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DENSITY ESTIMATES FOR A DEGENERATE/SINGULAR PHASE-TRANSITION MODEL*

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Abstract. We consider a Ginzburg–Landau type phase-transition model driven by a p-Laplacian type equation. We prove density estimates for absolute minimizers and we deduce the uniform convergence of level sets and the existence of plane-like minimizers in periodic media.

Key words. density estimates, p-Laplacian equation, phase-transition models, Ginzburg–Landau models, uniform convergence of level sets, plane-like minimizers

AMS subject classifications. 35J70, 35B45

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1. Introduction.

For a bounded domain Ω in $\mathbb{R}^n$ and $\varepsilon > 0$ consider an energy functional

$$\mathcal{J}_\varepsilon(u;\Omega) = \int_\Omega [A(x,\varepsilon\nabla u) + F(x,u)]\,dx, \tag{1.1}$$

where $A(x,\eta) \simeq |\eta|^p$, $1 < p < \infty$, and $F(x,u) \simeq |1-u^2|^\alpha$, $0 < \alpha \le p$ (see below for precise assumptions). Functionals of this type appear in the context of minimal surfaces, and it has been shown by Γ-convergence methods that sequences of minimizers converge in L^1_{loc} to suitable step functions satisfying a minimal interface property, as $\varepsilon \to 0+$ (see [MM77] for $p = 2$ and [Bou90] for the general case). Functionals of type (1.1) also have a physical relevance, since they appear in the study of the equilibrium of elastic rods under tension (see [Ant73]), in the context of fluid jets (see [AC81] and [ACF84]), and in the van der Waals–Cahn–Hilliard and Ginzburg–Landau theories of phase transition (see, for instance, [Row79]). In the phase-transition setting, the term $A(x,\varepsilon\nabla u)$ in the energy functional (1.1) can be seen as an interfacial energy contribution to the total energy, which penalizes the formation of interfaces (see [Gur85] for details).

The main purpose of this paper is to obtain Caffarelli–Córdoba [CC95] type density estimates for the absolute minimizers of the normalized functional

$$\mathcal{J}(u;\Omega) = \int_\Omega [A(x,\nabla u) + F(x,u)]\,dx. \tag{1.2}$$

Roughly speaking, such density estimates state that, if u is an absolute minimizer of $\mathcal{J}$, then the set $\{|u| < 1/2\}$ behaves in measure as an $(n-1)$-dimensional set, while $\{u > 1/2\}$ and $\{u < -1/2\}$ behave in measure as n-dimensional sets (a precise statement will be given in Theorem 1.1 below). As a consequence, we obtain the uniform convergence of the level sets of minimizers of $\mathcal{J}_\varepsilon$ to a surface of minimal

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"area" as $\varepsilon \to 0+$; see Theorem 7.1. Another application of the density estimates is the existence of plane-like minimizers of $\mathcal{J}$ in periodic media; see Theorem 7.3.

We now state in detail the assumptions required in this paper. We assume that $A : \Omega \times \mathbb{R}^n \ni (x, \eta) \longrightarrow \mathbb{R}$ is in $C^1(\Omega \times \mathbb{R}^n)$ and that

$$a(x, \eta) := D_\eta A(x, \eta)$$

is in $C(\Omega \times \mathbb{R}^n) \cap C^1(\Omega \times \mathbb{R}^n - \{0\})$. We require that

$$A(x, 0) = 0, \quad a(x, 0) = 0, \tag{1.3}$$

for every $x \in \Omega$ and that there exists $\Lambda > 0$ such that

$$\zeta \cdot D_\eta a(x, \eta)\, \zeta \geq \Lambda^{-1} |\zeta|^2\, |\eta|^{p-2} \text{ for any } \zeta \in \mathbb{R}^n, \tag{1.4}$$

$$|D_\eta a(x, \eta)| \leq \Lambda\, |\eta|^{p-2} \qquad \text{and} \tag{1.5}$$

$$|D_x a(x, \eta)| \leq \Lambda\, |\eta|^{p-1}\,, \tag{1.6}$$

$$\eta \cdot a(x, \eta) \geq \Lambda^{-1}\, |\eta|^p \tag{1.7}$$

for every $x \in \Omega$ and $\eta \in \mathbb{R}^n$.

Next, we assume that $F : \Omega \times \mathbb{R} \ni (x, u) \longrightarrow \mathbb{R}$ is a Carathéodory function, i.e., continuous in u for a.e. $x \in \Omega$ and measurable in x for every $u \in \mathbb{R}$, and satisfies

$$0 \leq F \leq M, \qquad F(x, \pm 1) = 0, \qquad \inf_{|u| \leq \theta} F(x, u) \geq \gamma(\theta) \tag{1.8}$$

for every $0 \leq \theta < 1$, where $\gamma(\theta)$ and M are positive constants. Here and below all structural inequalities on F are assumed to be uniform for a.e. $x \in \Omega$. Further, we assume that the partial derivative $F_u(x, u)$ exists for every $u \in (-1, 1)$ for a.e. $x \in \Omega$ and that

$$\sup_{|u| \leq \theta} |F_u(x, u)| \leq M(\theta) \tag{1.9}$$

for every $0 \leq \theta < 1$. We also assume the following growth condition near $u = \pm 1$: there exists $s_0 > 0$ and $d \leq p$ such that

$$F_u(x, -1 + s) \geq C s^{d-1}, \qquad F_u(x, 1 - s) \leq -C s^{d-1} \tag{1.10}$$

for every $s \in (0, s_0)$. Without loss of generality, we may and do assume $1 \leq d \leq p$. In the case $d = p$ we additionally require that

$$F_u \text{ is monotone increasing in } u \text{ for } u \in (-1, -1 + s_0) \text{ and } u \in (1 - s_0, 1). \tag{1.11}$$

Finally, if $1 < p \leq 2n/(n + 2)$, we require F to be uniformly Lipschitz in $u \in (-1, 1)$. More precisely, we assume that

$$\sup_{|u| < 1} |F_u(x, u)| \leq M \tag{1.12}$$

for a certain constant M.

We will refer to the constants that appear in (1.3)–(1.12), including n and p, as the *structural constants*. Quantities depending only on structural constants will be called *universal constants*.

A "typical" example of the functional $\mathcal{J}$, which satisfies the assumptions above, is given by

$$\int \Big(a_{i,j}(x)\,\partial_i u\,\partial_j u\Big)^{p/2} + Q(x)\,|1-u^2|^\alpha,$$

where $a_{i,j}$ is a C^1 symmetric positive definite matrix, $0 < Q_{\min} \le Q(x) \le Q_{\max}$ and $0 < \alpha \le p$. (The case $\alpha = 0$, which corresponds to $F(x,u) = Q(x)\chi_{(-1,1)}(u)$, has been treated recently in [PV03].)

We say that $u \in W^{1,p}(\Omega)$ is an *absolute minimizer* for $\mathcal{J}$ in Ω if $\mathcal{J}(u;\Omega) \le \mathcal{J}(u+\phi;\Omega)$ for any $\phi \in W_0^{1,p}(\Omega)$. In this paper, we will be concerned only with absolute minimizers u that satisfy $|u| \le 1$. Conditions (1.3)–(1.12) that we impose on $\mathcal{J}$ make it possible to apply the regularity results of Giaquinta and Giusti [GG82]. In particular, by Theorem 3.1 there, we will have that u is locally uniformly Hölder continuous in Ω. Moreover, in the region $\{|u| < 1\}$, the standard variational arguments show that u satisfies the Euler–Lagrange equation

$$\operatorname{div} a(x,\nabla u) = F_u(x,u)$$

in the weak sense. Then u is also $C^{1,\alpha}$ regular in $\{|u| < 1\}$ for some $0 < \alpha < 1$; see, e.g., [Tol84].

We will also denote by $\mathcal{L}^n$ the standard Lebesgue measure on $\mathbb{R}^n$.

The main result of this paper is as follows.

THEOREM 1.1. *For $1 < p < \infty$ assume that the hypotheses* (1.3)–(1.12) *hold. Fix $\theta \in (0,1)$ and let $|u| \le 1$ be an absolute minimizer for $\mathcal{J}$ in a bounded domain Ω, $x \in \{-\theta < u < \theta\}$ and $y \in \Omega$. Then, for every $\delta > 0$, there exist positive r_0, c, and C depending only on θ, on the structural constants, and on δ such that*

$$\mathcal{L}^n\big(B_r(x)\cap\{u>\theta\}\big) \ge c\,r^n \quad \textit{and} \quad \mathcal{L}^n\big(B_r(x)\cap\{u<-\theta\}\big) \ge c\,r^n, \tag{1.13}$$

$$\mathcal{L}^n\big(B_r(x)\cap\{|u|<\theta\}\big) \ge c\,r^{n-1} \quad \textit{and} \tag{1.14}$$

$$\mathcal{L}^n\big(B_r(y)\cap\{|u|<\theta\}\big) \le C\,r^{n-1}$$

for any $r \ge r_0$, provided $B_{r+\delta}(x)$, $B_{r+\delta}(y) \subset\subset \Omega$.

The density estimates of this type have been obtained originally in [CC95] for $p = 2$ and $A(x,\nabla u) = |\nabla u|^2$ and generalized in [Val04] to $A(x,\nabla u) = a_{i,j}\partial_i u\,\partial_j u$. The case of a general $p \in (1,\infty)$ with $A(x,\nabla u)$ satisfying the hypotheses above and $F(x,u) = Q(x)\,\chi_{(-1,1)}(u)$ has been considered in [PV03] as a model for non-Newtonian power-law fluid jets. The case treated here can be seen as a degenerate/singular phase-transition model driven by a p-Laplacian type equation.

We explicitly point out here that there is a restriction in Theorem 1.1 on the decay rate of the "double-well" potential $F(x,u)$ near $u = \pm 1$. In particular, for $F(x,u) = |1-u^2|^\alpha$ for some $\alpha > 0$, we must have $\alpha \le p$ by (1.10). The density estimates as in Theorem 1.1 are not known for $\alpha > p$. Thus, the larger we take p, the wider is the class of admissible potentials $F(x,u)$ for which the density estimates are known. In that sense, the perturbations with $A(x,\eta) \simeq |\eta|^p$ behave better for larger values of p.

We also note that additional difficulties appear in the case $1 < p < 2$. We need to require uniform Lipschitz continuity of the double-well potential $F(x,u)$ in $u \in (-1,1)$ in order to obtain the desired density estimates. This excludes the potentials $F(x,u) = |1-u^2|^\alpha$ with $0 < \alpha < 1$. However, we show that *at least* for the range of the exponents $2n/(n+2) < p < 2$, one can drop this uniform Lipschitz continuity assumption; see section 6.2.

The paper is organized as follows. In section 2 we collect some short-proof lemmas that will be of use in what follows. A Caccioppoli-type inequality is stated and proved in section 3. The proof of Theorem 1.1 is dealt with in section 4, and it makes use of an auxiliary result, namely, Lemma 4.1 below, which is interesting in itself and which roughly says that as soon as the density of sublevels of minimizers is positive in some ball, it must grow as r^n in balls of bigger radius r. We devote sections 5 and 6 to the proof of such an auxiliary result, considering the cases $p \geq 2$ and $1 < p \leq 2$ separately. In section 7 we point out some consequences that can be derived from Theorem 1.1, such as the uniform convergence of level sets of absolute minima to minimal interfaces and the existence of plane-like minimizers in periodic media.

2. Technical and elementary lemmas. We start this section with a recursive lemma.

LEMMA 2.1. *Let $v_k \geq 0$ and $a_k \geq 0$ be two nondecreasing sequences such that $v_1 + a_1 \geq c_0$,*

$$v_k^{(n-1)/n} \leq C_0 \left(v_{k+1} + a_{k+1} - v_k - a_k - c_1 a_k\right)^{1-\alpha} k^{\alpha(n-1)}$$

for any $k \in \mathbb{N}$ and some positive constants c_0, c_1, C_0, and $0 \leq \alpha < 1/n$. Then there exists $\gamma = \gamma(c_0, c_1, C_0, \alpha) > 0$ such that

$$v_k + a_k \geq \gamma\, k^n$$

for any $k \in \mathbb{N}$.

Proof. We start with an observation that it is enough to prove the estimate for $k \geq k_0$, since

$$v_k + a_k \geq v_1 + a_1 \geq c_0 \geq (c_0/k_0^n) k^n \quad \text{for } k \leq k_0.$$

The proof is by induction. Assume that $v_k + a_k \geq \gamma\, k^n$. Then either $v_k \geq (\gamma/2)k^n$ or $a_k \geq (\gamma/2)k^n$.

1. Assume first $v_k \geq (\gamma/2)k^n$. Then, using the recurrence relationship, we have

$$C_0 \left(v_{k+1} + a_{k+1} - v_k - a_k - c_1 a_k\right)^{1-\alpha} \geq (\gamma/2)^{(n-1)/n}\, k^{(1-\alpha)(n-1)}$$

and consequently

$$v_{k+1} + a_{k+1} \geq \gamma\, k^n + C\gamma^{\frac{1}{1-\alpha}\cdot\frac{n-1}{n}} k^{n-1}.$$

By our assumption, $\alpha < 1/n$, which implies that $\frac{1}{1-\alpha} \cdot \frac{n-1}{n} < 1$. Hence, if γ is sufficiently small,

$$v_{k+1} + a_{k+1} \geq \gamma(k^n + C_* k^{n-1})$$

for C_* as large as we wish. On the other hand, if we choose $C_* \geq 2^n$,

$$k^n + C_* k^{n-1} \geq (k+1)^n$$

and we obtain

$$v_{k+1} + a_{k+1} \geq \gamma(k+1)^n.$$

2. Assume now that $a_k \geq (\gamma/2)k^n$. Then

$$v_{k+1} + a_{k+1} \geq v_k + a_k + c_1 a_k \geq \gamma(k^n + (ck)k^{n-1}) \geq \gamma(k+1)^n$$

for sufficiently large k.

The proof is complete. ∎

The next lemma is similar in spirit. Its proof can be found on page 10 in [CC95] and is omitted here.

LEMMA 2.2. *Let $a_k \geq 0$ be a sequence such that $a_1 \geq c_0$, $a_k \leq C_0 L^n k^{n-1}$,*

$$\left(\sum_{1\leq j\leq k} a_j\right)^{(n-1)/n} \leq C_0 \left(a_{k+1} + \sum_{1\leq j\leq k} e^{-L(k+1-j)} a_j\right)$$

for any $k \in \mathbb{N}$ and some positive constants L, c_0, and C_0. Then, if $L = L(c_0, C_0)$ is suitably large, there exists $\gamma = \gamma(c_0, C_0) > 0$ such that

$$a_k \geq \gamma\, k^{n-1}$$

for any $k \in \mathbb{N}$. ∎

The next several lemmas are direct consequences of the structural hypotheses on $A(x,\eta)$ and $F(x,u)$.

LEMMA 2.3. *There exists a universal constant $\gamma > 0$ such that*

$$(a(x,\xi') - a(x,\xi)) \cdot (\xi' - \xi) \geq \gamma \cdot \begin{cases} (|\xi'| + |\xi|)^{p-2}|\xi' - \xi|^2 & \text{if } 1 < p \leq 2, \\ |\xi' - \xi|^p & \text{if } p \geq 2 \end{cases}$$

for every $\xi, \xi' \in \mathbb{R}^n$ and $x \in \Omega$.

Proof. For the reader's convenience we include a standard proof of this lemma. Set

$$\xi^s = s\,\xi' + (1-s)\,\xi, \quad 0 \leq s \leq 1. \tag{2.1}$$

Then $\xi^0 = \xi$ and $\xi^1 = \xi'$ and we have

$$a(x,\xi') - a(x,\xi) = \int_0^1 D_\eta a(x,\xi^s)(\xi' - \xi)ds.$$

By the hypothesis (1.4) we obtain

$$(a(x,\xi') - a(x,\xi)) \cdot (\xi' - \xi) \geq \Lambda^{-1}|\xi' - \xi|^2 \int_0^1 |\xi^s|^{p-2} ds.$$

Without loss of generality we may assume that $|\xi'| \leq |\xi|$. Then

$$(1/4)|\xi' - \xi| \leq |\xi^s| \leq |\xi'| + |\xi| \quad \text{for } 0 \leq s \leq 1/4.$$

Using the left-hand inequality for $p \geq 2$ and the right-hand inequality for $1 < p \leq 2$, we conclude the proof of the lemma. ∎

Lemma 2.4. *For any $p \geq 2$ there exists a universal constant $c > 0$ such that*

$$c\,|\xi' - \xi|^p \leq A(x,\xi') - A(x,\xi) - a(x,\xi)\cdot(\xi' - \xi) \tag{2.2}$$

for every $\xi, \xi' \in \mathbb{R}^n$ and $x \in \Omega$.

Proof. Let ξ^s be as in (2.1). Then

$$\begin{aligned}
&A(x,\xi') - A(x,\xi) = \int_0^1 a(x,\xi^s)\cdot(\xi'-\xi)ds\\
&= \int_0^1 (a(x,\xi^s) - a(x,\xi))\cdot(\xi'-\xi)ds + a(x,\xi)\cdot(\xi'-\xi)\\
&= \int_0^1 (a(x,\xi^s) - a(x,\xi))\cdot(\xi^s-\xi)\frac{ds}{s} + a(x,\xi)\cdot(\xi'-\xi).
\end{aligned}$$

From Lemma 2.3 for $p \geq 2$ we have that

$$(a(x,\xi^s) - a(x,\xi))\cdot(\xi^s - \xi) \geq \gamma\,|\xi^s - \xi|^p.$$

Hence

$$\begin{aligned}
&A(x,\xi') - A(x,\xi) \geq \gamma \int_0^1 |\xi^s - \xi|^p \frac{ds}{s} + a(x,\xi)\cdot(\xi'-\xi)\\
&= \gamma\,|\xi'-\xi|^p \int_0^1 s^{p-1} ds + a(x,\xi)\cdot(\xi'-\xi)\\
&= c\,|\xi'-\xi|^p + a(x,\xi)\cdot(\xi'-\xi). \qquad \square
\end{aligned}$$

The analogue of the preceding Lemma 2.4 for $1 < p \leq 2$ is as follows.

Lemma 2.5. *For any $1 < p \leq 2$ and $M \geq 0$ there exists a universal constant $c > 0$ such that*

$$c\,M^{p-2}\,|\xi' - \xi|^2 \leq A(x,\xi') - A(x,\xi) - a(x,\xi)\cdot(\xi' - \xi) \tag{2.3}$$

for every $\xi, \xi' \in \mathbb{R}^n$ with $|\xi| + |\xi'| \leq M$ and $x \in \Omega$.

Proof. The proof repeats the one for Lemma 2.4, except that we have to use the counterpart of Lemma 2.3 for $1 \leq p \leq 2$:

$$(a(x,\xi^s) - a(x,\xi))\cdot(\xi^s - \xi) \geq \gamma\,(|\xi^s| + |\xi|)^{p-2}|\xi^s - \xi|^2.$$

Then, also using $|\xi^s| + |\xi| \leq 2(|\xi'| + |\xi|)$, we will obtain

$$c\,(|\xi| + |\xi'|)^{p-2}|\xi' - \xi|^2 \leq A(x,\xi') - A(x,\xi) - a(x,\xi)\cdot(\xi'-\xi),$$

which implies (2.3) if $|\xi| + |\xi'| \leq M$. $\square$

The following result is elementary, and we omit the proof.

Lemma 2.6. *Let $d \geq 1$. There exists $c_d > 0$ such that*

$$(u+1)^d - (u'+1)^d \geq c_d\,(u - u')^d$$

for any $u \geq u' \geq -1$.

Next, we deduce an estimate on the double-well potential.

LEMMA 2.7. *There exists $c > 0$ such that, for any $-1 \le u' \le u \le \theta$,*

$$F(x,u) - F(x,u') \ge c\,(u-u')^d\,,$$

provided $1+\theta > 0$ is small enough.

We omit the proof of Lemma 2.7, which easily follows from (1.10) and Lemma 2.6. The proof of the next two lemmas is also elementary.

LEMMA 2.8. *Let us assume* (1.11). *Then, there exists $c > 0$ so that, for any $-1 \le u' \le u \le \theta$, $F(x,u) - F(x,u') \ge c\,(u'+1)^{d-1}\,(u-u')$, provided $1+\theta > 0$ is small enough.*

LEMMA 2.9. *Let us assume that F is uniformly Lipschitz continuous in u. Then, there exists $c > 0$ so that, for any $u \in [-1,1]$, $F(x,u) \le c(1+u)$.*

We now construct a barrier that will be of use during the proof of the main result.

LEMMA 2.10. *Fix $T \ge 1$, $\Theta \in (0,1]$, and $k \in \mathbb{N}$. Then, there exists a function $h \in C^2(B_{(k+1)T})$ so that $-1 \le h \le 1$, $h = 1$ on $\partial B_{(k+1)T}$,*

$$(h+1) + |\nabla h| + |D^2 h| \le C(h+1) \le Ce^{-\Theta T(k+1-j)} \tag{2.4}$$

in $B_{jT} - B_{(j-1)T}$ for $j = 1, \ldots, k$, and

$$|\nabla h| + |D^2 h| \le C\Theta(h+1) \tag{2.5}$$

in $B_{(k+1)T}$.

Proof. Define the following functions $\Phi : [0,1] \longrightarrow \mathbb{R}$, $\Psi : [1,(k+1)T] \longrightarrow \mathbb{R}$:

$$\Phi(t) = 2e^{\Theta[\frac{3}{8}t^6 - \frac{10}{8}t^4 + \frac{15}{8}t^2 - (k+1)T]} - 1 \quad \text{and}$$
$$\Psi(t) = 2e^{\Theta[t-(k+1)T]} - 1\,.$$

By explicit computations,

$$\Phi(1) = \Psi(1)\,, \quad \Phi'(1) = \Psi'(1), \quad \text{and} \quad \Phi''(1) = \Psi''(1)\,.$$

Thus, the function $\bar{h}$ agreeing with Φ in $[0,1]$ and with Ψ in $[1,(k+1)T]$ belongs to $C^2([0,(k+1)T])$. Define $h(x) = \bar{h}(|x|)$. Notice that $h \in C^2(B_{(k+1)T})$, since $\bar{h}'(0) = \Phi'(0) = 0$. Furthermore,

$$|\Phi'(t)| \le C\Theta t(\Phi+1)\,, \qquad |\Phi''(t)| \le C\Theta(\Phi+1) \tag{2.6}$$

in $[0,1]$ and

$$|\Psi'(t)| + |\Psi''(t)| \le C\Theta(\Psi+1) \tag{2.7}$$

in $[1,(k+1)T]$. Moreover,

$$(h+1) + |\nabla h| + |D^2 h| \le (\bar{h}+1) + \left(1 + \frac{2}{|x|}\right)|\bar{h}'| + |\bar{h}''|\,. \tag{2.8}$$

By means of (2.6), we bound the right-hand side of (2.8) in B_1 by

$$C(\Phi+1) \le Ce^{\Theta(C-(k+1)T)} \le Ce^{-\Theta T k}\,.$$

Similarly, using (2.7), we bound (2.8) by

$$C(\Psi+1) \le Ce^{-\Theta[(k+1)T-j]} \le Ce^{-\Theta T(k+1-j)}$$

in $B_{jT} - B_{(j-1)T}$ for $j = 2, \ldots, k$. This proves (2.4). In a similar way, one can prove (2.5). ∎

3. A Caccioppoli-type inequality. We now state and prove a weaker version of the Caccioppoli inequality.

LEMMA 3.1. *Fix $\delta > 0$. Let $|u| \le 1$ be an absolute minimizer for $\mathcal{J}$ in a domain Ω. Then, there exists $C > 0$, depending only on δ and on the structural constants, such that*

$$\int_{B_r(x_0)} |\nabla u|^p \le C\,(r+\delta)^n$$

for any $r > 0$ and any $x_0 \in \Omega$, provided $B_{r+\delta}(x_0) \subset \Omega$.

Proof. We start with a claim that

$$\int_\Omega a(x,\nabla u)\cdot\nabla\phi + \int_{\Omega\cap\{|u|\neq 1\}} F_u(x,u)\,\phi\,dx \ge 0 \tag{3.1}$$

for any nonnegative $\phi \in C_0^\infty(\Omega \cap \{u > -1\})$ and

$$\int_\Omega a(x,\nabla u)\cdot\nabla\psi + \int_{\Omega\cap\{|u|\neq 1\}} F_u(x,u)\,\psi\,dx \le 0 \tag{3.2}$$

for any nonnegative $\psi \in C_0^\infty(\Omega\cap\{u<1\})$. Let us show (3.2), the proof of (3.1) being analogous. For ψ as above and a small $\varepsilon > 0$, let

$$\psi_\varepsilon(x) = \psi(x)\chi_\varepsilon(u(x)),$$

where

$$\chi_\varepsilon(u) = \begin{cases} 0 & \text{if } u \le -1+\varepsilon, \\ (u+1)/\varepsilon - 1 & \text{if } -1+\varepsilon < u < -1+2\varepsilon, \\ 1 & \text{if } 1+u \ge 2\varepsilon. \end{cases}$$

Then $\psi_\varepsilon \in W^{1,p}(\Omega)$, $\operatorname{supp}\psi_\varepsilon \subset \Omega\cap\{|u|<1\}$, and therefore

$$\int_\Omega a(x,\nabla u)\cdot\nabla\psi_\varepsilon + F_u(x,u)\,\psi_\varepsilon = 0.$$

On the other hand,

$$\begin{aligned} &\int_\Omega a(x,\nabla u)\cdot\nabla\psi_\varepsilon \\ &= \int_\Omega [a(x,\nabla u)\cdot\nabla\psi]\chi_\varepsilon(u) + \frac{1}{\varepsilon}\int_{\Omega\cap\{\varepsilon<u+1<2\varepsilon\}} [a(x,\nabla u)\cdot\nabla u]\psi \\ &\ge \int_\Omega [a(x,\nabla u)\cdot\nabla\psi]\chi_\varepsilon(u) \to \int_\Omega a(x,\nabla u)\cdot\nabla\psi \end{aligned}$$

as $\varepsilon \to 0+$ and

$$\int_\Omega F_u(x,u)\psi_\varepsilon \to \int_{\Omega\cap\{u>-1\}} F_u(x,u)\psi.$$

The passage to the limit is legitimate, since

$$\int_\Omega |a(x,\nabla u)\cdot\nabla\psi| < \infty,$$

$\psi_\varepsilon \nearrow \psi\chi_{\{u>-1\}}$, and $F_u(x,u) \geq 0$ by (1.10) for u close to -1. Collecting the estimates above, we obtain (3.2).

Now fix $0 < \theta < 1$. If θ is sufficiently close to 1, the assumptions (1.9)–(1.10) and (3.1)–(3.2) above imply that

$$\int_\Omega a(x, \nabla u) \cdot \nabla\phi + K\,\phi \geq 0 \quad \text{and} \quad \int_\Omega a(x, \nabla u) \cdot \nabla\psi - K\,\psi \leq 0 \tag{3.3}$$

for any nonnegative $\phi \in C_0^\infty(\Omega \cap \{u > -\theta\})$ and $\psi \in C_0^\infty(\Omega \cap \{u < \theta\})$ with $K = M(\theta)$ as in (1.9). By standard density arguments, (3.3) also holds for nonnegative $\phi \in W_0^{1,p}(\Omega \cap \{u > -\theta\})$ and $\psi \in W_0^{1,p}(\Omega \cap \{u < \theta\})$.

Next, we observe that in light of Theorem 3.1 in [GG82], the distance between the level sets $\{u = -\theta\}$ and $\{u = \theta\}$ in $B_{r+\delta/2}(x_0)$ is bounded from below by some universal constant (depending only on θ, δ, and the structural constants). Therefore, by partition of unity, there exist two smooth functions η_- and η_+, supported in $B_{r+\delta/2}(x_0)$, so that $0 \leq \eta_-(x), \eta_+(x) \leq 1$ for any $x \in \Omega$, whose gradients are uniformly bounded by a universal constant and which satisfy

$$\begin{aligned} \eta_-(x) + \eta_+(x) \;=\; 1 \quad \forall x \in B_r(x_0)\,, \\ \operatorname{supp} \eta_- \subseteq \{-1 \leq u < \theta\}\,, \\ \operatorname{supp} \eta_+ \subseteq \{-\theta < u \leq 1\}\,, \\ \eta_-(x) + \eta_+(x) \leq 1 \quad \forall x \in \Omega\,. \end{aligned}$$

We set $\phi := (1-u)\,\eta_+^p$ and $\psi := (1+u)\,\eta_-^p$. By repeating the standard arguments in the proof of the Caccioppoli inequality (e.g., see Lemma 3.27 in [HKM93]), one infers that

$$\int_\Omega |\nabla u|^p \eta_-^p \;\leq\; C\,(r+\delta)^n \qquad \text{and} \qquad \int_\Omega |\nabla u|^p \eta_+^p \;\leq\; C\,(r+\delta)^n. \tag{3.4}$$

For the reader's convenience, we sketch the details of the proof of the second inequality in (3.4), the first being analogous. From (3.3),

$$0 \leq \int_\Omega -a(x, \nabla u) \cdot \nabla u \eta_+^p + p\,a(x, \nabla u)\,(1-u)\,\eta_+^{p-1} \nabla\eta_+ + K(1-u)\eta_+^p\,.$$

Therefore, introducing a parameter $\varepsilon \in (0,1)$, to be chosen suitably small in what follows, and using Young's inequality, we have

$$\begin{aligned} \int_\Omega |\nabla u|^p \eta_+^p &\leq C\,\left(\int_\Omega (|\nabla u|\eta_+)^{p-1} |\nabla\eta_+| + \eta_+^p \right) \\ &= C\,\left(\int_\Omega (\varepsilon|\nabla u|\eta_+)^{p-1} \frac{|\nabla\eta_+|}{\varepsilon^{p-1}} + \eta_+^p \right) \\ &\leq C\,\left(\int_\Omega (\varepsilon|\nabla u|\eta_+)^p + \frac{|\nabla\eta_+|^p}{\varepsilon^{p(p-1)}} + \eta_+^p \right) \\ &\leq C\varepsilon^p \int_\Omega (|\nabla u|\eta_+)^p + \frac{C}{\varepsilon^{p(p-1)}} \int_{B_{r+\delta}} (|\nabla\eta_+|^p + \eta_+^p) \\ &\leq C\varepsilon^p \int_\Omega |\nabla u|^p \eta_+^p + \frac{C}{\varepsilon^{p(p-1)}} (r+\delta)^n\,. \end{aligned}$$

Thus, the second inequality in (3.4) follows by choosing ε suitably small.

Using (3.4), we easily conclude the proof of the lemma:

$$\int_{B_r(x_0)} |\nabla u|^p = \int_{B_r(x_0)} |\nabla u|^p (\eta_- + \eta_+)^p$$
$$\leq C \int_{B_r(x_0)} |\nabla u|^p \eta_-^p + C \int_{B_r(x_0)} |\nabla u|^p \eta_+^p \leq C\,(r+\delta)^n . \qquad \square$$

4. Proof of Theorem 1.1. First, we point out that, since u is an absolute minimizer,

$$\mathcal{J}(u; B_r(y)) \leq C\, r^{n-1} \tag{4.1}$$

for any $r \geq r_0$, for a suitable universal r_0, provided $B_{r+\delta}(y) \subset \Omega$. To prove (4.1), with no loss of generality assume $y = 0$ and proceed as follows. Let h be a smooth function such that $h\big|_{B_{r-1}} = -1$ and $h\big|_{\partial B_r} = 1$. Let $u^* = \min\{u, h\}$. Then,

$$\begin{aligned}
\mathcal{J}(u; B_r) &\leq \mathcal{J}(u^*; B_r) \\
&\leq C \int_{B_r - B_{r-1}} (|\nabla u|^p + |\nabla h|^p) + r^{n-1} \\
&\leq C \int_{B_r - B_{r-1}} |\nabla u|^p + r^{n-1}.
\end{aligned}$$

Covering $B_r - B_{r-1}$ with balls of radius $\delta/2$, $\mathcal{B}_1, \ldots, \mathcal{B}_N$, with $N \leq C\, r^{n-1}$ and applying Lemma 3.1, we obtain

$$\int_{B_r - B_{r-1}} |\nabla u|^p \leq \sum_{i=1}^{N} \int_{\mathcal{B}_i} |\nabla u|^p \leq C\, r^{n-1} .$$

This completes the proof of (4.1).

We now focus our attention on the proof of (1.13). We will deal only with the first inequality in (1.13), the proof of the second one being analogous. To this end, we state the following result, the proof of which is deferred to sections 5 and 6.

LEMMA 4.1. *Let us assume the same hypotheses on A and F as in Theorem 1.1. Fix $\theta \in (-1, 1)$ and let u be an absolute minimizer for $\mathcal{J}$ in a domain Ω. Assume that there exist $\mu_1, \mu_2 > 0$ so that $B_{\mu_1}(x) \subset \Omega$ and $\mathcal{L}^n\big(B_{\mu_1}(x) \cap \{u > \theta\}\big) \geq \mu_2$. Then, for fixed $\delta > 0$, there exist positive r_0 and c depending only on θ, μ_1, μ_2, δ, and on the structural constants, such that $\mathcal{L}^n\big(B_r(x) \cap \{u > \theta\}\big) \geq c\, r^n$, for any $r \geq r_0$, provided $B_{r+\delta}(x) \subset\subset \Omega$.*

Analogously, if $\mathcal{L}^n\big(B_{\mu_1}(x) \cap \{u < \theta\}\big) \geq \mu_2$, then $\mathcal{L}^n\big(B_r(x) \cap \{u < \theta\}\big) \geq c\, r^n$ for any $r \geq r_0$, provided $B_{r+\delta}(x) \subset\subset \Omega$.

We now use the above result to prove (1.13). Let $\theta^\star = (1+\theta)/2 \in (\theta, 1)$. Since u is uniformly continuous (with a modulus of continuity depending only on the structural constants; see Theorem 3.1 in [GG82]) and $|u(x)| < \theta$, we have that $|u(x')| < \theta^\star$ for any $x' \in B_{\mu^\star}(x)$, for a suitable universal $\mu^\star > 0$. Hence, in view of Lemma 4.1, $\mathcal{L}^n\big(B_r(x) \cap \{u > -\theta^\star\}\big) \geq c\, r^n$ and $\mathcal{L}^n\big(B_r(x) \cap \{u < \theta^\star\}\big) \geq c\, r^n$. Therefore, by (1.8) and (4.1),

$$\begin{aligned}
&c\, r^n - \mathcal{L}^n\big(B_r(x) \cap \{u > \theta\}\big) \\
&\leq \mathcal{L}^n\big(B_r(x) \cap \{u > -\theta^\star\}\big) - \mathcal{L}^n\big(B_r(x) \cap \{u > \theta\}\big) \\
&\leq \mathcal{L}^n\big(B_r(x) \cap \{-\theta^\star < u \leq \theta\}\big) \\
&\leq C \int_{\{-\theta^\star < u \leq \theta\} \cap B_r(x)} F \\
&\leq C\, \mathcal{J}(u; B_r(x)) \leq C\, r^{n-1} .
\end{aligned}$$

Hence, if r is suitably large, $\mathcal{L}^n\big(B_r(x)\cap\{u>\theta\}\big)\geq c\,r^n$, thus proving (1.13).

We now deal with the proof of (1.14). The second inequality follows from (4.1); hence we focus on the proof of the first one. Let

$$u^\star(x)=\begin{cases}u(x) & \text{if} \quad |u(x)|<\theta\,,\\ \theta & \text{if} \quad u(x)\geq\theta\,,\\ -\theta & \text{if} \quad u(x)\leq-\theta.\end{cases}$$

Using a standard notation in geometric measure theory, we denote by $\mathcal{P}(E;U)$ the perimeter of the Borel set E in an open set U. Then, using the coarea formula, the isoperimetric inequality, and (1.13), we have

$$\begin{aligned}
&\int_{B_r(x)}|\nabla u^\star|\\
&\geq\int_{-\theta}^{\theta}\mathcal{P}\Big(\{u^\star<s\};\,B_r(x)\Big)\,ds\\
&\geq c\int_{-\theta}^{\theta}\min\Big\{\,\mathcal{L}^n\left(B_r(x)\cap\{u^\star<s\}\right),\,\mathcal{L}^n\left(B_r(x)\cap\{u^\star\geq s\}\right)\,\Big\}^{\frac{n-1}{n}}\,ds\\
&=c\int_{-\theta}^{\theta}\min\Big\{\,\mathcal{L}^n\left(B_r(x)\cap\{u<s\}\right),\,\mathcal{L}^n\left(B_r(x)\cap\{u\geq s\}\right)\,\Big\}^{\frac{n-1}{n}}\,ds\\
&\geq c\int_{-\theta}^{\theta}\min\Big\{\,\mathcal{L}^n\left(B_r(x)\cap\{u<-\theta\}\right),\,\mathcal{L}^n\left(B_r(x)\cap\{u\geq\theta\}\right)\,\Big\}^{\frac{n-1}{n}}\,ds\\
&\geq c\,r^{n-1}\,.
\end{aligned}$$

Let us now fix a suitably large parameter $K>0$. In view of the above estimate, denoting by p' the conjugated exponent of p, using Young's inequality and (4.1), we deduce that

$$\begin{aligned}
c\,r^{n-1}&\leq\frac{1}{K^p}\int_{B_r(x)}|\nabla u|^p+K^{p'}\,\mathcal{L}^n\left(B_r(x)\cap\{|u|<\theta\}\right)\\
&\leq\frac{C}{K^p}\mathcal{J}(u;B_r(x))\,+\,K^{p'}\,\mathcal{L}^n\left(B_r(x)\cap\{|u|<\theta\}\right)\\
&\leq\frac{C}{K^p}\,r^{n-1}\,+\,K^{p'}\,\mathcal{L}^n\left(B_r(x)\cap\{|u|<\theta\}\right)\,.
\end{aligned}$$

Then, (1.14) follows by choosing K large enough here above. ∎

5. Proof of Lemma 4.1. The case $p\geq2$. We will prove the first claim in Lemma 4.1, the second claim being analogous. We point out that it is enough to show the validity of the first claim of Lemma 4.1 for θ as close to -1 as we wish. Indeed, let us assume that the claim holds true for $\theta_\star$ and $-1<\theta_\star<\theta<1$. Then, if $\mathcal{L}^n\left(B_{\mu_1}(x)\cap\{u>\theta\}\right)\geq\mu_2$, then of course $\mathcal{L}^n\left(B_{\mu_1}(x)\cap\{u>\theta_\star\}\right)\geq\mu_2$, and so, since the claim holds for $\theta_\star$, $\mathcal{L}^n\left(B_r(x)\cap\{u>\theta_\star\}\right)\geq c\,r^n$. Thus, using the second part of (1.14) (which has already been proved via (4.1)),

$$\begin{aligned}
&\mathcal{L}^n\left(B_r(x)\cap\{u>\theta\}\right)\\
&\geq\mathcal{L}^n\left(B_r(x)\cap\{u>\theta_\star\}\right)\,-\,\mathcal{L}^n\left(B_r(x)\cap\{\theta_\star<u\leq\theta\}\right)\\
&\geq c\,r^n\,-\,C\,r^{n-1}\,\geq\,\tilde{c}\,r^n
\end{aligned}$$

if r is sufficiently big. This shows that we need only to prove the first claim of Lemma 4.1 for θ close to -1. Thus, we may assume that (1.10) is satisfied for $0<s\leq\theta+1$.

In this section we will assume $p \geq 2$. We will distinguish the cases $d < p$ and $d = p$, where d is the exponent that appears in (1.10).

5.1. The case $d < p$. For θ close to -1 let

$$\mathcal{V}_r = \mathcal{L}^n(\{u \geq \theta\} \cap B_r), \qquad \mathcal{A}_r = \int_{B_r} F(x,u)dx\,, \tag{5.1}$$

where B_r is short for $B_r(x)$. Then, we claim that

$$\mathcal{V}_r^{(n-1)/n} + \mathcal{A}_r \leq C_0\left(\mathcal{V}_{r+1} + \mathcal{A}_{r+1} - \mathcal{V}_r - \mathcal{A}_r\right). \tag{5.2}$$

With no loss of generality, we can take $r_0 \geq \mu_1$ and $\delta \geq 2$. Thus, by assumption, $\mathcal{V}_{r_0} \geq \mu_2 > 0$. Therefore, by means of Lemma 2.1, the above inequality implies that

$$\mathcal{V}_r \geq c\, r^n$$

for $r \geq 1$.

We now prove (5.2). We use a barrier function $h \in C^2(B_{r+1})$ such that

$$h\big|_{\partial B_{r+1}} = 1, \qquad h\big|_{B_r} = -1.$$

Let $\varepsilon = 1 + \theta$ and define $u^* = \min(u, h)$ and $\beta = \min(u - u^*, \varepsilon)$. Using the Sobolev inequality applied to β^p and then Young's inequality, we have

$$\begin{aligned}
&\left(\int_{B_{r+1}} |\beta|^{pn/(n-1)}\right)^{(n-1)/n} \\
&\leq C \int_{B_{r+1}\cap\{u-u^*<\varepsilon\}} |\beta|^{p-1}|\nabla\beta| \\
&\leq C\,K^p \int_{B_{r+1}\cap\{u-u^*<\varepsilon\}} |\nabla(u-u^*)|^p \\
&\quad + \frac{C}{K^{p'}} \int_{B_{r+1}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p\,.
\end{aligned} \tag{5.3}$$

Here, $K > 0$ is a free parameter that will be conveniently chosen in what follows. As customary, we also denoted the conjugated exponent of p by p'. Since $u^* = -1$ in B_r, $u - u^* \geq \varepsilon$ in $B_r \cap \{u \geq \theta\}$, the left-hand side of the inequality above is bounded from below by

$$c\,\mathcal{L}^n(\{u \geq \theta\} \cap B_r)^{(n-1)/n} = c\,\mathcal{V}_r^{(n-1)/n}.$$

Next, we apply (2.2) with $\xi = \nabla u^*$ and $\xi' = \nabla u$ to estimate $|\nabla(u-u^*)|^p$ in the right-hand side of (5.3). Thus, we obtain

$$\begin{aligned}
\mathcal{V}_r^{(n-1)/n} &\leq C\,K^p \int_{B_{r+1}} A(x,\nabla u) - A(x,\nabla u^*) \\
&\quad - C\,K^p \int_{B_{r+1}} a(x,\nabla u^*)\cdot\nabla(u-u^*) \\
&\quad + \frac{C}{K^{p'}} \int_{B_{r+1}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p.
\end{aligned} \tag{5.4}$$

Since $\operatorname{supp}(u-u^*) \subset B_{r+1} \subset\subset \Omega$, the minimality of u implies that $\mathcal{J}(u;\Omega) \leq \mathcal{J}(u^*;\Omega)$ or, equivalently,

$$\int_{B_{r+1}} A(x,\nabla u) - A(x,\nabla u^*) \leq \int_{B_{r+1}} F(x,u^*) - F(x,u).$$

Using this, and integrating by parts the term $a(x,\nabla u^*)\cdot\nabla(u-u^*)$ in the right-hand side of (5.4), we obtain

$$\begin{aligned} \mathcal{V}_r^{(n-1)/n} &\leq C\,K^p \int_{B_{r+1}} F(x,u^*) - F(x,u) \\ &\quad + C\,K^p \int_{B_{r+1}} \operatorname{div} a(x,\nabla u^*)(u-u^*) \\ &\quad + \frac{C}{K^{p'}} \int_{B_{r+1}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p. \end{aligned} \tag{5.5}$$

Notice also that, by definition of u^*,

$$\int_{B_{r+1}} \operatorname{div} a(x,\nabla u^*)(u-u^*) = \int_{B_{r+1}} \operatorname{div} a(x,\nabla h)(u-u^*).$$

Thus, to proceed, we recall that by (1.5) and (1.6) we have

$$\operatorname{div} a(x,\nabla h) \leq C|\nabla h|^{p-2}(|\nabla h| + |D^2 h|). \tag{5.6}$$

Now let h be a C^2 radial function, defined by

$$h(x) = -1 + 2(|x|-r)_+^\alpha,$$

with some fixed

$$\alpha > \max\left\{\frac{p}{p-d}, 2\right\}.$$

Then

$$|\nabla h| \leq C(h+1)^{(\alpha-1)/\alpha}, \qquad |D^2 h| \leq C(h+1)^{(\alpha-2)/\alpha}.$$

Hence,

$$\operatorname{div} a(x,\nabla h) \leq C(h+1)^{(p-2)(\alpha-1)/\alpha+(\alpha-2)/\alpha} \leq C(h+1)^{d-1} \tag{5.7}$$

and consequently

$$\begin{aligned} \mathcal{V}_r^{(n-1)/n} &\leq C\,K^p \int_{B_{r+1}} [F(x,u^*) - F(x,u)] \\ &\quad + C\,K^p \int_{B_{r+1}} (u^*+1)^{d-1}\,(u-u^*) \\ &\quad + \frac{C}{K^{p'}} \int_{B_{r+1}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p. \end{aligned} \tag{5.8}$$

We now split the right-hand side of the above inequality into three parts, namely, the contribution in B_r, the one in $\{u<\theta\}\cap(B_{r+1}-B_r)$, and the one in $\{u \geq \theta\}\cap(B_{r+1}-B_r)$.

1. The contribution in B_r. Here the second integrand in the right-hand side of (5.8) vanishes, as well as the term $F(x, u^*)$ of the first integrand. Besides, the third integral is taken over the region, where $u - u^* < \varepsilon$. In B_r, the latter condition is equivalent to $u < \theta$, since $u^* = -1$. Furthermore, if K is sufficiently large, for $u < \theta$ we have

$$-C\,K^p F(x,u) + C\,K^{-p'}(u+1)^p \le -c\,F(x,u),$$

since by our assumption $F(x,u) \ge c(u+1)^d \ge c(u+1)^p$ for $-1 \le u < \theta$. Hence, the contribution of the right-hand side of (5.8) in B_r is bounded from above by $-c\mathcal{A}_r$.

2. The contribution in $\{u < \theta\} \cap (B_{r+1} - B_r)$. Since $-1 \le u^* \le u < \theta$, from Lemma 2.7 we have that

$$F(x,u^*) - F(x,u) \le -c\,(u-u^*)^d.$$

Since both $u^* + 1 \le u + 1$ and $u - u^* \le u + 1$, we also have

$$(u^*+1)^{d-1}(u-u^*) \le (u+1)^d \le C\,F(x,u).$$

Thus,

$$K^p[F(x,u^*) - F(x,u) + (u^*+1)^{d-1}(u-u^*)] + K^{-p'}(u-u^*)^p \le C\,F(x,u),$$

and the total contribution of the right-hand side of (5.8) in $\{u < \theta\} \cap (B_{r+1} - B_r)$ is bounded from above by $C(\mathcal{A}_{r+1} - \mathcal{A}_r)$.

3. Finally, the contribution in $\{u \ge \theta\} \cap (B_{r+1} - B_r)$ is easily estimated by

$$C\,\mathcal{L}^n(\{u \ge \theta\} \cap (B_{r+1} - B_r)) = C(\mathcal{V}_{r+1} - \mathcal{V}_r),$$

since the terms inside the integrals are bounded.

Collecting the estimates from 1–3, we obtain (5.2), which completes the proof of Lemma 4.1 in the case $p \ge 2$, $d < p$. ∎

5.2. The case $d = p$. The proof is a refinement of the one in the case $d < p$. Here we use suitable positive parameters Θ and T: we will fix Θ small enough and then choose T suitably large (and in fact ΘT suitably large).

We define $\mathcal{V}_r$ as in (5.1) and set $a_k = \mathcal{V}_{kT} - \mathcal{V}_{(k-1)T}$. Then, we claim that

$$\left(\sum_{1\le j\le k} a_j\right)^{(n-1)/n} \le C\left(a_{k+1} + \sum_{1\le j\le k} e^{-L(k+1-j)} a_j\right), \tag{5.9}$$

with $L = \Theta T$ as large as we wish. With no loss of generality, we can take $r_0 \ge \mu_1$. Thus, by assumption, $\mathcal{V}_{r_0} \ge \mu_2 > 0$. Therefore, by means of Lemma 2.2, the above inequality implies that

$$\mathcal{V}_r \ge c\,r^n$$

for $r \ge 1$.

We now prove (5.9). We use the barrier function $h = h_k \in C^2(B_{(k+1)T})$ introduced in Lemma 2.10. Then, $-1 \le h \le 1$, $h = 1$, on $\partial B_{(k+1)T}$,

$$(h+1) + |\nabla h| + |D^2 h| \le C(h+1) \le Ce^{-\Theta T(k+1-j)} \tag{5.10}$$

in $B_{jT} - B_{(j-1)T}$, and

$$|\nabla h| + |D^2 h| \le C\Theta(h+1) \tag{5.11}$$

in $B_{(k+1)T}$.

Fix $\varepsilon \in (0, 1+\theta)$. Define $u^* = \min(u,h)$ and $\beta = \min(u-u^*, \varepsilon)$. Using the Sobolev inequality applied to β^p and then Young's inequality, we have

$$\begin{aligned}(5.12)\qquad & \left(\int_{B_{(k+1)T}} |\beta|^{pn/(n-1)}\right)^{(n-1)/n} \\ & \le C\,K^p \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} |\nabla(u-u^*)|^p \\ & \quad + \frac{C}{K^{p'}} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p\end{aligned}$$

with the parameter $K > 0$ to be chosen later. From (5.10),

$$u - u^* \ge \theta + 1 - Ce^{-\Theta T} > \varepsilon \tag{5.13}$$

in $B_{kT} \cap \{u \ge \theta\}$, provided ΘT is conveniently large; hence the left-hand side of the inequality above is bounded from below by

$$c\,\mathcal{L}^n(\{u \ge \theta\} \cap B_{kT})^{(n-1)/n} = c\,\mathcal{V}_{kT}^{(n-1)/n}.$$

Now, combining (5.12) and (2.2), using the minimality of u, and integrating by parts the term $a(x, \nabla u^*) \cdot \nabla(u-u^*)$ (for more details, see the respective part in section 5.1), we obtain

$$\begin{aligned}\mathcal{V}_{kT}^{(n-1)/n} &\le C\,K^p \int_{B_{(k+1)T}} F(x,u^*) - F(x,u) \\ &\quad + C\,K^p \int_{B_{(k+1)T}} \operatorname{div} a(x, \nabla u^*)(u-u^*) \\ &\quad + \frac{C}{K^{p'}} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p.\end{aligned}$$

Recalling (1.5) and (1.6), we have that

$$\operatorname{div} a(x, \nabla h) \le C|\nabla h|^{p-2}(|\nabla h| + |D^2 h|). \tag{5.14}$$

Hence,

$$\begin{aligned}\mathcal{V}_{kT}^{(n-1)/n} &\le C\,K^p \int_{B_{(k+1)T}} F(x,u^*) - F(x,u) \\ &\quad + C\,K^p \int_{B_{(k+1)T}} |\nabla u^*|^{p-2}\,(|\nabla u^*| + |D^2 u^*|)\,(u-u^*) \\ &\quad + \frac{C}{K^{p'}} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p\,.\end{aligned}$$

Thanks to the definition of u^*, we may replace u^* with h in the second integral in the previous inequality in order to gather

$$
\begin{aligned}
\mathcal{V}_{kT}^{(n-1)/n} &\le C\,K^p \int_{B_{(k+1)T}} F(x,u^*) - F(x,u) \\
&\quad + C\,K^p \int_{B_{(k+1)T}} |\nabla h|^{p-2}\,(|\nabla h| + |D^2 h|)\,(u-u^*) \\
&\quad + \frac{C}{K^{p'}} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p.
\end{aligned} \tag{5.15}
$$

We now split the right-hand side of the above inequality into three parts, namely, the contribution in $\{u<\theta\}$, the one in $\{u\ge\theta\}\cap(B_{(k+1)T}-B_{kT})$, and the one in $\{u\ge\theta\}\cap B_{kT}$.

1. The contribution in $\{u<\theta\}$ is estimated using (5.11) and Lemmas 2.7 and 2.8. We actually show that such contribution is negative. Indeed, using the above mentioned results and taking K suitably big (so as to kill the last term with the first one) and Θ suitably small (so as to kill the constant c in Lemma 2.8), the contribution in $\{u<\theta\}$ is bounded by

$$C\int_{B_{(k+1)T}\cap\{u<\theta\}} F(x,u^*) - F(x,u) + C\Theta(u^*+1)^{d-1}(u-u^*) \le 0\,.$$

2. The contribution in $\{u\ge\theta\}\cap(B_{(k+1)T}-B_{kT})$ of the right-hand side of (5.15) can be easily bounded by $C\,a_{k+1}$, since the terms inside the integrals are bounded.

3. We now estimate the contribution of the right-hand side of (5.15) in $\{u\ge\theta\}\cap B_{kT}$. First, notice that, by (5.13),

$$\int_{B_{kT}\cap\{u-u^*<\varepsilon\}\cap\{u\ge\theta\}} (u-u^*)^p = \int_{\emptyset}(u-u^*)^p = 0\,.$$

Also, from (1.9) and (1.11), it follows that F is uniformly Lipschitz continuous in u; thus, from Lemma 2.9, $F(x,h)\le c\,(1+h)$. Therefore, by (5.10), we bound the contribution in $\{u\ge\theta\}\cap B_{kT}$ by

$$
\begin{aligned}
&C\left(\sum_{j=1}^{k}\int_{(B_{jT}-B_{(j-1)T})\cap\{u\ge\theta\}} F(x,h) + |\nabla h| + |D^2 h|\right) \\
&\le C\left(\sum_{j=1}^{k} e^{-\Theta T(k+1-j)}\,\mathcal{L}^n\big((B_{jT}-B_{(j-1)T})\cap\{u\ge\theta\}\big)\right).
\end{aligned}
$$

In light of 1–3, we bound the right-hand side of (5.15) by

$$C\left(a_{k+1} + \sum_{1\le j\le k} e^{-L(k+1-j)}a_j\right).$$

This proves (5.9) and completes the proof of Theorem 1.1 in the case $p\ge 2$.

6. Proof of Lemma 4.1. The case $1 < p < 2$.

6.1. The case of uniformly Lipschitz F. Under the assumption (1.12) of the uniform Lipschitz continuity of the double-well potential F, every absolute minimizer u of $\mathcal{J}$ with $|u| \le 1$ will satisfy an equation

$$\operatorname{div} a(x, \nabla u) = g(x), \tag{6.1}$$

weakly in Ω for some $g \in L^\infty(\Omega)$. Indeed, if M is as in (1.12) and $\psi \in C_0^\infty(\{u < 1\})$ is nonnegative, using that $\mathcal{J}(u + \varepsilon\psi; \Omega) \ge \mathcal{J}(u; \Omega)$, we will easily obtain

$$\int_\Omega a(x, \nabla u) \cdot \nabla\psi + M\,\psi \ge 0.$$

On the other hand, by (3.2), we also have

$$\int_\Omega a(x, \nabla u) \cdot \nabla\psi - M\,\psi \le 0.$$

Hence (6.1) is satisfied with $|g| \le M$ in $\{u < 1\}$. Similarly, we prove (6.1) in $\{u > -1\}$ and consequently in Ω.

Note that $g(x) = F_u(x, u)$ a.e. in $\{|u| < 1\}$ and $g(x) = 0$ in $\Omega \setminus \overline{\{|u| < 1\}}$, but we have no information on $g(x)$ on the "free boundary" $\partial\{|u| < 1\} \cap \Omega$, except that it is bounded. However, that is sufficient for our purposes.

The equation (6.1) implies that u is locally uniformly $C^{1,\alpha}$ regular in Ω; see [Tol84]. Then the proof of Lemma 4.1 in the case $1 < p \le 2$ is a slight variation of the one for $p \ge 2$. The main difference is that we use Lemma 2.5 instead of Lemma 2.4. Technically, we should separately consider the cases $d < p$ and $d = p$. However, since the changes from the case $p \ge 2$ are similar in both cases, we sketch only the proof for the more subtle case $d = p$.

We consider suitable positive parameters Θ, T, and K (playing the same role as in section 5.2) and we define h, u^*, and β as we did in section 5.2 above. In analogy with (5.12), using the Sobolev inequality applied to β^2 and then Young's inequality, we have

$$\begin{aligned} &\left(\int_{B_{(k+1)T}} |\beta|^{2n/(n-1)}\right)^{(n-1)/n} \\ &\le C\,K^2 \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} |\nabla(u-u^*)|^2 \\ &\quad + \frac{C}{K^2} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^2\,. \end{aligned} \tag{6.2}$$

Arguing as in (5.13), we get that the left-hand side of the inequality above is estimated from below by

$$c\,\mathcal{L}^n(\{u \ge \theta\} \cap B_{kT})^{(n-1)/n} = c\,\mathcal{V}_{kT}^{(n-1)/n}.$$

Notice that $|\nabla u|$ is uniformly bounded by means of Theorem 1 in [Tol84] (and, indeed, u is $C^{1,\alpha}$ with uniform estimates in the interior of Ω). Thus, using (2.3), the minimality of u, and an integration by parts, we infer from (6.2) the following inequality:

$$
\begin{aligned}
\mathcal{V}_{kT}^{(n-1)/n} \leq C\,K^2 & \int_{B_{(k+1)T}} F(x,u^*) - F(x,u) \\
& + C\,K^2 \int_{B_{(k+1)T}} \operatorname{div} a(x,\nabla u^*)(u-u^*) \\
& + \frac{C}{K^2} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^2.
\end{aligned}
$$

In light of (5.14), we deduce that

$$
\begin{aligned}
\mathcal{V}_{kT}^{(n-1)/n} \leq C\,K^2 & \int_{B_{(k+1)T}} F(x,u^*) - F(x,u) \\
& + C\,K^2 \int_{B_{(k+1)T}} |\nabla u^*|^{p-2} \left(|\nabla u^*| + |D^2 u^*|\right)(u-u^*) \\
& + \frac{C}{K^2} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^2.
\end{aligned}
$$

By the definition of u^*, we may replace u^* with h in the second integral in the previous inequality, obtaining

$$
\begin{aligned}
\mathcal{V}_{kT}^{(n-1)/n} \leq C\,K^2 & \int_{B_{(k+1)T}} F(x,u^*) - F(x,u) \\
& + C\,K^2 \int_{B_{(k+1)T}} |\nabla h|^{p-2} \left(|\nabla h| + |D^2 h|\right)(u-u^*) \\
& + \frac{C}{K^2} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^2.
\end{aligned} \tag{6.3}
$$

As done in section 5.2, one splits the right-hand side of the above inequality into three parts, namely, the contribution in $\{u<\theta\}$, the one in $\{u\geq\theta\}\cap(B_{(k+1)T}-B_{kT})$, and the one in $\{u\geq\theta\}\cap B_{kT}$. Such estimates follow the lines of section 5.2. Namely, the contribution in $\{u\leq\theta\}$ is estimated by using (5.11) and Lemmas 2.7 and 2.8, obtaining, for big K and small $\Theta>0$, the bound

$$
C\int_{B_{(k+1)T}\cap\{u<\theta\}} F(x,u^*) - F(x,u) + C\Theta^{p-1}(u^*+1)^{d-1}(u-u^*) \leq 0\,,
$$

which is negative. The contribution in $\{u\geq\theta\}\cap(B_{(k+1)T}-B_{kT})$ of the right-hand side of (5.15) can be easily bounded by $C\,a_{k+1}$. As above, the contribution in $\{u\geq\theta\}\cap B_{kT}$ is bounded by using Lemma 2.9 and (5.10), obtaining

$$
C\left(\sum_{j=1}^{k} e^{-(p-1)\Theta T(k+1-j)}\,\mathcal{L}^n\big((B_{jT}-B_{(j-1)T})\cap\{u\geq\theta\}\big)\right).
$$

This proves (5.9) and hence completes the proof of Theorem 1.1 in the case $1<p\leq 2$ for potentials F satisfying (1.12). $\square$

6.2. The case $2n/(n+2) < p < 2$. We now show that the density estimate in Lemma 4.1 can be obtained at least for $2n/(n+2) < p < 2$ without the technical assumption (1.12) of uniform Lipschitz continuity of the double-well potential F in $u \in (-1, 1)$. This will be achieved with a more effective use of the inequality

$$c(|\xi'| + |\xi|)^{p-2}|\xi' - \xi|^2 \leq A(x, \xi') - A(x, \xi) - a(x, \xi) \cdot (\xi' - \xi) \tag{6.4}$$

for $1 < p < 2$; see the proof of Lemma 2.5.

Without loss of generality we may assume that $d < p$. Indeed, the additional hypothesis (1.11) in the case $d = p$ implies (1.12), contrary to our assumption.

We revisit the proof of Lemma 4.1 in section 5.1, now with $1 < p < 2$, and let h, u^*, and β be the same as there. We also introduce the weight

$$\omega = (|\nabla u| + |\nabla u^*|)^{1-p/2}.$$

Integrating (6.4) over B_r with $\xi' = \nabla u$ and $\xi = \nabla u^*$, we obtain that

$$c\int_{B_r} \frac{|\nabla\beta|^2}{\omega^2} \leq \int_{B_r} A(x, \nabla u) - A(x, \nabla u^*) - a(x, \nabla u^*) \cdot \nabla(u - u^*). \tag{6.5}$$

To avoid complications, related to the vanishing of ω, we also introduce its "regularization"

$$\omega_\varepsilon = (|\nabla u| + |\nabla u^*| + \varepsilon)^{1-p/2}, \quad \varepsilon > 0.$$

Observe that we always have $\omega_\varepsilon > \omega$ and $\omega_\varepsilon \searrow \omega$ as $\varepsilon \searrow 0$.

Analyzing the proof in section 5.1, we realize that one can improve the step when the Sobolev and Young inequalities are applied to the function β^p; see (5.3). Indeed, let κ and $\lambda(x)$ be a certain positive number and a function, to be chosen later. Then

$$c\left(\int_{B_r} \beta^{\kappa\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq \int_{B_r} \beta^{\kappa-1}|\nabla\beta| = \int_{B_r} \left(\frac{|\nabla\beta|}{\omega_\varepsilon}\lambda\right)\left(\beta^{\kappa-1}\frac{\omega_\varepsilon}{\lambda}\right),$$

where in the first step we have applied the Sobolev inequality to the function β^κ. Now we use Young's inequality, with a certain parameter q, $1 < q < 2$, and its conjugate $q' = q/(q-1)$. We obtain

$$c\int_{B_r} \left(\frac{|\nabla\beta|}{\omega_\varepsilon}\lambda\right)\left(\beta^{\kappa-1}\frac{\omega_\varepsilon}{\lambda}\right) \leq K^q \int_{B_r} \frac{|\nabla\beta|^q}{\omega_\varepsilon^q}\lambda^q + K^{-q'}\int_{B_r} \beta^{(\kappa-1)q'}\frac{\omega_\varepsilon^{q'}}{\lambda^{q'}}.$$

Applying the Hölder inequality with exponents $2/q$ and $2/(2-q)$ in both integrals on the right-hand side of the inequality above, we estimate it by

$$\begin{aligned} &K^q\left(\int_{B_r} \frac{|\nabla\beta|^2}{\omega_\varepsilon^2}\right)^{q/2}\left(\int_{B_r} \lambda^{2q/(2-q)}\right)^{1-q/2} \\ &+ K^{-q'}\left(\int_{B_r} \beta^{(\kappa-1)q'(2/q)}\right)^{q/2}\left(\int_{B_r} \left(\frac{\omega_\varepsilon}{\lambda}\right)^{2q'/(2-q)}\right)^{1-q/2}. \end{aligned}$$

Let us now choose λ so that

$$\lambda^{2q/(2-q)} = \left(\frac{\omega_\varepsilon}{\lambda}\right)^{2q'/(2-q)}.$$

A simple computation shows that

$$\lambda = \omega_\varepsilon^{q'/(q+q')} = \omega_\varepsilon^{1/q}.$$

Then the expression above transforms to

$$\left[K^q \left(\int_{B_r} \frac{|\nabla\beta|^2}{\omega_\varepsilon^2}\right)^{q/2} + K^{-q'} \left(\int_{B_r} \beta^{(\kappa-1)q'(2/q)}\right)^{q/2}\right] \left(\int_{B_r} \omega_\varepsilon^{1/(1-q/2)}\right)^{1-q/2},$$

which is bounded from above by

$$C\left[K^2 \int_{B_r} \frac{|\nabla\beta|^2}{\omega_\varepsilon^2} + K^{-2q'/q} \int_{B_r} \beta^{(\kappa-1)q'(2/q)}\right]^{q/2} \left(\int_{B_r} \omega_\varepsilon^{1/(1-q/2)}\right)^{1-q/2}.$$

Collecting the estimates above and then letting $\varepsilon \to 0$, we will arrive at the inequality

$$\left(\int_{B_r} \beta^{\kappa\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq$$
$$C\left[K^2 \int_{B_r} \frac{|\nabla\beta|^2}{\omega^2} + K^{-2q'/q} \int_{B_r} \beta^{(\kappa-1)q'(2/q)}\right]^{q/2} \left(\int_{B_r} \omega^{1/(1-q/2)}\right)^{1-q/2}.$$

Now observe that the term inside square brackets can be estimated similarly as in section 5.1, recalling also (6.5). We now have arrived at a point when we have to choose q. For that purpose we turn our attention to the term $\omega^{1/(1-q/2)}$. If we knew that ω is bounded, we could let $q \nearrow 2$. This is so, for instance, when F is uniformly Lipschitz in u, and we recover the proof in section 6.1 above. However, for non-Lipschitz F, we a priori know only the L^p integrability of $|\nabla u|$ and $|\nabla u^*|$. Moreover, we have

$$\int_{B_r} |\nabla u|^p + |\nabla u^*|^p \leq \int_{B_r} 2|\nabla u|^p + |\nabla h|^p \leq C\, r^{n-1},$$

by (4.1), for sufficiently large r. Thus, in order to obtain the desired density estimate, we choose q so that $\omega^{1/(1-q/2)} \simeq (|\nabla u|^p + |\nabla u^*|^p)$. Since $\omega = (|\nabla u| + |\nabla u^*|)^{1-p/2}$, we require

$$(1-p/2)/(1-q/2) = p \quad \Longleftrightarrow \quad q = 3 - \frac{2}{p}.$$

Observe that the condition $1 < q < 2$ is satisfied for $1 < p < 2$. As for the value of κ, we choose it to have

$$(\kappa - 1)q'(2/q) = p \quad \Longleftrightarrow \quad \kappa = p.$$

Thus, with this choice of constants we obtain

$$\left(\int_{B_r} \beta^{p\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C\left[K^2 \int_{B_r} \frac{|\nabla\beta|^2}{\omega^2} + K^{-p'} \int_{B_r} \beta^p\right]^{q/2} \left(r^{n-1}\right)^{1-q/2}.$$

Now, using (6.5) and repeating the arguments as in section 5.1, we can deduce the following recursive inequality:

$$c\,\mathcal{V}_r^{\frac{n-1}{n}} \leq \left[\mathcal{V}_{r+1} - \mathcal{V}_r + \mathcal{A}_{r+1} - \mathcal{A}_r - c\mathcal{A}_r\right]^{q/2} (r^{n-1})^{1-q/2}. \tag{6.6}$$

Now the question is whether we can infer from (6.6) that

$$\mathcal{V}_r \geq c\, r^n$$

for $r \geq 1$ if $\mathcal{V}_1 \geq \mu > 0$. The answer is affirmative when

$$\frac{n-1}{n} \cdot \frac{2}{q} < 1 \quad \Longleftrightarrow \quad p > \frac{2n}{n+2}.$$

This follows from Lemma 2.1 with $\alpha = 1 - q/2$. (Unfortunately, (6.6) alone does not imply the density estimate for $p \leq 2n/(n+2)$, since we do need $\alpha < 1/n$ in Lemma 2.1.)

Summarizing, we obtain that for the range of the exponents $2n/(n+2) < p < 2$, one can drop the assumption (1.12) to prove Lemma 4.1. This completes the proof of Theorem 1.1.

7. Consequences of the density estimates. We briefly show in this section two consequences that can be easily derived from Theorem 1.1, thanks to the techniques developed in the last years.

The first consequence is that level sets of absolute minimizers converge, up to subsequence, to minimal interfaces in L^∞_{loc}. More precisely, it has been proved in [Bou90] that minimizers u_ε of

$$\mathcal{J}_\varepsilon(u;\Omega) = \int_\Omega A(x, \varepsilon\, \nabla u) + F(x,u) \tag{7.1}$$

converge, up to subsequence, in L^1_{loc} to a step function u_0 which has a minimal interface with respect to a suitably weighted area. Indeed, from the above density estimates we have that level sets converge in L^∞_{loc}.

THEOREM 7.1. *Fix $\theta \in (0,1)$. Let $|u_\varepsilon| \leq 1$ be an absolute minimizer of* (7.1) *in a bounded domain Ω. Assume that, as ε tends to zero, u_ε converges in L^1_{loc} to*

$$u_0 := \chi_E - \chi_{\Omega - E}$$

for a suitable $E \subset \Omega$. Then, $\{|u_\varepsilon| \leq \theta\}$ converges locally uniformly to ∂E.

The latter convergence is understood in the sense that $\operatorname{dist}(x, \partial E) \to 0$ uniformly for $x \in \{|u_\varepsilon| \leq \theta\} \cap K$ for any $K \subset\subset \Omega$.

Proof. The proof repeats the one of Theorem 2 in [CC95]. Assume that the claim of the theorem is not correct. Then there is $\delta > 0$, $K \subset\subset \Omega$, and $\varepsilon_n \to 0$, such that there exist $x_n \in \{|u_{\varepsilon_n}| < \theta\}$ with, say, $B_\delta(x_n) \subset E \cap K$. Since the rescalings $\tilde{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x)$ are absolute minimizers of the normalized functional $\mathcal{J}$, applying the density estimates in Theorem 1.1 to $\tilde{u}_\varepsilon$ and then scaling back to u_ε, we will obtain that

$$\mathcal{L}^n(B_{\delta/2}(x_n) \cap \{u_{\varepsilon_n} < \theta\}) \geq c\, \delta^n$$

for some $c > 0$. But then,

$$\int_{B_{\delta/2}(x_n)} |u_{\varepsilon_n} - u_0| \geq c(1-\theta)\, \delta^n,$$

in contradiction with the hypothesis. This proves Theorem 7.1. ∎

Remark 7.2. We point out the following particular case of the above theorem. Let $F(x,u) = |1-u^2|^\alpha$ with $\alpha > 2$. Then it is unknown whether there is a uniform convergence of the level sets of minimizers in the singular perturbation problem

$$\int \varepsilon^2 |\nabla u|^2 + |1-u^2|^\alpha, \qquad \varepsilon \to 0+ .$$

However, if one perturbs with $\varepsilon^p |\nabla u|^p$ with $p \geq \alpha$, the uniform convergence follows from Theorem 7.1.

The second consequence of the density estimates is the existence of plane-like minimizers in the periodic setting. We say that u is a class A minimizer for $\mathcal{J}$ if it is an absolute minimizer for $\mathcal{J}$ in any ball B. With this setting, we can prove the following theorem.

THEOREM 7.3. *Assume that $A(x+e,\eta) = A(x,\eta)$ and $F(x+e,\eta) = F(x,\eta)$ for any $e \in \mathbb{Z}^n$. Fix $\theta \in (0,1)$. Then, there exists a positive constant M_0, depending only on θ and on the structural constants, such that, given any $\omega \in \mathbb{R}^n - \{0\}$, there exists a class A minimizer $u = u_\omega$ for the functional $\mathcal{J}$ for which the set $\{|u| \leq \theta\}$ is constrained in the strip $\{x \cdot \omega \in [0, M_0|\omega|]\}$.*

Furthermore, such u enjoys the following property of "quasi periodicity": if $\omega \in \mathbb{Q}^n - \{0\}$, then u is periodic (with respect to the identification induced by ω, i.e., $u(x+k) = u(x)$ for any $k \in \mathbb{Z}^n \cap \omega^\perp$); if $\omega \in \mathbb{R}^n - \mathbb{Q}^n$, then u can be approximated uniformly on compact sets by periodic class A minimizers.

Notice that M_0 above is *independent* of the frequency ω. These kinds of plane-like structures have been considered in [CdlL01] in the minimal surfaces case, and generalized to fluid jets and Ginzburg–Landau models in [Val04] and [PV03]. See also [Tor04] for a case with a degenerate metric. The proof of Theorem 7.3 is analogous to the one presented in section 8 of [PV03], with minor obvious changes, and we therefore omit the details.

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