $h^\bullet$ is called the *Yoneda functor*, and it is fully faithful.

Definition 1.7.5. *A DG $\mathcal{A}$-module $\mathcal{M}$ is called free if it isomorphic to a direct sum of shift of representable DG modules of the form $h^X[n]$, where $X \in \mathcal{A}$, $n \in \mathbb{Z}$.*

Definition 1.7.6. *A DG $\mathcal{A}$-module P is called semi-free if has a filtration:*

$$0 = \phi_0 \subset \phi_1 \subset \phi_2 \subset \ldots = P$$

such that each quotient ϕ_i/ϕ_{i-1} is free.

If $\phi_m = P$ for some m and ϕ_i/ϕ_{i-1} is a finite direct sum of DG modules of the form $h^Y[n]$, then we call P a *finitely generated semi-free* DG module. Denote by $\mathcal{SF}(\mathcal{A})$ the full DG subcategory of semi-free DG modules.

Definition 1.7.7. *Given a small DG category $\mathcal{A}$ we denote by $Perf(\mathcal{A})$ the DG category of perfect DG modules, that is the full DG subcategory of $\mathcal{SF}(\mathcal{A})$ consisting of all DG modules which are homotopy equivalent to a direct summand of a finitely generated semi-free DG module.*

Recall that, given two DG categories $\mathcal{A}$ and $\mathcal{B}$, their tensor product $\mathcal{A} \otimes \mathcal{B}$ is again a DG category. See [3] for references.

Let $\mathcal{A}$ and $\mathcal{B}$ be two DG categories, a $\mathcal{A}$-$\mathcal{B}$-*bimodule* is a DG $\mathcal{A}^{op} \otimes \mathcal{B}$-module. A *quasi-functor* from $\mathcal{A}$ to $\mathcal{B}$ is a $\mathcal{A}$-$\mathcal{B}$-bimodule $X \in \mathbf{D}(\mathcal{A}^{op} \otimes \mathcal{B})$ such that the tensor functor:

$$(-) \otimes_{\mathcal{A}} X : \mathbf{D}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B})$$

takes every representable $\mathcal{A}$-module to an object which is isomorphic to a representable $\mathcal{B}$-module.

Definition 1.7.8. *Let $\mathcal{T}$ be a triangulated category. An enhancement of $\mathcal{T}$ is a pair (B, ϵ), where $\mathcal{B}$ is a pretriangulated DG category and $\epsilon : H^0(\mathcal{B}) \longrightarrow \mathcal{T}$ is an equivalence of triangulated categories.*

Example 1.7.9. Let X be a quasi-projective scheme. Consider the triangulated category $\mathbf{D}_{\mathrm{qcoh}}(X)$. Let $\mathbf{C}(X)$ be the pretriangulated DG category of unbounded complexes over $\mathbf{QCoh}(X)$. Denote by $\mathcal{A}(X)$ the full subcategory of $\mathbf{C}(X)$ consisting of acyclic complexes. The *Drinfeld quotient* ([11]) of DG categories $\mathbf{C}(X)/\mathcal{A}(X)$ is a pretriangulated category and:

$$H^0(\mathbf{C}(X)/\mathcal{A}(X)) \xrightarrow{\cong} \mathbf{D}_{\mathrm{qcoh}}(X) \qquad (*)$$

Therefore we get an enhancement of $\mathbf{D}_{\mathrm{qcoh}}(X)$.

Example 1.7.10. Let X be a smooth projective variety. Let $\mathbf{P}(X)$ be the full DG subcategory of $\mathbf{C}(X)/\mathcal{A}(X)$ whose objects are sento to $\mathbf{Perf}(X)$ under the equivalence $(*)$ above. Then $\mathbf{P}(X)$ is an enhancement of $\mathbf{Perf}(X)$.

Example 1.7.11. Let X be a smooth projective variety. Consider the triangulated category $\mathbf{D}^b(X)$. Let $\mathbf{C}^+(X)$ be the pretriangulated DG category of bounded below complexes of $\mathcal{O}_X$-modules with bounded coherent cohomology, and let $\mathbf{I}(X)$ be the full pretriangulated subcategory of $\mathbf{C}^+(X)$ consisting of complexes of injective $\mathcal{O}_X$-modules. Then the functor:

$$\mathrm{H}^0(\mathbf{I}(X)) \longrightarrow \mathrm{H}^0(\mathbf{C}^+(X)) \longrightarrow \mathbf{D}^b(X)$$

is an equivalence, and thus we get an enhancement of $\mathbf{D}^b(X)$.

Definition 1.7.12. *A triangulated category $\mathcal{T}$ has a unique enhancement if it has one and for any two enhancements (B, ϵ) and (B', ϵ') of T the DG categories B and B' are quasi-equivalent, i.e. there exists a quasi-functor $\phi : B \longrightarrow B'$ which induces an equivalence:*

$$\mathrm{H}^0(\phi) : \mathrm{H}^0(B) \xrightarrow{\cong} \mathrm{H}^0(B').$$

In this case the enhancements (B, ϵ) and (B', ϵ') are called equivalent.

Let X be a a quasi-compact and quasi-separated scheme over k. We say that the scheme X has *enough locally free sheaves* if for any finitely presented sheaf $\mathcal{F}$ there is an epimorphism $\mathcal{E} \longrightarrow \mathcal{F}$ with a locally free sheaf $\mathcal{E}$ of finite type. Lunts and Orlov proved in [22] that:

Theorem 1.7.13. *Let X be a quasi-compact and separated scheme that has enough locally free sheaves. Then the derived category of quasi-coherent sheaves $\mathbf{D}_{\mathrm{qcoh}}(X)$ has a unique enhancement.*

And as a corollary:

Corollary 1.7.14. *Let X be a quasi-projective scheme over k.*

- *The derived category of quasi-coherent sheaves $\mathbf{D}_{\mathrm{qcoh}}(X)$ has a unique enhancement.*
- *The triangulated category of perfect complexes $\mathbf{Perf}(X) = \mathbf{D}^c_{\mathrm{qcoh}}(X)$ has a unique enhancement.*
- *The bounded derived category of coherent sheaves $\mathbf{D}^b(X)$ has a unique enhancement.*

Chapter 2

The double point

2.1 Background

As we mentioned in previous chapter, a highly non-trivial result by Lunts and Orlov in [22] shows, by the use of DG categories, that if X is projective such that the maximal torsion subsheaf of dimension zero $T_0(\mathcal{O}_X) \subset \mathcal{O}_X$ is trivial, Y is noetherian separated and $F : \mathbf{Perf}(X) \longrightarrow \mathbf{D}^b(Y)$ is an exact fully faithful functor, then F is of Fourier-Mukai type.

The hypothesis $T_0(\mathcal{O}_X) = 0$ is related with the use of ample sequences and it seems not to be a very natural assumption. What happens if we consider a projective scheme X such that $T_0(\mathcal{O}_X) \neq 0$? The simplest example of such scheme is given by $\operatorname{Spec} k$. Here the result is trivial (see [8] Remark 2.2) in view of the simple description of $\mathbf{D}^b(\operatorname{Spec} k)$. Thus, we could take in consideration a zero dimensional non-smooth scheme. In such way, the maximal torsion subsheaf of dimension zero is certainly not trivial. A basic model of such type of objects is given by the "double point scheme", which is the spectrum of the ring of dual numbers $A := k[\epsilon]/(\epsilon^2)$. Along the chapter we prove that, if:

$$F : \mathbf{Perf}(A) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(Y)$$

is a fully faithful functor, then there is an object $\mathcal{E} \in \mathbf{D}_{\mathrm{qcoh}}(\operatorname{Spec} A \times Y)$ such that:

$$\Phi_{\mathcal{E}}|_{\mathbf{Perf}(A)} \cong F.$$

Furthermore, if Y is noetherian and F sends $\mathbf{Perf}(A)$ to $\mathbf{D}^b(Y)$, then

$$\mathcal{E} \in \mathbf{D}^b(\operatorname{Spec} A \times Y).$$

Thus we show that the main result in [22] still holds in a case in which the maximal torsion subsheaf of dimension zero $T_0(\mathcal{O}_X)$ is not trivial, hence we do expect it is possible to avoid this hypothesis and prove the same result in a more general case.

We also deal with the problem of classifying all the stability conditions on the category $\mathbf{D}^b(A)$. Basically we prove that $\mathrm{Stab}(\mathbf{D}^b(A))$ is isomorphic to $\mathbb{C}$, the universal covering of $\mathbb{C}^*$.
In order to prove the results concerning such a classification, we will exploit the study on the category $\mathbf{D}^b(A)$ following an argument originally used by Jørgensen and Pauksztello in [18], Holm, Jørgensen and Yang in [14] for the category $\mathbf{Perf}(A)$.

2.2 Indecomposable complexes of $\mathbf{D}^b(A)$

Let $k = \mathbb{C}$ be a field and consider the ring $A = k[\epsilon]/(\epsilon^2)=\{a + \epsilon b \text{ s.t. } a, b \in k\}$. The spectrum of A consists of a single point, which corresponds to the maximal ideal (ϵ). We are interested in studying the subcategory $\mathbf{Perf}(\mathrm{Spec}\, A) \subset \mathbf{D}^b(\mathrm{Spec}\, A)$.

Recall that $\mathbf{D}^b(\mathrm{Spec}\, A)$ is the bounded derived category of coherent sheaves on $\mathrm{Spec}\, A$, and it is equivalent to $\mathbf{D}^b(\mathbf{mod}_{\mathrm{fg}} - A)$, the bounded derived category of finitely generated A-modules. On the other hand, the category $\mathbf{Perf}(\mathrm{Spec}\, A)$ is by definition the full subcategory of $\mathbf{D}^b(\mathrm{Spec}\, A)$ made by compact objects. Then, as we mentioned in previous chapter, $\mathbf{Perf}(\mathrm{Spec}\, A)$ is equivalent to $\mathbf{Perf}(A)$, the full subcategory of $\mathbf{D}^b(\mathrm{Spec}\, A) = \mathbf{D}^b(\mathbf{mod}_{\mathrm{fg}} - A) =: \mathbf{D}^b(A)$ consisting of bounded complexes of finitely generated projective modules modulo the homotopy relation.

We want to study in details objects and mophisms of $\mathbf{D}^b(A)$ and therefore we focus on indecomposable complexes: in an additive category, an object X is called *indecomposable* if $X \cong Y \oplus Z$ implies $Y \cong 0$ or $Z \cong 0$.
A good context to study indecomposable objects is provided by Krull-Schmidt categories, which are explained in [24].

Definition 2.2.1. *Let $\mathcal{C}$ be an additive category such that $\mathrm{End}_{\mathcal{C}}(X)$ is a semiperfect ring for all $X \in \mathcal{C}$ (in that case $\mathcal{C}$ is called a pre-Krull-Schmidt category). $\mathcal{C}$ is called a Krull-Schmidt category if every idempotent splits, i.e. for every X in $\mathcal{C}$ and for every $e \in \mathrm{End}_{\mathcal{C}}(X)$ such that $e^2 = e$, there exist Y in $\mathcal{C}$ and two morphisms $p : X \to Y$ and $q : Y \to X$ such that $qp = e$ and $pq = 1_Y$.*

An additive category in which every idempotent splits is also called *Karoubian*, hence a Krull-Schmidt category is a pre-Krull-Schmidt category that is also Karubian. Note that every abelian category is Karoubian.

In [34] one can find another definition of the split property: an idempotent $e : X \to X$ splits if and only if there exists a non trivial decomposition $X \cong Y \oplus Z$ with e corresponding to the projection on Y. These two definitions are equivalent in a triangulated category, which is our case. Thanks to the following result, proven in [24], we can limit ourselves to consider only indecomposable objects:

Theorem 2.2.2. *In a Krull-Schmidt category every object can be decomposed into a finite direct sum of indecomposable objects. Moreover this decomposition is unique up to isomorphism.*

Infact:

Proposition 2.2.3. *Let X be a projective variety. Then $\mathbf{Perf}(X)$ and $\mathbf{D}^b(X)$ are Krull-Schmidt categories.*

Proof. Since X is projective, the endomorphism ring of every object of $\mathbf{Perf}(X)$ and of $\mathbf{D}^b(X)$ is a finitely generated $k-$algebra of finite dimension, and then it is semiperfect. Moreover, $\mathbf{D}_{\mathrm{qcoh}}(X)$ is Karoubian, because it is a triangulated category with countably many direct sums. The subcategories $\mathbf{Perf}(X)$ and $\mathbf{D}^b(X)$ are thick and then Karoubian. □

Proposition 2.2.4. *Let $\mathcal{C}$ be a Karoubian triangulated category, $\mathcal{D}$ an additive category and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a fully faithful additive functor. Then F sends indecomposable objects of $\mathcal{C}$ to indecomposable objects of $\mathcal{D}$.*

Proof. Let $X^\bullet$ be an indecomposable object of $\mathcal{C}$. Since $\mathcal{C}$ is Karoubian, $\mathrm{Hom}(X^\bullet, X^\bullet)$ does not contain any idempotent except the identity and zero. Suppose $F(X^\bullet) \cong Y^\bullet \oplus Z^\bullet$, with $Y^\bullet$ and $Z^\bullet$ non zero. Since F is fully faithful and additive we have an isomorphism of rings:

$$\mathrm{Hom}(X^\bullet, X^\bullet) \cong \mathrm{Hom}(F(X^\bullet), F(X^\bullet)) \cong \mathrm{Hom}(Y^\bullet \oplus Z^\bullet, Y^\bullet \oplus Z^\bullet).$$

The last space contains the projection $Y^\bullet \oplus Z^\bullet \longrightarrow Y^\bullet$, which is an idempotent different from the identity and zero, giving a contraddiction. □

Definition 2.2.5. *For every $i \in \mathbb{N}$, $i > 0$ let:*

$$X_i := \{\, 0 \longrightarrow A_{(-i)} \xrightarrow{\epsilon} A_{(-i+1)} \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} A_{(-1)} \longrightarrow 0 \,\}.$$

$$X_\infty := \{\, \cdots \xrightarrow{\epsilon} A_{(-i)} \xrightarrow{\epsilon} A_{(-i+1)} \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} A_{(-1)} \longrightarrow 0 \,\}.$$

Where $A_{(l)}$ stands for the module A in the position $l \in \mathbb{Z}$.

We have that $\{X_i[h],\ \text{with } i > 0,\ \text{and } h \in \mathbb{Z}\}$ are the indecomposable objects of $\mathbf{Perf}(A)$. Moreover, $\{X_\infty[h],\ \text{with } h \in \mathbb{Z}\}$ are the indecomposable objects of $\mathbf{D}^b(A) \smallsetminus \mathbf{Perf}(A)$ (see [21], Section 3 or [20], example 3.7).

2.3 Maps between indecomposable complexes

In this section we will study in details the morphisms in the category $\mathbf{Perf}(A)$, which recall it is equivalent to the homotopy bounded category of complexes of finitely generated free $A-$modules.

Notice that for every complexes X_i, X_j and for every integers α, β:

$$\mathrm{Hom}(X_i[\alpha], X_j[\beta]) \cong \mathrm{Hom}(X_i, X_j[\beta - \alpha]).$$

Infact, morphisms between two indecomposable objects depend only on the mutual position of the two complexes.

We start with morphisms in $\mathbf{Perf}(A)$ by considering the space:

$$V := \mathrm{Hom}(X_i, X_j[\alpha])$$

with $i, j \neq \infty$.
If $i > j$, there are five cases, which go from $\alpha \leq -j$ to $\alpha \geq i$.

1. $\alpha \leq -j$. A morphism $X_i \longrightarrow X_j[\alpha]$ can be represented by such a diagram:

$$\begin{array}{ccccccccccccccccccc}
0 & \to & A & \to & \cdots & \to & A & \to & 0 & \to & \cdots & \to & \cdots & \to & \cdots & \to & \cdots & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \cdots & \to & \cdots & \to & \cdots & \to & \cdots & \to & 0 & \to & A & \to & \cdots & \to & A & \to & 0
\end{array}$$

   Where the vertical arrows are automorphisms of A, and therefore they must be of the form $a + \epsilon b$ where $a, b \in A$. However, it is clear that in this case all the vertical morphisms are zero and thus $V = 0$.

2. $-j < \alpha \leq 0$. In this case the vertical arrows must be of the form ϵb with $b \in A$, because of the commutativity of the squares:

$$\begin{array}{ccccccccccccccccc}
0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0 & \to & \cdots & \to & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \epsilon b_1} & & & & \downarrow{\scriptstyle \epsilon b_k} & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & \cdots & \to & 0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0
\end{array}$$

with $k = j + \alpha$. Define:

$$B := \sum_{l=1}^{k} (-1)^{l+1} b_{k-l+1}.$$

Up to homotopy we can reduce the diagram to be the following one:

$$\begin{array}{ccccccccccccccccc}
0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0 & \to & \cdots & \to & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle \epsilon B} & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & \cdots & \to & 0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0
\end{array}$$

This shows that in this case the space of morphisms V is isomorphic to k.

3. $0 < \alpha < i - j$. Again, by the commutativity of the squares we have:

$$\begin{array}{ccccccccccccccccc}
0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \epsilon b_1} & & & & \downarrow{\scriptstyle \epsilon b_k} & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & \cdots & \to & 0 & \to & A & \to & \cdots & \to & A & \to & 0 & \to & \cdots & \to & 0
\end{array}$$

Up to homotopy all the vertical morphisms become zero, hence $V = 0$.

4. $i - j \leq \alpha < i$ and $\alpha \neq 0$. This case is similar to (2). The commutativity of the squares implies that all the vertical morphisms must have the first same component:

$$\begin{array}{ccccccccccccccccc}
0 & \to & \cdots & \to & 0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a+\epsilon b_1} & & & & \downarrow{\scriptstyle a+\epsilon b_k} & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0 & \to & \cdots & \to & 0
\end{array}$$

Up to homotopy we can reduce the diagram to be the following one:

$$\begin{array}{ccccccccccccccccc}
0 & \to & \cdots & \to & 0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & A & \to & \cdots & \to & A & \to & \cdots & \to & A & \to & 0 & \to & \cdots & \to & 0
\end{array}$$

Thus V is still isomorphic to k.

5. $i \leq \alpha$. This case is similar to (1). Thus, V is equal to zero.

If $i = j$ and $\alpha \neq 0$, the calculations are analogous to the previous cases. Note however that (3) can not hold in this case.
If $\alpha = 0$, the commutativity of the square implies that we have a diagram of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \cdots & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle a+\epsilon b_1} & & & & \downarrow{\scriptstyle a+\epsilon b_h} & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & \cdots & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Define:

$$C := \sum_{l=1}^{i} (-1)^{l+1} b_{i-l+1}.$$

Up to homotopy we can reduce the diagram to be the following one:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \cdots & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a+\epsilon C} & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & \cdots & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

This shows that in this case the space of morphisms V is equal to $k \oplus k$.

If $i < j$, the results are similar to the case $i > j$.
We can sum up in the following:

Proposition 2.3.1. *Consider the space* $V = \mathrm{Hom}(X_i, X_j[\alpha])$*:*

- *If* $-j < \alpha \leq \min\{0, i-j\}$ *and* $(i-j, \alpha) \neq (0,0)$ *then* V *has dimension* 1 *and it is generated by* $\epsilon^i_{j[\alpha]}$. *These morphisms are named of* k_ϵ*-type.*

- *If* $\max\{0, i-j\} \leq \alpha < i$ *and* $(i-j, \alpha) \neq (0,0)$ *then* V *has dimension* 1 *and it is generated by* $1^i_{j[\alpha]}$. *These morphisms are named of* k_1*-type.*

- *If* $i = j$ *and* $\alpha = 0$ *then* V *has dimension* 2 *and it is generated by both* $\epsilon^i_{i[0]}$ *and* $1^i_{i[0]}$. *These morphisms are named of* k^2*-type*

- $V = \{0\}$ *for all the remaining cases.*

Remark 2.3.2. A morphism between two indecomposable objects can be described by a pair (a, b) of elements of k, where a is the coefficient of the generator 1 and b is the coefficient of the generator ϵ.

Remark 2.3.3. The results of this proposition can be extended to $\mathbf{D}^b(A)$; one can easily prove that:

- $\mathrm{Hom}(X_\infty, X_\infty[h])$ is generated by 1 if $h \geq 0$ and it is 0 otherwise.
- $\mathrm{Hom}(X_\infty, X_i[h])$ is generated by ϵ if $-i < h \leq 0$ and it is 0 otherwise.
- $\mathrm{Hom}(X_i, X_\infty[h])$ is generated by 1 if $0 \leq h < i$ and it is 0 otherwise.

As a consequence of the previous theorem, for all $X^\bullet$ and $Y^\bullet$ in $\mathbf{Perf}(A)$ there is the following isomorphism:

$$\mathrm{Hom}_{\mathbf{Perf}(A)}(X^\bullet, Y^\bullet) \cong \mathrm{Hom}_{\mathbf{Perf}(A)}(Y^\bullet, X^\bullet).$$

More generally, Serre duality holds in $\mathbf{Perf}(A)$. This is a particular case of [13], Theorem 6.7.

2.4 Compositions

We now want to describe compositions of morphisms in $\mathbf{Perf}(A)$. That is, given two morphisms between indecomposable objects:

$$f:\ X_i \longrightarrow X_j[\alpha] \ \text{ and } g:\ X_j[\alpha] \longrightarrow X_k[\beta],$$

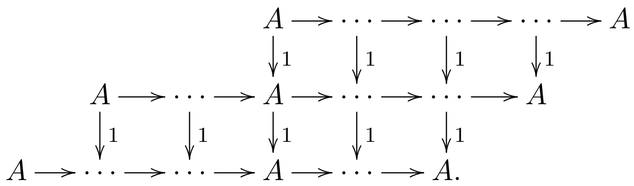
we want to find out what type of morphism is $g \circ f$. We proceed by studying compositions of generators of the morphisms described in Proposition 2.3.1. The situation is summed up in the following table. Clearly, if either f or g is the zero morphism, then also the composition $g \circ f$ is zero.

$\circ$	0	$1^i_{j[\alpha]}$	$\epsilon^i_{j[\alpha]}$
0	0	0	0
$1^{j[\alpha]}_{k[\beta]}$	0	(i)	(ii)
$\epsilon^{j[\alpha]}_{k[\beta]}$	0	(iii)	(iv)

Proposition 2.3.1 gives the conditions for the generators to be well defined:

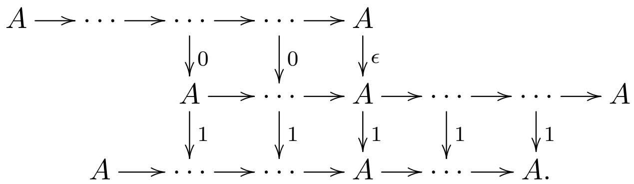
(i) holds when $\max\{0, i-j\} \leq \alpha < i$ and $\max\{0, j-k\} \leq \beta - \alpha < j$.
(ii) holds when $-j < \alpha \leq \min\{0, i-j\}$ and $\max\{0, j-k\} \leq \beta - \alpha < j$.
(iii) holds when $\max\{0, i-j\} \leq \alpha < i$ and $-k < \beta - \alpha \leq \min\{0, j-k\}$.
(iv) holds when $-j < \alpha \leq \min\{0, i-j\}$ and $-k < \beta - \alpha \leq \min\{0, j-k\}$.

(i) The composition of $1^{j[\alpha]}_{k[\beta]} \circ 1^{i}_{j[\alpha]}$ is a morphism from X_i to $X_k[\beta]$. If $\max\{0, i-k\} \leq \beta < i$ holds, that is the condition of having a morphism of k_1-type between X_i and $X_k[\beta]$, then $1^{j[\alpha]}_{k[\beta]} \circ 1^{i}_{j[\alpha]} = 1^{i}_{k[\beta]}$:



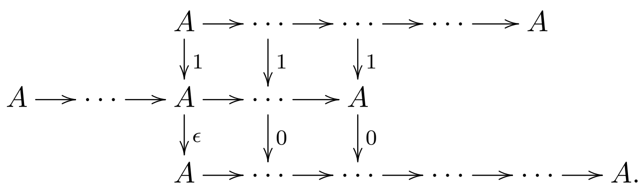
Otherwise $1^{j[\alpha]}_{k[\beta]} \circ 1^{i}_{j[\alpha]} = 0$

(ii) The composition $1^{j[\alpha]}_{k[\beta]} \circ \epsilon^{i}_{j[\alpha]}$ is a morphism from X_i to $X_k[\beta]$. If $-k < \beta \leq \min\{0, i-k\}$ holds, that is the condition of having a morphism of k_ϵ-type between X_i and $X_k[\beta]$, then $1^{j[\alpha]}_{k[\beta]} \circ \epsilon^{i}_{j[\alpha]} = \epsilon^{i}_{k[\beta]}$:



Otherwise $1^{j[\alpha]}_{k[\beta]} \circ \epsilon^{i}_{j[\alpha]} = 0$.

(iii) The composition of $\epsilon^{j[\alpha]}_{k[\beta]} \circ 1^{i}_{j[\alpha]}$ is a morphism from X_i to $X_k[\beta]$. If $-k < \beta \leq \min\{0, i-k\}$ holds, that is the condition of having a morphism of k_ϵ-type between X_i and $X_k[\beta]$, then $\epsilon^{j[\alpha]}_{k[\beta]} \circ 1^{i}_{j[\alpha]} = \epsilon^{i}_{k[\beta]}$:



Otherwise $\epsilon_{k[\beta]}^{j[\alpha]} \circ 1_{j[\alpha]}^{i} = 0$.

(iv) The composition of two morphisms of k_ϵ-type is always zero.

As in Remark 2.3.3, the above results hold, with the same inequalities, also in $\mathbf{D}^b(A)$.

2.5 Fully Faithful endofunctors of $\mathbf{Perf}(A)$

In this section we will deal with $k-$linear functors $F : \mathbf{Perf}(A) \longrightarrow \mathbf{Perf}(A)$ that commute with the shifts. For example, obviously the shift $[n]$ itself. Also the push forward $\mathbf{R}f_*$ along a proper morphism f of projective varieties gives such a type of functor. Furthermore these two functors are exact and of Fourier-Mukai type; see [16] for a deeper discussion.

For a more general analisys, in this section we will suppose F to be fully faithful but we will not require the functor to be exact.

Proposition 2.5.1. *Let $F : \mathbf{Perf}(A) \longrightarrow \mathbf{Perf}(A)$ be a fully faithful functor. On the objects, F is isomorphic to the shift functor $[n]$ for some integer n.*

Proof. F commutes with the shifts, so we can focus on the image of an indecomposable object X_i for any integer $i > 0$. By Proposition 2.2.4, F sends indecomposable objects to indecomposable objects, so $F(X_i) \cong X_j[\alpha]$ for some integer $j > 0$ and some α. F is also fully faithful, thus:

$$\mathrm{Hom}(X_i, X_i[\beta]) \cong \mathrm{Hom}(F(X_i), F(X_i)[\beta]) \cong$$

$$\cong \mathrm{Hom}(X_j[\alpha], X_j[\alpha+\beta]) \cong \mathrm{Hom}(X_j, X_j[\beta]).$$

It follows from Proposition 2.3.1 that $i = j$, and this proves that $F(X_i) \cong X_i[h_i]$ for some integer h_i. Actually h_i does not depend on i:

$$\mathrm{Hom}(X_i, X_j) \cong \mathrm{Hom}(F(X_i), F(X_j)) \cong$$

$$\cong \mathrm{Hom}(X_i[h_i], X_j[h_j]) \cong \mathrm{Hom}(X_i, X_j[h_j - h_i]).$$

Again, by Proposition 2.3.1, $h_j - h_i = 0$. □

Corollary 2.5.2. *Every fully faithful functor $F : \mathbf{Perf}(A) \longrightarrow \mathbf{Perf}(A)$ is an equivalence.*

Proof. It is clear from Proposition 2.5.1 that every fully faithful functor:

$$\mathbf{Perf}(A) \longrightarrow \mathbf{Perf}(A)$$

is also essentially surjective, hence it is an equivalence. □

With similar arguments, and by including the indecomposable objects X_∞, Proposition 2.5.1 and Corollary 2.5.2 can be extended to a fully faithful functor $F : \mathbf{D}^b(A) \longrightarrow \mathbf{D}^b(A)$.

Remark 2.5.3. Due to Proposition 2.5.1, $F(X_i)$ is isomorphic to $X_i[h]$ for a fixed $h \in \mathbb{Z}$. Up to composition with a shift $[-h]$, we can assume that F is isomorphic to the identity functor on the objects.

We now want to study the action of F on the morphisms between indecomposable elements.

Proposition 2.5.4. *Consider a morphism (a, b) as described in Remark 2.3.3 from an indecomposable object X_i to itself, that is:*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \cdots & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a+\epsilon\cdot b} & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \cdots & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

with $a,\ b \in k$. The action of the functor on the morphism (a, b) is given by an invertible matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & \delta_i \end{pmatrix}.$$

with $\delta_i \in k$.

Proof. Since F is a functor, it preserves compositions and the identity. By imposing these two conditions to a generic 2×2 matrix with coefficients in k, the result is straightforward. □

This shows that if (a, b) is a morphism of k^2-type from X_i to itself, then F acts only on its second component, which is the one generated by k_ϵ. It is also immediate to see that F can only act on the second component of a morphism of k_ϵ-type, as well as F can only act on the first component of a morphism of k_1-type. Hence we give the following definition:

Definition 2.5.5. *For all $i, j \in \mathbb{N}$ and $\alpha \in \mathbb{Z}$ we define $k^i_{j[\alpha]} \in k$ such that:*

- *if (a, b) is a morphism of k^2-type from X_i to X_i, then $F(a, b) = (a, k^i_{i[0]}b)$.*
- *if $(a, 0)$ is a morphism of k_1-type from X_i to $X_j[\alpha]$, then $F(a, 0) = (k^i_{j[\alpha]}a, 0)$.*
- *if $(0, b)$ is a morphism of k_ϵ-type from X_i to $X_j[\alpha]$, then $F(0, b) = (0, k^i_{j[\alpha]}b)$.*

Note that the element $k^i_{i[0]}$ corresponds to δ_i in Proposition 2.5.4. Furthermore, the functor F is fully faithful, hence all the coefficients $k^i_{j[\alpha]}$ are non zero.

Proposition 2.5.6. *$k^i_{i[0]}$ does not depend on $i \in \mathbb{N} \smallsetminus \{0\}$.*

Proof. We prove that $k^i_{i[0]} = k^1_{1[0]}$ for $i > 1$. Consider the following morphisms from X_i to X_1 and from X_1 to X_i:

$$\begin{array}{cccccccccc}
X_i : & & 0 & \longrightarrow & A & \longrightarrow \cdots \longrightarrow & A & \longrightarrow & 0 \\
\big\downarrow \epsilon^i_{1[0]} & & & & & & \big\downarrow \epsilon & & \\
X_1 : & & 0 & \longrightarrow & 0 & \longrightarrow 0 \longrightarrow & A & \longrightarrow & 0 \\
\big\downarrow 1^1_{i[0]} & & & & & & \big\downarrow 1 & & \\
X_i : & & 0 & \longrightarrow & A & \longrightarrow \cdots \longrightarrow & A & \longrightarrow & 0.
\end{array}$$

The functor F sends $\epsilon^i_{1[0]}$ to $k^i_{1[0]}\epsilon^i_{1[0]}$ and $1^1_{i[0]}$ to $k^1_{i[0]}1^1_{i[0]}$; moreover, the composition:

$$1^1_{i[0]} \circ \epsilon^i_{1[0]} = 1^i_{i[0]}$$

is a morphism between X_i and X_i and then it is sent by F to $k^i_{i[0]}1^i_{i[0]}$. As F preserves compositions:

$$F(1^1_{i[0]} \circ \epsilon^i_{1[0]}) = F(1^1_{i[0]}) \circ F(\epsilon^i_{1[0]}),$$

which means $k^i_{i[0]}1^i_{i[0]} = k^1_{i[0]}k^i_{1[0]}1^i_{i[0]}$.
It follows $k^i_{i[0]} = k^1_{i[0]}k^i_{1[0]}$. By composing these morphisms in the inverse order we get $\epsilon^i_{1[0]} \circ 1^1_{i[0]} = 1^1_{1[0]}$, a morphism between X_1 and X_1. It is sent by F to $(0, k^1_{1[0]}1^1_{1[0]})$. Again, F preserves compositions, hence $k^1_{1[0]} = k^1_{i[0]}k^i_{1[0]}$, that is $k^i_{i[0]} = k^1_{1[0]}$. ☐

Proposition 2.5.7. *Up to composing with a shift and a push forward along an automorphism of* Spec(A)*, the functor F is isomorphic to a functor which is the identity on the objects and has coefficients $k^i_{i[0]}$ equal to* 1.

Proof. Assume, as in Remark 2.5.3, that F is isomorphic to the identity on the objects. Moreover, it acts as multiplication by $\mu := k^i_{i[0]}$ on the morphisms of k^2-type, which is constant by Proposition 2.5.6. Now consider the map $\phi_\mu : A \longrightarrow A$ defined as follow:

$$a + \epsilon b \longmapsto a + \epsilon\mu b.$$

The induced push forward functor $(\phi_\mu)_*$ on $\mathbf{Perf}(A)$ is isomorphic to the identity on the objects and it acts as multiplication by μ^{-1} on morphism of k^2-type. Up to isomorphisms of functors, the composition $(\phi_\mu)_* \circ F$ is the identity on the objects and acts as the identity on morphisms of k^2-type. ☐

From now on, in view of Proposition 2.5.7, we can assume that the functor F satisfies the following condition:

(C1) F is the identity on the objects of $\mathbf{Perf}(A)$ and the coefficients $k^i_{i[0]}$ of F are equal to 1.

Lemma 2.5.8. *Let $k^i_{j[\alpha]}$ be the coefficient of a functor F satisfying (C1). The following relations hold:*

(R1) $k^j_{i[\alpha]}k^i_{j[-\alpha]} = 1$ *if* $-i < \alpha \leq \min\{0, j-i\}$ *or* $\max\{0, j-i\} \leq \alpha < j$.

(R2) $k^j_{i[\alpha]} = k^j_{i-1[\alpha]}k^{i-1}_{i[0]}$ $0 \leq \alpha < j \leq i$, $(i-j, \alpha) \neq (0,0), (1,0)$.

(R3) $k^j_{i[\alpha]} = k^j_{i-1[0]}k^{i-1}_{i[\alpha]}$ $j < i-1$ *and* $-i < \alpha \leq j-i$.

(R4) $k^{i-1}_{i[\alpha]} = k^{i-1}_{i-1[\alpha]}k^{i-1}_{i[0]}$ $1-i < \alpha < 0$.

(R5) $k^{i-1}_{i[2-i]} = k^{i-1}_{i-1[1]}k^{i-1}_{i[1-i]}$ $i > 2$.

Proof. (R1) For $i = j$ and $\alpha = 0$ the statement is trivial. In the other cases note that, when the first inequality holds, $k^j_{i[\alpha]}$ is related to a morphism of k_ϵ-type and $k^i_{j[-\alpha]}$ is related to a morphism of k_1-type. Also, the composition is a non zero morphism of k_ϵ-type between X_j and X_j. When the second inequality holds, the types are swapped and the composition is still non zero. So we have:

$$k^j_{i[\alpha]}k^i_{j[-\alpha]} = k^j_{j[0]} = 1.$$

(R2) The morphisms from X_j to $X_i[\alpha]$, from X_j to $X_{i-1}[\alpha]$ and from X_{i-1} to $X_i[0]$ are of k_1-type, hence case (i) of Section 4 implies that the composition:

$$1^j_{i[\alpha]} = 1^{i-1}_{i[0]} \circ 1^j_{i-1[\alpha]}$$

is non zero.

(R3) The morphism from X_j to $X_{i-1}[0]$ is of k_1-type, whilst the morphisms from X_j to $X_i[\alpha]$ and from X_{i-1} to $X_i[\alpha]$ are both of k_ϵ-type, hence case C of Section 4 implies that the composition:

$$\epsilon^j_{i[\alpha]} = \epsilon^{i-1}_{i[\alpha]} \circ 1^j_{i-1[0]}$$

is non zero.

(R4) The morphism from X_{i-1} to $X_i[0]$ is of k_1-type. The morphisms from X_{i-1} to $X_i[\alpha]$ and from X_{i-1} to $X_{i-1}[\alpha]$ are both of k_ϵ-type, hence case (ii) of Section 4 implies that the composition:

$$\epsilon^{i-1}_{i[\alpha]} = 1^{i-1}_{i[0]} \circ \epsilon^{i-1}_{i-1[\alpha]}$$

is non zero.

(R5) The morphism from X_{i-1} to $X_{i-1}[1]$ is of k_1-type. Again, the morphisms from X_{i-1} to $X_i[1-i]$ and from X_{i-1} to $X_i[2-i]$ are both of k_ϵ-type, hence case (iii) of Section 4 implies that the composition:

$$\epsilon^{i-1}_{i[1-i]} \circ 1^{i-1}_{i-1[1]} = \epsilon^{i-1}_{i[2-i]}$$

is non zero.

□

Lemma 2.5.9. *Let F be a functor satisfying (C1). The functor F is isomorphic to a functor F' such that the coefficients $k'^{i-1}_{i[0]}$ of F', are equal to 1 for all $i > 1$.*

Proof. The isomorphism of functors between F and F' is given by the coefficients:

$$\phi_1 = 1 \quad \phi_i = \prod_{h=1}^{i-1} (k^{h-1}_{h[0]})^{-1} : \; X_i \longrightarrow X_i$$

The following diagram is commutative, and shows that $k'^{i-1}_{i[0]} = 1$ concluding the proof of the lemma.

$$\begin{array}{ccc} X_{i-1} & \xrightarrow{\prod_{h=1}^{i-2} (k^{h-1}_{h[0]})^{-1}} & X_{i-1} \\ \Big\downarrow {\scriptstyle f k^{i-1}_{i[0]}} & & \Big\downarrow {\scriptstyle f \cdot k'^{i-1}_{i[0]}} \\ X_i & \xrightarrow{\prod_{h=1}^{i-1} (k^{h-1}_{h[0]})^{-1}} & X_i . \end{array}$$

□

From now on we suppose that the functor F satisfies both (C1) and, because of Lemma 2.5.9, the condition:

(C2) The coefficients $k^{i-1}_{i[0]}$ of F are equal to 1 for all $i > 1$.

Given a set of objects $\mathscr{E} \subset \mathrm{Ob}(\mathbf{Perf}(A))$ we denote by $\mathrm{add}\,\{\mathscr{E}\}$ the smallest full subcategory of $\mathbf{Perf}(A)$ containing $\mathscr{E}$ and closed under shifts and finite direct sums.

Theorem 2.5.10. *Let F be a functor satisfying (C1) and (C2). The action of F on the morphisms is completely determined by its coefficient $k^2_{1[1]} = \lambda$. In particular:*

$$k^i_{j[\alpha]} = \lambda^\alpha \tag{2.1}$$

for $-j < \alpha \leq \min\{0, i-j\}$ *or* $\max\{0, i-j\} \leq \alpha < i$.

Proof. We proceed by induction on the number of the indecomposable objects generating the subcategory add $\{X_1, \ldots, X_i\}$. On the subcategory add $\{X_1, X_2\}$ we have:

$$k^1_{2[0]} k^2_{1[0]} \overset{(R1)}{=} 1 \text{ and } k^2_{1[1]} k^1_{2[-1]} \overset{(R1)}{=} 1,$$

$$k^2_{2[1]} \overset{(R2)}{=} k^2_{1[1]} k^1_{2[0]} \overset{(C2)}{=} \lambda,$$

$$k^2_{2[-1]} \overset{(R1)}{=} (k^2_{2[1]})^{-1} = \lambda^{-1}.$$

Note that, by Proposition 2.3.1, these equalities determine the behaviour of F on all the coefficients and prove (2.1) for the subcategory add $\{X_1, X_2\}$.
Now assume that (2.1) holds true for the subcategory add $\{X_1, \ldots, X_{i-1}\}$. We prove it for the subcategory add $\{X_1, \ldots, X_i\}$. By assumption $k^i_{i[0]} = 1$. The description of morphisms in Proposition 2.3.1, implies that the following steps cover all the remaining coefficients of the functor on add $\{X_1, \ldots, X_i\}$.

(i) $k^j_{i[0]}$ for all $j < i$ (deducing the case of $k^i_{j[0]}$ by (R1)).

(ii) $k^j_{i[\alpha]}$ for all $0 < \alpha < j$, $j < i$ (deducing the case of $k^i_{j[-\alpha]}$ by (R1)).

(iii) $k^j_{i[\alpha]}$ for all $-i < \alpha \leq j - i$, $j < i$ (deducing the case $k^i_{j[-\alpha]}$ by (R1)).

(iv) $k^i_{i[\alpha]}$ for all $0 < \alpha < i$ (deducing the case $k^i_{i[-\alpha]}$ by (R1)).

As for the proof:

(i) If $j = i - 1$ one obtains $k^{i-1}_{i[0]} = 1$ by (C2).
For $j < i - 1$, by induction $k^j_{i-1[0]} = 1$. We have:

$$k^j_{i[0]} \overset{(R2)}{=} k^j_{i-1[0]} k^{i-1}_{i[0]} \overset{(C1)}{=} 1.$$

(ii) By induction $k^j_{i-1[\alpha]} = \lambda^\alpha$. The claim is true because:

$$k^j_{i[\alpha]} \overset{(R2)}{=} k^j_{i-1[\alpha]} k^{i-1}_{i[0]} = \lambda^\alpha.$$

(iii) If $j \neq i - 1$, by induction $k^j_{i-1[0]} = 1$, then:

$$k^j_{i[\alpha]} \overset{(R3)}{=} k^j_{i-1[0]} k^{i-1}_{i[\alpha]} = k^{i-1}_{i[\alpha]}.$$

Therefore we have to prove the claim only for $k^{i-1}_{i[\alpha]}$. In this case $1-i \leq \alpha < 0$.

- If $1-i < \alpha < 0$, by induction $k^{i-1}_{i-1[\alpha]} = \lambda^\alpha$, then:

$$k^{i-1}_{i[\alpha]} \overset{(R4)}{=} k^{i-1}_{i-1[\alpha]} k^{i-1}_{i[0]} \overset{(C2)}{=} \lambda^\alpha$$

- If $\alpha = 1-i$, it is sufficient to note that $k^{i-1}_{i[\alpha+1]} = k^{i-1}_{i[2-i]}$ belongs to the previous case and by induction $k^{i-1}_{i-1[1]} = \lambda$. Hence:

$$k^{i-1}_{i[1-i]} \overset{(R5)}{=} k^{i-1}_{i[2-i]} (k^{i-1}_{i-1[1]})^{-1} = \lambda^{\alpha+1}\lambda^{-1} = \lambda^\alpha.$$

(iv) We have:

$$k^{i}_{i[\alpha]} \overset{(R2)}{=} k^{i}_{i-1[\alpha]} k^{i-1}_{i[0]} = k^{i}_{i-1[\alpha]} \overset{(iii)}{=} \lambda^\alpha.$$

□

Corollary 2.5.11. *Let F be a functor satisfying (C1) and (C2). If F is exact, then it is isomorphic to the identity functor.*

Proof. It suffices to show that, if F is exact, then $\lambda = k^2_{1[1]} = 1$. Consider the following distinguished triangle:

$$X_1 \xrightarrow{\epsilon^1_{1[0]}} X_1 \xrightarrow{i} C(\epsilon^1_{1[0]}) \xrightarrow{p} X_1[1],$$

since the cone $C(\epsilon^1_{1[0]})$ on the morphism $\epsilon^1_{1[0]}$ is isomorphic to X_2, the triangle becomes:

$$X_1 \xrightarrow{\epsilon^1_{1[0]}} X_1 \xrightarrow{1^1_{2[0]}} X_2 \xrightarrow{1^2_{1[1]}} X_1[1]. \tag{2.2}$$

Now F sends the previous triangle in to the following one:

$$X_1 \xrightarrow{\epsilon^1_{1[0]}} X_1 \xrightarrow{1^1_{2[0]}} X_2 \xrightarrow{1^2_{1[1]}\lambda} X_1[1]. \tag{2.3}$$

Since F is exact, the triangle (2.3) is distinguished, hence it is isomorphic to the distinguished triangle (2.2). So we have:

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{\epsilon^1_{1[0]}} & X_1 & \xrightarrow{1^1_{2[0]}} & X_2 & \xrightarrow{\lambda 1^2_{1[1]}} & X_1[1] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle a+\epsilon b} & & \downarrow{\scriptstyle \mathrm{id}} \\
X_1 & \xrightarrow{\epsilon^1_{1[0]}} & X_1 & \xrightarrow{1^1_{2[0]}} & X_2 & \xrightarrow{1^2_{1[1]}} & X_1[1].
\end{array}$$

The diagram is commutative up to homotopy, hence:

$$\begin{cases} a = \lambda \\ a = 1 \end{cases}$$

Thus $\lambda = 1$ and, by Theorem 2.5.10, the functor F is the identity. □

Corollary 2.5.12. *Every exact autoequivalence of* $\mathbf{Perf}(A)$ *is of Fourier-Mukai type.*

Proof. [16], Proposition 5.10 shows that the composition of Fourier-Mukai functor is again of Fourier-Mukai type. Now, F is the identity up to shifts and push forwards functors, which are both of Fourier-Mukai type. Hence F itself is a Fourier-Mukai functor. □

Corollary 2.5.13. *If* $k \neq \mathbb{Z}_2$*, then there exists an autoequivalence* $\mathbf{Perf}(A)$ *that is not exact.*

Proof. Choose the coefficient $k^1_{2[1]} \neq 0, 1$, set all the coefficients as described in Theorem 2.5.10. The functor F is well defined since all the compositions are well posed:

$$k^i_{j[\alpha]} k^{j[\alpha]}_{l[\beta]} = k^i_{j[\alpha]} k^j_{l[\beta-\alpha]} = \lambda^{\alpha+\beta-\alpha} = \lambda^\beta = k^i_{l[\beta]}.$$

By Corollary 2.5.11, F is not exact. □

2.6 Fourier-Mukai functors on $\mathbf{Perf}(A)$

In the following we give a slight different version of [22], Corollary 9.13, which extends the results in 2.5.12.

Theorem 2.6.1. *Let* Y *be a quasi-compact and separated scheme. Let:*

$$F: \mathbf{Perf}(A) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(Y)$$

be a fully faithful functor. Then there is an object $\mathcal{E} \in \mathbf{D}_{\mathrm{qcoh}}(\operatorname{Spec} A \times Y))$ *such that:*

$$\Phi_{\mathcal{E}}|_{\mathbf{Perf}(A)} \cong F.$$

Furthermore, if Y *is noetherian and* F *sends* $\mathbf{Perf}(A)$ *to* $\mathbf{D}^b(Y)$*, then*

$$\mathcal{E} \in \mathbf{D}^b(\operatorname{Spec} A \times Y).$$

Proof. We know by ([22]) that there exist enhancements of the derived categories $\mathbf{D}_{\mathrm{qcoh}}(\operatorname{Spec} A))$ and $\mathbf{D}_{\mathrm{qcoh}}(Y))$, we call them $\mathcal{D}_{dg}(\mathrm{Qcoh}(\operatorname{Spec} A))$ and $\mathcal{D}_{dg}(\mathrm{Qcoh}(Y))$ respectively. Also, by [22]

Proposition 1.17, these enhancements are quasi equivalent to the DG categories $\mathcal{SF}(Perf(A))$ and $\mathcal{SF}(Perf(Y))$. Denote by:

$$\phi_A : \mathcal{D}_{dg}(\mathrm{Qcoh}A) \longrightarrow \mathcal{SF}(Perf(A))$$

$$\phi_Y : \mathcal{D}_{dg}(\mathrm{Qcoh}Y) \longrightarrow \mathcal{SF}(Perf(Y))$$

the corresponding quasi-functors. The functor F induces an equivalence:

$$\tilde{F} : \mathbf{Perf}(A) \xrightarrow{\sim} H^0(\mathcal{C})$$

where $\mathcal{C}$ is the full DG subcategory in $\mathcal{SF}(Perf(Y))$ consisting of all objects in the essential image of $H^0(\phi_Y) \circ F$. By [22], Theorem 6.4, there is a quasi-equivalence:

$$\mathcal{F} : Perf(A) \longrightarrow \mathcal{C}$$

which induces a quasi-equivalence:

$$\mathcal{F}^* : \mathcal{SF}(Perf(A)) \longrightarrow \mathcal{SF}(\mathcal{C}) .$$

Let $\mathcal{D} \subset \mathcal{SF}(Perf(Y))$ be a DG subcategory that contains $Perf(Y)$ and $\mathcal{C}$. Denote by $\mathcal{J} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{I} : Perf(Y) \longrightarrow \mathcal{D}$ the respective embeddings. Let:

$$\mathcal{H} := \phi_Y^{-1} \circ \mathcal{I}_* \circ \mathcal{J}^* \circ \mathcal{F}^* \circ \phi_A : \mathcal{D}_{dg}(\mathrm{Qcoh}A) \longrightarrow \mathcal{D}_{dg}(\mathrm{Qcoh}Y)$$

be the functor that makes the following diagram commutative:

$$\begin{array}{ccccccc}
\mathcal{D}_{dg}(\mathrm{Qcoh}A) & & & \xrightarrow{\mathcal{H}} & & & \mathcal{D}_{dg}(\mathrm{Qcoh}Y) \\
\downarrow{\scriptstyle \phi_A} & & & & & & \downarrow{\scriptstyle \phi_Y} \\
\mathcal{SF}(Perf(A)) & \xrightarrow{\mathcal{F}^*} & \mathcal{SF}(\mathcal{C}) & \xrightarrow{\mathcal{J}^*} & \mathcal{SF}(\mathcal{D}) & \xrightarrow{\mathcal{I}_*} & \mathcal{SF}(Perf(Y))
\end{array}$$

Notice that $H^0(\mathcal{H})$ commutes with direct sums, hence ([22], Theorem 9.10) the functor $H^0(\mathcal{H})$ is isomorphic to $\Phi_{\mathcal{E}}$ with $\mathcal{E} \in \mathbf{D}(\mathrm{Qcoh}(\mathrm{Spec}\, A \times Y))$.
As observed in the proof of [22], the restriction of $\mathcal{I}_* \circ \mathcal{J}^*$ on $\mathcal{C}$ is isomorphic to the inclusion $\mathcal{C} \longrightarrow \mathcal{SF}(Perf(Y))$, hence the restriction $\Phi_{\mathcal{E}}|_{\mathbf{Perf}(A)}$ is fully faithful.

Let $\mathfrak{A}$ be the full subcategory of $\mathbf{Perf}(A)$ whose object is only A, and let $j : \mathfrak{A} \to \mathbf{Perf}(A)$ be the natural embedding.

Define:

$$G := H^0(\mathcal{F})^{-1} \circ \tilde{F} : \ \mathbf{Perf}(A) \longrightarrow \mathbf{Perf}(A)$$

By [22], Theorem 6.4, there is an isomorphism of functors:

$$j \xrightarrow{\sim} G \circ j$$

on the category $\mathfrak{A}$. Hence, by Corollary 2.5.11, the functor G is the identity on the whole $\mathbf{Perf}(A)$. Therefore, the functors $H^0(\mathcal{F})$ and $\tilde{F}$ are isomorphic, that is:

$$(H^0(\phi_Y) \circ H^0(\mathcal{H}))|_{\mathbf{Perf}(A)} \cong (H^0(\phi_Y) \circ F) \Rightarrow \Phi_{\mathcal{E}}|_{\mathbf{Perf}(A)} \cong F.$$

Finally if Y is noetherian and F sends $\mathbf{Perf}(A)$ to $\mathbf{D}^b(Y)$, then [22], Corollary 9.13, implies:

$$\mathcal{E} \in \mathbf{D}^b(\mathrm{Coh}(\mathrm{Spec}\, A \times Y)).$$

□

Corollary 2.6.2. *Let Y be a quasi-compact and separated scheme. Let:*

$$F : \ \mathbf{D}^b(\mathrm{Spec}\, A) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(Y)$$

be a fully faithful functor that commutes with homotopy colimits. Then there is an object $\mathcal{E} \in \mathbf{D}_{\mathrm{qcoh}}(X \times Y))$ such that:

$$\Phi_{\mathcal{E}}|_{\mathbf{D}^b(A)} \cong F.$$

Proof. Corollary 9.14 in [22] shows a similar result: if X is a projective scheme such that $T_0(\mathcal{O}_X) = 0$ and Y is a quasi-compact separated scheme, then for every fully faithful functor that commutes with homotopy colimits:

$$F : \mathbf{D}^b(X) \longrightarrow \mathbf{D}_{\mathrm{qcoh}}(Y)$$

there is an object $\mathcal{E} \in \mathbf{D}_{\mathrm{qcoh}}(X \times Y))$ such that:

$$\Phi_{\mathcal{E}}|_{\mathbf{D}^b(X)} \cong F.$$

The authors assume $T_0(\mathcal{O}_X) = 0$ in order to prove that the restriction of the functor F to the subcategory of perfect complexes $\mathbf{Perf}(X)$ is of Fourier-Mukai type. In our case we have actually $T_0(\mathcal{O}_A) \neq 0$, but we have already shown in Theorem 2.6.1 that the restriction of F to $\mathbf{Perf}(A)$ is a Fourier-Mukai functor. Hence we do not need this hypothesis and the proof follows as in Corollary 9.14, [22].

□

2.7 t-structures on $\mathbf{D}^b(A)$

This section is devoted to the study of all the possible t-structures on $\mathbf{D}^b(A)$.

The first case we are interested in is the case of $\mathbf{Perf}(A)$. As we mentioned in previous chapter, all the t-structures on $\mathbf{Perf}(A)$ are trivial. Infact, similar arguments of those in Proposition 2.7.2 allow us to consider only the subcategory $\operatorname{add}\{X_1[h]\}$ as heart, and it is easy to verify that such a subcategory can not satisfy the property (2) of Proposition 1.6.2.

Let us turn to analyze the case of the category $\mathbf{D}^b(A)$ instead.

Remark 2.7.1. The subcategory $\mathcal{A} = \{X \in \mathbf{D}^b(A)$ s.t. $H^i(X) = 0$ for every $i \geq 0\}$ is the standard t-structure on $\mathbf{D}^b(A)$. Its heart is the subcategory $\operatorname{add}\{X_1, X_\infty\}$.

Proposition 2.7.2. *Up to shift, the unique t-structure on $\mathbf{D}^b(A)$ is the standard one.*

Proof. We look for all possible hearts satisfying the two properties of Proposition 1.6.2. Since the heart $\mathcal{A}$ is abelian, it is sufficient to check which indecomposable objects does $\mathcal{A}$ contain. Thanks to the first part of Proposition 1.6.2 it is easy to verify that, up to shifts, the only admissible candidates for hearts are $\mathcal{A} = \operatorname{add}\{X_1\}$, $\mathcal{A} = \operatorname{add}\{X_\infty\}$ and $\mathcal{A} = \operatorname{add}\{X_1, X_\infty\}$. The first case is not possible, since X_1 does not generate the whole category $\mathbf{D}^b(A)$. The distinguished triangle:

$$X_\infty \xrightarrow{\epsilon} X_1 \xrightarrow{1} X_\infty$$

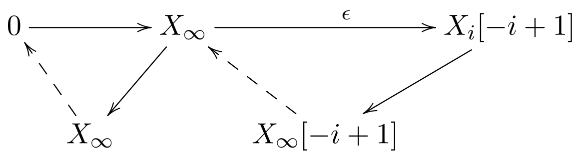
is an extension of X_1 by elements of $\operatorname{add}\{X_\infty\}$, and so if X_∞ is an element of $\mathcal{A}$, then X_1 is such. It follows that the unique possibility is $\mathcal{A} = \operatorname{add}\{X_1, X_\infty\}$. □

It could be interesting to look at the explicit construction of the filtration for the objects of $\mathbf{D}^b(A)$. The first step is to write the filtration of the indecomposable objects of $\mathbf{D}^b(A)$.

The filtration of X_1, X_∞ and all the other elements of the heart is provided by the distinguished triangle $0 \to \square \to \square$.
As for other indecomposables, by taking the cone one has the following exact triangle, for $1 < i < \infty$:

$$X_\infty \xrightarrow{\epsilon} X_i[-i+1] \to X_\infty[-i+1].$$

The filtration of the indecomposable object $X_i[-i+1]$ is the following:



By Remark 1.6.3, the filtration of other indecomposable objects can be obtained by shifting these ones above. Moreover, the filtration of every object X of $\mathbf{D}^b(A)$ can be constructed by taking direct sums of the filtration of indecomposable objects that generate X.

2.8 Stability conditions on $\mathbf{D}^b(A)$

In this section we will describe the space $\mathrm{Stab}(\mathbf{D}^b(A))$ of stability conditions on $\mathbf{D}^b(A)$. Thanks to the results of previous section, we know that all the t-structures on $\mathbf{D}^b(A)$ are given by shifts of the standard one. In particular all the possible hearts are $\mathcal{A}_h = \mathrm{add}\,\{X_1[h], X_\infty[h]\}$.

The exact sequence:

$$0 \to X_\infty \xrightarrow{\epsilon} X_1 \xrightarrow{1} X_\infty \to 0$$

gives a relation in the Grothendieck group $[X_1[h]] = 2\,[X_\infty[h]]$. It follows that the Grothendieck group is the free abelian group generated by $X_\infty[h]$.
In order to give the stability function, it suffices then to choose a vector v in H as the image of $X_\infty[h]$.

All objects of the hearts $\mathcal{A}_h$ are semi-stable.

Let $\mathcal{T}$ be a triangulated category. $\mathrm{Stab}(\mathcal{T})$ denotes the set of stability conditions which are locally finite. Note that, if $K(\mathcal{T})$ is discrete, as in the case we are dealing with, all the stability conditions are locally finite. Bridgeland proved that this space has a natural topology defined by a generalized metric. $\mathrm{Stab}(\mathcal{T})$ endowed with this topology, turns out to be a complex manifold. If the Grothendieck group is finitely generated, as in our case, this manifold is of finite dimension.

Proposition 2.8.1. *A stability condition on $\mathbf{D}^b(A)$ is given by an integer h and a vector $v \in H$.*

Proof. By Proposition 5.3 of [5], it is sufficient to provide a heart of a bounded t-structure and a stability function with the Harder-Narashiman property, but this property is already assured since the heart is artinian and noetherian. The integer h specifies the heart $\mathcal{A}_h$ as described above, and v describes the stability function $Z(X_\infty[h]) = v$. The Grothendieck group $K(\mathcal{A}_h)$ is, for every h, isomorphic to the Grothendieck group of the whole category $\mathbf{D}^b(A)$ (see [31] for details). The data (h, v) correspond to the stability condition $\sigma = (Z, \mathscr{P})$ where the group homomorphism Z is given by the stability function as observed above. Let ϕ be the phase of v; $\mathscr{P}$ is given by:

$$\mathscr{P}(\phi) = \operatorname{add}\{X_1[h], X_\infty[h]\}$$

and it is zero for all the other $\phi \in (0, 1]$. These data extend to all $\phi \in \mathbb{R}$. □

There are two group actions on the space $\operatorname{Stab}(\mathcal{T})$ (See [5], Lemma 8.2): a right action of $\widetilde{\mathrm{GL}^+}(2, \mathbb{R})$ and a left action by isometries of the group of auto-equivalences of the category $\mathscr{D}$.

Remark 2.8.2. The elements of $\widetilde{\mathrm{GL}^+}(2, \mathbb{R})$ (the universal covering of $\mathrm{GL}^+(2, \mathbb{R})$) are pairs (G, f) where $G \in \mathrm{GL}^+(2, \mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ such that:

- f is an increasing map with $f(x + 1) = f(x) + 1$ for every $x \in R$.
- $\frac{G \exp i\pi\phi}{|G \exp i\pi\phi|} = \exp i\pi f(\phi)$.

Let S be the subgroup of $\widetilde{\mathrm{GL}^+}(2, \mathbb{R})$ generated by rotations $(\exp i\pi\theta, f(x) = x + \theta)$, $\theta \in \mathbb{R}$ and scalings $(k, f(x) = x)$, $k \in \mathbb{R}^+$. We have:

$$S = \{(k \exp i\pi\theta, \quad f(x) = x + \theta) \quad \forall \theta \in \mathbb{R}, \quad k \in \mathbb{R}^+\}.$$

The action on $\operatorname{Stab}(\mathcal{T})$ is given by:

$$(G, f) \circlearrowleft (Z, \mathscr{P}(\phi)) = (G^{-1} \circ Z, \mathscr{P}(f(\phi)).$$

Lemma 2.8.3. *The action of S on $\operatorname{Stab}(\mathbf{D}^b(A))$ is free and transitive, hence $\operatorname{Stab}(\mathbf{D}^b(A)) \cong S$.*

Proof. Let $(Z, \mathscr{P})$ be the given stability condition, as in Proposition 2.8.1, by the pair (h, v). Note that $v = |v| \exp i\pi\phi_v$. $\mathscr{P}(\phi_v) = \operatorname{add}\{X_1[h], X_\infty[h]\}$ thus $\mathscr{P}(\phi_v - h) = \operatorname{add}\{X_1, X_\infty\}$. Let $\theta = -h - \phi_v$ and $k = (|v|)^{-1}$. By the element $(k \exp i\pi\theta, f(x) = x + \theta)$ of S one can see that $(Z, \mathscr{P})$ belongs to the same orbit of the stability condition $(0, -1)$. Then the action is transitive. Moreover it is straightforward to verify that the action is also free. □

Theorem 2.8.4. $\operatorname{Stab}(\mathbf{D}^b(A))$ *is isomorphic to* $\mathbb{C}$.

Proof. By Lemma 2.8.3, it is sufficient to verify the claim for the subgroup S. But S is the universal covering of the subgroup of $\mathrm{GL}^+(2, \mathbb{R})$ given by $\{k \exp(i\pi\theta), k \in \mathbb{R}^+, \theta \in \mathbb{R}\}$, which is isomorphic to $\mathbb{C}^*$. □

In [4] and [27], Bondarko and Pauksztello introduced the notion of co-t-structure. The definition is similar to Definition 1.6.1.

Definition 2.8.5. *Let $\mathcal{T}$ be a triangulated category. A co-t-structure on $\mathcal{T}$ is given by a full additive subcategory $\mathscr{F}$ such that:*

- $\mathscr{F}[-1] \subset \mathscr{F}$
- *For all objects E in $\mathcal{T}$, there exists a distinguished triangle:*

$$F \to E \to G$$

  where $F \in \mathscr{F}$ and $G \in \mathscr{F}^{\perp}$.

The co-heart of a co-t-structure is the subcategory $\mathcal{A} := \mathscr{F} \cap \mathscr{F}^{\perp}[-1]$.

There are important differences between t-structures and co-t-structures. One example is provided by the properties (1) and (2) of Remark 1.6.3: there are examples of co-hearts of a co-t-structures that are not abelian and in general the filtration (2) is not unique. Proposition 1.3.3 of [4] makes clear that the proof of Proposition 2.7.2 still works in the context of co-t-structures. The notion of co-stability conditions is also rather similar to the one of stability condition given in Definition 1.6.7. Note that the inequalities involving the shifts in the Harder-Narasimhan filtration are reversed:

- Any non zero object E admits a Harder-Narasimhan filtration, that is a finite number of inclusion:

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_{n-1} \hookrightarrow E_n = E$$

  such that $F_j = \mathrm{Cone}(E_{j-1} \hookrightarrow E_j)$ are semistable object with phase:

$$\phi(F_1) < \cdots < \phi(F_{n-1}) < \phi(F_n)$$

The space co-Stab$(\mathbf{D}^b(A))$ consisting of all the co-stability condition on a triangulated category T is a topological manifold. By following [18] and following the proof of Theorem 2.8.4 one obtains:

Proposition 2.8.6. co-Stab$(\mathbf{D}^b(A))$ *is isomorphic to* $\mathbb{C}$.

Chapter 3

Equivariant Sheaves

3.1 Motivations

The aim of this chapter is the study of the relationships between the category $\mathbf{D}^b_G(X)$, where X is a smooth projective variety over which acts a finite group G, and the category $\mathbf{Perf}(X/G)$ where X/G is the geomtrical quotient of X by G. The inspiration of this chapter is [19], where Kawamata proved Orlov's representability theorem also for derived categories of smooth stacks. More precisely the question is: *is it possible to obtain an autoequivalence* $\mathbf{Perf}(X/G) \longrightarrow \mathbf{Perf}(X/G)$ *starting from an autoequivalence* $\mathbf{D}^b_G(X) \longrightarrow \mathbf{D}^b_G(X)$?

Recall that we have a natural equivariant direct image functor:

$$\pi^G_* : \mathrm{Coh}^G(X) \longrightarrow \mathrm{Coh}(X/G)$$

defined as:

$$\pi^G_*(\mathcal{F}) := (\pi_*\mathcal{F})^G.$$

where $\pi : X \longrightarrow X/G$ and $\mathcal{F} \in \mathrm{Coh}^G(X)$.

Since π^G_* is an exact functor, it extends naturally to an exact functor of triangulated categories:

$$\Pi^G_* : \mathbf{D}^b_G(X) \longrightarrow \mathbf{D}^b(X/G).$$

Also, we have a left derived functor:

$$\mathbf{L}\pi^* := \mathbf{L}\pi^*_{|\mathbf{Perf}(X/G)} : \mathbf{Perf}(X/G) \longrightarrow \mathbf{D}^b_G(X).$$

where π^* is the usual pull-back functor:

$$\pi^* : \mathrm{Coh}(X/G) \longrightarrow \mathrm{Coh}^G(X).$$

Remark 3.1.1. If we restrict Π_*^G to the essential image of $\mathbf{L}\pi^*$, then $(\mathbf{L}\pi^*, \Pi_*^G)$ form an adjoint pair and $\Pi_*^G \circ \mathbf{L}\pi^*$ is isomorphic to the identity functor on $\mathbf{Perf}(X/G)$. The functor $\mathbf{L}\pi^*$ defines an equivalence between its essential image in $\mathbf{D}_G^b(X)$ and $\mathbf{Perf}(X/G)$.

Remark 3.1.2. Every autoequivalence of the equivariant derived category $\mathbf{D}_G^b(X)$ is of Fourier-Mukai type. That is, if $\Omega \in \mathrm{Aut}(\mathbf{D}_G^b(X))$, then there exists $\mathcal{P} \in \mathbf{D}_{G^2}^b(X \times X)$ such that $\Omega \sim \Phi_{\mathcal{P}}$. This is because the results proven by Kawamata in [19], and the fact that, if $[X/G]$ is the smooth stack associated to the normal projective variety X with quotient singularities, then $\mathrm{Coh}[X/G] \cong \mathrm{Coh}^G(X)$.

Thus if $\phi_{\mathcal{P}} : \mathbf{D}_G^b(X) \longrightarrow \mathbf{D}_G^b(X)$, is an autoequivalence, the more natural idea in order to get an autoequivalence of $\mathbf{Perf}(X/G)$ is to define the functor:

$$\Omega := \Pi_*^G \circ \Phi_{\mathcal{P}} \circ \mathbf{L}\pi^*.$$

For example, if we take $\mathcal{P} = \Delta_*(\mathcal{O}_X[d]) = \mathcal{O}_\Delta[d])$, $d \in \mathbb{Z}$, where $\mathcal{O}_\Delta[d]$ is the structure sheaf of the diagonal $\Delta(X) \subset X \times X$ and $\Delta : X \longrightarrow X \times X$, then the functor Ω constructed as above is just a shift and consequently it defines an autoequivalence of the category of perfect complexes on the singular quotient. Unfortunately this is not true in general, because it may happen that Ω does not takes value in $\mathbf{Perf}(X/G)$ and thus it cannot gives us a desired autoequivalence. We shall give an example of this situation:

Remark 3.1.3. Consider the projective plane $\mathbb{P}^2(\mathbb{C}) =: X$. There is an action by the finite group $G := \mathbb{Z}_2$ generated by the automorphism:

$$[z_0 : z_1 : z_2] \longrightarrow [z_0 : z_1 : -z_2].$$

Clearly this action is not free and the geometrical quotient $\mathbb{P}^2/\mathbb{Z}_2$ is the weighted projective plane $\mathbb{P}^2(1,1,2)$ that is isomorphic to the quadratic cone in $\mathbb{P}^3$ defined by:

$$\{[w_0 : w_1 : w_2 : w_3] \in \mathbb{P}^3 \text{ such that } w_0 w_1 = w_1^2\}$$

which is a surface with a unique singular point. The isomorphism is given by the map: $w_0 = z_0^2$, $w_1 = z_0 z_1$, $w_2 = z_2^2$, $w_3 = z_2^2$.

The fixed locus for the action of G is given by the set:

$$\{[z_0 : z_1 : z_2] \in \mathbb{P}^2 \text{ such that } z_2 = 0\} \quad \bigcup \quad \{[0:0:1]\}.$$

The G-equivariant derived category $\mathbf{D}^b_{\mathbb{Z}_2}(\mathbb{P}^2)$ has a nice description: it may be seen as the category of all bounded complexes of $\mathbb{Z}_2$-$k[x,y]$-free modules of finite type with generators having bounded degree, up to homotopy equivalence. See [33] for further details.
Consider the inclusion map $\Delta : X \longmapsto X \times X$. Let $\mathcal{O}(1)$ be the twisting sheaf of Serre on $\mathbb{P}^2$ and define $\mathcal{P} := \Delta_*(\mathcal{O}(1))$. The Fourier-Mukai functor associated to $\mathcal{P}$ is an autoequivalence of $\mathbf{D}^b_{\mathbb{Z}_2}(\mathbb{P}^2)$ and in particular:

$$\Phi_{\mathcal{P}}(\mathcal{E}^\bullet) = \mathcal{E}^\bullet \otimes \mathcal{O}(1)$$

for every $\mathcal{E}^\bullet \in \mathbf{D}^b_{\mathbb{Z}_2}(\mathbb{P}^2)$.
Then, it is easy to see that the functor $\Omega := \Pi^G_* \circ \Phi_{\mathcal{P}} \circ \mathbf{L}\pi^*$ sends the structure sheaf $\mathcal{O}_{X/G}$ into the twisting sheaf $\mathcal{O}_{X/G}(1)$ which is not a perfect object. Indeed it was shown in [9] that, on $\mathbb{P}(1,1,2)$, the sheaves $\mathcal{O}_{X/G}(d)$ are perfect if and only if d is an even number. Hence Ω is not an autoequivalence of the category of perfect complexes on the weighted projective plane.

Remark 3.1.4. The example above shows that if we choose $\mathcal{P} = \Delta_*(\mathcal{O}_X(d))$, with d being an even integer, then the functor Ω takes value into $\mathbf{Perf}(X/G)$. Infact, if $\mathcal{E}^\bullet \in \mathbf{Perf}(X/G)$, then:

$$\Omega(\mathcal{E}^\bullet) = \Pi^G_*(\mathcal{O}_X(d) \otimes \mathbf{L}\pi^*(\mathcal{E}^\bullet)) = \Pi^G_*(\mathbf{L}\pi^*(\mathcal{O}_{X/G}(d) \otimes \mathbf{L}\pi^*(\mathcal{E}^\bullet)) = \mathcal{O}_{X/G}(d) \otimes \mathcal{E}^\bullet.$$

Hence, in this case, Ω defines an autoequivalence of $\mathbf{Perf}(X/G)$.

We see therefore that, in order to reach our goal, first of all we have to give some 'descend' criteria for a G-locally free sheaf on X to descend to a locally free sheaf on the quotient X/G, and, consequently, for a complex of G-equivariant sheaves to descend to the category of perfect objects on X/G. Similar criteria were given in [33] in the case of a smooth quotient, and in [25] in a more general situation; we adapted those criteria to our case and proved that $\mathcal{F}^\bullet \in \mathbf{D}^b_G(X)$ descends to $\mathbf{Perf}(X/G)$ if and only if the the stabilizer G_x acts trivially on the $\mathcal{O}_X$-modules $\mathrm{H}^j(\mathcal{F}^\bullet \otimes^{\mathbf{L}} k(x))$ for every point $x \in X$.

3.2 Descent criteria

Definition 3.2.1. *Let X be a smooth projective variety. We say that a G-equivariant locally free sheaf of finite type $\mathcal{F} \in \mathrm{Coh}^G(X)$* descends *to $\mathrm{Coh}(X/G)$ if there exists a locally free sheaf $\mathcal{V} \in \mathrm{Coh}(X/G)$ such that $\pi^*(\mathcal{V})$ is isomorphic to $\mathcal{F}$.*

Definition 3.2.2. *Let X be a smooth projective variety. A G-equivariant complex of sheaves $\mathcal{F}^\bullet \in \mathbf{D}^b_G(X)$* descends *to $\mathbf{Perf}(X/G)$ if there exists a complex $\mathcal{V}^\bullet \in \mathbf{Perf}(X/G)$ such that $\mathbf{L}\pi^*(\mathcal{V}^\bullet)$ is quasi-isomorphic to $\mathcal{F}^\bullet$.*

Remark 3.2.3. Notice that a G-equivariant locally free sheaf of finite type $\mathcal{F}$ descends if and only if the canonical map:

$$\pi^*(\pi_*^G(\mathcal{F})) \longrightarrow \mathcal{F}$$

is an isomorphism.
Similarly, a G-equivariant complex of sheaves $\mathcal{F}^\bullet$ descends if and only if the canonical map:

$$\mathbf{L}\pi^*(\Pi_*^G(\mathcal{F})) \longrightarrow \mathcal{F}$$

is a quasi-isomorphism.

The following result gives us a starting point to investigate when a G-equivariant vector bundle descends.

Theorem 3.2.4. *Let $\mathcal{E}$ be a G-equivariant vector bundle on a smooth projective variety X. Then there exists a vector bundle $\mathcal{B}$ on X/G such that $\pi^*(\mathcal{B}) = \mathcal{E}$ if and only if for every $x \in X$ the stabilizer G_x acts trivially on the fiber $\mathcal{E}_x$ for every $x \in X$.*

Proof. See [10], Theorem 2.3. □

Notice that if the action of G is free, then every G-bundle descends to the quotient because all stabilizers are trivial. Also, there is an equivalence of categories:

$$\mathbf{D}^b_G(X) \xrightarrow{\sim} \mathbf{D}^b(X/G).$$

In our situation G does not act freely. However, in case we are dealing with invertible sheaves, things are much easier:

Lemma 3.2.5. *Let $\mathcal{L}$ be a G-line bundle on X. If n is a multiple of the order of G, then $\mathcal{L}^{\otimes n}$ descends to the quotient.*

Proof. The stabilizer is finite and its action on the fibers is represented by a one-dimensional homomorphism whose values must be roots of unity, hence after taking tensor powers for any multiple of the order of the group the action of G_x on fibers of $\mathcal{L}$ becomes trivial. □

Lemma 3.2.6. *If X is a smooth projective variety of dimension n and G is a finite group acting on X, then the category $\mathrm{Coh}^G(X)$ has homological dimension smaller than n. That is, for every pair of G-equivariant coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for every $i > n$.*

Proof. See [33], Lemma 1.1. □

Lemma 3.2.7. *Let X be a smooth projective variety. Let G be a finite group acting on X. Then every G-equivariant coherent sheaf admits a finite resolution of G-equivariant locally free sheaves of finite type.*

Proof. See [33], Lemma 1.2. □

Let $\mathbf{D}^b_G(X)$ be the bounded derived category of $\mathrm{Coh}^G(X)$. The results mentioned above lead to the following:

Corollary 3.2.8. *If $\mathcal{F}^\bullet \in \mathbf{D}^b_G(X)$ is a bounded complex of G-equivariant sheaves, then there exists a complex $\mathcal{G}^\bullet \in \mathbf{D}^b_G(X)$ such that $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{G}^\bullet$ and G^i is a G-equivariant locally free sheaf of finite type for every integer i.*

We now give a descent criterion at the level of derived categories:

Proposition 3.2.9. *Let $\mathcal{F}^\bullet \in \mathbf{D}^b_G(X)$. Then there exists a complex $\mathcal{B}^\bullet \in \mathbf{Perf}(X/G)$ such that $\mathbf{L}\pi^*(\mathcal{B}^\bullet) = \mathcal{F}^\bullet$ if and only if there exists a finite locally free G-resolution $\mathcal{E}^\bullet$ of $\mathcal{F}^\bullet$ such that for every $x \in X$ the stabilizer G_x acts trivially on the fiber $\mathcal{E}^i_x$ for every integer i.*

Proof. Suppose there exists a finite locally free G-resolution $\mathcal{E}^\bullet$ of $\mathcal{F}^\bullet$ such that for every $x \in X$ the stabilizer G_x acts trivially on the fiber $\mathcal{E}^i_x$ for every integer i. The functor π^G_* is exact and, since every $\mathcal{E}^i$ descends to a vector bundle on the quotient (see Theorem 3.2.4), it follows immediately that $\Pi^G_*(\mathcal{E}^\bullet)$ is a complex of vector bundles on X/G.
Viceversa, suppose there exists a complex $\mathcal{B}^\bullet \in \mathbf{Perf}(X/G)$ such that $\mathbf{L}\pi^*(\mathcal{B}^\bullet) = \mathcal{F}^\bullet$. Take a finite locally free G-resolution $\mathcal{E}^\bullet$ of $\mathcal{F}^\bullet$. The identity $\Pi^G_* \circ \mathbf{L}\pi^* = \mathrm{Id}$ implies that $\mathcal{B}^\bullet = \Pi^G_*(\mathcal{F}^\bullet)$. Take a resolution of $\mathcal{B}^\bullet$ by some vector bundles on X/G and denote it by $\mathcal{V}^\bullet = \{0 \longrightarrow V_1 \longrightarrow \ldots \longrightarrow V_n \longrightarrow 0\}$. We have the following quasi-isomorphisms:

$$\mathcal{F}^\bullet = \mathbf{L}\pi^*(\mathcal{B}^\bullet) = \mathbf{L}\pi^*(\mathcal{V}^\bullet) = \pi^*(\mathcal{V}^\bullet) = \{0 \longrightarrow \pi^*(V_1) \longrightarrow \ldots \longrightarrow \pi^*(V_n) \longrightarrow 0\}.$$

Notice that π^* preserves locally freeness hence $\pi^*(\mathcal{V}^\bullet)$ is a complex of vector bundles which is quasi-isomorphic to $\mathcal{F}^\bullet$. Furthermore G_x acts trivially on the fibers of $\pi^*(V_i)$ for every $i = 1, \ldots, n$ since $\pi^G_*(\pi^*(V_i)) = V_i$ is a vector bundle on X/G. □

In the future we will make strong use of the following result:

Lemma 3.2.10. *Let X be affine. For every G-equivariant coherent sheaf $\mathcal{F}$ on X, such that the action of the stabilizer G_x on the vector space $\mathcal{F} \otimes k(x)$ is trivial for every $x \in X$, there exists a G-equivariant locally free sheaf $\mathcal{E}$ on X with a surjective map $\mathcal{E} \longrightarrow \mathcal{F}$ such that $\mathcal{E}$ descends to the quotient.*

Proof. See [25], Lemma 2.14. □

Remark 3.2.11. Let V_i be $\mathbb{C}$-vector spaces for $i = 1, \dots, 3$ and let H be a finite group acting on V_i for all i. Suppose there is an exact sequence:

$$V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3$$

where β and α are H-maps of vector spaces. Also, suppose the action of H is trivial on V_1 and on V_3. Then the action of H on V_2 is trivial as well.
Infact, take $x \in V_2$. If $\beta(x) = 0$ then there exists $y \in V_1$ such that $\alpha(y) = x$. But then $hx = h\alpha(y) = \alpha(hy) = \alpha(y) = x$ for every $h \in H$.

Suppose now $\beta(x) = z \neq 0$ and take $h \in H$. Notice that there exists an integer n such that $h^n = \mathrm{Id}$ because the group H is finite. Now, $\beta(hx) = h\beta(x) = hz = z$, therefore $\beta(hx - x) = 0$ and then there exists $y \in V_1$ such that $x = hx + \alpha(y)$. Multiplicating by h we get $hx = h^2x + \alpha(y)$ and then $x = h^2x + 2\alpha(y)$. Iterating the same process we eventually obtain $x = h^n x + n\alpha(y) = x + n\alpha(y)$, which implies $\alpha(y) = 0$, that is $hx = x$.

The following result is an adaptation to our case of the replacement tool found in [25], Proposition 4.1.

Lemma 3.2.12. *Let X be a smooth projective variety. Let $\mathcal{F}^\bullet \in \mathbf{D}^b_G(X)$. Suppose $\mathcal{E}^\bullet$ is a G-equivariant resolution of $\mathcal{F}^\bullet$ such that:*

$$\mathcal{E}^\bullet : \{0 \longrightarrow \mathcal{E}^1 \xrightarrow{\alpha_1} \mathcal{E}^2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} \mathcal{E}^{n-1} \xrightarrow{\alpha_{n-1}} \mathcal{E}^n \longrightarrow 0\}$$

and the action of the stabilizer G_x is trivial on the $\mathcal{O}_X$-modules $\mathrm{H}^n(\mathcal{E}^\bullet \otimes k(x))$ for every point $x \in X$. Then there is a complex:

$$\mathcal{V}^\bullet := \{0 \longrightarrow \mathcal{V}^{-m} \longrightarrow \mathcal{V}^{-m+1} \longrightarrow \dots \longrightarrow \mathcal{V}^0 \longrightarrow \mathcal{V}^1 \longrightarrow \dots \longrightarrow \mathcal{V}^n \longrightarrow 0\}$$

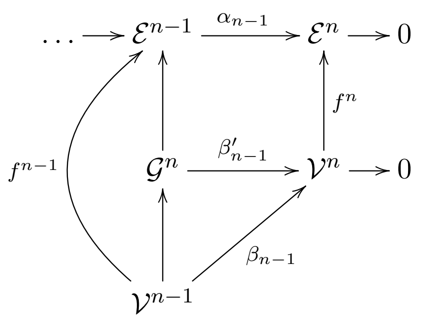
of G-equivariant locally free sheaves such that $\mathcal{V}^\bullet$ is quasi-isomorphic to $\mathcal{F}^\bullet$ and $\mathcal{V}^n$ descends to the quotient. In particular $\mathrm{H}^j(\mathcal{V}^\bullet) = 0$ for $j = -m, \dots, 0$.

Proof. The existence of a complex $\mathcal{V}^\bullet$ of G-equivariant locally free sheaves which is quasi-isomorphic to $\mathcal{F}^\bullet$ is trivial. We only need to prove that $\mathcal{V}^n$ descends but Remark 3.2.3 implies immediately that the descending properties must be checked locally, hence we can suppose that our variety X is affine.

We have a surjection $\pi^n : \mathcal{E}^n \longrightarrow \mathrm{coker}(\alpha_{n-1})$, which induces another surjection:

$$\mathcal{E}^n \otimes k(x) \longrightarrow \mathrm{coker}(\alpha_{n-1}) \otimes k(x)$$

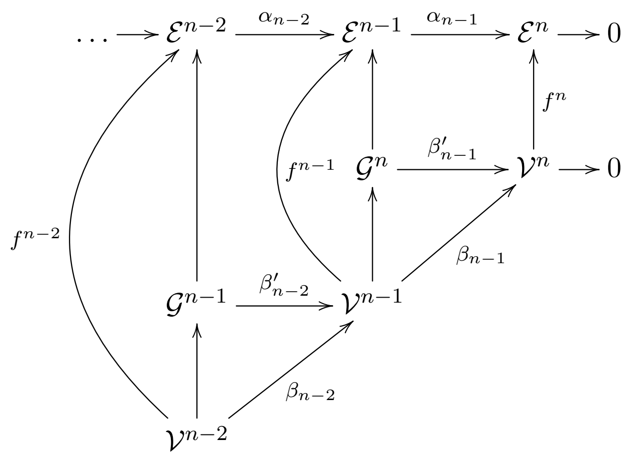
for all $x \in X$.
We know that $\mathrm{H}^n(\mathcal{F}^\bullet \otimes^{\mathbf{L}} k(x)) = \mathrm{H}^n(\mathcal{E}^\bullet \otimes k(x)) = \mathrm{coker}(\alpha_{n-1}) \otimes k(x)$ is G_x-invariant, then Lemma 3.2.10 implies that there is a G-equivariant locally free sheaf $\mathcal{V}^n$ with a surjective morphism $\gamma^n : \mathcal{V}^n \longrightarrow \mathrm{coker}(\alpha_{n-1})$ and a morphism $f^n : \mathcal{V}^n \longrightarrow \mathcal{E}^n$ such that $\pi^n \circ f^n = \gamma^n$.
We now consider the subsheaf $\mathcal{G}^n \subset \mathcal{E}^{n-1} \oplus \mathcal{V}^n$ made by those sections (e, v) such that $\alpha_{n-1}(e) = f^n(v)$. We have a natural morphism $\beta'_{n-1} : \mathcal{G}^n \longrightarrow \mathcal{V}^n$. Now take a G-equivariant locally free sheaf $\mathcal{V}^{n-1}$ which surjects onto $\mathcal{G}^n$. Then projecting on to the left, we obtain a map $f^{n-1} : \mathcal{V}^{n-1} \longrightarrow \mathcal{E}^{n-1}$, and composing with β'_{n-1} we obtain a morphism $\beta_{n-1} : \mathcal{V}^{n-1} \longrightarrow \mathcal{V}^n$.



Notice that the map: $\mathcal{V}^n/\mathrm{im}(\beta_{n-1}) \longrightarrow \mathrm{coker}(\alpha_{n-1})$ induced by f^n is an isomorphism. Infact, γ^n is surjective and $\ker(\gamma^n) = \mathrm{im}(\beta_{n-1})$: if a section v of V^n is such that $\gamma^n(v) = 0$ then there exist a section e of $\mathcal{E}^{n-1}$ such that $\alpha_{n-1}(e) = f^n(v)$, and thus $\beta'_{n-1}(e, v) = v$; now it is sufficient to take a section v' of $\mathcal{V}^{n-1}$ such that $\beta_{n-1}(v') = e$. Furthermore, if $v = \beta_{n-1}(v') = \beta_{n-1}(e, v)$ then $\gamma^n(v) = \pi^n(\alpha_{n-1}(e)) = 0$.

Notice that the map: $\gamma^{n-1} : \ker(\beta_{n-1}) \longrightarrow \ker(\alpha_{n-1})/\mathrm{im}(\alpha_{n-2})$ is surjective. If e' is a section of $\ker(\alpha_{n-1})/\mathrm{im}(\alpha_{n-2})$ then first we lift it to a section e of $\ker(\alpha_{n-1})$, then $(e, 0)$ is a section of $\ker(\beta'_{n-1})$ which maps to e, and thus it suffices to lift $(e, 0)$ to a section of $\ker(\beta_{n-1})$.

We now iterate the process described above. Consider the subsheaf $\mathcal{G}^{n-1} \subset \mathcal{E}^{n-2} \oplus \mathcal{V}^{n-1}$ made by those sections (e, v) such that $\alpha_{n-2}(e) = f^{n-1}(v)$. Again we have a natural morphism $\beta'_{n-2} : \mathcal{G}^{n-1} \longrightarrow \mathcal{V}^{n-1}$. Now take a G-equivariant locally free sheaf $\mathcal{V}^{n-2}$ which surjects onto $\mathcal{G}^{n-1}$. Then we obtain a map $f^{n-2} : \mathcal{V}^{n-2} \longrightarrow \mathcal{E}^{n-2}$, and composing with β'_{n-2} we obtain a morphism $\beta_{n-2} : \mathcal{V}^{n-2} \longrightarrow \mathcal{V}^{n-1}$. We just need to prove now that the kernel of γ^{n-1} corresponds to the image of β_{n-2}. If a section v of V^{n-1} is such that $\gamma^{n-1}(v) = 0$ then there exists a section e of $\mathcal{E}^{n-2}$ such that $\alpha_{n-2}(e) = f^{n-1}(v)$, and thus $\beta'_{n-2}(e, v) = v$; now it is sufficient to take a section v' of $\mathcal{V}^{n-2}$ such that $\beta_{n-2}(v') = e$. Furthermore, if $v = \beta_{n-2}(v') = \beta'_{n-2}(e, v)$ then $\gamma^{n-1}(v) = \pi^{n-1}(\alpha_{n-2}(e)) = 0$.



It is clear then that we can iterate this construction until we get a complex of G-equivariant vector bundles:

$$\mathcal{V}^\bullet : \{0 \longrightarrow \mathcal{V}^1 \xrightarrow{\beta_1} \mathcal{V}^2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-2}} \mathcal{V}^{n-1} \xrightarrow{\beta_{n-1}} \mathcal{V}^n \longrightarrow 0\}$$

such that $\gamma^j : \mathrm{H}^j(\mathcal{V}^\bullet) \xrightarrow{\sim} \mathrm{H}^j(\mathcal{E}^\bullet)$ is an isomorphism for $2 \leq j \leq n$ and γ^1 is surjective. Furthermore $\mathcal{V}^n$ descends to the quotient. Thus, if we consider the sheaf $\ker(\gamma^1) \overset{i}{\hookrightarrow} \mathcal{V}^1$ and a resolution:

$$0 \longrightarrow \mathcal{W}^{-m} \longrightarrow \mathcal{W}^{-m+1} \longrightarrow \dots \longrightarrow \mathcal{W}^{-1} \longrightarrow \mathcal{W}^0 \xrightarrow{\omega} \ker(\gamma^1) \longrightarrow 0$$

then the following complex:

$$0 \longrightarrow \mathcal{W}^{-m} \longrightarrow \dots \longrightarrow \mathcal{W}^0 \xrightarrow{i\circ\omega} \mathcal{V}^1 \xrightarrow{\beta_1} \mathcal{V}^2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-2}} \mathcal{V}^{n-1} \xrightarrow{\beta_{n-1}} \mathcal{V}^n \longrightarrow 0$$

is a complex of G-equivariant vector bundles quasi-isomorphic to $\mathcal{E}^\bullet$ such that $\mathcal{V}^n$ descends to the quotient. □

We are now ready to give the general descent criterion:

Theorem 3.2.13. *Let X be a smooth projective variety and take $\mathcal{F}^\bullet \in \mathbf{D}^b_G(X)$. Then $\mathcal{F}^\bullet$ descends to $\mathbf{Perf}(X/G)$ if and only if the the stabilizer G_x acts trivially on the $\mathcal{O}_X$-modules $\mathrm{H}^j(\mathcal{F}^\bullet \otimes^{\boldsymbol{L}} k(x))$ for every point $x \in X$.*

Proof. Suppose $\mathcal{F}^\bullet$ descends. Let $\mathcal{E}^\bullet$ be a finite resolution of G-locally free sheaves of finite type of $\mathcal{F}$. We have the following:

$$\mathrm{H}^j(\mathcal{F}^\bullet \otimes^{\mathbf{L}} k(x)) \xrightarrow{\sim} \mathrm{H}^j(\mathcal{E}^\bullet \otimes k(x)).$$

Since G_x acts trivially on $\mathcal{E}^j \otimes k(x)$ for all j, then it must act trivially also on $\mathrm{H}^j(\mathcal{E}^\bullet \otimes k(x))$, because the action of the group commutes with cohomology.

Viceversa, suppose that the stabilizer G_x acts trivially on the $\mathcal{O}_X$-modules $\mathrm{H}^j(\mathcal{F}^\bullet \otimes^{\mathbf{L}} k(x))$ for every point $x \in X$ and for every j. We divide the proof in two steps:

- Suppose $\mathcal{F}^\bullet = \mathcal{F}$ is just a coherent sheaf (i.e. a complex concentrated in degree zero).

We want to prove that $\mathcal{F}$ descends. Remember that we can suppose X to be affine because Remark 3.2.3 tells that descending property are local.

If the stabilizer G_x acts trivially on $\mathrm{H}^j(\mathcal{F}^\bullet \otimes^{\mathbf{L}} k(x))$ for every point $x \in X$, then in particular it acts trivially on $\mathcal{F} \otimes k(x)$ for every $x \in X$. Infact, if $\mathcal{E}^\bullet$ is a G-equivariant locally free resolution of $\mathcal{F}$, then $\mathrm{H}^0(\mathcal{E}^\bullet \otimes k(x)) = \mathrm{H}^0(\mathcal{F} \otimes k(x)) = \mathcal{F} \otimes k(x)$. Thus there exists a G-equivariant locally free sheaf $\mathcal{V}_1$ which descends to the quotient, and a surjective morphism: $\mathcal{V}_1 \longrightarrow \mathcal{F}$. Now, let $\mathcal{K}_1$ be the kernel of this morphism and consider the following exact sequence:

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{F} \longrightarrow 0.$$

We now take the long exact cohomology sequence:

$$\ldots \longrightarrow \mathrm{Tor}_j(\mathcal{F}, k(x)) \longrightarrow \mathrm{Tor}_{j-1}(\mathcal{K}_1, k(x)) \longrightarrow 0 \longrightarrow \mathrm{Tor}_{j-1}(\mathcal{F}, k(x)) \longrightarrow$$

$$\longrightarrow \mathrm{Tor}_{j-2}(\mathcal{K}_1, k(x)) \longrightarrow 0 \longrightarrow \mathrm{Tor}_{j-2}(\mathcal{F}, k(x)) \longrightarrow \ldots$$

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{Tor}_2(\mathcal{F}, k(x)) \longrightarrow \mathrm{Tor}_1(\mathcal{K}_1, k(x)) \longrightarrow 0 \longrightarrow$$

$$\longrightarrow \mathrm{Tor}_1(\mathcal{F}, k(x)) \longrightarrow \mathcal{K}_1 \otimes k(x) \longrightarrow \mathcal{V}_1 \otimes k(x) \longrightarrow \mathcal{F} \otimes k(x) \longrightarrow 0.$$

The vector space $\mathrm{H}^j(\mathcal{K}_1 \otimes^{\mathbf{L}} k(x))$ is always in between two G_x-invariant vector spaces and then Remark 3.2.11 implies that it is G_x-invariant as well. We now iterate the process: consider a G-equivariant locally free sheaf $\mathcal{V}_2$ which descends to the quotient, and a surjective morphism: $\mathcal{V}_2 \longrightarrow \mathcal{K}_1$ and let $\mathcal{K}_2$ be the kernel of this morphism. From the long exact cohomology sequence associated to the following exact sequence $0 \longrightarrow \mathcal{K}_2 \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{K}_1 \longrightarrow 0$ we deduce that $\mathrm{H}^j(\mathcal{K}_2 \otimes^{\mathbf{L}} k(x))$ is G_x-invariant, and in particular there exists a G-equivariant locally free sheaf $\mathcal{V}_3$ which descends to the quotient, and a surjective morphism: $\mathcal{V}_3 \longrightarrow \mathcal{K}_2$.

The process must end within a finite number of iterations, say $m \in \mathbb{Z}_{\geq 0}$, since the homological dimension of $\mathrm{Coh}^G(X)$ is finite. That is, at some point we obtain a kernel $\mathcal{K}_m$ which is a locally free sheaf such that the action of the stabilizer on the vector spaces $\mathcal{K}_m \otimes k(x)$ is trivial for every $x \in X$, and therefore (see Theorem 3.2.4) $\mathcal{K}_m$ descends. In other words, we obtain a finite G-equivariant locally free resolution of $\mathcal{F}$:

$$0 \longrightarrow \mathcal{V}_m \longrightarrow \ldots \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{F} \longrightarrow 0$$

such that every $\mathcal{V}_i$ descends to the quotient for $i = 1, \ldots m$.

- Suppose now that $\mathcal{F}^\bullet$ is a complex of G-equivariant sheaves.

We proceed by induction on the number n of non-zero cohomology groups of the complex $\mathcal{F}^\bullet$. More precisely, the induction statement is the following:

Let $\mathcal{F}^\bullet \in \mathbf{D}^b_G(X)$ be such that $\mathrm{H}^j(\mathcal{F}^\bullet) \neq 0$ for $j = 1, \ldots, n$ and such that the action of the stabilizer G_x is trivial on the $\mathcal{O}_X$-modules $\mathrm{H}^j(\mathcal{F}^\bullet \otimes^{\boldsymbol{L}} k(x))$ for every point $x \in X$. Then $\mathcal{F}^\bullet$ descends $\mathbf{Perf}(X/G)$.

If $n = 1$ the statement is true since it means that $\mathcal{F}^\bullet$ is just a sheaf, and therefore we go back to the previous case. Suppose now the statement is true for $n-1$. Take a resolution $\mathcal{E}^\bullet := \{0 \longrightarrow \mathcal{E}^1 \longrightarrow \ldots \longrightarrow \mathcal{E}^n \longrightarrow 0\}$ of $\mathcal{F}^\bullet$. By Lemma 3.2.12 there exists a complex:

$$\mathcal{V}^\bullet := \{0 \longrightarrow \mathcal{V}^{-m} \longrightarrow \mathcal{V}^{-m+1} \longrightarrow \ldots \longrightarrow \mathcal{V}^0 \longrightarrow \mathcal{V}^1 \longrightarrow \ldots \longrightarrow \mathcal{V}^n \longrightarrow 0\}$$

which is quasi-isomorphic to $\mathcal{E}$ and such that $\mathcal{V}^n$ descends to the quotient.

Now, we have the following exact triangle:

$$\mathcal{V}^n[-n] \otimes k(x) \longrightarrow \mathcal{V}^\bullet \otimes k(x) \longrightarrow \sigma_{\leq n-1}\mathcal{V}^\bullet \otimes k(x) \xrightarrow{[1]} .$$

where $\sigma_{\leq n-1}\mathcal{V}^\bullet$ is the stupid truncation of $\mathcal{V}^\bullet$ (see Definition 1.4.1).

From the long exact cohomology sequence:

$$0 \longrightarrow \mathrm{H}^{n-1}(\mathcal{V}^\bullet \otimes k(x)) \longrightarrow \mathrm{H}^{n-1}(\sigma_{\leq n-1}\mathcal{V}^\bullet \otimes k(x)) \longrightarrow$$

$$\longrightarrow \mathcal{V}^n[-n] \otimes k(x) \longrightarrow \mathrm{H}^n(\mathcal{V}^\bullet \otimes k(x)) \longrightarrow 0$$

and from Remark 3.2.11 we see that $\mathrm{H}^{n-1}(\sigma_{\leq n-1}\mathcal{V}^\bullet) \otimes k(x) = \mathrm{H}^{n-1}(\sigma_{\leq n-1}\mathcal{V}^\bullet \otimes k(x))$ is G_x-invariant. Also notice that $\mathrm{H}^j(\sigma_{\leq n-1}\mathcal{V}^\bullet \otimes k(x)) = \mathrm{H}^j(\mathcal{V}^\bullet \otimes k(x))$ for $j = 1, \ldots, n-2$. Hence $\sigma_{\leq n-1}\mathcal{V}^\bullet$ is a complex such that $\mathrm{H}^j(\sigma_{\leq n-1}\mathcal{V}^\bullet \otimes k(x))$ is G_x invariant for all j. Furthermore, $\mathrm{H}^j(\sigma_{\leq n-1}\mathcal{V}^\bullet) \neq 0$ for $j = 1, \ldots, n-1$, hence we can apply the induction hypothesis and conclude that $\sigma_{\leq n-1}\mathcal{V}^\bullet$ descends to $\mathbf{Perf}(X/G)$. We have then a commutative diagram:

$$\begin{array}{ccccc} \mathcal{V}^n[-n] & \longrightarrow & \mathcal{V}^\bullet & \longrightarrow & \sigma_{\leq n-1}\mathcal{V}^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{L}\pi^* \circ \Pi^G_*(\mathcal{V}^n[-n]) & \longrightarrow & \mathbf{L}\pi^* \circ \Pi^G_*(\mathcal{V}^\bullet) & \longrightarrow & \mathbf{L}\pi^* \circ \Pi^G_*(\sigma_{\leq n-1}\mathcal{V}^\bullet) \end{array}$$

The first and third vertical map are quasi-isomorphism, and this implies that also the second vertical map is a quasi-isomorphism, that is $\mathcal{V}^\bullet$ descends to $\mathbf{Perf}(X/G)$. □

3.3 Equivariant Fourier-Mukai transforms

In this section we give an example which shows how to build an autoequivalence of $\mathbf{Perf}(X/G)$ starting from a Fourier-Mukai autoequivalence of $\mathbf{D}^b_G(X)$. We deal with the case of projective varieties with ample canonical sheaf or ample anticanonical sheaf, since all the autoequivalences of the bounded G-equivariant derived category are classified:

Theorem 3.3.1. *Let X be a smooth normal projective variety with ample canonical sheaf or ample anticanonical sheaf. Then the group of isomorphism classes of exact autoequivalence $\mathbf{D}^b_G(X)$ is generated by automorphisms, tensor products with invertible sheaves and shifts.*

Proof. See [19], Theorem 7.2. □

Let X be projective with $\mathcal{K}_X$ or $-\mathcal{K}_X$ ample. Then, if we want to study whenever an autoequivalence $\Phi_{\mathcal{P}}$ of $\mathbf{D}^b_G(X)$ induces and autoequivalence of $\mathbf{Perf}(X/G)$, it suffices to study what happens when $\Phi_{\mathcal{P}}$ is a generator of the group of exact autoequivalences of $\mathbf{D}^b_G(X)$.

If $\Phi_{\mathcal{P}}$ is a shift, that is $\mathcal{P} = \Delta_*(\mathcal{O}_X[d])$ for a certain integer $d \in \mathbb{Z}$, where $\Delta : X \longrightarrow X \times X$, then it is straightforward to see that $\Phi_{\mathcal{P}}$ always induces an autoequivalence of $\mathbf{Perf}(X/G)$.

Suppose now Φ_P is a tensor product with an invertible sheaf $\mathcal{L}$ on X, that is $\mathcal{P} = \Delta_*(\mathcal{L})$ where $\Delta : X \longrightarrow X \times X$. Hence the Fourier-Mukai transform associated to the kernel $\Delta_*(\mathcal{L})$

gives rise to a functor:

$$\Omega := \Pi_*^G \circ \Phi_P \circ \mathbf{L}\pi^* : \mathbf{Perf}(X/G) \longrightarrow \mathbf{Perf}(X/G)$$

$$\Omega(\mathcal{A}^\bullet) = \Pi_*^G(\mathcal{L} \otimes \mathbf{L}\pi^*(\mathcal{A}^\bullet)).$$

If $\mathcal{L}$ descends to the quotient, i.e. the stabilizer G_x acts trivially on $\mathcal{L} \otimes k(x)$ for every $x \in X$, then Ω defines an autoequivalence of $\mathbf{Perf}(X/G)$.
If $\mathcal{L}$ does not descends, then it suffices to take the tensor product of $\mathcal{L}$ to the order of the group G. Infact, according to Lemma 3.2.5, the stabilizer G_x acts trivially on every fiber of $\mathcal{L}^{\otimes |G|}$.

If Φ_P is such that $\mathcal{P} = \mathcal{O}_{\Gamma_f}$ for an automorphism $f : X \longrightarrow X$ then:

$$\Omega := \Pi_*^G \circ \Phi_P \circ \mathbf{L}\pi^* : \mathbf{Perf}(X/G) \longrightarrow \mathbf{Perf}(X/G)$$

$$\Omega(\mathcal{A}^\bullet) = \Pi_*^G(f_*(\mathbf{L}\pi^*(\mathcal{A}^\bullet))).$$

Then, in this case Ω defines always an autoequivalence, because automorphisms of X preserve the descending property, infact:

$$\mathrm{H}^j(f_*(\mathbf{L}\pi^*(\mathcal{A}^\bullet)) \otimes^{\mathbf{L}} k(x)) = \mathrm{H}^j(f_*(\mathbf{L}\pi^*(\mathcal{A}^\bullet) \otimes^{\mathbf{L}} k(f^{-1}(x))))$$

because of the projection formula.

In the more general situation we have the following:

Theorem 3.3.2. *Let X be a smooth projective variety and let $\Phi_{\mathcal{P}}$ be an autoequivalence of $\mathbf{D}^b_G(X)$. Then the functor:*

$$\Omega := \Pi_*^G \circ \Phi_P \circ \boldsymbol{L}\pi^* : \mathbf{Perf}(X/G) \longrightarrow \mathbf{Perf}(X/G)$$

is an autoequivalence of $\mathbf{Perf}(X/G)$ if and only if the stabilizer G_x acts trivially on the $\mathcal{O}_X$-modules $\mathrm{H}^j(\Phi_{\mathcal{P}}(\mathbf{L}\pi^*(\mathcal{A}^\bullet)) \otimes k(x))$ for every $x \in X$ and $j \in \mathbb{Z}$, where $\mathcal{A}^\bullet \in \mathbf{Perf}(X/G)$.

This result comes from the descending criterion that we proved previously. Infact, Ω defines an equivalence if and only if $\Phi_{\mathcal{P}}(\mathbf{L}\pi^*(\mathcal{A}^\bullet)$ descends to the quotient, and this happens, by Theorem 3.2.13, if and only if the stabilizer G_x acts trivially on $\mathrm{H}^j(\Phi_{\mathcal{P}}(\mathbf{L}\pi^*(\mathcal{A}^\bullet)) \otimes k(x))$ for every $x \in X$ and $j \in \mathbb{Z}$.

We believe that is it possible to find a deeper description for those autoequivalences $\Phi_{\mathcal{P}}$ of

$\mathbf{D}^b_G(X)$ such that the induced functor Ω becomes an autoequivalence of $\mathbf{Perf}(X/G)$. In particular, we hope that it is achievable to find a descent criterion involving only the kernel $\mathcal{P}$ associated to such autoequivalences $\Phi_{\mathcal{P}}$. This is still an open problem on which we want to dedicate ourselves in the future.

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