Focusing on contraction

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Abstract. Focusing [1] is a proof-theoretic device to structure proof search in the sequent calculus: it provides a normal form to cut-free proofs in which the application of invertible and non-invertible inference rules is structured in two separate and disjoint phases. It is commonly believed that every "reasonable" sequent calculus has a natural focused version. Although stemming from proof-search considerations, focusing has not been thoroughly investigated in actual theorem proving, in particular w.r.t. termination, if not for the folk observations that only negative formulas need to be duplicated (or contracted if seen from the top down) in the focusing phase. We present a contraction-free (and hence terminating) focused proof system for multi-succedent propositional intuitionistic logic, which refines the **G4ip** calculus of Vorob'ev, Hudelmeier and Dyckhoff. We prove the completeness of the approach semantically and argue that this offers a viable alternative to other more syntactical means.

1 Introduction and related work

Focusing [1] is a proof-theoretic device to structure proof search in the sequent calculus: it provides a normal form to cut-free proofs in which the application of invertible and non-invertible inference rules is structured in two separate and disjoint phases. In the first, called the negative or asynchronous phase, we apply (reading the proof bottom up) all invertible inference rules in whatever order, until none is left. The second phase, called the positive or synchronous phase, "focuses" on a formula, by selecting a not necessarily invertible inference rule. If after the (reverse) application of that introduction rule, a sub-formula of that focused formula appears that also requires a non-invertible inference rule, then the phase continues with that sub-formula as the new focus. The phase ends either with success or when only formulas with invertible inference rules are encountered and phase one is re-entered. Certain "structural" rules are used to recognize this switch. Compare this to standard presentation of proof search, such as [22], where Waaler and Wallen describe a search strategy for the intuitionistic multi-succedent calculus **LB** by dividing rules in groups to be applied following some priorities and a set of additional constraints. This without a proof of completeness. Focusing internalizes in the proof-theory a stringent strategy, and a provably complete one, from which many additional optimizations follow.

Contraction (or duplication, seen from the bottom up) is one of Gentzen's original structural rules permitting the reuse of some formula in the antecedent or succedent of a sequent:

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{Contr}\ L \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ \text{Contr}\ R$$

We are interested in proof search for propositional logics and from this standpoint contraction is a rather worrisome rule: it can be applied at any time making termination problematic even for decidable logics, thus forcing the use of potentially expensive and non-logical methods like loop detection. It is therefore valuable to ask whether contraction can be removed, in particular in the context of focused proofs.

As it emerged from linear logic, focusing naturally fits other logics with strong dualities, such as classical logic. As such, it is maybe not surprising that issue of contraction has not been fully investigated: in linear logic contraction (and weakening) are tagged by exponentials, while in classical logic duplication does not affect completeness. As far as intuitionistic logic, an important corollary of the completeness of focusing is that contraction is exactly located in between the asynchronous and synchronous phases and can be restricted to negative formulas[3]. This is a beginning, but it is well-known (see the system **G3ip** [21]) that the only propositional connective we *do* need to contract is *implication*.

There is a further element: Gentzen's presentation of intuitionistic logic is obtained from his classical system **LK** by means of a cardinality restriction imposed on the succedent of every sequent: at most one formula occurrence. This has been generalized by Maehara (see [15]), who retained a multiple-conclusion version, provided that the rules for right implication (and universal quantification) can only be performed if there is a single formula in the succedent of the premise to which these rules are applied. As these are the same connectives where in the Kripke semantics a world jump is required, this historically opened up a fecund link with tableaux systems. Moreover, Maehara's **LB** (following [22]'s terminology) has more symmetries from the permutation point of view and therefore may seem a better candidate for focusing than mono-succedent **LJ**. The two crucial rules are:

$$\frac{\Gamma, A \to B \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \to L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B, \Delta} \to R$$

Interestingly here, in opposition to **LJ**, the $\to L$ rule *is* invertible, while $\to R$ is not. According to the focusing diktat, $\to L$ would be classified as left asynchronous and eagerly applied, and this makes the asynchronous phase endless. While techniques such as freezing [4] or some form of loop checking could be used, we exploit a well-known formulation of a contraction-free calculus, known as **G4ip** [21], following Vorob'ev, Hudelmeier and Dyckhoff, where the $\to L$ rule is replaced by a series of rules that originate from the analysis of the shape of

[3] Recall that in **LJ** a formula is negative (positive) if its right introduction rule is invertible (non-invertible).

the subformula A of the main formula $A \to B$ of the rule. It is then routine that such a system is indeed terminating, in the sense that any bottom-up derivation of any given sequent is of finite length[4]. It is instead *not* routine to focalize such a system, called **G4ipf**, and this is the main result of the present paper.

As the focusing strategy severely restricts proofs construction, it is paramount to show that we do not lose any proof – in other terms that focusing is complete w.r.t. standard intuitionistic logic. There are in the literature several ways to prove that, all of them proof-theoretical and none of them completely satisfactory for our purposes:

1. The permutation-based approach, dating back to Andreoli [1], works by proving inversion properties of asynchronous connectives and postponement properties of synchronous ones. This is very brittle and particularly problematic for contraction-free calculi: in fact, it requires to prove at the same time that contraction is admissible and in the focusing setting this is far from trivial.
2. One can establish admissibility of the *cut* and of the non-atomic *initial* rule in the focused calculus and then show that all ordinary rules are admissible in the latter using cut. This has been championed in [8]. While a syntactic proof of cut-elimination is an interesting result per se, the sheer number of the judgments involved and hence of the cut reductions (principal, focus, blur, commutative and preserving cuts in the terminology of the cited paper) makes the well founded-ness of the inductive argument very delicate and hard to extend.
3. The so-called "grand-tour through linear logic" strategy of Miller and Liang [14]. Here, to show that a refinement of an intuitionistic proof system such as ours is complete, we have to provide an embedding into **LLF** (the canonical focused system for full linear logic) and then show that the latter translation is entailed by Miller and Liang's 1/0 translation. The trouble here is that contraction-free systems cannot be faithfully encoded in **LLF** [18]. While there are refinements of **LLF**, namely linear logic with sub-exponentials [20], which may be able to faithfully encode such systems, a "grand-tour" strategy in this context is uncharted territory. Furthermore, sub-exponential encodings of focused systems tend to be very, very prolix, which makes closing the grand-tour rather unlikely.
4. Finally, Miller and Saurin propose a direct proof of completeness of focusing in linear logic in [19] based on the notion of focalization graph. Again, this seems hard to extend to asymmetric calculi such as intutionism, let alone those contraction-free.

In this paper, instead, we prove completeness adapting the traditional Kripke semantic argument. While this is well-worn in tableaux-like systems, it is the first time that the model-theoretic semantics of focusing has been considered. The highlights of our proof are explained in Section 3.3.

[4] With some additional effort, one can prove that contraction is admissible in the contraction-free calculus [10].

Although stemming from proof-search considerations, focusing has still to make an impact in actual theorem proving. Exceptions are:

- Inverse-based systems such as Imogen [16] and LIFF [7]: because the inverse method is forward and saturation-based, the issue of contraction does not come into play – in fact it exhibits different issues w.r.t. termination (namely subsumption) and is in general not geared towards finite failure.
- TAC [5] is a prototype of a family of focused systems for automated inductive theorem proving, including one for **LJF**. Because the emphasis is on the automation of inductive proofs and the objective is to either succeed or quickly fail, most care is applied to limit the application of the induction rule by means of *freezing*. Contraction is handled heuristically, by letting the user set a bound for how many time an assumption can be duplicated for each initial goal; once the bound is reached, the system becomes essentially linear.
- Henriksen's [13] presents an analysis of contraction-free classical logic: here contraction has an impact only in the presence of two kinds of disjunction/conjunctions, namely positive vs. negative, as in linear logic. The author shows that contraction can be disposed of by viewing the introduction rule for positive disjunction as a *restart* rule, similar to Gabbay's [12]:

$$\frac{\vdash \Theta, pos(A) \Downarrow B}{\vdash \Theta \Downarrow A \vee^{+} B} \qquad \text{plus dual}$$

where $pos(A) = A \wedge^{+} t^{+}$ delays the non-chosen branch if A is negative (Θ is positive only), and the focus left rule does not make any contraction. This is neat, but not helpful as far as **LB** is concerned.

2 The proof system

We consider a standard propositional language based on a denumerable set of atoms, the constant $\bot$ and the connectives $\wedge$, $\vee$ and $\rightarrow$; $\neg A$ stands for $A \rightarrow \bot$. Our aim is to give a focalized version of the well-known contraction-free calculus **G4ip** of Vorob'ev, Hudelmeier and Dyckhoff [21]. To this end, one starts with a classification of formulas in the (a)synchronous categories. In focused versions of **LJ** such as **LJF** [14], an asynchronous formula has a right invertible rule and a non-invertible left one – and dually for synchronous. The contraction-free approach does not enjoy this symmetry – the idea is in fact to consider the possible shape that the antecedent of an implication can have and provide a specialized left (and here right[5]) introduction rule, yielding a finer view of implicational connectives, which now come in pairs. As we shall see shortly, formulas of the kind $(A \rightarrow B) \rightarrow C$ have non-invertible left and right rules, while the intro rules for $(A \wedge B) \rightarrow C$ and $(A \vee B) \rightarrow C$ are *both* invertible. Formulas

[5] And in this sense our calculus is reminiscent of Avron's decomposition proof systems [3].

$a \to B$, with a an atom, have a peculiar behaviour: right rule is non-invertible, left rule is invertible, but can be applied only if the left context contains the atom a. This motivates the following, slight unusual, classification of formulas – we discuss the issue of *polarization* of atoms in Section 4.

$$\begin{array}{rcl} \text{Async Formula (AF)} & ::= & \bot \mid A \wedge B \mid A \vee B \mid \bot \to B \mid (A \wedge B) \to C \mid (A \vee B) \to C \\ \text{Sync Formula (SF)} & ::= & a \mid a \to B \mid (A \to B) \to C \qquad \text{where } a \text{ is an atom} \\ \text{AF}^+ & ::= & a \mid \text{AF} \\ \text{SF}^- & ::= & \text{a non-atomic SF} \end{array}$$

The calculus is based on the following judgments, whose rules are displayed in Figure 1:

- $\Theta; \Gamma \Longrightarrow \Delta; \Psi$. Active sequent;
- $\Theta; A \ll \Psi$. Left-focused sequent;
- $\Theta \gg A; \Psi$. Right-focused sequent.

Γ and Δ denote multisets of formulas, while Θ and Ψ denote multisets of SF. We use the standard notation of [21]; for instance, by Γ, Δ we mean multiset union of Γ and Δ.

Proof search alternates between an *asynchronous phase*, where asynchronous formulas are considered, and a *synchronous phase*, where synchronous ones are. The dotted lines highlights the rule that govern the phase change. In the asynchronous phase we eagerly apply the asynchronous rules to active sequents $\Theta; \Gamma \Longrightarrow \Delta; \Psi$. If the main formula is an AF, the formula is decomposed; otherwise, it is moved to one of the outer contexts Θ and Ψ (rule Act^{L} or Act^{R}). When the inner contexts are emptied (namely, we get a sequent of the form $\Theta; \cdot \Longrightarrow \cdot; \Psi$), no asynchronous rule can be applied and the synchronous phase starts by selecting a formula H in Θ, Ψ for focus (rule Focus^{L} or Focus^{R}). Differently from the asynchronous phase, the rules to be applied are determined by the formula under focus. Note that the choice of H determines a backtracking point: if proof search yields a sequent where Θ only contains atoms and Ψ is empty, no formula can be picked and the construction of the derivation fails; to continue proof search, one has to backtrack to the last applied Focus^{L} or Focus^{R} rule and select, if possible, a new formula for focus. The left-focused phase is started by the application of rule Focus^{L} and involves left-focused sequents of the form $\Theta; A \ll \Psi$. Here we analyze implications whose antecedents are either a or $A \to B$. In the first case (rule $\to$at), we perform a sort of forward application of modus ponens, provided that $a \in \Theta$, otherwise we backtrack. The application of rule $\to\to L$ determines a transition to a new asynchronous phase in the left premise, while focus is maintained in the right premise. The phase terminates when an AF^+ formula is produced with a call to rule Blur^{L}. Alternatively, a right-focused phase begins by selecting a formula H in Ψ (rule Focus^{R}). Let us assume that H is an atom. If $H \in \Theta$, we apply the axiom-rule Init and the construction of a closed branch succeeds; otherwise, we get a failure and we have to backtrack. If $H = K \to B$, we apply $\to$R, which ends the synchronous phase and starts a new asynchronous phase. This is similar to the **LJQ** system [9].

$$\frac{}{\Theta;\Gamma,\bot\Longrightarrow\Delta;\Psi}\ \bot L \qquad \frac{\Theta;\Gamma\Longrightarrow\Delta;\Psi}{\Theta;\Gamma\Longrightarrow\bot,\Delta;\Psi}\ \bot R$$

$$\frac{\Theta;\Gamma,A,B\Longrightarrow\Delta;\Psi}{\Theta;\Gamma,A\wedge B\Longrightarrow\Delta;\Psi}\ \wedge L \qquad \frac{\Theta;\Gamma\Longrightarrow A,\Delta;\Psi \qquad \Theta;\Gamma\Longrightarrow B,\Delta;\Psi}{\Theta;\Gamma\Longrightarrow A\wedge B,\Delta;\Psi}\ \wedge R$$

$$\frac{\Theta;\Gamma,A\Longrightarrow\Delta;\Psi \qquad \Theta;\Gamma,B\Longrightarrow\Delta;\Psi}{\Theta;\Gamma,A\vee B\Longrightarrow\Delta;\Psi}\ \vee L \qquad \frac{\Theta;\Gamma\Longrightarrow A,B,\Delta;\Psi}{\Theta;\Gamma\Longrightarrow A\vee B,\Delta;\Psi}\ \vee R$$

$$\frac{\Theta;\Gamma\Longrightarrow\Delta;\Psi}{\Theta;\Gamma,\bot\to B\Longrightarrow\Delta;\Psi}\ \bot{\to}L \qquad \frac{}{\Theta;\Gamma\Longrightarrow\bot\to B,\Delta;\Psi}\ \bot{\to}R$$

$$\frac{\Theta;\Gamma,A\to B\to C\Longrightarrow\Delta;\Psi}{\Theta;\Gamma,(A\wedge B)\to C\Longrightarrow\Delta;\Psi}\ \wedge{\to}L \qquad \frac{\Theta;\Gamma\Longrightarrow A\to B\to C,\Delta;\Psi}{\Theta;\Gamma\Longrightarrow(A\wedge B)\to C,\Delta;\Psi}\ \wedge{\to}R$$

$$\frac{\Theta;\Gamma,A\to C,B\to C\Longrightarrow\Delta;\Psi}{\Theta;\Gamma,(A\vee B)\to C\Longrightarrow\Delta;\Psi}\ \vee{\to}L \qquad \frac{\Theta;\Gamma\Longrightarrow A\to C,\Delta;\Psi \qquad \Theta;\Gamma\Longrightarrow B\to C,\Delta;\Psi}{\Theta;\Gamma\Longrightarrow(A\vee B)\to C,\Delta;\Psi}\ \vee{\to}R$$

$$\frac{\Theta,S;\Gamma\Longrightarrow\Delta;\Psi}{\Theta;\Gamma,S\Longrightarrow\Delta;\Psi}\ \mathrm{Act^L} \qquad \frac{\Theta;\Gamma\Longrightarrow\Delta;S,\Psi}{\Theta;\Gamma\Longrightarrow S,\Delta;\Psi}\ \mathrm{Act^R}$$

$$\frac{\Theta;S^-\ll\Psi}{\Theta,S^-;\cdot\Longrightarrow\cdot;\Psi}\ \mathrm{Focus^L} \qquad \frac{\Theta\gg S;\Psi}{\Theta;\cdot\Longrightarrow\cdot;S,\Psi}\ \mathrm{Focus^R} \qquad \frac{\Theta;T\Longrightarrow\cdot;\Psi}{\Theta;T\ll\Psi}\ \mathrm{Blur^L}$$

$$\frac{}{\Theta,a\gg a;\Psi}\ \mathrm{Init} \qquad \frac{\Theta;K\Longrightarrow B;\cdot}{\Theta\gg K\to B;\Psi}\ {\to}\mathrm{R}$$

$$\frac{\Theta,a;B\ll\Psi}{\Theta,a;a\to B\ll\Psi}\ {\to}\mathrm{at} \qquad \frac{\Theta;A,B\to C\Longrightarrow B;\cdot \qquad \Theta;C\ll\Psi}{\Theta;(A\to B)\to C\ll\Psi}\ {\to\to}L$$

A, B and C are any formulas, S is a SF, S^- is a SF$^-$, T is a AF$^+$ and $K\to B$ is a SF.

Fig. 1. The **G4ipf** calculus

We remark that the main difference between **G4ipf** and a standard focused calculus such as **LJF** is that the rule Focus$^{\mathrm{L}}$ does not require the contraction of the formula selected for focus. This is a crucial point to avoid the generation of branches of infinite length and to guarantee the termination of the proof search procedure outlined above (see Section 3.1).

A *derivation* $\mathcal{D}$ of a sequent σ in **G4ipf** is a tree of sequents built bottom-up starting from σ and applying backward the rules of **G4ipf**. A *branch* of $\mathcal{D}$ is a sequence of sequents corresponding to the path from the root σ of $\mathcal{D}$ to a leaf σ_l of $\mathcal{D}$. If σ_l is the conclusion of one of the axiom-rules $\bot L$, $\bot\to R$ and Init (the rules with no premises), the branch is *closed*. A derivation is closed if all its branches are closed. A sequent σ is *provable in* **G4ipf** if there exists a closed derivation of σ; a formula A is provable if the active sequent $\cdot;\cdot\Longrightarrow A;\cdot$ with empty contexts Θ, Γ and Ψ is provable.

Example 1. Here we provide an example of a **G4ipf**-derivation of the formula $\neg\neg(a \vee \neg a)$. Recall that a derivation of such a formula in the standard calculus requires an application of contraction.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a;\bot \Longrightarrow \cdot;}\,\bot L}{a;\bot \ll \cdot}\,\mathrm{Blur^L}}{a;\neg a \ll \cdot}\,{\to}\mathrm{at}}{\neg a, a;\cdot \Longrightarrow \cdot;\cdot}\,\mathrm{Focus^L}}{\neg a; a, \bot \to \bot \Longrightarrow \bot;\cdot}\,[\bot R, \bot \to L, \mathrm{Act^L}] \qquad \dfrac{\dfrac{}{\neg a;\bot \Longrightarrow \cdot;\cdot}\,\bot L}{\neg a;\bot \ll \cdot}\,\mathrm{Blur^L}}{\neg a;\neg\neg a \ll \cdot}\,{\to\to}\,L}{\neg a, \neg\neg a;\cdot \Longrightarrow \cdot;\cdot}\,\mathrm{Focus^L}}{\cdot;\neg(a \vee \neg a) \Longrightarrow \bot;\cdot}\,[\bot R, \vee \to L, \mathrm{Act^L} \times 2]}{\cdot \gg \neg\neg(a \vee \neg a);\cdot}\,{\to}\mathrm{R}}{\cdot;\cdot \Longrightarrow \cdot;\neg\neg(a \vee \neg a)}\,\mathrm{Focus^R}}{\cdot;\cdot \Longrightarrow \neg\neg(a \vee \neg a);\cdot}\,\mathrm{Act^R}
$$

The double line corresponds to an asynchronous phase where more than one rule is applied. The only backtracking point is the choice of the formula for left-focus in the active sequent $\neg a, \neg\neg a;\cdot \Longrightarrow \cdot;\cdot$. If we select $\neg a$ instead of $\neg\neg a$, we get the sequent $\neg\neg a;\neg a \ll \cdot$ and the construction of the derivation immediately fails.

3 Meta-theory

We show that proof search in **G4ipf** can be performed in finite time. We define a well-founded relation $\prec$ such that, if σ is the conclusion of a rule R of **G4ipf** and σ' any of the premises of R, then $\sigma' \prec \sigma$. As a consequence, branches of infinite length cannot be generated in proof search and the provability of σ in **G4ipf** can be decided in finite time.

3.1 Termination

We assign to any formula A a *weight* $wg(A)$ following [21]:

$$
\begin{array}{ll}
\mathrm{wg}(a) \,=\, \mathrm{wg}(\bot) \,=\, 2 & \mathrm{wg}(A \wedge B) \,=\, \mathrm{wg}(A) + \mathrm{wg}(A) \cdot \mathrm{wg}(B) \\
\mathrm{wg}(A \vee B) \,=\, 1 + \mathrm{wg}(A) + \mathrm{wg}(B) & \mathrm{wg}(A \to B) \,=\, 1 + \mathrm{wg}(A) \cdot \mathrm{wg}(B)
\end{array}
$$

The weight $\mathrm{wg}(\sigma)$ of a sequent σ is the sum of $\mathrm{wg}(A)$, for every A in σ. One can easily prove that the following properties hold:

- $\mathrm{wg}(A \to (B \to C)) \,<\, \mathrm{wg}((A \wedge B) \to C)$;
- $\mathrm{wg}(A \to C) + \mathrm{wg}(B \to C) \,<\, \mathrm{wg}((A \vee B) \to C)$;
- $\mathrm{wg}(A) + \mathrm{wg}(B \to C) + \mathrm{wg}(C) \,<\, \mathrm{wg}((A \to B) \to C)$.

The above properties suffice to prove that proof search in the calculus **G4ip** terminates. Indeed, if R is a rule of **G4ip**, σ_1 the conclusion of R and σ_2 any of the premises of R, it holds that $\mathrm{wg}(\sigma_2) < \mathrm{wg}(\sigma_1)$; since weights are positive numbers, we cannot generate branches of infinite length. On the other hand, in **G4ipf** we cannot use the weight of the whole sequent as a measure, since we have rules where the conclusion and the premise have the same weight (Focus, Act and Blur).

Let $\prec_s$ ($\prec_d$) be the smallest relation between two sequents related by a rule of the same (different) judgment such that $\sigma_1 \prec_s \sigma_2$ ($\sigma_1 \prec_d \sigma_2$) if there exists a rule R of **G4ipf** such that σ_2 is the conclusion of R and σ_1 is any of the premises of R. For instance:

$$(\,\Theta;\Gamma, A \Longrightarrow \Delta;\Psi\,) \prec_s (\,\Theta;\Gamma, A \vee B \Longrightarrow \Delta;\Psi\,) \quad (\,\Theta, a; B \ll \Psi\,) \prec_s (\,\Theta, a; a \to B \ll \Psi\,)$$

$$(\,\Theta; A \Longrightarrow B;\cdot\,) \prec_d (\,\Theta \gg A \to B;\Psi\,) \prec_d (\,\Theta;\cdot \Longrightarrow \cdot; A \to B, \Psi\,)$$

Note that $\sigma_1 \prec_s \sigma_2$ implies $\mathrm{wg}(\sigma_1) \leq \mathrm{wg}(\sigma_2)$; moreover, if $\sigma_1 \prec_d \sigma_2$ then $\mathrm{wg}(\sigma_1) = \mathrm{wg}(\sigma_2)$.

Using as a measure the lexicographic ordering of $\langle \mathrm{wg}(A), \mathrm{wg}(\Gamma), \mathrm{wg}(\Delta)\rangle$ we can show (see the proof in the Appendix):

Lemma 1. *$\prec_s$ is a well-founded relation.*

The relation $\prec_d$ corresponds to the application of a rule which starts or ends a synchronous phase. Note that a synchronous phase cannot start by selecting an atom (indeed, the formula S^- chosen for focus by $\mathrm{Focus}^{\mathrm{L}}$ must be a SF^-), otherwise we could generate an infinite loop where an atom a is picked for focus by $\mathrm{Focus}^{\mathrm{L}}$ and immediately released by $\mathrm{Blur}^{\mathrm{L}}$. As a consequence, we cannot have chains of the form $\sigma_1 \prec_d \sigma_2 \prec_d \sigma_3$, but between two $\prec_d$ at least an $\prec_s$ must occur. In the following lemma we show that two active sequents immediately before and after a synchronous phase have decreasing weights.

Lemma 2. *Let σ_a and σ_b be two active sequents, let $\sigma_1, \ldots, \sigma_n$ be $n \geq 1$ focused sequents such that $\sigma_a \prec_d \sigma_1 \prec_s \cdots \prec_s \sigma_n \prec_d \sigma_b$. Then $wg(\sigma_a) < wg(\sigma_b)$.*

Proof. By definition of $\prec_d$, σ_n is obtained by applying $\mathrm{Focus}^{\mathrm{L}}$ or $\mathrm{Focus}^{\mathrm{R}}$ to σ_b, σ_a is obtained by applying $\mathrm{Blur}^{\mathrm{L}}$ or $\to$R to σ_1, while in $\sigma_1, \ldots, \sigma_n$ only synchronous rules are applied. If $n = 1$, we have two possible cases:

1. $\sigma_a = \Theta; A, B \to C \Longrightarrow B;\cdot$
   $\sigma_1 = \Theta; (A \to B) \to C \ll \Psi$
   $\sigma_b = \Theta, (A \to B) \to C;\cdot \Longrightarrow \cdot;\Psi$;
2. $\sigma_a = \Theta; A \Longrightarrow B;\cdot$
   $\sigma_1 = \Theta \gg A \to B;\Psi$
   $\sigma_b = \Theta;\cdot \Longrightarrow \cdot; A \to B, \Psi$ (where A is an atom or an implication).

In both cases $\mathrm{wg}(\sigma_a) < \mathrm{wg}(\sigma_b)$. Let $n > 1$. We have:

$$\sigma_a = \Theta; H_1 \Longrightarrow \cdot;\Psi, \quad \sigma_1 = \Theta; H_1 \ll \Psi, \quad \ldots \quad \sigma_n = \Theta; H_n \ll \Psi$$
$$\sigma_b = \Theta, H_n;\cdot \Longrightarrow \cdot;\Psi$$

Since $\mathrm{wg}(H_1) < \mathrm{wg}(H_n)$, it holds that $\mathrm{wg}(\sigma_a) < \mathrm{wg}(\sigma_b)$. □

Let $\prec$ be the transitive closure of the relation $\prec_s \cup \prec_d$. Note that $\sigma_1 \prec \sigma_2$ implies $\mathrm{wg}(\sigma_1) \leq \mathrm{wg}(\sigma_2)$. Using lemmas 1 and 2, one can prove that (see the proof in the Appendix):

Proposition 1. *$\prec$ is a well-founded order relation.*

By Proposition 1, every branch of a derivation of **G4ipf** has finite length. Indeed, let $\mathcal{D}$ be a (possibly open) derivation of σ_1 and let $\sigma_1, \sigma_2, \ldots$ be a branch of $\mathcal{D}$. We have $\sigma_{i+1} \prec \sigma_i$ for every $i \geq 1$, hence the branch has finite length.

3.2 Semantics

A Kripke model is a structure $\mathcal{K} = \langle P, \leq, \rho, V \rangle$, where $\langle P, \leq, \rho \rangle$ is a finite poset with minimum element ρ; V is a function mapping every $\alpha \in P$ to a subset of atoms such that $\alpha \leq \beta$ implies $V(\alpha) \subseteq V(\beta)$. We write $\alpha < \beta$ to mean $\alpha \leq \beta$ and $\alpha \neq \beta$. The *forcing relation* $\mathcal{K}, \alpha \Vdash H$ (α forces H in $\mathcal{K}$) is defined as follows:

- $\mathcal{K}, \alpha \nVdash \bot$;
- for every atom a, $\mathcal{K}, \alpha \Vdash a$ iff $a \in V(\alpha)$;
- $\mathcal{K}, \alpha \Vdash A \wedge B$ iff $\mathcal{K}, \alpha \Vdash A$ and $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \vee B$ iff $\mathcal{K}, \alpha \Vdash A$ or $\mathcal{K}, \alpha \Vdash B$;
- $\mathcal{K}, \alpha \Vdash A \to B$ iff, for every $\beta \in P$ such that $\alpha \leq \beta$, $\mathcal{K}, \beta \nVdash A$ or $\mathcal{K}, \beta \Vdash B$.

Monotonicity property holds for arbitrary formulas, i.e.: $\mathcal{K}, \alpha \Vdash A$ and $\alpha \leq \beta$ imply $\mathcal{K}, \beta \Vdash A$. A formula A *is valid in* $\mathcal{K}$ iff $\mathcal{K}, \rho \Vdash A$. It is well-known that intuitionistic propositional logic **Int** coincides with the set of formulas valid in all (finite) Kripke models [6].

Given a Kripke model $\mathcal{K} = \langle P, \leq, \rho, V \rangle$, a world $\alpha \in P$ and a sequent σ, the relation $\mathcal{K}, \alpha \rhd \sigma$ ($\mathcal{K}$ *realizes* σ at α) is defined as follows:

- $\mathcal{K}, \alpha \rhd \Theta; \Gamma \Longrightarrow \Delta; \Psi$ iff
  $\mathcal{K}, \alpha \Vdash A$ for every $A \in \Theta, \Gamma$ and $\mathcal{K}, \alpha \nVdash B$ for every $B \in \Delta, \Psi$.
- $\mathcal{K}, \alpha \rhd \Theta; A \ll \Psi$ iff $\mathcal{K}, \alpha \rhd \Theta; A \Longrightarrow \cdot; \Psi$.
- $\mathcal{K}, \alpha \rhd \Theta \gg A; \Psi$ iff $\mathcal{K}, \alpha \rhd \Theta; \cdot \Longrightarrow A; \Psi$.

A sequent $\sigma = \Theta; \Gamma \Longrightarrow \Delta; \Psi$ is *realizable* if there exists a model $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ such that $\mathcal{K}, \rho \rhd \sigma$; in this case we say that $\mathcal{K}$ is a *model* of σ. We point out that σ is realizable iff the formula $\bigwedge(\Theta, \Gamma) \to \bigvee(\Delta, \Psi)$ is not intuitionistically valid. Moreover, it is easy to check that, if σ is the conclusion of one of the axiom-rules $\bot L$, $\bot \to R$ and Init, then σ is not realizable. A rule R is *sound* iff, if the conclusion of R is realizable, then at least one of its premises is realizable. We can esaily proof that (see the Appendix):

Proposition 2. *The rules of* **G4ipf** *are sound.*

By Proposition 2 the soundness of **G4ipf** follows (see the proof in the Appendix):

Theorem 1 (Soundness). *If σ is provable in* **G4ipf** *then σ is not realizable.*

3.3 Completeness

We show that, if proof search for a sequent σ fails, we can build a model $\mathcal{K}$ of σ, and this proves the completeness of **G4ipf**. Henceforth, by *unprovable* we mean 'not provable in **G4ipf**'.

A left-focused sequent $\Theta; H \ll \Psi$ is *strongly unprovable* iff one of the following conditions holds:

(i) H is an AF^+ and the sequent $\Theta; H \Longrightarrow \cdot; \Psi$ is unprovable;
(ii) $H = A \to B$ and $\Theta; B \ll \Psi$ is strongly unprovable.

By definition of the rules of **G4ipf**, we immediately get:

Lemma 3. *If $\sigma = \Theta; H \ll \Psi$ is strongly unprovable, then σ is unprovable.*

Let $\sigma = \Theta; H \ll \Psi$ be a left-focused sequent.

- σ is *at-unprovable w.r.t.* $a \to B$ iff, for some $m \geq 0$, it holds that $H = H_1 \to \cdots \to H_m \to a \to B$ and $a \notin \Theta$ (if $m = 0$, then $H = a \to B$);
- σ is *at-unprovable* if, for some $a \to B$, σ is at-unprovable w.r.t. $a \to B$;
- σ is $\to$*-unprovable w.r.t.* $(A \to B) \to C$ iff, for some $m \geq 0$, it holds that $H = H_1 \to \cdots \to H_m \to (A \to B) \to C$ and $\Theta; A, B \to C \Longrightarrow B; \cdot$ is unprovable (if $m = 0$, then $H = (A \to B) \to C$);
- σ is $\to$*-unprovable* if, for some $(A \to B) \to C$, σ is $\to$-unprovable w.r.t. $(A \to B) \to C$.

Note that a sequent can match the above definitions in more than one way. For instance, let $\sigma = \cdot; a_1 \to (a_2 \to a_3) \to a_4 \to a_5 \ll a_6$; then:

- σ is at-unprovable w.r.t. $a_1 \to (a_2 \to a_3) \to a_4 \to a_5$ and w.r.t. $a_4 \to a_5$;
- σ is $\to$-unprovable w.r.t. $(a_2 \to a_3) \to a_4 \to a_5$.

Lemma 4. *Let $\sigma = \Theta; H \ll \Psi$ be an unprovable sequent. Then, σ is strongly unprovable or at-unprovable or $\to$-unprovable.*

Proof. By induction on $\prec$. Let us assume that, for every $\sigma' \prec \sigma$, the lemma holds for σ'; we prove the lemma for σ by a case analysis.

- Let H be an AF^+. Since the sequent σ is unprovable then $\Theta; H \Longrightarrow \cdot; \Psi$ is unprovable. Hence by definition σ is strongly unprovable.
- Let $H = a \to B$. If $a \notin \Theta$ then σ is at-unprovable w.r.t. $a \to B$. Let $a \in \Theta$ and let $\sigma' = \Theta; B \ll \Psi$. Then σ' is unprovable. Since $\sigma' \prec \sigma$, by IH σ' is strongly unprovable or at-unprovable or $\to$-unprovable. If σ' is strongly unprovable, by definition σ is strongly unprovable. Let us assume that σ' is at-unprovable w.r.t. $a' \to C$. Then $B = H_1 \to \cdots \to H_m \to a' \to C$ and $a' \notin \Theta$. This implies that σ is at-unprovable w.r.t. $a' \to C$. Finally, let us assume that σ' is $\to$-unprovable w.r.t. $(C \to D) \to E$. Then $B = H_1 \to \cdots \to H_m \to (C \to D) \to E$ and the sequent $\Theta; C, D \to E \Longrightarrow D; \cdot$ is unprovable. If follows that σ is $\to$-unprovable w.r.t. $(C \to D) \to E$.

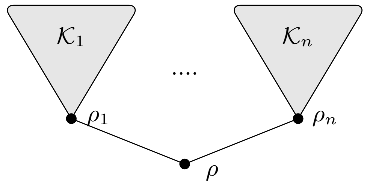

Fig. 2. The model Model(At, $\{\mathcal{K}_1, \ldots, \mathcal{K}_n\}$)

– Let $H = (B \to C) \to D$. If $\Theta; B, C \to D \Longrightarrow C; \cdot$ is unprovable, then by definition σ is $\to$-unprovable w.r.t. $(B \to C) \to D$. Otherwise, let $\Theta; B, C \to D \Longrightarrow C; \cdot$ be provable. Then $\sigma' = \Theta; D \ll \Psi$ is unprovable. Since $\sigma' \prec \sigma$, by IH σ' is strongly unprovable or at-unprovable or $\to$-unprovable. Reasoning as above, the lemma holds for σ. □

Let $\mathcal{S} = \{\mathcal{K}_1, \ldots \mathcal{K}_n\}$ be a (possibly empty) set of models $\mathcal{K}_i = \langle P_i, \leq_i, \rho_i, V_i \rangle$ $(1 \leq i \leq n)$, let At be a set of atoms such that, for every $1 \leq i \leq n$, $\text{At} \subseteq V_i(\rho_i)$; without loss of generality, we can assume that the sets P_i are pairwise disjoint. By Model(At, $\mathcal{S}$) we denote the Kripke model $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ defined as follows:

1. If $\mathcal{S}$ is empty, then $\mathcal{K}$ is the Kripke model consisting of only the world ρ and $V(\rho) = \text{At}$.
2. Let $n \geq 1$. Then (see Fig. 2):
   - ρ is new (namely, $\rho \notin \bigcup_{i \in \{1,\ldots,n\}} P_i$) and $P = \{\rho\} \cup \bigcup_{i \in \{1,\ldots,n\}} P_i$;
   - $\leq = \{ (\rho, \alpha) \mid \alpha \in P \} \cup \bigcup_{i \in \{1,\ldots,n\}} \leq_i$;
   - $V(\rho) = \text{At}$ and, for every $i \in \{1, \ldots, n\}$ and $\alpha \in P_i$, $V(\alpha) = V_i(\alpha)$.

It is easy to check that $\mathcal{K}$ is a well-defined Kripke model. In Point 2, for every $1 \leq i \leq n$, every $\alpha \in P_i$ and every formula A, it holds that $\mathcal{K}, \alpha \Vdash A$ iff $\mathcal{K}_i, \alpha \Vdash A$. A world β of a model $\mathcal{K}$ is an immediate successor of α if $\alpha < \beta$ and, for every γ such that $\alpha \leq \gamma \leq \beta$, either $\gamma = \alpha$ or $\gamma = \beta$.

Lemma 5. *Let $H = H_1 \to \cdots \to H_m \to A \to B$ $(m \geq 0)$, let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be a model such that $\mathcal{K}, \rho \nVdash A$ and, for every immediate successor α of ρ, it holds that $\mathcal{K}, \alpha \Vdash H$. Then $\mathcal{K}, \rho \Vdash H$.*

In the next lemma we show how to build a Kripke model of an unprovable sequent.

Lemma 6. *Let $\sigma = \Theta; \cdot \Longrightarrow \cdot; \Psi$ be an unprovable sequent such that, for every non-atomic $H \in \Theta$, the sequent $\Theta \setminus \{H\}; H \ll \Psi$ is at-unprovable or $\to$-unprovable. Let* At *be the set of atoms of Θ and let Θ_1 be the set of non-atomic formulas H of Θ such that the sequent $\Theta \setminus \{H\}; H \ll \Psi$ is not at-unprovable. Let $\mathcal{S}$ be a (possibly empty) set of models satisfying the following conditions:*

(i) For every $H \in \Theta_1$, let $(A \to B) \to C$ such that $\Theta \setminus \{H\}; H \ll \Psi$ is $\to$-unprovable w.r.t. $(A \to B) \to C$; then $\mathcal{S}$ contains a model of the sequent $\Theta \setminus \{H\}; A, B \to C \Longrightarrow B; \cdot$.

(ii) For every $A \to B \in \Psi$, $\mathcal{S}$ contains a model of the sequent $\Theta; A \Longrightarrow B; \cdot$.
(iii) Every model of $\mathcal{S}$ is of type (i) or (ii).

Then, $\mathrm{Model}(\mathrm{At}, \mathcal{S})$ *is a model of* σ*.*

Proof. Let us assume that the set of models $\mathcal{S}$ is empty. Then Θ_1 is empty and Ψ only contains atoms not belonging to At. By definition, $\mathcal{K} = \mathrm{Model}(\mathrm{At}, \mathcal{S})$ has only the world ρ. Since $V(\rho) = \mathrm{At}$, we immediately get $\mathcal{K}, \rho \Vdash a$, for every $a \in \mathrm{At}$, and $\mathcal{K}, \rho \nVdash a'$, for every $a' \in \Psi$. Let H be a non-atomic formula of Θ. Since $\Theta_1 = \emptyset$, the sequent $\Theta \setminus \{H\}; H \ll \Psi$ is at-unprovable. This means that $H = H_1 \to \cdots \to H_m \to a \to B$, where $a \notin \mathrm{At}$, hence $\mathcal{K}, \rho \Vdash H$. This proves that $\mathcal{K}, \rho \rhd \sigma$, thus $\mathcal{K}$ is a model of σ.

Let us assume that $\mathcal{S}$ contains the models $\mathcal{K}_1 = \langle P_1, \leq_1, \rho_1, V_1 \rangle$, ..., $\mathcal{K}_n = \langle P_n, \leq_n, \rho_n, V_n \rangle$ $(n \geq 1)$ and let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be the model $\mathrm{Model}(\mathrm{At}, \mathcal{S})$; we show that $\mathcal{K}$ is a model of σ.

If $a \in \mathrm{At}$, then $\mathcal{K}, \rho \Vdash a$ by definition of V.

Let H be a non-atomic formula of Θ. If $H \notin \Theta_1$, then the sequent $\Theta \setminus \{H\}; H \ll \Psi$ is at-unprovable, namely $H = H_1 \to \cdots \to H_m \to a \to B$, where $a \notin \mathrm{At}$. Firstly, we note that $\mathcal{K}_i, \rho_i \Vdash H$, for every $1 \leq i \leq n$; indeed, by (i)–(iii), $\mathcal{K}_i$ is a model of a sequent of the form $\Theta'; \Gamma' \Longrightarrow \Delta'; \cdot$ such that $H \in \Theta'$. It follows that $\mathcal{K}_i, \rho_i \Vdash H$, for every $1 \leq i \leq n$; hence $\mathcal{K}, \rho_i \Vdash H$. By definition of V, we have $\mathcal{K}, \rho \nVdash a$. By Lemma 5, we get $\mathcal{K}, \rho \Vdash H$.

Let $H \in \Theta_1$ and let $\Theta \setminus \{H\}; H \ll \Psi$ be $\to$-unprovable w.r.t. $(A \to B) \to C$. This mean that $H = H_1 \to \cdots \to H_m \to (A \to B) \to C$ and, by (i), $\mathcal{S}$ contains a model $\mathcal{K}_j$ of $\Theta \setminus \{H\}; A, B \to C \Longrightarrow B; \cdot$. This implies that:

(P1) $\mathcal{K}_j, \rho_j \Vdash A$;
(P2) $\mathcal{K}_j, \rho_j \Vdash B \to C$;
(P3) $\mathcal{K}_j, \rho_j \nVdash B$.

By (P1) and (P2) it follows that $\mathcal{K}_j, \rho_j \Vdash (A \to B) \to C$, which implies $\mathcal{K}_j, \rho_j \Vdash H$. Moreover, if $i \in \{1, \ldots, n\}$ and $i \neq j$, then by (i)– (iii) $\mathcal{K}_i$ is a model of a sequent $\Theta'; \Gamma' \Longrightarrow \Delta'; \cdot$ such that $H \in \Theta'$, hence $\mathcal{K}_i, \rho_i \Vdash H$. Thus, for every $1 \leq i \leq n$, it holds that $\mathcal{K}_i, \rho_i \Vdash H$, which implies $\mathcal{K}, \rho_i \Vdash H$. By (P1) and (P3), we have $\mathcal{K}, \rho_j \Vdash A$ and $\mathcal{K}, \rho_j \nVdash B$. Since $\rho < \rho_j$ in $\mathcal{K}$, we get $\mathcal{K}, \rho \nVdash A \to B$. By Lemma 5, we conclude $\mathcal{K}, \rho \Vdash H$.

Let $H \in \Psi$. If H is an atom, then $H \notin \mathrm{At}$, otherwise σ would be provable; hence $\mathcal{K}, \rho \nVdash H$. Let $H = A \to B$. By (ii), $\mathcal{S}$ contains a model $\mathcal{K}_j$ of $\Theta; A \Longrightarrow B; \cdot$. Thus, $\mathcal{K}_j, \rho_j \Vdash A$ and $\mathcal{K}_j, \rho_j \nVdash B$, which implies $\mathcal{K}, \rho \nVdash A \to B$. We conclude that $\mathcal{K}$ is a model of σ. □

We can now prove the completeness of **G4ipf**.

Proposition 3 (Completeness). *Let $\sigma = \Theta; \Gamma \Longrightarrow \Delta; \Psi$. If σ is unprovable, then σ is realizable.*

Proof. By induction on $\prec$. If Γ, Δ is not empty, the proposition easily follows by the induction hypothesis. For instance, let $\sigma = \Theta; \Gamma, A \vee B \Longrightarrow \Delta; \Psi$. By definition of the rule $\vee L$, one of the sequents $\sigma_A = \Theta; \Gamma, A \Longrightarrow \Delta; \Psi$ or $\sigma_B = \Theta; \Gamma, B \Longrightarrow \Delta; \Psi$ is unprovable. Since $\sigma_A \prec \sigma$ and $\sigma_B \prec \sigma$, by induction hypothesis there exists a model $\mathcal{K}$ of σ_A or of σ_B. In either case $\mathcal{K}$ is a model of σ, hence σ is realizable.

Let $\sigma = \Theta; \cdot \Longrightarrow \cdot; \Psi$. We distinguish two cases (C1) and (C2).

(C1) There is a non-atomic formula $H \in \Theta$ such that $\sigma' = \Theta \setminus \{H\}; H \ll \Psi$ is strongly unprovable.

By Lemma 3, σ' is unprovable. Since $\sigma' \prec \sigma$, by induction hypothesis there exists a model $\mathcal{K}$ of σ'; since $\mathcal{K}$ is also a model of σ, we conclude that σ is realizable.

(C2) For every non-atomic $H \in \Theta$, the sequent $\sigma' = \Theta \setminus \{H\}; H \ll \Psi$ is not strongly unprovable.

We build a model of σ by applying Lemma 6. We point out that the hypothesis of Lemma 6 are satisfied. Indeed, for every non-atomic $H \in \Theta$, since $\sigma' = \Theta \setminus \{H\}; H \ll \Psi$ is not strongly unprovable, by Lemma 4 σ' is at-unprovable or $\rightarrow$-unprovable. The (possibly empty) set of models $\mathcal{S}$ can be defined as follows:

(a) For every $H \in \Theta_1$, let us assume that $\Theta \setminus \{H\}; H \ll \Psi$ is $\rightarrow$-unprovable w.r.t. $(A \to B) \to C$. Then $H = H_1 \to \cdots \to H_m \to (A \to B) \to C$ and the sequent $\sigma_H = \Theta \setminus \{H\}; A, B \to C \Longrightarrow B; \cdot$ is unprovable. Since $\sigma_H \prec \sigma$, by induction hypothesis there exists a model of σ_H.

(b) For every $K = A \to B \in \Psi$, the sequent $\sigma_K = \Theta; A \Longrightarrow B; \cdot$ is unprovable (otherwise σ would be provable). Since $\sigma_K \prec \sigma$, by induction hypothesis there exists a model of σ_K.

Thus, we can define $\mathcal{S}$ as the set of models $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ mentioned in (a) and in (b); note that, since $\mathrm{At} \subseteq \Theta$, we have $\mathrm{At} \subseteq V(\rho)$. By Lemma 6, $\mathrm{Model}(\mathrm{At}, \mathcal{S})$ is a model of σ, hence σ is realizable. □

The above proof shows how to build a model of an unprovable sequent (see in particular points (a) and (b)). We remark that, in the model construction, only active sequents are relevant, while focused sequents are skipped. This justifies why standard model construction techniques are not directly applicable and a more involved machinery is needed.

By soundness and completeness of **G4ipf**, a sequent σ is provable in **G4ipf** iff σ is not realizable. By definition, $A \in \mathbf{Int}$ iff the sequent $\cdot; \cdot \Longrightarrow A; \cdot$ is not realizable. We conclude that $A \in \mathbf{Int}$ iff A is provable in **G4ipf**.

4 Conclusions and future work

We have presented a focused version of the contraction-free calculus **G4ip** [21]. Essentially, every treatment of focusing [14] extends the (a)synchronous classification of connectives to *atoms*, assigning them a *bias* or *polarity*. Different polarizations of atoms do not affect provability, but do influence significantly the *shape* of the derivation, allowing one to informally characterize forward and backward reasoning via respectively positive and negative bias assignments. Unfortunately, the contraction-free approach is essentially forward and negative bias do not work as expected. Here is why: standard presentations, where contraction on focus is allowed, use the following rules

$$\frac{}{\Theta; n \ll n, \Psi}\ \mathrm{Init}^{\mathrm{L}} \qquad \frac{\Theta; P \Longrightarrow \cdot; \Psi}{\Theta; P \ll \Psi}\ \mathrm{Blur}^{\mathrm{L}}$$

$$\frac{\Theta; \cdot \Longrightarrow \cdot; n \qquad \Theta; B \ll \Psi}{\Theta; n \to B \ll \Psi} \to \mathrm{at}- \qquad \frac{\Theta, p; B \ll \Psi}{\Theta, p; p \to B \ll \Psi} \to \mathrm{at}+$$

where n is a negative atom, p is a positive atom, P an AF or a positive atom. These rules without contraction give rise to an incomplete calculus. For instance, let us consider the non-realizable sequent $\sigma = n \to p, (n \to p) \to n; \cdot \Longrightarrow \cdot; p$. The only rule applicable to σ is $\mathrm{Focus}^{\mathrm{L}}$. If we select $n \to p$ we get:

$$\frac{(n \to p) \to n; \cdot \Longrightarrow \cdot; n \qquad \begin{array}{c}\vdots\\ (n \to p) \to n; p \ll p\end{array}}{(n \to p) \to n; n \to p \ll p} \to \mathrm{at}-$$

But the left premise is unprovable. On the other hand, if we choose $(n \to p) \to n$ we get:

$$\frac{\begin{array}{c}\vdots\\ n \to p; n, p \to n \Longrightarrow p; \cdot\end{array} \qquad n \to p; n \ll p}{n \to p; (n \to p) \to n \ll p} \to\to L$$

But the right premise is unprovable because there is no rule that can blur a negative atom from focus. To get a complete calculus we should allow $\mathrm{Blur}^{\mathrm{L}}$ on negative atoms, but in this case the calculus does not properly capture "backward chaining".

This paper is but a beginning of our investigation of focusing:

- It is commonly believed that every "reasonable" sequent calculus has a natural focused version. We aim to test this "universality" hypothesis further by investigating its applicability to a rather peculiar logic, Gödel-Dummett's, which is well-known to lead a double life as a super-intuitionistic (but not constructive) and as a quintessential fuzzy logic [17].
- We plan to investigate *counterexample* search in focused systems. The natural question is: considering that focused calculi restrict the shape of derivations, what kind of counter models do they yield, upon failure? How do they compare to calculi such as [2] or the calculus [11] designed to yield models of *minimal* depth?
- There seems to be a connection between contraction-free calculi and Gabbay's restart rule [12], a technique to make goal oriented provability with diminishing resources complete for intuitionistic provability. Focusing could be the key to understand this.

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Appendix

Proof of Lemma 1

To prove that $\prec_s$ is a well-founded relation, we have to show that there is no infinite descending $\prec_s$-chain of the form

$$\cdots \prec_s \sigma_3 \prec_s \sigma_2 \prec_s \sigma_1$$

Note that all the sequents in the $\prec_s$-chain have the same kind. Thus, either all the sequents in the $\prec_s$-chain are focused or all are active.

Let $\sigma_1 = \Theta_1; A_1 \ll \Psi_1$ and $\sigma_2 = \Theta_2; A_2 \ll \Psi_2$ be two focused sequents such that $\sigma_1 \prec_s \sigma_2$. Then, $\Theta_1 = \Theta_2$, $\Psi_1 = \Psi_2$ and $\mathrm{wg}(A_1) < \mathrm{wg}(A_2)$, hence $\mathrm{wg}(\sigma_1) < \mathrm{wg}(\sigma_2)$. Since the weight of a sequent is a positive number, every descending $\prec_s$-chains containing focused sequents has finite length.

Let $\sigma_1 = \Theta_1; \Gamma_1 \Longrightarrow \Delta_1; \Psi_1$ and $\sigma_2 = \Theta_2; \Gamma_2 \Longrightarrow \Delta_2; \Psi_2$ be two active sequents such that $\sigma_1 \prec_s \sigma_2$. Then, one of the following conditions holds:

1. $\mathrm{wg}(\sigma_1) < \mathrm{wg}(\sigma_2)$;
2. $\mathrm{wg}(\sigma_1) = \mathrm{wg}(\sigma_2)$ and $\mathrm{wg}(\Gamma_1, \Delta_1) < \mathrm{wg}(\Gamma_2, \Delta_2)$.

Thus, every descending $\prec_s$-chains containing active sequents has finite length.

Proof of Proposition 1

We have to prove that $\prec$ is a well-founded order relation. By definition, $\prec$ is transitive. We show that there exists no infinite descending $\prec$-chain; this also implies that $\prec$ is not reflexive. Let us assume, by absurd, that there exists an infinite $\prec$-chain $\mathcal{C}$ of sequents σ_i $(i \geq 1)$ such that $\sigma_{i+1} \prec \sigma_i$ for every $i \geq 1$. We have $\mathrm{wg}(\sigma_{i+1}) \leq \mathrm{wg}(\sigma_i)$ for every $i \geq 1$. Since, by Lemma 1, the relation $\prec_s$ is well-founded, $\mathcal{C}$ contains infinitely many occurrences of $\prec_d$. By Lemma 2, from $\mathcal{C}$ we can extract an infinite sequence of active sequents σ'_i such that $\mathrm{wg}(\sigma'_{i+1}) < \mathrm{wg}(\sigma'_i)$ for every $i \geq 1$, a contradiction. We conclude that every descending $\prec$-chain has finite length, hence $\prec$ well-founded.

Proof of Proposition 2

We have to prove that the rules of **G4ipf** are sound. All the cases except the one for $\to\to L$ and $\to$R rules are immediate.

Let R be the rule $\to$R, let $\sigma = \Theta \gg A \to B; \Psi$ be the conclusion of R and let $\mathcal{K} = \langle P, \leq, \rho, V \rangle$ be a Kripke model such that $\mathcal{K}, \rho \rhd \sigma$. Since $\mathcal{K}, \rho \nVdash A \to B$, there exists $\beta \in P$ such that $\mathcal{K}, \beta \Vdash A$ and $\mathcal{K}, \beta \nVdash B$. It follows that the submodel of $\mathcal{K}$ having root β realizes the premise $\Theta; A \Longrightarrow B; \cdot$ of R.

Let R be the rule $\to\to L$, let $\sigma = \Theta; (A \to B) \to C \ll \Psi$ be the conclusion of R and let us assume $\mathcal{K}, \rho \rhd \sigma$. If $\mathcal{K}, \rho \Vdash C$, we get $\mathcal{K}, \rho \rhd \Theta; C \ll \Psi$, hence the right-most premise of R is realizable. Let us assume $\mathcal{K}, \rho \nVdash C$. Since $\mathcal{K}, \rho \Vdash (A \to B) \to C$, we have $\mathcal{K}, \rho \nVdash A \to B$. Then, there exists $\beta \in P$ such that $\mathcal{K}, \beta \Vdash A$ and $\mathcal{K}, \beta \nVdash B$. It follows that $\mathcal{K}, \beta \Vdash B \to C$, and this implies $\mathcal{K}, \beta \rhd \Theta; A, B \to C \Longrightarrow B; \cdot$; thus, the left-most premise of R is realizable.

Proof of Theorem 1 (Soundness of G4ipf)

Let $\mathcal{D}$ be a closed derivation of σ and let us assume that σ is realizable. By Proposition 2, one of the initial sequents σ of $\mathcal{D}$ is realizable. Since σ is the conclusion of an axiom-rule, we get a contradiction.