ZEROS OF BRAUER CHARACTERS AND LINEAR ACTIONS OF FINITE GROUPS: SMALL PRIMES

SILVIO DOLFI AND EMANUELE PACIFICI

Dedicated to the memory of David Chillag

ABSTRACT. We describe the finite groups whose p-Brauer character table, for $p = 2$ or $p = 3$, does not contain any zero. This completes the analysis in [6], where we considered the case $p \geq 5$.

INTRODUCTION

In a recent paper ([6]), motivated by the study of G. Malle in [10], we considered the problem of describing the structure of the finite groups whose p-Brauer character table, for a fixed prime p, does not contain any zero.

As Malle shows ([10, Theorem 1.3]), the groups satisfying this condition are solvable if $p \neq 2$, whereas for $p = 2$ nonsolvable examples occur. The nonabelian simple composition factors of such groups are classified ([10, Theorem 1.2]) as the simple groups in the class

$$\mathcal{L} = \{L_2(2^a),\ a \geq 2;\quad L_2(q),\ q = 2^a + 1 \geq 5;\quad {}^2B_2(2^{2a+1}),\ a \geq 1;\quad S_4(2^a),\ a \geq 2\}\ .$$

While in [6] we considered characteristics $p \geq 5$, here we complete the analysis by addressing the cases $p = 2$ and $p = 3$. In the following statements, $\mathbf{F}(G)$ denotes the Fitting subgroup of the group G and $l_{p'}(G)$ the p'-length of G (p a prime). Note also that, as $\mathbf{O}_p(G)$ lies in the kernel of every Brauer character, we are allowed to assume $\mathbf{O}_p(G) = 1$.

Theorem A. *Let G be a finite group and $p \in \{2, 3\}$. Assume that $\mathbf{O}_p(G) = 1$ and that the p-Brauer character table of G does not contain any zero. Then the following conclusions hold.*

(a) *If $p = 2$ and G is solvable, then $G/\mathbf{F}(G)$ is a $\{2, 3\}$-group with elementary abelian Sylow 3-subgroups and $2'$-length at most 1.*
(b) *If $p = 2$ and G is nonsolvable, then there exist normal subgroups R, N of G, $R \leq N$, with R solvable, $l_{2'}(R) \leq 4$, N/R a direct product of simple groups belonging to the class $\mathcal{L}$ and G/N a group of 2-power order.*

Key words and phrases. Finite groups; Brauer characters.
The authors are partially supported by the MIUR project "Teoria dei gruppi e applicazioni".

(c) *If $p = 3$, then $G/\mathbf{F}(G)$ is a subgroup of a direct product $A \times B$, where A is a $\{2, 3\}$-group with elementary abelian Sylow 2-subgroups and $3'$-length at most 1 and $B \simeq (\mathrm{Sym}(3) \wr \mathrm{Sym}(3)) \wr P$, where P is a 3-group. In particular, $G/\mathbf{F}(G)$ is a $\{2, 3\}$-group of $3'$-length at most 2.*

Theorem B. *Let G be a finite solvable group and $p \in \{2, 3\}$. Assume that the p-Brauer character table of G does not contain any zero. Then the following conclusions hold.*

(a) *If $p = 2$, then $l_{2'}(G) \leq 2$, $l_2(G/\mathbf{O}_2(G)) \leq 2$ and $\mathbf{O}^2(G/\mathbf{O}_2(G)) = A \times B$, where A is a $\{2, 3\}$-group and B is a nilpotent $\{2, 3\}'$-group.*
(b) *If $p = 3$, then $l_{3'}(G) \leq 3$, $l_3(G/\mathbf{O}_3(G)) \leq 3$ and $\mathbf{O}^3(G/\mathbf{O}_3(G)) = A \times B$, where A is a $\{2, 3\}$-group and B is a nilpotent $\{2, 3\}'$-group.*

Theorem A and Theorem B should be paired, respectively, with Theorem A and Corollary B of [6]. Note that in Theorem B above the solvability assumption is redundant for $p = 3$, but needed if $p = 2$.

We remark that we have no examples of groups with no zeros in the 3-Brauer character table and with $3'$-length greater than 2. So, part (b) of Theorem B can possibly be improved.

The study of groups whith no zeros in the Brauer character table can be approached by considering some particular linear actions. Namely, denoting by V a faithful irreducible module over a prime field for a finite solvable group G, it is relevant to keep under control the situation when *every p'-element of G fixes at least one element in each G-orbit on V*. As customary, the analysis of the "primitive case" turns out to be a crucial step. The following statement encloses Theorems 2.1 and 3.1 of this paper and, paired with Theorem 3.1 of [6], should be compared with Corollary 10.6 of [11] (see Theorem 1.3).

As usual, we denote by $\Gamma(q^n)$ the semilinear group on the field of q^n elements.

Theorem C. *Let G be a finite solvable group, $p \in \{2, 3\}$ and V a faithful primitive G-module of order q^d, q a prime. Assume that G is not a p-group, and that every p'-element of G fixes an element in each G-orbit on V. Then one of the following conclusions hold.*

(a) *$p = 2$, $q^d = 3^2$, and G is isomorphic either to $\mathrm{GL}(2, 3)$ or to $\mathrm{SL}(2, 3)$, acting naturally on V.*
(b) *$p = 3$, $q = 2$, $d \in \{2, 6\}$, and G is isomorphic to $\Gamma(2^2)$ if $d = 2$, whereas G is isomorphic to a Hall $\{2, 3\}$-subgroup of $\Gamma(2^6)$ if $d = 6$. In both cases, G acts naturally on V.*

By means of the previous result, we are able to classify the *irreducible* actions with the relevant orbit property for $p \in \{2, 3\}$. This is done in Theorem 2.2 and Theorem 3.3, which complete the analysis of [6, Theorem C].

1. Preliminaries

Every group considered throughout the paper is meant to be a finite group. The preliminary notation and results that are relevant for our purposes are essentially those of [6], and here we shall recall only few of them. We start by introducing an orbit property that will play a central role in our discussion.

Definition 1.1. Let Σ be a finite nonempty set, and let G be a subgroup of $\mathrm{Sym}(\Sigma)$. Also, let $\mathcal{O}$ be an orbit of the action of G on Σ, and π a set of prime numbers. We say that the orbit $\mathcal{O}$ is *π-deranged* if there exists a π-element of G which does not fix any element in $\mathcal{O}$.

For the purposes of the present paper, we will have $\pi = p'$, that is the set of prime numbers different from a fixed prime p. We shall consider p'-deranged orbits both in the general case of permutation actions and in the case of linear actions on modules.

We recall that a G-module V is said to be *primitive* if V is irreducible and V is not induced by a submodule of any proper subgroup of G. By Clifford Theory, it follows that if V is a primitive G-module, then the restriction V_N is a homogeneous module (i.e. direct sum of isomorphic modules) for every normal subgroup N of G.

Also, following [11], we say that a (not necessarily irreducible) G-module V is *pseudo-primitive* if V_N is homogeneous for every characteristic subgroup N of G.

The structure of primitive solvable groups of linear transformations is rather tight. We collect the relevant facts in the following proposition (see for instance [14, Theorem 2.2], [11, Lemma 0.5 and Theorem 1.9]).

Proposition 1.2. *Let G be a solvable group, and V a faithful primitive G-module over a finite field $\mathbb{K}$. Then there exist subgroups $Z \leq U \leq F \leq A$, and E, all normal in G, with the following properties.*

(a) *U is cyclic, and E is a product of subgroups $E_i \trianglelefteq G$ of pairwise coprime orders, such that, for every i, E_i is cyclic of prime order p_i or an extraspecial p_i-group (of exponent p_i if $p_i \neq 2$, and of order $p_i^{2n_i+1}$ for a suitable integer n_i). Also, $F = EU$ is a central product, $Z = E \cap U = \mathbf{Z}(E)$ and $\mathbf{C}_G(F) \leq F$. Moreover, $F = \mathbf{F}(G) \cap A$ and $|\mathbf{F}(G) : F| \leq 2$.*

(b) *$F/U \simeq E/Z$ is a direct sum of completely reducible G/F-modules.*

(c) *$A = \mathbf{C}_G(U)$, so that G/A embeds into the abelian group $\mathrm{Aut}(U)$.*

(d) *A/F acts faithfully on E/Z, and $A/\mathbf{C}_A(E_i/\mathbf{Z}(E_i))$ embeds into the symplectic group $\mathrm{Sp}(2n_i, p_i)$.*

(e) *Setting $e = \sqrt{|E : Z|}$, we have that e divides $\dim_{\mathbb{K}}(V)$.*

If q is a prime and V is a finite vector space of order q^n, then $\Gamma(V)$ denotes a subgroup of $\mathrm{Aut}(V)$ isomorphic to the *semilinear group* $\Gamma(q^n)$, obtained by identifying V with $\mathrm{GF}(q^n)$ (see [11, page 38]). We shall write $\Gamma_0(V)$ for the subgroup of $\Gamma(V)$ consisting of the multiplication maps. In the setting of Proposition 1.2, if $e = 1$, then Corollary 2.3(a) of [11] yields that G can be identified with a subgroup of $\Gamma(V)$ acting naturally on V; it is easy to see that, conversely, $G \leq \Gamma(V)$ implies $e = 1$.

We recall next a result by T. Wolf, concerning module actions where all orbits have size not divisible by any prime in a fixed set π. Observe that, if π is the set of prime numbers different from a given prime p, this is a special kind of action without p'-deranged orbits. In fact, the two conditions coincide when the acting group has cyclic Hall p'-subgroups. Moreover, somewhat surprisingly, the two conditions turn out to be equivalent also when the action is primitive, as one can check by comparing the following Theorem 1.3 with Theorem 3.1 of [6] and Theorem C of the present paper.

Theorem 1.3 ([11], Corollary 10.6)**.** *Let V be a finite faithful and pseudo-primitive G-module, for a solvable group G. Let π be a set of primes such that $\pi \cap \pi(G) \neq \emptyset$, and assume that $\mathbf{C}_G(v)$ contains a Hall π-subgroup of G for all $v \in V$. Then V is an irreducible G-module and one of the following occurs.*

(a) $G \leq \Gamma(V)$.

(b) $\pi = \{3\}$, $|V| = 3^2$ *and G is isomorphic either to* $\mathrm{GL}(2,3)$ *or to* $\mathrm{SL}(2,3)$.

(c) $\pi = \{2\}$, $|V| = 2^6$, $F = \mathbf{F}(G)$ *is extraspecial of order* 3^3 *and exponent* 3 *and G/F is a group of order* 2 *that acts inverting all elements of* $F/\mathbf{Z}(F)$ *and trivially on* $\mathbf{Z}(F)$.

We stress that the group G in part (c) of Theorem 1.3 is determined up to conjugation in $\mathrm{GL}(6,2)$ (see [11, Example 10.3]) and that the module V is not primitive (in fact G has a non-cyclic normal subgroup of order 9).

We now prove a proposition concerning semilinear groups acting with no p'-deranged orbits on finite vector spaces, with $p = 2$ or $p = 3$; this completes the analysis carried out in Theorem 3.1(b) of [6], where the case $p \geq 5$ is treated.

Proposition 1.4. *Let q be a prime, V a vector space of order q^d, and G a subgroup of $\Gamma(V)$ acting irreducibly on V. For a fixed prime p, assume that G is not a p-group, and that there are no p'-deranged orbits for the action of G on V. Then the following conclusions hold.*

(a) $p \neq 2$.

(b) *If $p = 3$, then $q = 2$ and $d \in \{2, 6\}$. More precisely, if $d = 2$ then $G = \Gamma(2^2)$, whereas if $d = 6$ then either G is a Hall $\{2,3\}$-subgroup of $\Gamma(2^6)$, or $G \leq \Gamma(2^6)$ is a Frobenius group of order* 18.

Conversely, the groups in conclusion (b) *act irreducibly and with no $3'$-deranged orbits on the natural module V.*

Proof. Set $\Gamma_0 = \Gamma_0(V)$, and $G_0 = G \cap \Gamma_0$; we know that $G/G_0 \simeq \Gamma_0 G/\Gamma_0$ is cyclic. Now, if R is a Hall p'-subgroup of G, we get $R \cap G_0 = 1$, because G_0 acts fixed-point freely on V; thus G_0 is a cyclic p-group. This implies that $R \simeq G_0R/G_0$ is cyclic, and also that $r := |R|$ divides $|\Gamma(V) : \Gamma_0| = d$.

Next, we observe that $\mathbf{C}_V(R) \cap \mathbf{C}_V(R^g) = \{0\}$ for every $g \in G$ such that $R^g \neq R$. In fact, if $v \in V \setminus \{0\}$, then $\mathbf{C}_G(v) \cap G_0 = 1$; therefore $\mathbf{C}_G(v)$ is cyclic, and it cannot contain two distinct Hall p'-subgroups of G. Since R is cyclic, the assumption on p'-deranged orbits implies that the centralizer of every nontrivial element of V contains one (and hence only one) Hall p'-subgroup of G. Thus, $V \setminus \{0\}$ is partitioned by the sets $\mathbf{C}_V(R) \setminus \{0\}$ for $R \in \mathrm{Hall}_{p'}(G)$. It follows that $q^d - 1 = h(|\mathbf{C}_V(R)| - 1) = h(q^{d/r} - 1)$, where h is the cardinality of the set of Hall p'-subgroups of G and the second equality follows from Lemma 3(ii) of [4].

By coprimality, $G_0 = \mathbf{C}_{G_0}(R) \times [G_0, R]$ and, G_0 being a cyclic p-group not centralized by R, it follows $\mathbf{C}_{G_0}(R) = 1$ and $h = |G_0| = p^a$, for a suitable integer a. Hence

$$p^a = \frac{q^d - 1}{q^{d/r} - 1}.$$

Next, assume that there exists a Zsigmondy prime divisor of $q^d - 1$ (see [11, Theorem 6.2]), i.e., p is in fact the unique Zsigmondy prime divisor of $q^d - 1$. In particular, d divides $p - 1$. Since d cannot be 1 (otherwise G would be a p-group), p is not 2. If $p = 3$, then d, r and q must be 2, and so G is a subgroup of $\Gamma(2^2)$. As G is not a 3-group, it must be the whole $\Gamma(2^2)$.

On the other hand, assume there does not exist a Zsigmondy prime divisor of $q^d - 1$. If $d = 2$, then also r must be 2, and this yields $p \neq 2$ (recall that r is a p'-number). Also, if $d = 2$ and $p = 3$, then $3^a = q + 1$ is a power of 2 (by Zsigmondy's Theorem [11, 6.2]), a contradiction. It remains to treat the case $d = 6$ and $q = 2$. In this situation too, p is clearly not 2. If $p = 3$, then $r = a = 2$ (as $2^6 - 1$ and $(2^6 - 1)/(2^2 - 1)$ are not powers of 3). Recalling that $|G_0| = h = 3^2$, we have $|G| \in \{2 \cdot 3^2, 2 \cdot 3^3\}$, and conclusions (a), (b) follow. The last claim of the statement is straightforward. ∎

Remark 1.5. We observe that, if $G \leq \Gamma(2^6)$ has order 18, then the action of G on the natural module V is not primitive. In fact, as can be easily checked, G has a subgroup $H \simeq \mathrm{Sym}(3)$, and V_H has a submodule W of dimension 2, such that $V \simeq W^G$.

The other two groups appearing in conclusion (b) of Proposition 1.4 do act primitively on the natural module V.

2. Linear actions with no $2'$-deranged orbits

In the next result, we describe the solvable groups acting faithfully, primitively and without $2'$-deranged orbits on a finite vector space over a prime field. In fact, it turns out that there are only two of them.

Theorem 2.1. *Let G be a solvable group, q a prime number, and V a faithful primitive G-module of order q^d. Assume that G is not a 2-group and that there are no $2'$-deranged orbits for the action of G on V, i.e., for every $v \in V$ and $x \in G$ of odd order, there exists $g \in G$ such that $x \in \mathbf{C}_G(v^g)$. Then $q = 3$, $d = 2$, and G is isomorphic either to* $\mathrm{GL}(2,3)$ *or to* $\mathrm{SL}(2,3)$*, acting naturally on V.*

Proof. We shall take into account the fact that, by Lemma 2.9 of [6], the Fitting subgroup of G is a 2-group; also, we shall use the description of the structure of G provided by Proposition 1.2, and the notation introduced therein. Observe that, since in the present context U is a cyclic 2-group, the factor group G/A embeds in the abelian 2-group $\mathrm{Aut}(U)$; in particular, F is properly contained in A, as otherwise G would be a 2-group.

Let N be a subgroup of A such that N/F is a chief factor of G, say of order r^k for a suitable prime r. Note that $r \neq 2$, as otherwise $N \leq F$. Now, denote by X a subgroup of order r of N. By Lemma 2.2 of [5] (and as $\mathbf{C}_A(E) \leq F$), the group $[X, E]$ is an extraspecial 2-group of order 2^{2m+1} for a suitable $m \in \{1, \ldots, n\}$, where $2^{2n+1} = |E|$. Moreover, as can be deduced from the proof of Lemma 2.4 of [14] (or [5, Lemma 2.4]), we have $|\mathbf{C}_V(X)| \leq |V|^{\alpha_m}$, where

$$\alpha_m = \begin{cases} \frac{1}{r}\left(\frac{2^m+r-1}{2^m}\right) & \text{if} \quad r \mid 2^m - 1; \\ \\ \frac{1}{r}\left(\frac{2^m+1}{2^m}\right) & \text{if} \quad r \mid 2^m + 1. \end{cases}$$

(Observe that X acts fixed-point freely on $[X, E]/Z$, therefore r is a divisor of $2^{2m} - 1$.) In any case, α_m is at most $1/2$. Observe also that the number of N-conjugates of X is at most 2^{2m}; in fact

$$|N : \mathbf{N}_N(X)| = |F : \mathbf{N}_F(X)| \leq |F : \mathbf{C}_F(X)| = |E : \mathbf{C}_E(X)| =$$
$$= |E/Z : \mathbf{C}_E(X)/Z| = |[X, E]/Z| = 2^{2m}.$$

Denote by R a Sylow r-subgroup of N (say $|R| = r^k$), and by λ_m the number of subgroups X of R such that $|X| = r$ and $|[X, E]| = 2^{2m+1}$. We conclude that, if

$$\sum_{m=1}^{n} \lambda_m 2^{2m} |V|^{\alpha_m} < |V| \tag{1}$$

holds, then in particular the centralizers in V of the elements of order r in N do not cover the whole V. In other words there exists an element v in V such that, for $x \in N$ with $\mathrm{o}(x) = r$, we have $x \notin \bigcup_{g \in G} \mathbf{C}_G(v^g)$, against our assumptions.

Taking into account that $\sum_{m=1}^{n} \lambda_m$ equals $\frac{r^k-1}{r-1}$ (the total number of subgroups of order r in R), and that $\alpha_m \leq 1/2$, the left-hand side of Inequality (1) is bounded above by $\frac{r^k-1}{r-1} \cdot e^2 \cdot |V|^{1/2} = \frac{r^k-1}{r-1} \cdot e^2 \cdot q^{\frac{ef}{2}}$, for a suitable $f \in \mathbb{N}$. Thus, also the inequality

$$\frac{r^k - 1}{r - 1} \cdot e^2 < q^{\frac{ef}{2}} \tag{2}$$

would yield a contradiction. This will be enough for most instances of the following analysis.

Since G is not a 2-group and it has no $2'$-deranged orbits on V, then G cannot have regular orbits on V. Hence, as e is a power of the prime 2, Theorem 3.1 of [14] yields $e \in \{1, 2, 4, 8, 16\}$. (But the value 1 is of course not a possibility, as in that case we would get $F = U = A$, thus G/F would be a 2-group, as well as G itself). We shall work to show that $e = 2$ is in fact the only possible value, and in that case we get either $G \simeq \mathrm{GL}(2, 3)$ or $G \simeq \mathrm{SL}(2, 3)$.

We start by considering the case $e = 2$. In this situation, we have $A/F \leq \mathrm{Sp}(2, 2) \simeq \mathrm{Sym}(3)$, therefore the prime r must be 3 and the elementary abelian 3-subgroup N/F of A/F is in fact cyclic. Also, since G/A is a 2-group, the Sylow 3-subgroups of G have order 3. Now, our assumptions imply that every $v \in V$ is centralized by a Sylow 3-subgroup of G: therefore we are in a position to apply Theorem 1.3, obtaining $q^d = 3^2$ and $G \simeq \mathrm{GL}(2, 3)$ or $G \simeq \mathrm{SL}(2, 3)$, as wanted.

Since $N \trianglelefteq G$ and V is a primitive G-module, then V_N is a pseudo-primitive N-module. So, by Theorem 1.3 we can assume that N/F is non-cyclic.

Set now $e = 4$, so that $A/F \leq \mathrm{Sp}(4, 2) \simeq \mathrm{Sym}(6)$. More precisely, $A/F \leq \mathrm{O}^-(4, 2) \simeq S_5$ if $E \simeq Q_8 \curlyvee D_8$, and $A/F \leq \mathrm{O}^+(4, 2) \simeq S_3 \wr S_2$ if $E \simeq Q_8 \curlyvee Q_8$ (see for instance [13, Theorem 2.4.6 and Appendix B]). Since the Sylow r-subgroups of S_5 are cyclic for $r \neq 2$, by the paragraph above we can rule out the case $E \simeq Q_8 \curlyvee D_8$, therefore we can assume $E \simeq Q_8 \curlyvee Q_8$ and $r = 3$. Also, the rank k of N/F is at most 2, hence Inequality (2) is satisfied (and our assumptions are not) for every $q^f \geq 9$. We conclude that $f = 1$ and $q \in \{3, 5, 7\}$, i.e., G is isomorphic to a subgroup of $\mathrm{GL}(4, 3)$, or $\mathrm{GL}(4, 5)$, or $\mathrm{GL}(4, 7)$. Now, consider a subgroup $E \simeq Q_8 \curlyvee Q_8$ of $\mathrm{GL}(4, q)$, for $q \in \{3, 5, 7\}$ (by [13, Theorem 2.4.7], there is exactly one conjugacy class of such subgroups), and let $H = \mathbf{N}_{\mathrm{GL}(4,q)}(E)$. One checks with GAP ([8]) that for $q \in \{5, 7\}$ there exists an element x of the natural module V such that $\mathbf{C}_H(x)$ is a $3'$-group. As $G \leq H$, this yields a $2'$-deranged orbit for the action of G on V, against the assumption. For $q = 3$, one checks that H has two conjugacy classes C_1, C_2 of elements of order 3 and that they are real. Also, there exists an element x of the natural module V such that $\mathbf{C}_H(x)$ has Sylow 3-subgroups of order 3. Thus, $\mathbf{C}_H(x)$ intersects just one class among C_1 and C_2. As $|H : G|$ is coprime to 3, $G \cap C_1$ and $G \cap C_2$ are both nonempty. It follows that x lies in a $2'$-deranged orbit for the action of G on V, a contradiction.

As for the case $e = 8$, we get $A/F \leq \mathrm{Sp}(6, 2)$. Using the information in the Atlas [1], we see that the prime r lies in $\{3, 5, 7\}$, and the rank k of N/F is at most 3 for $r = 3$, whereas it is 1 for $r \neq 3$. In fact, in the latter situation, the r-part of $|\mathrm{Sp}(6, 2)|$ is r, and we can exclude this case as above. On the other hand, for $r = 3$, Inequality (2) is satisfied whenever $q^f \geq 7$; therefore we have $f = 1$ and $q \in \{3, 5\}$, i.e., G is isomorphic to a subgroup of $\mathrm{GL}(8, 3)$ or $\mathrm{GL}(8, 5)$. Assume first that $G \leq \mathrm{GL}(8,3)$ and let E be an extraspecial 2-subgroup of $\mathrm{GL}(8,3)$, with $|E| = 2^7$ (there are just two of them, up to conjugation). Let $H = \mathbf{N}_{\mathrm{GL}(8,3)}(E)$ and Q a Sylow 3-subgroup of H. Then with GAP [8] one checks that for every $N = ET$, where T varies among the representatives of conjugacy classes of elementary abelian subgroups of Q, there exists an element $v \in V$ with $3 \nmid |\mathbf{C}_N(v)|$, against our assumptions.

If $G \leq \mathrm{GL}(8,5)$, then we have to consider the finer Inequality (1): the maximum value of $2^{2m}|V|^{\alpha_m} = 2^{2m}5^{8\alpha_m}$, for $m \in \{1,2,3\}$, is attained when $m = 2$ (and it is $2^4 \cdot 5^4$). Nevertheless, even if m is 2 for every $X \leq N$ with $|X| = 3$, the left-hand side of Inequality (1) (that is, $2^4 \cdot 5^4 \cdot 13$) is still smaller than $|V| = 5^8$, again a contradiction.

Finally, consider the case $e = 16$. We have $A/F \leq \mathrm{Sp}(8,2)$, thus $r \in \{3,5,7,17\}$. Moreover, the r-part of $|\mathrm{Sp}(8,2)|$ is r if $r \in \{7,17\}$, so these cases cannot occur. As for $r = 3$ or $r = 5$, by Atlas [1] we have $k \leq 4$ and $k \leq 2$ respectively. If $r = 5$, then Inequality (2) is satisfied for every $q^f \geq 3$, therefore also this case does not occur. It remains to consider the case $r = 3$: here Inequality (2) holds whenever $q^f \geq 5$, or $q^f \geq 3$ and $k \leq 3$, so only the case $k = 4$, $f = 1$ and $q = 3$ is left. In other words, G embeds into $\mathrm{GL}(16,3)$. Now, consider Inequality (1): the maximum value of $2^{2m}|V|^{\alpha_m} = 2^{2m}3^{16\alpha_m}$, for $m \in \{1,2,3,4\}$, is attained when $m = 4$ (and it is $2^8 \cdot 3^6$). But even if m is set to be 4 for every $X \leq N$ with $|X| = 3$, the left-hand side of Inequality (1) (that is, $2^{11} \cdot 3^6 \cdot 5$) is still smaller than $|V| = 3^{16}$. This is the final contradiction, and the proof is complete. ∎

We are now ready to describe the structure of solvable groups acting irreducibly and with no $2'$-deranged orbits (compare with [6, Theorem C]). In the following, we will denote by $\mathcal{P}(\Sigma)$ the set consisting of the subsets of a set Σ.

Theorem 2.2. *Let G be a solvable group, q a prime number, and V a faithful irreducible G-module over* $\mathrm{GF}(q)$. *Assume that G is not a 2-group, and that there are no $2'$-deranged orbits for the action of G on V. Then $q = 3$ and G is isomorphic to a subgroup of $H \wr K$, where H is isomorphic either to* $\mathrm{GL}(2,3)$ *or to* $\mathrm{SL}(2,3)$*, and K is a (possibly trivial) 2-group.*

Proof. Choose a subgroup T of G and a primitive submodule W of V_T such that $V = W^G$ (possibly $T = G$). Denoting by H the factor group $T/\mathbf{C}_T(W)$, we first observe that, by Lemma 2.7 of [6], there does not exist any $2'$-deranged orbit for the action of H on W. Therefore, by Lemma 2.9 in [6], $\mathbf{F}(H)$ is a 2-group; moreover, if $H \neq \mathbf{F}(H)$, then Theorem 2.1 yields $H \simeq \mathrm{GL}(2,3)$ or $H \simeq \mathrm{SL}(2,3)$.

In what follows, we shall keep in mind remarks 2.1 and 2.3 of [6]. In particular, denoting by Σ a right transversal for T in G, we recall that G can be identified with a subgroup of $H \wr K$, where K is a transitive subgroup of $\mathrm{Sym}(\Sigma)$; also, setting

$s = |\Sigma|$, the group $H \wr K$ (thus G) acts naturally on the direct sum $W^{\oplus s}$ of s copies of W, and the G-modules V and $W^{\oplus s}$ are isomorphic.

We first show that K is a 2-group. Assume, working for a contradiction, that K is not a 2-group. By Lemma 2.8 of [6], there exists a subset A of Σ such that $(A, \Sigma \setminus A)$ lies in a $2'$-deranged orbit for the action of K on $\mathcal{P}(\Sigma)$. Now, take a nonzero element $w \in W$, and consider the element v of $W^{\oplus s}$ whose ith component is w if $i \in A$, whereas it is 0 if $i \notin A$. We claim that v lies in a $2'$-deranged orbit for the action of G on $W^{\oplus s}$. In fact, let $k \in K$ be a $2'$-element which does not fix any element in the K-orbit of $(A, \Sigma \setminus A)$, and let $x \in G$ be a $2'$-element that is a preimage of k along the top projection of G onto K. Now, it is easy to see that x does not fix any element in the G-orbit of v. Our claim is proved, yielding a contradiction.

Hence, K is a 2-group. As G is not a 2-group and G is isomorphic to a subgroup of $H \wr K$, we conclude that H is not a 2-group. Therefore, as observed above, Theorem 2.1 applies to the action of H on W, and we are done. ■

3. Linear actions with no $3'$-deranged orbits

In this section we deal with the case $p = 3$. As in the previous section, a key step is the analysis of the primitive case, which is carried out in the following theorem.

Theorem 3.1. *Let G be a solvable group, q a prime number, and V a faithful primitive G-module of order q^d. Assume that G is not a 3-group, and that there are no $3'$-deranged orbits for the action of G on V. Then $q = 2$, $d \in \{2, 6\}$, and G is isomorphic to a subgroup of $\Gamma(V)$ acting naturally on V. More precisely, G is isomorphic to $\Gamma(2^2)$ if $d = 2$, whereas G is isomorphic to a Hall $\{2, 3\}$-subgroup of $\Gamma(2^6)$ if $d = 6$.*

Proof. As in the proof of Theorem 2.1, we shall keep in mind Lemma 2.9 of [6], together with Proposition 1.2 and the relevant notation. In particular, here $F = \mathbf{F}(G)$ is a 3-group.

Since G is not a 3-group and it has no $3'$-deranged orbits on V, in particular G has no regular orbits on V. Hence, as e is a power of 3, by Theorem 3.1 of [14] we get $e \in \{1, 3, 9\}$. As already mentioned, the condition $e = 1$ is equivalent to the fact that G is isomorphic to a subgroup of $\Gamma(V)$ acting naturally on V; therefore, in this case, we are in a position to apply Proposition 1.4 (taking also into account Remark 1.5), achieving the desired conclusion. In view of that, the rest of the proof aims to exclude the other two possibilities for the value of e.

Let N/F be a chief factor of G. We claim that, in both cases $e = 3$ and $e = 9$, there exist $v \in V$ and a $3'$-element $x \in N$ such that x does not fix any element in the G-orbit of v. In other words, we prove the existence of a $3'$-deranged orbit for the action of G on V, against the assumption.

First we show that N/F cannot be cyclic. In fact, let r be a prime divisor of $|N/F|$ (clearly $r \neq 3$); if N/F is cyclic, then our assumption concerning $3'$-deranged orbits implies that r does not divide $|N : \mathbf{C}_N(v)|$ for all $v \in V$. Since $N \trianglelefteq G$ and V is a primitive G-module, we have that V is a pseudo-primitive N-module. As N is not a subgroup of $\Gamma(V)$, and V has characteristic different from 3, we are in the situation described in case (c) of Theorem 1.3; in particular, $e = 3$ and $G \leq \mathrm{GL}(6, 2)$. Now, $\mathrm{GL}(6, 2)$ has just one conjugacy class of extraspecial 3-groups of order 27 and exponent 3; let E be a representative of this class and let $H = \mathbf{N}_{\mathrm{GL}(6,2)}(E)$. One sees that H, which is an extension of E by a group isomorphic to $\mathrm{GL}(2, 3)$, has just two orbits on the nonzero elements of the natural module V: one of size 36 and one of size 27. The centralizers in H of vectors in the orbit of size 36 are isomorphic to $S_3 \times S_3$. It follows that every subgroup of H that contains an element of order 4 has a $3'$-deranged orbit. It is easily checked (for instance with GAP ([8])) that if $G \leq H$ has Sylow 2-subgroups of exponent 2, then V is not primitive as a G-module. This contradiction excludes the possibility that N/F is cyclic.

In particular, we get that $N \leq A$, as otherwise N/F is a group of automorphisms of the cyclic 3-group U and hence it is cyclic.

Also, if $e = 3$, we have $A/F \leq \mathrm{Sp}(2, 3) = \mathrm{SL}(2, 3)$, and again N/F would be cyclic. Therefore, $e \neq 3$.

We henceforth assume $e = 3^2$. In this case, A/F is isomorphic to a subgroup of $\mathrm{Sp}(4, 3)$. Write $N = FR$, where R is a Sylow r-subgroup of N, with $r \neq 3$. As $|\mathrm{Sp}(4, 3)| = 2^7 \cdot 3^4 \cdot 5$, it follows that R is an elementary abelian 2-group. Now, if T is a Sylow 2-subgroup of $\mathrm{Sp}(4, 3)$, then $T \simeq (Q_8 \times Q_8) : C_2$ and hence $|R|$ must be 4; so R contains three involutions. Let $x \in R$ be an involution and let $X = \langle x \rangle$. By Lemma 2.2 of [5], $[X, E]$ is an extraspecial 3-group. Write $|[X, E]| = 3^{2m+1}$. As in the proof of part (3) of Lemma 2.4 in [5] (or of Lemma 2.4 in [14]), one sees that $|\mathbf{C}_V(x)| \leq |V|^{\alpha}$, where

$$\alpha \leq \frac{1}{2}\left(\frac{3^m + 1}{3^m}\right).$$

The eigenvalues of x are either 1 or -1. If x is not the central involution $-I$ of $\mathrm{Sp}(4, 3)$, then not all the eigenvalues of x are -1, so (as x has determinant 1) x has exactly two eigenvalues 1 and two eigenvalues -1. Thus, if $x \neq -I$, we get that $|[X, E]| = 3^3$, $\alpha \leq 2/3$ and there are at most 3^2 involutions conjugate to x in N (since $|\mathbf{C}_E(X)| = 3^3$). In particular, in N there are at most $3 \cdot 3^2$ involutions distinct from $-I$; in fact, if $y \in N$ is an involution, then y is N-conjugate to an element of R.

On the other hand, if $x = -I$, we have that $|[X, E]| = 3^5$, $\alpha \leq 5/9$ and there are at most 3^4 involutions conjugate to x in N.

Now,

$$27q^{6f} + 81q^{5f} < q^{9f}$$

is always satisfied, as $q^f \geq 4$ (this follows from the fact that, denoting by W a simple submodule of V_U, q^f is a power of $|W|$, and clearly U acts fixed-point freely on W; see [14, Theorem 2.2 part (6) and (7)]). Therefore, there is a $v \in V$ which is not centralized by any involution of N, giving a $3'$-deranged orbit of G on V, the final contradiction. ■

We shall also make use of the following notation.

Definition 3.2. Let Σ be a finite nonempty set. Given a positive integer k, we define $\mathcal{P}_k(\Sigma)$ to be the set of ordered $(k+1)$-tuples $(\Xi_1, \Xi_2, \ldots, \Xi_{k+1})$, where the Ξ_j are (possibly empty) subsets of Σ such that $\Xi_j \cap \Xi_l = \emptyset$ whenever $j \neq l$, and $\bigcup_{j=1}^{k+1} \Xi_j = \Sigma$. If G is a permutation group on Σ, then an action of G on $\mathcal{P}_k(\Sigma)$ can be defined in a natural way. Note that, by associating $\Xi \subseteq \Sigma$ with the pair $(\Xi, \Sigma \setminus \Xi)$, we will identify $\mathcal{P}(\Sigma)$ with $\mathcal{P}_1(\Sigma)$.

We are now ready to deal with the irreducible case for $p = 3$. The following statement should also be compared with Theorem C of [6].

Theorem 3.3. *Let G be a solvable group, q a prime number, and V a faithful irreducible G-module over* $\mathrm{GF}(q)$. *Assume that G is not a 3-group, and that there are no $3'$-deranged orbits for the action of G on V. Then $q = 2$ and one of the following conclusions hold.*

(a) *G is isomorphic to a subgroup of $H \wr K$, where either $H = \Gamma(2^2)$ or H is a Hall $\{2,3\}$-subgroup of $\Gamma(2^6)$, and K is a (possibly trivial) 3-group.*

(b) *G is isomorphic to a subgroup of $\Gamma(2^2) \wr (\mathrm{Sym}(3) \wr P)$, where P is a (possibly trivial) 3-group.*

Proof. As in Theorem 2.2, choose a subgroup T of G and a primitive submodule W of V_T such that $V = W^G$ (possibly $T = G$). Denoting by H the factor group $T/\mathbf{C}_T(W)$, there are no $3'$-deranged orbits for the action of H on W. Therefore $\mathbf{F}(H)$ is a 3-group; moreover, if H is not a 3-group, then $H \leq \Gamma(W)$ is one of the two groups in the conclusions of Theorem 3.1.

Again, denoting by Σ a right transversal for T in G, we identify G with a subgroup of $H \wr K$, where K is a transitive subgroup of $\mathrm{Sym}(\Sigma)$.

Let us consider the case when K is a 3-group. As G is not a 3-group and G is isomorphic to a subgroup of $H \wr K$, we conclude that H is not a 3-group. Therefore, as observed above, Theorem 3.1 applies to the action of H on W, and we get conclusion (a).

In view of that, we shall henceforth assume that K is not a 3-group. If there exists a subset A of Σ such that $(A, \Sigma \setminus A)$ lies in a $3'$-deranged orbit for the action of K on $\mathcal{P}(\Sigma)$, then we can argue as in the third paragraph of the proof of Theorem 2.2, getting a contradiction. We conclude that there does not exist any $3'$-deranged orbit for the action of K on $\mathcal{P}(\Sigma)$.

As K is assumed not to be a 3-group, we are in a position to apply Lemma 2.8 of [6], getting that $K \simeq \mathrm{Sym}(3) \wr P$ where P is a (possibly trivial) 3-group, and that there exists a $3'$-deranged orbit for the action of K on $\mathcal{P}_2(\Sigma)$. Let (A, B, C) be an element of $\mathcal{P}_2(\Sigma)$ lying in such an orbit. Assume that H is not transitive on $W \setminus \{0\}$, and choose two elements w, z of $W \setminus \{0\}$ lying in distinct H-orbits. Set now v to be the element of $W^{\oplus s}$ whose ith component is w if $i \in A$, it is z if $i \in B$, and it is 0 if $i \in C$. It is not difficult to see that v lies in a $3'$-deranged orbit for the action of G on $W^{\oplus s}$, again contradicting our assumptions.

Therefore, the action of H on $W\setminus\{0\}$ must be transitive. As we already observed, $\mathbf{F}(H)$ is a 3-group. Now, set $|W| = q^n$: if $H = \mathbf{F}(H)$, then $q^n - 1$ is a power of 3, so we get $q = n = 2$ and $|\mathbf{F}(H)| = 3$ (whence we get conclusion (b) of the statement). On the other hand, if $H \neq \mathbf{F}(H)$ then Theorem 3.1 applies to the action of H on W but, among the two groups in the conclusions of that theorem, only $\Gamma(2^2)$ acts transitively on $W \setminus \{0\}$. Therefore we get conclusion (b) as well. ■

4. The nonsolvable case

We conclude our analysis of groups whose p-Brauer character table does not contain any zero by considering *nonsolvable* groups satisfying this condition. As mentioned in the Introduction, only for $p = 2$ this class of groups turns out to be non-empty.

Our first task is to keep under control the $2'$-length of the solvable radical (i.e. the largest solvable normal subgroup) in a group of this kind. Before stating and proving the relevant results, it will be convenient to fix the following notation: for a given group G, we set $\mathbf{D}_k(G)$ to be the $(2k+1)$th term of the 2-series of G (i.e., $\mathbf{D}_k(G) = \mathbf{O}_{2,2',2,\ldots,2',2}(G)$, where $2'$ appears k times).

Lemma 4.1. *Let Ω be a finite nonempty set, and let G be a primitive solvable subgroup of* $\mathrm{Sym}(\Omega)$. *Then there exist three subsets Ω_1, Ω_2 and Ω_3 of Ω, lying in pairwise distinct G-orbits of $\mathcal{P}(\Omega)$, such that every $2'$-element of the stabilizer G_{Ω_i} lies in $\mathbf{D}_2(G)$ for $i \in \{1, 2, 3\}$.*

Proof. Since G is a solvable group acting faithfully and primitively on Ω, we know that G has a unique minimal normal subgroup V and, denoting by S the stabilizer in G of a point, we have $G = VS$ with $V \cap S = 1$ and $\mathbf{C}_S(V) = 1$. Moreover, V acts regularly on Ω (so that $|\Omega| = |V| = p^n$ for a suitable prime p and $n \in \mathbb{N}$), and the action of S on Ω is equivalent to the action by conjugation of S on V.

Following [11, Lemma 5.1], for $g \in G$, we denote by $n(g)$ the number of $\langle g\rangle$-orbits on Ω, and by $s(g)$ the number of fixed points of g on Ω. We claim that, for every nontrivial $2'$-element g of G, we have

$$n(g) \leq \frac{2}{3}|\Omega|.$$

In fact, if $s(g) = 0$, then $n(g) \leq |\Omega|/3$ because every $\langle g\rangle$-orbit on Ω has size at least 3. On the other hand, if $s(g) \neq 0$, then we can assume $g \in S$ and we get $s(g) = |\mathbf{C}_V(g)| \leq |\Omega|/p$ (where the last inequality holds because S acts faithfully by conjugation on V). Thus,

$$n(g) \leq s(g) + \frac{|\Omega| - s(g)}{3} \leq \frac{|\Omega|}{3} + \frac{2}{3} \cdot \frac{|\Omega|}{p} \leq \frac{2}{3}|\Omega|,$$

as desired.

Now, a subset Ξ of Ω is (setwise) stabilized by an element $g \in G$ if and only if Ξ is a union of $\langle g\rangle$-orbits. Therefore, an element $g \in G$ stabilizes $2^{n(g)}$ subsets of Ω, and if g is a nontrivial $2'$-element, then $2^{n(g)} \leq 2^{\frac{2}{3}|\Omega|}$. As a consequence, if

$$2^{|\Omega|} - |G| \cdot 2^{\frac{2}{3}|\Omega|} \geq 3|G|, \tag{3}$$

then there are at least three subsets Ω_1, Ω_2 and Ω_3 of Ω, lying in pairwise distinct G-orbits of $\mathcal{P}(\Omega)$, such that G_{Ω_i} is a 2-group for every $i \in \{1, 2, 3\}$. Taking into account that, by [11, Corollary 3.6], we have $|G| \leq \frac{1}{2}|\Omega|^{\frac{13}{4}}$, it can be checked that inequality (3) holds provided $|\Omega| > 49$.

We can therefore assume that $|\Omega|$ is either a prime $p \leq 47$, or a prime power in $\{2^2, 2^3, 2^4, 2^5, 3^2, 3^3, 5^2, 7^2\}$. In the former case, as S embeds in $\mathrm{Aut}(V)$, the group G is metacyclic, whence $l_{2'}(G) \leq 1$. In the latter case, in view of [11, Theorem 2.11, Theorem 2.12 and Corollary 2.15], we get $l_{2'}(G) \leq 2$. Therefore, in any case, $\mathbf{D}_2(G) = G$ and the desired conclusion follows. ∎

Proposition 4.2. *Let Ω be a finite nonempty set, and let G be a transitive solvable subgroup of* $\mathrm{Sym}(\Omega)$*. Then there exists $\Delta \subseteq \Omega$ such that every $2'$-element of G_Δ lies in $\mathbf{D}_2(G)$.*

Proof. We can clearly assume $|\Omega| > 1$. Let Γ be a minimal nontrivial block for the action of G on Ω (i.e. $|\Gamma| > 1$, but we allow $\Gamma = \Omega$) and, denoting by L the pointwise stabilizer of Γ in G, set $H = G_\Gamma/L$. In this situation, H can be identified with a primitive subgroup of $\mathrm{Sym}(\Gamma)$. Also, let Σ be a right transversal for G_Γ in G; in view of remarks 2.1 and 2.2 of [6], G can be identified with a subgroup of $H \wr K$, where $K \leq \mathrm{Sym}(\Sigma)$ is a homomorphic image of G acting transitively on Σ. Furthermore, the group $H \wr K$ (thus G, as well) acts naturally on the cartesian product $\Gamma \times \Sigma$, and the G-sets Ω and $\Gamma \times \Sigma$ are equivalent. If $|\Sigma| = s$, then we identify Σ with $\{1, 2, \ldots, s\} \subseteq \mathbb{N}$.

An application of [3, Corollary 4] to the action of K on Σ yields two disjoint subsets Ξ_1, Ξ_2 of Σ such that $K_{\Xi_1} \cap K_{\Xi_2}$ is a 2-group, and we can consider the map $\theta : \Sigma \to \{1, 2, 3\}$ defined by $\theta(i) = j$ if $i \in \Xi_j$ (for $j \in \{1, 2\}$), whereas $\theta(i) = 3$ if $i \in \Sigma \setminus (\Xi_1 \cup \Xi_2)$. Also, applying Lemma 4.1 to the primitive action of H on Γ, we obtain three subsets Γ_1, Γ_2, Γ_3 of Γ which satisfy the conclusions of that proposition. Now, set

$$\Delta = \{(\gamma, i) \in \Gamma \times \Sigma \ : \ \gamma \in \Gamma_{\theta(i)}\};$$

we claim that every $2'$-element of G_Δ lies in $\mathbf{D}_2(G)$. In fact, let g be a $2'$-element in G_Δ, and (according to the identification of G with a subgroup of $H \wr K$) write g as $(h_1, h_2, \ldots, h_s) \cdot k$, where $h_r \in H$ for every $r \in \{1, \ldots, s\}$ and $k \in K$; it is easily seen that k must lie in $K_{\Xi_1} \cap K_{\Xi_2}$, and that every h_r must lie in one of the H_{Γ_j}. We conclude that $k = 1$ and $h_r \in \mathbf{D}_2(H)$ for every $r \in \{1, \ldots, s\}$ (note that k and the h_r are elements of odd order). Our claim follows, and the proof is complete. ■

Next, two results concerning group actions on modules over finite fields.

Lemma 4.3. *Let G be a solvable group, and V a faithful primitive G-module over a prime field. If there are less than five regular orbits for the action of G on V, then $l_{2'}(G) \leq 2$.*

Proof. Taking into account the main result of [14], and following the notation introduced in Proposition 1.2 (that we freely use throughout the proof), our assumption on the number of regular orbits implies $e \leq 10$ or $e = 16$.

If $e = 1$ then, by [11, Corollary 2.3(a)], the group G is isomorphic to a subgroup of the semilinear group on V, therefore it is metabelian, and $l_{2'}(G) \leq 1$.

Assume now $e \in \{2, 3, 5, 7\}$. In this case, by Proposition 1.2(d), the factor group A/F embeds into $\mathrm{Sp}(2, e) = \mathrm{SL}(2, e)$ (more specifically, into a subgroup which is maximal among the solvable subgroups of $\mathrm{SL}(2, e)$). Using for instance [1] for the cases $e = 5$ and $e = 7$, and setting $T/F = \mathbf{O}_{2,2'}(A/F)$, it can be checked that $|A/T| \leq 2$. In particular, A/T is central in G/T. This, together with the fact that F is nilpotent and G/A is abelian (by Proposition 1.2), yields the desired conclusion.

For the case $e = 6$, the factor group A/F embeds into $\mathrm{SL}(2, 2) \times \mathrm{SL}(2, 3)$ and we get the same situation as in the paragraph above.

As regards the cases $e \in \{4, 8, 16\}$, we refer to the proof of Lemma 3.3 in [5] (part (a), (b) and (c) respectively). It turns out that, setting $T/F = \mathbf{O}_{2',2,2'}(A/F)$, we get $|A/T| \leq 2$ (whence A/T is central in G/T). In any case, again taking into account that F is nilpotent and G/A is abelian, we are done as well.

Finally, if $e = 9$, then A/F is isomorphic to a solvable and completely reducible subgroup of $\mathrm{Sp}(4, 3)$, and the possible structure of A/F is described in Lemma 3.2 of [5]. In particular it is easily checked that, setting $T/F = \mathbf{O}_{2',2,2'}(A/F)$, we get again $|A/T| \leq 2$, and the desired conclusion follows as above. ■

Proposition 4.4. *Let G be a solvable group, and V a direct sum of irreducible G-modules over prime fields (possibly not in the same characteristic) such that $\mathbf{C}_G(V) = 1$. Then there exists $v \in V$ such that every $2'$-element of $\mathbf{C}_G(v)$ lies in $\mathbf{D}_3(G)$.*

Proof. We start by proving the result under the additional assumption that V is an irreducible G-module. As in Theorem 2.2, choose a subgroup T of G and a primitive submodule W of V_T such that $V = W^G$. Denoting by H the factor group $T/\mathbf{C}_T(W)$ and by Σ a right transversal for T in G, we identify G with a subgroup of $H \wr K$, where K is a transitive solvable subgroup of $\mathrm{Sym}(\Sigma)$. Also, if $|\Sigma| = s$, then we identify Σ with the set $\{1, \ldots, s\} \subseteq \mathbb{N}$.

By [3, Corollary 4] there exist two disjoint subsets Ξ_1, Ξ_2 of Σ such that $K_{\Xi_1} \cap K_{\Xi_2}$ is a 2-group. As in Proposition 4.2, we define a map $\theta : \Sigma \to \{1, 2, 3\}$ by $\theta(i) = j$ if $i \in \Xi_j$ (for $j \in \{1, 2\}$), whereas $\theta(i) = 3$ if $i \in \Sigma \setminus (\Xi_1 \cup \Xi_2)$.

If we can find three regular orbits for the action of H on W, then we choose an element from each of them and we denote by x_1, x_2, x_3 the relevant elements.

Now, consider the vector $v = w_1 + w_2 + \cdots + w_s \in W^{\oplus s}$ whose ith component is $x_{\theta(i)}$. It is easy to see that, given an element $g = (h_1, \ldots, h_s) \cdot k \in G$ (with $h_r \in H$ and $k \in K$, according to the identification of G with a subgroup of $H \wr K$), the element g fixes v if and only if k stabilizes both Ξ_1 and Ξ_2, and each h_r centralizes w_r. In other words, $\mathbf{C}_G(v)$ is a 2-group and we are done in this case.

Assume now that there do not exist three regular orbits for the action of H on W; in this situation, by Lemma 4.3, we get $l_{2'}(H) \leq 2$. If there exist at least two H-orbits in $W \setminus \{0\}$, then take non-zero elements $u_1, u_2 \in W$ lying in distinct H-orbits and $u_3 = 0$. Consider the vector $v = w_1 + w_2 + \cdots + w_s \in W^{\oplus s}$ whose ith component is $u_{\theta(i)}$. As above, if $g = (h_1, \ldots, h_s) \cdot k$ is a $2'$-element in $\mathbf{C}_G(v)$, then $k = 1$ and hence $g \in \mathbf{D}_2(G)$.

Finally, assume that H is transitive on $W \setminus \{0\}$. By Theorem 6.8 of [11], then $l_{2'}(H) \leq 1$. An application of Proposition 4.2 to the action of K on Σ yields a subset Δ of Σ such that every $2'$-element of K_Δ lies in $\mathbf{D}_2(K)$. Choose any nonzero element $x \in W$, and define the vector $v = w_1 + w_2 + \cdots + w_s \in W^{\oplus s}$ by setting $w_i = x$ if $i \in \Delta$ and $w_i = 0$ otherwise. It is easily seen that, if $g = (h_1, \ldots, h_s) \cdot k \in G$ centralizes v, then k stabilizes Δ. As a consequence, if g has odd order, then k lies in $\mathbf{D}_2(K)$ and therefore g lies in $\mathbf{D}_3(G)$, as wanted. This concludes the proof for the irreducible case.

Finally, we go back to the general statement: assume $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, where the V_i are irreducible G-modules. By our analysis in the irreducible case, for every $i \in \{1, \ldots, n\}$ there exists $v_i \in V_i$ such that every $2'$-element of $\mathbf{C}_{G/\mathbf{C}_G(V_i)}(v_i)$ lies in $\mathbf{D}_3(G/\mathbf{C}_G(V_i))$. Setting $v = v_1 + \cdots + v_n$ and taking a $2'$-element g in $\mathbf{C}_G(v)$, we clearly get $g \in \bigcap \mathbf{C}_G(v_i)$. In particular, the image of g under the canonical embedding of G into $G/\mathbf{C}_G(V_1) \times \cdots \times G/\mathbf{C}_G(V_n)$ lies in $\mathbf{D}_3(G/\mathbf{C}_G(V_1)) \times \cdots \times \mathbf{D}_3(G/\mathbf{C}_G(V_n))$, and the conclusion easily follows. ■

We are now in a position to provide an upper bound for the $2'$-length of the solvable radical of G, under the assumption that the 2-Brauer character table of G has no zeros.

Theorem 4.5. *Let G be a group, and assume that the 2-Brauer character table of G does not contain any zero. Then the solvable radical of G has $2'$-length at most 4.*

Proof. Since the hypothesis is clearly inherited by the factor group $G/\mathbf{O}_2(G)$, we can assume $\mathbf{O}_2(G) = 1$. Therefore, denoting by S the solvable radical of G, the group $V = \mathbf{F}(S)/\mathbf{\Phi}(S)$ has odd order. Moreover, by Gaschütz Theorem ([9, III.4.5]), V is a direct sum of irreducible S-modules; the same holds for $\widehat{V} := \mathrm{Irr}(V) = \mathrm{IBr}_2(V)$, where the last equality is due to the fact that $|V|$ is an odd number, and we have $\mathbf{C}_S(\widehat{V}) = \mathbf{C}_S(V) = \mathbf{F}(S)$.

Now, by Proposition 4.4, we can find $\mu \in \widehat{V}$ such that every $2'$-element in $\mathbf{C}_{S/\mathbf{F}(S)}(\mu)$ lies in $\mathbf{D}_3(S/\mathbf{F}(S))$. Thus, regarding μ as an element of $\mathrm{IBr}_2(\mathbf{F}(S))$, every $2'$-element in $I_S(\mu)$ lies in $\mathbf{D}_4(S)$.

Working for a contradiction, assume $\mathbf{D}_4(S) \neq S$; then, as can be easily seen, there exists $g \in S \setminus \mathbf{D}_4(S)$ such that g is a $2'$-element. Clearly g does not lie in $I_S(\mu)$, but, for every $x \in G$, it also does not lie in $I_S(\mu^x)$ (this follows from the fact that $\mathbf{D}_4(S)$ is a normal subgroup of G). Therefore we have

$$g \notin \bigcup_{x \in G} I_G(\mu)^x.$$

Finally, take $\phi \in \mathrm{IBr}_2(G)$ lying over μ; since ϕ is induced by an irreducible 2-Brauer character of $I_G(\mu)$, it vanishes on every $2'$-element in $G \setminus \bigcup_{x \in G} I_G(\mu)^x$. But then we get $\phi(g) = 0$, the final contradiction which completes the proof. ∎

In order to complete our analysis, we will need one last result on permutation actions. Note that in the following lemma we are not requiring that the permutation group is transitive (although we are going to apply it to a transitive action).

Lemma 4.6. *Let Ω be a finite nonempty set and let G be a subgroup of* $\mathrm{Sym}(\Omega)$. *Assume that G is not a 2-group. Then there exists an odd prime divisor p of $|G|$ and two disjoint subsets Δ_1 and Δ_2 of Ω such that p does not divide $|G_{\Delta_1} \cap G_{\Delta_2}|$.*

Proof. By Theorem 2 of [3], it is enough to show that if G is primitive on Ω, then there exist (at least) three *p-regular orbits* of G on $\mathcal{P}_2(\Omega)$, i.e., orbits of size divisible by the full p-part of $|G|$.

If $\mathrm{Alt}(\Omega) \not\leq G$ or $|\Omega| \leq 4$, this follows from part (b) of Lemma 1 in [3].

So we can assume that $\Omega = \{1, 2, \ldots, n\}$ with $n \geq 5$, and that G is either A_n or S_n. As a consequence of Bertrand's Theorem, there exists a prime $p \leq n$, such that $p > m = \lceil n/2 \rceil$ (so $m = n/2$ if n is even, and $m = (n+1)/2$ if n is odd). Then $(\{1, \ldots, m\}, \{m+1, \ldots, n\}, \emptyset)$, $(\{1, \ldots, m\}, \emptyset, \{m+1, \ldots, n\})$ and $(\emptyset, \{1, \ldots, m\}, \{m+1, \ldots, n\})$ are elements of three distinct p-regular orbits of G on $\mathcal{P}_2(\Omega)$. ∎

In the proof of the following theorem, we will make use of the results in [10]. As mentioned in the Introduction, $\mathcal{L}$ will denote the class of the simple groups defined as follows:

$$\mathcal{L} = \{L_2(2^a),\ a \geq 2;\quad L_2(q),\ q = 2^a + 1 \geq 5;\quad {}^2B_2(2^{2a+1}),\ a \geq 1;\quad S_4(2^a),\ a \geq 2\}.$$

Theorem 4.7. *Let G be a group with no nontrivial normal solvable subgroups. Assume that the 2-Brauer character table of G does not contain any zero. Then the generalized Fitting subgroup $\mathbf{F}^*(G)$ of G is a direct product of simple groups in $\mathcal{L}$ and $G/\mathbf{F}^*(G)$ is a 2-group.*

Proof. Set $N = \mathbf{F}^*(G)$: as $\mathbf{F}(G) = 1$, N is a direct product of nonabelian minimal normal subgroups. Let M be a minimal normal subgroup of G, thus $M = S_1 \times S_2 \cdots \times S_m$ where the subgroups S_i are isomorphic to a nonabelian simple group S.

If $S \notin \mathcal{L}$, then by [10, Theorem 1.2] there exists a character $\phi \in \mathrm{IBr}_2(S)$ and a $2'$-element $x \in S$ such that $\phi^a(x) = 0$ for all $a \in \mathrm{Aut}(S)$. Let $\psi = \phi \times \phi \times \cdots \times \phi \in \mathrm{IBr}_2(M)$ and let $\eta \in \mathrm{IBr}_2(I_G(\psi))$ be a character lying above ψ. Then, by Clifford Correspondence, $\theta = \eta^G \in \mathrm{IBr}_2(G)$. Let $g = x \times x \times \cdots \times x \in M$. Since $\theta(g)$ is a sum of products whose factors are of the type $\phi^a(x)$ for some $a \in \mathrm{Aut}(S)$, we get the contradiction $\theta(g) = 0$.

Thus, N is a direct product of groups in the class $\mathcal{L}$. It remains to prove that G/N is a 2-group. Write $N = M_1 \times M_2 \times \cdots \times M_k$, where the M_i are the minimal normal subgroups of G. If $k > 1$, then by induction $G/M_i\mathbf{C}_G(M_i)$ is a 2-group for each $i = 1, 2, \ldots, k$. Since G/N is isomorphic to a subgroup of the direct product of the factor groups $G/M_i\mathbf{C}_G(M_i)$, we can assume that $N = M$ is the unique minimal normal subgroup of G. Recalling that $M = S_1 \times S_2 \times \cdots \times S_m$, with $S_i \simeq S \in \mathcal{L}$, set $L = \bigcap_{i=1}^m \mathbf{N}_G(S_i)$. We will first show that L/M is a 2-group. Now, L/M is a subgroup of a direct product of copies of the outer automorphism group $\mathrm{Out}(S)$ of S. If $S = L_2(q)$ with $q = 2^a + 1$, then either q is a Fermat prime or $q = 9$ and hence $\mathrm{Out}(S)$ is a 2-group (either C_2 or $C_2 \times C_2$, respectively). Therefore, we can assume that S is either $L_2(2^a)$, ${}^2B_2(2^{2a+1})$ or $S_4(2^a)$. In this case $O = \mathrm{Out}(S)$ is cyclic (and S has no nontrivial diagonal automorphism). We claim that O has a regular orbit on $\mathrm{IBr}_2(S)$. To show this, we recall that by Theorem 3.1 of [7], there exists an odd order element $g \in S$ such that $\mathbf{C}_{\mathrm{Aut}(S)}(g) = \langle g \rangle$. It follows that the S-conjugacy class g^S of g is fixed only by inner automorphisms of S (in fact, if $\alpha \in \mathrm{Aut}(S)$ fixes g^S, then there exist an element $x \in S$ such that $g^\alpha = g^x$; so $\alpha x^{-1} \in \mathbf{C}_{\mathrm{Aut}(S)}(g) \leq S$ and hence α is an inner automorphism of S). Since O is cyclic and the 2-Brauer character table is a non-singular matrix, by Brauer Permutation Lemma there exists a character $\phi \in \mathrm{IBr}_2(S)$ such that $I_{\mathrm{Aut}(S)}(\phi) \leq S$. Let $\psi = \phi \times \phi \times \cdots \times \phi \in \mathrm{IBr}_2(M)$. Then $I_G(\psi) \cap L = M$. Let $\hat{\psi} \in \mathrm{IBr}_2(I_G(\psi))$ be a character lying above ψ and $\theta = \hat{\psi}^G$. Thus $\theta \in \mathrm{IBr}_2(G)$ and $\theta(y) = 0$ for every $y \in L \setminus M$, and hence we conclude that L/M is a 2-group.

Finally, we show that G/L is a 2-group, too. First, we observe that, for every simple group $S \in \mathcal{L}$, there are at least three distinct degrees for irreducible 2-Brauer characters. For the groups in characteristic 2, this follows by considering that the Steinberg character gives an irreducible Brauer character (by restriction to the elements of odd order) and that not all nonprincipal characters can have 2-defect zero. For $S = L_2(q)$, where $q = 2^a + 1$, $\mathrm{IBr}_2(S)$ has both a characters of degree 2^a (i.e. of 2-defect zero) and of degree 2^{a-1} (see, for instance, [2, Section VIII (a)]). So, let a and b be distinct degrees of nonlinear characters in $\mathrm{IBr}_2(S)$. Assume, working by contradiction, that G/L is not a 2-group. Now, $\overline{G} = G/L$ is a permutation group on $\Omega = \{S_1, S_2, \dots, S_m\}$ and by Lemma 4.6 there exist an odd prime divisor p of $|\overline{G}|$ and disjoint subsets $\Delta_1, \Delta_2 \subseteq \Omega$ such that p does not divide $|\overline{G}_{\Delta_1} \cap \overline{G}_{\Delta_2}|$. Consider $\psi = \phi_1 \times \phi_2 \times \cdots \times \phi_m$, where $\phi_i \in \mathrm{IBr}_2(S_i)$ is such that $\phi_i(1) = a$ if $S_i \in \Delta_1$, $\phi_i(1) = b$ if $S_i \in \Delta_2$, and $\phi_i = 1_{S_i}$ otherwise. Since $I_G(\psi)L/L \leq \overline{G}_{\Delta_1} \cap \overline{G}_{\Delta_2}$ is a p'-group, a p-element $g \in G \setminus L$ cannot lie in any G-conjugate of $I_G(\psi)$. Therefore, if $\theta \in \mathrm{IBr}_2(G)$ lies over ψ, we get $\theta(g) = 0$, a contradiction. Hence G/L is a 2-group, and the proof is complete. ■

5. Brauer character tables with no zeros

As an application of the results in the previous sections, we can now derive Theorem A, that was stated in the Introduction. For solvable groups, the argument is essentially the same as in Theorem A of [6], except for the fact that the results about linear actions obtained there have to be replaced with those of the present paper. However, for the convenience of the reader, we give here a complete proof.

We start with a preliminary remark.

Remark 5.1. Let B and C be groups acting on disjoint sets Γ and Δ. Then $B \times C$ acts in a natural way on the union $\Gamma \cup \Delta$ and, for any group A, the wreath product $A \wr (B \times C)$ is isomorphic to $(A \wr B) \times (A \wr C)$.

Proof of Theorem A. We first consider the case of a solvable group G. Observe that our assumption on the Brauer character table is obviously inherited by factor groups. In view of this fact, it will be enough to prove Theorem A in the case when the Frattini subgroup $\mathbf{\Phi}(G)$ of G is trivial; this extra assumption ensures that $F := \mathbf{F}(G)$ is a completely reducible G-module (possibly in mixed characteristic).

Let V be a minimal normal subgroup of G. Then $\widehat{V} = \mathrm{Irr}(V) = \mathrm{IBr}_p(V)$ (recall that p does not divide $|V|$) is a faithful irreducible $G/\mathbf{C}_G(V)$ module. Take $\mu \in \widehat{V}$ and let $\phi \in \mathrm{IBr}_p(G)$ lying over μ. By Clifford correspondence (see for instance [12, (8.9)]), ϕ is induced from an irreducible Brauer character of $I_G(\mu)$, and therefore it vanishes on every p'-element not belonging to the set $S = \bigcup_{x \in G} I_G(\mu^x)$. Since the Brauer character ϕ has by assumption no value equal to zero, every p'-element of G lies in S and hence we conclude that there are no p'-deranged orbits for the action of $G/\mathbf{C}_G(V)$ on $\widehat{V}$.

If $p = 2$, then either $G/\mathbf{C}_G(V)$ is a 2-group or we can apply Theorem 2.2 and conclude that $G/\mathbf{C}_G(V)$ is a $\{2,3\}$-group with elementary abelian Sylow 3-subgroups (moreover, V is a 3-group).

Assume now that $p = 3$ and that $G/\mathbf{C}_G(V)$ is not a 3-group. Then Theorem 3.3 yields that $G/\mathbf{C}_G(V)$ is a $\{2,3\}$-group (and V a 2-group); moreover, $G/\mathbf{C}_G(V)$ has elementary abelian Sylow 2-subgroups and $l_{3'}(G/\mathbf{C}_G(V)) \leq 1$, unless $G/\mathbf{C}_G(V)$ is isomorphic to a subgroup of $\Gamma(2^2) \wr (\mathrm{Sym}(3) \wr P) \simeq (\mathrm{Sym}(3) \wr \mathrm{Sym}(3)) \wr P$, where P is a 3-group.

Writing $F = V_1 \times \cdots \times V_n$ where the V_i are minimal normal subgroups of G, and observing that $F = \bigcap_{i=1}^n \mathbf{C}_G(V_i)$, conclusions (a) and (c) now follow (also taking into account Remark 5.1) because G/F can be regarded as a subgroup of $G/\mathbf{C}_G(V_1) \times \cdots \times G/\mathbf{C}_G(V_n)$.

Assume now that G is nonsolvable. Then, by Theorem 1.3 of [10], $p = 2$. Let R be the solvable radical of G. By Theorem 4.5, we get that $l_{2'}(R) \leq 4$. An application of Theorem 4.7 to the factor group G/R yields conclusion (b). ■

We believe that the structure description in case (b) of Theorem A could be improved. One possible strategy is studying the action on the generalized Fitting subgroup. However, we did not pursue this line of analysis, as we did not have sufficient information on primitive module actions of nonsolvable groups.

Finally, we prove Theorem B.

Proof of Theorem B. The bounds on $l_p(G)$ and $l_{p'}(G)$, for both $p = 2$ and $p = 3$, follow easily from (a) and (c) of Theorem A.

Let $p = 2$ and assume (by factoring out $\mathbf{O}_2(G)$) that $\mathbf{O}_2(G) = 1$. Write $H = \mathbf{O}^2(G)$. Note that $F = \mathbf{F}(G) = \mathbf{F}(H)$ is a $2'$-group and let $F = T \times B$, where T is a 3-group and B is a $\{2,3\}'$-group. Let Q be a Sylow q-subgroup of F, for some prime divisor q of F (so, $q \neq 2$).

Let N and M be normal subgroups of G such that $\mathbf{\Phi}(Q) \leq N \leq M \leq Q$ and M/N is a chief factor of G. Let $V = \mathrm{IBr}_2(M/N)$ be the dual group of M/N. As in the second paragraph of the proof of Theorem A, one gets that the action of $G/\mathbf{C}_G(V)$ on V has no $2'$-deranged orbits. So, by applying Theorem 2.2 to the action of G on the dual groups $V_1, \ldots V_n$ of a G-chief series of $Q/\mathbf{\Phi}(Q)$ we conclude that G/C, where $C = \bigcap_{i=1}^n \mathbf{C}_G(V_i)$ coincides with the stabilizer of the series, is a 2-group if $q \neq 3$ and a subgroup of a direct product of copies of $\mathrm{GL}(2,3)$ if $q = 3$.

Therefore, recalling that $C/\mathbf{C}_G(Q)$ is a q-group, we conclude that $H/\mathbf{C}_H(Q)$ is a (possibly trivial) q-group if $q \neq 3$ and that it is a $\{2,3\}$-group if $q = 3$. We deduce that

$$\frac{H}{F} = \frac{F\mathbf{C}_H(B)}{F} \times \frac{F\mathbf{C}_H(T)}{F}$$

where $F\mathbf{C}_H(B)/F$ is a $\{2,3\}$-group and $F\mathbf{C}_H(T)/F$ is a nilpotent $\{2,3\}'$-group. For a prime divisor $q \neq 3$ of $|F|$, let Q_0 be a Sylow q-subgroup of H. Then $Q_0F \trianglelefteq H$ and Q_0 acts trivially on the q-complement of F. It follows that Q_0 is normal in Q_0F and hence $Q_0 = Q$. So, q does not divide $|H/F|$ for all primes $q \neq 2, 3$. Hence, we conclude that $H = F\mathbf{C}_H(B)$ and hence that $H = A \times B$ where A is a $\{2,3\}$-group and B is a nilpotent $\{2,3\}'$-group.

When $p = 3$, one argues similarly, using Theorem 3.3. ■

References

[1] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of Finite Groups*, Oxford University Press, London, 1984.

[2] R. Burkhardt, *Die Zerlegungsmatrizen der Gruppen $PSL(2, p^f)$*, J. Algebra **40** (1976), 75–96.

[3] S. Dolfi, *Orbits of permutation groups on the power set*, Arch. Math. (Basel) **75** (2000), 321–327.

[4] S. Dolfi, *On independent sets in the class graph of a finite group*, J. Algebra **303** (2006), 216–224.

[5] S. Dolfi, *Large orbits in coprime actions of solvable groups*, Trans. Amer. Math. Soc. **360** (2008), 135–152.

[6] S. Dolfi, E. Pacifici, *Zeros of Brauer characters and linear actions of finite groups*, J. Algebra **340** (2011), 104–113.

[7] W. Feit and G.M. Seitz, *On finite rational groups and related topics*, Illinois J. Math. **33** (1989),103–131.

[8] [GAP] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.5.7; 2012. (http://www.gap-system.org)

[9] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1983.

[10] G. Malle, *Zeros of Brauer characters and the defect zero graph*, J. Group Theory **13** (2010), 171–187.

[11] O. Manz, T.R. Wolf, *Representations of Solvable Groups*, Cambridge University Press, Cambridge, 1993.

[12] G. Navarro, *Characters and blocks of finite groups*, Cambridge University Press, Cambridge, 1998.

[13] M.W. Short, *The Primitive Soluble Permutation Groups of Degree less than 256*, Lecture Notes in Mathematics **1519**, Springer-Verlag, 1992.

[14] Y. Yang, *Regular orbits of finite primitive solvable groups*, J. Algebra **323** (2010), 2735–2755.

Silvio Dolfi, Dipartimento di Matematica U. Dini,
Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.
E-mail address: `dolfi@math.unifi.it`

Emanuele Pacifici, Dipartimento di Matematica F. Enriques,
Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy.
E-mail address: `emanuele.pacifici@unimi.it`