Rigidity Results for Elliptic PDEs with Uniform Limits: an Abstract Framework with Applications

ALBERTO FARINA & ENRICO VALDINOCI

ABSTRACT. We provide an abstract framework for a symmetry result arising in a conjecture of G.W. Gibbons and we apply it to the fractional Laplace operator, to the elliptic operators with constant coefficients, to the quasilinear operators, and to elliptic fully nonlinear operators with possible gradient dependence.

1. INTRODUCTION

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the problem

$$\begin{cases} Lu(x) = f(u(x)) & \text{for any } x \in \mathbb{R}^n, \\ \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, & \text{uniformly in } x' \in \mathbb{R}^{n-1}. \end{cases} \tag{1.1}$$

Here, L is an operator (not necessarily linear) acting on a space $\mathbb{X}$ of smooth (say, C^r with $r \geqslant 1$) functions and commuting with the translations, i.e.,

$$L(u(x+y)) = (Lu)(x+y) \quad \text{for any } y \in \mathbb{R}^n, \tag{1.2}$$

whose precise assumptions will be listed below.

The space $\mathbb{X}$ is supposed to contain functions from $\mathbb{R}^n$ to $\mathbb{R}$ and to be translation invariant (with respect to the translations in $\mathbb{R}^n$), that is

$$\text{if } u \in \mathbb{X}, \text{ then the functions } x \mapsto u(x+y) \text{ lie in } \mathbb{X} \text{ too, for any } y \in \mathbb{R}^n. \tag{1.3}$$

We remark that we need that the space $\mathbb{X}$ is translation invariant, as in (1.3), because we want to study the operator on both the function and on its translation.

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On the other hand, we do not need that the operator itself is translation invariant, but only that it commutes with the translations, as in (1.2). This technical detail will allow us to deal with nonlinear operators too.

As for the nonlinearity, we suppose that $f \in C^1(\mathbb{R})$, with

$$\inf_{r\in(-\infty,-1]\cup[1,+\infty)} f'(r) > 0. \tag{1.4}$$

A paradigmatic example of nonlinearity satisfying the above assumptions is the function $f(r) = r^3 - r$.

The goal of this paper is to prove that u possesses one-dimensional symmetry, that is, that there exists $u_0 : \mathbb{R} \to \mathbb{R}$ such that

$$u(x', x_n) = u_0(x_n) \quad \text{for any } (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{1.5}$$

For this, the following hypotheses are taken on L:

H1. (Linearization): If $\varphi \in \mathbb{X}$ satisfies $L\varphi = f(\varphi)$ in $\mathbb{R}^n$, then there exists an operator $\tilde{L}$ acting on some space of functions $\tilde{\mathbb{X}}$, with $\tilde{\mathbb{X}}$ translation invariant in the sense of (1.3), such that $\partial_\omega \varphi \in \tilde{\mathbb{X}}$ for any $\omega \in S^{n-1}$ (where, as usual, ∂_ω denotes the directional derivative) and

$$\tilde{L}(\partial_\omega \varphi) = f'(\varphi)\, \partial_\omega \varphi \quad \text{in } \mathbb{R}^n.$$

H2. (Compactness): If $\varphi \in \mathbb{X}$ satisfies (1.1), $x^{(k)} \in \mathbb{R}^n$ and $\varphi^{(k)}(x) := \varphi(x + x^{(k)})$, we have that there exists a function $\varphi^{(\infty)} \in \mathbb{X}$ such that, up to a subsequence,

$$\begin{aligned} &\lim_{k\to+\infty} \varphi^{(k)}(x) = \varphi^{(\infty)}(x), \\ &\lim_{k\to+\infty} \nabla\varphi^{(k)}(x) = \nabla\varphi^{(\infty)}(x), \\ &\lim_{k\to+\infty} L\varphi^{(k)} = L\varphi^{(\infty)}, \end{aligned}$$

for any $x \in \mathbb{R}^n$.

H3. (Maximum Principle for the linearized operator): If $w \in \tilde{\mathbb{X}}$ satisfies $\tilde{L}w = c(x)w$ in $\mathbb{R}^n$, with

$$w(x) \geqslant 0 \text{ if } |x_n| \leqslant M \quad \text{and} \quad c(x) \geqslant \kappa \text{ if } |x_n| \geqslant M,$$

for some $\kappa > 0$ and $M > 0$, then

$$w(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}^n.$$

H4. (Strong Maximum Principle for the linearized equation): If $v \in \tilde{\mathbb{X}}$ satisfies $\tilde{L}v = f'(\varphi)v$, for some $\varphi \in \mathbb{X}$, and $v \geqslant 0$ in $\mathbb{R}^n$ with $v(0) = 0$, then v vanishes identically.

H5. (Maximum Principle for the difference operator): Given $\varphi \in \mathbb{X}$, let

$$\underline{L}_\varphi w(x) := L(\varphi + w)(x) - L\varphi(x).$$

Let U be an open set contained in $\{x_n \leqslant \mu_-\} \cup \{x_n \geqslant \mu_+\}$, for some $\mu_+ > \mu_- \in \mathbb{R}$. If $w \in \mathbb{X}$ satisfies $\underline{L}_\varphi w = c(x)w$ in $\mathbb{R}^n$, with

$$w(x) \geqslant 0 \text{ in } \mathbb{R}^n \setminus U \quad \text{and} \quad c(x) \geqslant \kappa \text{ if } x \in U,$$

for some $\kappa > 0$, then

$$w(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}^n.$$

H6. (Strong Maximum Principle for the difference equation): If $v \in \mathbb{X}$ satisfies $\underline{L}_\varphi v = f(\varphi + v) - f(\varphi)$ for some $\varphi \in \mathbb{X}$, and $v \geqslant 0$ in $\mathbb{R}^n$ with $v(0) = 0$, then v vanishes identically.

We remark that assumption (H1) is almost harmless (it boils down to the standard linearization procedure if the operator L is differentiable). Similarly, (H2) is a very weak condition and it does not even require, in principle, a regularity theory for (1.1) (for instance one can suitably choose the space $\mathbb{X}$ in order to control enough derivatives of u to obtain the required compactness).

Under the above assumptions, we may state our general result as follows:

Theorem 1.1. *Let $u \in \mathbb{X}$ be a solution of* (1.1)*, with $\|u\|_{C^{1,\beta}(\mathbb{R}^n)}$ finite, for some $\beta \in (0,1)$. Let L satisfy* (H1)–(H6) *and f satisfy* (1.4)*. Then u possesses one-dimensional symmetry, that is,* (1.5) *holds.*

Theorem 1.1 is motivated by a famous conjecture of Gibbons when L is the Laplace operator (see [14, 48]), which was motivated by the cosmological problem of detecting the shape of the interfaces which "separate" the different regions of the universe which possibly arose from the big bang. Such conjecture was proved independently and with different methods by [4, 8, 36]. See [37, 38] for the case of discontinuous nonlinearities. In [45] it is also shown that the uniform control of only one limit in (1.1) is enough to obtain that u is one-dimensional under the additional assumption that u is a minimal solution.

In this sense, Theorem 1.1 may be seen as a generalization of the results of [4, 8, 36] to a more general class of operators. Such generalization is performed in order to apply Theorem 1.1 to concrete cases of interest. As an application, we consider the case in which L is a fractional power of the Laplacian:

Theorem 1.2. *Let $L = -(-\Delta)^s$, with $s \in (0,1)$. Let f satisfy* (1.4)*. If $u \in W^{3,\infty}(\mathbb{R}^n)$ is a solution of* (1.1)*, then u possesses one-dimensional symmetry.*

We refer to [54, 63, 64] for the basics of fractional Laplacian theory. We would like to recall that the fractional Laplacian is a very important operator, since it naturally surfaces in many different areas, such as: the thin obstacle problem [13], optimization [33], finance [27], phase transitions [1, 2, 24, 67], stratified materials [66], anomalous diffusion [55], crystal dislocation [57, 68], soft thin films [53], some models of semipermeable membranes and flame propagation [21], conservation laws [9], the ultrarelativistic limit of quantum mechanics [40], quasigeostrophic flows [20, 56], multiple scattering [18, 31, 47], minimal surfaces [22, 28], materials science [3], probability [5, 7, 51, 52, 69], and water waves [17, 19, 25, 26, 29, 34, 35, 46, 50, 59, 60, 65, 71, 72].

When $s = \frac{1}{2}$, Theorem 1.2 was proven, by different methods, in [24], and an extension of that proof to any $s \in (0, 1)$ is given in [23]. Also, we recall that in dimension $n = 2$ the uniform limit assumption may be dropped in (1.1) and Theorem 1.2 still holds true for monotone solutions, as proved in [23, 67]. The case $n = 3$ has also been recently treated for some values of s, see [16], but many fundamental questions are still open.

Now, as another consequence of Theorem 1.1, we give a very general result on (possibly nonlinear) elliptic operators. For this, we denote by Sym^n the space of $(n \times n)$-symmetric matrices.

Theorem 1.3. *Let* $F = F(M, p) \in C^1(\mathrm{Sym}^n \times \mathbb{R}^n)$. *Assume that there exists* $\lambda \in C(\mathrm{Sym}^n \times \mathbb{R}^n, (0, +\infty))$ *such that*

$$F(M + N, p) - F(M, p) \geqslant \lambda(M, p)\|N\| \tag{1.6}$$

for any nonnegative definite $(n \times n)$*-symmetric matrix* N. *Let* $Lu = F(D^2u, \nabla u)$. *Let* f *satisfy* (1.4) *and* $\beta \in (0, 1)$. *If* $u \in C^{3,\beta}(\mathbb{R}^n)$ *is a solution of* (1.1), *then* u *possesses one-dimensional symmetry.*

We remark that condition (1.6) is an ellipticity assumption: compare, for instance, with Definition 2.1 on page 12 of [15]—we remark, in fact, that, by (1.6), F is uniformly elliptic when restricted to any compact subset of $\mathrm{Sym}^n \times \mathbb{R}^n$ and, since u is assumed to be smooth, then F is evaluated in a bounded set of parameters where it is uniformly elliptic.

The application of Theorem 1.3 is very wide, since it comprises, for instance:

- The Laplace operator, with the choice
$$F(M, p) = \mathrm{Tr}\, M,$$
- Elliptic operators with constant coefficients, take
$$F(M, p) = a_{ij}M_{ij} + b \cdot p,$$
- Quasilinear operators, such as
$$F(M, p) = (a + |p|^2)^{(m-2)/2}\,\mathrm{Tr}\, M + (m - 2)(a + |p|^2)^{(m-4)/2}M_{ij}p_ip_j,$$
with $a > 0$ and $m > 1$,
- The mean curvature operator
$$F(M, p) = (1 + |p|^2)^{-1/2}\,\mathrm{Tr}\, M - (1 + |p|^2)^{-3/2}M_{ij}p_ip_j,$$

- The elliptic fully nonlinear operators.

We point out that the assumption that u is smooth in Theorem 1.3 is not very restrictive, since it may be obtained in many cases via elliptic regularity theory once $u \in C^{1,\beta}(\mathbb{R}^n)$, see [15] and references therein (in fact, a stronger regularity theory holds if $n = 2$, see [58]).

We recall that a result similar to Theorem 1.3 for the uniformly elliptic fully nonlinear operators of the form $F(M,p) = F(M)$ has also been obtained in [62] by using the theory of viscosity solutions under the additional assumption on the existence of a suitable one-dimensional profile[1]. Thus, the viscosity setting of [62] and the classical one that we deal with here are related but different in spirit (though, under additional assumptions on the operator, viscosity solutions do become classical, see Chapters 8 and 9 in [15]).

It would be interesting to treat also the case of assumptions even more general than (H1)–(H6). For instance, it would be interesting to deal with operators in which elliptic singularities and degeneracies occur (see e.g. [45] and also [32, 41, 70] for some results in this direction and [42] for related problems and further references). Other cases of interest that one would like to deal with are the subelliptic operators and the operators arising in hyperbolic geometry (see, e.g., [6, 10–12, 43, 44] for results in these frameworks, and also Section 2.8 of [42] for further details). Moreover, after this work was completed, we have received the interesting paper [30] in which related symmetry results have been obtained for viscosity solutions of some fully nonlinear PDEs of a special form.

The proof of Theorem 1.1 that we give makes use of the technique of [36, 39], suitably modified in order to comprise our general case. For this, in Section 2, we give an intermediate result based on monotonicity cones. In Section 3, we complete the proof of Theorem 1.1, while Theorems 1.2 and 1.3 are proven in Sections 4 and 5 respectively, by showing that the operators under consideration fulfill assumptions (H1)–(H6).

2. A First Symmetry Result Via Monotonicity Cones

The proof of Theorem 1.1 makes use of a first provisional statement, which goes as follows:

Lemma 2.1. *Let $u \in \mathbb{X}$ be a bounded and uniformly Lipschitz solution of* (1.1), *with L satisfying* (H1)–(H4) *and f satisfying* (1.4). *Assume also that there exists $a \in (0,1)$ such that*

(2.1)
$$\partial_\nu u(x) > 0 \text{ for any } x \in \mathbb{R}^n \text{ and any } \nu = (\nu_1, \dots, \nu_n) \in \mathrm{S}^{n-1} \text{ with } \nu_n \geqslant a.$$

Then u possesses one-dimensional symmetry.

[1] It may be worth to remark that we do not need to assume that any one-dimensional profile exists. In fact, if no one-dimensional profile exists, our result may be seen as a non-existence result: namely, if we prove that the solution must be one-dimensional and no one-dimensional solution exists, then we have that there are no solutions at all.

Of course, Lemma 2.1 is just Theorem 1.1 with the additional hypothesis on the monotonicity cone in (2.1): in Section 3 we will show that such additional assumption is, in fact, not needed and so we will be able to derive Theorem 1.1 from Lemma 2.1.

Proof. In order to prove Lemma 2.1, we show that

$$\partial_\nu u(x) > 0 \ \text{ for any } x \in \mathbb{R}^n \text{ and any } \nu = (\nu_1, \dots, \nu_n) \in \mathrm{S}^{n-1} \text{ with } \nu_n > 0. \tag{2.2}$$

To prove (2.2), we take

$$\underline{a} := \inf\{a > 0 \text{ for which (2.1) holds}\}. \tag{2.3}$$

If $\underline{a} = 0$, then (2.2) is proved, so we assume, by contradiction, that

$$\underline{a} > 0. \tag{2.4}$$

Given $S > 0$, we define

$$i_S := \inf_{\substack{x' \in \mathbb{R}^{n-1}, |x_n| \leqslant S \\ \nu_n \geqslant \underline{a}}} \partial_\nu u(x', x_n).$$

By construction,

$$i_S \geqslant 0; \tag{2.5}$$

we claim that, in fact

$$i_S > 0. \tag{2.6}$$

To prove (2.6), we argue by contradiction and we suppose that there exist a sequence of $\nu^{(k)} \in S^{n-1}$ and $x^{(k)} \in \mathbb{R}^n$ with

$$|x_n^{(k)}| \leqslant S, \tag{2.7}$$

$\nu^{(k)} \geqslant \underline{a}$, and

$$\lim_{k \to +\infty} \partial_{\nu^{(k)}} u(x^{(k)}) = 0. \tag{2.8}$$

From (2.5),

$$\partial_{\nu^{(k)}} u(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}^n. \tag{2.9}$$

We define

$$u^{(k)}(x) := u(x + x^{(k)}). \tag{2.10}$$

Then, (2.8) becomes

$$\lim_{k\to+\infty} \nabla u^{(k)}(0)\cdot \nu^{(k)} = 0. \tag{2.11}$$

Analogously, (2.9) writes

$$\nabla u^{(k)}(x)\cdot \nu^{(k)} \geqslant 0 \quad \text{for any } x\in\mathbb{R}^n. \tag{2.12}$$

Notice also that

$$|Lu^{(k)}| = |f(u^{(k)})| \leqslant \sup_{r\in[-\|u\|_{L^\infty(\mathbb{R}^n)},\, \|u\|_{L^\infty(\mathbb{R}^n)}]} |f(r)|,$$

where (1.2) has been used. Thus, from (2.11), (2.12) and (H2), there exist $u^{(\infty)} \in \mathbb{X}$ and $\nu^{\infty} \in S^{n-1}$, with

$$\nu^{(\infty)} \geqslant \underline{a}, \tag{2.13}$$

such that

$$\lim_{k\to+\infty} u^{(k)}(x) = u^{(\infty)}(x) \quad \text{for any } x\in\mathbb{R}^n, \tag{2.14}$$

and

$$Lu^{(\infty)} = f(u^{(\infty)}),\ \nabla u^{(\infty)}\cdot\nu^{(\infty)} \geqslant 0 \text{ in } \mathbb{R}^n, \quad \text{with } \nabla u^{(\infty)}(0)\cdot \nu^{(\infty)} = 0.$$

Therefore, by (H1), the function $v := \partial_{\nu^{(\infty)}} u^{(\infty)} \in \tilde{\mathbb{X}}$ satisfies

$$\tilde{L}v = f'(u)v,\ v(x) \geqslant 0 = v(0) \quad \text{for any } x\in\mathbb{R}^n.$$

As a consequence, from (H4), v vanishes identically. Accordingly,

$$u^{(\infty)}(\nu^{(\infty)}t) - u^{(\infty)}(-\nu^{(\infty)}t) = \int_{-t}^{t} v(\nu^{(\infty)}s)\,ds = 0 \quad \text{for any } t\geqslant 0. \tag{2.15}$$

Recalling the uniform limit assumption in (1.1), we now take $M>0$ in such a way that

$$u(x) \geqslant \frac{1}{2} \quad \text{if } x_n \geqslant M, \qquad u(x) \leqslant -\frac{1}{2} \quad \text{if } x_n \leqslant -M.$$

Then, recalling (2.7) and (2.10),

$$u^{(k)}(x) \geqslant \frac{1}{2} \quad \text{if } x_n \geqslant M+S, \qquad u^{(k)}(x) \leqslant -\frac{1}{2} \quad \text{if } x_n \leqslant -M-S.$$

Hence, from (2.14),

$$u^{(\infty)}(x) \geqslant \frac{1}{2} \quad \text{if } x_n \geqslant M + S, \qquad u^{(\infty)}(x) \leqslant -\frac{1}{2} \quad \text{if } x_n \leqslant -M - S. \tag{2.16}$$

We recall that, from (2.4) and (2.13), $\nu_n^{(\infty)} \geqslant \underline{a} > 0$, so (2.16) implies that

$$u^{(\infty)}(\nu^{(\infty)}t) \geqslant \frac{1}{2} \text{ and } u^{(\infty)}(-\nu^{(\infty)}t) \leqslant -\frac{1}{2} \quad \text{if } t \geqslant \frac{M+S}{\underline{a}}.$$

This and (2.15) give that

$$0 = u^{(\infty)}(\nu^{(\infty)}t) - u^{(\infty)}(-\nu^{(\infty)}t) \geqslant 1.$$

This contradiction proves (2.6).

Now, we use (1.4), to see that $f'(r) \geqslant \kappa$, for a suitable $\kappa > 0$, if $|r-1| \leqslant \eta^\star$, for a suitable $\eta^\star \in (0, \frac{1}{4})$. Also, the uniform limit assumption in (1.1) enables us to take $M_\star > 0$ such that $u(x) \geqslant 1 - \eta^\star$ if $x_n \geqslant M_\star$ and $u(x) \leqslant -1 + \eta^\star$ if $x_n \leqslant -M_\star$. We also define $c(x) := f'(u(x))$. Hence, $c(x) \geqslant \kappa$ when $|x_N| \geqslant M_\star$. Let also

$$\varepsilon := \frac{i_{M_\star}}{2(1 + \|\nabla u\|_{L^\infty(\mathbb{R}^n)})}.$$

Notice that $\varepsilon > 0$, thanks to (2.6).

If $|x_n| \leqslant M_\star$ and $\nu \in S^{n-1}$ with $\nu_n \in [\underline{a} - \varepsilon, \underline{a}]$, then

$$\begin{aligned} \partial_\nu u(x) &= \nabla u(x) \cdot \nu \geqslant \nabla u(x) \cdot (\nu', \underline{a}) - |\nabla u(x) \cdot (0, \underline{a} - \nu_n)| \\ &\geqslant i_{M_\star} - \|\nabla u\|_{L^\infty(\mathbb{R}^n)} \varepsilon \geqslant \frac{i_{M_\star}}{2} > 0. \end{aligned}$$

Therefore, by (H3), $\partial_\nu u(x) \geqslant 0$ for any $x \in \mathbb{R}^n$ and any $\nu \in S^{n-1}$ with $\nu_n \in [\underline{a} - \varepsilon, \underline{a}]$. In fact, by (H4), we see that $\partial_\nu u(x) > 0$ for any $x \in \mathbb{R}^n$ and any $\nu \in S^{n-1}$ with $\nu_n \in [\underline{a} - \varepsilon, \underline{a}]$. This is in contradiction with (2.3), and so it proves (2.2).

Then, from (2.2), by taking $\mu = -\nu$, we obtain that

$$\partial_\mu u(x) < 0 \quad \forall\, x \in \mathbb{R}^n \text{ and } \forall\, \mu = (\mu_1, \ldots, \mu_n) \in S^{n-1} \text{ with } \mu_n < 0. \tag{2.17}$$

By taking limits of ν_n and μ_n to 0 in (2.2) and (2.17), we deduce that

$$\partial_\omega u(x) = 0 \quad \forall\, x \in \mathbb{R}^n \text{ and } \forall\, \omega = (\omega_1, \ldots, \omega_n) \in S^{n-1} \text{ with } \omega_n = 0.$$

Hence, $\partial_{x_1} u(x) = \cdots = \partial_{x_{n-1}} u(x) = 0$ for any $x \in \mathbb{R}^n$, which ends the proof of Lemma 2.1. □

3. Proof of Theorem 1.1

Some of the arguments needed to proof Theorem 1.1 will be appropriate modifications of the ones used in the proof of Lemma 2.1, by taking into account the difference operator $\underline{L}_u$ instead of the linearized operator $\tilde{L}$. In order to prove Theorem 1.1, first of all, we show that

$$\partial_n u(x) > 0 \quad \text{for any } x \in \mathbb{R}^n. \tag{3.1}$$

To prove (3.1), we take $h \geqslant 0$, we let

$$\mathcal{T}_h u(x) := u(x + he_n) - u(x)$$

and we observe that

$$f(u(x + he_n)) - f(u(x)) = c_h(x)\mathcal{T}_h u(x), \tag{3.2}$$

where

$$c_h(x) := \int_0^1 f'(tu(x) + (1-t)u(x + he_n))\,\mathrm{d}t. \tag{3.3}$$

Now, recalling (1.4), we take $\delta \in (0, \frac{1}{2})$ such that

$$f' \geqslant \kappa \quad \text{in } (-\infty, -1+\delta] \cup [1-\delta, +\infty), \text{ for some } \kappa > 0. \tag{3.4}$$

Then, by the uniform limit assumption in (1.1), we take $M > 0$ such that

$$u(x) \geqslant 1 - \delta \quad \text{if } x_n \geqslant M, \qquad u(x) \leqslant -1 + \delta \quad \text{if } x_n \leqslant -M. \tag{3.5}$$

Now, we observe the following useful property:

$$\text{if } x \in \{\mathcal{T}_h u < 0\} \cap \{|x_n| \geqslant M\}, \quad \text{then } c_h(x) \geqslant \kappa. \tag{3.6}$$

Indeed, on the one hand, if $x \in \{\mathcal{T}_h u < 0\} \cap \{x_n \leqslant -M\}$,

$$u(x + he_n) < u(x) \leqslant -1 + \delta$$

and therefore

$$tu(x) + (1-t)u(x + he_n) \leqslant -1 + \delta \tag{3.7}$$

for any $x \in \{\mathcal{T}_h u < 0\} \cap \{x_n \leqslant -M\}$ and $t \in [0, 1]$. On the other hand, if $x \in \{\mathcal{T}_h u < 0\} \cap \{x_n \geqslant M\}$, then $u(x) > u(x + he_n) \geqslant 1 - \delta$, and therefore

$$tu(x) + (1-t)u(x + he_n) \geqslant 1 - \delta \tag{3.8}$$

for any $x \in \{\mathcal{T}_h u < 0\} \cap \{x_n \geqslant M\}$ and $t \in [0,1]$. From (3.3), (3.4), (3.7), and (3.8), we obtain that (3.6) holds true.

We claim that

$$\text{if } h \geqslant 2M, \quad \text{then } \mathcal{T}_h u(x) > 0 \text{ for any } x \in \mathbb{R}^n. \tag{3.9}$$

To prove (3.9), fix $h \geqslant 2M$ and let $U := \{\mathcal{T}_h u < 0\}$. Then,

$$\text{if } x_n = -M,\ \mathcal{T}_h u(x) \geqslant \inf_{x_n \geqslant M} u(x) - \sup_{x_n \leqslant -M} u(x) \geqslant (1-\delta) - (-1+\delta) > 0,$$

and so

$$U = U_+ \cup U_-, \tag{3.10}$$

where U_+ (respectively U_-) is an open set contained in the half-space $\{x_n \geqslant -M\}$ (respectively $\{x_n \leqslant -M\}$). Then, (3.2), (3.10) and (3.6), together with (H5), imply that if $h \geqslant 2M$, then $\mathcal{T}_h u(x) \geqslant 0$ for any $x \in \mathbb{R}^n$. Hence, (3.9) follows from (H6).

Now, we define

$$h_0 := \inf\Big\{h > 0 \text{ such that } \mathcal{T}_h u(x) > 0 \text{ for any } x \in \mathbb{R}^n \text{ with } |x_n| \leqslant M\Big\}.$$

Note that this definition is well-posed, due to (3.9).

We prove that

$$h_0 = 0. \tag{3.11}$$

The proof of (3.11) is by contradiction. Suppose $h_0 > 0$. We have that

$$u(x + (h_0+\varepsilon)e_n) - u(x) \geqslant 0 \quad \text{for any } x \in \{|x_n| \leqslant M\} \text{ and any } \varepsilon > 0$$

and

$$u(x^{(k)} + (h_0 - \varepsilon^{(k)})e_n) - u(x^{(k)}) \leqslant 0 \quad \text{for some } x^{(k)} \in \{|x_n| \leqslant M\},$$

where $\varepsilon^{(k)} \geqslant 0$ is an infinitesimal sequence. As a consequence,

$$\mathcal{T}_{h_0} u(x) = \lim_{\varepsilon \to 0^+} u(x + (h_0+\varepsilon)e_n) - u(x) \geqslant 0 \quad \text{for any } x \in \{|x_n| \leqslant M\}.$$

Therefore, recalling (3.6) and (H5),

$$\mathcal{T}_{h_0} u(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}^n. \tag{3.12}$$

Now we define $u^{(k)}(x) := u(x + x^{(k)})$, and we deduce from (H2) that, up to a subsequence, $u^{(k)}$ approaches some $u^{(\infty)}$, with

$$\underline{L}_u(\mathcal{T}_{h_0} u^{(\infty)}) = f(\mathcal{T}_{h_0} u^{(\infty)} + u) - f(u).$$

By (3.12), we see that $\mathcal{T}_{h_0} u^{(\infty)}(x) \geqslant 0$ for any $x \in \mathbb{R}^n$. Also,

$$\begin{aligned}\mathcal{T}_{h_0} u^{(\infty)}(0) &= \lim_{k\to+\infty} u(x^{(k)} + h_0 e_n) - u(x^{(k)}) \\ &\leqslant \lim_{k\to+\infty} u(x^{(k)} + (h_0 - \varepsilon^{(k)})e_n) - u(x^{(k)}) + \varepsilon^{(k)} \|u\|_{C^{1,\beta}(\mathbb{R}^n)} \leqslant 0,\end{aligned}$$

hence $\mathcal{T}_{h_0} u^{(\infty)}(0) = 0$. Consequently, by (H6), we get that $\mathcal{T}_{h_0} u^{(\infty)}$ vanishes identically. Therefore, $u^{(\infty)}(x + h_0 e_n) = u^{(\infty)}(x)$ for any $x \in \mathbb{R}^n$ and so, by iterating,

$$u^{(\infty)}(x + j h_0 e_n) = u^{(\infty)}(x) \quad \text{for any } x \in \mathbb{R}^n \text{ and any } j \in \mathbb{Z}. \tag{3.13}$$

Now, if $j \in \mathbb{N} \cap [2M/h_0, +\infty)$, we have that $jh_0 + x_n^{(k)} \geqslant M$ and $-jh_0 + x_n^{(k)} \leqslant -M$, so $u(jh_0 e_n + x^{(k)}) \geqslant 1 - \delta$ and $u(-jh_0 e_n + x^{(k)}) \leqslant -1 + \delta$. Then, for such a j,

$$\begin{aligned}2(1-\delta) &\geqslant \lim_{k\to+\infty} u(jh_0 e_n + x^{(k)}) - u(-jh_0 e_n + x^{(k)}) \\ &= u^{(\infty)}(jh_0 e_n) - u^{(\infty)}(-jh_0 e_n).\end{aligned}$$

Since this contradicts (3.13), we have proved (3.11). That is, $\mathcal{T}_h u(x) \geqslant 0$ for any x with $\{|x_n| \leqslant M\}$ and any $h \geqslant 0$. Consequently, by (3.6) and (H5), we deduce that $\mathcal{T}_h u(x) \geqslant 0$ for any $x \in \mathbb{R}^n$. Accordingly, $\partial_n u(x) \geqslant 0$ for any $x \in \mathbb{R}^n$, and then (3.1) follows from (H4).

Now we show that for any $S > 0$ there exists $a(S) \in (0,1)$ such that

$$\begin{aligned}\partial_\nu u(x) > 0 \quad &\text{for any } x \in \mathbb{R}^n \cap \{|x_n| \leqslant S\} \\ &\text{and any } \nu = (\nu_1, \dots, \nu_n) \in \mathrm{S}^{n-1} \text{ with } \nu_n \geqslant a(S).\end{aligned} \tag{3.14}$$

The proof of (3.14) is by contradiction. Suppose that, for a fixed S, there exist sequences $x^{(k)} \in \{|x_n| \leqslant S\}$ and $\nu^{(k)} \in \mathrm{S}^{n-1}$ such that $\nu_n^{(k)} \geqslant 1 - (1/k)$ and

$$\partial_{\nu^{(k)}} u(x^{(k)}) \leqslant 0. \tag{3.15}$$

Let $u^{(k)}(x) := u(x + x^{(k)})$. Notice that $\nu^{(k)}$ approaches e_n for large k, therefore, by (H2), we obtain that, up to a subsequence, $u^{(k)}$ approaches some $u^{(\infty)}$ together with its derivative, with $Lu^{(\infty)} = f(u^{(\infty)})$ and $\tilde{L}(\partial_n u^{(\infty)}) = f'(u^{(\infty)})\,\partial_n u^{(\infty)}$. We remark that, by (3.1),

$$\partial_n u^{(\infty)} \geqslant 0,$$

while, by (3.15),

$$\partial_n u^{(\infty)}(0) \leqslant 0.$$

Accordingly, (H4) says that $\partial_n u^{(\infty)}$ vanishes identically. Thus, if $t - S$ is large enough (hence $te_n - |x_n^{(k)}|$ is large enough), the uniform limit in (1.1) gives that

$$\begin{aligned}\frac{9}{10} &\leqslant \lim_{k\to+\infty} u(te_n + x^{(k)}) = u^{(\infty)}(te_n)\\ &= u^{(\infty)}(-te_n) = \lim_{k\to+\infty} u(-te_n + x^{(k)})\\ &\leqslant -\frac{9}{10}.\end{aligned}$$

This contradiction proves (3.14).

Now, recalling the definition of M given in (3.4) and (3.5), in the notation of (3.14), we define

$$a := a(M).$$

Then, as a consequence of (3.14), (H3), and (3.4), we have that $\partial_\nu u(x) \geqslant 0$ for any $x \in \mathbb{R}^n$, if $\nu_n \geqslant a$. Then, by (3.14) and (H4), we conclude that $\partial_\nu u(x) > 0$ for any $x \in \mathbb{R}^n$, if $\nu_n \geqslant a$. That is, condition (2.1) holds true. Therefore, the proof of Theorem 1.1 is completed thanks to Lemma 2.1. □

4. PROOF OF THEOREM 1.2

We will deduce Theorem 1.2 from Theorem 1.1, in which $L = \tilde{L} = L_\varphi := -(-\Delta)^s$, $\mathbb{X} := W^{3,\infty}(\mathbb{R}^n)$, and $\tilde{\mathbb{X}} := W^{2,\infty}(\mathbb{R}^n)$. For this, we need to check hypotheses (H1)–(H6). First, we claim that, if $u \in W^{3,\infty}(\mathbb{R}^n)$, then

$$\partial_\omega(-(-\Delta)^s u) = -(-\Delta)^s(\partial_\omega u). \tag{4.1}$$

Indeed, (4.1) is obvious if u belongs to the Schwartz class of rapidly decreasing functions, since, in this case, one can represent $(-\Delta)^s$ via a Fourier transform (see, for instance, [54, 63, 64, 69]) and check (4.1).

If, on the other hand, $u \in W^{3,\infty}(\mathbb{R}^n)$, we have that for any $h > 0$,

$$\begin{aligned}&\frac{u(h\omega+y)+u(h\omega-y)-2u(h\omega)}{|y|^{n+2s}} - \frac{u(y)+u(-y)-2u(0)}{|y|^{n+2s}}\\ &\quad= \frac{u(h\omega+y)-u(y)+u(h\omega-y)-u(-y)-2u(h\omega)+2u(0)}{|y|^{n+2s}}\\ &\quad\leqslant 5\|u\|_{W^{3,\infty}(\mathbb{R}^n)} h\Big[|y|^{-(n+2s)}\chi_{\mathbb{R}^n\setminus B_1}(y) + |y|^{2-(n+2s)}\chi_{B_1}(y)\Big] \in L^1(\mathbb{R}^n).\end{aligned}$$

Thus, the Dominated Convergence Theorem gives that

$$\begin{aligned}
&\partial_\omega\left(\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy\right)_{x=0}\\
&=\lim_{h\to0^+}\int_{\mathbb{R}^n}\frac{u(h\omega+y)+u(h\omega-y)-2u(h\omega)}{h|y|^{n+2s}}\,dy\\
&\quad-\int_{\mathbb{R}^n}\frac{u(y)+u(-y)-2u(0)}{h|y|^{n+2s}}\,dy\\
&=\lim_{h\to0^+}\int_{\mathbb{R}^n}\frac{u(h\omega+y)-u(y)+u(h\omega-y)-u(-y)-2u(h\omega)+2u(0)}{h|y|^{n+2s}}\,dy\\
&=\int_{\mathbb{R}^n}\frac{\partial_\omega u(y)+\partial_\omega u(-y)-2\partial_\omega u(0)}{|y|^{n+2s}}\,dy.
\end{aligned}$$

Thus, via the integral representation of the fractional Laplacian (see [54, 63, 64, 69]), the above identity reads

$$\partial_\omega\Big(-(-\Delta)^s u(x)\Big)_{x=0}=-\Big((-\Delta)^s(\partial_\omega u)\Big)(0),$$

which proves (4.1) at $x=0$ (and analogously at any point).

Then, hypothesis (H1) follows from (4.1). Hypothesis (H2) follows from the fact that $\mathbb{X}=W^{3,\infty}$, using the Theorem of Ascoli and the integral representation of the fractional Laplacian. We now prove (H3), by arguing by contradiction. We suppose that

$$i:=\inf_{\mathbb{R}^n} w<0,$$

and we take $x^{(k)}\in\mathbb{R}^n$ such that

$$\lim_{k\to+\infty} w(x^{(k)})=i<0.$$

In particular, we may suppose that $w(x^{(k)})\leqslant i/2<0$, and therefore $x^{(k)}\in\{|x_n|\geqslant M\}$, so $c(x^{(k)})\geqslant\kappa>0$. Thus, if we set $w^{(k)}(x):=w(x+x^{(k)})$, we see that

$$-(-\Delta)^s w^{(k)}(x)=-(-\Delta)^s w(x+x^{(k)})=c(x+x^{(k)})w(x+x^{(k)}).$$

In particular,

$$\begin{aligned}
(4.2)\qquad C(n,s)\int_{\mathbb{R}^n}\frac{w^{(k)}(y)-w^{(k)}(0)}{|y|^{n+2s}} &= -(-\Delta)^s w^{(k)}(0)\\
&= c(x^{(k)})w(x^{(k)})\leqslant\frac{\kappa i}{2},
\end{aligned}$$

for a suitable $C(n,s) > 0$. In the first equality in (4.2), we have used one of the classical representations of the fractional Laplacian, see, e.g., [54, 63, 64, 69] for details.

Since $w \in \tilde{\mathbb{X}} := W^{2,\infty}(\mathbb{R}^n)$, we have that $w^{(k)}$ converges locally uniformly to some $w^{(\infty)}$, up to a subsequence, due to Theorem of Ascoli, and so, by taking the limit in (4.2), we have

$$C(n,s)\int_{\mathbb{R}^n} \frac{w^{(\infty)}(y) - w^{(\infty)}(0)}{|y|^{n+2s}} \leqslant \frac{\kappa i}{2}. \tag{4.3}$$

On the other hand,

$$\begin{aligned} w^{(\infty)}(0) &= \lim_{k\to+\infty} w^{(k)}(0) = \lim_{k\to+\infty} w(x^{(k)}) \\ &= \inf_{\mathbb{R}^n} w \leqslant w(y + x^{(k)}) = w^{(k)}(y) \end{aligned} \tag{4.4}$$

for any $y \in \mathbb{R}^n$, and so

$$w^{(\infty)}(0) \leqslant w^{(\infty)}(y)$$

for any $y \in \mathbb{R}^n$. As a consequence, (4.3) gives that

$$0 \leqslant C(n,s)\int_{\mathbb{R}^n} \frac{w^{(\infty)}(y) - w^{(\infty)}(0)}{|y|^{n+2s}} \leqslant \frac{\kappa i}{2} < 0.$$

This contradiction proves (H3).

Take now v as requested in (H4): then, the integral representation of the fractional Laplacian gives that, for a suitable $C(n,s) > 0$,

$$\begin{aligned} 0 &= f'(u(0))v(0) = -(-\Delta)^s v(0) \\ &= C(n,s)\int_{\mathbb{R}^n} \frac{v(y) - v(0)}{|y|^{n+2s}}\,\mathrm{d}y = C(n,s)\int_{\mathbb{R}^n} \frac{v(y)}{|y|^{n+2s}}\,\mathrm{d}y, \end{aligned} \tag{4.5}$$

with the integral taken in the Cauchy principal value sense. Since $v \geqslant 0$, (4.5) implies that v is identically zero, thus checking (H4).

The proof of (H5) (respectively (H6)) is analogous to the one of (H3) (respectively (H4)): just take U instead of $\{|x_n| \geqslant M\}$ (respectively $f(u+v) - f(u)$ instead of $f'(u)v$). The proof of Theorem 1.2 is thus complete. ◻

5. PROOF OF THEOREM 1.3

We take $\mathbb{X} := C^{3,\beta}(\mathbb{R}^n)$ and $\tilde{\mathbb{X}} := C^{2,\beta}(\mathbb{R}^n)$. Notice that, for any $v \in \tilde{\mathbb{X}}$,

$$f(u+v) - f(u) = \underline{c}v, \quad \text{with } \underline{c}(x) := \int_0^1 f'(u(x) + tv(x))\,\mathrm{d}t. \tag{5.1}$$

Also,

$$
\begin{aligned}
\tilde{L}v &= \sum_{i,j=1}^{n} \tilde{a}_{ij}\,\partial^2_{ij}v + \tilde{b}\cdot\nabla v,\\
\underline{L}_u v &= F(D^2u + D^2v, \nabla u + \nabla v) - F(D^2u, \nabla u + \nabla v)\\
&\quad + F(D^2u, \nabla u + \nabla v) - F(D^2u, \nabla u)\\
&= \sum_{i,j=1}^{n} \underline{a}_{ij}\,\partial^2_{ij}v + \underline{b}\cdot\nabla v,
\end{aligned}
\tag{5.2}
$$

with

$$
\begin{aligned}
\tilde{a}_{ij}(x) &:= \frac{\partial F}{\partial M_{ij}}(D^2u(x), \nabla u(x)),\\
\tilde{b}(x) &:= \frac{\partial F}{\partial p}(D^2u(x), \nabla u(x)),\\
\underline{a}_{ij}(x) &:= \int_0^1 \frac{\partial F}{\partial M_{ij}}(D^2u(x) + tD^2v(x), \nabla u(x))\,\mathrm{d}t,
\end{aligned}
$$

and

$$
\underline{b}(x) := \int_0^1 \frac{\partial F}{\partial p}(D^2u(x), \nabla u(x) + t\nabla v(x))\,\mathrm{d}t.
$$

In this way, (H1) is obviously satisfied and (H2) is a consequence of the Theorem of Ascoli. We observe that, by construction

$$
\tilde{a}_{ij},\ \underline{a}_{ij},\ \tilde{b},\ \underline{b},\ \underline{c} \in C^{0,\beta}(\mathbb{R}^n) \subset L^\infty_{\mathrm{loc}}(\mathbb{R}^n). \tag{5.3}
$$

Moreover, from (1.6)

$$
\sum_{i,j=1}^{n} \frac{\partial F}{\partial M_{ij}}(M,p)N_{ij} = \lim_{s\to 0^+} \frac{F(M+sN,p) - F(M,p)}{s} \geqslant \lambda(M,p)\|N\|,
$$

for any nonnegative definite matrix N. In particular, given any $\xi \in \mathbb{R}^n$, taking $N_{ij} := \xi_i\xi_j$,

$$
\begin{aligned}
\sum_{i,j=1}^{n} \frac{\partial F}{\partial M_{ij}}(M,p)\xi_i\xi_j &\geqslant \lambda(M,p)\sqrt{\sum_{i,j=1}^{n}(\xi_i\xi_j)^2} \geqslant \lambda(M,p)\sqrt{\sum_{i=1}^{n}(\xi_i\xi_i)^2}\\
&\geqslant \frac{\lambda(M,p)}{n^2}\|\xi\|^2.
\end{aligned}
$$

Therefore, given any $R > 0$, there exists $\lambda^\star_{R,u,v} > 0$ such that

$$\inf_{\substack{x\in B_R\\ \tau,\sigma\in[0,1]}} \sum_{i,j} \frac{\partial F}{\partial M_{ij}}(D^2u(x)+\tau D^2v(x), \nabla u(x)+\sigma\nabla v(x))\xi_i\xi_j \geqslant \lambda^\star_{R,u,v}\|\xi\|^2.$$

As a consequence, for any $\xi \in \mathbb{R}^n$,

$$\inf_{x\in B_R}\sum_{i,j=1}^n \tilde{a}_{ij}\xi_i\xi_j \geqslant \lambda^\star_{R,u,v}\|\xi\|^2 \quad\text{and}\quad \inf_{x\in B_R}\sum_{i,j=1}^n \underline{a}_{ij}\xi_i\xi_j \geqslant \lambda^\star_{R,u,v}\|\xi\|^2. \tag{5.4}$$

In particular,

$$\sum_{i,j=1}^n \tilde{a}_{ij}\xi_i\xi_j \geqslant 0 \quad\text{and}\quad \sum_{i,j=1}^n \underline{a}_{ij}\xi_i\xi_j \geqslant 0. \tag{5.5}$$

Then, (H4) and (H6) are a consequence of (5.1), (5.2), (5.3), (5.4), and Hopf Strong Maximum Principle (see, for instance, [49] or Theorem 2.1.2 of [61]). Therefore, in order to complete the proof of Theorem 1.3, it remains to prove (H3) and (H5) (and then invoke Theorem 1.1).

We prove (H5) (the proof of (H3) is completely analogous). Suppose, by contradiction, that the conditions on w in (H5) hold, but

$$i := \inf_{\mathbb{R}^n} w < 0.$$

We take $x^{(k)} \in \mathbb{R}^n$ such that

$$\lim_{k\to+\infty} w(x^{(k)}) = i < 0.$$

In particular, we may suppose that $w(x^{(k)}) \leqslant i/2 < 0$, and therefore $x^{(k)} \in U$, and so $c(x^{(k)}) \geqslant \kappa > 0$.

We set $w^{(k)}(x) := w(x + x^{(k)})$ and we use the definition of $\tilde{\mathbb{X}}$ and the Theorem of Ascoli to obtain, up to a subsequence, that $w^{(k)}$ approaches some $w^{(\infty)}$ locally uniformly together with two derivatives. This also gives the convergence of the coefficients $\underline{a}_{ij} = \underline{a}^{(k)}_{ij}$ and $\underline{b} = \underline{b}^{(k)}$ obtained for $w^{(k)}$ via (5.2) to suitable $\underline{a}^{(\infty)}_{ij}$ and $\underline{b}^{(\infty)}$. Notice that, from (5.5), we have

$$\sum_{i,j=1}^n \underline{a}^{(\infty)}_{ij}\xi_i\xi_j \geqslant 0 \quad \text{for any } \xi\in\mathbb{R}^n. \tag{5.6}$$

Also, 0 is a minimum for $w^{(\infty)}$ (see the computation in (4.4)), and so $\nabla w^{(\infty)}(0) = 0$ and $D^2 w^{(\infty)}(0)$ is nonnegative definite.

As a consequence, recalling (5.6),

$$\begin{aligned}\frac{\kappa \iota}{2} &\geqslant \lim_{k\to+\infty} c(x^{(k)})w(x^{(k)}) = \lim_{k\to+\infty} \underline{L}_u w(x^{(k)})\\ &= \lim_{k\to+\infty} \sum_{i,j=1}^{n} \underline{a}_{ij}^{(k)}\, \partial^2_{ij} w(x^{(k)}) + \underline{b}^{(k)} \cdot \nabla w(x^{(k)})\\ &= \lim_{k\to+\infty} \sum_{i,j=1}^{n} \underline{a}_{ij}^{(k)}\, \partial^2_{ij} w^{(k)}(0) + \underline{b}^{(k)} \cdot \nabla w^{(k)}(0)\\ &= \sum_{i,j=1}^{n} \underline{a}_{ij}^{(\infty)}\, \partial^2_{ij} w^{(\infty)}(0) + \underline{b}^{(\infty)} \cdot \nabla w^{(\infty)}(0)\\ &\geqslant 0.\end{aligned}$$

Since $\iota < 0$, this is a contradiction and it proves (H5). The proof of Theorem 1.3 is thus completed. □

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ALBERTO FARINA:
LAMFA – CNRS UMR 6140
Université de Picardie Jules Verne
Faculté des Sciences
33, rue Saint-Leu, Amiens, France
E-MAIL: alberto.farina@u-picardie.fr

ENRICO VALDINOCI:
Università di Roma Tor Vergata
Dipartimento di Matematica
Rome, Italy
E-MAIL: enrico.valdinoci@uniroma2.it

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