Balanced superprojective varieties

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A B S T R A C T

We first review the definition of superprojective spaces from the functor-of-points perspective. We derive the relation between superprojective spaces and supercosets in the framework of the theory of sheaves. As an application of the geometry of superprojective spaces, we extend Donaldson's definition of balanced manifolds to supermanifolds and we derive the new conditions of a balanced supermanifold. We apply the construction to superpoints viewed as submanifolds of superprojective spaces. We conclude with a list of open issues and interesting problems that can be addressed in the present context.

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1. Introduction

Supermanifolds are rather well known in supersymmetric theories and in string theory. They provide a very natural ground to understand the supersymmetry and supergravity from a geometric point of view. Indeed, a supermanifold contains the anticommuting coordinates which are needed to construct the superfields whose natural environment is the graded algebras [1,2]. However, the best way to understand the supermanifold is using the theory of sheaves [2,3]. In the present notes we review this approach and its usefulness in theoretical physics and in particular in the latest developments (twistor string theory [4] and pure spinor string theory [5]).

In the case of twistor string theory, the target space is indeed the supermanifold $\mathbb{CP}^{(3|4)}$ which can be described in two ways: as a supercoset of the supergroup $PSU(4|4)/SU(3|4)$ or as a quotient of the quadratic hypersurface in the superspace $\mathbb{C}^{(4|4)}$ given by

$$\sum_{\alpha\dot{\alpha}} |Z^{\alpha\dot{\alpha}}|^2 + \sum_A \bar{\psi}_A \psi^A = 1 \tag{1.1}$$

where $(Z^{\alpha\dot{\alpha}}, \psi^A)$ are the supertwistor coordinates. Obviously, this equation needs a clarification: the commuting coordinates $Z^{\alpha\dot{\alpha}}$ cannot be numbers for the above equation to have a non-trivial meaning. One way to interpret the above equation is using the sheaf point of view where $Z^{\alpha\dot{\alpha}}, \psi^A$ are the generators of a sheaf of super-commuting algebra over open sets on $\mathbb{CP}^3$. In this way, the supermanifold can be viewed as

$$(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(Z^{\alpha\dot{\alpha}}, \psi^A)) \tag{1.2}$$

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and Eq. (1.1) makes sense (see also [6]). The second way is using the *functor of points*. This is a functor between the category of sets and the category of supermanifolds and, as is well explained in [7] and the forthcoming sections, it assigns a point in a supermanifold in terms of a set of coordinates. The easiest way to realize the functor of point is to map a superspace into a supermanifold and describe the latter in terms of points identified by morphisms. Concretely, this amounts to choosing a graded algebra with N generators and representing the generators of the sheaf $\mathcal{O}_{\mathbb{CP}^3}(Z^{\alpha\dot{\alpha}}, \psi^A)$ in terms of them. Then inserting this decomposition in (1.1), one gets a set of numerical equations for the coefficients of the decomposition and they can be solved or studied by the conventional means of algebraic geometry.

Of course the hypersurface (1.1) is one example of the manifold that can be realized in terms of the generators of $\mathcal{O}_{\mathbb{CP}^3}(Z^{\alpha\dot{\alpha}}, \psi^A)$ and that can be studied by means of the functor of points [8,9]. Notice that also from the supercoset point of view, the technique of the functor of point gives us a representation of the supercoset in terms of the generators of a sheaf. Indeed, by multiplying supermatrices (whose entries are the generators of the sheaf) one finds that the entries cannot be numbers and they have to be promoted to the generator of a sheaf. Therefore the multiplication between matrices and the group multiplication of a supergroup has to be understood as a morphism of a ringed space. This point of view has been emphasized by Manin [2] and recently by [7,10]. We provide here a more elementary explanation of the role of functor of point in the case of supergroup and supercosets. The purpose of this is to use the functor of point to define the superprojective spaces (as $\mathbb{CP}^{(3|4)}$ above) and to prove the isomorphism with the supercoset point of view as in the purely bosonic case.

In the second part of the paper, we develop two applications for superprojective spaces. Following the recent analysis of Donaldson [11] on balanced manifold, we extend his definition to supermanifolds.[1] One ingredient is the definition of balanced submanifold of a projective space (for example a point or a line). For that we extend the integral equation given in [11] to an integration on the supermanifold. The definition of the integral of a superform in a supermanifold is not an obvious extension since a regularization is needed. This can be done using the projection forms as illustrated in [14] and discussed in more detail in [15]. We briefly discuss this point in the text, but we refer to [16] for a more detailed account.

After discussing the general theory, we provided a simple example of the embedding of $\mathbb{P}^{1|2}$ into the superprojective space $\mathbb{P}^{2m-1|2m}$ of sections $H^0(\mathbb{P}^{1|2}, L^{\otimes m})$ where $L^{\otimes m}$ is the m-power of a line bundle L over $\mathbb{P}^{1|2}$. In this case both the base manifold and the sections $\mathbb{P}^{2m-1|2m}$ are super-Calabi–Yau spaces (in the sense that they are super-Kähler spaces with vanishing Ricci tensor and an holomorphic top form Ω_{CY}) and for those there is a natural measure for integrating superforms provided by $\Omega_{CY} \wedge \bar{\Omega}_{CY}$. It is shown that there are two types of conditions emerging from the extension of the Donaldson equations to the supermanifold case and therefore this restricts the number of supermanifolds that can be balanced subvarieties of superprojective spaces. In generalizing the analysis of Donaldson we have taken into account the extension of the Kodaira embedding theorem discussed in [17].

The second application is to consider a set of points $\mathbb{C}^{0|N}$ immersed in the superprojective space $\mathbb{P}^{1|N}$ as a subvariety. In this case we computed explicitly the general expression for the case $\mathbb{C}^{0|2}$ embedded into $\mathbb{P}^{1|2}$ and we found the condition for the balancing of a point. We found also how the supermanifold case generalizes the classical embedding condition and we argued how one can recover the classical balancing in addition to the requirements on the parameter of the superembeddings. We showed that this is tied to the choice of the integration measure for superforms.

A concluding remark: we have not explored all possible implications of our extension neither have we discussed the relation with the stability of points in the sense of Geometric Invariant Theory (GIT) [18]. Nevertheless we have found rather interesting that some applications admit a non-trivial generalization of the usual geometric setting. These results open new questions about the geometry of sheaves and their functor-of-point interpretation.

The paper is organized as follows: in Section 2 we define the supermanifolds from a sheaf theory point of view. We discuss the basic architecture and the set of morphisms. In Section 3 we define superprojective spaces and in Section 4 we provide a functor-of-point interpretation. Part of this material is a summary of notes [7]. This allows us to use the local coordinates and to define the concept of a point in a supermanifold. In Section 5, we study supergroups and superdeterminant (Berezinians) from the functor-of-point perspective needed to see the definition of superprojective space as supercosets of supergroups discussed in Section 5.1. In Section 6, we extend the construction of Donaldson to supermanifold and we define balanced supermanifolds. Finally, in Section 6.3 we discuss the balancing of points in superprojective spaces.

2. Supermanifolds

2.1. Definitions

A **super-commutative** ring is a $\mathbb{Z}_2$-graded ring $A = A_0 \oplus A_1$ such that if $i, j \in \mathbb{Z}_2$, then $a_i a_j \in A_{i+j}$ and $a_i a_j = (-1)^{i+j} a_j a_i$, where $a_k \in A_k$. Elements in A_0 (resp. A_1) are called **even** (resp. **odd**).

A **superspace** is a super-ringed space such that the stalks are local super-commutative rings (Manin–Varadarajan). Since the odd elements are nilpotent, this reduces to require that the even component reduces to a local commutative ring.

A **super-domain** $U^{p|q}$ is the super-ringed space $(U^p, \mathcal{C}^{\infty p|q})$, where $U^p \subseteq \mathbb{R}^p$ is open and $\mathcal{C}^{\infty p|q}$ is the sheaf of super-commutative rings given by:

$$V \mapsto \mathcal{C}^{\infty}(V) \left[\theta^1, \theta^2, \ldots, \theta^q \right], \tag{2.1}$$

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[1] Recently, the techniques developed by Donaldson [12] have been used to compute the Calabi–Yau metric numerically [13].

where $V \subseteq U^p$ is and $\theta^1, \theta^2, \ldots, \theta^q$ are generators of a Grassmann algebra. The grading is the natural grading in even and odd elements. The notation is taken from [19] and from the notes [7].

Every element of $\mathcal{C}^{\infty p|q}(V)$ may be written as $\sum_I f_I \theta^I$, where I is a multi-index. A **supermanifold** of dimension $p|q$ is a super-ringed space locally isomorphic, as a ringed space, to $\mathbb{R}^{p|q}$. The coordinates x_i of $\mathbb{R}^p$ are called the even coordinates (or bosonic), while the coordinates θ^j are called the odd coordinates (or fermionic). We will denote by $(M, \mathcal{O}_M)$ the supermanifold whose underlying topological space is M and whose sheaf of super-commutative rings is $\mathcal{O}_M$.

To a section s of $\mathcal{O}_M$ on an open set containing x one may associate the **value** of s in x as the unique real number $s^\sim(x)$ such that $s - s^\sim(x)$ is not invertible on every neighborhood of x. The sheaf of algebras $\mathcal{O}^\sim$, whose sections are the functions $s^\sim$, defines the structure of a differentiable manifold on M, called the **reduced manifold** and denoted $M^\sim$.

2.2. Morphisms

In order to understand the structure of supermanifolds it is useful to study their morphisms. Here we describe how a morphism of supermanifolds looks like locally. A **morphism** ψ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is given by a smooth map map $\psi^\sim$ from $X^\sim$ to $Y^\sim$ together with a sheaf map:

$$\psi_V^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\psi^{-1}(V)), \tag{2.2}$$

where V is open in Y. The homomorphisms ψ_V^* must commute with the restrictions and they must be compatible with the super-ring structure. Moreover they satisfy

$$\psi_V^*(s)^\sim = s^\sim \circ \psi^\sim.$$

We illustrate this with an example taken from [7]. Given $M = \mathbb{R}^{1|2}$, we describe a morphism ψ of M into itself such that $\psi^\sim$ is the identity. Let ψ^* be the pull-back map defined previously. We denote $\{t, \theta^1, \theta^2\}$ the coordinates on M, where t can be interpreted both as the coordinate on $M^\sim = \mathbb{R}$ or as an even section of the sheaf. Since the sheaf map must be compatible with the $\mathbb{Z}_2$-grading, ψ^*t is an even section and $(\psi^*t)^\sim = t$. Then,

$$\psi^*(t) = t + f(t)\theta^1\theta^2.$$

Similarly,

$$\psi^*(\theta^j) = g_j(t)\theta^1 + h_j(t)\theta^2.$$

It is important to observe that this defines uniquely ψ^* for sections of the form

$$a + b_1\theta^1 + b_2\theta^2$$

where a, b_1 and b_2 are polynomials in t. It is therefore reasonable to expect that ψ^* is uniquely defined. Let us take, for simplicity, the case where

$$\psi^*(t) = t + \theta^1\theta^2,$$

and

$$\psi^*(\theta^j) = \theta^j. \tag{2.3}$$

If g is a smooth function of t on an open set $U \subseteq \mathbb{R}$, we want to define $\psi_U^*(g)$.

Let us expand $g(t + \theta^1\theta^2)$ as a formal Taylor series:

$$g(t + \theta^1\theta^2) = g(t) + g'(t)\theta^1\theta^2.$$

The series does not continue because $(\theta^1\theta^2)^2 = 0$. Then, we define

$$\psi_U^*(g) = g(t) + g'(t)\theta^1\theta^2.$$

If

$$g = g_0 + g_1\theta^1 + g_2\theta^2 + g_{12}\theta^1\theta^2,$$

then we must define

$$\psi_U^*(g) = \psi_U^*(g_0) + \psi_U^*(g_1)\theta^1 + \psi_U^*(g_2)\theta^2 + \psi_U^*(g_{12})\theta^1\theta^2$$

where we have used (2.3). The family (ψ_U^*) then defines a morphism between $\mathbb{R}^{1|2}$ and itself. This method can be extended to the general case.

Let us recall some fundamental local properties of morphisms. A morphism ψ between two super-domains $U^{p|q}$ and $V^{r|s}$ is given by a smooth map $\psi^\sim : U \to V$ and a homomorphism of super-algebras

$$\psi^* : \mathcal{C}^{\infty r|s}(V) \to \mathcal{C}^{\infty p|q}(U).$$

It must satisfy the following properties:

- If $t = (t_1, \ldots, t_r)$ are coordinates on V^r, each component t_j can also be interpreted as a section of $\mathcal{C}^{\infty \, r|s}(V)$. If $f_i = \psi^*(t_i)$, then f_i is an even element of the algebra $\mathcal{C}^{\infty \, p|q}(U)$.
- The smooth map $\psi^\sim : U \to V$ must be $\psi^\sim = (f_1^\sim, \ldots, f_r^\sim)$, where the $f_i^\sim$ are the values of the even elements above.
- If θ_j is a generator of $\mathcal{C}^{\infty \, r|s}(V)$, then $g_j = \psi^*(\theta_j)$ is an odd element of the algebra $\mathcal{C}^{\infty \, p|q}(U)$.

The following fundamental theorem (see for example [7]) gives a local characterization of morphisms:

Theorem 1 (*Structure of Morphisms*). *Suppose $\phi : U \to V$ is a smooth map and f_i, g_j, with $i = 1, \ldots, r, j = 1, \ldots, s$, are given elements of $\mathcal{C}^{\infty \, p|q}(U)$, with f_i even, g_j odd and satisfying $\phi = (f_1^\sim, \ldots, f_r^\sim)$. Then there exists a unique morphism $\psi : U^{p|q} \to V^{r|s}$ with $\psi^\sim = \phi$ and $\psi^*(t_i) = f_i$ and $\psi^*(\theta_j) = g_j$.*

Remark. If V is a vector bundle over a smooth manifold M, then we can form its **exterior bundle** $E = \Lambda^{\max} V$. Let $\mathcal{O}(E)$ be the sheaf of sections of E. Then, locally on M, the sheaf is isomorphic to $U^{p|q}$ where $p = \dim(M)$ and $q = \operatorname{rank}(V)$. This is clearly true whenever V is restricted to some open subset of M over which it is trivial. Consequently, $(M, \mathcal{O}(E))$ is a supermanifold, denoted by $E^\flat$. Every supermanifold is locally isomorphic to a supermanifold of the form $E^\flat$. However we should note the important fact that $E^\flat$, as a supermanifold, has many more morphisms than the corresponding exterior bundle E, because of the possibility that the even and odd coordinates can be mixed under transformations. This is well illustrated by the previous simple example. Another way to say the same thing is that there are less morphisms which preserve the bundle structure than morphisms which preserve the supermanifold structure.

2.3. Local charts on supermanifolds

We describe how supermanifolds can be constructed by patching local charts. Let $X = \bigcup_i X_i$ be a topological space, with $\{X_i\}$ open, and let $\mathcal{O}_i$ be a sheaf of rings on X_i, for each i. We write (see [19]) $X_{ij} = X_i \cap X_j$, $X_{ijk} = X_i \cap X_j \cap X_k$, and so on. We now introduce isomorphisms of sheaves which represent the "coordinate changes" on our supermanifold. They allow us to glue the single pieces to get the final supermanifold. Let

$$f_{ij} : \left(X_{ji}, \mathcal{O}_j|_{X_{ji}} \right) \longrightarrow \left(X_{ij}, \mathcal{O}_i|_{X_{ij}} \right)$$

be an isomorphisms of sheaves with

$$f_{ij}^\sim = Id.$$

This means that these maps represent differentiable coordinate changes on the underlying manifold.

To say that we glue the ringed spaces $(X_i, \mathcal{O}_i)$ through the f_{ij} means that we are constructing a sheaf of rings $\mathcal{O}$ on X and for each i a sheaf isomorphism

$$f_i : (X_i, \mathcal{O}|_{X_i}) \longrightarrow (X_i, \mathcal{O}_i),$$
$$f_i^\sim = Id_{X_i}$$

such that

$$f_{ij} = f_i f_j^{-1},$$

for all i and j.

The following usual cocycle conditions are necessary and sufficient for the existence of the sheaf $\mathcal{O}$:

i. $f_{ii} = Id$ on $\mathcal{O}_i$;
ii. $f_{ij} f_{ji} = Id$ on $\mathcal{O}_i|_{X_i}$;
iii. $f_{ij} f_{jk} f_{ki} = Id$ on $\mathcal{O}_i|_{X_{ijk}}$.

3. Projective superspaces

Due to their importance in physical applications we now give a detailed description of projective superspaces. One can work either on $\mathbb{R}$ or on $\mathbb{C}$, but we choose to stay on $\mathbb{C}$. Let X be the complex projective space of dimension n. The superprojective space will be called Y. The homogeneous coordinates are $\{z_i\}$. Let us consider the underlying topological space as X, and let us construct the sheaf of super-commutative rings on it. For any open subset $V \subseteq X$ we denote by V' its preimage in $\mathbb{C}^{n+1} \setminus \{0\}$. Then, let us define $A(V') = H(V') [\theta^1, \theta^2, \ldots, \theta^q]$, where $H(V')$ is the algebra of holomorphic functions on V' and $\{\theta^1, \theta^2, \ldots, \theta^q\}$ are the odd generators of a Grassmann algebra. $\mathbb{C}^*$ acts on this super-algebra by:

$$t : \sum_I f_I(z) \, \theta^I \longrightarrow \sum_I t^{-|I|} f_I \left(t^{-1} z \right) \theta^I. \tag{3.1}$$

The superprojective space has a ring over V given by:

$$\mathcal{O}_Y(V) = A(V')^{\mathbb{C}^*}$$

which is the subalgebra of elements invariant by this action. This is the formal definition of a projective superspace (see for example [7]), however we would like to construct the same space more explicitly from gluing different super-domains as in Section 2.3.

Let X_i be the open set where the coordinate z_i does not vanish. Then the super-commutative ring $\mathcal{O}_Y(X_i)$ is generated by elements of the type

$$f_0\left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right), \qquad f_r\left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)\frac{\theta^r}{z_i}, \quad r = 1, \ldots, q.$$

In fact, to be invariant with respect to the action of $\mathbb{C}^*$, the functions f_I in Eq. (3.1) must be homogeneous of degree $-|I|$. Then, it is obvious that the only coordinate we can divide by, on X_i, is z_i: all functions f_I are of degree $-|I|$ and holomorphic on X_i. If we put, on X_i, for $l \neq i$, $\Xi_l^{(i)} = \frac{z_l}{z_i}$ and $\Theta_r^{(i)} = \frac{\theta^r}{z_i}$, then $\mathcal{O}_Y(X_i)$ is generated, as a super-commutative ring, by the objects of the form

$$F_0^{(i)}\left(\Xi_0^{(i)}, \Xi_1^{(i)}, \ldots, \Xi_{i-1}^{(i)}, \Xi_{i+1}^{(i)}, \ldots, \Xi_n^{(i)}\right), \qquad F_a^{(i)}\left(\Xi_0^{(i)}, \Xi_1^{(i)}, \ldots, \Xi_{i-1}^{(i)}, \Xi_{i+1}^{(i)}, \ldots, \Xi_n^{(i)}\right)\Theta_a^{(i)},$$

where $F_0^{(i)}$ and the $F_a^{(i)}$'s are analytic functions on $\mathbb{C}^n$. In order to avoid confusion we have put the index i in parenthesis: it just denotes the fact that we are defining objects over the local chart X_i. In the following, for convenience in the notation, we also adopt the convention that $\Xi_i^{(i)} = 1$ for all i.

To explain the "coordinate change" morphisms let us recall what happens in the ordinary complex projective spaces.

If we consider $\mathbb{P}^n(\mathbb{C})$ with the ordinary complex analytic structure, then, over the affine open set X_i where $z_i \neq 0$, we can define the affine coordinates $w_a^{(i)} = \frac{z_a}{z_i}$, $a \neq i$. The sheaf of rings over X_i is $H(X_i)$, the ring of analytic functions over X_i. Every element f of $H(X_i)$ can also be expressed as a function in homogeneous coordinates $F(z_0, z_1, \ldots, z_n)$. Two functions, $F^{(i)}$ on X_i and $F^{(j)}$ on X_j, represent "the same function" on the intersection $X_i \cap X_j$ if, when expressed in homogeneous coordinates, they give the same function F. The isomorphism between $(X_i \cap X_j, H(X_i)|_{X_j})$ and $(X_j \cap X_i, H(X_j)|_{X_i})$ sends $F^{(i)}$ to $F^{(j)}$, i.e. expresses $F^{(i)}$ with respect to the affine coordinates $w_a^{(j)} = \frac{z_a}{z_j}$. The total manifold is obtained by gluing these domains X_i as in the previous section.

We now return to considering the superprojective spaces. We have the two sheaves $\mathcal{O}_Y(X_i)|_{X_j}$ and $\mathcal{O}_Y(X_j)|_{X_i}$. In the same way as before, we have the morphisms given by the "coordinate changes". So, on $X_i \cap X_j$, the isomorphism simply affirms the equivalence between the objects of the super-commutative ring expressed either by the first system of affine coordinates, or by the second one. So for instance we have that $\Xi_l^{(j)} = \frac{z_l}{z_j}$ and $\Theta_r^{(j)} = \frac{\theta^r}{z_j}$ can be also expressed as

$$\Xi_l^{(j)} = \frac{\Xi_l^{(i)}}{\Xi_j^{(i)}}, \qquad \Theta_r^{(j)} = \frac{\Theta_r^{(i)}}{\Xi_j^{(i)}}.$$

Which, in the language used in the previous section, means that the morphism ψ_{ji} gluing $(X_i \cap X_j, \mathcal{O}_Y(X_i)|_{X_j})$ and $(X_j \cap X_i, \mathcal{O}_Y(X_j)|_{X_i})$ is such that $\widetilde{\psi_{ji}}$ is the usual change of coordinates map on projective space and

$$\psi_{ji}^*(\Xi_l^{(j)}) = \frac{\Xi_l^{(i)}}{\Xi_j^{(i)}}, \qquad \psi_{ji}^*(\Theta_r^{(j)}) = \frac{\Theta_r^{(i)}}{\Xi_j^{(i)}}.$$

The supermanifold is obtained by observing that the coordinate changes satisfy the cocycle conditions of the previous section.

4. The functor of points

We now wish to explain how the physicists' interpretation of the z_i's as "even coordinates" and the θ_j's as "odd coordinates" can be obtained from the "super-ringed space" interpretation of supermanifolds through the concept of "functor of points". The key to understanding this is Theorem 1.

Given two supermanifolds X and S, the S-points of X (or the points of X parametrized by S) are given by the set

$$X(S) = \text{Hom}(S, X) = \{\text{set of morphisms } S \to X\}.$$

X is the supermanifold we want to describe and S is the model on which we base the description of X. Changing S modifies the description of X. The functor which associates S to $X(S)$ is a functor between the category of supermanifolds and the category of sets (which are the "points" of the supermanifolds). See also [10] for more details.

Let us interpret this in the case when $X = V^{r|s}$ and $S = U^{p|q}$. According to Theorem 1, a morphism $\psi \in \text{Hom}(U^{p|q}, V^{r|s})$ is uniquely determined by a choice of r even sections and s odd sections of $C^{\infty p|q}(U)$, i.e. morphisms are in one to one correspondence with $(r + s)$-tuples $(f_1, \ldots, f_r, g_1, \ldots, g_s)$, where f_j's are even and g_j's are odd in the algebra $C^{\infty p|q}(U)$. If

we denote by $\Gamma_q^0(U)$ and $\Gamma_q^1(U)$ respectively the set of even and odd sections of $C^{\infty \, p|q}(U)$, then the above fact is expressed as

$$\text{Hom}(U^{p|q}, V^{r|s}) = (\Gamma_q^0(U))^r \times (\Gamma_q^1(U))^s. \tag{4.1}$$

The sub-index q denotes the "number of odd generators" of the algebra we are considering.

In particular, if $S = \mathbb{R}^{0|q}$, then

$$\text{Hom}(\mathbb{R}^{0|q}, V^{r|s}) = (\Gamma_q^0)^r \times (\Gamma_q^1)^s \tag{4.2}$$

where (Γ_q^0) and (Γ_q^1) represent the even and the odd component of a Grassmann algebra with q generators, respectively.

One could say that the "super-ringed space" structure of X encodes the information of how the even and odd coordinates (z, θ) glue together, but independently of the number of generators of the underlying super-algebra. The number of generators (q in the above case) can be fixed by taking a supermanifold S and constructing $\text{Hom}(S, X)$. We will see some examples shortly.

4.1. Coordinates of superprojective spaces

We are going to consider the superprojective space

$$\mathbb{P}^{p|q} = \left(\mathbb{P}^p, \mathcal{O}_{\mathbb{P}^p} \right) \tag{4.3}$$

which is defined as in Section 3 as a ringed space and $\dim(\mathcal{O}_{\mathbb{P}^p}) = q$. We want to describe the set of $\mathbb{C}^{0|N}$-points of this space. The space $\mathbb{C}^{0|N}$ can be viewed as the super-commutative ring $\mathcal{O}_{\mathbb{C}^0}$ over the single point (denoted by P) of the corresponding topological space $\mathbb{C}^0$ and can be identified precisely with the Grassmann algebra with N generators that we denote by Γ_N.

Let us consider the open subsets $\{X_i\}$, $i = 0, 1, 2, \ldots, p$, of $\mathbb{P}^p$ where $z_i \neq 0$, with the corresponding super-commutative ring $\mathcal{O}_{X_i}$. A morphism between $\mathbb{C}^{0|N}$ and $(X_i, \mathcal{O}_{X_i})$ is completely defined by the pull-back for each generator of the ring $\mathcal{O}_{X_i}$

$$\tau_{(i)} \in \text{Hom}(\mathbb{C}^{0|N}, (X_i, \mathcal{O}_{X_i})), \quad \tau_{(i)}^* : \mathcal{O}_{X_i} \to \mathcal{O}_{\mathbb{C}^0} \tag{4.4}$$

where $\mathcal{O}_{\mathbb{C}^0} = \mathbb{C}[\theta_1, \ldots, \theta_N] = \Gamma_N$. To clarify this point, we take the generators of $\mathcal{O}_{X_i}$: $\Theta_j^{(i)}, j = 1, \ldots, q$ and the affine coordinates $\Xi_j^{(i)}$ on X_i and we map into $\mathbb{C}[\theta_1, \ldots, \theta_N]$ as follows

$$\tau_{(i)}^*(\Xi_j^{(i)}) = f_j^{(i)} \quad j = 1, \ldots, p \tag{4.5}$$
$$\tau_{(i)}^*(\Theta_r^{(i)}) = \eta_r^{(i)} \quad r = 1, \ldots, q$$

where the $f_j^{(i)}$ (resp. $\eta_r^{(i)}$) are even (resp. odd) elements of the Grassmann algebra Γ_N. It is clear that $\tau_{(i)}^{\sim}(P) = ((f_1^{(i)})^{\sim}, \ldots, (f_p^{(i)})^{\sim})$. We therefore see that for every i, $\text{Hom}(\mathbb{C}^{0|N}, (X_i, \mathcal{O}_{X_i}))$ can be identified with a copy of $(\Gamma_N^0)^p \times (\Gamma_N^1)^q$.

To obtain all the possible morphisms from $\mathbb{C}^{0|N}$ to $\mathbb{P}^{p|q}$, we must take into account that the latter is built by "gluing" super-domains by means of the "coordinate change isomorphisms", this corresponds to gluing together all copies of $(\Gamma_N^0)^p \times (\Gamma_N^1)^q$ for all possible i's. Since a morphism in $\text{Hom}(\mathbb{C}^{0|N}, \mathbb{P}^{p|q})$ must be compatible with the restriction maps, it must commute with the "coordinate changes". This means that, if $\tau_{(j)}^*$ is the pull-back of a morphism to X_j, and $\psi_{ij}^* : \mathcal{O}_{X_j}|_{X_i \cap X_j} \longrightarrow \mathcal{O}_{X_i}|_{X_i \cap X_j}$ is the isomorphism which represents "coordinate changes", then

$$\tau_{(j)}^* = \tau_{(i)}^* \circ \psi_{ij}^*.$$

This then induces a map between subsets of the ith and jth copy of $(\Gamma_N^0)^p \times (\Gamma_N^1)^q$ as follows

$$(f_1^{(i)}, \ldots, f_p^{(i)}) \mapsto (f_j^{(i)})^{-1}(f_1^{(i)}, \ldots, 1, \ldots, f_p^{(i)}),$$
$$(\eta_1^{(i)}, \ldots, \eta_q^{(i)}) \mapsto (f_j^{(i)})^{-1}(\eta_1^{(i)}, \ldots, \eta_q^{(i)}).$$

By means of this map we glue the two copies together. Performing all these gluings gives a model for $\text{Hom}(\mathbb{C}^{0|N}, \mathbb{P}^{p|q})$, consisting of the $\mathbb{C}^{0|N}$-points of $\mathbb{P}^{p|q}$.

Another way to interpret this model is as follows. We consider a set of "homogeneous" (even and odd) generators $z_0, \ldots, z_p, \theta_1, \ldots, \theta_q$, where the z_j's are in Γ_N^0 and at least one them is invertible and the θ_j's are in Γ_N^1. One obtains the local generators on each X_i simply "dividing" by z_i (exactly like in the standard projective case, when one looks for the "affine coordinates"). This way we see that we can identify

$$\text{Hom}(\mathbb{C}^{0|N}, \mathbb{P}^{p|q}) = \frac{\left((\Gamma_N^0)^{p+1} \setminus B^{p+1} \right) \times (\Gamma_N^1)^q}{(\Gamma_N^0)^*},$$

where $(\Gamma_N^0)^*$ is the set of the even invertible elements and $B = (\Gamma_N^0) \setminus (\Gamma_N^0)^*$. This model is exactly the generalization of the projective space as a supermanifold in the sense of Rogers, Bruzzo and others (see book [3] for a complete discussion).

5. Supergroups and superdeterminants

As another illustration of the meaning of the functor of points we consider the case of supergroups. For simplicity we will just look at the cases of $GL(1|1)$, $SL(1|1)$ and finally we will give another construction of the superprojective space as the quotient space $SU(n|m)/U(n-1|m)$.

Let us now consider the simplest case of supergroup $GL(1|1)$. As a supermanifold, $GL(1|1)$ is isomorphic to the super-domain $U^{2|2} = (U^2, \mathcal{C}^{\infty 2|2})$, where $U^2 = (\mathbb{C}^*)^2$. If (z_1, z_2) are the coordinates on U^2 and θ_1, θ_2 are the generators of the Grassmann algebra, it is convenient to use the notation in matrix form

$$\begin{pmatrix} z_1 & \theta_1 \\ \theta_2 & z_2 \end{pmatrix}. \tag{5.1}$$

We can define the "product" on $GL(1|1)$ as a morphism

$$\psi \in \mathrm{Hom}(GL(1|1) \times GL(1|1), GL(1|1))$$

such that

$$\psi^\sim : GL(1|1)_0 \times GL(1|1)_0 \longrightarrow GL(1|1)_0$$
$$(z_1, z_2) \times (z_3, z_4) \longmapsto (z_1 z_3, z_2 z_4),$$
$$\psi^* : \mathcal{C}^\infty(U^2)[\vartheta_1, \vartheta_2] \longrightarrow \mathcal{C}^\infty(U^2 \times U^2)[\theta_1, \theta_2, \theta_3, \theta_4]$$
$$\begin{pmatrix} w_1 & \vartheta_1 \\ \vartheta_2 & w_2 \end{pmatrix} \longmapsto \begin{pmatrix} z_1 z_3 + \theta_1 \theta_4 & \theta_1 z_4 + z_1 \theta_3 \\ \theta_2 z_3 + z_2 \theta_4 & z_2 z_4 + \theta_2 \theta_3 \end{pmatrix},$$

where the action of the pull-back morphism ψ^* has been specified only for the generators of the algebra (see Theorem 1).

We now apply the functor of points to recover the usual interpretation of $GL(1|1)$ as the set of "invertible supermatrices". Take as model space $S = \mathbb{C}^{0|q}$, then $\mathrm{Hom}(S, \mathrm{GL}(1|1))$ can be identified with the set of matrices

$$g = \begin{pmatrix} \psi^* z_1 & \psi^* \theta_1 \\ \psi^* \theta_2 & \psi^* z_2 \end{pmatrix}, \tag{5.2}$$

where $\psi^* z_i$ are even elements of the Grassmann algebra Γ_q, whose value is different from zero, and $\psi^* \theta_i$ are odd elements. To simplify notation we denote $\psi^* z_i$ (resp. $\psi^* \theta_i$) by z_i (resp. θ_i).

The above "product" becomes the usual multiplication of supermatrices as follows. A morphism from S to $GL(1|1) \times GL(1|1)$ is given by a pair of matrices, g_1 and g_2, as above. Composition with the product morphism gives a morphism from S to $GL(1|1)$, represented by a matrix g_3, which can be seen to be given by the usual multiplication of matrices:

$$g_1 = \begin{pmatrix} z_1 & \theta_1 \\ \theta_2 & z_2 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} z_3 & \theta_3 \\ \theta_4 & z_4 \end{pmatrix} \tag{5.3}$$
$$g_3 = g_1 g_2 = \begin{pmatrix} z_1 z_3 + \theta_1 \theta_4 & \theta_1 z_4 + z_1 \theta_3 \\ \theta_2 z_3 + z_2 \theta_4 & z_2 z_4 + \theta_2 \theta_3 \end{pmatrix}.$$

Recall the classical formula for the superdeterminant (or Berezinian) of a supermatrix in GL(1|1):

$$sdet(g) = Ber(g) = \frac{z_1}{z_2}\left(1 + \frac{\theta_1 \theta_2}{z_1 z_2}\right) \tag{5.4}$$

which is well defined if $z_2 \neq 0$. The Berezinian can also be understood from the sheaf point of view, as a morphism Ber from $GL(1|1)$ to $\mathbb{C}^{1|0}$:

$$Ber^\sim : GL(1|1)_0 \longrightarrow \mathbb{C} \qquad Ber^* : \mathcal{C}^{\infty 1|0} \longrightarrow \mathcal{C}^{\infty 2|2}$$
$$(z_1, z_2) \longmapsto z_1/z_2, \qquad w \longmapsto \frac{z_1}{z_2}\left(1 + \frac{\theta_1 \theta_2}{z_1 z_2}\right).$$

Next, we consider a subset of supermatrices $GL(1|1)$ with the property that "the superdeterminant is 1". They are denoted by $SL(1|1)$. We want to describe this space using the sheaf theoretic interpretation of supermanifolds, by restricting the base manifold and considering an appropriate quotient sheaf. We need to give a meaningful interpretation of the relation

$$\frac{z_1}{z_2}\left(1 + \frac{\theta_1 \theta_2}{z_1 z_2}\right) = 1. \tag{5.5}$$

We do it as follows. Let $\mathcal{J}$ be the ideal in $\mathcal{C}^{\infty 2|2}$ generated by

$$\frac{z_1}{z_2}\left(1 + \frac{\theta_1 \theta_2}{z_1 z_2}\right) - 1. \tag{5.6}$$

The base manifold of $SL(1|1)$ is the support of this ideal, i.e. the subset $X \subset \mathbb{C}^* \times \mathbb{C}^*$ of points around which no element of $\mathcal{J}$ is invertible. Clearly X is the diagonal in $\mathbb{C}^* \times \mathbb{C}^*$, i.e. the set where $z_1 = z_2$. The sheaf $\mathcal{O}_X$ is the restriction to X of the quotient sheaf

$$\mathcal{C}^{\infty\,2|2}/\mathcal{J}. \tag{5.7}$$

It remains to show that this ringed manifold $SL(1|1)$ is really a supermanifold, i.e. it is obtained by pasting super-domains. In fact observe that the relation (5.6) tells us that over an open set $V \subset X$, $z_1 = z_2 - \frac{\theta_1 \theta_2}{z_2}$. Therefore the ring over V is $\mathcal{C}^\infty(z_2 - \frac{\theta_1 \theta_2}{z_2}, z_2)[\theta_1, \theta_2]$ which can be seen to be isomorphic to a super-ring of the type $\mathcal{C}^\infty(z)[\Psi_1, \Psi_2]$, so locally $SL(1|1)$ is isomorphic to a super-domain. What we have done here is to show explicitly that $SL(1|1)$ is a sub–supermanifold of dimension 1|2 of $GL(1|1)$ in the sense of [7].

To conclude the description of $X = SL(1|1)$, we present its interpretation by means of the functor of points, using the model space $S = \mathbb{C}^{0|N}$. Then, the morphisms in $\mathrm{Hom}(S, X)$ can be viewed as the morphisms in $\mathrm{Hom}(S, GL(1|1))$ such that:

 (i) the map between the underlying topological spaces has image contained in the diagonal of $\mathbb{C}^* \times \mathbb{C}^*$;
(ii) the pull-back map descends to the quotient, i.e. $\psi^*(j) = 0$ for any $j \in \mathcal{J}$.

Then, the set $\mathrm{Hom}(S, X)$ can be viewed as the matrices of the form

$$g = \begin{pmatrix} \psi^* z_1 & \psi^* \theta_1 \\ \psi^* \theta_2 & \psi^* z_2 \end{pmatrix}, \tag{5.8}$$

with the conditions that: (1) the value of $\psi^* z_1$ is equal to the value of $\psi^* z_2$ (by (i)), and (2) the superdeterminant of the matrix in (5.8) is 1 (by (ii)).

We start describing $GL(1|1, \mathbb{C})$ in a different way, by passing to real supergroups. We use the following idea: think about $\mathbb{C}^n$, with a complex basis $\{v_1; v_2; \ldots; v_n\}$. Then, $\mathbb{C}^n$ can be viewed as a real $2n$-dimensional space, with a real basis

$$\{v_1; iv_1; v_2; iv_2; \ldots; v_n; iv_n\}.$$

So, let us take $V^2 = \{(a; b) \in \mathbb{R}^2 : a^2 + b^2 \neq 0\}$. We take, as a base manifold, $(V^2)^2$, and the ring over this total space is $\mathcal{C}_\mathbb{R}^\infty (z_1; z_1'; z_2; z_2') [\theta; \theta'; \psi; \psi']$. Note that z_1' and z_2' (as even generators), and θ' and ψ' (as odd generators) play the role of the vectors (written above) iv_j. So, the primed generators are independent from the real point of view; sums like $xz_1 + yz_1'$ (or $\theta + \theta'$) represent complex even (respectively, odd) elements, since they put together the real and the imaginary part.

We can view the generators in the following matrix form, which helps us in writing the product morphism.

$$\begin{pmatrix} z_1 + z_1' & \theta + \theta' \\ \psi + \psi' & z_2 + z_2' \end{pmatrix}. \tag{5.9}$$

The product morphism is defined by: $m : (V^2)^2 \times (V^2)^2 \longrightarrow (V^2)^2$, and one can describe exactly the pull-back morphism by performing explicitly the matrix multiplication:

$$\begin{pmatrix} z_1^1 + z_1'^1 & \theta^1 + \theta'^1 \\ \psi^1 + \psi'^1 & z_2^1 + z_2'^1 \end{pmatrix} \cdot \begin{pmatrix} z_1^2 + z_1'^2 & \theta^2 + \theta'^2 \\ \psi^2 + \psi'^2 & z_2^2 + z_2'^2 \end{pmatrix}.$$

Then, we try to define a "complex conjugation". It is a morphism, such that its pull-back map takes the form:

$$\rho^* : \mathcal{C}_\mathbb{R}^\infty (z_1; z_1'; z_2; z_2') [\theta; \theta'; \psi; \psi'] \longrightarrow \mathcal{C}_\mathbb{R}^\infty (z_1; z_1'; z_2; z_2') [\theta; \theta'; \psi; \psi'],$$

and it is completely determined once one knows the behavior of the generators. In fact, it sends $z_1; z_2; \theta; \psi$ to themselves, while the primed elements $z_1'; z_2'; \theta'; \psi'$ undergo a change of sign, going respectively to $-z_1'; -z_2'; -\theta'; -\psi'$.

Next, we define the "Hermitian transpose", which will be denoted by the symbol $\mathcal{H}$. Its pull-back map sends $z_1; z_2$ to themselves; $z_1'; z_2'$ to $-z_1'; -z_2'$, respectively; $\theta; \theta'$ to $\psi'; \psi$, respectively, and $\psi'; -\psi$ to $\theta'; -\theta$, respectively. Note that the exchange of the θ with the ψ is due to the transposition operation, while the exchange of a non-primed generator with a primed one is due to a multiplication by $-i$. We will see this better when we pass to Grassmann algebras by the functor of points. We have again to require that $\mathcal{H}$ is a morphism. Note that it is not true that $(\mathcal{H}^*)^2 = \mathrm{Id}$.

Now, we can describe the functor of points with respect to a particular model space. We choose, as a model space, $\mathbb{C}^{0|q}$. So, $\mathbb{C}^{0|q} = (P; \mathbb{C}[\xi_1; \xi_2; \ldots; \xi_q])$. The conjugation is a map, defined, in a standard way (since $\mathbb{C}^{0|q}$ is simply a Grassmann algebra) $\sigma : \mathbb{C}^{0|q} \longrightarrow \mathbb{C}^{0|q}$, with pull-back defined by: $\sigma^*((x + iy)\xi_i) = (x - iy)\xi_i$.

When we construct the morphisms $\overline{\mathrm{Hom}}(\mathbb{C}^{0|q}; GL(1|1))$, we only take the ones which are $\mathbb{R}$-linear and compatible with conjugations, which means that $\sigma^* \circ \varphi^* = \varphi^* \circ \rho^*$. This means for example, that, if we choose an odd element of $\mathbb{C}^{0|q}$ as the pull-back of θ, the pull-back of θ' is consequently i times the pull-back of θ. (In general this is not true, but we have assumed here that there is a complex structure J on the anticommuting coordinates and we consider only those morphisms which commute with J.)

In this way, we can describe the $\mathbb{C}^{0|q}$-points of $GL(1|1)$ as the set of matrices of the form

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix},$$

where a; b are even elements of a complex Grassmann algebra with q generators, α; β are odd elements of the same algebra, and the Berezinian is invertible.

Now, we can pass to construct the supergroup $U(1|1)$. Note that it is a real supermanifold. We have to perform the quotient of $C_{\mathbb{R}}^{\infty}\left(z_1; z_1'; z_2; z_2'\right)\left[\theta; \theta'; \psi; \psi'\right]$, by the four relations obtained from:

$$A \cdot \mathcal{H}^*(A) = \mathbb{I}, \tag{5.10}$$

where A is a sheaf element written in the matrix form (5.9). By explicitly writing the elements on the left, we get four generators of an ideal I. Taking the quotient

$$C_{\mathbb{R}}^{\infty}(z_1; z_1'; z_2; z_2')[\theta; \theta'; \psi; \psi']/I,$$

we finally get the sheaf corresponding to $U(1|1)$. The base manifold is the support of this sheaf. Applying the functor-of-points interpretation and using $\mathbb{C}^{0|q}$ as a model, it is easy to see that the elements of $U(1|1)$ correspond, as a set, to the matrices of the form

$$B = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}. \tag{5.11}$$

They preserve the "scalar" product $\langle (z, \theta), (z, \theta) \rangle = z\bar{z} + i\theta\bar{\theta}$. Note that the bar is the conjugation in $\mathbb{C}^{0|q}$, i.e. $\bar{a} = \sigma^*(a)$. It follows that $B^\dagger \cdot B = \mathbb{I}$, where $\dagger$ represents the usual "adjoint" of supermatrices. It represents the correspondence between a matrix and its Hermitian transpose from the $\mathbb{C}^{0|q}$-point of view, since we can see that for every element $\varphi \in \overline{Hom}\left(\mathbb{C}^{0|q}; GL(1|1)\right)$, $\varphi^* \circ \mathcal{H}^*$ is related to φ^* by exactly performing the $\dagger$ operation. More precisely,

$$B^\dagger = \begin{pmatrix} \bar{a} & -i\bar{\beta} \\ -i\bar{\alpha} & \bar{b} \end{pmatrix}. \tag{5.12}$$

For the sake of completeness, we write explicitly the $\mathbb{C}^{0|q}$-points of $U(1|1)$. They are in bijective correspondence with the matrices of the form

$$U = \begin{pmatrix} 1 - \dfrac{i}{2}\gamma\bar{\gamma} & -ie^{i\psi}\bar{\gamma} \\ \gamma & e^{i\psi}\left(1 + \dfrac{i}{2}\gamma\bar{\gamma}\right) \end{pmatrix}, \tag{5.13}$$

where ψ is a real phase and γ is a generic odd element of the Grassmann algebra.

A similar construction applies to $U(n|m)$ supergroups. To get the supergroups $SU(n|m)$ we have to quotient with respect to the Berezinian equal to one. The body part of $SU(n|m)$ is $U(1) \times SU(n) \times SU(m)$. The odd part belongs to the fundamental representation of $SU(n) \times U(m)$.

5.1. Superprojective spaces as supercosets

Here we show that, using the functor-of-points framework, the superprojective spaces can be described in three different and equivalent ways. We first remind the reader the three methods to define the classical projective space and then we extend it to superprojective ones.

In the classical case, let $z_i, i = 1, \ldots, n + 1$ be the coordinates on $\mathbb{C}^{n+1} - \{0\}$ and define the projective space $\mathbb{P}^n$ by the quotient

$$(z_1, \ldots, z_{n+1}) \sim \lambda(z_1, \ldots, z_{n+1}), \quad \lambda \in \mathbb{C}^*. \tag{5.14}$$

This is the standard definition of $\mathbb{P}^n$. Alternatively, one can fix the modulus of λ by setting

$$\sum_{i=1}^{n+1} |z_i|^2 = r > 0, \quad z_i \sim e^{i\phi}z_i, \forall i, \phi \in \mathbb{R} \tag{5.15}$$

up to the phase ϕ. The first equation fixes the modulus $|\lambda|^2 = 1$ and the second equation removes its phase. Let us choose $r = 1$, this implies that the vector z_i has modulus equal to one.

The $SU(n+1)$ symmetry of (5.15) is used to bring the vector z_i in the form $(1, 0, \ldots, 0)$ which has modulus equal to one. This vector has a stability group which is $U(n)$. A stability group is the subgroup of transformations which leaves $(1, 0, \ldots, 0)$ invariant. Therefore, we can define the projective space as the coset

$$SU(n + 1)/U(n). \tag{5.16}$$

The three ways to define a $\mathbb{P}^n$ are easily seen to be equivalent.

Let us consider now the superprojective spaces $\mathbb{P}^{n|m}$. The definition (5.14) can be repeated as follows

$$(z_1, \ldots, z_{n+1}, \theta_1, \ldots, \theta_m) \sim \lambda(z_1, \ldots, z_{n+1}, \theta_1, \ldots, \theta_m), \tag{5.17}$$

with λ not belonging to $\mathbb{C}^*$, but to the space of even quantities which are invertible (see Section 4.1). Again, we can use the alternative definition of $\mathbb{P}^{n|m}$ [6,20,4,7]

$$\sum_{i=1}^{n+1} |z_i|^2 + i \sum_{A=1}^{m} \bar{\theta}_A \theta_A = r > 0, \quad z_i \sim e^{i\phi} z_i, \theta_A \sim e^{i\phi} \theta_A, \forall i, A, \phi \in \mathbb{R}. \tag{5.18}$$

Eq. (5.18) is the correct extension of (5.15), but its interpretation needs some comment. As we have seen there are two ways to describe supermanifolds: (i) using the sheaf description and (ii) using the functor of points. According to the first framework, Eq. (5.18) can be seen as an algebraic equation among the generators of the sheaves of the supermanifolds. Eq. (5.18) is consistent with the projection (5.17) and it defines an hypersurface in $\mathbb{C}^{n+1|m}$. According to the second description, one has to decompose the coordinates z_i and θ_A on the basis of the generators of the super-domain and the coefficients need to satisfy a set of algebraic equations.

Note that $SU(n+1|m)$ acts on $\mathbb{P}^{n|m}$ as follows:

$$\psi : SU(n+1|m) \times \mathbb{P}^{n|m} \longrightarrow \mathbb{P}^{n|m} \tag{5.19}$$

with the pull-back defined by

$$\psi^*(z_i) = \sum_{j=1}^{n+1} A_{ij} z_j + \sum_{A=1}^{m} \alpha_{iA} \theta_A, \qquad \psi^*(\theta_A) = \sum_{j=1}^{n+1} \beta_{Aj} z_j + \sum_{B=1}^{m} B_{BA} \theta_B, \tag{5.20}$$

where A_{ij}, B_{AB} are the even generators of $SU(n+1|m)$ and α_{iA}, β_{Aj} are the odd ones. It is easy to see that the action is transitive like in the classical case. So, as in the classical case one can define the supercoset $SU(n+1|m)/SU(n|m)$ which can be identified with the superprojective space $\mathbb{P}^{n|m}$. An analysis of the supercosets can be found in the book [2] and recently it has been discussed in [10].

Therefore applying the construction above and starting from a vector $((1, \ldots, 0), (0, \ldots, 0))$ (where the first set of components are the even coordinates and the second set the odd ones) we end up with Eq. (5.18). The odd part of (5.18) is obtained by acting with the odd part of the supergroup on the unit vector. Notice that this is not the only possibility, indeed we can start from an odd vector $((0, \ldots, 0), (1, \ldots, 0))$ which has the following norm

$$\|((0, \ldots, 0), (1, \ldots, 0))\|^2 = i \theta_1 \bar{\theta}_1.$$

In this case, acting with the supergroup on it (and preserving the subgroup $U(n|m)$), we end up with the new equation

$$\sum_{i=1}^{n+1} |z_i|^2 + i \sum_{A=1}^{m} \bar{\theta}_A \theta_A = r, \tag{5.21}$$

where r is an even element of the algebra. For example, starting from the vector

$$((1, \ldots, 0), (1, \ldots, 0))$$

we have $r = r_0 + i \theta_1 \bar{\theta}_1$ whose body r_0 is positive.

6. Balanced supermanifolds

In this section, we propose a possible extension of notion of *balanced manifold* (see [11] or [25] for a survey) to the supermanifolds. We found it appropriate to report the present results since they call for a functor-of-point interpretation and for the definition of stable supermanifolds.

6.1. Integration of superforms

The integration of superforms has been studied in the literature and it is based on the notion of the integral superforms (see for example [2] for a discussion based on sheaf technique and the references therein). This is due to the fact that the superforms are built with basic 1-superform $d\theta$ and their wedge products. However, those products $d\theta \wedge \cdots \wedge d\theta$ form a differential complex

$$0 \xrightarrow{d} \Omega^{(0|0)} \xrightarrow{d} \Omega^{(1|0)} \cdots \xrightarrow{d} \Omega^{(n|0)} \xrightarrow{d} \cdots \tag{6.1}$$

(the notation $\Omega^{(a|b)}$ denotes the form degree a and the *picture number* b that will be explained few lines below) which is bounded from below, but not from above. Therefore, there is no notion of a top form to be integrated on the supermanifold $\mathbb{C}^{p+1|q}$.

While the integration of functions on the supermanifold is clear since it is obtained by the Berezin integral [19], the integration of the superforms is obtained by using the method of the projection forms

$$\mathrm{d}\mu_V = \frac{1}{p!}\Omega^p_{FB} \wedge \mathcal{U}_q \tag{6.2}$$

where $\mathcal{U}_q$ is the Thom class obtained by viewing the supermanifold $\mathbb{P}^{p|q}$ as modeled on $\mathbb{P}^p$. For a nice review see [21]. The construction of $\mathcal{U}_q$ for $\mathbb{P}^{p|q}$ will be given in [16].

One simple way to define the integration forms is to introduce the new basic forms $\delta(\mathrm{d}\theta)$ where the symbol δ has the usual properties of Dirac's delta distribution. They satisfy in addition some simple rules such as

$$\delta(\mathrm{d}\theta) \wedge \delta(\mathrm{d}\theta') = -\delta(\mathrm{d}\theta') \wedge \delta(\mathrm{d}\theta), \qquad \mathrm{d}\theta\delta(\mathrm{d}\theta) = 0, \qquad \mathrm{d}\theta\delta'(\mathrm{d}\theta) = -\delta(\mathrm{d}\theta). \tag{6.3}$$

The systematic exposition of these rules can be found in [22] and they can be put in a more mathematical framework using the results of [3].

We introduce the *picture number* by counting the number of delta functions and we denote by $\Omega^{r|s}$ the space of r-forms with picture s. For example, in the case of $\mathbb{C}^{p+1|q}$, the integral form

$$\mathrm{d}x^{[K_1} \wedge \cdots \wedge \mathrm{d}x^{K_l]}\mathrm{d}\theta^{[i_{l+1}} \wedge \cdots \wedge \mathrm{d}\theta^{i_r} \wedge \delta(\mathrm{d}\theta^{i_1}) \wedge \cdots \wedge \delta(\mathrm{d}\theta^{i_s]}) \tag{6.4}$$

is an r-form with picture s. All indices K_i and i_j are antisymmetrized among themselves. We denote by $[I_1 \dots I_s]$ the antysimmetrization of the indices. Notice that we can also add derivatives of delta functions $\delta^{(n)}(\mathrm{d}\theta)$ and they effectively reduce the form degree. Indeed, even negative form degree can be considered by forms of the type

$$\delta^{(n_1)}(\mathrm{d}\theta^{i_1}) \wedge \cdots \wedge \delta^{(n_s)}(\mathrm{d}\theta^{i_s}) \tag{6.5}$$

which is a $-(n_1 + \cdots + n_s)$-form with picture s. The integral forms form a new complex as follows

$$\cdots \xrightarrow{d} \Omega^{(r|q)} \xrightarrow{d} \Omega^{(r+1|q)} \cdots \xrightarrow{d} \Omega^{(p+1|q)} \xrightarrow{d} 0 \tag{6.6}$$

where $\Omega^{(p+1|q)}$ is the top form $\mathrm{d}x^{[K_1} \wedge \cdots \wedge \mathrm{d}x^{K_{p+1}]}\delta(\mathrm{d}\theta^{[i_1}) \wedge \cdots \wedge \delta(\mathrm{d}\theta^{i_q]})$ which can be integrated on the supermanifold. As in the usual commuting geometry, there is an isomorphism between the cohomologies $H^{(0|0)}$ and $H^{(p+1|q)}$. In addition, one can define two operations acting on the cohomology groups $H^{(r|s)}$ which change the picture number s (see for example [22]).

Given a function $f(x, \theta)$ on the supermanifold $\mathbb{C}^{(p+1|q)}$, we define its integral by the super top form $\omega^{(p+1|q)} = f(x, \theta)\mathrm{d}^{p+1}x\delta(\mathrm{d}\theta^1) \wedge \cdots \wedge \delta(\mathrm{d}\theta^q)$ belonging to $\Omega^{(p+1|q)}$ as follows

$$\int_{\mathbb{C}^{(p+1|q)}} \omega^{(p+1|q)} = \epsilon^{i_1\cdots i_q}\partial_{\theta^{i_1}} \cdots \partial_{\theta^{i_q}} \int_{\mathbb{C}^{p+1}} f(x, \theta)|_{\theta=\bar{\theta}=0} \tag{6.7}$$

where the last equality is obtained by integrating on the delta functions and selecting the bosonic top form. The remaining integrals are the usual integral of densities and the Berezin integral. The latter can be understood in terms of the Berezinian sheaf [23]. It is easy to show that indeed the measure is invariant under general coordinate changes and the density transform as a Berezinian with the superdeterminant.

Before defining Donaldson's balanced supervarieties, we need to define the integration on the projective space which will involve the technology of integral forms. This can be done in two ways:

1. We use the fact that for a projective space there is a gauge symmetry defined by the rescaling of the coordinates. So, given X the vector field generating such a symmetry, the induced measure on the projective space is given by

$$\iota_X \Omega^{(p+1|q)}_{\mathbb{C}} = \Omega^{(p|q)}_{\mathbb{P}} \tag{6.8}$$

where $\Omega^{(p+1|q)}_{\mathbb{C}}$ is the top form in the supermanifold $\mathbb{C}^{(p+1|q)}$ and $\Omega^{(p|q)}_{\mathbb{P}}$ is the top form in the projective superspace $\mathbb{P}^{p|q}$. As we have discussed in the above sections, the scale symmetry is generated by the holomorphic vector field

$$X = \sum_I z_I \partial_{z_I} + \sum_i \theta^i \partial_{\theta^i}. \tag{6.9}$$

To illustrate these statements, we focus on the case of the super-Calabi–Yau $\mathbb{P}^{1|2}$. The holomorphic top form for $\mathbb{C}^{(p+1|q)}$ is given by

$$\Omega^{(p+1|q)}_{\mathbb{C}} = \epsilon^{IJ}\mathrm{d}z_I \wedge \mathrm{d}z_J\epsilon_{ij}\delta(\mathrm{d}\theta^i)\delta(\mathrm{d}\theta^j), \tag{6.10}$$

and we act with the contraction ι_X to get

$$\iota_X \Omega^{(p+1|q)}_{\mathbb{C}} = \epsilon^{IJ}z_I\mathrm{d}z_J\epsilon_{ij}\delta(\mathrm{d}\theta^i) \wedge \delta(\mathrm{d}\theta^j) \tag{6.11}$$

where we have used the relation $\iota_{\theta^i \partial_{\theta^i}} \delta(\mathrm{d}\theta^i) = \theta^i \delta'(\mathrm{d}\theta^i)$ (where the index i is not summed) and the fact that $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ on $\mathbb{P}^1$ vanishes. Using affine coordinates $z = z_1/z_0$, $\psi^i = \theta^i/z_0$ we get

$$\Omega_{CY} = \mathrm{d}z\, \epsilon_{ij} \delta(\mathrm{d}\psi^i) \wedge \delta(\mathrm{d}\psi^j). \tag{6.12}$$

It is important to underline that Ω_{CY} is not a differential form, but rather an integral form (see also [4]) and it can be identified with the measure associated with the Fubini–Study metric for the supermanifold $\mathbb{P}^{1|2}$. To see this, we re-derive (6.12) in a different way.

2. We use the algebraic equation

$$\sum_{i=1}^{p+1} |z_i|^2 + \mathrm{i} \sum_{l=1}^{q} \bar{\theta}_l \theta_l = r \tag{6.13}$$

where $r \in \mathbb{C}$ and we derive the conditions for a measure on the sub–supermanifold. For the super-Calabi–Yau $\mathbb{P}^{1|2}$ the Eq. (5.21) (or (6.13)) becomes

$$|z_0|^2 + |z_1|^2 + \mathrm{i}\,\delta_{ij} \bar{\theta}^i \theta^j = r \tag{6.14}$$

and by the symmetry $z_I \to \mathrm{e}^{\mathrm{i}\alpha} z_I$, $\theta_i \to \mathrm{e}^{\mathrm{i}\alpha} \theta_i$. Here we have used the homogeneous coordinates z_I with $I = 0, 1$ and the affine coordinate z is defined by $z = z_1/z_0$. Acting with the ∂ differential operator on (6.14) we get

$$\delta^{IJ} \bar{z}_I \mathrm{d}z_J + \mathrm{i}\,\delta_{ij} \bar{\theta}^i \mathrm{d}\theta^j = 0. \tag{6.15}$$

The measure for the super-Calabi–Yau space can be determined by choosing the ansatz

$$\mathrm{d}z_I \epsilon_{ij} \delta(\mathrm{d}\theta^i) \wedge \delta(\mathrm{d}\theta^j) = \Omega_{CY} F_I(z_I, \theta^i) \tag{6.16}$$

where $F_I(z_I, \theta^i)$ is a function of the coordinates z_I and θ^i which is needed to satisfy the constraint (6.15). Indeed, by contracting both sides of (6.16) with $\delta^{IJ} \bar{z}_I$ and using the identity $\mathrm{d}\theta^i \delta(\mathrm{d}\theta^i) = 0$ for each i, we find that $F_I = f(\theta) \epsilon_{IJ} \bar{z}^J$ where $f(\theta)$ is a scalar function needed to guarantee the holomorphicity of Ω_{CY}. By simple algebraic manipulations, one finds

$$\Omega_{CY} = \epsilon^{IJ} z_I \mathrm{d}z_J \epsilon_{ij} \delta(\mathrm{d}\theta^i) \wedge \delta(\mathrm{d}\theta^j) \tag{6.17}$$

which coincides with the above derivation.

Finally the integration of a function F on the superspace is given by

$$\int_{\mathbb{P}^{1|2}} \Omega_{CY} \wedge \bar{\Omega}_{CY} F(z, \bar{z}, \theta^i, \bar{\theta}^i) = \int_{\mathbb{P}^1} \mathrm{d}z \wedge \mathrm{d}\bar{z}\, \epsilon_{ij} \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j} \epsilon_{ij} \frac{\partial}{\partial \bar{\theta}^i} \frac{\partial}{\partial \bar{\theta}^j} F(z, \bar{z}, \theta^i, \bar{\theta}^i)|_{\theta = \bar{\theta} = 0}. \tag{6.18}$$

In the second line we have taken the four derivatives with respect to the fermionic coordinates θ_i and $\bar{\theta}_i$ and then set them to zero. It remains to perform the usual integration on the $\mathbb{P}^1$.

6.2. Donaldson's balanced superprojective spaces

Let us consider the superprojective space $\mathbb{P}^{p|q}$ with standard coordinates $[z_0, \ldots, z_p, \theta_1, \ldots, \theta_q]$ and the matrix valued function on $\mathbb{P}^{p|q}$ given by

$$B_{ik} = \frac{z_i \bar{z}_k}{\sum\limits_{l=0}^{p} |z_l|^2 + \mathrm{i} \sum\limits_{l=1}^{q} \theta_l \bar{\theta}_l},$$

$$B_{iK} = \frac{z_i \bar{\theta}_K}{\sum\limits_{l=0}^{p} |z_l|^2 + \mathrm{i} \sum\limits_{l=1}^{q} \theta_l \bar{\theta}_l}, \qquad B_{Ik} = \frac{\theta_I \bar{z}_k}{\sum\limits_{l=0}^{p} |z_l|^2 + \mathrm{i} \sum\limits_{l=1}^{q} \theta_l \bar{\theta}_l}, \tag{6.19}$$

$$B_{IK} = \frac{\mathrm{i}\theta_I \bar{\theta}_K}{\sum\limits_{l=0}^{p} |z_l|^2 + \mathrm{i} \sum\limits_{l=1}^{q} \theta_l \bar{\theta}_l}.$$

If we denote by V a projective sub–supervariety of $\mathbb{P}^{p|q}$, we define the $(p + q + 1) \times (p + q + 1)$-matrix by the block matrix

$$M(V)_{AB} = \begin{pmatrix} \int_V B_{ik} \mathrm{d}\mu_V & \int_V B_{iK} \mathrm{d}\mu_V \\ \int_V B_{Ik} \mathrm{d}\mu_V & \int_V B_{IK} \mathrm{d}\mu_V \end{pmatrix} \tag{6.20}$$

where the indices A, B run over $p + 1 + q$ values. Notice that $\overline{M(V)}_{ij} = M_{ji}$, $\overline{M(V)}_{iK} = M(V)_{Ik}$ and $\overline{M(V)}_{IK} = -M_{KI}$. The measure $d\mu_V$ is defined above. In the case of the super-Calabi–Yau, we use the definition given in (6.18).

Applying the rules described above for the integration, one immediately gets

$$M(V)_{iK} = M(V)_{Ik} = 0 \tag{6.21}$$

and there are only non-trivial blocks M_{ik} and M_{IK}. Since the computation of $\mathcal{U}_q$ in (6.2) requires new ingredients, we consider in the following only super-Calabi–Yau spaces. For them we can use a different measure provided by the holomorphic form.

Following Donaldson [11], we define a balanced supermanifold if $M(V)$ is a multiple of the identity matrix. However, notice that the identity matrix in the supermanifold has the block structure $\mathbb{I} = (\delta_{i\bar{j}}, \delta_{I\bar{K}})$. Hence, a supermanifold is balanced iff there exist two real numbers λ and η such that

$$\int_V B_{i\bar{j}}\, d\mu_V = \lambda\, \delta_{i\bar{j}}, \qquad \int_V B_{i\bar{J}}\, d\mu_V = 0, \qquad \int_V B_{I\bar{J}}\, d\mu_V = \eta\, \delta_{I\bar{J}}. \tag{6.22}$$

If $p = q$, then we must have $\lambda = -\eta$. This is due to the presence of a additional $U(1)$ subgroup of the stability group $SU(p|q)$ (which is the group of isometries of the supermetric $\mathbb{I}$) and this reduces the supergroup to $PSU(p|p)$. Notice that the integration over the fermionic coordinates produces two terms: one is coming from the expansion of the denominator in $B_{i\bar{j}}$ or $B_{I\bar{K}}$ and the second is coming from the expansion of the measure $d\mu_V$. The second source of interest in (6.22) is the presence of the additional constraints (the second and the third relations) on the bosonic manifold.

In the case of super-Calabi–Yau space we use the integration measure obtained by the nowhere-vanishing holomorphic form Ω_{CY} described above. Notice that in the super-Calabi–Yau case, the Fubini–Study measure on the superprojective space coincides with the Ricci flat supermetric (see also for the explicit computation [24] or using group theoretical arguments in [4]). This simplifies the construction.

Let us now consider a generic polarized supermanifold (M, L) with L an holomorphic super line bundle where the transition functions are elements of $GL(1|1)$. In addition, we require that the super line bundle has its first Chern class $c_1(L)$ represented by a Kähler form of the supermanifold ω. For a positive m, we construct the tensor power series $L^{\otimes m}$ of the super line bundle and we denote by $H^0(M, L^{\otimes m})$ the space of holomorphic sections (as clarified above) of $L^{\otimes m}$. The holomorphic section could be odd or even. We use the extension of the Kodaira embedding theorem [17] asserting that for a sufficiently large m the holomorphic sections define a projective embedding

$$i_m : M \to \mathbb{P}\Big(H^0(M, L^{\otimes m})\Big). \tag{6.23}$$

A choice of holomorphic sections $(s_{0|0}, \ldots, s_{p_m|q_m})$ in $H^0(M, L^{\otimes m})$ identifies $\mathbb{P}\Big(H^0(M, L^{\otimes m})\Big)$ with a superprojective $\mathbb{P}^{p_m|q_m}$ where the superdimensions p_m and q_m are due to the choice of even and odd sections.

Let us consider an example. We consider $\mathbb{P}^{1|2}$ (which is a super-Calabi–Yau). Chosen m, $H^0(\mathbb{P}^{1|2}, L^{\otimes m})$ is spanned by

$$H^0(\mathbb{P}^{1|2}, L^{\otimes m}) = \Big\{ z_0^a z_1^{m-a},\, z_0^a z_1^{m-a-1}\theta_I,\, z_0^a z_1^{m-a-2}\theta_1\theta_2 \Big\} \tag{6.24}$$

$p_m = 2m - 1$ and $q_m = 2m$. We can see that each space $H^0(\mathbb{P}^{1|2}, L^{\otimes m})$ is again a super-Calabi–Yau space. This can be verified easily using the formulas given in [24] and it amounts to see that the number of anticommuting coordinates must exceed of one w.r.t. the commuting ones. In analogy with the bosonic case, we define the supermanifold $(M, L^{\otimes m})$ superbalanced if one can choose a basis in $\mathbb{P}\Big(H^0(M, L^{\otimes})\Big)$ such that the $V = \iota_m(M)$ is a superbalanced variety.

On the space $\mathbb{P}\Big(H^0(M, L^{\otimes m})\Big)$ we can define the Kähler form induced by the Fubini–Study form on $\mathbb{P}^{p_m|q_m}$, namely

$$\omega_m = \frac{i}{2\pi} \partial\bar{\partial} \log \sum_{l=0}^{p_m+q_m} \left| \frac{s_l(x)}{\sigma(x)} \right|^2 \tag{6.25}$$

where $\sigma(x)$ is an invertible even section of $H^0(\mathbb{P}^{1|2}, L^{\otimes m})$. For $H^0(\mathbb{P}^{1|2}, L^{\otimes m})$ we have

$$\omega_m = \frac{i}{2\pi} \partial\bar{\partial} \log\Big(\sum_{l=0}^{m} |z_0^l z_1^{m-l}|^2 + i \sum_{l=0}^{m-1} |z_0^l z_1^{m-1-l}|^2 (\theta_0 + \theta_1)(\bar{\theta}_0 + \bar{\theta}_1) - \sum_{l=0}^{m-2} |z_0^l z_1^{m-2-l}|^2 \theta_1\bar{\theta}_1\theta_2\bar{\theta}_2 \Big). \tag{6.26}$$

The $\theta - \bar{\theta}$-sections are absent in the usual geometry and it appeared in physics in the context of supertwistor geometry. The differentials ∂ and $\bar{\partial}$ are natural extensions of the one dimensional case. In the case of $H^0(\mathbb{P}^{1|2}, L^{\otimes m})$, we can define an holomorphic form by separating the commuting sections s_i with $i = 1, \ldots, 2m - 1$ from the anticommuting ones $\widehat{s}_I$ with $I = 1, \ldots, 2m$ and following the prescription given above in Section 6.3.

The next step is to consider a super Hermitian metric $L^{\otimes m} \times L^{\otimes m} \to \mathbb{C}^{1|0}$ defined by the formula

$$h_m(q, q') = \frac{1}{\lambda} \frac{\frac{q}{\sigma(x)} \overline{\frac{q'}{\sigma(x)}}}{\sum\limits_{l=0}^{p_m+q_m} |s_l(x)|^2}. \tag{6.27}$$

In the denominator, we have both commuting and anticommuting sections and they have to be taken into account to define an L^2-product and an orthonormal basis for $\mathbb{P}\left(H^0(M, L^{\otimes m})\right)$ as follows

$$\langle s_i, s_j \rangle_h = \int_M h_m(s_i(x), s_j(x)) \Omega_{CY} \wedge \bar{\Omega}_{CY} = (\delta_{ij}, \epsilon_{ij}) \tag{6.28}$$

where δ_{ij} is the diagonal metric for even sections and ϵ_{ij} is the off-diagonal metric for odd sections. The metric $\langle \cdots, \cdots \rangle_h$ reduced the symmetry group from $GL(p_m|q_m)$ to the supergroup $SU(p_m|q_m)$.

To finish this paragraph, we analyze in detail the balancing of $\mathbb{P}^{1|2}$ into $H^0(\mathbb{P}^{1|2}, L^{\otimes 2})$, namely into the space of homogeneous sections of degree 2. They are given by the set

$$\left\{ z_0^a z_1^{2-a}, z_0^{a'} z_1^{1-a'} \theta_I, \theta_1 \theta_2 \right\} \tag{6.29}$$

where $a = 0, 1, 2$ and $a' = 0, 1$. So, we can form the following integrals (where z is the affine coordinate on $\mathbb{P}^1$)

$$B_{a,b} = \int_{\mathbb{P}^{1|2}} \Omega_{CY} \wedge \bar{\Omega}_{CY} \frac{z^{2-a} \bar{z}^{2-b}}{1 + \sum\limits_{a=0,1,2} |z|^{4-2a} + \mathrm{i} \sum\limits_{a'=0,1} |z|^{2-2a'} \theta_I \bar{\theta}^I + \theta_1 \bar{\theta}_1 \theta_2 \bar{\theta}_2}$$

$$B_{a'I,b'J} = \int_{\mathbb{P}^{1|2}} \Omega_{CY} \wedge \bar{\Omega}_{CY} \frac{z^{1-a'} \bar{z}^{1-b'} \theta_I \bar{\theta}_J}{1 + \sum\limits_{a=0,1,2} |z|^{4-2a} + \mathrm{i} \sum\limits_{a'=0,1} |z|^{2-2a'} \theta_I \bar{\theta}^I + \theta_1 \bar{\theta}_1 \theta_2 \bar{\theta}_2}$$

$$B_{12,12} = \int_{\mathbb{P}^{1|2}} \Omega_{CY} \wedge \bar{\Omega}_{CY} \frac{\theta_1 \theta_2 \bar{\theta}_1 \bar{\theta}_2}{1 + \sum\limits_{a=0,1,2} |z|^{4-2a} + \mathrm{i} \sum\limits_{a'=0,1} |z|^{2-2a'} \theta_I \bar{\theta}^I + \theta_1 \bar{\theta}_1 \theta_2 \bar{\theta}_2} \tag{6.30}$$

where the integrals are easily performed by the previous instructions. In the last integral, the Berezin integration removes those θ's and it leaves a bosonic integral on $\mathbb{P}^1$ which is similar to the classical integrals in the bosonic balanced manifolds. However, here we see that we have new conditions coming from the other integrals. For example, from the first one we need to expand the denominator to soak up enough θ's. And this leads to new conditions on the embeddings. Notice that in general the form of the embedding given by (6.29) is not balanced, but one needs to adjust some numerical coefficients in front of each given section.

6.3. Balancing of points

As a second application, we consider the problem of the stability of point of the type $\mathbb{C}^{0|n}$ into the superprojective space $\mathbb{P}^{1|n}$. Since the superspace $\mathbb{C}^{0|n}$ is not a Calabi–Yau, we need to use the measure defined in (6.2). However, in this simple case it reduces to the product of delta functions: $\prod_{i=1}^n \wedge \delta(\mathrm{d}\theta^i)$. Before doing that, we specify the embedding as follows: we denote by $\eta_i, \bar{\eta}_i$ the anticommuting generators of $\mathbb{C}^{0|n}$, we construct the morphism between the two superspace by the map

$$P : \mathbb{C}^{0|n} \longrightarrow \mathbb{P}^{1|n}$$

$$X_I = P^*(x_I) = \alpha_I + \alpha_{I,[jk]} \eta_j \eta_k + \cdots \tag{6.31}$$

$$\Theta_i = P^*(\theta_i) = \beta_{i,j} \eta_j + \beta_{i,jkl} \eta_j \eta_k \eta_l + \cdots \tag{6.32}$$

where x_I, θ_i are the homogeneous coordinates on $\mathbb{P}^{1|n}$, with $I = 0, 1$ and $i = 1, \ldots, n$. The capital letters denote the pull-backs of the sheaf generators.

First we construct the matrices for the bosonic embeddings. For each single point $[x_0 : x_1]$ the map discussed in (6.19)–(6.20) gives

$$B[x_0 : x_1] = \frac{1}{|x_0|^2 + |x_1|^2} \begin{pmatrix} |x_0|^2 & x_0 \bar{x}_1 \\ x_1 \bar{x}_0 & |x_1|^2 \end{pmatrix} \tag{6.33}$$

and to extend it to supermanifold, we substitute the pull-backs X_I in place of the coordinates x_I. In this way, the momentum map becomes a superfield of the anticommuting coordinates η_i and therefore we need to integrate over them to get a numerical value. For that reason we define the following new quantity

$$M(P) = \int \prod_{i=1}^n \mathrm{d}\eta_i \mathrm{d}\bar{\eta}_i \left(\frac{1}{|X_0|^2 + |X_1|^2 + \mathrm{i} \sum\limits_j \Theta_j \bar{\Theta}_j} \begin{pmatrix} |X_0|^2 & X_0 \bar{X}_1 \\ X_1 \bar{X}_0 & |X_1|^2 \end{pmatrix} \right) \tag{6.34}$$

where P is the point in the superprojective space $\mathbb{P}^{1|N}$. We also define $\sigma_{ij} = \frac{\partial}{\partial \eta_i} \Theta_j(\eta_i)$ as the embedding matrix

$$\sigma_{ij} = \beta_{i,j} + \beta_{i,jkl} \eta_k \eta_l + \cdots. \tag{6.35}$$

The function $M(P)$ is the generalization of the usual moment map $B[x_0 : x_1]$, where we embed the point $[x_0 : x_1]$ into a $su(2)$ matrix. On the other hand, for supermanifolds $M(P)$ is the embedding of the point P into the upper-left corner of the supermatrix $su(2|N)$ which is the Lie algebra of the isometry group of $\mathbb{P}^{1|N}$ which is represented by an $su(2)$ matrix. For that reason the normalization term $|X_0|^2 + |X_1|^2 + i\sum_j \Theta_j \bar{\Theta}_j$ acquires the supplementary summand $i\sum_j \Theta_j \bar{\Theta}_j$. Notice that we have also to take into account the embedding of the point into the $su(N)$ part of the supermatrix needed to implement the third type of condition in Eqs. (6.22) and this will be done later.

So, the final condition for the stability of a set of points $\mathbb{C}^{0|n}$ immersed into $\mathbb{P}^{1|n}$ is

$$\sum_P \prod_{i=1}^n \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \bar{\eta}_i} \mathcal{M}_P|_{\eta_i = \bar{\eta}_i = 0} = \lambda \mathbf{1},$$

$$\mathcal{M}_P = \frac{1}{|X_{P,0}|^2 + |X_{P,1}|^2 + i\sum_j \Theta_{P,j}\bar{\Theta}_{P,j}} \begin{pmatrix} |X_{P,0}|^2 & X_{P,0}\bar{X}_{P,1} \\ X_{P,1}\bar{X}_{P,0} & |X_{P,1}|^2 \end{pmatrix} \tag{6.36}$$

where $X_{P,I}$, $\Theta_{P,i}$ are respectively the embeddings for the point P and the sum is extended over all points.

Before discussing the third type of condition in Eqs. (6.22), let us analyze the condition (6.36) for a specific example, when $\mathbb{C}^{0|2}$ is embedded into $\mathbb{P}^{1|2}$. For that we consider the embedding

$$\begin{aligned} P &: \mathbb{C}^{0|2} \longrightarrow \mathbb{P}^{1|2} \\ X_i &= P^*(x_i) = \alpha_i + \widetilde{\alpha}_i \eta_1 \eta_2 \\ \Theta_i &= P^*(\theta_i) = \sigma_{i,j}\eta_j \end{aligned} \tag{6.37}$$

and we compute explicitly the expression in (6.36). After few manipulations, we get

$$\begin{aligned} \sum_P \Bigg\{ \Upsilon_P^3 \left(2|\sum_i \bar{\alpha}_{P,i}\widetilde{\alpha}_{P,i}|^2 - \sum_i |\alpha_{P,i}|^2 \sum_i |\widetilde{\alpha}_{P,i}|^2 - 2\,(\det|\sigma|^2) \right) \begin{pmatrix} |\alpha_{P,0}|^2 & \alpha_{P,0}\bar{\alpha}_{P,1} \\ \alpha_{P,1}\bar{\alpha}_{P,0} & |\alpha_{P,1}|^2 \end{pmatrix} \\ + \Upsilon_P^2 \left[\left(\sum_i \alpha_{P,i}\widetilde{\bar{\alpha}}_{P,i} \right) \begin{pmatrix} \alpha_{P,0}\widetilde{\bar{\alpha}}_{P,0} & \alpha_{P,1}\widetilde{\bar{\alpha}}_{P,0} \\ \alpha_{P,0}\widetilde{\bar{\alpha}}_{P,1} & \alpha_{P,1}\widetilde{\bar{\alpha}}_{P,1} \end{pmatrix} + \text{h.c.} \right] - \Upsilon_P \begin{pmatrix} |\widetilde{\alpha}_{P,0}|^2 & \widetilde{\alpha}_{P,0}\widetilde{\bar{\alpha}}_{P,1} \\ \widetilde{\alpha}_{P,1}\widetilde{\bar{\alpha}}_{P,0} & |\widetilde{\alpha}_{P,1}|^2 \end{pmatrix} \Bigg\} = \lambda\mathbf{1}. \end{aligned} \tag{6.38}$$

where $\Upsilon_P = 1/\sum_i |\alpha_{P,i}|^2$. For example, in the case of only a single point $P = [1 : 0]$, we get the simplified equation

$$\begin{pmatrix} -|\widetilde{\alpha}_1|^2 + \widetilde{\alpha}_0^2 + \widetilde{\bar{\alpha}}_0^2 - 2\det|\sigma|^2 & \widetilde{\alpha}_0\widetilde{\alpha}_1 - \widetilde{\alpha}_0\widetilde{\bar{\alpha}}_1 \\ \widetilde{\bar{\alpha}}_0\widetilde{\alpha}_1 - \widetilde{\alpha}_0\widetilde{\alpha}_1 & -|\widetilde{\alpha}_1|^2 \end{pmatrix} = \lambda\mathbf{1}. \tag{6.39}$$

From the up-right corner we get $\widetilde{\alpha}_1$ is real and this fixes the constant λ. Then, we get the condition, $\widetilde{\alpha}_0^2 + \widetilde{\bar{\alpha}}_0^2 - 2\det|\sigma|^2 = 0$ which can be solved in terms of $\widetilde{\alpha}_0$. Therefore, there is a single point whose embedding into $\mathbb{P}^{1|2}$ is balanced. The logic can be repeated for several points and other solutions can be also found. Notice that the non-numerical part of X_i, namely the part which is parametrized by $\widetilde{\alpha}_i$ plays a fundamental role and serves for the balancing. We can also recover the classical solution by setting all $\widetilde{\alpha}_i$ to zero. This implies the classical balancing condition and therefore we found that there are also the classical solutions with anticommuting coordinates.

It remains to compute the contribution for the embedding in the second $su(2)$ of the supermatrix $su(2|2)$ and for that we have

$$M(P)_{kl} = \int \prod_{i=1}^n d\eta_i d\bar{\eta}_i \left(\frac{i\Theta_k\bar{\Theta}_l}{|X_0|^2 + |X_1|^2 + i\sum_j \Theta_j\bar{\Theta}_j} \right) \tag{6.40}$$

as follows from (6.22). It is easy to evaluate the Berezin integrals to get

$$M(P)_{kl} = \lambda\delta_{kl} \tag{6.41}$$

where $\lambda = \det|\sigma|^2/(\sum_i |\alpha_i|^2)^2$ and in the case of the point $P = [1 : 0]$ we have $\lambda = \det|\sigma|^2$. In the way, we notice that this part of the embedding is automatically balanced and it does not yield a new condition on the parameters of the embedding.

To our knowledge, the present discussion is a way to formulate the balancing of points into a superprojective space. Of course, one can add further condition, for example inserting in the integral the factor $\exp(i\sum_j \Theta_j\bar{\Theta}_j)$. This term reproduces the previous results, but in addition it leads to a further condition that coincides with the classical requirement of balanced points into a projective space. So, instead of imposing by hand the additional condition of classical stability, the modification of the integration measure yields all possible sets of conditions. Moreover, for the case with more than 2 anticommuting coordinates the exponential factor $\exp(i\sum_j \Theta_j\bar{\Theta}_j)$ will lead to new conditions on the embedding. This makes sense since adding new anticommuting coordinates requires new embedding parameters to be fixed.

There are several questions that can be addressed in the same framework, for example: can all points be made balanced after an $SL(2|2)$ transformation? Is there a relation between our definition of balanced supermanifolds and a suitable notion of stability, such as in GIT, in the supermanifold context (see [18])? We will leave these questions to forthcoming publications.

Acknowledgments

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