Complete duality for quasiconvex dynamic risk measures on modules of the L^p-type

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Abstract

In the conditional setting we provide a complete duality between quasiconvex risk measures defined on L^0 modules of the L^p type and the appropriate class of dual functions. This is based on a general result which extends the usual Penot-Volle representation for quasiconvex real valued maps.

Keywords: quasiconvex functions, dual representation, complete duality, L^0-modules, dynamic risk measures.
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1 Introduction

The already-fifteen-years-old theory of risk measures is still originating many questions and springing out lots of new problems which trigger off the interest of researchers. Recently Kupper and Schachermayer [KS10] showed that in a dynamic framework only the entropic risk measure is in agreement with all the usual assumptions such as cash additivity, monotonicity, convexity, law invariance and time consistency. It's thus natural to question if these assumptions are too restrictive and indeed cash additivity was the first to be doubted and weakened to cash subadditivity, by El Karoui and Ravanelli [ER09].

Currently a debate between convexity and quasiconvexity is trying to give a better explanation to the concept of diversification, see Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio [CV10]. On one hand quasiconvexity can be considered as the mathematical translation of the principle of diversification; on the other, under the cash additivity assumption, convexity and quasiconvexity are equivalent. Once we give up cash additivity we are automatically induced to enlarge the class of feasible risk measures to the class of quasiconvex funtionals. In [FMP12] the authors show that on the level of distributions there do not exist any convex lower semicontinuous risk measure, but they provide a huge class of quasiconvex lower semicontinuous risk measures which contains as particular cases the Value at Risk, the Worst Case risk measure and the Certainty Equivalents. In [CV10], the representation of a quasiconvex cash subadditive (real valued) risk measure ρ is written in terms of the quasiaffine dual function R.

A *complete duality* for real valued quasiconvex functionals has been firstly established in [CV09]: the idea is to prove a *one to one* relationship between quasiconvex monotone functionals ρ and the function R in the dual representation. Obviously R will be unique only in an opportune class of maps satisfying certain properties. In Decision Theory the function R can be interpreted as the decision maker's index of uncertainty aversion: the uniquesness of R becomes crucial (see [CV09] and [DK10]) if we want to guarantee a robust dual representation of ρ characterized in terms of the unique R.

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In the conditional setting, where the maps take values in a set of random variables - for example, $\rho : L^p(\Omega, \mathcal{F}_T, \mathbb{P}) \to L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, $t < T$ - the representation of dynamic quasiconvex maps is obtained adopting a similar function R (see [FM11b]). The particular case of the Conditional Certainty Equivalent is treated in Frittelli Maggis [FM11a]. We stress that this framework is very relevant in all applications involving dynamic features and as far as we know a complete duality in this framework was lacking in literature.

As described in [DK10] topological vector spaces are the utmost general environment in which we are naturally led to embed the theory of risk preferences in the static case. On the other hand once we shift the problem to the conditional case (as in [FM11b]) the mathematical challenges become harder and harder so that topological vector spaces appears as unsuitable structures. Recently Filipovic, Kupper and Vogelpoth [FKV10] discussed many advantages of working in a module framework whenever dealing with the conditional setting. The intuition behind the use of modules is simple and natural: given a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ and a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$, suppose that a set L of time-T maturity contingent claims is fixed (for concreteness let $L = L^p(\mathcal{F}_T)$) and an agent is computing the risk of a portfolio at an intermediate time $t < T$. All the $\mathcal{F}_t$-measurable random variables are going to be known at time t, thus the $\mathcal{F}_t$ measurable random variables will act as scalars in the process of diversification of our portfolio, forcing to consider the new set

$$\begin{aligned} L^p_{\mathcal{F}_t}(\mathcal{F}_T) &:= L^0(\Omega, \mathcal{F}_t, \mathbb{P}) \cdot L^p(\Omega, \mathcal{F}_T, \mathbb{P}) \\ &= \left\{ YX \mid Y \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}),\ X \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}) \right\} \end{aligned}$$

as the domain of the risk measures. This product structure is exactly the one that constitutes the nature of L^0-modules. The most significant contribution on this topic comes from the extensive research produced by Guo from 1992 until today. An useful reference is [Gu11] on which the most important results are resumed and compared with the present literature and in particular with the recent development provided by Filipovic Kupper and Vogelpoth [FKV09]. The key point in both [FKV09] and [Gu10] is to provide a conditional form of the Hyperplane Separation Theorems. It is well known that many fundamental results in Mathematical Finance rely on these: for instance Arbitrage Theory and the duality results on risk measure or utility maximization.

In [FKV09] and [Gu10] the authors brilliantly succeed in the task of giving a topological structure to L^0-modules and to extend those theorems from functional analysis, which are relevant for financial applications. Once this rigorous analytical background has been carefully built up, it is possible to develop it further and obtain other interesting results and applications.

The overall aim of this paper is the establishment of a complete duality for evenly quasiconvex *conditional* risk measures (Theorem 15). Our findings may be adapted in a *dynamic framework* in Decision Theory (see [CV09]). As explained in Section 3 evenly quasiconvexity of the map ρ is an assumption weaker than lower semicontinuity and quasiconvexity.
As already mentioned, uniqueness of the representation is a delicate issue in the conditional case: once embedded in the the of L^0-modules the complete duality for conditional risk measures (see Theorem 15 for the precise statement), perfectly matches what had been obtained in [CV09] for the static case and provide great evidences of the power of the module approach.

Let $\mathcal{G} \subset \mathcal{F}$ be two sigma algebras we deduce under suitable conditions that $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$ is an evenly quasiconvex conditional risk measure *if and only if*

$$\rho(X) = \sup_{Q \in \mathcal{P}^q} R\left(E_Q\left[-X|\mathcal{G}\right], Q\right) \tag{1}$$

where $\mathcal{P}^q$ is a subset of probabilities Q such that $E[\frac{dQ}{d\mathbb{P}}|\mathcal{G}] = 1$ and R is *unique* in the class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ (see the Definition 14). In particular R will take the form

$$R(Y, Q) = \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{ \rho(\xi) \mid E_Q\left[-\xi|\mathcal{G}\right] = Y \right\}$$

A posteriori if we add the assumption $\rho(X+\alpha)=\rho(X)-\alpha$ for every $\alpha \in L^0(\mathcal{G})$, then the quasiconvex map ρ is automatically convex and $R(Y,Q)=Y-\rho^*(-Q)$ (see Corollary 17) so that we recover the dual representation proved in [DS05].
The function R has also an interesting interpretation related to the dual representation of convex risk measures. It's not hard to show that for every $X \in L^p_{\mathcal{G}}(\mathcal{F})$, $Q \in \mathcal{P}^q$ and any map $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$ we have:

$$R\left(E_Q\left[-X|\mathcal{G}\right], Q\right) \geq E_Q\left[-X|\mathcal{G}\right] - \rho^*(-Q), \tag{2}$$

where ρ^* is the convex conjugate of ρ.
From equation (1) we deduce that whenever the preferences of an agent are described by a quasiconvex - not convex - risk measure we cannot recover the risk only taking a *supremum* of the Fenchel conjugate, i.e. of the RHS of (2), over all the possible probabilistic scenarios. We shall need a more cautious approach represented by the new penalty function $R\left(E_Q\left[-X|\mathcal{G}\right], Q\right)$. The quantity $R(Y,Q)$ is therefore the reserve amount required at the intermediate time t ($\mathcal{F}_t = \mathcal{G}$) under the scenario Q, to cover an expected loss $Y \in L^0(\mathcal{G})$ in the future.

The paper is organized as follows. In Section 2 we provide some preliminary notions and facts: a short review about L^0-modules of the L^p type and the concept of conditionally evenly convex set. Section 3 is devoted to the regularity, quasiconvexity and continuity assumptions of the maps $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$. In Section 3.1 we state the complete duality for quasiconvex conditional risk measures. We include in Section 3.2 some complementary results. Section 4 is devoted to the proofs of the main contributions of the paper. Two more technical lemmas are deferred to the Appendix.

2 Notations, setting and topological properties

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed throughout this chapter and $\mathcal{G} \subseteq \mathcal{F}$ is any sigma algebra contained in $\mathcal{F}$. We denote with $L^0(\Omega, \mathcal{F}, \mathbb{P}) = L^0(\mathcal{F})$ (resp. $L^0(\mathcal{G})$) the space of $\mathcal{F}$ (resp. $\mathcal{G}$) measurable random variables that are $\mathbb{P}$ a.s. finite, whereas by $\bar{L}^0(\mathcal{F})$ the space of extended random variables which may take values in $\mathbb{R} \cup \{\infty\}$. In general since $(\Omega, \mathbb{P})$ are fixed we will always omit them. We define $L^0_+(\mathcal{F}) = \{X \in L^0(\mathcal{F}) \mid X \geq 0\}$ and $L^0_{++}(\mathcal{F}) = \{X \in L^0(\mathcal{F}) \mid X > 0\}$. We remind that all equalities/inequalities among random variables are meant to hold $\mathbb{P}$-a.s.. As the expected value $E_{\mathbb{P}}[\cdot]$ is mostly computed w.r.t. the reference probability $\mathbb{P}$, we will often omit $\mathbb{P}$ in the notation.
Moreover the essential ($\mathbb{P}$ almost surely) *supremum ess* $\sup_\lambda(X_\lambda)$ of an arbitrary family of random variables $X_\lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ will be simply denoted by $\sup_\lambda(X_\lambda)$, and similarly for the essential *infimum*. The symbol $\vee$ (resp. $\wedge$) denotes the essential ($\mathbb{P}$ almost surely) *maximum* (resp. the essential *minimum*) between two random variables, which are the usual lattice operations.

On L^0 modules of the L^p type. We now introduce the structure of normed module of the L^p type which play a key role in the financial applications and are studied in detail in [KV09] Section 4.2.

Consider the generalized conditional expectation of $\mathcal{F}$-measurable non negative random variables: $E[\cdot|\mathcal{G}] : L^0_+(\mathcal{F}) \to \bar{L}^0_+(\mathcal{G})$

$$E[X|\mathcal{G}] =: \lim_{n \to +\infty} E[X \wedge n|\mathcal{G}].$$

The basic properties of conditional expectation still hold true. In particular for every $X, X_1, X_2 \in L^0_+(\mathcal{F})$ and $Y \in L^0(\mathcal{G})$

(i) $YE[X|\mathcal{G}] = E[YX|\mathcal{G}]$;

(ii) $E[X_1 + X_2|\mathcal{G}] = E[X_1|\mathcal{G}] + E[X_2|\mathcal{G}]$;

(iii) $E[X] = E[E[X|\mathcal{G}]]$.

$L^0(\mathcal{G})$ equipped with the order of the almost sure dominance is a lattice ordered ring. Let $p \in [1, \infty]$ and consider the algebraic module over the ring $L^0(\mathcal{G})$ defined as

$$L^p_{\mathcal{G}}(\mathcal{F}) =: \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \|X|\mathcal{G}\|_p \in L^0(\Omega, \mathcal{G}, \mathbb{P})\}$$

where $\|\cdot|\mathcal{G}\|_p$ is assigned by

$$\|X|\mathcal{G}\|_p =: \begin{cases} E[|X|^p|\mathcal{G}]^{\frac{1}{p}} & \text{if } p < +\infty \\ \inf\{Y \in \bar{L}^0(\mathcal{G}) \mid Y \geq |X|\} & \text{if } p = +\infty \end{cases} \tag{3}$$

By this definition $L^p_{\mathcal{G}}(\mathcal{F})$ inherits the product structure i.e.

$$L^p_{\mathcal{G}}(\mathcal{F}) = L^0(\mathcal{G})L^p(\mathcal{F}) = \{YX \mid Y \in L^0(\mathcal{G}),\ X \in L^p(\mathcal{F})\}.$$

This last property allows the conditional expectation to be well defined for every $\widetilde{X} \in L^p_{\mathcal{G}}(\mathcal{F})$; indeed, if $\widetilde{X} = YX$ with $Y \in L^0(\mathcal{G})$ and $X \in L^p(\mathcal{F})$, then $E[\widetilde{X}|\mathcal{G}] = YE[X|\mathcal{G}]$ is a finite valued random variable. Moreover $\|\cdot|\mathcal{G}\|_p$ is a $L^0(\mathcal{G})$-norm according to the following definition.

Definition 1 *A map $\|\cdot\| : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0_+$ is a L^0-norm on $L^p_{\mathcal{G}}(\mathcal{F})$ if*

(i) $\|\gamma X\| = |\gamma|\|X\|$ *for all $\gamma \in L^0$ and $X \in L^p_{\mathcal{G}}(\mathcal{F})$,*

(ii) $\|X_1 + X_2\| \leq \|X_1\| + \|X_2\|$ *for all $X_1, X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$.*

(iii) $\|X\| = 0$ *implies $X = 0$.*

If we endow $L^0(\mathcal{G})$ with a topology τ_0 we may automatically induce a topology τ on $L^p_{\mathcal{G}}(\mathcal{F})$ by

$$X_\alpha \xrightarrow{\tau} X \quad \text{if and only if} \quad \|X_\alpha - X\|_p \xrightarrow{\tau_0} 0$$

Two natural choices for τ_0 are the topology of the convergence in probability (as used in [Gu10]) or the uniform topology as introduced in [FKV09]. In the following Remark we recall the second one since is non-standard in the literature.

Remark 2 *For every $\varepsilon \in L^0_{++}(\mathcal{G})$, the ball $B_\varepsilon := \{Y \in L^0(\mathcal{G}) \mid |Y| \leq \varepsilon\}$ centered in $0 \in L^0(\mathcal{G})$ gives the neighborhood basis of 0. A set $V \subset L^0(\mathcal{G})$ is a neighborhood of $Y \in L^0(\mathcal{G})$ if there exists $\varepsilon \in L^0_{++}(\mathcal{G})$ such that $Y + B_\varepsilon \subset V$. A set V is open if it is a neighborhood of all $Y \in V$. $(L^0(\mathcal{G}), |\cdot|)$ stands for $L^0(\mathcal{G})$ endowed with this topology: in this case the space looses the property of being a topological vector space. It is easy to see that a net converges in this topology, namely $Y_N \xrightarrow{|\cdot|} Y$ if for every $\varepsilon \in L^0_{++}(\mathcal{G})$ there exists $\overline{N}$ such that $|Y - Y_N| < \varepsilon$ for every $N > \overline{N}$.*

Given the pair $(L^p_{\mathcal{G}}(\mathcal{F}), \tau)$, $(L^0(\mathcal{G}), \tau_0)$ the dual module of $(L^p_{\mathcal{G}}(\mathcal{F}))^*$ will be the collection of continuous functional $\mu : (L^p_{\mathcal{G}}(\mathcal{F}), \tau) \to (L^0(\mathcal{G}), \tau_0)$ which are $L^0(\mathcal{G})$-linear i.e.

$$\mu(\alpha X + \beta Y) = \alpha\mu(X) + \beta\mu(Y)$$

for every $\alpha, \beta \in L^0(\mathcal{G})$ and $X, Y \in L^p_{\mathcal{G}}(\mathcal{F})$.
From now on we will consider on $L^0(\mathcal{G})$ any topology τ_0 such that the dual module can be identified with a random variable

$$Z \in L^q_{\mathcal{G}}(\mathcal{F}) \rightleftarrows \mu(\cdot) = E[Z \cdot |\mathcal{G}]$$

where $\frac{1}{p} + \frac{1}{q} = 1$. So in general we can identify the dual modules of $L^p_{\mathcal{G}}(\mathcal{F})$ with $L^q_{\mathcal{G}}(\mathcal{F})$. This occurs in particular if τ_0 is the one defined in Remark 2 or the topology of convergence in probability (see [Gu10] and [FKV09]).

On conditionally evenly convex sets. We recall that a subset V of a locally convex topological vector space is *evenly convex* if it is the intersection of a family of open half spaces, or equivalently, if every $X \notin V$ can be separated from V by a continuous linear functional. Obviously an evenly convex set is necessarily convex and moreover the whole topological vector space is a trivial case of evenly convex set. In this subsection we recall the generalization of the concept of evenly convexity as introduced in [FM12], which is tailor made for the conditional setting. We refer the reader to [FM12] for further details and motivations.

Definition 3 *Let $\mathcal{C}$ be a subset of $L^p_{\mathcal{G}}(\mathcal{F})$.*

(CSet) *$\mathcal{C}$ has the countable concatenation property if for every countable partition $\{A_n\}_n \subseteq \mathcal{G}$ and for every countable collection of elements $\{X_n\}_n \subset \mathcal{C}$ we have $\sum_n \mathbf{1}_{A_n} X_n \in \mathcal{C}$.*

We denote by $\mathcal{C}^{cc}$ the countable concatenation hull of $\mathcal{C}$, namely the smallest set $\mathcal{C}^{cc} \supset \mathcal{C}$ which satisfies (CSet).

We notice that an arbitrary set $\mathcal{C} \subset L^p_{\mathcal{G}}(\mathcal{F})$ may present some components which degenerate to the entire module. Basically it might occur that for some $A \in \mathcal{G}$, $\mathcal{C}\mathbf{1}_A = L^p_{\mathcal{G}}(\mathcal{F})\mathbf{1}_A$, i.e., for each $\xi \in L^p_{\mathcal{G}}(\mathcal{F})$ there exists $\eta \in \mathcal{C}$ such that $\eta \mathbf{1}_A = \xi \mathbf{1}_A$. In this case there are no chances to guarantee a separation on the set Ω as for the results given in [FKV09]. Thus we need to determine the maximal $\mathcal{G}$-measurable set on which $\mathcal{C}$ reduces to $L^p_{\mathcal{G}}(\mathcal{F})$. The existence of the maximal element has been proved in [FM12] (see Remark 24 in the Appendix of the present paper) and the following definition is well posed.

Notation 4 *Fix a set $\mathcal{C} \subseteq L^p_{\mathcal{G}}(\mathcal{F})$. As the class $\mathcal{A}(\mathcal{C}) := \{A \in \mathcal{G} \mid \mathcal{C}\mathbf{1}_A = L^p_{\mathcal{G}}(\mathcal{F})\mathbf{1}_A\}$ is closed with respect to countable union, we denote with $A_{\mathcal{C}}$ the $\mathcal{G}$-measurable maximal element of the class $\mathcal{A}(\mathcal{C})$ and with $D_{\mathcal{C}}$ the (P-a.s. unique) complement of $A_{\mathcal{C}}$ (see also the Remark 24). Hence $\mathcal{C}\mathbf{1}_{A_{\mathcal{C}}} = L^p_{\mathcal{G}}(\mathcal{F})\mathbf{1}_{A_{\mathcal{C}}}$.*

Definition 5 *Let $\mathcal{C}$ be a subset of $L^p_{\mathcal{G}}(\mathcal{F})$. We will say that*

(i) *$X \in L^p_{\mathcal{G}}(\mathcal{F})$ is outside $\mathcal{C}$ if $\mathbf{1}_A\{X\} \cap \mathbf{1}_A\mathcal{C} = \varnothing$ for every $A \in \mathcal{G}$ with $A \subseteq D_{\mathcal{C}}$ and $\mathbb{P}(A) > 0$.*

(ii) *$\mathcal{C}$ is* L^0-convex *if $\Lambda X_1 + (1-\Lambda)X_2 \in \mathcal{C}$ for every $X_1, X_2 \in \mathcal{C}$, $\Lambda \in L^0(\mathcal{G})$ and $0 \leq Lambda \leq 1$.*

(iii) *$\mathcal{C}$ is* conditional evenly convex *if $\mathcal{C}$ satisfies (CSet) and for every X outside $\mathcal{C}$ there exists a $\mu \in L^q_{\mathcal{G}}(\mathcal{F})$ such that*

$$\mu(X) > \mu(\xi) \text{ on } D_{\mathcal{C}},\ \forall \xi \in \mathcal{C}.$$

In [FM12] it is showed that any conditional evenly convex set is also L^0-convex and it can be characterized as intersection of half spaces.

3 Conditional Risk Measures defined on $L^p_{\mathcal{G}}(\mathcal{F})$.

In this section we summarize the properties of the risk maps that we will consider and recall the Penot-Volle type dual representation of quasiconvex conditional maps as proved in [FM12].

Definition 6 *A map $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ is*

(REG) *regular if for every $X_1, X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$ and $A \in \mathcal{G}$,*

$$\rho(X_1\mathbf{1}_A + X_2\mathbf{1}_{A^C}) = \rho(X_1)\mathbf{1}_A + \rho(X_2)\mathbf{1}_{A^C}.$$

Remark 7 *It is well known that (REG) is equivalent to:*

$$\rho(X\mathbf{1}_A)\mathbf{1}_A = \rho(X)\mathbf{1}_A,\ \forall A \in \mathcal{G},\ \forall X \in L^p_{\mathcal{G}}(\mathcal{F}).$$

In the module setting it is even true that (REG) is equivalent to countably regularity, i.e.

$$\rho(\sum_{i=1}^{\infty} X_i \mathbf{1}_{A_i}) = \sum_{i=1}^{\infty} \rho(X_i)\mathbf{1}_{A_i} \text{ on } \cup_{i=1}^{\infty} A_i$$

for $X_i \in L^p_{\mathcal{G}}(\mathcal{F})$ *and* $\{A_i\}_i$ *a sequence of disjoint* $\mathcal{G}$ *measurable sets. Indeed, from the module properties,* $X := \sum_{i=1}^{\infty} X_i \mathbf{1}_{A_i} \in L^p_{\mathcal{G}}(\mathcal{F})$ *and* $\sum_{i=1}^{\infty} \rho(X_i)\mathbf{1}_{A_i} \in \bar{L}^0(\mathcal{G})$*; (REG) then implies* $\rho(X)\mathbf{1}_{A_i} = \rho(X\mathbf{1}_{A_i})\mathbf{1}_{A_i} = \rho(X_i\mathbf{1}_{A_i})\mathbf{1}_{A_i} = \rho(X_i)\mathbf{1}_{A_i}$.

Let $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ be (REG). There might exist a set $A \in \mathcal{G}$ on which the map ρ is infinite, in the sense that $\rho(\xi)\mathbf{1}_A = +\infty\mathbf{1}_A$ for every $\xi \in L^p_{\mathcal{G}}(\mathcal{F})$. For this reason we introduce

$$\mathcal{A} := \{A \in \mathcal{G} \mid \rho(\xi)\mathbf{1}_A = +\infty\mathbf{1}_A \;\forall\, \xi \in L^p_{\mathcal{G}}(\mathcal{F})\}.$$

Applying Lemma 25 in Appendix with $F := \left\{\rho(\xi) \mid \xi \in L^p_{\mathcal{G}}(\mathcal{F})\right\}$ and $Y_0 = +\infty$ we can deduce the existence of two maximal sets $T_\rho \in \mathcal{G}$ and $\Upsilon_\rho \in \mathcal{G}$ for which $P(T_\rho \cap \Upsilon_\rho) = 0$, $P(T_\rho \cup \Upsilon_\rho) = 1$ and

$$\begin{aligned} &\rho(\xi) = +\infty \quad \text{on } \Upsilon_\rho \quad \text{for every } \xi, \eta \in L^p_{\mathcal{G}}(\mathcal{F}), \\ &\rho(\zeta) < +\infty \quad \text{on } T_\rho \quad \text{for some } \zeta \in L^p_{\mathcal{G}}(\mathcal{F}). \end{aligned} \tag{4}$$

Suppose that a map ρ satisfies: $\mathbb{P}(\Upsilon_\rho) > 0$ so that $\rho(\xi)\mathbf{1}_{\Upsilon_\rho} = +\infty\mathbf{1}_{\Upsilon_\rho}$ for every $\xi \in L^p_{\mathcal{G}}(\mathcal{F})$. Then its lower level sets $\{X \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \rho(X) \le \eta\}$, $\eta \in L^0(\mathcal{G})$, are all empty (and so convex and closed). This would imply that any such map is quasiconvex and lower semicontinuous, regardless of its behavior on (the relevant set) T_ρ. This explains the need of introducing the set T_ρ in the following definition of (QCO), (LSC) and (EVQ).
Hereafter we state the conditional version of some relevant properties of the maps under investigation. To this aim we define, for $Y \in L^0(\mathcal{G})$,

$$U^Y_\rho := \{\xi \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \rho(\xi)\mathbf{1}_{T_\rho} \le Y\}.$$

Definition 8 *Let* $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$*. The map* ρ *is:*

(QCO) *quasiconvex if the sets* U^Y_ρ *are* $L^0(\mathcal{G})$*-convex* $\forall Y \in L^0(\mathcal{G})$.

(EVQ) *evenly quasiconvex if the sets* U^Y_ρ *are conditional evenly convex* $\forall Y \in L^0(\mathcal{G})$.

(LSC) *lower semicontinuous if the sets* U^Y_ρ *are closed* $\forall Y \in L^0(\mathcal{G})$.

Remark 9 *Let* $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$.

(i) *The quasiconvexity of* ρ *is equivalent to the condition*

$$\rho(\Lambda X_1 + (1-\Lambda)X_2) \le \rho(X_1) \vee \rho(X_2), \tag{5}$$

for every $X_1, X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$, $\Lambda \in L^0(\mathcal{G})$ *and* $0 \le \Lambda \le 1$.

(ii) *If the map* ρ *is (REG) then* U^Y_ρ *satisfies (CSet).*

Regularity also guarantees that evenly quasiconvex maps are indeed quasiconvex. Moreover the next Lemma shows that the property (EVQ) is weaker than (QCO) plus (LSC) (see [FM12] Section 5 for further details).

Lemma 10 *Let* $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ *be (REG).*

(i) *If* ρ *is (EVQ) then it is (QCO).*

(ii) *If* ρ *is (QCO) and (LSC) then it is (EVQ).*

Theorem 11 ([FM12] Theorem 16) *If $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ is (REG) and (EVQ) then*

$$\rho(X) = \sup_{\mu \in L^q_{\mathcal{G}}(\mathcal{F})} \mathcal{R}(\mu(X), \mu), \tag{6}$$

where for $Y \in L^0(\mathcal{G})$ and $\mu \in L^q_{\mathcal{G}}(\mathcal{F})$,

$$\mathcal{R}(Y, \mu) := \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{ \rho(\xi) \mid \mu(\xi) \geq Y \right\}. \tag{7}$$

Definition 12 *We say that a map $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ is:*

($\downarrow$ MON) *monotone decreasing if $X_1 \geq X_2 \Rightarrow \rho(X_1) \leq \rho(X_2)$.*

Definition 13 *A* quasiconvex conditional risk measure *is a map $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ satisfying (REG), ($\downarrow$MON) and (QCO).*

Recall that the principle of diversification states that "diversification should not increase the risk", i.e. the diversified position $\Lambda X + (1 - \Lambda)Y$ is less risky than either the positions X or Y. Thus the mathematical formulation of this priciple is exactly quasiconvexity, i.e. the property (5). Under the cash additivity axiom

(CAS) $\rho(X + \Lambda) = \rho(X) - \Lambda$, for any $\Lambda \in L^0(\mathcal{G})$ and $X \in L^p_{\mathcal{G}}(\mathcal{F})$,

convexity and quasiconvexity are equivalent, so that they both provide the right interpretation of this principle. As vividly discussed by El Karoui and Ravanelli [ER09] the lack of liquidity of zero coupon bonds is the primary reason of the failure of cash additivity. In addition, in the *time consistent* case, the cash subadditivity property

(CSA) $\rho(X + \Lambda) \geq \rho(X) - \Lambda$, for any $\Lambda \in L^0_+(\mathcal{G})$ and $X \in L^p_{\mathcal{G}}(\mathcal{F})$,

is the adequate property of a conditional risk measure *for processes* (see the discussion in Section 5, [FP06]).

Thus it is unavoidable in the dynamic setting to relax the convexity axiom to quasiconvexity (and (CAS) to (CSA)) in order to regain the best modeling of diversification.

3.1 Complete Duality

This section is devoted to main result of this paper: a complete quasiconvex conditional duality between the risk measure ρ and the dual map R. Given any ρ we guarantee the existence of a unique map R in the class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ which allows a dual representation of the form given by equation (10). On the other hand every $R \in \mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ will identify a unique quasiconvex conditional risk measure by the mean of the representation (10).

As discussed in the previous section, the duality concerning conditional quasiconvex risk measures holds only under an additional continuity assumption (either (EVQ) or (LSC)). For the analysis of the complete duality in this section, we chose the weakest assumption, i.e. (EVQ), and we leave the (LSC) case for further investigation. We stress that, in virtue of Lemma 10, any map satisfying the assumptions of Theorem 15 is a conditional quasiconvex risk measure.

Due to the assumption that ρ is monotone *decreasing*, we modify, with just a difference in the sign, the definition of the dual function and rename it as:

$$R(Y, Z) := \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{ \rho(\xi) \mid E\left[-\xi Z | \mathcal{G}\right] \geq Y \right\}. \tag{8}$$

The function R is well defined on the domain

$$\Sigma = \{(Y, Z) \in L^0(\mathcal{G}) \times L^q_{\mathcal{G}}(\mathcal{F}) \mid \exists \xi \in L^p_{\mathcal{G}}(\mathcal{F}) \text{ s.t. } E[-Z\xi|\mathcal{G}] \geq Y\}. \tag{9}$$

Let also introduce the following set:

$$\mathcal{P}^q =: \{Z \in L^q_{\mathcal{G}}(\mathcal{F}) \mid Z \geq 0,\ E[Z|\mathcal{G}] = 1\}$$

and with a slight abuse of notation we will write that the probability $Q \in \mathcal{P}^q$ instead of $Z = \frac{dQ}{d\mathbb{P}} \in \mathcal{P}^q$ and $R(Y, Q)$ instead of $R\left(Y, \frac{dQ}{d\mathbb{P}}\right)$. In addition for every $Q \in \mathcal{P}^q$ we have $E\left[\frac{dQ}{d\mathbb{P}} X|\mathcal{G}\right] = E_Q[X|\mathcal{G}]$.

Definition 14 *The class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ is composed by maps $K : L^0(\mathcal{G}) \times \mathcal{P}^q \to \bar{L}^0(\mathcal{G})$ s.t.*

(i) *K is increasing in the first component.*

(ii) *$K(Y\mathbf{1}_A, Q)\mathbf{1}_A = K(Y, Q)\mathbf{1}_A$ for every $A \in \mathcal{G}$ and $(Y, \frac{dQ}{d\mathbb{P}}) \in \Sigma$.*

(iii) *$\inf_{Y\in L^0(\mathcal{G})} K(Y, Q) = \inf_{Y\in L^0(\mathcal{G})} K(Y, Q')$ for every $Q, Q' \in \mathcal{P}^q$.*

(iv) *K is $\diamond$-evenly $L^0(\mathcal{G})$-quasiconcave: for every $(Y^*, Q^*) \in L^0(\mathcal{G}) \times \mathcal{P}^q$, $A \in \mathcal{G}$ and $\alpha \in L^0(\mathcal{G})$ such that $K(Y^*, Q^*) < \alpha$ on A, there exists $(S^*, X^*) \in L^0_{++}(\mathcal{G}) \times L^p_{\mathcal{G}}(\mathcal{F})$ with*

$$Y^*S^* + E\left[X^* \frac{dQ^*}{d\mathbb{P}} \mid \mathcal{G}\right] < YS^* + E\left[X^* \frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] \text{ on } A$$

for every (Y, Q) such that $K(Y, Q) \geq \alpha$ on A.

(v) *the set $\mathcal{K}(X) = \left\{K(E[X\frac{dQ}{d\mathbb{P}}|\mathcal{G}], Q) \mid Q \in \mathcal{P}^q\right\}$ is upward directed for every $X \in L^p_{\mathcal{G}}(\mathcal{F})$.*

(vi) *$K(Y, Q_1)\mathbf{1}_A = K(Y, Q_2)\mathbf{1}_A$, if $\frac{dQ_1}{d\mathbb{P}}\mathbf{1}_A = \frac{dQ_2}{d\mathbb{P}}\mathbf{1}_A$, $Q_i \in \mathcal{P}^q$, and $A \in \mathcal{G}$.*

We will show in Lemma 21 that the class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ is not empty.

Theorem 15 *The map $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$ satisfies (REG), (↓MON), (EVQ) if and only if*

$$\rho(X) = \sup_{Q\in\mathcal{P}^q} R\left(E_Q[-X|\mathcal{G}], Q\right) \tag{10}$$

where

$$R(Y, Q) = \inf_{\xi\in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi) \mid E_Q[-\xi|\mathcal{G}] = Y\}$$

is unique in the class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$.

Proof. In Section 4.2. ■

3.2 Complements

From Theorem 15 we can deduce the following proposition which confirm what was obtained in [FM11b].

Proposition 16 *Suppose that ρ satisfies the same assumptions of Theorem 15. Then the restriction $\widehat{\rho} := \rho\mathbf{1}_{L^p(\mathcal{F})}$ defined by $\widehat{\rho}(X) = \rho(X)$ for every $X \in L^p(\mathcal{F})$ can be represented as*

$$\widehat{\rho}(X) = \sup_{Q\in\mathcal{P}^q} \inf_{\xi\in L^p(\mathcal{F})} \{\widehat{\rho}(\xi) \mid E_Q[-\xi|\mathcal{G}] = E_Q[-X|\mathcal{G}]\}.$$

Proof. For every $X \in L^p(\mathcal{F})$, $Q \in \mathcal{P}^q$ we have

$$\begin{aligned} \widehat{\rho}(X) &\geq \inf_{\xi\in L^p(\mathcal{F})} \{\widehat{\rho}(\xi) \mid E_Q[-\xi|\mathcal{G}] = E_Q[-X|\mathcal{G}]\} \\ &\geq \inf_{\xi\in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi) \mid E_Q[-\xi|\mathcal{G}] = E_Q[-X|\mathcal{G}]\} \end{aligned}$$

and hence the thesis. ■

The following result is meant to confirm that the dual representation chosen for quasiconvex maps is indeed a good generalization of the convex case.

Corollary 17 *Let $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$.*
(i) If $Q \in \mathcal{P}^q$ and if ρ is (MON), (REG) and (CAS) then

$$R(E_Q(-X|\mathcal{G}), Q) = E_Q(-X|\mathcal{G}) - \rho^*(-Q)$$

where

$$\rho^*(-Q) = \sup_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{E_Q[-\xi|\mathcal{G}] - \rho(\xi)\}.$$

(ii) Under the same assumptions of Theorem 15 and if ρ satisfies in addition (CAS) then

$$\rho(X) = \sup_{Q \in \mathcal{P}^q} \{E_Q(-X|\mathcal{G}) - \rho^*(-Q)\}.$$

Proof. Denote $\mu(\cdot) =: E_Q[\cdot \mid \mathcal{G}]$. By definition of R

$$\begin{aligned} R(E_Q(-X|\mathcal{G}), Q) &= \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi) \mid \mu(-\xi) = \mu(-X)\} \\ &= \mu(-X) + \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi) - \mu(-X) \mid \mu(-\xi) = \mu(-X)\} \\ &= \mu(-X) + \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi) - \mu(-\xi) \mid \mu(-\xi) = \mu(-X)\} \\ &= \mu(-X) - \sup_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi) - \mu(-X) \mid \mu(-\xi) = \mu(-X)\} \\ &= \mu(-X) - \rho^*(-Q), \end{aligned}$$

where the last equality follows from

$$\begin{aligned} \rho^*(-Q) &\overset{(CAS)}{=} \sup_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{\mu(-\xi - \mu(X-\xi)) - \rho(\xi + \mu(X-\xi))\} \\ &= \sup_{\eta \in L^p_{\mathcal{G}}(\mathcal{F})} \{\mu(-\eta) - \rho(\eta) \mid \eta = \xi + \mu(X-\xi)\} \\ &\leq \sup_{\eta \in L^p_{\mathcal{G}}(\mathcal{F})} \{\mu(-\eta) - \rho(\eta) \mid \mu(-\eta) = \mu(-X)\} \leq \rho^*(-Q). \end{aligned}$$

(ii) It is a consequence of (i) and Theorem 15. ■

3.3 A characterization *via* the risk acceptance family

In this subsection we assume for the sake of simplicity that $\rho(0) \in L^0(\mathcal{G})$ which implies that $\mathbb{P}(T_\rho) = 1$. In this way we do not loose any generality imposing $\rho(0) = 0$ (if not, just define $\tilde{\rho}(\cdot) := \rho(\cdot) - \rho(0)$). We remind that if $\rho(0) = 0$ then (REG) is equivalent to the condition

$$\rho(X\mathbf{1}_A) = \rho(X)\mathbf{1}_A, \ A \in L^0(\mathcal{G}).$$

Given a risk measure one can always define for every $Y \in L^0(\mathcal{G})$ the risk acceptance set of level Y as

$$\mathcal{A}^Y_\rho = \{X \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \rho(X) \leq Y\}.$$

This set represents the collection of financial positions whose risk is smaller of the fixed level Y and are strictly related to the Acceptability Indices [CM09]. Given a risk measure ρ we can associate a family of risk acceptance sets, namely $\{\mathcal{A}^Y_\rho \mid Y \in L^0(\mathcal{G})\}$, as it was suggested in the static case in [DK10].

Definition 18 *A family $\mathbb{A} = \{\mathcal{A}^Y \mid Y \in L^0(\mathcal{G})\}$ of subsets $\mathcal{A}^Y \subset L^p_{\mathcal{G}}(\mathcal{F})$ is called risk acceptance family if the following properties hold:*
(i) convexity: $\mathcal{A}^Y$ is $L^0(\mathcal{G})$-convex for every $Y \in L^0(\mathcal{G})$;
(ii) monotonicity:

· $X_1 \in \mathcal{A}^Y$ *and* $X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$, $X_2 \geq X_1$ *implies* $X_2 \in \mathcal{A}^Y$;
· $\mathcal{A}^{Y_1} \subseteq \mathcal{A}^{Y_2}$ *for any* $Y_1 \leq Y_2$, $Y_i \in L^0(\mathcal{G})$;
(iii) regularity: fix $X \in \mathcal{A}^Y$ *then for every* $G \in \mathcal{G}$ *we have*

$$\inf\{Y\mathbf{1}_G \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X \in \mathcal{A}^Y\} = \inf\{Y \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X\mathbf{1}_G \in \mathcal{A}^Y\}$$

These three properties allow us to induce a one to one relationship between quasiconvex conditional risk measures and risk acceptance families as we prove in the following

Proposition 19 *For any quasiconvex conditional risk measure* $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ *the family*

$$\mathbb{A}_\rho = \{\mathcal{A}^Y_\rho \mid Y \in L^0(\mathcal{G})\}$$

with $\mathcal{A}^Y_\rho = \{X \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \rho(X) \leq Y\}$ *is a risk acceptance family.*
Viceversa *for every risk acceptance family* $\mathbb{A}$ *the map*

$$\rho_{\mathbb{A}}(X) = \inf\{Y \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X \in \mathcal{A}^Y\}$$

is a well defined quasiconvex conditional risk measure $\rho_{\mathbb{A}} : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ *such that* $\rho(0) = 0$. *Moreover,* $\rho_{\mathbb{A}_\rho} = \rho$ *and if* $\mathcal{A}^Y = \bigcap_{Y'>Y} \mathcal{A}^{Y'}$ *for every* $Y \in L^0(\mathcal{G})$ *then* $\mathbb{A}_{\rho_{\mathbb{A}}} = \mathbb{A}$.

Proof. The proof is an extension from the static case provided in [CM09] and [DK10]. ($\downarrow$MON) and (QCO) of ρ imply that $\mathcal{A}^Y_\rho$ is convex and monotone. Also notice that

$$\begin{aligned} & \inf\{Y \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X\mathbf{1}_G \in \mathcal{A}^Y_\rho\} = \inf\{Y \mid \rho(X\mathbf{1}_G) \leq Y \text{ for } Y \in L^0(\mathcal{G})\} \\ = \; & \rho(X\mathbf{1}_G) = \rho(X)\mathbf{1}_G = \inf\{Y\mathbf{1}_G \mid Y \in L^0(\mathcal{G}) \text{ s.t. } \rho(X) \leq Y\} \\ = \; & \inf\{Y\mathbf{1}_G \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X \in \mathcal{A}^Y_\rho\}, \end{aligned}$$

i.e. $\mathcal{A}^Y_\rho$ is regular.

Viceversa: we first prove that $\rho_{\mathbb{A}}$ is (REG). For every $G \in \mathcal{G}$

$$\begin{aligned} \rho_{\mathbb{A}}(X\mathbf{1}_G) = \; & \inf\{Y \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X\mathbf{1}_G \in \mathcal{A}^Y\} \\ \overset{(iii)}{=} \; & \inf\{Y\mathbf{1}_G \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X \in \mathcal{A}^Y\} = \rho_{\mathbb{A}}(X)\mathbf{1}_G \end{aligned}$$

Now consider $X_1, X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$, $X_1 \leq X_2$. Let $G^C = \{\rho_{\mathbb{A}}(X_1) = +\infty\}$ so that $\rho_{\mathbb{A}}(X_1\mathbf{1}_{G^C}) \geq \rho_{\mathbb{A}}(X_2\mathbf{1}_{G^C})$. Otherwise consider the collection of Ys such that $X_1\mathbf{1}_G \in \mathcal{A}^Y$. Since $\mathcal{A}^Y$ is monotone we have that $X_2\mathbf{1}_G \in \mathcal{A}^Y$ if $X_1\mathbf{1}_G \in \mathcal{A}^Y$ and this implies that

$$\begin{aligned} \rho_{\mathbb{A}}(X_1)\mathbf{1}_G &= \inf\{Y\mathbf{1}_G \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X_1 \in \mathcal{A}^Y\} \\ &= \inf\{Y \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X_1\mathbf{1}_G \in \mathcal{A}^Y\} \\ &\geq \inf\{Y \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X_2\mathbf{1}_G \in \mathcal{A}^Y\} \\ &= \inf\{Y\mathbf{1}_G \mid Y \in L^0(\mathcal{G}) \text{ s.t. } X_2 \in \mathcal{A}^Y\} = \rho(X_2)\mathbf{1}_G, \end{aligned}$$

i.e. $\rho_{\mathbb{A}}(X_1\mathbf{1}_G) \geq \rho_{\mathbb{A}}(X_2\mathbf{1}_G)$. And this shows that $\rho_{\mathbb{A}}(\cdot)$ is ($\downarrow$MON).
Let $X_1, X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$ and take any $\Lambda \in L^0(\mathcal{G})$, $0 \leq \Lambda \leq 1$. Define the set $B =: \{\rho_{\mathbb{A}}(X_1) \leq \rho_{\mathbb{A}}(X_2)\}$. If $X_1\mathbf{1}_{B^C} + X_2\mathbf{1}_B \in \mathcal{A}^{Y'}$ for some $Y' \in L^0(\mathcal{G})$ then for sure $Y' \geq \rho_{\mathbb{A}}(X_1) \vee \rho_{\mathbb{A}}(X_2) \geq \rho(X_i)$ for $i = 1, 2$. Hence also $\rho(X_i) \in \mathcal{A}^{Y'}$ for $i = 1, 2$ and by convexity we have that $\Lambda X_1 + (1-\Lambda)X_2 \in \mathcal{A}^{Y'}$. Then $\rho_{\mathbb{A}}(\Lambda X_1 + (1-\Lambda)X_2) \leq \rho_{\mathbb{A}}(X_1) \vee \rho_{\mathbb{A}}(X_2)$.
If $X_1\mathbf{1}_{B^C} + X_2\mathbf{1}_B \notin \mathcal{A}^{Y'}$ for every $Y' \in L^0(\mathcal{G})$ then from property (iii) we deduce that $\rho_{\mathbb{A}}(X_1) = \rho_{\mathbb{A}}(X_2) = +\infty$ and the thesis is trivial.
Now consider $B = \{\rho(X) = +\infty\}$: $\rho_{\mathbb{A}_\rho}(X) = \rho(X)$ follows from

$$\begin{aligned} \rho_{\mathbb{A}_\rho}(X)\mathbf{1}_B &= \inf\{Y\mathbf{1}_B \mid Y \in L^0(\mathcal{G}) \text{ s.t. } \rho(X) \leq Y\} = +\infty\mathbf{1}_B \\ \rho_{\mathbb{A}_\rho}(X)\mathbf{1}_{B^C} &= \inf\{Y\mathbf{1}_{B^C} \mid Y \in L^0(\mathcal{G}) \text{ s.t. } \rho(X) \leq Y\} \\ &= \inf\{Y \mid Y \in L^0(\mathcal{G}) \text{ s.t. } \rho(X)\mathbf{1}_{B^C} \leq Y\} = \rho(X)\mathbf{1}_{B^C} \end{aligned}$$

For the second claim notice that if $X \in \mathcal{A}^Y$ then $\rho_{\mathbb{A}}(X) \leq Y$ which means that $X \in \mathcal{A}^Y_{\rho_{\mathbb{A}}}$. Conversely if $X \in \mathcal{A}^Y_{\rho_{\mathbb{A}}}$ then $\rho_{\mathbb{A}}(X) \leq Y$ and by monotonicity this implies that $X \in \mathcal{A}^{Y'}$ for every $Y' > Y$. From the right continuity we take the intersection and get that $X \in \mathcal{A}^Y$. ■

4 Proofs

4.1 General properties of $\mathcal{R}(Y, \mu)$

Following the path traced in [FM11b] and [FM12], we adapt (without giving a proof) to the L^p module framework the foremost properties holding for the function $\mathcal{R} : L^0(\mathcal{G}) \times L^q_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ defined in (7). Let the effective domain of the function $\mathcal{R}$ be:

$$\Sigma_{\mathcal{R}} := \{(Y, \mu) \in L^0(\mathcal{G}) \times L^q_{\mathcal{G}}(\mathcal{F}) \mid \exists \xi \in L^p_{\mathcal{G}}(\mathcal{F}) \text{ s.t. } \mu(\xi) \geq Y\}. \tag{11}$$

Lemma 20 *Let $\mu \in L^q_{\mathcal{G}}(\mathcal{F})$, $X \in L^p_{\mathcal{G}}(\mathcal{F})$ and $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to \bar{L}^0(\mathcal{G})$ satisfy (REG).*
i) $\mathcal{R}(\cdot, \mu)$ is monotone non decreasing.
ii) $\mathcal{R}(\Lambda\mu(X), \Lambda\mu) = \mathcal{R}(\mu(X), \mu)$ for every $\Lambda \in L^0(\mathcal{G})$.
iii) For every $Y \in L^0(\mathcal{G})$ and $\mu \in L^q_{\mathcal{G}}(\mathcal{F})$, the set

$$\mathcal{A}_\mu(Y) \doteq \{\rho(\xi) \,|\, \xi \in L^p_{\mathcal{G}}(\mathcal{F}),\ \mu(\xi) \geq Y\}$$

is downward directed in the sense that for every $\rho(\xi_1), \rho(\xi_2) \in \mathcal{A}_\mu(Y)$ there exists $\rho(\xi^*) \in \mathcal{A}_\mu(Y)$ such that $\rho(\xi^*) \leq \min\{\rho(\xi_1), \rho(\xi_2)\}$.

In addition, if $\mathcal{R}(Y, \mu) < \alpha$ for some $\alpha \in L^0(\mathcal{G})$ then there exists ξ such that $\mu(\xi) \geq Y$ and $\rho(\xi) < \alpha$.
iv) For every $A \in \mathcal{G}$, $(Y, \mu) \in \Sigma_{\mathcal{R}}$

$$\mathcal{R}(Y, \mu)\mathbf{1}_A = \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi)\mathbf{1}_A \mid Y \geq \mu(X)\} \tag{12}$$

$$= \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi)\mathbf{1}_A \mid Y\mathbf{1}_A \geq \mu(X\mathbf{1}_A)\} = \mathcal{R}(Y\mathbf{1}_A, \mu)\mathbf{1}_A \tag{13}$$

v) For every $X_1, X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$
(a) $\mathcal{R}(\mu(X_1), \mu) \wedge \mathcal{R}(\mu(X_2), \mu) = \mathcal{R}(\mu(X_1) \wedge \mu(X_2), \mu)$
(b) $\mathcal{R}(\mu(X_1), \mu) \vee \mathcal{R}(\mu(X_2), \mu) = \mathcal{R}(\mu(X_1) \vee \mu(X_2), \mu)$
vi) The map $\mathcal{R}(\mu(X), \mu)$ is quasi-affine with respect to X in the sense that for every $X_1, X_2 \in L^p_{\mathcal{G}}(\mathcal{F})$, $\Lambda \in L^0(\mathcal{G})$ and $0 \leq \Lambda \leq 1$, we have
$\mathcal{R}(\mu(\Lambda X_1 + (1 - \Lambda)X_2), \mu) \geq \mathcal{R}(\mu(X_1), \mu) \wedge \mathcal{R}(\mu(X_2), \mu)$ (quasiconcavity)
$\mathcal{R}(\mu(\Lambda X_1 + (1 - \Lambda)X_2), \mu) \leq \mathcal{R}(\mu(X_1), \mu) \vee \mathcal{R}(\mu(X_2), \mu)$ (quasiconvexity).
vii) $\inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu_1) = \inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu_2)$ for every $\mu_1, \mu_2 \in L^q_{\mathcal{G}}(\mathcal{F})$.

4.2 Proofs of the complete duality stated in Section 3.1

We need some preliminary results

Lemma 21 *Let ρ be (REG). The function R defined in (8) belongs to $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$*

Proof. We check the items in Definition 14.
i) and ii) can be easily shown.
iii) Observe that $R(Y, Q) \geq \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \rho(\xi)$, for all $(Y, Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q$, so that

$$\inf_{Y \in L^0(\mathcal{G})} R(Y, Q) \geq \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \rho(\xi).$$

Conversely notice that the set $\{\rho(\xi) \mid \xi \in L^p_{\mathcal{G}}(\mathcal{F})\}$ is downward directed and then there exists $\rho(\xi_n) \downarrow \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \rho(\xi)$. For every $Q \in \mathcal{P}^q$ we have

$$\rho(\xi_n) \geq R\left(E\left[-\xi_n \frac{dQ}{d\mathbb{P}} |\mathcal{G}\right], Q\right) \geq \inf_{Y \in L^0(\mathcal{G})} R(Y, Q)$$

and therefore

$$\inf_{Y\in L^0(\mathcal{G})} R(Y,Q) \le \inf_{\xi\in L^p_{\mathcal{G}}(\mathcal{F})} \rho(\xi).$$

iv) For $\alpha \in L^0(\mathcal{G})$ and $A \in \mathcal{G}$ define $U^A_\alpha = \{(Y,Q) \in L^0(\mathcal{G})\times\mathcal{P}^q \mid R(Y,Q) \ge \alpha \text{ on } A\}$, and suppose $\emptyset \ne U^A_\alpha \ne L^0(\mathcal{G})\times\mathcal{P}^q$. Let $(Y^*,Q^*) \in L^0(\mathcal{G})\times\mathcal{P}^q$ such that $R(Y^*,Q^*) < \alpha$ on A.

As mentioned in Lemma 20 (iii) there exists $X^* \in L^p_{\mathcal{G}}(\mathcal{F})$ such that $E[-X^*\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}] \ge Y^*$ and $\rho(X^*) < \alpha$ on A. Since $R(Y,Q) \ge \alpha$ on A for every $(Y,Q) \in U^A_\alpha$, we deduce that $E[-X^*\frac{dQ}{d\mathbb{P}}|\mathcal{G}] < Y$ on A, for every $(Y,Q) \in U_\alpha$. Otherwise we could define $B = \{\omega \in A \mid E[-X^*\frac{dQ}{d\mathbb{P}}|\mathcal{G}] \ge Y\} \subseteq A$, $\mathbb{P}(B) > 0$ and then Lemma 20 (iv) would imply $R(Y\mathbf{1}_B,Q) < \alpha$ on B.

Finally we can conclude that for every $(Y,Q) \in U^A_\alpha$

$$Y^* + E\left[X^*\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}\right] \le 0 < Y + E\left[X^*\frac{dQ}{d\mathbb{P}}|\mathcal{G}\right] \text{ on } A.$$

v) Take $Q_1, Q_2 \in \mathcal{P}^q$ and define $F = \{R(E[X\frac{dQ_1}{d\mathbb{P}}|\mathcal{G}], Q_1) \ge R(E[X\frac{dQ_2}{d\mathbb{P}}|\mathcal{G}], Q_2)\}$ and let $\widehat{Q}$ given by

$$\frac{d\widehat{Q}}{d\mathbb{P}} := \mathbf{1}_F\frac{dQ_1}{d\mathbb{P}} + \mathbf{1}_{F^C}\frac{dQ_2}{d\mathbb{P}} \in \mathcal{P}^q.$$

It is easy to show, using an argument similar to the one in [FM11b], Lemma 3.5 v) that

$$R\left(E\left[X\frac{d\widehat{Q}}{d\mathbb{P}}|\mathcal{G}\right], \widehat{Q}\right) = R\left(E\left[X\frac{dQ_1}{d\mathbb{P}}|\mathcal{G}\right], Q_1\right) \vee R\left(E\left[X\frac{dQ_2}{d\mathbb{P}}|\mathcal{G}\right], Q_2\right),$$

which shows that the set $\left\{R(E[X\frac{dQ}{d\mathbb{P}}|\mathcal{G}],Q) \mid Q \in \mathcal{P}^q\right\}$ is upward directed.

vi) It follows from the same argument used in [FM11b], Lemma 3.5 iv). ■

Lemma 22 *If $Q \in \mathcal{P}^q$ and if ρ is ($\downarrow$ MON) and (REG) then*

$$R(Y,Q) = \inf_{\xi\in L^p_{\mathcal{G}}(\mathcal{F})}\left\{\rho(\xi) \mid E\left[-\xi\frac{dQ}{d\mathbb{P}}|\mathcal{G}\right] = Y\right\}. \tag{14}$$

Proof. For sake of simplicity we denote by $\mu(\cdot) = E[\cdot\frac{dQ}{d\mathbb{P}}|\mathcal{G}]$ and $r(Y,\mu)$ the right hand side of equation (14). Notice that $R(Y,\mu) \le r(Y,\mu)$. By contradiction, suppose that $\mathbb{P}(A) > 0$ where $A =: \{R(Y,\mu) < r(Y,\mu)\}$. From Lemma 20, there exists a r.v. $\xi \in L^p_{\mathcal{G}}(\mathcal{F})$ satisfying the following conditions

- $\mu(-\xi) \ge Y$ and $\mathbb{P}(\mu(-\xi) > Y) > 0$.
- $R(Y,\mu)(\omega) \le \rho(\xi)(\omega) < r(Y,\mu)(\omega)$ for $\mathbb{P}$-almost every $\omega \in A$.

Then $Z := \mu(-\xi) - Y \in L^0(\mathcal{G}) \subseteq L^p_{\mathcal{G}}(\mathcal{F})$ satisfies $Z \ge 0$, $\mathbb{P}(Z > 0) > 0$ and, thanks to ($\downarrow$ MON), $\rho(\xi) \ge \rho(\xi + Z)$. From $\mu(-(\xi+Z)) = Y$ we deduce:

$$R(Y,\mu)(\omega) \le \rho(\xi)(\omega) < r(Y,\mu)(\omega) \le \rho(\xi+Z)(\omega) \text{ for } \mathbb{P}\text{-a.e. } \omega \in A,$$

which is a contradiction. ■

4.2.1 Proof of Theorem 15

During the whole proof we fix an arbitrary $X \in L^p_{\mathcal{G}}(\mathcal{F})$.

ONLY IF. For the proof of the 'only if 'we here repeat for sake of completeness some arguments used in [FM12].
There might exist a set $A \in \mathcal{G}$ on which the map ρ is constant, in the sense that $\rho(\xi)\mathbf{1}_A = \rho(\eta)\mathbf{1}_A$ for every $\xi, \eta \in E$. For this reason we introduce

$$\mathcal{A} := \{B \in \mathcal{G} \mid \rho(\xi)\mathbf{1}_B = \rho(\eta)\mathbf{1}_B \; \forall \xi, \eta \in L^p_{\mathcal{G}}(\mathcal{F})\}.$$

Applying Lemma 25 in Appendix with $F := \left\{\rho(\xi) - \rho(\eta) \mid \xi, \eta \in L^p_{\mathcal{G}}(\mathcal{F})\right\}$ (we consider the convention $+\infty - \infty = 0$) and $Y_0 = 0$ we can deduce the existence of two maximal sets $A \in \mathcal{G}$ and $A^\vdash \in \mathcal{G}$ for which $P(A \cap A^\vdash) = 0$, $P(A \cup A^\vdash) = 1$ and

$$\begin{aligned} \rho(\xi) = \rho(\eta) \quad & \text{on } A \quad \text{for every } \xi, \eta \in L^p_{\mathcal{G}}(\mathcal{F}), \\ \rho(\zeta_1) < \rho(\zeta_2) \quad & \text{on } A^\vdash \quad \text{for some } \zeta_1, \zeta_2 \in L^p_{\mathcal{G}}(\mathcal{F}). \end{aligned} \tag{15}$$

Recall that $\Upsilon_\rho \in \mathcal{G}$ is the maximal set on which $\rho(\xi)\mathbf{1}_{\Upsilon_\rho} = +\infty\mathbf{1}_{\Upsilon_\rho}$ for every $\xi \in L^p_{\mathcal{G}}(\mathcal{F})$ and T_ρ its complement. Notice that $\Upsilon_\rho \subset A$.
Fix $X \in L^p_{\mathcal{G}}(\mathcal{F})$ and $G = \{\rho(X) < +\infty\}$. For every $\varepsilon \in L^0_{++}(\mathcal{G})$ we set

$$Y_\varepsilon =: 0\mathbf{1}_{\Upsilon_\rho} + \rho(X)\mathbf{1}_{A\setminus\Upsilon_\rho} + (\rho(X) - \varepsilon)\mathbf{1}_{G\cap A^\vdash} + \varepsilon\mathbf{1}_{G^C\cap A^\vdash} \tag{16}$$

and for every $\varepsilon \in L^0(\mathcal{G})_{++}$ we set the evenly convex set

$$\mathcal{C}_\varepsilon =: \{\xi \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \rho(\xi)\mathbf{1}_{T_\rho} \leq Y_\varepsilon\} \neq \emptyset$$

This may be separated from X by $\mu_\varepsilon \in L^q_{\mathcal{G}}(\mathcal{F})$ i.e.

$$\mu_\varepsilon(X) > \mu_\varepsilon(\xi) \quad \text{on } D_{\mathcal{C}_\varepsilon}, \; \forall \xi \in \mathcal{C}_\varepsilon. \tag{17}$$

Let $\eta \in L^p_{\mathcal{G}}(\mathcal{F})$, $\eta \geq 0$. If $\xi \in \mathcal{C}_\varepsilon$ then ($\downarrow$ MON) implies $\xi + n\eta \in \mathcal{C}_\varepsilon$ for every $n \in \mathbb{N}$. Since $\mu_\varepsilon(\cdot) = E[Z_\varepsilon \cdot |\mathcal{G}]$ for some $Z_\varepsilon \in L^q_{\mathcal{G}}(\mathcal{F})$, from (17) we deduce that the following holds on the set $D_{\mathcal{C}_\varepsilon}$:

$$E[Z_\varepsilon(\xi + n\eta) \mid \mathcal{G}] < E[Z_\varepsilon X \mid \mathcal{G}] \Longrightarrow E[-Z_\varepsilon\eta \mid \mathcal{G}] > \frac{E[Z_\varepsilon(\xi - X) \mid \mathcal{G}]}{n}, \quad \forall n \in \mathbb{N}$$

i.e. $E[Z_\varepsilon\eta \mid \mathcal{G}]\mathbf{1}_{D_{\mathcal{C}_\varepsilon}} \leq 0$ for every $\eta \in L^p_{\mathcal{G}}(\mathcal{F})$, $\eta \geq 0$. This implies, as $\mathbf{1}_{\{Z_\varepsilon > 0\}} \in L^p_{\mathcal{G}}(\mathcal{F})$, that $Z_\varepsilon\mathbf{1}_{D_{\mathcal{C}_\varepsilon}} \leq 0$.

We now show that $Z_\varepsilon < 0$ on $D_{\mathcal{C}_\varepsilon}$. Suppose there existed a $\mathcal{G}$-measurable set $G \subset D_{\mathcal{C}_\varepsilon}$, $\mathbb{P}(G) > 0$, on which $Z_\varepsilon = 0$ and fix $\xi \in \mathcal{C}_\varepsilon$. From $E[Z_\varepsilon\xi \mid \mathcal{G}] < E[Z_\varepsilon X \mid \mathcal{G}]$ on $D_{\mathcal{C}_\varepsilon}$ we can find a $\delta_\xi \in L^0_{++}(\mathcal{G})$ such that $E[Z_\varepsilon\xi \mid \mathcal{G}] + \delta_\xi < E[Z_\varepsilon X \mid \mathcal{G}]$ on $D_{\mathcal{C}_\varepsilon}$ which implies

$$\delta_\xi\mathbf{1}_G = E[Z_\varepsilon\mathbf{1}_G\xi \mid \mathcal{G}] + \delta_\xi\mathbf{1}_G \leq E[Z_\varepsilon\mathbf{1}_G X \mid \mathcal{G}] = 0.$$

which is a contradiction since $\mathbb{P}(\delta_\xi\mathbf{1}_G > 0) > 0$.
We deduce that $E[Z_\varepsilon\mathbf{1}_B] = E[E[Z_\varepsilon \mid \mathcal{G}]\mathbf{1}_B] < 0$ for every $B \in \mathcal{G}$, $B \subseteq D_{\mathcal{C}_\varepsilon}$ and then $E[Z_\varepsilon \mid \mathcal{G}] < 0$ on $D_{\mathcal{C}_\varepsilon}$. Consider any $W \in L^q_{\mathcal{G}}(\mathcal{F})$ with $\mathbb{P}(W > 0) = 1$, $E[W \mid \mathcal{G}] = 1$ and define $\frac{dQ_\varepsilon}{d\mathbb{P}} \in L^1(\mathcal{F})$ as

$$\frac{dQ_\varepsilon}{d\mathbb{P}} = \frac{Z_\varepsilon}{E[Z_\varepsilon \mid \mathcal{G}]}\mathbf{1}_{D_{\mathcal{C}_\varepsilon}} + W\mathbf{1}_{D^\vdash_{E[Z_\varepsilon|\mathcal{G}]}}.$$

Following the idea of the proof of Theorem 11 (see [FM12]) we can easily deduce that

$$\begin{aligned} \rho(X) \quad & \geq \quad \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \{\rho(\xi) \mid \mu_\varepsilon(\xi) \geq \mu_\varepsilon(X)\} \\ & \geq \quad \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{\rho(\xi) \mid E\left[-\xi\frac{dQ_\varepsilon}{d\mathbb{P}}|\mathcal{G}\right] \geq E\left[-X\frac{dQ_\varepsilon}{d\mathbb{P}}|\mathcal{G}\right]\right\} \\ & \geq \quad (\rho(X) - \varepsilon)\mathbf{1}_G + \varepsilon\mathbf{1}_{G^C} \end{aligned} \tag{18}$$

and hence

$$\rho(X) = \sup_{Q \in \mathcal{P}^q} \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{\rho(\xi) \mid E\left[-\xi\frac{dQ}{d\mathbb{P}}|\mathcal{G}\right] \geq E\left[-X\frac{dQ}{d\mathbb{P}}|\mathcal{G}\right]\right\}.$$

Applying Lemma 22 we can substitute $=$ in the constraint.

IF. We assume that $\rho(X) = \sup_{Q\in\mathcal{P}^q} R(E[-X\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}], Q)$ holds for some $R \in \mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$. Since R is monotone in the first component and $R(Y\mathbf{1}_A, Q)\mathbf{1}_A = R(Y, Q)\mathbf{1}_A$ for every $A \in \mathcal{G}$ we easily deduce that ρ is (MON) and (REG). We need to show that ρ is (EVQ).
Let $\mathcal{V}_\alpha = \{\xi \in L^p_{\mathcal{G}}(\mathcal{F}) \mid \rho(\xi)\mathbf{1}_{T_\rho} \leq \alpha\}$ where $\alpha \in L^0(\mathcal{G})$ and recall that $D_{\mathcal{V}_\alpha}$ is the complementary of the set provided in Definition 4. Notice that $D_{\mathcal{V}_\alpha} \subseteq T_\rho$. Take $X^* \in L^p_{\mathcal{G}}(\mathcal{F})$ satisfying $X^*\mathbf{1}_A \cap V_\alpha \mathbf{1}_A = \emptyset$ for every $A \in \mathcal{G}$, $A \subseteq D_{\mathcal{V}_\alpha}$, $P(A) > 0$. Hence

$$\rho(X^*) = \sup_{Q\in\mathcal{P}^q} R(E[-X^*\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}], Q) > \alpha$$

on the set $D_{\mathcal{V}_\alpha}$. Since the set $\{R(E[-X^*\frac{dQ}{d\mathbb{P}}|\mathcal{G}], Q) \mid Q \in \mathcal{P}^q\}$ is upward directed there exists $\{Q_m\} \subset \mathcal{P}^q$ s.t.

$$R\left(E\left[-X^*\frac{dQ_m}{d\mathbb{P}} \mid \mathcal{G}\right], Q_m\right) \uparrow \rho(X^*) \quad \text{ as } m \uparrow +\infty.$$

Let $\delta \in L^0_{++}(\mathcal{G})$ satisfies $\delta < \rho(X) - \alpha$ and consider the sets

$$F_m = \left\{R(E\left[-X^*\frac{dQ_m}{d\mathbb{P}}|\mathcal{G}\right], Q_m) > \rho(X) - \delta\right\}$$

and the partition of Ω given by $G_1 = F_1$ and $G_m = F_m \setminus G_{m-1}$. We have from the properties of the module $L^q_{\mathcal{G}}(\mathcal{F})$ that

$$\frac{dQ^*}{d\mathbb{P}} = \sum_{m=1}^{\infty} \frac{dQ_m}{d\mathbb{P}}\mathbf{1}_{G_m} \in L^q_{\mathcal{G}}(\mathcal{F})$$

and then $Q^* \in \mathcal{P}^q$ with $R(E[-X^*\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}], Q^*) > \alpha$ on the set $D_{\mathcal{V}_\alpha}$.

Let $\xi \in V_\alpha$. It remains to show that this Q^* separates X^* from V_α on the set $D_{\mathcal{V}_\alpha}$. If there existed $A \subseteq D_{\mathcal{V}_\alpha} \in \mathcal{G}$ such that $E[\xi\frac{dQ^*}{d\mathbb{P}}\mathbf{1}_A|\mathcal{G}] \leq E[X^*\frac{dQ^*}{d\mathbb{P}}\mathbf{1}_A|\mathcal{G}]$ on A then $\rho(\xi\mathbf{1}_A) \geq R(E[-\xi\frac{dQ^*}{d\mathbb{P}}\mathbf{1}_A|\mathcal{G}], Q^*) \geq R(E[-X^*\frac{dQ^*}{d\mathbb{P}}\mathbf{1}_A|\mathcal{G}], Q^*) > \alpha$ on A. This implies $\rho(\xi) > \alpha$ on A which is a contradiction unless $\mathbb{P}(A) = 0$. Hence $E[\xi\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}] > E[X^*\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}]$ on $D_{\mathcal{V}_\alpha}$ for every $\xi \in V_\alpha$.

UNIQUENESS. First we need the following preliminary result. Define the set

$$\mathcal{A}(Y, Q) = \left\{\xi \in L^p_{\mathcal{G}}(\mathcal{F}) \mid E\left[-\xi\frac{dQ}{d\mathbb{P}}|\mathcal{G}\right] \geq Y\right\}.$$

Lemma 23 *If $K \in \mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$, then for each $(Y^*, Q^*) \in L^0(\mathcal{G}) \times \mathcal{P}^q$*

$$K(Y^*, Q^*) = \sup_{Q\in\mathcal{P}^q} \inf_{X\in\mathcal{A}(Y^*,Q^*)} K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathcal{G}\right], Q\right) \tag{19}$$

Proof. Consider

$$\psi(Q, Q^*, Y^*) = \inf_{X\in\mathcal{A}(Y^*,Q^*)} K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathcal{G}\right], Q\right)$$

Notice that $E[-X\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}] \geq Y^*$ for every $X \in \mathcal{A}(Y^*, Q^*)$ implies

$$\psi(Q^*, Q^*, Y^*) = \inf_{X\in\mathcal{A}(Y^*,Q^*)} K\left(E\left[-X\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}\right], Q^*\right) \geq K(Y^*, Q^*)$$

On the other hand $E[Y^*\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}] = Y^*$ so that $-Y^* \in \mathcal{A}(Y^*, Q^*)$ and the second inequality is actually an equality

$$\psi(Q^*, Q^*, Y^*) \leq K\left(E\left[-(-Y^*)\frac{dQ^*}{d\mathbb{P}}|\mathcal{G}\right], Q^*\right) = K(Y^*, Q^*).$$

If we show that $\psi(Q, Q^*, Y^*) \leq \psi(Q^*, Q^*, Y^*)$ for every $Q \in \mathcal{P}^q$ then (19) is proved. To this aim we define

$$\mathcal{A} := \left\{ A \in \mathcal{G} \mid E\left[X \frac{dQ^*}{d\mathbb{P}} | \mathcal{G}\right] 1_A = E\left[X \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right] 1_A, \ \ \forall\, X \in L^p_{\mathcal{G}}(\mathcal{F}) \right\}$$

For every $A \in \mathcal{A}$ and every $X \in L^p_{\mathcal{G}}(\mathcal{F})$

$$\begin{aligned} K\left(E\left[-X \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right], Q\right) \mathbf{1}_A &= K\left(E\left[-X \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right] \mathbf{1}_A, Q\right) \mathbf{1}_A \\ = K\left(E\left[-X \frac{dQ^*}{d\mathbb{P}} | \mathcal{G}\right] \mathbf{1}_A, Q^*\right) \mathbf{1}_A &= K\left(E\left[-X \frac{dQ^*}{d\mathbb{P}} | \mathcal{G}\right], Q^*\right) \mathbf{1}_A \end{aligned}$$

which implies

$$\psi(Q, Q^*, Y^*)\mathbf{1}_A = \psi(Q^*, Q^*, Y^*)\mathbf{1}_A. \tag{20}$$

Notice that $\mathcal{A} = \{A \in \mathcal{G} \mid Y = 0 \text{ on } A, \forall Y \in F\}$, where

$$F := \left\{ E\left[X \frac{dQ^*}{d\mathbb{P}} | \mathcal{G}\right] - E\left[X \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right] \mid X \in L^p_{\mathcal{G}}(\mathcal{F}) \right\}.$$

As the conditional expectation is (REG), we may apply Lemma 25 and deduce the existence of two maximal sets $A_M \in \mathcal{A}$ and $A^{\vdash}_M \in \mathcal{A}^{\vdash}$ such that: $P(A_M \cap A^{\vdash}_M) = 0$, $P(A_M \cup A^{\vdash}_M) = 1$; $E\left[X \frac{dQ^*}{d\mathbb{P}} | \mathcal{G}\right] \mathbf{1}_{A_M} = E\left[X \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right] \mathbf{1}_{A_M}$, $\forall\, X \in L^p_{\mathcal{G}}(\mathcal{F})$; and $E\left[-X^* \frac{dQ^*}{d\mathbb{P}} | \mathcal{G}\right] \neq E\left[-X^* \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right]$ on $A^{\vdash}_M$, for some $X^* \in L^p_{\mathcal{G}}(\mathcal{F})$. Considering $A_M \in \mathcal{A}$ we then deduce from (20)

$$\psi(Q, Q^*, Y^*)\mathbf{1}_{A_M} = \psi(Q^*, Q^*, Y^*)\mathbf{1}_{A_M}.$$

Now we consider $A^{\vdash}_M \in \mathcal{A}^{\vdash}$ and define $Z^* := X^* - E\left[-X^* \frac{dQ^*}{d\mathbb{P}} | \mathcal{G}\right]$. Surely $E\left[Z^* \frac{dQ^*}{d\mathbb{P}} | \mathcal{G}\right] = 0$ and $E\left[Z^* \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right] \neq 0$ on $A^{\vdash}_M$. We deduce that for every $\alpha \in L^0(\mathcal{G})$, $-Y^* + \alpha Z^* \in \mathcal{A}(Y^*, Q^*)$. Also notice that any $Y \in L^0(\mathcal{G})$ can be written as $Y\mathbf{1}_{A^{\vdash}_M} = E[(-Y^* + \alpha_Y Z^*) \frac{dQ}{d\mathbb{P}} | \mathcal{G}] \mathbf{1}_{A^{\vdash}_M}$, with $\alpha_Y \in L^0(\mathcal{G})$. Finally

$$\begin{aligned} \psi(Q, Q^*, Y^*)\mathbf{1}_{A^{\vdash}_M} &\leq \inf_{\alpha \in L^0(\mathcal{G})} K\left(E\left[-(-Y^* + \alpha Z^*) \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right], Q\right) \mathbf{1}_{A^{\vdash}_M} \\ &= \inf_{Y \in L^0(\mathcal{G})} K\left(Y \mathbf{1}_{A^{\vdash}_M}, Q\right) \mathbf{1}_{A^{\vdash}_M} = \inf_{Y \in L^0(\mathcal{G})} K\left(Y \mathbf{1}_{A^{\vdash}_M}, Q^*\right) \mathbf{1}_{A^{\vdash}_M} \\ &\leq K(Y^*, Q^*)\, \mathbf{1}_{A^{\vdash}_M}. \end{aligned}$$

As $P(A_M \cup A^{\vdash}_M) = 1$, we conclude that $\psi(Q, Q^*, Y^*) \leq \psi(Q^*, Q^*, Y^*) = K(Y^*, Q^*)$ and the claim is proved. ■

To prove the uniqueness we show that for every $K \in \mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ such that

$$\rho(X) = \sup_{Q \in \mathcal{P}^q} K\left(E\left[-X \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right], Q\right),$$

K must satisfy

$$K(Y, Q) = \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \left\{ \rho(\xi) \mid E\left[-\xi \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right] \geq Y \right\}.$$

By the Lemma 23

$$\begin{aligned} K(Y^*, Q^*) &= \sup_{Q \in \mathcal{P}^q} \inf_{X \in \mathcal{A}(Y^*, Q^*)} K\left(E\left[-X \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right], Q\right) \\ &\leq \inf_{X \in \mathcal{A}(Y^*, Q^*)} \sup_{Q \in \mathcal{P}^q} K\left(E\left[-X \frac{dQ}{d\mathbb{P}} | \mathcal{G}\right], Q\right) = \inf_{X \in \mathcal{A}(Y^*, Q^*)} \rho(X). \end{aligned}$$

It remains to prove the reverse inequality, i.e.

$$K(Y^*,Q^*) \le \inf_{X\in\mathcal{A}(Y^*,Q^*)} \rho(X). \tag{21}$$

Consider the class:

$$\mathcal{A} := \left\{A \in \mathcal{G} \mid K(Y,Q)\mathbf{1}_A \le K(Y^*,Q^*)\mathbf{1}_A \;\forall\,(Y,Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q\right\}.$$

Notice that $\mathcal{A} = \{A \in \mathcal{G} \mid Z \le Z_0 \text{ on } A, \forall Z \in F\}$, where

$$F := \left\{K(Y,Q) \mid (Y,Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q\right\}.$$

and $Z_0 = K(Y^*,Q^*)$. In order to apply Lemma 25, let $A_i \in \mathcal{A}^{\vdash}$ be a sequence of disjoint sets and $Z_i = K(Y_i,Q_i)$ be the corresponding element in F. From $K(Y\mathbf{1}_A,Q)\mathbf{1}_A = K(Y,Q)\mathbf{1}_A$ we deduce (as in Remark 9 i) that $K(\sum_{i=1}^{\infty} Y_i\mathbf{1}_{A_i},Q_j)\mathbf{1}_{A_\infty} = \sum_{i=1}^{\infty} K(Y_i,Q_j)\mathbf{1}_{A_i}$, with $A_\infty = \cup A_i$. From $K(Y,Q_1)\mathbf{1}_A = K(Y,Q_2)\mathbf{1}_A$ if $\frac{dQ_1}{d\mathbb{P}}\mathbf{1}_A = \frac{dQ_2}{d\mathbb{P}}\mathbf{1}_A$, $A \in \mathcal{G}$, we obtain

$$\begin{aligned} K\Big(\sum_{i=1}^{\infty} Y_i\mathbf{1}_{A_i}, \sum_{j=1}^{\infty} Q_j\mathbf{1}_{A_j}\Big)\mathbf{1}_{A_\infty} &= \sum_{i=1}^{\infty} K\Big(Y_i, \sum_{j=1}^{\infty} Q_j\mathbf{1}_{A_j}\Big)\mathbf{1}_{A_i} \\ &= \sum_{i=1}^{\infty} K(Y_i,Q_i)\mathbf{1}_{A_i} = \sum_{i=1}^{\infty} Z_i\mathbf{1}_{A_i} \end{aligned}$$

showing that $\sum_{i=1}^{\infty} Z_i\mathbf{1}_{A_i} \in F$. From Lemma 25 we may deduce the existence of two maximal sets $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$ such that: $P(A_M \cap A_M^{\vdash}) = 0$, $P(A_M \cup A_M^{\vdash}) = 1$; $K(Y,Q)\mathbf{1}_{A_M} \le K(Y^*,Q^*)\mathbf{1}_{A_M}$ $\forall\,(Y,Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q$; and

$$K(Y^*,Q^*) < K(\overline{Y},\overline{Q}) \text{ on } A_M^{\vdash}, \tag{22}$$

for some $(\overline{Y},\overline{Q}) \in L^0(\mathcal{G}) \times \mathcal{P}^q$. On $A_M \in \mathcal{A}$ the inequality (21) is obviously true and we need only to show it on the set $A_M^{\vdash}$.

From (22) we can easily build a $\beta \in L^0(\mathcal{G})$ such that $K(Y^*,Q^*) < \beta \le K(\overline{Y},\overline{Q})$ on $A_M^{\vdash}$ and β is arbitrarily close to $K(Y^*,Q^*)$ on $A_M^{\vdash}$. An example of such β is obtained by taking $\lambda \downarrow 0$ in the family:

$$\begin{aligned} \beta_\lambda &:= 1_{A_M^{\vdash}}\left[\lambda K(\overline{Y},\overline{Q}) + (1-\lambda)K(Y^*,Q^*)\right] 1_{\{K(\overline{Y},\overline{Q})<\infty\}\cap\{K(Y^*,Q^*)>-\infty\}} \\ &+ 1_{A_M^{\vdash}} 1_{\{K(\overline{Y},\overline{Q})=\infty\}}\left[(K(Y^*,Q^*)+\lambda)1_{\{K(Y^*,Q^*)>-\infty\}} - \frac{1}{\lambda}1_{\{K(Y^*,Q^*)=-\infty\}}\right]. \end{aligned}$$

Since the set $U_\beta := \{(Y,Q) \in L^0(\mathcal{G}) \times \mathcal{P}^q \mid K(Y,Q) \ge \beta \text{ on } A_M^{\vdash}\}$ is not empty, the assumption that K is $\diamond$-evenly $L^0(\mathcal{G})$-quasiconcave implies the existence of $(S^*,X^*) \in L^0_{++}(\mathcal{G}) \times L^p_{\mathcal{G}}(\mathcal{F})$ with

$$Y^*S^* + E\left[X^*\frac{dQ^*}{d\mathbb{P}} \mid \mathcal{G}\right] < YS^* + E\left[X^*\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] \text{ on } A_M^{\vdash}$$

for every $(Y,Q) \in U_\beta$.

We claim that for every $(Y,Q) \in U_\beta$

$$Y + E\left[\widehat{X}\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] > 0 \text{ on } A_M^{\vdash}\ ,$$

where $\widehat{X} := \frac{X^*}{S^*} + \Lambda$ and $\Lambda := -Y^* - E[\frac{X^*}{S^*}\frac{dQ^*}{d\mathbb{P}} \mid \mathcal{G}]$. Indeed, for every $(Y,Q) \in U_\beta$

$$\begin{aligned} & Y^*S^* + E\left[X^*\frac{dQ^*}{d\mathbb{P}} \mid \mathcal{G}\right] < YS^* + E\left[X^*\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] \text{ on } A_M^{\vdash}\ , \\ \text{implies}\quad & Y^* + E\left[\left(\frac{X^*}{S^*}+\Lambda\right)\frac{dQ^*}{d\mathbb{P}} \mid \mathcal{G}\right] < Y + E\left[\left(\frac{X^*}{S^*}+\Lambda\right)\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] \text{ on } A_M^{\vdash}\ , \\ \text{implies}\quad & Y^* + E\left[\widehat{X}\frac{dQ^*}{d\mathbb{P}} \mid \mathcal{G}\right] < Y + E\left[\widehat{X}\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] \text{ on } A_M^{\vdash}\ , \end{aligned}$$

i.e. the claim holds, as $E[\widehat{X}\frac{dQ^*}{d\mathbb{P}} \mid \mathcal{G}] = -Y^*$.

For every $Q \in \mathcal{P}^q$ define $Y_Q := E\left[-\widehat{X}\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right]$. We show that

$$K(Y_Q, Q) < \beta \text{ on } A_M^{\vdash}. \tag{23}$$

Suppose by contradiction that there exists $B \subseteq A_M^{\vdash}$, $B \in \mathcal{G}$, $P(B) > 0$, such that $K(Y_Q, Q) \geq \beta$ on B. Take $(Y_1, Q_1) \in U_\beta$ and define $\widetilde{Y} := Y_Q\mathbf{1}_B + Y_1\mathbf{1}_{B^C}$ and $\widetilde{Q} \in \mathcal{P}^q$ by

$$\frac{d\tilde{Q}}{d\mathbb{P}} = \frac{dQ}{d\mathbb{P}}\mathbf{1}_B + \frac{dQ_1}{d\mathbb{P}}\mathbf{1}_{B^C}.$$

Thus $K(\widetilde{Y}, \widetilde{Q}) \geq \beta$ on $A_M^{\vdash}$ and $\widetilde{Y} + E\left[\widehat{X}\frac{d\tilde{Q}}{d\mathbb{P}} \mid \mathcal{G}\right] > 0$ on $A_M^{\vdash}$, which implies $Y_Q + E\left[\widehat{X}\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] > 0$ on B and this is impossible and (23) is proven.

Since $\widehat{X} \in \mathcal{A}(Y^*, Q^*)$ we can conclude that

$$\begin{aligned} K(Y^*, Q^*)\mathbf{1}_{A_M^{\vdash}} &\leq \inf_{X \in \mathcal{A}(Y^*,Q^*)} \sup_{Q \in \mathcal{P}^q} K\left(E\left[-X\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right], Q\right)\mathbf{1}_{A_M^{\vdash}} \\ &\leq \sup_{Q \in \mathcal{P}^q} K\left(E\left[-\widehat{X}\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right], Q\right)\mathbf{1}_{A_M^{\vdash}} \leq \beta\mathbf{1}_{A_M^{\vdash}}. \end{aligned}$$

As β is arbitrarily close to $K(Y^*, Q^*)$, the equality must hold and then we obtain:

$$K(Y^*, Q^*) = \inf_{X \in \mathcal{A}(Y^*,Q^*)} \rho(X) \text{ on } A_M^{\vdash}.$$

This concludes the proof of Theorem 15.

5 Appendix

Remark 24 *By Lemma 2.9 in [FKV09], we know that any non-empty class $\mathcal{A}$ of subsets of a sigma algebra $\mathcal{G}$ has a supremum* $\operatorname{ess.sup}\{\mathcal{A}\} \in \mathcal{G}$ *and that if $\mathcal{A}$ is closed with respect to finite union (i.e. $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}$) then there is a sequence $A_n \in \mathcal{A}$ such that* $\operatorname{ess.sup}\{\mathcal{A}\} = \bigcup_{n\in\mathbb{N}} A_n$*. Obviously, if $\mathcal{A}$ is closed with respect to countable union then* $\operatorname{ess.sup}\{\mathcal{A}\} = \bigcup_{n\in\mathbb{N}} A_n := A_M \in \mathcal{A}$ *is the maximal element in $\mathcal{A}$.*

The next Lemma is used several times in the proofs of the paper. It says that for any subset $F \subset L^0(\mathcal{G})$ that is "closed w.r.to pasting" it is possible to determine a maximal set $A_M \in \mathcal{G}$ (which may have zero probability) such that $Y1_{A_M} \geq 0\ \forall Y \in F$ and one element $\overline{Y} \in F$ for which $\overline{Y} < 0$ on the complement of A_M.

Lemma 25 *With the symbol $\trianglerighteq$ denote any one of the binary relations $\geq, \leq, =, >, <$ and with $\lhd$ its negation. Consider a class $F \subseteq \bar{L}^0(\mathcal{G})$ of random variables, $Y_0 \in \bar{L}^0(\mathcal{G})$ and the classes of sets*

$$\begin{aligned} \mathcal{A} :=&\ \{A \in \mathcal{G} \mid \forall Y \in F\ Y \trianglerighteq Y_0 \text{ on } A\}, \\ \mathcal{A}^{\vdash} :=&\ \{A^{\vdash} \in \mathcal{G} \mid \exists Y \in F \text{ s.t. } Y \lhd Y_0 \text{ on } A^{\vdash}\}. \end{aligned}$$

Suppose that for any sequence of disjoint sets $A_i^{\vdash} \in \mathcal{A}^{\vdash}$ and the associated r.v. $Y_i \in F$ we have $\sum_1^\infty Y_i 1_{A_i^{\vdash}} \in F$. Then there exist two maximal sets $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$ such that $P(A_M \cap A_M^{\vdash}) = 0$, $P(A_M \cup A_M^{\vdash}) = 1$ and

$$\begin{aligned} &Y \trianglerighteq Y_0 \text{ on } A_M,\ \forall Y \in F \\ &\overline{Y} \lhd Y_0 \text{ on } A_M^{\vdash}, \text{ for some } \overline{Y} \in F. \end{aligned}$$

Proof. Notice that $\mathcal{A}$ and $\mathcal{A}^{\vdash}$ are closed with respect to *countable* union. This claim is obvious for $\mathcal{A}$. For $\mathcal{A}^{\vdash}$, suppose that $A_i^{\vdash} \in \mathcal{A}^{\vdash}$ and that $Y_i \in F$ satisfies $P(\{Y_i \lhd Y_0\} \cap A_i^{\vdash}) = P(A_i^{\vdash})$. Defining $B_1 := A_i^{\vdash}$, $B_i := A_i^{\vdash} \setminus B_{i-1}$, $A_\infty^{\vdash} := \bigcup_{i=1}^{\infty} A_i^{\vdash} = \bigcup_{i=1}^{\infty} B_i$ we see that B_i are disjoint elements of $\mathcal{A}^{\vdash}$ and that $Y^* := \sum_1^\infty Y_i 1_{B_i} \in F$ satisfies $P(\{Y^* \lhd Y_0\} \cap A_\infty^{\vdash}) = P(A_\infty^{\vdash})$ and so $A_\infty^{\vdash} \in \mathcal{A}^{\vdash}$.

The Remark 24 guarantees the existence of two sets $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$ such that:

(a) $P(A \cap (A_M)^C) = 0$ for all $A \in \mathcal{A}$,

(b) $P(A^{\vdash} \cap (A_M^{\vdash})^C) = 0$ for all $A^{\vdash} \in \mathcal{A}^{\vdash}$.

Obviously, $P(A_M \cap A_M^{\vdash}) = 0$, as $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$. To show that $P(A_M \cup A_M^{\vdash}) = 1$, let $D := \Omega \setminus \left\{A_M \cup A_M^{\vdash}\right\} \in \mathcal{G}$. By contradiction suppose that $P(D) > 0$. As $D \subseteq (A_M)^C$, from condition (a) we get $D \notin \mathcal{A}$. Therefore, $\exists \overline{Y} \in F$ s.t. $P(\left\{\overline{Y} \unrhd Y_0\right\} \cap D) < P(D)$, i.e. $P(\left\{\overline{Y} \lhd Y_0\right\} \cap D) > 0$. If we set $B := \left\{\overline{Y} \lhd Y_0\right\} \cap D$ then it satisfies $P(\left\{\overline{Y} \lhd Y_0\right\} \cap B) = P(B) > 0$ and, by definition of $\mathcal{A}^{\vdash}$, B belongs to $\mathcal{A}^{\vdash}$. On the other hand, as $B \subseteq D \subseteq (A_M^{\vdash})^C$, $P(B) = P(B \cap (A_M^{\vdash})^C)$, and from condition (b) $P(B \cap (A_M^{\vdash})^C) = 0$, which contradicts $P(B) > 0$. ■

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