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p-adic regulators and p-adic families of modular forms

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Abstract

The theme of this Thesis is Iwasawa theory of Hida p-adic analytic families of modular forms. Our main goal is to describe special values of Hida's p-adic L-functions in the context of a p-adic Birch and Swinnerton-Dyer conjecture for the weight variable.

Let $E/\mathbb{Q}$ be an elliptic curve with ordinary reduction at a prime p, corresponding to a weight two newform f. The exceptional zero formulas of Bertolini-Darmon [**BD07**] and Greenberg-Stevens [**GS93**] establish deep relations between the arithmetic of E and the behaviour at $(k, s) = (2, 1)$ of the Mazur-Kitagawa two-variable p-adic L-function $L_p(f_\infty, k, s)$ attached to the Hida family f_∞ containing f. The goal of Part 1 is to give an interpretation of these formulas in the framework of a p-adic Birch and Swinnerton-Dyer conjecture for $L_p(f_\infty, k, s)$. The main problem consists in the construction of *p-adic regulators* encoding the arithmetic of the special L-values of the classical specializations of f_∞. We address this problem by appealing to Nekovář's theory of Selmer complexes [**Nek06**]. More precisely, the key ingredient in the definition of the p-adic regulator is the *p-adic weight pairing*, defined on the extended Mordell-Weil group of $E/\mathbb{Q}$. This pairing comes from Nekovář duality for a suitable big Selmer complex attached to Hida's universal ordinary deformation $\Lambda(f_\infty)$ of the p-adic Tate module of E.

In Part 2 we consider the algebraic side of the matter, i.e. we study the special values of *algebraic Hida's p-adic L-functions*. These are defined as characteristic ideals of 'big' Selmer groups (or complexes) attached to $\Lambda(f_\infty)$ and $\mathbb{Z}_p$-power extensions of number fields. Making use of Mazur-Rubin theory of organizing modules and of a general theory of *abstract height pairings* that we develop in Appendix C, we deduce various several-variable algebraic p-adic Birch and Swinnerton-Dyer formulas, generalizing well known results of Schneider, Perrin-Riou, Jones, Nekovář *et. al.* Via the Main Conjectures of Iwasawa theory for GL_2, recently proved thanks to the work of Kato-Rohrlich, Bertolini-Darmon, Vatsal, Ochiai, Skinner-Urban *et. al.*, these formulas provide strong evidence in support of the conjectures proposed in Part 1.

In Part 3 we study the Mazur-Tate-Teitelbaum conjecture for elliptic curves $E/\mathbb{Q}$ with *split* multiplicative reduction at p (i.e. in the presence of an exceptional zero for the cyclotomic p-adic L-function in the sense of [**MTT86**]). Making use of Nekovář's theory we prove exceptional zero formulas for the p-adic L-functions arising from norm-compatible systems of cohomology classes via Perrin-Riou-Coleman 'big' logarithm. Applying this formulas to Kato's Euler system for the p-adic Tate module of E, we are able to reprove the main result of [**GS93**], and to relate the second derivative of the cyclotomic p-adic L-function of E to the cyclotomic p-adic regulator. Besides providing evidence in support of the Mazur-Tate-Teitelbaum conjecture, our result suggests an analogue in the multiplicative setting of a conjecture of Perrin-Riou, relating Beilinson-Kato elements to Heegner points.

More detailed descriptions of the results are given in the introduction to each part of the Thesis.

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Contents

Part 1

p-adic regulators and Hida p-adic L-functions

Introduction

Hida theory of p-adic analytic families of modular forms has proved to be a powerful tool in the study of the arithmetic of (p-ordinary) elliptic curves. The proof by Greenberg and Stevens of the Mazur-Tate-Teitelbaum exceptional zero conjecture is a well-known example. As another remarkable example, recently Bertolini and Darmon [**BD07**] proved a p-adic Gross-Zagier formula allowing us to produce (in some cases) rational points on elliptic curves from derivatives of certain Hida p-adic L-functions. In this note we relate the 'analytic' results of [**BD07**] to Nekovář 'algebraic' theory of Selmer complexes [**Nek06**]. This leads us to propose a p-adic Birch and Swinnerton-Dyer conjecture for the 'weight variable', placing the results of [**BD07**] in a more general and natural setting (cfr. [**BD07**, pag. 375, Rem. 8]).

Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor $N_E = pM$, having a prime $p \geq 5$ of multiplicative reduction, and let f_E be the newform of weight 2 on $\Gamma_0(N_E)$ attached to $E/\mathbb{Q}$. Hida theory allows us to consider $f_E = f_2$ as the weight 2 element in a p-adic analytic family of modular forms $f_\infty = \{f_k\}_{k \in U_E \cap \mathbb{Z}^{\geq 2}}$. Here k runs through the even integers in a p-adic disc $U_E \subset \mathbb{Z}_p$ centered at 2, and f_k is the p-stabilisation of a normalized eigenform of weight k and level $\Gamma_1(M)$. For every $k \in U_E \cap \mathbb{Z}^{\geq 2}$, we denote by $\alpha_p(k) := a_p(f_k)$ the p-th Fourier coefficient of f_k.

Fix a quadratic character χ, of conductor coprime with N_E. We denote by $L_p(f_\infty, \chi, k, s)$ the Mazur-Kitagawa two-variable p-adic L-function, attached to f_E, χ and the choice of 'Shimura periods' $\Omega_k \in \mathbb{C}$ ($k \in U_E \cap \mathbb{Z}^{\geq 2}$). It is a $\mathbb{C}_p$-valued p-adic analytic function defined on $U_E \times \mathbb{Z}_p$, interpolating the critical values of the Hecke L-series $L(f_k, \chi, s)$ of f_k twisted by χ (see [**BD07**] or Section (2.2)). We consider the restriction

$$L_p(f_\infty, \chi, k, k/2) : U_E \longrightarrow \mathbb{C}_p$$

of $L_p(f_\infty, \chi, k, s)$ to the central critical line $s = k/2$. It satisfies the following interpolation properties: for every $k \in U_E$

$$L_p(f_\infty, \chi, k, k/2) \doteq (1 - \chi(p)\alpha_p(k)^{-1}p^{k/2-1}) \cdot L(f_k, \chi, k/2), \tag{1}$$

where $\doteq$ denotes equality up to a non-zero scalar. The 'Euler factor' $(1 - \chi(p)\alpha_p(k)^{-1}p^{k/2-1})$ appearing in (1) is zero precisely when $k = 2$ and

$$\chi(p) = \alpha_p(2). \tag{2}$$

In this case $L_p(f_\infty, \chi, k, s)$ has an *exceptional zero* at $(k, s) = (2, 1)$, meaning that $L_p(f_\infty, \chi, 2, 1) = 0$ independently on whether $L(E/\mathbb{Q}, s)$ vanishes or not at $s = 1$. In this note we are especially interested in this exceptional zero situation, and we assume for the rest of this introduction that (2) is satisfied.

The arithmetic of 'the data' (f_∞, p, χ) strongly depends on the sign $\mathrm{sign}(E, \chi) \in \{\pm 1\}$ appearing in the functional equation satisfied by $L(E, \chi, s) := L(f_E, \chi, s)$. If $\mathrm{sign}(E, \chi) = +1$, then $L_p(f_\infty, \chi, k, k/2) \equiv 0$ vanishes identically (see Section (2.2)). This is the situation considered (for $\chi = 1$) by Greenberg and Stevens in [**GS93**]. If

$$\mathrm{sign}(E, \chi) = -1, \tag{3}$$

a conjecture of Greenberg predicts that $L_p(f_\infty, \chi, k, k/2)$ is not identically zero, i.e. that $L(f_k, \chi, k/2) \neq 0$ for almost all $k \in U_E$. Assume for the rest of the introduction that (3) is satisfied.

The assumptions (2) and (3) imply that $L_p(f_\infty, \chi, k, k/2)$ vanishes to order at least 2 at $k = 2$. For the second derivative, we have the following result. Write $K_\chi/\mathbb{Q}$ for the quadratic field attached to χ (resp., $K_\chi := \mathbb{Q}$), if $\chi \neq 1$ (resp., $\chi = 1$). There is a global point $\mathbf{P}_\chi \in E(K_\chi)^\chi$ and a rational number $\ell \in \mathbb{Q}^*$ such that

$$\frac{d^2}{dk^2} L_p(f_\infty, \chi, k, k/2)_{k=2} = \ell \cdot \log_E(\mathbf{P}_\chi)^2, \tag{4}$$

where $\log_E : E(\overline{\mathbb{Q}_p}) \to \mathbb{G}_a(\overline{\mathbb{Q}_p})$ is the formal group logarithm on $E/\mathbb{Q}_p$. The point $\mathbf{P}_\chi$ is a Heegner point, coming from an appropriate Shimura curve parametrisation of $E/\mathbb{Q}$, and it is of infinite order if and only if $L'(E, \chi, 1) \neq 0$. This result has been proved by Bertolini and Darmon in [**BD07**] (assuming an extra hypothesis subsequently removed by Mok in [**Mok11**]).

As remarked by the authors in [**BD07**], it would be worthwhile to understand (4) in the framework of a Birch and Swinnerton-Dyer conjecture for the Hida p-adic L-function $L_p(f_\infty, \chi, k, k/2)$. This faces us with the problem of constructing a *regulator term* 'compatible' with (4). The aim of this note is to show that we can indeed construct such a regulator *via* Nekovář duality for the Selmer complex attached to a suitable Hida big Galois representation, interpolating the Deligne representations of the elements of f_∞ (see Sec. (2.3)).

More precisely, Nekovář's construction of abstract Cassels-Tate pairings (recalled in Sec. 3) produces a cohomologically-defined *p-adic weight pairing*

$$\langle -, -\rangle^{\mathrm{Nek}}_{K_\chi,p} : E^\dagger(K_\chi) \otimes \mathbb{Q}_p \times E^\dagger(K_\chi) \otimes \mathbb{Q}_p \longrightarrow \mathbb{Q}_p,$$

where $E^\dagger(K_\chi)$ is the extended Mordell-Weil group of E/K_χ. We can think of $\langle -, -\rangle^{\mathrm{Nek}}_{K_\chi,p}$ as an analogue in this context of the canonical p-adic height considered in [**MTT86**], [**BD96**] and [**PR92**], and as a p-adic variant of the classical Neron-Tate height, with the essential difference that $\langle -, -\rangle^{\mathrm{Nek}}_{K_\chi,p}$ is alternating.

Write $E(K_\chi)^\chi$ and $E^\dagger(K_\chi)^\chi$ for the subgroups of $E(K_\chi)$ and $E^\dagger(K_\chi)$ on which $Gal(K_\chi/\mathbb{Q})$ acts via χ. Under our assumptions (2) and (3)

$$\mathrm{rank}_{\mathbb{Z}} E^\dagger(K_\chi)^\chi = \mathrm{rank}_{\mathbb{Z}} E(K_\chi)^\chi + 1.$$

More precisely, $E^\dagger(K_\chi)^\chi$ modulo torsion is generated by a basis of $E(K_\chi)^\chi$ modulo torsion and a suitable 'Tate period' $q_\chi \in E^\dagger(K_\chi)^\chi$ (see Sec. (5.5)). This is the 'algebraic manifestation' of the presence of an exceptional zero for the p-adic L-function. In Section (4.4) (see also the proof of Prop. (5.6)) we prove the following 'explicit formula':

THEOREM 0.1. *For every $P \in E(K_\chi)^\chi$ we have*

$$\langle q_\chi, P\rangle^{\mathrm{Nek}}_{K_\chi,p} = c \cdot \log_E(P), \tag{5}$$

where $c = 1$ if $\chi \neq 1$ and $c = 1/2$ if $\chi = 1$.

Using (5), we can rephrase (4) in the following way, emphasizing the analogy with the classical Gross-Zagier formula: there is a scalar $\ell \in \mathbb{Q}^*$ such that

$$\frac{d^2}{dk^2} L_p(f_\infty, \chi, k, k/2)_{k=2} = \ell \cdot \left(\langle q_\chi, \mathbf{P}_\chi\rangle^{\mathrm{Nek}}_{K_\chi,p}\right)^2.$$

This result, combined with Nekovář theory, suggests a close relation between the dominant term in the Taylor expansion of $L_p(f_\infty, \chi, k, k/2)$ at $k = 2$ and the determinant of $\langle -, -\rangle^{\mathrm{Nek}}_{K_\chi,p}$. This leads us to propose in Sec. 11.8 a p-adic Birch and Swinnerton-Dyer conjecture for the weight variable, in the spirit of the conjectures formulated in [**MTT86**] and [**BD96**].

More generally: let $E/\mathbb{Q}$ be an elliptic curve of conductor N_E *ordinary* at a prime $p \geq 5$, and χ a primitive quadratic character of conductor coprime with $p \cdot N_E$. The constructions of f_∞ and $L_p(f_\infty, \chi, k, s)$ generalize to this setting and we can consider (cfr. Sec. (2.2)) the *generic part* $L_p^{\mathrm{gen}}(f_\infty, \chi, k)$ of the restriction of $L_p(f_\infty, \chi, k, s)$ to the central critical line $s = k/2$. This is a p-adic analytic function on U_E, which is conjecturally not identically zero. Conjecture (6.1) relates the leading term of $L_p^{\mathrm{gen}}(f_\infty, \chi, k)$ to a *p-adic regulator*, defined in terms of $\langle -, -\rangle^{\mathrm{Nek}}_{K_\chi,p}$ and the Mazur-Tate-Teitelbaum p-adic cyclotomic height.

We finally mention the work of Delbourgo, related to the subject of this note. In [**Del08**, Ch. 10], a 'two-variable' big Selmer group is attached to the 'cyclotomic' and 'Hida' deformation of the p-adic Tate module of $E/\mathbb{Q}$. Assuming that $E/\mathbb{Q}_p$ does *not* have split multiplicative reduction, the leading term of its characteristic power series is expressed in term of a certain p-adic regulator. Moreover, a main conjecture is formulated, relating this power series to the Mazur-Kitagawa p-adic L-function. It is likely that analogues of the results (resp., conjectures) of [**Del08**, Ch. 10] can be proved (resp., formulated) also in the exceptional case, in terms of the cyclotomic deformation of the big Selmer complex $\widetilde{H}^2_f(\mathbb{Q}, T(\mathcal{P}))$ defined in Sec. (3) and the p-adic regulator of Sec. (5). This 'Iwasawa theoretic' point of view may also serve as a motivation for Conjecture (6.1), in the same way as the main conjecture of Iwasawa theory [**Gre94b**], together with the algebraic p-adic BSD formulas of Schneider et al. (see for example [**BD95**]) motivate the p-adic Birch and Swinnerton-Dyer conjectures of [**MTT86**] and [**BD96**].

Notations. The following notations will be used throughout this note:

- $E/\mathbb{Q}$ is an elliptic curve defined over $\mathbb{Q}$ of conductor N_E;
- $f_E = \sum_{n\geq 1} a_n(E)\cdot q^n \in S_2(\Gamma_0(N_E),\mathbb{Z})$ is the newform attached to $E/\mathbb{Q}$ by the modularity theorem;
- $p \geq 5$ is a rational prime of *ordinary* (i.e. good ordinary or multiplicative) reduction for E;
- $K/\mathbb{Q}$ is a number field of discriminant d_K; we write $D_K = |d_K|$;
- $S_f \supset \{v|p\cdot N_E\cdot D_K\}$ is a finite set of finite primes of K;
- $\overline{K} = \overline{\mathbb{Q}}$ (resp. $\overline{K_v} = \overline{\mathbb{Q}_l}$, $S_f \ni v|l$) is a fixed algebraic closure of $\mathbb{Q}$ (resp. of the completion K_v of K at $v \in S_f$);
- $G_{K,S} := Gal(K_S/K)$ is the Galois group of the maximal algebraic extension $K_S \subset \overline{K}$ of K which is unramified outside $S_f \cup \{v|\infty\}$;
- $\rho_v : \overline{K} \hookrightarrow \overline{K_v}$ (for $v \in S_f$) is a fixed embedding which extends $K \hookrightarrow K_v$;
- $\rho_v^* : G_v := Gal(\overline{K_v}/K_v) \hookrightarrow G_K$ (or $G_{\mathbb{Q}_l} \hookrightarrow G_{\mathbb{Q}}$ if $v|l$) is the morphism attached to ρ_v;
- if M is a $\mathbb{Z}[G_{K,S}]$-module, we write M_v for the $\mathbb{Z}[G_v]$-module M, on which G_v acts via ρ_v^*;
- if L is a field, we write $T_p(L^*) := \varprojlim \mu_{p^n}(\overline{L})$;
- if M is a $\mathbb{Z}_p[Gal(\overline{L}/L)]$-module, $M(1) := M_L(1) := M \otimes_{\mathbb{Z}_p} T_p(L^*)$ (with diagonal $Gal(\overline{L}/L)$-action).

1. Kummer theory

In this section we recall some results from Kummer theory. Given a profinite group G and a finite dimensional $\mathbb{Q}_p$-vector space M, endowed with a continuous $\mathbb{Q}_p$-linear action of G (for the p-adic topology on M), $H^*(L,M) = H^*\left(C^\bullet_{\mathrm{cont}}(G,M)\right)$ denotes the continuous cohomology group of G_L with values in M, as defined in [**Tat76**] or [**Jan88**]. If $G_L := Gal(\overline{L}/L)$ is the absolute Galois group of a field L, we use the notation $H^*(L,M)$ for $H^*(G_L,M)$.

1.1. The multiplicative group. In this paragraph $L/\mathbb{Q}_p$ is a local field and π_L is a uniformizer in the maximal order $\mathcal{O}_L$ of L.

By Hilbert Satz 90, the connecting morphism attached to the short exact sequence of discrete G_L-modules $0 \to \mu_{p^n} \to \overline{L}^* \xrightarrow{p^n} \overline{L}^* \to 0$, defines an isomorphism $L^*/(L^*)^{p^n} \xrightarrow{\sim} H^1(L,\mu_{p^n})$. Taking the inverse limit $n\to\infty$ and extending scalars to $\mathbb{Q}_p$, we obtain the Kummer isomorphism

$$L^*\widehat{\otimes}\mathbb{Q}_p \xrightarrow{\sim} H^1(L,\mathbb{Q}_p(1)).$$

Given $x \in L^*$ we write γ_x^L (or simply γ_x if L is fixed) for the image of $x\otimes 1$ under the Kummer map. By local Tate duality, we have a (perfect) duality

$$\langle\ ,\ \rangle_L := inv_L\,(x\cup y) : H^1(L,\mathbb{Q}_p(1)) \times H^1(L,\mathbb{Q}_p) \to H^2(L,\mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p,$$

where $\cup$ is induced by multiplication $\mathbb{Q}_p(1)\times\mathbb{Q}_p \to \mathbb{Q}_p(1)$ and inv_L is the invariant map of local class field theory. Note that $H^1(L,\mathbb{Q}_p) = \mathrm{Hom}_{cts}\left(G_L^{ab},\mathbb{Q}_p\right)$ ($G^{ab} := G/[G:G]'$ for the closure $[G:G]'$ of the commutator subgroup of G). Recall also the reciprocity map [**Ser67**]:

$$\mathrm{rec}_L : L^* \to G_L^{ab},$$

normalized in such a way that $\mathrm{rec}_L(\pi_L)^{-1} \in \mathrm{Fr}_L$ is an arithmetic Frobenius in G_L^{ab}. We write also $\mathrm{rec}_p := \mathrm{rec}_{\mathbb{Q}_p}$.

Proposition 1.1. *a) For every $q \in L^\times$ and $\chi \in H^1(L,\mathbb{Q}_p)$ we have*

$$\langle\gamma_q,\chi\rangle_L = \chi\left(\mathrm{rec}_L(q)\right).$$

b) Let $\chi_{cy} : G_{\mathbb{Q}_p}^{ab} \twoheadrightarrow \mathbb{Z}_p^*$ be the p-adic cyclotomic character. For every $q = p^{ord_p(q)}\cdot u \in \mathbb{Q}_p^*$

$$\chi_{cy}(\mathrm{rec}_p(q)) = u \in \mathbb{Z}_p^*.$$

PROOF. *a*) Follows by [**Ser67**, Sec. (2.3)] (see also [**Nek06**, Sec. (11.3.5)]). *b*) Recalling our normalization of rec_p, this follows by [**Ser67**, Sec. (3.1)]. □

1.2. Elliptic curves. Let us consider the elliptic curve $E/\mathbb{Q}$ fixed above. Denote by $\mathrm{Ta}_p(E) := \varprojlim E(\overline{\mathbb{Q}})[p^n]$ the Tate module of $E/\mathbb{Q}$ and by $V_p(E) := \mathrm{Ta}_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. As S_f contains every prime of bad reduction for $E/\mathbb{Q}$, $\mathrm{Ta}_p(E)$ is a (continuous) $G_{K,S}$-module [**Sil86**, Ch. VII]. For every $v \in S_f$, the embedding ρ_v induces an isomorphism of G_v-modules (again denoted ρ_v) $\rho_v : \left(E(\overline{\mathbb{Q}})[p^n]\right)_v \xrightarrow{\sim} E(\overline{K_v})[p^n]$. Taking limits we obtain an isomorphism of $\mathbb{Q}_p[G_v]$-modules $\rho_v : V_p(E)_v \xrightarrow{\sim} V_v(E) := \left(\varprojlim E(\overline{K_v})[p^n]\right) \otimes \mathbb{Q}_p$. When the context makes it clear, we write simply $V_p(E)$ for the G_v-module $V_p(E)_v$.

Recall the (injective) global Kummer map

$$E(K) \otimes \mathbb{Q}_p = \left(\varprojlim E(K)/p^n E(K)\right) \otimes \mathbb{Q}_p \overset{\lim \kappa_n}{\longleftrightarrow} \left(\varprojlim H^1(G_{K,S}, E[p^n])\right) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1(G_{K,S}, V_p(E)),$$

where κ_n is the usual Kummer map on $E(K)/p^n$ ant the last isomorphism is obtained (extending scalars to $\mathbb{Q}_p$) from $H^1(G_{K,S}, \mathrm{Ta}_p(E)) \xrightarrow{\sim} \varprojlim H^1(G_{K,S}, \mathrm{Ta}_p(E)/p^n)$ (see for example [**Nek06**, Lemma (4.2.2)] or [**Jan88**]). For every $P \in E(K)$ we write γ_P for the image of $P \otimes 1$ under this map. Replacing $G_{K,S}$ by G_v ($v \in S_f$) we obtain also a local Kummer map $E(K_v) \otimes \mathbb{Q}_p \hookrightarrow H^1(K_v, V_v(E))$. Given $P_v \in E(K_v)$, we write again γ_{P_v} for the image of $P_v \otimes 1$ and we consider $E(K_v) \otimes \mathbb{Q}$ as a subspace of $H^1(K_v, V_v(E))$. We have $res_v(\gamma_P) = \rho_v^{-1}\left(\gamma_{\rho_v(P)}\right)$: here we have written (by abuse of notation) again ρ_v^{-1} for the map induced in cohomology by the isomorphism $\rho_v^{-1} : V_v(E) \xrightarrow{\sim} V_p(E)_v$ and

$$res_v : H^1(G_{K,S}, V_p(E)) \to H^1(K_v, V_p(E)) := H^1(G_v, V_p(E)_v)$$

for the 'restriction' map induced in cohomology by the morphism of pairs (ρ_v^*, id). When there is no risk of confusion, we identify $V_p(E)_v$ with $V_v(E)$ and res_v with $\rho_v \circ res_v$. Furthermore, we consider $E(K_v) \otimes \mathbb{Q}_p$ also as a submodule of $H^1(K_v, V_p(E))$ under $\rho_v^{-1} : H^1(K_v, V_v(E)) \xrightarrow{\sim} H^1(K_v, V_p(E))$.

Writing $Sel_{\mathbb{Q}_p}(E; K) \subset H^1(G_{K,S}, V_p(E))$ for the Selmer group defined by the local conditions $E(K_v) \otimes \mathbb{Q}_v$ (for $v \in S_f$), we have a short exact sequence

$$0 \to E(K) \otimes \mathbb{Q}_p \to Sel_{\mathbb{Q}_p}(E; K) \to \mathrm{Ta}_p(Ш(E/K)) \otimes \mathbb{Q}_p \to 0, \tag{6}$$

where $Ш(E/K)$ is the Tate-Shafarevich group of E/K. As shown by R. Greenberg, we can also describe this Selmer group in terms of the ordinary filtration on the Galois representation $\mathrm{Ta}_p(E)$, in the following way.

If $E/\mathbb{Q}$ has good ordinary reduction we have a short exact sequence of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules

$$0 \to V_p(E)^+ \overset{i_v^+}{\to} V_p(E)_v \overset{p_v^-}{\to} V_p(E)^- \to 0, \tag{7}$$

with $\dim_{\mathbb{Q}_p} V_p(E)^{\pm} = 1$. Here $V_p(E)^- := \mathrm{Ta}_p(\overline{E}_p) \otimes \mathbb{Q}_p$ (resp., $V_p(E)^+ := \mathrm{Ta}_p(\widehat{E}) \otimes \mathbb{Q}_p$) is the p-adic Tate module of the reduction $\overline{E}_p/\mathbb{F}_p$ of $E/\mathbb{Q}_p$ (resp., of the formal group $\widehat{E}$ of $E/\mathbb{Q}_p$ [**Sil86**, Ch. VII]) with $\mathbb{Q}_p$-coefficients. The map p_v^- is induced by ρ_v and the reduction map $E(\overline{\mathbb{Q}_p}) \to \overline{E}_p(\overline{\mathbb{F}_p})$. In particular $V_p(E)^-$ is unramified at p.

If $E/\mathbb{Q}_p$ has multiplicative reduction, Tate's p-adic analytic uniformisation gives us a group isomorphism

$$\Phi_{Tate} : \overline{\mathbb{Q}_p}^* / q_E^{\mathbb{Z}} \xrightarrow{\sim} E(\overline{\mathbb{Q}_p}), \tag{8}$$

where $q_E \in p\mathbb{Z}_p$ is Tate p-adic period of $E/\mathbb{Q}_p$ [**Sil94**, Ch. V]. We have $\Phi_{Tate}(x^g) = \chi(g) \cdot \Phi_{Tate}(x)^g$ for every $g \in G_{\mathbb{Q}_p}$, where $\chi : G_{\mathbb{Q}_p} \to \{\pm 1\}$ is the quadratic unramified character (resp., the trivial character) if $E/\mathbb{Q}_p$ has non-split (resp., split) multiplicative reduction [**Sil94**, Ch. V]. As $q_E \in p\mathbb{Z}_p$, we have an exact sequence of $G_{\mathbb{Q}_p}$-modules $0 \to \mu_{p^n}(\chi) \overset{\Phi_{Tate}}{\to} E(\overline{\mathbb{Q}_p})[p^n] \overset{P_{Tate}}{\to} (\mathbb{Z}/p^n\mathbb{Z})(\chi) \to 0$. Taking the inverse limit on n and extending scalars to $\mathbb{Q}_p$, we obtain the *fundamental exact sequence* of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules

$$0 \to \mathbb{Q}_p'(1) \overset{i_v^+}{\to} V_p(E)_v \overset{p_v^-}{\to} \mathbb{Q}_p' \to 0. \tag{9}$$

Here $\mathbb{Q}_p' := \mathbb{Q}_p(\chi)$ and i_v^+ (resp. p_v^-) is induced by (the limit of) $\rho_v^{-1} \circ \Phi_{Tate}$ (resp. $P_{Tate} \circ \rho_v$). We will write $V_p(E)^+ := \mathbb{Q}_p'(1)$ and $V_p(E)^- := \mathbb{Q}_p'$, so that we obtain (7) also in this case.

Define, for every $v|p$, $H^1_f(K_v, V_p(E)) \subset H^1(K_v, V_p(E))$ as the image of $H^1(K_v, V_p(E)^+)$ under the map induced in cohomology by i_v^+, and put $H^1_f(K_v, V_p(E)) = 0$ for $S_f \ni v \nmid p$. These local conditions define a Selmer group $H^1_f(K, V_p(E)) \subset H^1(G_{K,S}, V_p(E))$ (independent on the choice of $S_f \supset \{v|p \cdot N_E\}$). The following Lemma is proved in [**Gre97**, Sec. 2] (see also [**Nek06**, Lemma (9.6.7.3)]).

LEMMA 1.2. *$E(K_v) \otimes \mathbb{Q}_p = H^1_f(K_v, V_p(E))$ for every $v \in S_f$. In particular $Sel_{\mathbb{Q}_p}(E; K) = H^1_f(K, V_p(E))$.*

REMARK 1.3. Suppose that E/K_v has split multiplicative reduction, i.e. $\mathbb{Q}_p' = \mathbb{Q}_p$ as G_v-modules:

a) if $\Phi_{Tate}(\widetilde{P}) = P \in E(K_v)$, for a $\widetilde{P} \in K_v^*$, then $(\Phi_{Tate})_*(\gamma_{\widetilde{P}}) = \gamma_P$ (we write again $\Phi_{Tate} : \mathbb{Q}_p(1) \to V_v(E)$ for the map induced by Φ_{Tate}). This follows from the definitions and proves (the first assertion of) the preceding Lemma in this case.

b) Let $\partial_v : \mathbb{Q}_p \to H^1(K_v, \mathbb{Q}_p(1))$ be the connecting morphism attached to (9). A short inspections reveals that $\partial_v(1) = \gamma_{q_E}$ $(:= \gamma_{q_E}^{K_v})$. In other words, under the identifications of elements in $H^1(K_v, \mathbb{Q}_p(1))$ with (continuous) extensions classes in $\mathrm{Ext}^1_{\mathbb{Q}_p[G_v]}(\mathbb{Q}_p, \mathbb{Q}_p(1))$, γ_{q_E} corresponds to the class of (9).

1.3. Products. Let $v \in S_f$ be a prime which divide p and let

$$\begin{aligned} (\ ,\ )_W &: \mathrm{Ta}_p(E) \times \mathrm{Ta}_p(E) \longrightarrow T_p(\mathbb{Q}^*) \\ (\textit{resp.},\quad (\ ,\ )_v &: T_p(E) \times T_p(E) \longrightarrow T_p(\mathbb{Q}_p^*)\quad) \end{aligned}$$

be the Weil pairing on $\mathrm{Ta}_p(E)$ (resp., on $T_p(E) := \varprojlim E(\overline{\mathbb{Q}_p})[p^n]$). (We use the opposite 'sign convetion' to that of [**Sil86**], so that our Weil pairing is minus that defined in [**Sil86**, Ch. III].) It is a perfect, alternating and $G_{K,S}$-equivariant (resp., $G_{\mathbb{Q}_p}$-equivariant) $\mathbb{Z}_p$-bilinear form.

If $E/\mathbb{Q}_p$ has multiplicative reduction and writing again $\rho_v : T_p(\mathbb{Q}^*)_v \xrightarrow{\sim} T_p(\mathbb{Q}_p^*)$ for the isomorphism of $\mathbb{Z}_p[G_v]$-modules induced by ρ_v, we have

$$\rho_v\left((x,y)_W\right) = (\rho_v(x), \rho_v(y))_v; \quad (\Phi_{Tate}(\alpha), \beta)_v = \alpha \times P_{Tate}(\beta) \tag{10}$$

for every $x, y \in \mathrm{Ta}_p(E)$, $\alpha \in T_p(\mathbb{Q}_p^*)$ and $\beta \in T_p(E)$ (here $\times$ is multiplication, once we identify $T_p(\mathbb{Q}_p^*)$ with $\mathbb{Z}_p$ as $\mathbb{Z}_p$-modules). The first equality follows from the definition of the Weil pairing, while the second can be proved using the description of principal divisors on $E_{q_E} := \overline{\mathbb{Q}_p}^*/q_E^{\mathbb{Z}}$ in terms of p-adic theta functions (see for example [**Tat95**]).

Write $\cup_W : C^\bullet_{\mathrm{cont}}(G_v, V_p(E)_v) \otimes C^\bullet_{\mathrm{cont}}(G_v, V_p(E)_v) \to C^\bullet_{\mathrm{cont}}(G_v, T_p(\mathbb{Q}^*)_v \otimes \mathbb{Q}_p)$ for the cup-product induced on cochains by $(\ ,\ )_W$. If $E/\mathbb{Q}_p$ is multiplicative, it follows by (10) that

$$\rho_v\left(y \cup_W i_v^+(x)\right) = -p_v^-(y) \cup x \in C^2_{\mathrm{cont}}(K_v, \mathbb{Q}_p(1)) \tag{11}$$

for every $x \in C^1_{\mathrm{cont}}(K_v, V_p(E)^+)$ and $y \in C^1_{\mathrm{cont}}(K_v, V_p(E))$. In (11) $\cup$ is the cup-product pairing induced by multiplication $V_p(E)^+ \times V_p(E)^- \to \mathbb{Q}_p(1)$ and we have written again ρ_v for the isomorphism induced on cochains by $\rho_v \otimes \mathbb{Q}_p$.

2. Hida theory

In this section we recall some fundamental results of Hida Theory. We use [**NP00**], [**Nek06**, Sec. 12.7] and [**BD07**] as main references.

2.1. Hida families. Let $N := N_E/p^{\mathrm{ord}_p(N_E)}$ be the *tame* conductor of E and fix an embedding $\rho_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Writing ψ for the trivial character modulo N_E, let

$$X^2 - a_p(E)X + \psi(p)p = (X - \alpha_p) \cdot (X - \beta_p),$$

with $\alpha_p, \beta_p \in \overline{\mathbb{Q}}$. Since E is ordinary at p, we have $\alpha_p, \beta_p \in \mathbb{Z}_p$ (under ρ_p) and we can assume $\alpha_p \in \mathbb{Z}_p^*$ and $\beta_p \in p\mathbb{Z}_p$. We define the *p-stabilization* $f_E^0 \in S_2(\Gamma_0(Np), \mathbb{Z}_p)$ of f_E by

$$f_E^0(z) := f_E(z) + \beta_p \cdot f_E(pz). \tag{12}$$

(In particular $f_E^0 = f_E$ if $E/\mathbb{Q}_p$ has multiplicative reduction.) As follows by [**Hid85**, Lemma 3.3], f_E^0 is the unique normalized eigenform on $\Gamma_0(Np)$ such that $a_l(f_E^0) = a_l(E)$ for every prime $l \neq p$. Moreover $a_p(f_E^0) = \alpha_p \in \mathbb{Z}_p^*$.

Consider Hida universal ordinary Hecke algebra of tame conductor N

$$\mathfrak{h}_\infty^{\mathrm{ord}} = \mathfrak{h}_\infty^{\mathrm{ord}}(N) := \varprojlim_{r\geq 1} \mathfrak{h}_{2,r}^{ord}.$$

Here $\mathfrak{h}_{2,r}^{ord} := e_{ord} \cdot (\mathfrak{h}(\Gamma_1(Np^r)) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, where $\mathfrak{h}(\Gamma_1(Np^r)) \subset \mathrm{End}_{\mathbb{Z}}(S_2(\Gamma_1(Np^r), \mathbb{Z}))$ is the algebra generated by the Hecke operators T_l, for every prime l and the diamond operator $\langle a \rangle$, for every $a \in (\mathbb{Z}/Np^r\mathbb{Z})^*$, and $e_{ord} := \lim_{n\to\infty} T_p^{n!}$ is Hida's ordinary projector. We will write also U_p for T_p. We have a morphism $\langle\ \rangle_r : \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^*] \to \mathfrak{h}_{2,r}^{ord}$. Putting $\Gamma := 1 + p\mathbb{Z}_p$ and taking the (inverse) limit $r \to \infty$, we obtain the "diamond" morphism (with the normalization of [**NP00**, §(1.4)])

$$\langle\ \rangle : \Lambda := \mathbb{Z}_p[[\Gamma]] \to \mathbb{Z}_p[[\mathbb{Z}_{N,p}^*]] \longrightarrow \mathfrak{h}_\infty^{\mathrm{ord}},$$

where $\mathbb{Z}_{N,p}^* := \mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*$. $\langle\ \rangle$ gives $\mathfrak{h}_\infty^{\mathrm{ord}}$ a structure of Λ-algebra. By [**Hid86a**], $\mathfrak{h}_\infty^{\mathrm{ord}}$ is a free Λ-module of finite rank. It follows that $\mathfrak{h}_\infty^{\mathrm{ord}} = \prod_{\mathfrak{m}_j} \mathfrak{h}_{\infty,\mathfrak{m}_j}^{ord}$ decomposes as a (finite) direct sum of its completions at maximal ideals $\mathfrak{m}_j$. As $\alpha_p \in \mathbb{Z}_p^*$, f_E (or better f_E^0) gives rise to a morphism of $\mathbb{Z}_p$-algebras

$$\eta_{f_E} : \mathfrak{h}_\infty^{\mathrm{ord}} \to \mathbb{Z}_p \tag{13}$$

defined sending $\mathbb{Z}_{N,p}^*$ to 1 and T_l to $a_l(f_E^0)$ for every prime l. η_{f_E} factorizes through $\mathfrak{h}_\infty^{\mathrm{ord}} \twoheadrightarrow \mathfrak{h}_{\infty,\mathfrak{m}}^{ord}$ for a unique maximal ideal $\mathfrak{m} = \mathfrak{m}_j$. Write $\mathcal{P} := Ker(\eta_{f_E}) \in Spec(\mathfrak{h}_{\infty,\mathfrak{m}}^{ord})$: by [**Hid86a**, Cor. (1.4)] (see also [**Nek06**, §(12.7.5)]) the localization of $\mathfrak{h}_{\infty,\mathfrak{m}}^{ord}$ at $\mathcal{P}$ is a discrete valuation ring, unramified over $\Lambda_{\overline{\mathfrak{p}}}$, where $\overline{\mathfrak{p}} = (\gamma - 1)$ for a topological generator γ of Γ. Then $\mathfrak{h}_{\infty,\mathfrak{m}}^{ord}$ contains a unique minimal prime $\mathcal{P}_{min}$ s.t. η_{f_E} factorizes through the local domain

$$R = R_E := \mathfrak{h}_{\infty,\mathfrak{m}}^{ord}/\mathcal{P}_{min}.$$

We will write from now on $R = R_E$ and $\overline{\mathcal{P}} := \mathcal{P}/\mathcal{P}_{min} \in Spec(R)$. The localization $R_{\overline{\mathcal{P}}}$ is a discrete valuation ring, unramified over $\Lambda_{\overline{\mathfrak{p}}}$. Fix a topological generator γ of Γ (e.g. $\gamma = 1 + p \in \Gamma$) and the corresponding uniformizer of $R_{\overline{\mathcal{P}}}$

$$\varpi := (\gamma - 1) \in R_{\overline{\mathcal{P}}}.$$

We write again T_l and U_p for the image of the Hecke operators in R. As η_E takes values in $\mathbb{Z}_p$ (i.e. E is defined over $\mathbb{Q}$) the residue field $\mathrm{Frac}(R/\overline{\mathcal{P}}) = R_{\overline{\mathcal{P}}}/\varpi R_{\overline{\mathcal{P}}}$ of $R_{\overline{\mathcal{P}}}$ is identified with $\mathbb{Q}_p$. With the terminology of [**Hid86a**], $\mathscr{R} := Frac(R)$ is the (primitive) local component to which f_E belongs. $\mathfrak{h}_{\infty,\mathfrak{m}}^{ord}$ is the *Hida family* attached to f_E and R is the *branch* of the Hida family in which f_E lives. This terminology is justified by the following analytic interpretations of the results above, given in [**GS93**] (see also the next section).

Let $\mathcal{A} \subset \overline{\mathbb{Q}}_p[[w-2]]$ be the ring of formal power series in $w - 2$ converging for w in some p-adic neighborhood of 2. The ring $\mathcal{A}$ is endowed with a structure of Λ-algebra, defined as follows: let $\wp \mapsto f_\wp(X)$ be the isomorphism $\Lambda \xrightarrow{\sim} \mathbb{Z}_p[[X]]$ determined by $f_\gamma(X) = X+1$. We associate to $\wp \in \Lambda$ the analytic function on $\mathbb{Z}_p$ given by $w \mapsto f_\wp(\gamma^{w-2} - 1)$. Since $\mathcal{A}$ is Henselian and since the augmentation ideal $(\gamma - 1) \subset \Lambda$ is unramified in $R_{\overline{\mathcal{P}}}$, there exists a unique morphism of Λ-algebras

$$\eta_{f_\infty} : R_{\overline{\mathcal{P}}} \to \mathcal{A} \tag{14}$$

such that $(\eta_{f_\infty}(r))_{w=2} = \eta_{f_E}(r)$ for every $r \in R$. Define, for every positive integer n, $\alpha_n(w) := \eta_{f_\infty}(T_n) \in \mathcal{A}$, where T_n is the n-th Hecke operator, defined in terms of the T_l's by the usual relations ([**Shi71**, Ch. III]). As R is finite over Λ, there exists a p-adic neighborhood $2 \in U$ such that $\alpha_n(w) \in \mathcal{A}_U$ for every $n \in \mathbb{N}$, where $\mathcal{A}_U \subset \mathcal{A}$ is the ring of analytic functions on U. Consider the formal q-expansion

$$f_\infty := \sum_{n\geq 1} \alpha_n(w) \cdot q^n \in \mathcal{A}_U[[q]].$$

For every even integer $k \in U \cap \mathbb{Z}^{\geq 2}$, the *weight k-specialization* $f_k := \sum_{n\geq 1} \alpha_n(k) \cdot q^n$ is the q-expansion of a normalized eigenform on $\Gamma_1(Np)$ and $f_2 = f_E^0$. If $k \equiv 2 \bmod (p-1)$, then f_k has trivial character, i.e. it is a normalized eigenform on $\Gamma_0(Np)$. As follows by [**Hid86a**, Cor. (1.3)], f_k is new at the primes dividing the tame level N and is not p-new for $k>2$. More precisely, for every $k>2$ such that $k\equiv 2 \bmod (p-1)$ there exists a (unique) newform $f_k^{\#}$ on $\Gamma_0(N)$ such that $a_l(f_k^{\#})=a_l(f_k)$ for every prime $l\neq p$. (With the terminology used above, $f_k=(f_k^{\#})^0$ is the p-stabilization of $f_k^{\#}$ and they satisfy a relations analogous to (12) for $k=2$.)

Let $\mathbb{Z}':=\{z\in\mathbb{Z}^{\geq 2} : z\equiv 2 \bmod (p-1)\}$ and $f_2^{\#}:=f_E$. We call $\{f_k^{\#}\}_{k\in U\cap\mathbb{Z}'}$ the Hida family attached to $E/\mathbb{Q}$.

2.2. p-adic L-functions. For more details on the results and constructions recalled in this section, we refer the reader to [**BD07**, Sec. 1].

Let $\chi:(\mathbb{Z}/m\mathbb{Z})^*\to\{\pm 1\}$ be a quadratic Dirichlet character, of conductor m coprime with p, with $\chi(-1)=:w_\infty$, and let $\tau_\chi=\sum_{j=1}^m \chi(j)e^{2\pi i j/m}$ be the associated Gauss sum. For every $k\in U\cap\mathbb{Z}'$, the complex L-function $L(f_k^{\#},\chi,s):=\sum_{n\geq 1}\chi(n)a_n(f_k^{\#})\cdot n^{-s}$ (defined for $\mathrm{Re}(s)>(k+1)/2$) extends to an entire function on $\mathbb{C}$. Recall our convention: $f_E=f_2^{\#}$.

For every integer $1\leq j\leq k-1$, we define the *algebraic part* of $L(f_k^{\#},\chi,j)$ by

$$L^*(f_k^{\#},\chi,j):=\frac{(j-1)!\tau_\chi}{(-2\pi i)^{j-1}\,\Omega_k}L(f_k^{\#},\chi,j). \tag{15}$$

Here we fix, for every $k\in U\cap\mathbb{Z}'$, 'Shimura periods' $\Omega^{\pm}_{f_k^{\#}}\in\mathbb{C}^*$ as in [**BD07**, Prop. (1.1)], [**Shi77a**, Sec. 1], and we write $\Omega_k:=\Omega^{\mathrm{sign}(w_\infty)}_{f_k^{\#}}$. If $\chi(-1)=(-1)^{j-1}w_\infty$, (15) belongs to the field generated by the Fourier coefficient of $f_k^{\#}$, and we consider it as an element of $\overline{\mathbb{Q}}_p\subset\mathbb{C}_p$ under the embedding $\rho_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}_p}$ (fixed in the preceding section), where $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}_p$.

Remark 2.1. In [**BD07**], the periods are chosen in such a way that $\Omega^+_{f_k^{\#}}\Omega^-_{f_k^{\#}}=\langle f_k^{\#},f_k^{\#}\rangle$ is the Petersson scalar product of $f_k^{\#}$ with itself. We do not impose this 'normalization' here, as we will fix Ω_2 later in a 'convenient way' (cfr. Sec. (11.8)).

Up to shrinking U if necessary, Sec. 1 of [**BD07**] constructs a $\mathbb{C}_p$-valued function

$$L_p(f_\infty,\chi,k,s):U\times\mathbb{Z}_p\longrightarrow\mathbb{C}_p$$

which interpolates the Mazur-Tate-Teitelbaum p-adic L-functions $L_p(f_k^{\#},\chi,s)$ attached in [**MTT86**] to the elements of f_∞ and the periods Ω_k. More precisely $L_p(f_\infty,\chi,k,s)$ satisfies the following properties:

1. $L_p(f_\infty,\chi,k,s)$ is (locally) analytic on each variable;
2. for every $k\in U\cap\mathbb{Z}'$ and every odd integer $1\leq j\leq k-1$, there exists a scalar $\lambda(k)\in\mathbb{C}_p^*$ such that

$$L_p(f_\infty,\chi,k,j)=\lambda(k)(1-\chi(p)\alpha_p(k)^{-1}p^{j-1})(1-\chi(p)\alpha_p(k)^{-1}p^{k-j-1})^{\epsilon_k}\cdot L^*(f_k^{\#},\chi,j), \tag{16}$$

   where $\epsilon_k=1$ if $f_k\neq f_k^{\#}$ and $\epsilon_k=0$ otherwise;
3. $\lambda(2)=1$.

Remark 2.2. Using the terminology of [**BD07**], $L_p(f_\infty,\chi,k,s)$ is determined by the choice of an ordinary, $\Gamma_0(N)$-equivariant modular symbol μ_* with values in the space of measures on $(\mathbb{Z}_p^2)'$ (the set of primitive vectors in $\mathbb{Z}_p\times\mathbb{Z}_p$), *interpolating* the classical modular symbol attached to f_k in weight k (see [**BD07**, Sec. (1.3)]). The existence of such a modular symbol follows from [**GS93**, Th. 5.13], and the scalars $\{\lambda(k)\}$ come from the interpolation process [**BD07**, Th. (1.5)]. Once we have fixed the periods $\{\Omega_k\}$ as above (depending on $\chi(-1)=w_\infty$), μ_*, and then $L_p(f_\infty,\chi,k,s)$, is unique up to multiplication by a nowhere vanishing analytic function α on U, satisfying $\alpha(2)=1$. Here we fix such a μ_* and call $L_p(f_\infty,\chi,k,s)$ *the* Mazur-Kitagawa p-adic L-function attached to χ.

Note that taking $j=k/2$ in (16) we obtain (for $k\equiv 2 \bmod 2(p-1)$)

$$L_p(f_\infty,\chi,k,k/2)=\lambda(k)(1-\chi(p)\alpha_p(k)^{-1}p^{k/2-1})^{\beta}L^*(f_k^{\#},\chi,k/2), \tag{17}$$

where $\beta = 2$ if $f_k \neq f_k^\#$ and $\beta = 1$ otherwise (i.e. if $k = 2$ and $E/\mathbb{Q}_p$ is multiplicative). This shows that, if $\chi(p) = \alpha_p$, $L_p(f_\infty, \chi, k, s)$ has an *exceptional zero* at $(k, s) = (2, 1)$, i.e. $L_p(f_\infty, \chi, 2, 1) = 0$ independently on whether $L(E/\mathbb{Q}, s)$ vanishes or not at $s = 1$.

Write w_k for the sign in the functional equation satisfied by the Hecke L-series $L(f_k^\#, s)$, for $k \in U \cap \mathbb{Z}'$. It is known [**NP00**, Sec. (3.4.4)] that $w_k =: w_{gen}$ is constant for every $k > 2$, and that

$$w_2 = \mathrm{sign}(E, \mathbb{Q}) = \begin{cases} w_{gen} & \text{if } \; p \nmid N_E; \\ \\ -\alpha_p \cdot w_{gen} & \text{if } \; p | N_E. \end{cases}$$

Recalling that $f_k^\#$ $(k > 2)$ is a newform on $\Gamma_0(N)$, by [**Shi71**, Th. (3.66)] we see that $L(f_k^\#, \chi, s)$ has constant sign

$$\mathrm{sign}(f_\infty, \chi) := \chi(-N) \cdot w_{gen} \tag{18}$$

in its functional equation, for every $k > 2$. Note that $\mathrm{sign}(f_\infty, \chi)$ is opposite to the sign (see again *loc. cit.*)

$$\mathrm{sign}(E, \chi) = \chi(-N_E) \cdot w_2$$

of $L(f_E, \chi, s)$ if and only if $\chi(p) = \alpha_p$, i.e. if and only if $L_p(f_\infty, \chi, k, s)$ has an exceptional zero. Moreover this happens precisely when the twist $E^\chi/\mathbb{Q}$ of $E/\mathbb{Q}$ has *split* multiplicative reduction at p [**MTT86**].

As follows by (18) and the interpolation formula (17), $L_p(f_\infty, \chi, k, k/2) \equiv 0$ vanishes identically if $\mathrm{sign}(f_\infty, \chi) = -1$. We define the *generic part* of the restriction of $L_p(f_\infty, \chi, k, s)$ to the central critical line by

$$L_p^{\mathrm{gen}}(f_\infty, \chi, k) := \left. \frac{L_p(f_\infty, \chi, k, s)}{(s - k/2)^{e_{\mathrm{gen}}(\chi)}} \right|_{s=k/2}; \quad e_{\mathrm{gen}}(\chi) := \begin{cases} 0 & \text{if } \; \mathrm{sign}(f_\infty, \chi) = +1; \\ \\ 1 & \text{if } \; \mathrm{sign}(f_\infty, \chi) = -1. \end{cases}$$

It is a $\mathbb{C}_p$-valued, p-adic analytic function on U. The terminology is justified by Greenberg conjecture, predicting that $L_p^{\mathrm{gen}}(f_\infty, \chi, k)$ is not identically zero (see Sec. 7 for results in this direction). When $\chi = \chi_{\mathrm{triv}}$ is the trivial character, we write simply $L_p(f_\infty, k, s)$, $L_p^{\mathrm{gen}}(f_\infty, k) = L_p^{\mathrm{gen}}(f_\infty/\mathbb{Q}, k)$ and e_{gen} for the objects attached to χ_{triv}.

Let K be a quadratic field such that $(D_K, p) = 1$ and let $\epsilon_K : (\mathbb{Z}/D_K\mathbb{Z})^* \to \{\pm\}$ be the associated quadratic character. Putting $e_{\mathrm{gen}}(K) := e_{\mathrm{gen}} + e_{\mathrm{gen}}(\epsilon_K)$, we define the *Hida p-adic L-function* of E/K by

$$L_p^{\mathrm{gen}}(f_\infty/K, k) := \left. \frac{L_p(f_\infty, k, s) \cdot L_p(f_\infty, \epsilon_K, k, s)}{(s - k/2)^{e_{\mathrm{gen}}(K)}} \right|_{s=k/2} = L_p^{\mathrm{gen}}(f_\infty, k) \cdot L_p^{\mathrm{gen}}(f_\infty, \epsilon_K, k).$$

2.3. Big Galois representations. Let $\mathbb{Q}_{Np} \subset \overline{\mathbb{Q}}$ be the maximal algebraic extension of $\mathbb{Q}$ which is unramified outside $p \cdot N \cdot \infty$, and let $\mathfrak{G} := Gal(\mathbb{Q}_{Np}/\mathbb{Q})$. In this section we recall briefly how we can construct a self-dual big Galois representation of $\mathfrak{G}$ which *interpolates* $V_p(E)$ in weight two and, more generally, a suitable self-dual twist of the Deligne representation of $f_k^\#$ in weight $k \in U \cap \mathbb{Z}'$. For more details and references, see [**Nek06**, Sec. (12.7)] or [**NP00**].

As explained in [**Nek06**, Sec. (12.7.8)-(12.7.10)] there exists a *continuous* $R_{\overline{\mathcal{P}}}[\mathfrak{G}]$-module $T(R_{\overline{\mathcal{P}}})$, free of rank two over $R_{\overline{\mathcal{P}}}$, such that: for every prime $\ell \nmid Np$

$$\mathrm{trace}\left(\mathrm{Fr}(\ell) | \, T(R_{\overline{\mathcal{P}}})\right) = T_\ell; \quad \det\left(\mathrm{Fr}(\ell) | \, T(R_{\overline{\mathcal{P}}})\right) = \ell \cdot \langle \ell \rangle \,, \tag{19}$$

where $\mathrm{Fr}(\ell) \in G_\mathbb{Q}$ is an arithmetic Frobenius and $\langle \; \rangle : \mathbb{Z}_p[[\mathbb{Z}_{N,p}^*]] \to \mathfrak{h}_\infty^{\mathrm{ord}} \twoheadrightarrow R$ is the diamond morphism. The term continuous means that

$$T(R_{\overline{\mathcal{P}}}) \in \left({}^{ad}_{R[\mathfrak{G}]}\mathrm{Mod} \right)$$

is an *admissible* $R[\mathfrak{G}]$-modulo, as defined in [**Nek06**, Sec. (3.2)] (see also the next section). The representation $T(R_{\overline{\mathcal{P}}})$ can be constructed as follows [**Hid86a**],[**NP00**].

Let $X_r := X_1(Np^r)_{/\mathbb{Q}}$ be the modular curve over $\mathbb{Q}$ (as defined, for example, in [**Roh97**]) and $J_r := \mathrm{Pic}^0(X_r)$. The Hecke algebra $\mathfrak{h}(\Gamma_1(Np^r))$ acts on J_r via algebraic correspondences and this action commutes with that of $G_\mathbb{Q}$. Let $\pi_1 : X_{r+1} \to X_r$ be the morphism defined by the inclusion $\Gamma_1(Np^{r+1}) \subset \Gamma_1(Np^r)$ and $\pi_1^* : J_r(\overline{\mathbb{Q}})_{p^\infty} \to J_{r+1}(\overline{\mathbb{Q}})_{p^\infty}$ be the map induced by (contravariant) functoriality. Write $J_\infty :=$

$\varinjlim_{\pi_1^*} J_r(\overline{\mathbb{Q}})_{p^\infty}$ and $J_\infty^{ord} = e_{ord} \cdot J_\infty$ for its ordinary part. By a fundamental theorem of Hida, J_∞^{ord} is an $\mathfrak{h}_\infty^{\mathrm{ord}}[G_\mathbb{Q}]$-module whose Pontrjagin dual is free of finite rank over Λ. Define (with the notations of Sec. (2.1))

$$\mathrm{Ta}_\infty^{ord} := \mathrm{Hom}_{\mathbb{Z}_p}\left(J_\infty^{ord}, \mu_{p^\infty}\right) \otimes_{\mathfrak{h}_\infty^{\mathrm{ord}}} \mathfrak{h}_{\infty,\mathfrak{m}}^{ord}; \qquad \mathrm{Ta}_\infty(R) := \mathrm{Ta}_\infty^{ord} \otimes_{\mathfrak{h}_{\infty,\mathfrak{m}}^{ord}} R.$$

With these notations, $T(R_{\overline{\mathcal{P}}}) := \mathrm{Ta}_\infty(R) \otimes_R R_{\overline{\mathcal{P}}}$ is the localization of $\mathrm{Ta}_\infty(R)$ at $\overline{\mathcal{P}}$. As J_∞^{ord} is unramified at every rational place $l \nmid Np\infty$, the same is true for $T(R_{\overline{\mathcal{P}}})$ (i.e. it is a $R_{\overline{\mathcal{P}}}[\mathfrak{G}]$-module). The identity (19) is a manifestation of the Eichler-Shimura congruence relation [**Roh97**, page 72].

To obtain a *self-dual* representation, we consider a suitable twist of $T(R_{\overline{\mathcal{P}}})$. More precisely define the character

$$\Psi_\mathbb{Q} = \text{“}\langle\chi_{cy}\rangle^{-1/2}\text{”} : G_\mathbb{Q} \to R^*$$

as follows: let $\chi_{cy} : G_\mathbb{Q} \twoheadrightarrow Gal(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_p^\times$ be the p-adic cyclotomic character and $\kappa : \mathbb{Z}_p^* \to 1 + p\mathbb{Z}_p$ the projection on principal units. For every $g \in G_\mathbb{Q}$ we put $\Psi_\mathbb{Q}(g) := \left\langle (\kappa \circ \chi_{cy}(g))^{-1/2} \right\rangle$ (as $p \neq 2$ every element of $\Gamma = 1+p\mathbb{Z}_p$ has a unique square root in Γ). Note that, writing $\chi_{cy,N} : G_\mathbb{Q} \twoheadrightarrow Gal(\mathbb{Q}(\mu_{Np^\infty})/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_{N,p}^*$, $\Psi_\mathbb{Q}(g)^{-2} = \langle\chi_{cy,N}(g)\rangle$. This follows by the fact that f_E (or f_E^0) has trivial character, i.e. the *character* of R is trivial (with the terminology of [**Hid86a**]). We can finally define

$$T(R) := \mathrm{Ta}_\infty(R) \otimes_R \Psi_\mathbb{Q}; \qquad T(\mathcal{P}) := T(R) \otimes_R R_{\overline{\mathcal{P}}}.$$

As $\Psi_\mathbb{Q} \equiv 1 \bmod (\varpi)$ and $V_p(E)$ is irreducible, the Eichler-Shimura relation (19), combined with the Chebotarev density theorem (and the definition of $\mathcal{P}$ as the kernel of (13)) gives us an isomorphism of $\mathbb{Q}_p[\mathfrak{G}]$-modules

$$T(\mathcal{P})_{k=2} := T(\mathcal{P}) \otimes R_{\overline{\mathcal{P}}}/(\varpi) \xrightarrow{\sim} V_p(E). \tag{20}$$

Furthermore, again by (19) (and the discussion above) the determinant of $T(\mathcal{P})$ is the p-adic cyclotomic character. In [**NP00**, Sec. (1.6)] it is shown how these properties imply the existence of an $R_{\overline{\mathcal{P}}}$-bilinear, alternating and $\mathfrak{G}$-equivariant form

$$\pi := \pi_{R_{\overline{\mathcal{P}}}} : T(\mathcal{P}) \otimes_{R_{\overline{\mathcal{P}}}} T(\mathcal{P}) \to R_{\overline{\mathcal{P}}}(1) := R_{\overline{\mathcal{P}}} \otimes T_p(\mathbb{Q}^*) \tag{21}$$

which induces an isomorphism of $R_{\overline{\mathcal{P}}}[\mathfrak{G}]$-modules

$$\mathrm{adj}(\pi) : T(\mathcal{P}) \xrightarrow{\sim} \mathrm{Hom}_{R_{\overline{\mathcal{P}}}}(T(\mathcal{P}), R_{\overline{\mathcal{P}}}(1)) =: T(\mathcal{P})^*(1).$$

(As remarked in [**Nek06**, Section 12.7.13.6], the geometric construction of (21) given in [**NP00**] was done earlier by Ohta; see the reference given in *loc. cit.*) Write 'mod ϖ' for the compositions $T(\mathcal{P}) \twoheadrightarrow T(\mathcal{P})_{k=2} \xrightarrow{\sim} V_p(E)$ and $R_{\overline{\mathcal{P}}} \twoheadrightarrow R_{\overline{\mathcal{P}}}/\varpi \xrightarrow{\sim} \mathbb{Q}_p$. Multiplying π by a unit in $R_{\overline{\mathcal{P}}}^*$ if necessary, we can assume, as we do from now on, that

$$\pi(x \otimes y) \bmod \varpi = (x \bmod \varpi, y \bmod \varpi)_W \tag{22}$$

for every $x, y \in T(\mathcal{P})$. This follows by the facts that π and $(\ ,\ )_W$ are perfect and alternating.

2.3.1. *Ramification at p.* Fix a prime $v \in S_f$ which divide p. Recall our fixed embedding $\rho_v : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p} \hookrightarrow G_\mathbb{Q}$ be the corresponding inertia and decomposition groups. As described in [**Nek06**, Sec. (12.7.5)], [**NP00**] or [**MT90**], the restriction $T(\mathcal{P})_v$ of $T(\mathcal{P})$ to $G_{\mathbb{Q}_p}$ is reducible. More precisely: write $\Psi = \Psi_{\mathbb{Q}_p}$ for the 'restriction' of $\Psi_\mathbb{Q}$ to $G_{\mathbb{Q}_p}$. There exists a short exact sequence of $R_{\overline{\mathcal{P}}}[G_{\mathbb{Q}_p}]$-modules

$$0 \to T(\mathcal{P})^+ \xrightarrow{i_v^+} T(\mathcal{P})_v \xrightarrow{p_v^-} T(\mathcal{P})^- \to 0, \tag{23}$$

with $T(\mathcal{P})^\pm$ free of rank one over $R_{\overline{\mathcal{P}}}$. Furthermore $G_{\mathbb{Q}_p}$ acts on $T(\mathcal{P})^+$ (resp., $T(\mathcal{P})^-$) via the character $\phi_R^{-1} \cdot \chi_{cy} \cdot \Psi^{-1}$ (resp., $\Psi \cdot \phi_R$), where

$$\phi_R : G_{\mathbb{Q}_p} \twoheadrightarrow G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p} \to R^*$$

is the unramified character which sends an arithmetic Frobenius $\mathrm{Fr}(p)$ at p to the p-th Hecke operator U_p. In other words, if we fix a splitting of $R_{\overline{\mathcal{P}}}$-modules $T(\mathcal{P}) \xrightarrow{\sim} T(\mathcal{P})^+ \oplus T(\mathcal{P})^- \xrightarrow{\sim} R_{\overline{\mathcal{P}}}^2$, the action of $G_{\mathbb{Q}_p}$ on $T(\mathcal{P})_v$ is described by the matrix

$$\begin{pmatrix} \chi_{cy} \cdot \Psi^{-1} \cdot \phi_R^{-1} & \star \\ 0 & \Psi \cdot \phi_R \end{pmatrix} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(R_{\overline{\mathcal{P}}}).$$

Putting $(T(\mathcal{P})^{\pm})_{k=2} := T(\mathcal{P})^{\pm} \otimes R_{\overline{\mathcal{P}}}/(\varpi)$, (20) extends to an isomorphism of short exact sequences of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules

(24)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & (T(\mathcal{P})^+)_{k=2} & \longrightarrow & (T(\mathcal{P})_{k=2})_v & \longrightarrow & (T(\mathcal{P})^-)_{k=2} & \longrightarrow & 0 \\ & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \\ 0 & \longrightarrow & V_p(E)^+ & \longrightarrow & V_p(E)_v & \longrightarrow & V_p(E)^- & \longrightarrow & 0, \end{array}$$

where the bottom row is the exact sequence (7) or (9). Write $T(\mathcal{P})^*(1)^{\pm} := \mathrm{Hom}_{R_{\overline{\mathcal{P}}}}(T(\mathcal{P})^{\mp}, R_{\overline{\mathcal{P}}})(1)$. As π is alternating, $\mathrm{adj}(\pi)$ induces an isomorphism of short exact sequences of $R_{\overline{\mathcal{P}}}[G_{\mathbb{Q}_p}]$-modules

(25)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\mathcal{P})^+ & \longrightarrow & T(\mathcal{P})_v & \longrightarrow & T(\mathcal{P})^- & \longrightarrow & 0 \\ & & \downarrow\sim & & \downarrow \mathrm{adj}(\pi) & & \downarrow\sim & & \\ 0 & \longrightarrow & T(\mathcal{P})^*(1)^+ & \longrightarrow & T(\mathcal{P})^*_v(1) & \longrightarrow & T(\mathcal{P})^*(1)^- & \longrightarrow & 0 \end{array} .$$

(Using the notations of [**Nek06**, §(6.8)], this means that $T(\mathcal{P})^+ \perp\!\!\perp_{\pi} T(\mathcal{P})^+$, after replacing $R_{\overline{\mathcal{P}}}$ with its dualizing complex $\omega_{R_{\overline{\mathcal{P}}}} := [Frac(R) \to Frac(R)/R_{\overline{\mathcal{P}}}]$ in (21)).

If $w \in S_f$ is another prime dividing p, then there exists $\sigma_w \in G_{\mathbb{Q}}$ and $\alpha_w \in G_{\mathbb{Q}_p}$ s.t. $\rho_w = \alpha_w \circ \rho_v \circ \sigma_w$. Putting $i_w^+ := \sigma_w^{-1} \circ i_v^+ \circ \alpha_w^{-1}$ and $p_w^- := \alpha_w \circ p_v^- \circ \sigma_w$, we obtain an exact sequence of $R_{\overline{\mathcal{P}}}[G_{\mathbb{Q}_p}]$-modules

$$0 \to T(\mathcal{P})^+ \xrightarrow{i_w^+} T(\mathcal{P})_w \xrightarrow{p_w^-} T(\mathcal{P})^- \to 0$$

and analogues of (24) and (25).

3. Nekovář duality

In this section we introduce Nekovář's Selmer complexes attached to the big ordinary representation $T(\mathcal{P})$, and the (abstract) Cassels-Tate pairing in this setting. Every notation or 'sign convention' regarding complexes which is not explicitly defined is as in [**Nek06**, Ch. 1].

3.1. Selmer complexes. Let $T(\mathcal{P})$ be the big Galois representation considered in the preceding section. $T(\mathcal{P}) \in {}^{ad}_{R[G_{K,S}]}\mathrm{Mod}$ is an *admissible* $R[G_{K,S}]$-module (as defined in [**Nek06**, Sec. (3.2)]) and we can define, for $G \in \{G_{K,S}; G_v,\ v \in S_f\}$, the complex [**Nek06**, Def. (3.4.1.1)]

$$C^{\bullet}_{\mathrm{cont}}(G, T(\mathcal{P})) := \varinjlim_{T_\alpha \in \mathcal{S}(T(\mathcal{P}))} C^{\bullet}_{\mathrm{cont}}(G, T_\alpha);$$

here $T_\alpha \in \mathcal{S}(T(\mathcal{P}))$ if $T_\alpha \subset T(\mathcal{P})$ is an $R[G]$-submodule such that *a)* T_α is a finitely generated R-module and *b)* the action of G is continuous for the profinite topology on G and the $\mathfrak{m}_R$-adic topology on T_α ($\mathfrak{m}_R$ is the maximal ideal of the local ring $R = R_E$). For $T_\alpha \in \mathcal{S}(T(\mathcal{P}))$, $C^{\bullet}_{\mathrm{cont}}(G, T_\alpha) = \varprojlim C^{\bullet}_{\mathrm{cont}}(G, T_\alpha/\mathfrak{m}_R^n T_\alpha)$ is the usual complex defined in degree n by the set $C^n_{\mathrm{cont}}(G, T_\alpha)$ of continuous maps $G^n \to T_\alpha$. (To be precise: if $v \in S_f$, then $T(\mathcal{P})_v \in {}^{ad}_{R[G_v]}\mathrm{Mod}$ is an admissible $R[G_v]$-module and $C^{\bullet}_{\mathrm{cont}}(G_v, T(\mathcal{P})) := C^{\bullet}_{\mathrm{cont}}(G_v, T(\mathcal{P})_v)$). We write also $C^{\bullet}_{\mathrm{cont}}(K_v, T(\mathcal{P}))$ for $C^{\bullet}_{\mathrm{cont}}(G_v, T(\mathcal{P}))$. As $T(\mathcal{P}) = T(R) \otimes R_{\overline{\mathcal{P}}}$ and $T(R)$ is finite over R, it follows from [**Nek06**, Prop. (3.4.4)] that the natural morphism of complexes

$$C^{\bullet}_{\mathrm{cont}}(G, T(R)) \otimes_R R_{\overline{\mathcal{P}}} \xrightarrow{\sim} C^{\bullet}_{\mathrm{cont}}(G, T(\mathcal{P}))$$

is an isomorphism and $C^{\bullet}_{\mathrm{cont}}(G, T(R))$ has the usual meaning.

We have, for every $v \in S_f$, a natural *restriction* map

$$res_v : C^\bullet_{\rm cont}(G_{K,S}, T(\mathcal{P})) \to C^\bullet_{\rm cont}(K_v, T(\mathcal{P}))$$

induced by the morphism of pairs (ρ_v^*, id). By the results of the preceding section, we also have the (admissible) $R[G_{\mathbb{Q}_p}]$-module $T(\mathcal{P})^\pm$ and we define as above the complex $C^\bullet_{\rm cont}(K_v, T(\mathcal{P})^\pm) := C^\bullet_{\rm cont}(G_v, T(\mathcal{P})^\pm)$, for every $v|p$.

In the same way we can consider the (continuous) $\mathbb{Q}_p[G_{K,S}]$-module $V_p(E)$ and the $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules $V_p(E)^\pm$. In this case $C^\bullet_{\rm cont}(G, V_p(E)) \xrightarrow{\sim} C^\bullet_{\rm cont}(G, {\rm Ta}_p(E)) \otimes \mathbb{Q}_p$ and $C^\bullet_{\rm cont}(K_v, V_p(E)^\pm)$ (for $v|p$) are the usual complexes of continuous cochains (for the p-adic topology).

Let $X \in \{T(\mathcal{P}), V_p(E)\}$. We define, as in [**Nek06**, 12.7.13], [**NP00**], local conditions for $v \in S_f$ by

$$U_v^+(X) := \begin{cases} C^\bullet_{\rm cont}(K_v, X^+) & \text{if} \quad v|p; \\ 0 & \text{if} \quad v \nmid p \end{cases}$$

and the corresponding Nekovář *Selmer complex*

$$\widetilde{C}^\bullet_f(G_{K,S}, X) := {\rm Cone}\left(C^\bullet_{\rm cont}(G_{K,S}, X) \oplus \bigoplus_{v \in S_f} U_v^+(X) \overset{res_{S_f} - i^+_{S_f}}{\to} \bigoplus_{v \in S_f} C^\bullet_{\rm cont}(K_v, X) \right)[-1].$$

Here $res_{S_f} := \bigoplus_{v \in S_f} res_v$, $i^+_{S_f} := \bigoplus_{v \in S_f} i^+_v$ and $i^+_v : U^+_v(X) \to C^\bullet_{\rm cont}(K_v, X)$ (by abuse of notation) is the map induced by the inclusion of G_v-modules $i^+_v : X^+ \hookrightarrow X_v$ (i.e. zero if $v \nmid p$). Write $R_X = R_{\overline{\mathcal{P}}}$ (resp. $\mathbb{Q}_p$) for $X = T(\mathcal{P})$ (resp. $X = V_p(E)$), $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X)$ for the image of $\widetilde{C}^\bullet_f(G_{K,S}, X)$ in the derived category $\mathcal{D}(R_X) := \mathcal{D}({}_{R_X}{\rm Mod})$ of complexes of R_X-modules and

$$\widetilde{H}^*_f(G_{K,S}, X) := H^*\left(\widetilde{C}^\bullet_f(G_{K,S}, X)\right)$$

for the cohomology of $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X)$. We collect in the following propositions some important facts we will use below.

PROPOSITION 3.1. *a)* $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \in \mathcal{D}^b_{ft}(R_X)$ *(i.e. has 'bounded' cohomology of finite type over* R_X*). Furthermore it is independent (up to isomorphism) on the choice of the finite set* S_f*. We write* $\widetilde{\mathbf{R}\Gamma}_f(K, X) = \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X)$ *and* $\widetilde{H}^*_f(K, X) := \widetilde{H}^*_f(G_{K,S}, X)$.

b) there exists an exact triangle in $\mathcal{D}^b(R_{\overline{\mathcal{P}}})$

$$\widetilde{\mathbf{R}\Gamma}_f(K, T(\mathcal{P})) \xrightarrow{\varpi} \widetilde{\mathbf{R}\Gamma}_f(K, T(\mathcal{P})) \to \widetilde{\mathbf{R}\Gamma}_f(K, V_p(E))$$

inducing short exact sequences

$$0 \to \widetilde{H}^q_f(K, T(\mathcal{P}))/(\varpi) \to \widetilde{H}^q_f(K, V_p(E)) \xrightarrow{i_{\mathcal{P}}} \widetilde{H}^{q+1}_f(K, T(\mathcal{P}))[\varpi] \to 0.$$

c) $\widetilde{H}^1_f(K, T(\mathcal{P}))$ *is a free* $R_{\overline{\mathcal{P}}}$*-module.*

PROOF. All these statements are special cases of [**Nek06**, Prop. (12.7.13.4)]. For future reference, we recall how to prove *b*). Let $G \in \{G_{K,S}, G_v,\ v \in S_f\}$ and $\dagger = \emptyset, +$. Combining the exact sequences (when defined) of complexes of $R_{\overline{\mathcal{P}}}$-modules [**Nek06**, Prop. (3.4.2)]

$$0 \to C^\bullet_{\rm cont}(G, T(\mathcal{P})^\dagger) \xrightarrow{\varpi} C^\bullet_{\rm cont}(G, T(\mathcal{P})^\dagger) \to C^\bullet_{\rm cont}(G, V_p(E)^\dagger) \to 0, \tag{26}$$

associated to the specialization maps $T(\mathcal{P})^\dagger \twoheadrightarrow \left(T(\mathcal{P})^\dagger\right)_{w=2} \xrightarrow{\sim} V_p(E)^\dagger$, we obtain a short exact sequence

$$0 \to \widetilde{C}^\bullet_f(G_{K,S}, T(\mathcal{P})) \xrightarrow{\varpi} \widetilde{C}^\bullet_f(G_{K,S}, T(\mathcal{P})) \to \widetilde{C}^\bullet_f(G_{K,S}, V_p(E)) \to 0, \tag{27}$$

which defines the exact triangle above. $i_{\mathcal{P}}$ is then the connecting morphism attached to (27). $\square$

Write $H^*(G,-) := H^*\left(C^\bullet_{\rm cont}(G,-)\right)$ and $C^\bullet_{\rm cont}(K_v,X^-) := C^\bullet_{\rm cont}(K_v,X)$ for $S_f \ni v \nmid p$. Noting that $\mathrm{Cone}\left(C^\bullet_{\rm cont}(K_v,X^+) \overset{-i_v^+}{\to} C^\bullet_{\rm cont}(K_v,X)\right) \overset{\sim}{\to} C^\bullet_{\rm cont}(K_v,X^-)$ in the derived category, we obtain an exact triangle in $\mathcal{D}^b(R_X)$

$$\bigoplus_{v\in S_f} C^\bullet_{\rm cont}(K_v,X^-)[-1] \to \widetilde{\mathbf{R}\Gamma}_f(K,X) \to C^\bullet_{\rm cont}(G_{K,S},X).$$

We note that $H^0(K_v,X^-)=0$ unless $v|p$, $X=V_p(E)$ and E/K_v has *split* multiplicative reduction. (For $X=V_p(E)$ this follows easily by the discussion in Sec. (1.2). The result for $X=T(\mathcal{P})$ follows easily from this and Sec. (2.3).) We then obtain in cohomology a short exact sequence of R_X-modules

$$0 \to \bigoplus_{v\in S_f^{sp}} H^0(K_v,X^-) \to \widetilde{H}^1_f(K,X) \to H^1_f(K,X) \to 0, \tag{28}$$

where $S_f^{sp} := \{v|p : E/K_v \text{ has split multiplicative reduction}\}$ and $H^1_f(K,X) \subset H^1(G_{K,S},X)$ is the Selmer group attached to the local conditions $i_v^+\left(H^1(U_v^+(X))\right) \subset H^1(K_v,X)$. Specializing (28) to $X=V_p(E)$, we obtain by Lemma (1.2) an exact sequence

$$0 \to \bigoplus_{v\in S_f^{sp}} \mathbb{Q}_p \overset{\iota}{\to} \widetilde{H}^1_f(K,V_p(E)) \to Sel_{\mathbb{Q}_p}(E;K) \to 0. \tag{29}$$

3.2. Class field theory. Let M be an R-module, considered as an admissible $R[G_{K,S}]$-module with trivial $G_{K,S}$-action. Define

$$\mathcal{K}_M := \mathrm{Cone}\left(\tau_{\geq 2}C^\bullet_{\rm cont}(G_{K,S},M(1)) \overset{res_{S_f}}{\to} \bigoplus_{v\in S_f} \tau_{\geq 2}C^\bullet_{\rm cont}(K_v,M(1))\right)[-1],$$

where $M(1) := M\otimes_{\mathbb{Z}_p} T_p(\mathbb{Q}^*)$ and $\tau_{\geq 2}X^\bullet$ is the good filtration of $X^\bullet$ in degree two [**Nek06**, page 33]. (Note that, for $v\in S_f$, $C^\bullet_{\rm cont}(K_v,M(1)) := C^\bullet_{\rm cont}(G_v,M(1)_v)$). By class field theory [**Nek06**, Sec. (5.4.1)] $H^q(\mathcal{K}_M)=0$ for every $q\neq 3$ and the sum of the invariant maps of local class field theory induces an isomorphism of M-modules

$$\underline{inv}_{S_f}(M) : H^3(\mathcal{K}_M) \overset{\sim}{\to} M,$$

which is functorial in M. We can describe $\underline{inv}_{S_f}$ explicitly as follows.

First of all, we have for every $v\in S_f$ an isomorphism $inv_v(M) : H^2(K_v,M(1)) \overset{\sim}{\to} M$, obtained as the composition $H^2(G_v,M(1)_v) \overset{\sim}{\to} H^2(G_v, M\otimes_{\mathbb{Z}_p} T_p(K_v^*)) \overset{\sim}{\to} M$. Here the first isomorphism is induced by $id\otimes\rho_v$, and the second is defined (taking limits) by the invariant map of local class field theory (as in [**Nek06**, §(5.2)]).

Let $\boldsymbol{x} = (x,(y_v)) \in \mathcal{K}^3_M$ be a 3-cocycle, for $x\in C^3_{\rm cont}(G_{K,S},M(1))$ and $(y_v) \in \bigoplus_{v\in S_f} C^2_{\rm cont}(K_v,M(1))$ (to be precise we should write $([y_v])$ for the second component in $\boldsymbol{x}$, where $[y_v]$ denotes the class of y_v modulo the image of $\delta : C^1_{\rm cont}(K_v,M(1)) \to C^2_{\rm cont}(K_v,M(1))$). Since $\boldsymbol{x}$ is a cocycle we have

$$0 = d_{\mathcal{K}_M}(\boldsymbol{x}) = \left(\delta(x), -\left(\delta(y_v) + res_v(x)\right)\right),$$

where δ is the differential in $C^\bullet_{\rm cont}(-,-)$. As $H^3(G_{K,S},M(1)) = 0$ [**Mil04**, Ch. I], there exists $\vartheta \in C^2_{\rm cont}(G_{K,S},M(1))$ such that $\delta(\vartheta)=x$, so $[\boldsymbol{x}] = [(0,(y_v+res_v(\vartheta))_v)] \in H^3(\mathcal{K}_M)$. We have

$$\underline{inv}_{S_f}(M)([\boldsymbol{x}]) = \sum_{v\in S_f} inv_v(M)\left([y_v+res_v(\vartheta)]\right).$$

The facts that this expression does not depend on the choice of ϑ and that $\underline{inv}_{S_f}$ is an isomorphism is essentially a restatement of the fundamental exact sequence of global class field theory (for more details, see [**Nek06**, Ch. 5], in particular the exact sequence (5.3.1.2)).

We can consider $\mathbb{Q}_p$ as an R-module, identifying it with the residue field $R_{\overline{\mathcal{P}}}/\varpi$ of $R_{\overline{\mathcal{P}}}$ (see Sec. (2.1)). By [**Nek06**, Prop. (3.5.10)], $C^\bullet_{\mathrm{cont}}(G,\mathbb{Q}_p(1))$ is then identified with $C^\bullet_{\mathrm{cont}}(G,R_{\overline{\mathcal{P}}}/\varpi(1))$. By the functoriality of $\underline{inv}_{S_f}$ we have

$$\left(\underline{inv}_{S_f}(R_{\overline{\mathcal{P}}})(x)\right) \bmod \varpi = \underline{inv}_{S_f}(\mathbb{Q}_p)(x \bmod \varpi) \tag{30}$$

for every $x \in \mathcal{K}_{R_{\overline{\mathcal{P}}}}$ (and 'mod ϖ' : $R_{\overline{\mathcal{P}}} \twoheadrightarrow R_{\overline{\mathcal{P}}}/\varpi \overset{\sim}{\to} \mathbb{Q}_p$).

We write from now on $\mathcal{K}$ for $\mathcal{K}_{R_{\overline{\mathcal{P}}}}$.

3.3. Products. The morphism $\pi : T(\mathcal{P})\otimes_{R_{\overline{\mathcal{P}}}} T(\mathcal{P}) \to R_{\overline{\mathcal{P}}}(1)$ induces, for $G \in \{G_{K,S}, G_v\}$, (truncated) cup-products

$$\cup_\pi : C^\bullet_{\mathrm{cont}}(G,T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} C^\bullet_{\mathrm{cont}}(G,T(\mathcal{P})) \to C^\bullet_{\mathrm{cont}}(G,R_{\overline{\mathcal{P}}}(1)) \overset{\tau_{\geq 2}}{\to} \tau_{\geq 2} C^\bullet_{\mathrm{cont}}(G,R_{\overline{\mathcal{P}}}(1)).$$

The first map is the composition of the cup product $C^\bullet_{\mathrm{cont}}(G,T(\mathcal{P})) \otimes C^\bullet_{\mathrm{cont}}(G,T(\mathcal{P})) \to C^\bullet_{\mathrm{cont}}(G,T(\mathcal{P}) \otimes T(\mathcal{P}))$ (defined by the usual formulas on cochains [**Nek06**, Sec. (3.4.5.1)]) with the map induced by π. When the context is clear, we write $\cup_\pi$ also for the usual (non-truncated) cup-product.

We will write $\widetilde{C}^\bullet_f(T(\mathcal{P})) := \widetilde{C}^\bullet_f(G_{K,S},T(\mathcal{P}))$ and $(x_n, x_n^+, x_{n-1}) \in \widetilde{C}^n_f(G_{K,S},T(\mathcal{P}))$ for an n-cochain, where $x_n \in C^n_{\mathrm{cont}}(G_{K,S},T(\mathcal{P}))$, $x_n^+ = (x^+_{n,v})_{v|p} \in \bigoplus_{v|p} C^n_{\mathrm{cont}}(K_v,T(\mathcal{P})^+)$ and $x_{n-1} = (x_{n-1,v})_{v\in S_f} \in \bigoplus_{v\in S_f} C^{n-1}_{\mathrm{cont}}(K_v,T(\mathcal{P}))$. Given $\alpha = (\alpha_v)_{v\in S_f}, \beta = (\beta_v)_{v\in S_f} \in \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(K_v,T(\mathcal{P}))$, we write

$$\alpha \cup_\pi \beta := \oplus_{v\in S_f} \alpha_v \cup_\pi \beta_v.$$

Let $r, s \in R$. A simple direct computation [**Nek06**, Prop. (1.3.2)] shows that the formula

$$\begin{aligned} &(x_n, x_n^+, x_{n-1}) \cup_{\pi,r} (y_m, y_m^+, y_{m-1}) := \\ &\Big(x_n \cup_\pi y_m,\ x_{n-1} \cup_\pi \left(r \cdot res_{S_f}(y_m) + (1-r)\cdot i^+_{S_f}(y^+_m)\right) \\ &\qquad +(-1)^n \left((1-r)\cdot res_{S_f}(x_n) + r \cdot i^+_{S_f}\left(x^+_n\right)\right) \cup_\pi y_{m-1}\Big) \end{aligned} \tag{31}$$

defines a morphism of complexes of $R_{\overline{\mathcal{P}}}$-modules

$$\cup_{\pi,r} : \widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \widetilde{C}^\bullet_f(T(\mathcal{P})) \to \mathcal{K}.$$

Moreover the formula $k_{r,s}\left((x_n, x_n^+, x_{n-1}) \otimes (y_m, y_m^+, y_{m-1})\right) = (0, (-1)^n (r-s)\cdot x_{n-1} \cup_\pi y_{m-1})$ defines a homotopy $k_{r,s} : \cup_{\pi,r} \rightsquigarrow \cup_{\pi,s}$.

3.4. Generalized Cassels-Tate pairings. Define $\mathscr{R} := \mathrm{Frac}(R_{\overline{\mathcal{P}}})$ and $\overline{R_{\overline{\mathcal{P}}}} := [R_{\overline{\mathcal{P}}} \overset{-i}{\to} \mathscr{R}]$, concentrated in degrees $[0,1]$. The morphism

$$v_{R_{\overline{\mathcal{P}}}} : \overline{R_{\overline{\mathcal{P}}}} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}} = [R_{\overline{\mathcal{P}}} \overset{(-i,-i)}{\to} \mathscr{R} \oplus \mathscr{R} \overset{(-\mathrm{id},\mathrm{id})}{\to} \mathscr{R}] \to \overline{R_{\overline{\mathcal{P}}}}$$

defined by the identity (resp. the projection on the first factor) in degree zero (resp. one) is a quasi-isomorphism. Write $\widetilde{C}^\bullet_f(T(\mathcal{P})) := \widetilde{C}^\bullet_f(G_{K,S},T(\mathcal{P}))$ and let $r \in R$. We define a morphism of complexes

$$\left(\widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \otimes_{R_{\overline{\mathcal{P}}}} \left(\widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \longrightarrow \mathcal{K} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}. \tag{32}$$

by the composition

$$\begin{aligned} &\left(\widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \otimes_{R_{\overline{\mathcal{P}}}} \left(\widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \overset{s_{23}}{\to} \left(\widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \widetilde{C}^\bullet_f(T(\mathcal{P}))\right) \otimes_{R_{\overline{\mathcal{P}}}} \left(\overline{R_{\overline{\mathcal{P}}}} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \to \\ &\overset{id\otimes v_{R_{\overline{\mathcal{P}}}}}{\to} \left(\widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \widetilde{C}^\bullet_f(T(\mathcal{P}))\right) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}} \overset{\cup_{\pi,r}\otimes id}{\to} \mathcal{K} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}, \end{aligned}$$

with $\cup_{\pi,r}$ as in preceding section and $s_{23}\left((a\otimes b)\otimes(c\otimes d)\right) := (-1)^{deg(b)deg(c)}\left((a\otimes c)\otimes(b\otimes d)\right)$. The cup-product (32) induces in cohomology a morphism of $R_{\overline{\mathcal{P}}}$-modules

$$\cup_{\pi,2,2} : H^2\left(\widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \otimes_{R_{\overline{\mathcal{P}}}} H^2\left(\widetilde{C}^\bullet_f(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \longrightarrow H^4\left(\mathcal{K} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right), \tag{33}$$

which is independent on the choice of $r \in R$.

Let Z be a complex of $R_{\overline{\mathcal{P}}}$-modules with cohomology of finite type over $R_{\overline{\mathcal{P}}}$. The cohomology sequence of the exact triangle in $\mathcal{D}(R_{\overline{\mathcal{P}}})$

$$Z \xrightarrow{i} Z \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{R} \to \left(Z \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right)[1]$$

splits into short exact sequences of $R_{\overline{\mathcal{P}}}$-modules

$$0 \to H^{q-1}(Z) \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{R}/R_{\overline{\mathcal{P}}} \to H^q\left(Z \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \to H^q(Z)_{R_{\overline{\mathcal{P}}}-Tor} \to 0, \tag{34}$$

where $M_{R_{\overline{\mathcal{P}}}-Tor} := \mathrm{Ker}\left(M \to M \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{R}\right)$. Taking $Z = \mathcal{K}$ and $q = 4$ we obtain from Section (0.4) an isomorphism

$$H^4(\mathcal{K} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}) \xrightarrow{\sim} H^3(\mathcal{K}) \otimes_{R_{\overline{\mathcal{P}}}} \mathscr{R}/R_{\overline{\mathcal{P}}} \xrightarrow{\sim} \mathscr{R}/R_{\overline{\mathcal{P}}}, \tag{35}$$

where the last map is given by $\underline{inv}_{S_f} \otimes id$.

Note that every term in (34) is a torsion $R_{\overline{\mathcal{P}}}$-module and the first is $\overline{\mathcal{P}}$-divisible. Taking $Z = \widetilde{C}^{\bullet}_f(T(\mathcal{P}))$ and $q = 2$, it follows that the cup product in (33) factorizes through the projection $H^2\left(Z \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right) \to H^2(Z)_{R_{\overline{\mathcal{P}}}-Tor}$. Composing $\cup_{\pi,2,2}$ with (35) we then obtain an $R_{\overline{\mathcal{P}}}$-bilinear form

$$\cup_{CT} : \widetilde{H}^2_f(K, T(\mathcal{P}))_{R_{\overline{\mathcal{P}}}-Tor} \times \widetilde{H}^2_f(K, T(\mathcal{P}))_{R_{\overline{\mathcal{P}}}-Tor} \longrightarrow \mathscr{R}/R_{\overline{\mathcal{P}}}. \tag{36}$$

We have the following fundamental:

THEOREM 3.2. *$\cup_{CT}$ is non-degenerate and alternating.*

PROOF. This is a special case of [**Nek06**, Prop. (12.7.13.4)] or [**Nek06**, Th. (10.4.4)]. □

3.5. Behaviour under Galois conjugation. We assume in this paragraph that $K/\mathbb{Q}$ is a Galois extension. Let S^0_f be the set of rational primes dividing $p \cdot N_E \cdot D_K$. Fix for every prime $l \in S^0_f$ an embedding $\rho_l : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_l}$, inducing $\rho^*_l : G_{\mathbb{Q}_l} \hookrightarrow G_{\mathbb{Q}}$. We also fix elements $\sigma_{j,l} \in G_{\mathbb{Q}}$ ($\sigma_{1,l} := id$) which represent the coset space $G_K \backslash G_{\mathbb{Q}} / \rho^*_l(G_{\mathbb{Q}_l})$, and assume that $\{\rho_l \circ \sigma_j^{-1}\}_j = \{\rho_v\}_{v|l}$ (where ρ_v is the embedding fixed at the beginning of this note).

As described in [**Nek06**, §(8.8)], $Gal(K/\mathbb{Q})$ acts on $\widetilde{H}^q_f(K, X)$, for $X \in \{T(\mathcal{P}), V_p(E)\}$ and $q \geq 0$. More precisely: for every $g \in G_{\mathbb{Q}}$, we can define a morphism of complexes

$$\mathrm{Ad}_f(g) := (\mathrm{Ad}(g), \mathrm{Ad}^+(g), F(g); m_g) : \widetilde{C}^{\bullet}_f(G_{K,S}, X) \to \widetilde{C}^{\bullet}_f(G_{K,S}, X)$$

as follows. First of all, $\mathrm{Ad}(g) : C^{\bullet}_{\mathrm{cont}}(G_{K,S}, X) \to C^{\bullet}_{\mathrm{cont}}(G_{K,S}, X)$ denotes the usual action of g by Galois conjugation, i.e. the map induced by the morphism of pairs $x \mapsto g(x)$ ($x \in X$), $\sigma \mapsto g^{-1}\sigma g$ ($\sigma \in G_K$) between (G_K, X) and itself. In a similar way, $\mathrm{Ad}^+(g)$ (resp., $F(g) = (F(g)_l)_{l \in S^0_f}$) denotes the action of g by Galois conjugation on the 'semilocal complex' $\bigoplus_{v|p} C^{\bullet}_{\mathrm{cont}}(K_v, X^+)$ (resp., $\bigoplus_{l \in S^0_f} \bigoplus_{v|l} C^{\bullet}_{\mathrm{cont}}(K_v, X_v)$). We have $F(g) \circ i^+_{S_f} = i^+_{S_f} \circ \mathrm{Ad}^+(g)$ and there exists a homotopy $m_g = m_g(X) : res_{S_f} \circ \mathrm{Ad}(g) \rightsquigarrow F(g) \circ res_{S_f}$, which is functorial in X (see [**Nek06**, §(8.1.7.3)] for an explicit description of the homotopy m_g). It follows that the formula

$$\mathrm{Ad}_f(g)(x_n, x^+_n, x_{n-1}) := (\mathrm{Ad}(\sigma)(x_n), \mathrm{Ad}^+(g)(x^+_n), F(g)(x_{n-1}) + m_g(x_n)) \tag{37}$$

defines a morphism of complexes $\widetilde{C}^{\bullet}_f(G_{K,S}, X) \to \widetilde{C}^{\bullet}_f(G_{K,S}, X)$. By [**Nek06**, Lemma (8.6.4.4)] this map induces in cohomology the action of $Gal(K/\mathbb{Q})$ on $\widetilde{H}^q_f(K, X)$ alluded to above. We denote by x^g or $g(x)$ the action of $g \in Gal(K/\mathbb{Q})$ on $x \in \widetilde{H}^q_f(K, X)$. In [**Nek06**, Prop. (8.8.9)] (or *loc.cit.*, formula (10.3.2.2)) it is proved that $\cup_{CT}$ is $Gal(K/\mathbb{Q})$-equivariant, i.e.

$$g(x) \cup_{CT} g(y) = x \cup_{CT} y \tag{38}$$

for every $x, y \in \widetilde{H}^2_f(K, T(\mathcal{P}))_{R_{\overline{\mathcal{P}}}-Tor}$ and $g \in Gal(K/\mathbb{Q})$. (This follows essentially by the Galois invariance of the local invariants). For more details on the constructions above, we refer the reader to [**Nek06**, Ch. VIII], especially to paragraphs (8.1.7.3), (8.6) and (8.8).

4. The p-adic weight pairing

In this Section we apply the constructions recalled above to define the *p-adic weight pairing* $\langle -,-\rangle_{K,p}^{\mathrm{Nek}}$ on the extended Mordell-Weil group $E^\dagger(K)$ of E/K. For every prime $v|p$ of K at which E has *split* multiplicative reduction, we have a 'Tate period' $q_v \in E^\dagger(K) - E(K)$. Given $P \in E(K)$, we can compute $\langle q_v, P\rangle_{K,p}^{\mathrm{Nek}}$ explicitly in terms of the formal group logarithm on $E/\mathbb{Q}$ (cfr. Cor. (4.6)). As explained in the introduction (see also Sec. (7)), this computation is the key for relating the algebraic constructions of Nekovář to the analytic results of Bertolini and Darmon.

4.1. The extended Mordell-Weil group. Let $S_f^{sp} \neq \emptyset$ and let $K_p' := \prod_{v \in S_f^{sp}} K_v$. We write again (by abuse of notation)

$$\Phi_{Tate} : \left(K_p'\right)^* \to \bigoplus_{v \in S_f^{sp}} E(K_v)$$

for the direct sum of the Tate parametrisations (8). Following [**MTT86**] and [**BD96**], we define the *extended Mordell-Weil group*

$$E^\dagger(K) := \left\{(P,\widetilde{P}) \mid P \in E(K),\ \widetilde{P} \in \left(K_p'\right)^* \ \textit{and}\ \Phi_{Tate}(\widetilde{P}) = (\rho_v(P))_{v \in S_f^{sp}}\right\}.$$

Given $v \in S_f^{sp}$, we write $q_v := (0,(1,\ldots,q_E,\ldots,1)) \in E^\dagger(K)$ (with q_E as v-component). We have a short exact sequence

$$0 \to \bigoplus_{v \in S_f^{sp}} \mathbb{Z} \to E^\dagger(K) \to E(K) \to 0, \tag{39}$$

where the first map sends the v-th generator to q_v and the second is projection. If $S_f^{sp} = \emptyset$, define $E^\dagger(K) := E(K)$.

We have a natural map

$$i_E^\dagger : E^\dagger(K) \to \widetilde{H}_f^1(K, V_p(E)), \tag{40}$$

defined in the following manner. Let $(P,\widetilde{P}) \in E^\dagger(K)$, with $\widetilde{P} = (\widetilde{P}_v)_{v \in S_f^{sp}} \in \left(K_p'\right)^*$. Since $res_v(\gamma_P) = i_v^+(\gamma_{\widetilde{P}_v})$ (see Remark (1.3)), for every representatives $\gamma_P^0 \in C^1_{\mathrm{cont}}(G_{K,S}, V_p(E))$ and $\gamma^0_{\widetilde{P}_v} \in C^1_{\mathrm{cont}}(K_v, \mathbb{Q}_p(1))$ of γ_P and $\gamma_{\widetilde{P}_v}$ respectively, there exists a unique $\varepsilon_v^0 \in C^0_{\mathrm{cont}}(K_v, V_p(E))$ such that $res_v(\gamma_P^0) = i_v^+(\gamma^0_{\widetilde{P}_v}) - \delta(\varepsilon_v^0)$, where δ is the differential in $C^\bullet_{\mathrm{cont}}(K_v, V_p(E))$ (ε_v^0 is unique since $H^0(K_v, V_p(E)) = 0$, by [**Sil86**, page 118]). In the same way, for every $v \notin S_f^{sp}$, there exists a unique $\gamma_v \in H^1(U_v^+(V_p(E)))$ s.t. $i_v^+(\gamma_v) = res_v(\gamma_P)$. For $v|p$ (resp. $v \nmid p$) this follows from $H^0(K_v, V_p(E)^-) = 0$ (resp. $U_v^+(V_p(E)) := 0$) and Lemma (1.2). In particular, for every representative $\gamma_v^0 \in U_v^+(V_p(E))$ of γ_v, we can find a unique $\varepsilon_v^0 \in C^0_{\mathrm{cont}}(K_v, V_p(E))$ such that $res_v(\gamma_P^0) = i_v^+(\gamma_v^0) - \delta(\varepsilon_v^0)$. Recalling the definition of the differential in the Selmer complex $\widetilde{C}_f^\bullet(G_{K,S}, V_p(E))$,

$$(P,\widetilde{P})^0 := \left(\gamma_P^0, (\gamma^0_{\widetilde{P}_v})_{v \in S_f^{sp}} + (\gamma_v^0)_{v \notin S_f^{sp}}, (\varepsilon_v^0)_{v \in S_f}\right) \in \widetilde{C}_f^1(G_{K,S}, V_p(E))$$

is a 1-cocycle. Furthermore it is easily seen that a different choice $\gamma_P^1 = \gamma_P^0 + \delta(\vartheta_P)$, $\gamma^1_{\widetilde{P}_v} = \gamma^0_{\widetilde{P}_v} + \delta(\vartheta_v)$ and $\gamma_v^1 = \gamma_v^0 + \delta(\vartheta_v)$ of representatives leads to the 1-cocycle $(P,\widetilde{P})^1 = (P,\widetilde{P})^0 + d_{\widetilde{C}_f^\bullet}(\vartheta_P, (\vartheta_v), 0)$. We can then define in (40) $i_E^\dagger(P,\widetilde{P})$ as the image in cohomology of $(P,\widetilde{P})^0$.

Lemma 4.1. *Let $i_E^\dagger : E^\dagger(K) \otimes \mathbb{Q}_p \to \widetilde{H}_f^1(K, V_p(E))$ be the map induced by* (40)*. Then $i_E^\dagger$ is injective and is an isomorphism provided that $\text{Ш}(E/K)_{p^\infty}$ is finite.*

Proof. This follows easily from the exact sequences (29) and (6). ☐

4.2. Definition of $\langle\ ,\ \rangle_{K,p}^{\mathrm{Nek}}$. Consider the following composition:

$$\phi_E : E^\dagger(K) \xrightarrow{i_E^\dagger} \widetilde{H}^1_f(K, V_p(E)) \xrightarrow{i_{\mathcal{P}}} \widetilde{H}^2_f(K, T(\mathcal{P}))[\varpi],$$

where $i_{\mathcal{P}}$ is defined in Proposition (3.1). Given $x = [s/\varpi] \in \mathscr{R}/R_{\overline{\mathcal{P}}}[\varpi]$, we write $\varpi \cdot (x \bmod 1) \in \mathbb{Q}_p$ for the image of $s \bmod \varpi \in R_{\overline{\mathcal{P}}}/\varpi$ under the isomorphism $R_{\overline{\mathcal{P}}}/\varpi \xrightarrow{\sim} \mathbb{Q}_p$ (cfr. Sec. (2.1)). Define the *p-adic weight pairing*

$$\langle -, - \rangle_{K,p}^{\mathrm{Nek}} : E^\dagger(K) \times E^\dagger(K) \longrightarrow \mathbb{Q}_p$$

by the formula:

$$\log_p(\gamma)^{-1} \cdot \langle x, y \rangle_{K,p}^{\mathrm{Nek}} := \varpi \cdot (\phi_E(x) \cup_{CT} \phi_E(y) \bmod 1),$$

for every $x, y \in E^\dagger(K)$. The multiplicative factor $\log_p(\gamma)$ serves the purpose of removing the dependence of $\langle -, - \rangle_{K,p}^{\mathrm{Nek}}$ on the choice of a topological generator $\gamma \in \Gamma$. We use the same notation for the extension of $\langle -, - \rangle_{K,p}^{\mathrm{Nek}}$ to $E^\dagger(K) \otimes \mathbb{Q}_p$.

PROPOSITION 4.2. *a) $\langle x, x \rangle_{K,p}^{\mathrm{Nek}} = 0$ for every $x \in E^\dagger(K)$. b) Suppose that $Ш(E/K)_{p^\infty}$ is finite. Then $\langle -, - \rangle_{K,p}^{\mathrm{Nek}}$ is non-degenerate (on $E^\dagger(K) \otimes \mathbb{Q}_p$) if and only if $\widetilde{H}^1_f(K, T(\mathcal{P})) = 0$ and $\widetilde{H}^2_f(K, T(\mathcal{P}))$ is a semi-simple $R_{\overline{\mathcal{P}}}$-module.*

PROOF. *a*) Follows from the corresponding property of $\cup_{CT}$ (Theorem (3.2)). *b*) Write for simplicity $N := \widetilde{H}^2_f(K, T(\mathcal{P}))_{R_{\overline{\mathcal{P}}}-Tor}$. By the structure theorem for finite modules over discrete valuation rings, we have an isomorphism of $R_{\overline{\mathcal{P}}}$-modules $N \xrightarrow{\sim} \bigoplus_{j=0}^n R_{\overline{\mathcal{P}}}/(\varpi)^{e_j}$, for integers $1 \leq e_1 \leq \cdots \leq e_n$. Since $\cup_{CT}$ is non-degenerate by Theorem (3.2), it follows easily that the right (or left) kernel of the restriction $\cup'_{CT}$ of $\cup_{CT}$ to $N[\varpi] \times N[\varpi]$ is $\varpi \cdot N \cap N[\varpi]$. In particular $\cup'_{CT}$ is non-degenerate if and only if $e_j = 0$ for every $e_j > 1$, i.e. if and only if N is semi-simple over $R_{\overline{\mathcal{P}}}$. The claim in *b*) follows combining this observation, Lemma (4.1) and Proposition (3.1). □

4.3. Behaviour under Galois conjugation. We assume in this Section that $K/\mathbb{Q}$ is Galois. The notations are those introduced in Sec. (3.5).

Let $S_f^{sp} \neq \emptyset$ and write $\rho := \rho_p$, $\sigma_j := \sigma_{j,p}$, $\rho_j := \rho_p \circ \sigma_j^{-1}$ and $K_j := \rho_j(K)$ $(= K_i)$. As $K/\mathbb{Q}$ is Galois, this implies that $S_f^{sp} = \{v|p\}$ and $K_p := K \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} K'_p$, under the $\mathbb{Q}_p$-linear map which sends $x \otimes 1$ to $(\rho_j(x))_j$. We consider on K'_p the $Gal(K/\mathbb{Q})$-action coming from the diagonal action on K_p. As $G_K \backslash G_{\mathbb{Q}} / \rho^*(G_{\mathbb{Q}_p})$ represents all the prime $v|p$, we have, for every j, $g^{-1} \cdot \sigma_j = u_j \cdot \sigma_{g(j)} \cdot \rho^*(\overline{g}_j)$, where $u_j \in G_K$ and $\overline{g}_j \in G_{\mathbb{Q}_p}$. This 'decomposition' is unique if we require (as we do) that $\overline{g}_j$ belongs to a fixed set of representatives of $Gal(K_j/\mathbb{Q}_p)$. Then $g(y) = (g(y)_j)$, with

$$g(y)_j = \overline{g}_j^{-1}\left(y_{g(j)}\right).$$

We write $\alpha_j(g) = \alpha_p$ (resp., $\alpha_j(g) = 1$) if $\chi_{un}(\overline{g}_j) = -1$ (resp., $\chi_{un}(g) = +1$), where $\chi_{un} : G_{\mathbb{Q}_p} \to \{\pm 1\}$ is the quadratic unramified character. Then the formula

$$(P, (y_j))^g := (g(P), (g(y)_j^{\alpha_j(g)}))$$

defines an action of $Gal(K/\mathbb{Q})$ on $E^\dagger(K)$. The twist in the Galois action is forced by the fact that Φ_{Tate} is not defined over $\mathbb{Q}_p$ when $\alpha_p = -1$. If $S_f^{sp} = \emptyset$, then $E^\dagger(K) = E(K)$ and we consider the natural $Gal(K/\mathbb{Q})$-action on $E^\dagger(K)$.

LEMMA 4.3. *$i_E^\dagger(x^g) = i_E^\dagger(x)^g$ for every $x \in E^\dagger(K)$ and $g \in Gal(K/\mathbb{Q})$.*

PROOF. When $S_f^{sp} = \emptyset$, we have an isomorphism of $\mathbb{Q}_p[Gal(K/\mathbb{Q})]$-modules $\iota : \widetilde{H}^1_f(K, V_p(E)) \xrightarrow{\sim} Sel_{\mathbb{Q}_p}(E; K)$ (cfr. (29)). Then $i_E^\dagger$ is the composition of ι^{-1} with the Kummer map $E(K) \to Sel_{\mathbb{Q}_p}(E; k)$, which is $Gal(K/\mathbb{Q})$-equivariant. Assume now $S_f^{sp} = \{v|p\}$.

In general, given $(x_j) \in \bigoplus_j C^\bullet_{\mathrm{cont}}(K_j, V_p(E)^+)$ (and with the notations of Sec. (3.5)), the map $\mathrm{Ad}^+(g)$ sends $x_{g(j)}$ (on the $g(j)$-component) to $\mathrm{Ad}(\overline{g}_j^{-1})(x_{g(j)}) \in C^\bullet_{\mathrm{cont}}(K_j, V_p(E)^+)$ (on the j-component), for any j (see [**Nek06**, §(8.1.7.3)]). Let $x = (P, (y_j))$ and write $x_0 := (\gamma_P^0, (\gamma_j^0), \varepsilon) \in \widetilde{C}_f^1(G_{K,S}, V_p(E))$ for a representative of $i_E^\dagger(x)$ (here $\gamma_j^0 := \gamma_{y_j}^0$ and we use the notations following (40) in Sec. (4.1)). We recall that, once we have fixed γ_P^0 and (γ_j^0), ε is uniquely determined by the requirement that x_0 is a cocycle.

Assume first that $\alpha_p = +1$, so that $V_p(E)^+ = \mathbb{Q}_p(1)$ as $G_{\mathbb{Q}_p}$-modules. By the definition of $i_E^\dagger$ and the fact that, in cohomology, $\gamma_{h(*)} = \mathrm{Ad}(h)(\gamma_*)$ for the (local and global) Kummer maps, we see that $i_E^\dagger(x^g)$ is represented by

$$x_0^g = (\mathrm{Ad}(g)(\gamma_P^0), \mathrm{Ad}^+(g)(\gamma_{y_j}^0), \varepsilon^g), \tag{41}$$

for an $\varepsilon^g = (\varepsilon_v^g)_{v \in S_f} \in \bigoplus_{v \in S_f} C^0_{\mathrm{cont}}(K_v, V_p(E))$. Since x_0^g is a cocycle, we must have

$$\begin{aligned}
\delta(\epsilon^g) &= i_{S_f}^+ \circ \mathrm{Ad}^+(g)((\gamma_j^0)) - res_{S_f} \circ \mathrm{Ad}(g)(\gamma_P^0) \\
&= F(g)\left(i_{S_f}^+(\gamma_j^0) - res_{S_f}(\gamma_P^0)\right) + m_g \circ \delta(\gamma_P^0) + \delta \circ m_g(\gamma_P^0) \\
&= \delta\left(F(g)(\varepsilon) + m_g(\gamma_P^0)\right).
\end{aligned}$$

As remarked above, this implies that $\varepsilon^g = F(g)(\varepsilon) + m_g(\gamma_P^0)$, so $i_E^\dagger(x^g) = [x_0^g] = [\mathrm{Ad}_f(g)(x_0)] = i_E^\dagger(x)^g$.

Suppose now that $\alpha_p = -1$. Then $V_p(E)^+ = \mathbb{Q}_p(1) \otimes \chi_{un} \in {}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}\mathrm{Mod}$, so

$$\overline{g}_j^{-1}\left(\gamma_{(y_{g(j)})}\right) = \chi_{un}(\overline{g}_j) \cdot \gamma_{(\overline{g}_j^{-1}(y_{g(j)}))} = \gamma_{\left(g(y)_j^{\alpha_j(g)}\right)} \in H^1(K_j, \mathbb{Q}_p(1)).$$

In other words, we can take again (41) as a representative of $i_E^\dagger(x^g)$, and the above argument works. □

As a corollary we obtain the following

Proposition 4.4. *$\langle x^g, y^g\rangle_{K,p}^{\mathrm{Nek}} = \langle x, y\rangle_{K,p}^{\mathrm{Nek}}$ for every $x, y \in E^\dagger(K)$ and $g \in Gal(K/\mathbb{Q})$.*

Proof. By the definition of $\langle\ ,\ \rangle_{K,p}^{\mathrm{Nek}}$, formula (38) and the preceding Lemma, it is sufficient to note that $i_{\mathcal{P}} : \widetilde{H}_f^1(K, V_p(E)) \to \widetilde{H}_f^2(K, T(\mathcal{P}))$ is Galois equivariant. This follows from the definition of $i_{\mathcal{P}}$ (in the proof of Prop. (3.1)) and the functoriality of $\mathrm{Ad}(g)$, $\mathrm{Ad}^+(g)$, $F(g)$ and m_g (cfr. formula (37)). □

4.4. Height computations in the exceptional case. We assume in this section that $S_f^{sp} \neq \emptyset$, i.e. that E/K_v has split multiplicative reduction at a prime v dividing p. We also fix such a prime $v|p$.

We identify as usual $R_{\overline{\mathcal{P}}}/\varpi \xrightarrow{\sim} \mathbb{Q}$ (cfr. Sec. (2.1)). For every $\wp \in R_{\overline{\mathcal{P}}}$, write $\wp(0) = \wp \bmod \varpi \in \mathbb{Q}_p$ and $d\wp/d\varpi := \left(\varpi^{-1} \cdot (\wp - \wp(0))\right) \bmod \varpi \in \mathbb{Q}_p$. Let us define the morphism

$$\chi_E^{wt} := (\phi_R \bmod \varpi) \cdot \frac{d}{d\varpi}(\Psi \cdot \phi_R) \in \mathrm{Hom}_{cts}\left(G_{\mathbb{Q}_p}, \mathbb{Q}_p\right).$$

The additivity of χ_E^{wt} follows by: $\phi_R^2 \bmod \varpi = 1$ (since $\phi_R(\mathrm{Fr}(p)) = U_p$ and $U_p \bmod \varpi = \alpha_p = \pm 1$) and $\Psi(g) \bmod \varpi = 1$. We have the following

Theorem 4.5. *For every $(P, \widetilde{P}) \in E^\dagger(K)$ and $v \in S_f^{sp}$, we have*

$$\left\langle q_v, (P, \widetilde{P})\right\rangle_{K,p}^{\mathrm{Nek}} = -\log_p(\gamma) \cdot \chi_E^{wt}\left(\mathrm{rec}_p(N_{K_v/\mathbb{Q}_p}(\widetilde{P}_v))\right),$$

where $\widetilde{P}_v$ is the v-component of $\widetilde{P}$ and $\mathrm{rec}_p : \mathbb{Q}_p^* \to G_{\mathbb{Q}_p}^{ab}$ is the reciprocity map.

Before beginning the proof, we give two corollaries. For the first, write $\log_{q_E} : \mathbb{G}_m(\mathbb{Q}_p) \to \mathbb{G}_a(\mathbb{Q}_p)$ for the branch of the p-adic logarithm which vanishes at $q_E \in p\mathbb{Z}_p$.

Corollary 4.6. *For every $(P, \widetilde{P}) \in E^\dagger(K)$ and $v \in S_f^{sp}$ we have*

$$\left\langle q_v, (P, \widetilde{P})\right\rangle_{K,p}^{\mathrm{Nek}} = \frac{1}{2} \cdot \log_{q_E}\left(N_{K_v/\mathbb{Q}_p}(\widetilde{P}_v)\right).$$

In particular $\langle q_v, q_w\rangle_{K,p}^{\mathrm{Nek}} = 0$ for every $v, w \in S_f^{sp}$.

PROOF. Let $\mathrm{ord}_v : K_v^* \twoheadrightarrow \mathbb{Z}$ be the normalized valuation attached to the prime v and let $\mathcal{O}_v$ be the ring of integers in K_v. Put $Q := \mathrm{ord}_v(q_E) \cdot P$, $\widetilde{Q}_v^* := q_E^{-\mathrm{ord}_v(\widetilde{P}_v)} \widetilde{P}_v^{\mathrm{ord}_v(q_E)} \in \mathcal{O}_v^*$ and $\widetilde{Q}_w^* := \widetilde{P}_w^{\mathrm{ord}_v(q_E)}$ for every $v \neq w \in S_f^{sp}$. Then $\mathrm{ord}_v(q_E) \cdot (P, \widetilde{P}) = (Q, \widetilde{Q}^*) + \mathrm{ord}_v(\widetilde{P}_v) \cdot q_v \in E^\dagger(K)$. As $\langle q_v, q_v \rangle_{K,p}^{\mathrm{Nek}} = 0$ (Prop. (4.2)) we have

$$\mathrm{ord}_v(q_E) \cdot \left\langle q_v, (P, \widetilde{P}) \right\rangle_{K,p}^{\mathrm{Nek}} = \left\langle q_v, (Q, \widetilde{Q}^*) \right\rangle_{K,p}^{\mathrm{Nek}}. \tag{42}$$

Writing $u_v := N_{K_v/\mathbb{Q}_p}(\widetilde{Q}_v^*) \in \mathbb{Z}_p^*$, it follows from Prop. (1.1) that $\mathrm{rec}_p(u_v) \overset{\sim}{\mapsto} (1, u_v) \in G_{\mathbb{Q}_p}^{un} \times \mathbb{Z}_p^*$, where we identify $G_{\mathbb{Q}}^{ab} \overset{\sim}{\to} G_{\mathbb{Q}}^{un} \times \mathbb{Z}_p^*$ under the p-adic cyclotomic character χ_{cy} on the "second component". In particular we have $\phi_R(\mathrm{rec}_p(u_v)) = 1$ and $\Psi(\mathrm{rec}_p(u_v)) := \left\langle (\kappa \circ \chi_{cy}(\mathrm{rec}_p(u_v)))^{-1/2} \right\rangle = \left\langle \kappa(u_v)^{-1/2} \right\rangle$ ($\kappa : \mathbb{Z}_p^* \twoheadrightarrow \Gamma$ is the projection on principal units). We then obtain from the preceding theorem

$$\left\langle q_v, (Q, \widetilde{Q}^*) \right\rangle_{K,p}^{\mathrm{Nek}} = -\log_p(\gamma) \cdot \frac{d}{d\varpi} \left(\left\langle \kappa(u_v)^{-1/2} \right\rangle \right) = \frac{1}{2} \cdot \log_p(u_v). \tag{43}$$

The second equality follows from the fact: $d\wp/d\varpi = \log_p(\wp)/\log_p(\gamma)$ for every $\wp \in \Gamma$. (This can be easily proved noting that $\ell_\gamma(-) := \frac{\log_p(-)}{\log_p(\gamma)}$ gives an isomorphism of Γ to the additive group $\mathbb{Z}_p$, with inverse $z \mapsto \gamma^z$.) Combining (42) with (43) we see that $2 \cdot \langle q_v, P \rangle_{K,p}^{\mathrm{Nek}}$ equals

$$\log_p(N_{K_v/\mathbb{Q}_p}(\widetilde{P}_v)) - \frac{\mathrm{ord}_v(\widetilde{P}_v)}{\mathrm{ord}_v(q_E)} \cdot [K_v : \mathbb{Q}_p] \cdot \log_p(q_E) = \log_p(N_{K_v/\mathbb{Q}_p}(\widetilde{P}_v)) - \frac{\mathrm{ord}_p(N_{K_v/\mathbb{Q}_p}(\widetilde{P}_v))}{\mathrm{ord}_p(q_E)} \cdot \log_p(q_E),$$

as was to be shown. ☐

It follows that $\left\langle q_v, (P, \widetilde{P}) \right\rangle_{K,p}^{\mathrm{Nek}}$ does not depend on the choice of $\widetilde{P} \in (K_p')^*$ such that $\Phi_{Tate}(\widetilde{P}) = (\rho_v(P))$. We then write simply $\langle q_v, P \rangle_{K,p}^{\mathrm{Nek}}$ for $\left\langle q_v, (P, \widetilde{P}) \right\rangle_{K,p}^{\mathrm{Nek}}$ from now on.

As another interesting corollary of Prop. (4.5) we can recognize from $\langle q_v, q_v \rangle_{K,p}^{wt} = 0$ the well-known formula of Greenberg-Stevens [**GS93**], relating the derivative of the Hecke operator U_p to the L-invariant of $E/\mathbb{Q}_p$, defined by

$$\mathscr{L}_E := \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}.$$

COROLLARY 4.7. $-2\alpha_p \cdot \alpha_p'(2) = \mathscr{L}_E$.

PROOF. Let $v \in S_f^{sp}$. Write $n_v := [K_v : \mathbb{Q}_p]$ and $q_E = p^n \cdot u$, with $u \in \mathbb{Z}_p^*$. We have $\mathrm{rec}_p(N_{K_v/\mathbb{Q}_p}(q_E)) = (\mathrm{Fr}(p)^{-n \cdot n_v}, u^{n_v}) \in G_{\mathbb{Q}_p}^{un} \times \mathbb{Z}_p^*$, where $\mathrm{Fr}(p)$ is an arithmetic Frobenius in $G_{\mathbb{Q}_p}^{un}$. As $\phi_R(\mathrm{Fr}(p)) = U_p$ and $U_p(0) = \alpha_p$, we obtain (as in the preceding proof)

$$0 = \langle q_v, q_v \rangle_{K,p}^{\mathrm{Nek}} = \alpha_p^{-n \cdot n_v} \cdot \frac{d}{d\varpi} \left(U_p^{-n \cdot n_v} \cdot \left\langle \kappa(u)^{-n_v/2} \right\rangle \right). \tag{44}$$

Since $\log_p(\gamma) \cdot \frac{dU_p}{d\varpi} = \alpha_p'(2)$ (as follows easily looking at the power series expansion of $\gamma^{w-2} - 1$), a simple calculation using the 'product formula' for the derivative in (44) concludes the proof. ☐

We now begin the proof of Th. (4.5). Write

$$\chi_v^{wt} := \mathrm{Res}_{K_v/\mathbb{Q}_p}(\chi_E^{wt}) = \frac{d\,(\Psi \cdot \phi_R)}{d\varpi} \in \mathrm{Hom}_{cts}\,(G_v, \mathbb{Q}_p)$$

(noting that $\phi_R \bmod \varpi = 1$ on G_v, since E/K_v is split multiplicative). Recalling the exact sequence (29), let $\mathcal{Q}_v \in \widetilde{H}_f^1(K, V_p(E))$ be the image under ι of $(0, \ldots, 1, \ldots, 0) \in \bigoplus_{v \in S_f^{sp}} \mathbb{Q}_p$ (with 1 as v-component). The following Lemma reduces the computation to local class field theory. The notations are those introduced in Sec. (4.2).

LEMMA 4.8. *For every* $P_f = [(P, P^+, \varepsilon_P)] \in \widetilde{H}^1_f(K, V_p(E))$ *we have*

$$\varpi \cdot (i_{\mathcal{P}}(\mathcal{Q}_v) \cup_{CT} i_{\mathcal{P}}(P_f) \bmod 1) = -\langle [P_v^+], \chi_v^{wt}\rangle_{K_v},$$

where $[P_v^+] \in H^1(K_v, \mathbb{Q}_p(1))$ *is the cohomology class of the v-component of* $P^+ \in \bigoplus_{v|p} C^1_{\mathrm{cont}}(K_v, V_p(E)^+)$.

PROOF. Write $\overline{\boldsymbol{x}} := i_{\mathcal{P}}(\mathcal{Q}_v)$ and $\overline{\boldsymbol{y}} := i_{\mathcal{P}}(P_f)$. To avoid heavy notations, we fix in this proof a splitting of $\mathbb{Q}_p$-modules (resp. $R_{\overline{\mathcal{P}}}$-modules) $V_p(E)_v \xrightarrow{\sim} V_p(E)^+ \oplus V_p(E)^- = \mathbb{Q}_p^2$ (resp. $T(\mathcal{P})_v \xrightarrow{\sim} T(\mathcal{P})^+ \oplus T(\mathcal{P})^- \xrightarrow{\sim} R^2_{\overline{\mathcal{P}}}$) in (9) (resp., (23)). We then identify $V_p(E)_v$ (resp., $T(\mathcal{P})_v$) with $\mathbb{Q}_p^2$ (resp., $R^2_{\overline{\mathcal{P}}}$), with G_v-action given by the matrix

$$\begin{pmatrix} \chi_{cy} & \star_2 \\ 0 & 1 \end{pmatrix} \quad (resp. \quad \begin{pmatrix} \chi_{cy} \cdot \Psi^{-1} \cdot \phi_R^{-1} & \star \\ 0 & \Psi \cdot \phi_R \end{pmatrix}). \tag{45}$$

Write 'mod ϖ' for the compositions $T(\mathcal{P})^\dagger \twoheadrightarrow \left(T(\mathcal{P})^\dagger\right)_{k=2} \xrightarrow{\sim} V_p(E)^\dagger$ ($\dagger \in \{\emptyset, +, -\}$), defined in (20) and (24). We assume that the splittings are compatible under 'mod ϖ' (this amounts to requiring that $(0,1) \in T(\mathcal{P})_v$ specializes to $(0,1) \in V_p(E)_v$ under 'mod ϖ').

By construction

$$\mathcal{Q}_v = [(0, \star_2, (0,1))] \in \widetilde{H}^1_f(K, V_p(E)), \tag{46}$$

where $\star_2 : G_v \to \mathbb{Q}_p(1) = V_p(E)^+$ is as in (45) and $(0,1) \in V_p(E)_v$. Let $\widetilde{\chi}_v^{wt} \in C^1_{\mathrm{cont}}(K_v, T(\mathcal{P}))$ be the 1-cochain defined by

$$\widetilde{\chi}_v^{wt}(g) := \left(0, \varpi^{-1} \cdot (1 - \Psi(g) \cdot \phi_R(g))\right) \in T(\mathcal{P})_v.$$

Note that $p_v^- (\widetilde{\chi}_v^{wt} \bmod \varpi) = -\chi_v^{wt}$ for the projection $p_v^- : C^1_{\mathrm{cont}}(K_v, V_p(E)) \to C^1_{\mathrm{cont}}(K_v, \mathbb{Q}_p)$ in (9). We easily obtain

$$\overline{\boldsymbol{x}} := i_{\mathcal{P}}(\mathcal{Q}_v) = [0, ?, \widetilde{\chi}_v^{wt}] \in \widetilde{H}^2_f(K, T(\mathcal{P})), \tag{47}$$

where $? \in C^2_{\mathrm{cont}}(K_v, T(\mathcal{P})^+)$ is a 2-cocycle which will be not involved in the computations below. Indeed $i_{\mathcal{P}}(\mathcal{Q}_v)$ is represented by a 2-cocycle $! \in \widetilde{C}^2_f(G_{K,S}, T(\mathcal{P}))$ s.t. $\varpi \cdot ! = d_{\widetilde{C}^\bullet_f}(\widetilde{\boldsymbol{x}})$, where $\widetilde{\boldsymbol{x}} \in \widetilde{C}^1_f(G_{K,S}, T(\mathcal{P}))$ is any 1-cochain which lifts a representative of $\mathcal{Q}_v$ under $\widetilde{C}^1_f(G_{K,S}, T(\mathcal{P})) \to \widetilde{C}^1_f(G_{K,S}, V_p(E))$ (see (27)). By (46) we can take $\widetilde{\boldsymbol{x}} := (0, \star, (0,1))$, where $(0,1) \in C^0_{\mathrm{cont}}(K_v, T(\mathcal{P}))$ and $\star$ in (45) is considered as a 1-cochain on G_v with values in the 'first component' $T(\mathcal{P})^+$ of $T(\mathcal{P})_v$. Then the first component of $d_{\widetilde{C}^\bullet_f}(\widetilde{\boldsymbol{x}})$ is zero, while (using (45) and the definition of the differential $d_{\widetilde{C}^\bullet_f}$) the third is the 1-cochain

$$G_p \ni g \mapsto i_v^+(\star)(g) - \delta\left((0,1)\right)(g) = (\star(g), 0) - (\star(g), \Psi(g) \cdot \phi_R(g)) + (0,1).$$

Putting everything together we obtain (47).

Write $\widetilde{\boldsymbol{y}} = (\widetilde{P}, \widetilde{P}^+, \widetilde{\varepsilon}_P) \in \widetilde{C}^1_f(G_{K,S}, T(\mathcal{P}))$ for a lift of the 1-cocycle $(P, P^+, \varepsilon_P) \in \widetilde{C}^1_f(G_{K,S}, V_p(E))$. To prove the Lemma it is sufficient to prove the formula

$$\varpi \cdot (\overline{\boldsymbol{x}} \cup_{CT} \overline{\boldsymbol{y}} \quad \bmod 1) \overset{?}{=} inv_v(\mathbb{Q}_p)\left(\left[\left(\widetilde{\chi}_v^{wt} \cup_\pi i_v^+(\widetilde{P}_v^+)\right) \bmod \varpi\right]\right) \tag{48}$$

($inv_v(\mathbb{Q}_p)$ is as in Sec. (0.4)). Indeed, using our normalization (22) for π, $\left(\widetilde{\chi}_v^{wt} \cup_\pi i_v^+(\widetilde{P}_v^+)\right) \bmod \varpi$ is equal to $(\widetilde{\chi}_v^{wt} \bmod \varpi) \cup_W i_v^+(P_v^+)$. By the definitions of $inv_v(\mathbb{Q}_p)$, $\widetilde{\chi}_v^{wt}$, and formula (11), we can then rewrite (48) as

$$\varpi \cdot (\overline{\boldsymbol{x}} \cup_{CT} \overline{\boldsymbol{y}} \quad \bmod 1) = inv_{K_v}\left(\left[\chi_v^{wt} \cup P_v^+\right]\right) = -\left\langle [P_v^+], \chi_v^{wt}\right\rangle_{K_v}.$$

The last equality can be proved easily using the 'transposition operators' defined in [**Nek06**, Sec. (3.4.5.3)], or Kummer theory (see for example formula (11.3.5.2) in [**Nek06**]). It then remains to prove (48). For this we simply retrace the constructions of section (3.4).

a) We have to choose 2-cocycles $\overline{\boldsymbol{x}}' \in \left(\widetilde{C}_f^{\bullet}(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right)^2$ and $\overline{\boldsymbol{y}}' \in \left(\widetilde{C}_f^{\bullet}(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right)^2$ which lift (a representative of) $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{y}}$ respectively in the exact sequence (34). By the definitions we can take (writing for simplicity again $\overline{*}$ for a representative of the class $\overline{*}$)

$$\overline{\boldsymbol{x}}' := \left(\overline{\boldsymbol{x}}, \widetilde{\boldsymbol{x}} \otimes \varpi^{-1}\right); \quad \overline{\boldsymbol{y}}' := \left(\overline{\boldsymbol{y}}, \widetilde{\boldsymbol{y}} \otimes \varpi^{-1}\right).$$

b) Recalling the quasi-isomorphism $v_{R_{\overline{\mathcal{P}}}} : \overline{R_{\overline{\mathcal{P}}}} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}} \to \overline{R_{\overline{\mathcal{P}}}}$, we have

$$\left(\left(id \otimes v_{R_{\overline{\mathcal{P}}}}\right) \circ s_{23}\right)\left(\overline{\boldsymbol{x}}' \otimes \overline{\boldsymbol{y}}'\right) = \left(\overline{\boldsymbol{x}} \otimes \overline{\boldsymbol{y}}, (\overline{\boldsymbol{x}} \otimes \widetilde{\boldsymbol{y}}) \otimes \varpi^{-1}\right) \in \left(\left(\widetilde{C}_f^{\bullet}(T(\mathcal{P})) \otimes_{R_{\overline{\mathcal{P}}}} \widetilde{C}_f^{\bullet}(T(\mathcal{P}))\right) \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right)^4. \tag{49}$$

c) We have to compute the image of (49) under the morphism $\cup_{\pi,r} \otimes id$. We take $r = 0$, the representative in (47) for $\overline{\boldsymbol{x}}$, and $\widetilde{\boldsymbol{y}}$ as above. By formula (160) we obtain the 4-cocycle

$$\left(\left(0, \widetilde{\chi}_v^{wt} \cup_\pi i_v^+(!!)\right), \left(0, \left(\widetilde{\chi}_v^{wt} \cup_\pi i_v^+(\widetilde{P}_v^+)\right) \otimes \varpi^{-1}\right)\right) \in \left(\mathcal{K} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}\right)^4, \tag{50}$$

where $!! \in C^2_{\rm cont}(K_v, T(\mathcal{P})^+)$ is the second component of the representative of $\overline{\boldsymbol{y}}$ determined by $\widetilde{\boldsymbol{y}}$.

d) We have to apply the isomorphism (35), followed by $\mathscr{R}/R_{\overline{\mathcal{P}}}[\varpi] \xrightarrow{\sim} \mathbb{Q}_p$ to the cohomology class of (50). Write

$$\mathcal{X} := \left(0, \widetilde{\chi}_v^{wt} \cup_\pi i_v^+(!!)\right); \quad \mathcal{Y} := \left(0, \widetilde{\chi}_v^{wt} \cup_\pi i_v^+(\widetilde{P}_v^+)\right).$$

It follows immediately by the definitions (and the fact that $\cup_\pi \circ (i_v^+ \otimes i_v^+)$ is the zero map) that $\mathcal{X}$ and $\mathcal{Y}$ satisfy the hypothesis of Lemma (4.9) below. We then obtain

$$\varpi \cdot (\overline{\boldsymbol{x}} \cup_{CT} \overline{\boldsymbol{y}} \bmod 1) = \underline{inv}_{S_f}(\mathbb{Q}_p)\left([\mathcal{Y} \bmod \varpi]\right) := inv_v(\mathbb{Q}_p)\left(\left[\left(\widetilde{\chi}_v^{wt} \cup_\pi i_v^+(\widetilde{P}_v^+)\right) \bmod \varpi\right]\right).$$

We have proved (48) and with it the Lemma. □

LEMMA 4.9. *Let $\mathcal{X} \in \mathcal{K}^4$ and $\mathcal{Y} \in \mathcal{K}^3$ be cochains such that $d_{\mathcal{K}}(\mathcal{Y}) = \varpi \cdot \mathcal{X}$, so that $[\mathcal{X}, \mathcal{Y} \otimes \varpi^{-1}] \in H^4(\mathcal{K} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}})[\varpi]$. Writing $I_{S_f} : H^4(\mathcal{K} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}})[\varpi] \xrightarrow{\sim} \mathbb{Q}_p$ for the composition of the isomorphism* (35) *with $\mathscr{R}/R_{\overline{\mathcal{P}}}[\varpi] \xrightarrow{\sim} \mathbb{Q}_p$, we have*

$$I_{S_f}\left([\mathcal{X}, \mathcal{Y} \otimes \varpi^{-1}]\right) = \underline{inv}_{S_f}(\mathbb{Q}_p)\left([\mathcal{Y} \bmod \varpi]\right).$$

PROOF. Since $d_{\mathcal{K}}(\mathcal{X}) = 0$ and $H^4(\mathcal{K}) = 0$, we have $\mathcal{X} = d_{\mathcal{K}}(\mathcal{T})$ for a 3-cochain $\mathcal{T} \in \mathcal{K}^3$. By the definition of the differential in $\mathcal{K} \otimes_{R_{\overline{\mathcal{P}}}} \overline{R_{\overline{\mathcal{P}}}}$, it follows that $[\mathcal{X}, \mathcal{Y} \otimes \varpi^{-1}] = [0, (\mathcal{Y} - \varpi \cdot \mathcal{T}) \otimes \varpi^{-1}]$. By construction, the image of this element under the isomorphism (35) is given by $\underline{inv}_{S_f}(R_{\overline{\mathcal{P}}})\left([\mathcal{Y} - \varpi \cdot \mathcal{T}]\right) \cdot [\varpi^{-1}] \in \mathscr{R}/R_{\overline{\mathcal{P}}}[\varpi]$. It follows that

$$I_{S_f}([\mathcal{X}, \mathcal{Y} \otimes \varpi^{-1}]) = \underline{inv}_{S_f}(R_{\overline{\mathcal{P}}})\left([\mathcal{Y} - \varpi \cdot \mathcal{T}]\right) \bmod \varpi = \underline{inv}_{S_f}(\mathbb{Q}_p)\left([\mathcal{Y} \bmod \varpi]\right)$$

(the last equality from (30)). □

We can now prove Th. (4.5).

PROOF OF THEOREM (4.5). By Remark (1.3) $\gamma_{q_E} = \partial_v(1) \in H^1(K_v, \mathbb{Q}_p(1))$. Identifying $V_p(E)_v \xrightarrow{\sim} \mathbb{Q}_p(1) \oplus \mathbb{Q}_p$ (as $\mathbb{Q}_p$-modules) as in the proof of Lemma (4.8), and using directly the construction of the connecting homomorphism ∂_v, we see that γ_{q_E} is represented by the 1-cocycle $\star_2 \in C^1_{\rm cont}(K_v, \mathbb{Q}_p(1))$ (corresponding to the choice of the lift $(0,1) \in C^0_{\rm cont}(K_v, V_p(E))$ of $1 \in H^0(K_v, V_p(E)^-)$). Recalling the definitions of $i_E^\dagger$ and $\mathcal{Q}_v$ we obtain

$$i_E^\dagger(q_v) = [(0, \star_2, (0,1))] = \mathcal{Q}_v \in \widetilde{H}^1_f(K, V_p(E)).$$

Lemma (4.8) gives

$$(\ddagger) := \varpi \cdot \left(\phi_E(q_v) \cup_{CT} \phi_E(P, \widetilde{P}) \bmod 1\right) = -\left\langle \gamma_{\widetilde{P}_v}, \chi_v^{wt} \right\rangle_{K_v} = -\left\langle \gamma_{\widetilde{P}_v}, \mathrm{Res}_{K_v/\mathbb{Q}_p}(\chi_E^{wt}) \right\rangle_{K_v}, \tag{51}$$

where $\phi_E := i_{\mathcal{P}} \circ i_E^\dagger$ (cfr. Sec. (4.2)). Using the compatibility of Tate local duality under restriction and corestriction in (finite) field extentions and the identity $\mathrm{Cor}_{K_v/\mathbb{Q}_p}(\gamma_{\widetilde{P}_v}) = \gamma^{\mathbb{Q}_p}_{N_{K_v/\mathbb{Q}_p}(\widetilde{P}_v)}$ (i.e. the image of

$N_{K_v/\mathbb{Q}_p}(\widetilde{P}_v)\otimes 1$ under the Kummer isomorphism $\gamma_*^{\mathbb{Q}_p}:\mathbb{Q}_p^*\widehat{\otimes}\mathbb{Q}_p\xrightarrow{\sim}H^1(\mathbb{Q}_p,\mathbb{Q}_p(1)))$, it follows from (51) and Prop. (1.1) that

$$(\ddagger)=-\chi_E^{wt}\left(\mathrm{rec}_p\left(N_{K_v/\mathbb{Q}_p}(\widetilde{P}_v)\right)\right).$$

We finish the proof multiplying this equation by $\log_p(\gamma)$. □

5. The regulator term

In this section we introduce the *p-adic regulator* which appears in the 'right hand side' of the p-adic Birch and Swinnerton-Dyer conjecture of Section (11.8) below. More precisely: given a quadratic character χ, Conj. (6.1) expresses the leading term of $L_p^{\mathrm{gen}}(f_\infty,\chi,k)$ as the product of arithmetics invariants attached to $E^\chi/\mathbb{Q}$ and the p-adic regulator. We now briefly motivate the definitions below, assuming for simplicity that $\chi=\chi_{\mathrm{tiv}}$ is the trivial character.

We consider the two-variable Mazur-Kitagawa p-adic L-function $L_p(f_\infty,k,s)$. When $e_{\mathrm{gen}}(\chi_{\mathrm{triv}})=0$, we have $L_p^{\mathrm{gen}}(f_\infty,k):=L_p(f_\infty,k,k/2)$. Moreover, assuming the finiteness of $\text{Ш}(E/\mathbb{Q})$, the parity conjecture proved in [**Nek06**, Section 12] tells us that $E^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p$ has *even* $\mathbb{Q}_p$-dimension. In this case we expect $\langle -,-\rangle_{\mathbb{Q},p}^{\mathrm{Nek}}$ to be non-degenerate (cfr. Prop. 4.2). Moreover, the results of [**BD07**] (cfr. Sec. (7.1)) and the analogy with Iwasawa theory (discussed at the end of the introduction) lead us to define the regulator as the determinant of $\langle -,-\rangle_{\mathbb{Q},p}^{\mathrm{Nek}}$.

Assume now $e_{\mathrm{gen}}(\chi_{\mathrm{triv}})=1$. In this case we know that $L_p(f_\infty,k,k/2)\equiv 0$. Assuming again the finiteness of $\text{Ш}(E/\mathbb{Q})$, the parity conjecture and Prop. 4.2 imply that $\langle -,-\rangle_{\mathbb{Q},p}^{\mathrm{Nek}}$ is degenerate. We then consider $L_p^{\mathrm{gen}}(f_\infty,k)$, obtained by differentiating $L_p(f_\infty,k,s)$ with respect to the *cyclotomic* variable s, and restricting the resulting function to the central critical line $s=k/2$. In light of the conjectures formulated in [**MTT86**], we expect that both the p-adic weight pairing and the p-adic cyclotomic height play a role in the computation of the leading term of $L_p^{\mathrm{gen}}(f_\infty,k)$ at $k=2$. This leads us to introduce a sort of 'derived regulator' (cfr. [**BD95**]), with the p-adic cyclotomic height as 'derived height' on the radical of $\langle -,-\rangle_{\mathbb{Q},p}^{\mathrm{Nek}}$. (For more details and further motivation, see Sec. (5.4), Rem. (6.2.2) and Th. (7.3) below.)

5.1. The p-adic cyclotomic height pairing. We now very briefly recall some construction from [**MTT86**], needed in the definition of the regulator below. In Ch. II, Sec. 4 of *loc. cit.*, the *analytic λ-height*

$$\langle -,-\rangle_{K,\lambda}^{\mathrm{MTT}}:E(K)\otimes\mathbb{Q}_p\times E(K)\otimes\mathbb{Q}_p\longrightarrow\mathbb{Q}_p$$

is associated to any continuous morphism $\lambda:K^*\backslash\mathbb{A}_K^*/\prod_{v\nmid p}U_v\to\mathbb{Q}_p$, where $\mathbb{A}_K^*$ is the group of idèles of K, U_v is the group of units of K_v if v is finite, and $U_v=K_v^*$ if $v|\infty$. We fix $\lambda:=\lambda_0$ such that its v-component for $v|p$ is given by $\lambda_v:=\log_p\circ N_{K_v/\mathbb{Q}_p}$, and we write $\langle -,-\rangle_{K,p}^{\mathrm{MTT}}:=\langle -,-\rangle_{K,\lambda_0}^{\mathrm{MTT}}$. It is a symmetric bilinear form.

If E/K_v has split multiplicative reduction at some prime $v|p$, we extend the definition of $\langle -,-\rangle_K^{\mathrm{MTT}}$ to $E^\dagger(K)\otimes\mathbb{Q}_p$ as follows, following [**MTT86**]. With the notations of Sec. (4.1), write $E_0(K)\subset E(K)$ for the finite index subgroup consisting of points P such that of $\rho_v(P)\in\Phi_{Tate}(\mathcal{O}_{K_v}^*)$, for every $v\in S_f^{sp}$. Let $E_0^\dagger(K)\subset E^\dagger(K)$ be the inverse image of $E_0(K)$ under the natural projection $E^\dagger(K)\to E(K)$. In order to lift the λ-height to a bilinear form

$$\langle -,-\rangle_{K,p}^{\mathrm{MTT}}:E^\dagger(K)\otimes\mathbb{Q}_p\times E^\dagger(K)\otimes\mathbb{Q}_p\longrightarrow\mathbb{Q}_p \tag{52}$$

it is sufficient to define $\langle -,-\rangle_{K,p}^{\mathrm{MTT}}$ on $E_0^\dagger(K)$. Consider the short exact sequence

$$0\to\Lambda_p\to E_0^\dagger(K)\to E_0(K)\to 0,$$

where $\Lambda_p:=\bigoplus_{v\in S_f^{sp}}q_v^{\mathbb{Z}}$. Define $E_0(K)\to E_0^\dagger(K)$ sending P to $(P,(y_v^*))\in E_0^\dagger(K)$, where $y_v^*:=y_v^*(P)$ is the unique lift in $\mathcal{O}_v^*$ of $\rho_v(P)$ under Φ_{Tate} (recalling that $q_E\in p\mathbb{Z}_p$). This map defines a splitting of the exact sequence above, and we obtain $E_0^\dagger(K)\xrightarrow{\sim}\Lambda_p\oplus E_0(K)$. We finally define (52) to be the unique $\mathbb{Q}_p$-bilinear, symmetric extension of $\langle -,-\rangle_{K,p}^{\mathrm{MTT}}$ which satisfies the following conditions. Given $P=(P,(y_v^*))\in E_0^\dagger(K)$ and $q_v,q_w\in\Lambda_p$ $(v\neq w)$,

$$\langle q_v,P\rangle_{K,p}^{\mathrm{MTT}}:=\log_p\left(N_{K_v/\mathbb{Q}_p}\left(y_v^*(P)\right)\right);\quad \langle q_v,q_v\rangle_{K,p}^{\mathrm{MTT}}:=\log_p(N_{K_v/\mathbb{Q}_p}(q_E));\quad \langle q_v,q_w\rangle_{K,p}^{\mathrm{MTT}}:=0.$$

If $K/\mathbb{Q}$ is Galois, $\langle -,-\rangle_{K,p}^{\mathrm{MTT}}$ is $Gal(K/\mathbb{Q})$-equivariant: $\langle x^\sigma,y^\sigma\rangle_K^{\mathrm{MTT}}=\langle x,y\rangle_K^{\mathrm{MTT}}$ for every $x,y\in E^\dagger(K)\otimes\mathbb{Q}_p$ and $\sigma\in Gal(K/\mathbb{Q})$. We refer the reader to [**MTT86**, Ch. II] for more details.

5.2. The regulator. Let $E_\infty^\dagger(K)\subset E^\dagger(K)\otimes\mathbb{Z}_p$ be the (left=right) radical of

$$\langle -,-\rangle_{K,p}^{\mathrm{Nek}}:E^\dagger(K)\otimes\mathbb{Z}_p\times E^\dagger(K)\otimes\mathbb{Z}_p\longrightarrow\mathbb{Q}_p.$$

Write $\widetilde{r}_\infty:=\mathrm{rank}_{\mathbb{Z}_p}E_\infty^\dagger(K)$, $\widetilde{r}:=\mathrm{rank}_{\mathbb{Z}}E^\dagger(K)$ and $t=\widetilde{r}-\widetilde{r}_\infty$. Let $(P_1,\dots,P_{\widetilde{r}_\infty})$ be a $\mathbb{Z}_p$-basis of $E_\infty^\dagger(K)$ modulo torsion. We note that the quotient $E^\dagger(K)\otimes\mathbb{Z}_p/E_\infty^\dagger(K)$ is a free $\mathbb{Z}_p$-module (since $E_\infty^\dagger(K)$ is p-adically saturated in $E^\dagger(K)\otimes\mathbb{Z}_p$). We can then complete $(P_j)_{j=1}^{\widetilde{r}_\infty}$ to a $\mathbb{Z}_p$-basis $(P_1,\dots,P_{\widetilde{r}_\infty},Q_1,\dots,Q_t)$ of $E^\dagger(K)\otimes\mathbb{Z}_p$ modulo torsion. We define 'partial regulators'

$$\mathcal{R}_{K,p}^{\infty}:=\det\left(\langle P_i,P_j\rangle_{K,p}^{\mathrm{MTT}}\right);\quad \mathcal{R}_{K,p}^{\mathrm{Nek}}:=\det\left(\langle Q_i,Q_j\rangle_{K,p}^{\mathrm{Nek}}\right). \tag{53}$$

These are well defined elements of the multiplicative monoid $\mathbb{Q}_p/\mathbb{Z}_p^*$. Finally, any $\mathbb{Z}$-basis $(T_1,\dots,T_{\widetilde{r}})$ of $E^\dagger(K)/\mathrm{tors}$ gives rise to a $\mathbb{Z}_p$-basis of $E^\dagger(K)\otimes\mathbb{Z}_p$ modulo torsion. Take $M\in\mathrm{GL}_{\widetilde{r}}(\mathbb{Z}_p)$ which sends $(T_1,\dots,T_{\widetilde{r}})$ to $(P_1,\dots,Q_t)$ and define the *p-adic regulator* of E/K

$$\mathcal{R}_{K,p}(E)=\mathcal{R}_{K,p}:=\det(M)^{-2}\cdot\mathcal{R}_{K,p}^{\infty}\cdot\mathcal{R}_{K,p}^{\mathrm{Nek}}.$$

This definition is independent on the choices made above, and is therefore a well defined element of $\mathbb{Q}_p$.

5.3. Regulators over quadratic fields. We assume for the rest of this section that $K/\mathbb{Q}$ is either $\mathbb{Q}$ or a quadratic field which is unramified at p. Given a $\mathbb{Z}_p[Gal(K/\mathbb{Q})]$-module M, let $M^\pm$ be the $\pm$-eigenspace for the action of $Gal(K/\mathbb{Q})$, so that $M=M^+\oplus M^-$, since $p\neq 2$. (Clearly $M=M^+$ when $K=\mathbb{Q}$.)

Since $\langle -,-\rangle_{K,p}^{\mathrm{Nek}}$ is $Gal(K/\mathbb{Q})$-equivariant (Prop. (4.4)), $\langle x^+,x^-\rangle_{K,p}^{\mathrm{Nek}}=0$ for every $x^\pm\in(E^\dagger(K)\otimes\mathbb{Z}_p)^\pm$. In particular, letting $w\in\{\pm\}$, $E_\infty^\dagger(K)^w\subset(E^\dagger(K)\otimes\mathbb{Z}_p)^w$ is the radical of the pairing

$$\langle -,-\rangle_{K,p}^{\mathrm{Nek},w}:(E^\dagger(K)\otimes\mathbb{Z}_p)^w\times(E^\dagger(K)\otimes\mathbb{Z}_p)^w\longrightarrow\mathbb{Q}_p$$

induced by restricting $\langle -,-\rangle_{K,p}^{\mathrm{Nek}}$ to the w-eigensapce. Writing $\widetilde{r}_\infty^w:=\mathrm{rank}_{\mathbb{Z}_p}E_\infty^\dagger(K)^w$, take a $\mathbb{Z}_p$-basis $(P_j^w)_{1\le j\le\widetilde{r}_\infty^w}$ of $E_\infty^\dagger(K)^w/\mathrm{tors}$, and complete it to a $\mathbb{Z}_p$-basis $(P_j^w,Q_i^w)_{i,j}$ of $(E^\dagger(K)\otimes\mathbb{Z}_p)^w$ modulo torsion, with $1\le j\le\widetilde{r}^w-\widetilde{r}_\infty^w$ and $\widetilde{r}^w:=\mathrm{rank}_{\mathbb{Z}}E^\dagger(K)^w$. Then $(P_i^+,P_j^-,Q_s^+,Q_t^-)_{i,j,s,t}$ is a $\mathbb{Z}_p$-basis of $E^\dagger(K)\otimes\mathbb{Z}_p$ modulo torsion, which can be used to compute the partial regulators (53). Using again the $Gal(K/\mathbb{Q})$-equivariance of $\langle -,-\rangle_{K,p}^{\mathrm{Nek}}$ and $\langle -,-\rangle_{K,p}^{\mathrm{MTT}}$, we have the factorization in $\mathbb{Q}_p/\mathbb{Z}_p^*$

$$\mathcal{R}_{K,p}^{\infty}=\mathcal{R}_{K,p}^{\infty,+}\cdot\mathcal{R}_{K,p}^{\infty,-};\quad \mathcal{R}_{K,p}^{\mathrm{Nek}}=\mathcal{R}_{K,p}^{\mathrm{Nek},+}\cdot\mathcal{R}_{K,p}^{\mathrm{Nek},-},$$

where

$$\mathcal{R}_{K,p}^{\infty,w}:=\det\left(\langle P_i^w,P_j^w\rangle_{K,p}^{\mathrm{MTT}}\right);\quad \mathcal{R}_{K,p}^{\mathrm{Nek},w}:=\det\left(\langle Q_i^w,Q_j^w\rangle_{K,p}^{\mathrm{Nek}}\right).$$

As above, take any $\mathbb{Z}$-basis (T_j^w) of $E^\dagger(K)^w$ modulo torsion, and let $M_w\in\mathrm{GL}_{\widetilde{r}^w}(\mathbb{Z}_p)$ be a matrix which sends (T_j^w) to (P_j^w,Q_i^w). Defining $\mathcal{R}_{K,p}^w:=\det(M_w)^{-2}\cdot\mathcal{R}_{K,p}^{\infty,w}\cdot\mathcal{R}_{K,p}^{\mathrm{Nek},w}\in\mathbb{Q}_p$, we obtain

$$\mathcal{R}_{K,p}\doteq\mathcal{R}_{K,p}^+\cdot\mathcal{R}_{K,p}^-,$$

with $\doteq$ denoting equality (in $\mathbb{Q}_p$) up to some power of 2.

5.4. Non-vanishing conjectures. Let $K/\mathbb{Q}$ be as in the preceding section and let $w \in \{\pm\}$. Writing again $\langle -,-\rangle_{K,p}^{\mathrm{Nek},w}$ for the restriction of $\langle -,-\rangle_{K,p}^{\mathrm{Nek}}$ to $(E^\dagger(K)\otimes\mathbb{Q}_p)^w$, we have the following more precise version of Prop (4.2).

PROPOSITION 5.1. *Assume that the p-primary part of $\text{Ш}(E/K)^w$ is finite. Then $\langle -,-\rangle_{K,p}^{\mathrm{Nek},w}$ is non-degenerate if and only if $\widetilde{H}^1_f(K,T(\mathcal{P}))^w = 0$ and $\widetilde{H}^2_f(K,T(\mathcal{P}))^w$ is semi-simple.*

PROOF. Since $i_{\mathcal{P}}$ and 'mod ϖ' are morphisms of $Gal(K/\mathbb{Q})$-modules, we obtain from Prop. (3.1) the exact sequence

$$0 \to \widetilde{H}^1_f(K,T(\mathcal{P}))^w/\varpi \to \widetilde{H}^1_f(K,V_p(E))^w \to \widetilde{H}^2_f(K,T(\mathcal{P}))^w[\varpi] \to 0.$$

Write for simplicity $N := \widetilde{H}^2_f(K,T(\mathcal{P}))_{R_{\overline{\mathcal{P}}}-Tor}$. Since $\cup_{CT}$ is $Gal(K/\mathbb{Q})$-equivariant (see Section (3.5)), N^w is orthogonal to N^{-w} under $\cup_{CT}$. By Th. (3.2), the restriction of $\cup_{CT}$ to N^w is non-degenerate. The argument used in the proof of Prop. (4.2) tells us that the restriction of $\cup_{CT}\circ(i_{\mathcal{P}}\times i_{\mathcal{P}})$ to $\widetilde{H}^1_f(K,V_p(E))^w$ is non-degenerate if and only if $\widetilde{H}^1_f(K,T(\mathcal{P}))^w = 0$ and $\widetilde{H}^2_f(K,T(\mathcal{P}))^w$ is semi-simple. Now, if the p-primary part of $\text{Ш}(E/K)^w$ is finite, $i_E^\dagger$ induces an isomorphism $\left(E^\dagger(K)\otimes\mathbb{Q}_p\right)^w \xrightarrow{\sim} \widetilde{H}^1_f(K,V_p(E))^w$ and we conclude. □

As suggested by the *low-rank* cases discussed in Sec. (5.5) below, we expect that $\widetilde{H}^2_f(K,T(\mathcal{P}))^w$ is always semi-simple. Assuming this, the behaviour of $\langle -,-\rangle_{K,p}^{\mathrm{Nek},w}$ is 'determined' by the module $\widetilde{H}^1_f(K,T(\mathcal{P}))^w$ (which represents the analogue in this context of the module of universal norms in Iwasawa theory). We know that this is a free $R_{\overline{\mathcal{P}}}$-module. Its rank is predicted by the following conjecture. With the notations of Sec. (2.2), write $e_{\mathrm{gen}}(+) := e_{\mathrm{gen}}$. If K is quadratic and ϵ_K is the associated quadratic character, we write $e_{\mathrm{gen}}(-) := e_{\mathrm{gen}}(\epsilon_K)$.

CONJECTURE 5.2. $\mathrm{rank}_{R_{\overline{\mathcal{P}}}}\widetilde{H}^1_f(K,T(\mathcal{P}))^w = e_{\mathrm{gen}}(w) = \widetilde{r}^w_\infty$.

REMARK 5.3. Conj. (5.2) is intimately connected with Greenberg conjecture, predicting $L_p^{\mathrm{gen}}(f_\infty,\chi,k) \not\equiv 0$. More precisely, in **[NP00]** it is proved (for $K=\mathbb{Q}$) that Greenberg conjecture implies the equality $\mathrm{rank}_{R_{\overline{\mathcal{P}}}}\widetilde{H}^1_f(K,T(\mathcal{P}))^w = e_{\mathrm{gen}}(w)$. Assuming this and the finiteness of $\text{Ш}(E/K)$, the second equality of Conj. (5.2) is equivalent to the semi-simplicity of $\widetilde{H}^2_f(K,T(\mathcal{P}))^w$ (by Prop. (5.1)).

We note that the parity conjecture predicts $e_{\mathrm{gen}}(w) \overset{?}{\equiv} \widetilde{r}^w \bmod 2$ ($\widetilde{r}^w := \mathrm{rank}_{\mathbb{Z}}E^\dagger(K)^w$). In particular we expect that $\mathcal{R}^w_{K,p} = \mathcal{R}^{\mathrm{Nek},w}_{K,p}$ is the determinant of $\langle -,-\rangle_{K,p}^{\mathrm{Nek},w}$ when $E^\dagger(K)^w$ has *even* rank. When $E^\dagger(K)^w$ has *odd* rank the above conjecture predicts that $E^\dagger_\infty(K)^w/\mathrm{tors}$ is generated by a vector $P^w \in (E^\dagger(K)\otimes\mathbb{Z}_p)^w$. (We note that, assuming the parity conjecture, $\widetilde{r}^w$ is odd and $\langle -,-\rangle_{K,p}^{\mathrm{Nek},w}$ is degenerate, since it is alternating.) In this case we have

$$\mathcal{R}^w_{K,p} \doteq \langle P^w,P^w\rangle_{K,p}^{\mathrm{MTT}}\cdot\mathcal{R}^{\mathrm{Nek},w}_{K,p},$$

with $\doteq$ denoting equality up to a p-adic unit. Finally, guided by Conj. (5.2), the conjectural non-degeneracy of $\langle -,-\rangle_{K,p}^{\mathrm{MTT}}$ and the analogy with the Galois case considered in **[BD96]**, **[MTT86]**, we propose the following non-vanishing conjecture.

CONJECTURE 5.4. $\mathcal{R}^w_{K,p} \neq 0$ *for any* $w \in \{\pm\}$*. In particular:* $\mathcal{R}_{K,p} \neq 0$.

5.5. Examples of *low-rank*. Assume that K is as in the preceding sections. If K is quadratic, let $\chi \in \{1,\epsilon_K\}$ and write $\mathcal{R}^\chi_{K,p} := \mathcal{R}^{w(\chi)}_{K,p}$, where $w(1) := +$ and $w(\epsilon_K) := -$. In a similar way, we write $\langle -,-\rangle_{K,p}^{\mathrm{Nek},\chi} := \langle -,-\rangle_{K,p}^{\mathrm{Nek},w(\chi)}$. If $K=\mathbb{Q}$, we put $\chi = 1$, $\mathcal{R}^\chi_{K,p} := \mathcal{R}_{\mathbb{Q},p}$ and $\langle -,-\rangle_{K,p}^{\mathrm{Nek},\chi} := \langle -,-\rangle_{\mathbb{Q},p}^{\mathrm{Nek}}$. We now give some 'low-rank' examples in which we can compute explicitly these regulators.

We say that (E,χ) is *exceptional* if the following conditions are satisfied:

1. $E/\mathbb{Q}_p$ has multiplicative reduction;

2. $\chi(p) = \alpha_p$.

In the exceptional case, we can define a 'Tate period' $q_\chi \in E^\dagger(K)^\chi$ as follows. Note that E/K_v has split multiplicative reduction at every prime $v|p$ (since p is unramified in K). First of all: if $K = \mathbb{Q}$, then $\chi = 1$ and we write $q_\chi = q_E := (0, q_E) \in E^\dagger(\mathbb{Q})$. Assume now $K/\mathbb{Q}$ quadratic. If p is inert in K, write again $q_E = (0, q_E) \in E^\dagger(K)$. If p splits in K, let $q_E^+ := (0, (q_E, q_E)) \in E^\dagger(K)^+$ and $q_E^- := (0, (q_E, q_E^{-1})) \in E^\dagger(K)^-$. Define

$$q_\chi := \begin{cases} q_E^+ & \text{if} \quad \epsilon_K(p) = 1, \quad \chi = 1; \\ q_E^- & \text{if} \quad \epsilon_K(p) = 1, \quad \chi = \epsilon_K; \\ q_E & \text{if} \quad \epsilon_K(p) = -1. \end{cases}$$

LEMMA 5.5. *If (E, χ) is exceptional, $q_\chi \in E^\dagger(K)^\chi$ and* $\mathrm{rank}_{\mathbb{Z}} E^\dagger(K)^\chi = \mathrm{rank}_{\mathbb{Z}} E(K)^\chi + 1$.

PROOF. Follows by the definition of the Galois action on $E^\dagger(K)$ in Section (4.3). □

The even case. The computations of Sec. (4.4) allow us to write explicitly the regulator when (E, χ) is exceptional and $E(K)^\chi$ has rank one. This is a significant case, in light of the results of **[BD07]** (see Sec. (7.1) or the Introduction). Write

$$\log_E : E(\overline{\mathbb{Q}}_p) \overset{\Phi_{Tate}^{-1}}{\longrightarrow} \overline{\mathbb{Q}}_p^* \overset{\log_{q_E}}{\longrightarrow} \overline{\mathbb{Q}}_p$$

for the formal group logarithm on $E/\mathbb{Q}_p$. Identifying $E(\overline{\mathbb{Q}}) \subset E(\overline{\mathbb{Q}_p})$ under the embedding ρ_p (of Sec. (4.3)), we can consider the logarithm $\log_E(P)$ of a global point $P \in E(\overline{\mathbb{Q}})$.

PROPOSITION 5.6. *Assume that (E, χ) is exceptional and that* $\mathrm{rank}_{\mathbb{Z}} E(K)^\chi = 1$. *Then*

$$\mathcal{R}_{K,p}^\chi = c \cdot \log_E(P_\chi)^2 \in \mathbb{Q}_p^*,$$

where P_χ is a generator of $E(K)^\chi/\mathrm{tors}$ and $c := 1/4$ if $K = \mathbb{Q}$ and 1 otherwise.

PROOF. We consider the case $K/\mathbb{Q}$ quadratic. The other case is similar and simpler. By the preceding Lemma $\{q_\chi, P_\chi\}$ is a basis of $E^\dagger(K)^\chi$ modulo torsion. Since $\langle -, - \rangle_{K,p}^{\mathrm{Nek}}$ is alternating, to prove that $\langle -, - \rangle_{K,p}^{\mathrm{Nek},\chi}$ is non degenerate, we have to prove $\langle q_\chi, P_\chi \rangle_{K,p}^{\mathrm{Nek}} \neq 0$.

Suppose first that p splits in K, so that $\alpha_p = 1$. Take $y_\chi \in \mathbb{Q}_p^*$ such that $\Phi_{Tate}(y_\chi) = \rho_p(P_\chi)$. If $\chi = 1$ (resp., $\chi = \epsilon_K$), $(P_\chi, (y_\chi, y_\chi))$ (resp., $(P_\chi, (y_\chi, y_\chi^{-1}))$) is in $E^\dagger(K)^\chi$. It follows by Corollary (4.6) that

$$\langle q_\chi, P_\chi \rangle_{K,p}^{\mathrm{Nek}} = \frac{1}{2} \left(\log_{q_E}(y_\chi) \pm \log_{q_E}(y_\chi^{\pm 1}) \right) = \log_{q_E}(y_\chi) =: \log_E(P_\chi) \neq 0$$

(since P_χ has infinite order).

Assume now that p is inert in K. It follows again by Corollary (4.6) and the properties of the Tate parametrisation that

$$\langle q_E, P_\chi \rangle_{K,p}^{\mathrm{Nek}} = \frac{1}{2} \cdot \log_E(P_\chi + \alpha_p P_\chi^\sigma), \tag{54}$$

where σ is the non-trivial element in $Gal(K/\mathbb{Q})$. Recalling that $P_\chi \in E(K)^\chi$, our hypothesis $\chi(p) = \alpha_p$ implies that (54) is again $\log_E(P_\chi) \neq 0$.

Since $\langle -, - \rangle_{K,p}^{\mathrm{Nek},\chi}$ is non-degenerate, $E_\infty^\dagger(K)^\chi = 0$. By the definition of $\mathcal{R}_{K,p}^\chi$ we thus have

$$\mathcal{R}_{K,p}^\chi = \mathcal{R}_{K,p}^{\mathrm{Nek},\chi} = \det \begin{pmatrix} 0 & \langle q_\chi, P_\chi \rangle_{K,p}^{\mathrm{Nek}} \\ - \langle q_\chi, P_\chi \rangle_{K,p}^{\mathrm{Nek}} & 0 \end{pmatrix},$$

concluding the proof. □

COROLLARY 5.7. *Assume (E, χ) exceptional and* $\mathrm{ord}_{s=1} L(f_E, \chi, s) = 1$. *Then $\widetilde{H}_f^1(K_\chi, T(\mathcal{P}))^\chi = 0$ and $\widetilde{H}_f^2(K_\chi, T(\mathcal{P}))^\chi$ is semi-simple.*

PROOF. Kolyvagin's theorem [**Kol90**] (see also [**Dar04**, Ch. 10]), applied to $E^\chi/\mathbb{Q}$, implies that $\mathrm{rank}_{\mathbb{Z}} E(K)^\chi = 1$ and $Ш(E/K)^\chi$ is finite. Moreover, by the preceding proposition, $\mathcal{R}^\chi_{K,p}$ is non-zero. The statement follows combining this with Prop. (5.1). □

The odd case. Turning to the odd case, assume that $\mathrm{rank}_{\mathbb{Z}} E^\dagger(K)^\chi = 1$. In this case $\langle -,-\rangle^{\mathrm{Nek},\chi}_{K,p}$ is clearly the trivial map, and $E^\dagger_\infty(K)^\chi = (E^\dagger(K)\otimes\mathbb{Z}_p)^\chi$. By definition, $\mathcal{R}^\chi_{K,p} = \mathcal{R}^{\infty,\chi}_{K,p} = \langle P_\chi, P_\chi\rangle^{\mathrm{MTT}}_{K,p}$, where P_χ is a generator of $E^\dagger(K)^\chi$ modulo torsion. It is conjectured in [**MTT86**] that this is always non-zero.

When (E,χ) is exceptional, we can take $P_\chi = q_\chi$. Writing $c = 2$ (resp., $c = 1$) if K is quadratic (resp., $K = \mathbb{Q}$) we have

$$\mathcal{R}^\chi_{K,p} = c \cdot \log_p(q_E) \in \mathbb{Q}_p^*,$$

which is known to be non-zero by [**BSDGP96**].

Applications to Conj. (5.2) *and* (5.4). The computations above can be used to prove Conj. (5.2) and (5.4) (at least) in some simple case. To give a quite general example, let us write $r_{\min}(\chi) := 1$ (resp., $r_{\min}(\chi) = 0$) if $\mathrm{sign}(E,\chi) = -1$ (resp, $\mathrm{sign}(E,\chi) = 1$). (We recall that $\mathrm{sign}(E,\chi) = \chi(-N_E)\cdot \mathrm{sign}(E,\mathbb{Q})$ is the sign in the functional equation satisfied by $L(f_E,\chi,s)$.) We put $r_{\min}(\mathbb{Q}) = r_{\min}(1)$ and $r_{\min}(K) = r_{\min}(1) + r_{\min}(\epsilon_K)$ if $K/\mathbb{Q}$ is quadratic.

Consider the following conditions:

I. (E,χ) is exceptional and $\mathrm{ord}_{s=1} L(f_E,\chi,1) = r_{\min}(\chi)$;

II. $L(f_E,\chi,1) \neq 0$.

Given $K/\mathbb{Q}$ quadratic (resp., $K = \mathbb{Q}$), we say that (E,K) is *exceptional of low-rank* if (E,χ) satisfies I or II for both $\chi \in \{1,\epsilon_K\}$ (resp., $\chi = 1$). In this case, Kolyvagin theorem implies that $Ш(E/K)$ is finite and that $\mathrm{rank}_{\mathbb{Z}} E(K)^\chi = r_{\min}(\chi) \le 1$. Combining the computations above with Prop. (5.1), we obtain the following:

LEMMA 5.8. *Conjectures* (5.2) *and* (5.4) *are true if* (E,K) *is exceptional of low-rank.*

(We note that, if (E,K) is exceptional of low-rank, then

$$\mathrm{rank}_{\mathbb{Z}} E^\dagger(K) = \widetilde{r}_{\min} := r_{\min} + \#S_f^{sp} \le 4.)$$

6. A p-adic Birch and Swinnerton-Dyer conjecture

Guided by the p-adic Birch and Swinnerton-Dyer conjectures formulated in [**MTT86**] and [**BD96**], we propose a conjecture relating the leading term of Hida p-adic L-functions to the regulator defined above. Evidence supporting it, coming from the main results of [**BD07**] and [**GS93**], will be given in the next section.

6.1. Definitions and notations. Given a quadratic Dirichlet character χ of conductor coprime with p, define

$$M_p(\chi) := \begin{cases} (1-\chi(p)\alpha_p^{-1})^\beta & \text{if} \quad \chi(p) \neq \alpha_p; \\ \\ \eta(\chi)\cdot \mathrm{ord}_p(q_E)^{-1} & \text{if} \quad \chi(p) = \alpha_p. \end{cases}$$

where $\eta(\chi) = 1$ (resp., $\eta(\chi) = 2$) if $e_{\mathrm{gen}}(\chi) = 1$ (resp., $e_{\mathrm{gen}}(\chi) = 0$), and β is defined in (17).

Let $F/\mathbb{Q}$ be a number field. For every finite prime v of F, $c_v = c_v(E/F) := [E(F_v) : E_0(F_v)]$ is the local Tamagawa number of E/F_v ([**Sil86**, Ch. VII]). Assuming the finiteness of $Ш(E/F)$, we write

$$\mathbf{BSD}(E,F) := \#Ш(E/F)\cdot \prod_v c_v \cdot \left(\#E(F)_{\mathrm{tors}}\right)^{-2}.$$

We recall that the definition of $L_p(f_\infty,\chi,k,s)$ depends on the choice of complex periods $\Omega_k := \Omega^{\mathrm{sign}(\chi)}_{f_k^\#} \in \mathbb{C}^*$, satisfying the following property: the $\mathrm{sign}(\chi)$-part of the modular symbol $\widetilde{I}_{f_k^\#}$ attached to $f_k^\#$ [**BD07**, Sec. 1] takes values in $\Omega_k\cdot\mathbb{Q}$. For $k = 2$ we can choose Ω_2 'explicitly' as follows. Let $\Omega_E^+ := \int_{E(\mathbb{R})} |\omega_E|$ and $\Omega_E := \iint_{E(\mathbb{C})} |\omega_E \wedge \overline{i\omega_E}|$ be the real and complex periods of $E/\mathbb{Q}$. (We write ω_E for the Néron differential attached to a minimal Weierstrass equation for $E/\mathbb{Q}$.) By the discussion in [**MTT86**, Ch. II], we can take $\Omega^{\pm}_{f_E}$ such that:

$$\Omega^{+}_{f_E} := \Omega^{+}_{E}; \quad \Omega^{+}_{f_E} \cdot \Omega^{-}_{f_E} = i \cdot \Omega_E.$$

We fix this choice for the rest of this note.

6.2. The conjecture. Let $K/\mathbb{Q}$ be either $\mathbb{Q}$ or a quadratic field of discriminant coprime with $p \cdot N_E$. We write $M_p(\mathbb{Q}) := M_p(1)$ and $M_p(K) = M_p(1) \cdot M_p(\epsilon_K)$ if K is quadratic. The p-adic Birch and Swinnerton-Dyer conjecture alluded to in the introduction can be formulated as follows.

Conjecture 6.1. *Let* $\widetilde{r}_{\mathrm{gen}}(K) := \mathrm{rank}_{\mathbb{Z}} E^{\dagger}(K) - e_{\mathrm{gen}}(K)$. *Then* $\mathrm{ord}_{k=2} L_p^{\mathrm{gen}}(f_\infty/K, k) = \widetilde{r}_{\mathrm{gen}}(K)$ *and the following equality holds in* $\mathbb{Q}_p^*$*:*

$$\left.\frac{L_p^{\mathrm{gen}}(f_\infty/K, k)}{(k-2)^{\widetilde{r}_{\mathrm{gen}}(K)}}\right|_{k=2} = M_p(K) \cdot \mathbf{BSD}(E, K) \cdot \mathcal{R}_{K,p}(E). \tag{55}$$

Remarks 6.2. 1. Assume $e_{\mathrm{gen}}(K)$ and $\mathrm{rank}_{\mathbb{Z}} E^{\dagger}(K) = 0$. Recalling our choice of complex periods, the interpolation formula (17) reduces the conjecture to the classical Birch and Swinnerton-Dyer conjecture.

2. Assuming $\widetilde{r}_{\mathrm{gen}}(K) = 0$ and $e_{\mathrm{gen}}(K) > 0$, Conj. (6.1) is a variant of the conjecture in [**MTT86**]. For example, let $K = \mathbb{Q}$, so that $L_p(f_\infty, k, k/2) \equiv 0$ is identically zero. Looking at the Taylor expansion of $L_p(f_\infty, k, k/2)$ at $(k, s) = (2, 1)$ (cfr. Th. (7.3)), we see that

$$L_p^{\mathrm{gen}}(f_\infty, 2) = \frac{d}{ds} L_p(f_\infty, 2, s)_{s=1}.$$

Since $L_p(f_\infty, 2, s)$ is the Mazur-Tate-Teitelbaum p-adic L-function attached to $E/\mathbb{Q}$ and the complex period Ω^{+}_{E}, we recover the conjecture in Ch. II, §11 of *loc. cit.*.

3. Assume that $K/\mathbb{Q}$ is quadratic and $\chi \in \{1, \epsilon_K\}$. We note that, as both $L_p^{\mathrm{gen}}(f_\infty/K, k)$ and $\mathcal{R}_{K,p}$ factorize into the product of their χ-pats, Conj. (6.1) gives also a conjectural formula relating the leading term of $L_p^{\mathrm{gen}}(f_\infty, \epsilon_K, k)$ to $\mathcal{R}^{-}_{K,p}$.

4. When $K/\mathbb{Q}$ is a generic Galois extension (of discriminant coprime with $p \cdot N_E$), the regulator $\mathcal{R}_{K,p}$ is defined. We can define $L_p^{\mathrm{gen}}(f_\infty/K, k)$ as the products of the L-functions $L_p^{\mathrm{gen}}(f_\infty, \chi, k)$, for χ running through the characters of $Gal(K/\mathbb{Q})$ (the constructions of [**BD07**, Sec. 1] work also in this cae). It is interesting to understand if the conjecture, as stated above, is a 'good prediction' in this generality.

5. In [**BD07**] a two-variable p-adic L-function $\mathscr{L}_p(f_\infty/K, k, s)$ is attached to a quadratic imaginary field $K/\mathbb{Q}$ satisfying a suitable Heegner condition. (See in particular Rem. (3.6) of *loc. cit.*, where $\mathscr{L}_p(f_\infty/K, k, s)$ is denoted $L_p(f_\infty/K, k, s)$.) This is a p-adic analytic function defined on $U \times \mathbb{Z}_p$. Its restriction to the central critical line $s = k/2$ is (essentially) $L_p(f_\infty, k, k/2) \cdot L_p(f_\infty, \epsilon_K, k, k/2)$, while its restriction to the weight two line $k = 2$ is the *anticyclotomic* p-adic L-function attached to E/K in [**BD96**]. In this case, we can again define a regulator term $\mathcal{R}^{\mathrm{acy}}_{K,p}$, replacing in the constructions above the p-adic cyclotomic height $\langle -, - \rangle^{\mathrm{MTT}}_{K,p}$ with its anticyclotomic counterpart [**BD96**]. By the results in [**BD07**], we think that an analogue of Conj. (6.1) in term of $\mathcal{R}^{\mathrm{acy}}_{K,p}$ and $\mathscr{L}_p(f_\infty/K, k, s)$ should be valid.

6. When K is imaginary quadratic, the definition of the p-adic L-function $\mathscr{L}_p(f_\infty/K, k, s)$ alluded to in the preceding remark relies on the construction, also given in [**BD07**], of a 'square-root' p-adic L-function $\mathscr{L}_p(f_\infty/K, k)$. It is a p-adic analytic function on U, satisfying $\mathscr{L}_p(f_\infty/K, k)^2 = \eta(k) \cdot L_p(f_\infty, k, k/2) \cdot L_p(f_\infty, \epsilon_K, k, k/2)$, where $\eta(k)$ is analytic and $\eta(2) \in \mathbb{Q}^*$ (see Corollary 5.3 of *loc. cit.*). When K is real quadratic (and satisfies suitable Heegner conditions), analogues of this construction are given by the same authors in [**BD09**], and by Shahabi in [**Sha08**]. Assume also that $e_{\mathrm{gen}}(K) = 0$, so that Greenberg conjecture predicts that $\mathscr{L}_p(f_\infty/K, k)$ is not identically zero, and Conj. (5.2) predicts $\mathcal{R}_{K,p} \stackrel{?}{=} \mathcal{R}^{\mathrm{Nek}}_{K,p}$. Moreover, since $\langle -, - \rangle^{\mathrm{Nek}}_{K,p}$ is alternating, we see that $\mathcal{R}^{\mathrm{Nek}}_{K,p}$ is a square. In these cases, we can 'refine' the above conjecture in terms of $\mathscr{L}_p(f_\infty/K, k)$ and a 'square-root' regulator. (See also Remarks (7.2.2) and (7.7).)

7. Results on Conjecture (6.1)

We now recall some result supporting Conj. (6.1). More precisely, thanks to the results of Bertolini and Darmon [**BD07**] and the exceptional zero formula proved by Greenberg and Stevens [**GS93**], we can prove Conj. (6.1) in some exceptional case, at least up to a non-zero rational number.

Assume in this Section that $E/\mathbb{Q}_p$ has multiplicative reduction.

7.1. The main result of [BD07]. Let χ be a primitive quadratic character of conductor coprime with N_E. If χ is non-trivial, we write as usual $K = K_\chi$ for the quadratic field attached to χ and $\mathcal{R}^\chi_{K,p} = \mathcal{R}^-_{K,p}$. If χ is trivial, put $K = \mathbb{Q}$ and $\mathcal{R}^\chi_{K,p} = \mathcal{R}_{\mathbb{Q},p}$. Taking into account Prop. (5.6) (and its proof), we can rephrase [**BD07**, Th. 5.4], as generalized in [**Mok11**, Sec. 6], in the following:

THEOREM 7.1. *Assume that* $\mathrm{sign}(E,\chi) = -1$ *and* $\chi(p) = \alpha_p$. *Then*

a) $L^{\mathrm{gen}}_p(f_\infty,\chi,k) = L_p(f_\infty,\chi,k,k/2)$ *vanishes to order at least* 2 *at* $k = 2$*;*

b) there exists a global point $\mathbf{P}_\chi \in (E(K)\otimes\mathbb{Q})^\chi$ *and a rational number* $t \in \mathbb{Q}^*$ *such that*

$$\frac{d^2}{dk^2} L^{\mathrm{gen}}_p(f_\infty,\chi,k)_{k=2} = t \cdot \left(\langle q_\chi, \mathbf{P}_\chi\rangle^{\mathrm{Nek}}_{K,p}\right)^2 ; \tag{56}$$

c) $\mathbf{P}_\chi$ *is of infinite order if and only if* $L'(E/\mathbb{Q},\chi,1) \neq 0$. *In this case, there is a rational number* $\ell \in \mathbb{Q}^*$ *such that*

$$\frac{d^2}{dk^2} L^{\mathrm{gen}}_p(f_\infty,\chi,k)_{k=2} = \ell \cdot \mathcal{R}^\chi_{K,p} \in \mathbb{Q}^*_p.$$

REMARKS 7.2. 1. As explained in [**BD07**] and [**Mok11**], the point $\mathbf{P}_\chi$ in the preceding Theorem is a Heegner point, coming from an appropriate Shimura curve parametrization of $E/\mathbb{Q}$. The first statement in *c*) is then a consequence of the work of Zhang, generalizing the classical Gross-Zagier formula.

2. We assume (for simplicity) in this remark $\chi = 1$, and we write $\mathbf{P} = \mathbf{P}_\chi$. We also use the notations of Rem. (6.2.6). The proof of [**BD07**, Th. 5.4] uses the 'square-root' p-adic L-function $\mathscr{L}_p(f_\infty/K,k)$ attached to an auxiliary complex quadratic field in which p is inert, and chosen in such a way that $L_p(f_\infty,\epsilon_K,2,1) \in \mathbb{Q}^*$. Exploiting the ideas introduced in [**BD98**], the authors prove that the first derivative of $\mathscr{L}_p(f_\infty/K,k)$ at $k = 2$ is equal to the formal group logarithm of $\mathbf{P}$. We can then state the following 'refined version' of (56):

$$\frac{d}{dk}\mathscr{L}_p(f_\infty/K,k)_{k=2} = \langle q_E, \mathbf{P}\rangle^{\mathrm{Nek}}_{K,p}.$$

Furthermore (17) gives us the formula $t^{-1} = \eta(2)\cdot L^*(f_E,\epsilon_K,1) \in \mathbb{Q}^*$ for the scalar appearing in (56).

3. As conjectured in [**Mok11**, Sec. 6] (and proved in [**BD07**, Th. 5.4] under the assumptions considered there), the scalars ℓ and t should satisfy the congruence:

$$\ell \equiv t \overset{?}{\equiv} L^*(f_E,\psi,1) \bmod (\mathbb{Q}^*)^2, \tag{57}$$

where ψ is any quadratic Dirichlet character of conductor coprime with that of χ and satisfying:

a) $\psi(-1) = \chi(-1)$ and $\psi(l) = \chi(l)$ for every prime $l|N = N_E/p$;

b) $\psi(p) = -\chi(p)$;

c) $L(f_E,\psi,1) \neq 0$.

4. Under the hypothesis of the preceding Theorem, assume $\chi = 1$ and $L'(E/\mathbb{Q},1) \neq 1$. In this case $\mathrm{rank}_{\mathbb{Z}} E^\dagger(\mathbb{Q}) = 2$ and point *c*) of the Theorem shows that Conj. (6.1) holds up to a nonzero rational scalar. Moreover, combining Conj. (6.1) with (57), we should have

$$L^*(f_E,\psi,1)\cdot \mathbf{BSD}(E,\mathbb{Q}) \overset{?}{\equiv} c_p \bmod (\mathbb{Q}^*)^2.$$

Here ψ is the quadratic character attached to any real quadratic field K_ψ such that: p is inert in K_ψ, every prime $l|N$ splits in K_ψ and $\mathrm{ord}_{s=1} L(E/K_\psi,s) = 1$. This is consistent with the classical Birch and Swinnerton-Dyer conjecture (for $E/\mathbb{Q}$ and E/K_ψ), predicting

$$L^*(f_E,\psi,1)\cdot \mathbf{BSD}(E,\mathbb{Q}) \overset{?}{\equiv} \mathbf{BSD}(E,K_\psi) \equiv c_p \bmod (\mathbb{Q}^*)^2$$

(the second congruence by the finiteness of Ш(E/K), following by Kolyvagin theorem).

7.2. The exceptional zero formula [GS93]. Assuming again that $L_p(f_\infty,\chi,k,s)$ has an exceptional zero at $(2,1)$, we now consider the case of even order of vanishing for $L(f_E,\chi,s)$ at $s=1$. In this case $\mathrm{sign}(f_\infty,\chi)=-1$ and we are in the situation considered (for $\chi=1$) by Greenberg and Stevens. The following is a variant of the main result of [**GS93**], thanks to the generalizations of the constructions of *loc. cit.* given in [**BD07**].

THEOREM 7.3. *Assume that* $\mathrm{sign}(E,\chi)=+1$ *and* $\chi(p)=\alpha_p$. *Then*

$$L_p^{\mathrm{gen}}(f_\infty,\chi,2)=\mathscr{L}_E\cdot L^*(f_E,\chi,1).$$

PROOF. The assumptions imply that $\mathrm{sign}(f_\infty,\chi)=-\mathrm{sign}(E,\chi)=-1$, so that $L_p(f_\infty,\chi,k,k/2)\equiv 0$ is identically zero for $k\in U$ (cfr. Sec. 2.2). In particular, the Taylor expansion of $L_p(f_\infty,\chi,k,s)$ at $(2,1)$ is of the form

$$L_p(f_\infty,\chi,k,s)=c\cdot(s-1)-\frac{c}{2}\cdot(k-2)+(\cdots),$$

for $c\in\mathbb{C}_p$ and $(\cdots)$ denoting higher order terms. It follows that

$$L_p^{\mathrm{gen}}(f_\infty,\chi,2)=c=-2\cdot\frac{d}{dk}L_p(f_\infty,\chi,k,1)_{k=2}. \tag{58}$$

By [**BD07**, Remark 1.13], $L_p(f_\infty,\chi,k,1)=(1-\chi(p)\alpha_p(k)^{-1})\cdot L_p^*(f_\infty,\chi,k)$. Here the *improved p-adic L-function* $L_p^*(f_\infty,\chi,k)$ is a p-adic analytic function on U, satisfying [**BD07**, Prop. 1.3]

$$L_p^*(f_\infty,\chi,2)=L^*(f_E,\chi,1). \tag{59}$$

Moreover, as $\chi(p)=\alpha_p=\alpha_p(2)=\pm1$

$$(1-\chi(p)\alpha_p(k)^{-1})_{k=2}=0;\quad \frac{d}{dk}(1-\chi(p)\alpha_p(k)^{-1})_{k=2}=\alpha_p\cdot\alpha_p'(2)=-\frac{1}{2}\mathscr{L}_E, \tag{60}$$

the second relation by Cor. (4.7). The Theorem follows combining (58), (59) and (60). □

REMARK 7.4. Assume $\chi=1$ and $L(E/\mathbb{Q},1)\neq 0$, so that $\widetilde{r}_{\mathrm{gen}}=\mathrm{rank}_{\mathbb{Z}}E^\dagger(\mathbb{Q})-e_{\mathrm{gen}}=0$. By the definitions, the preceding Theorem gives

$$L_p^{\mathrm{gen}}(f_\infty,2)=\ell\cdot\mathcal{R}_{\mathbb{Q},p}\in\mathbb{Q}_p^*;\quad \ell:=M_p(1)\cdot\frac{L(E/\mathbb{Q},1)}{\Omega_E^+}\in\mathbb{Q}^*,$$

which is consistent with Conj. (6.1), via the classical Birch and Swinnerton-Dyer conjecture.

7.3. Other applications to conjecture (6.1)**.** Let $K/\mathbb{Q}$ be as in Sec. (6.2). We say that Conj. (6.1) holds up to $\mathbb{Q}^*$ if $\mathrm{ord}_{k=2}L_p^{\mathrm{gen}}(f_\infty/K,k)=\widetilde{r}_{\mathrm{gen}}(K)$ and (55) holds up to a non-zero rational number.

We recall that, when $K/\mathbb{Q}$ is quadratic, both $L_p^{\mathrm{gen}}(f_\infty/K,k)$ and $\mathcal{R}_{K,p}$ admit factorizations into '$\pm$-parts'. Using this fact, and with the terminology introduced in Sec. (5.5), the results of the preceding sections give the following:

THEOREM 7.5. *Assume that* (E,K) *is exceptional of low-rank. Then Conjecture* (6.1) *holds up to* $\mathbb{Q}^*$.

To give a significant example, let K be a quadratic field and assume that (E,p,K) satisfies:

1. p splits in K;
2. $E/\mathbb{Q}_p$ has split multiplicative reduction;
3. $\mathrm{sign}(E,\mathbb{Q})=\mathrm{sign}(E,\epsilon_K)=-1$.

In this case both $L_p(f_\infty,\chi,k,s)$ ($\chi\in\{1,\epsilon_K\}$) have an exceptional zero at $(s,k)=(2,1)$, so $\mathrm{sign}(f_\infty,\chi)=1$. In particular $e_{\mathrm{gen}}(K)=0$ and

$$L_p^{\mathrm{gen}}(f_\infty/K,k)=L_p(f_\infty,k,k/2)\cdot L_p(f_\infty,\epsilon_K,k,k/2). \tag{61}$$

Moreover, both 1 and ϵ_K satisfy the hypothesis of Th. (7.1). Using the factorizations (61) and $L(E/K,s)=L(E/\mathbb{Q},s)\cdot L(E/\mathbb{Q},\epsilon_K,s)$, and noting that $\mathrm{ord}_{s=1}L(E/\mathbb{Q},\chi,1)\geq 1$ (by the assumptions above), we thus obtain:

A. $\mathrm{ord}_{k=2} L_p^{\mathrm{gen}}(f_\infty/K,k) \geq 4$;

B. there exist $\mathbf{P}^+ \in E(\mathbb{Q}) \otimes \mathbb{Q}$ and $\mathbf{P}^- \in (E(K) \otimes \mathbb{Q})^-$, and a scalar $t \in \mathbb{Q}^*$ such that

$$\left.\frac{L_p^{\mathrm{gen}}(f_\infty/K,k)}{(k-2)^4}\right|_{k=2} = t \cdot \left(\left\langle q_E^+, \mathbf{P}^+\right\rangle_{K,p}^{\mathrm{Nek}} \cdot \left\langle q_E^-, \mathbf{P}^-\right\rangle_{K,p}^{\mathrm{Nek}}\right)^2, \tag{62}$$

where $q_E^\pm$ are defined in Sec. (5.5).

C. $\mathbf{P}^+$ and $\mathbf{P}^-$ are simultaneously of infinite order if and only if $L''(E/K,1) \neq 0$. In this case $\widetilde{r}_{\mathrm{gen}}(K) = \mathrm{rank}_{\mathbb{Z}} E^\dagger(K) = 4$ and there is a scalar $\ell \in \mathbb{Q}^*$ such that

$$\left.\frac{L_p^{\mathrm{gen}}(f_\infty/K,k)}{(k-2)^4}\right|_{k=2} = \ell \cdot \mathcal{R}_{K,p} \in \mathbb{Q}_p^*.$$

(Recalling the factorization for $\mathcal{R}_{K,p}$ (cfr. Sec. (5.3)), the last assertion in *C.* follows combining Th. (7.1) with the computations of Sec. (5.5).)

Remark 7.6. By Rem. (7.2), we expect $t \overset{?}{\equiv} L^*(f_E,\psi_1,1) \cdot L^*(f_E,\psi_2,1) \bmod (\mathbb{Q}^*)^2$, where ψ_1 (resp. ψ_2) is the quadratic character attached to a real quadratic field K_1 (resp., K_2) in which p is inert, and every prime $p \neq l | N_E$ splits (resp., for every prime $p \neq l|N_E$, $\psi_2(l) = \epsilon_K(l)$). Assuming the classical Birch and Swinnerton-Dyer conjecture (for $E/\mathbb{Q}$ and E/K_j), we obtain (multiplying t by the square of the 'algebraic part' of $L'(E/\mathbb{Q},1)$):

$$\ell \equiv t \overset{?}{\equiv} \mathbf{BSD}(E,K_1) \cdot \mathbf{BSD}(E,K_2) \equiv \prod_{l|N,\ \epsilon_K(l)=-1} c_l \bmod (\mathbb{Q}^*)^2$$

(with $N = N_E/p$). Recalling the definitions, this is in line with the prediction of Conj. (6.1).

Remark 7.7. Assume that K is *real* quadratic, satisfying the (classical) Heegner hypothesis: every prime $l|N_E$ splits in K. Under these assumptions, in [**Sha08**] a square root p-adic L-function $\mathscr{L}_p(f_\infty/K,k)$ is constructed, satisfying (‡) $\mathscr{L}_p(f_\infty/K,k)^2 = D_K^{\frac{k-2}{2}} \cdot L_p(f_\infty,k,k/2) \cdot L_p(f_\infty,\epsilon_K,k,k/2)$ (cfr. Remark (6.2.6)). Assuming that $E/\mathbb{Q}$ has a prime of semistable reduction other than p, we can then rephrase [**Sha08**, Th. *B*] in the following way. There is a scalar $q \in \mathbb{Q}^*$ such that

$$\frac{d^2}{dk^2}\mathscr{L}_p(f_\infty/K,k)_{k=2} = q \cdot \left\langle q_E^+, \mathbf{P}^+\right\rangle_{K,p}^{\mathrm{Nek}} \cdot \left\langle q_E^-, \mathbf{P}^-\right\rangle_{K,p}^{\mathrm{Nek}}, \tag{63}$$

obtaining a more precise version of formula (62). (We remark that, once $\mathscr{L}_p(f_\infty/K,k)$ satisfying (‡) is constructed, we obtain (63) from (62) taking $q := \pm 2 \cdot \sqrt{t}$, since [**BD07**, Th. 5.4] gives us $t \in (\mathbb{Q}^*)^2$.)

Part 2

Organizing modules for Hida families

8. Introduction

Let us fix the data (E, K, p), consisting of a number field K, a rational prime $p \geq 5$ and an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at p. Let $\mathcal{K}/K$ be the maximal $\mathbb{Z}_p$-power extensions of K. Given $K \subset L \subset \mathcal{L} \subset \mathcal{K}$, we write $\mathbb{Z}_p(L) := \mathbb{Z}_p[[\mathrm{Gal}(L/K)]]$ and $I_{\mathcal{L}/L} := \ker(\mathbb{Z}_p(\mathcal{L}) \twoheadrightarrow \mathbb{Z}_p(L))$. We write $\iota : \mathbb{Z}_p(L) \to \mathbb{Z}_p(L)$ for the involution induced by inversion on $\mathrm{Gal}(L/K)$, and for every $\mathbb{Z}_p(L)$-module $\ddagger$, $\ddagger^\iota$ denotes the $\mathbb{Z}_p(L)$-module obtained composing the original $\mathbb{Z}_p(L)$-action with ι.

8.1. Galois deformations. For every integer $n \geq 1$ and every finite subextension L/K of $\mathcal{K}/K$, Kummer theory associates to (E, K, p) the p^n-Selmer group

$$\mathrm{Sel}_{p^n}(L, E) := \ker\left[H^1(L, E_{p^n}) \to \prod_v H^1(L_v, E)_{p^n}\right],$$

where v runs over the (finite) places of L. Let us write

$$\mathrm{Sel}_{p^\infty}(L, E) := \varinjlim_{n\geq 1} \mathrm{Sel}_{p^n}(L, E); \quad M_p(L, E) := \varprojlim_{n\geq 1} \mathrm{Sel}_{p^n}(L, E),$$

where the direct (resp., inverse) limit is taken with respect to the maps induced on Galois cohomology by the inclusion $E_{p^n}(\overline{\mathbb{Q}}) \subset E_{p^{n+1}}(\overline{\mathbb{Q}})$ (resp., by multiplication by p: $E_{p^{n+1}}(\overline{\mathbb{Q}}) \twoheadrightarrow E_{p^n}(\overline{\mathbb{Q}})$). These groups fits into short exact sequences

$$0 \to E(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(L, E) \to Ш(E/L)_{p^\infty} \to 0;$$
$$0 \to E(L)\widehat{\otimes}\mathbb{Z}_p \to M_p(L, E) \to \mathrm{Ta}_p\left(Ш(E/L)\right) \to 0,$$

where $Ш(E/L) \subset H^1(L, E)$ is the Tate-Shafarevich group of everywhere locally trivial cocycle, and $\mathrm{Ta}_p(*) := \varprojlim_{n\geq 1} *_{p^n}$ is the Tate module of the $\mathbb{Z}_p$-module $*$. For an arbitrary tower of extensions $K \subset L \subset \mathcal{K}$ we also write:

$$\mathrm{Sel}_{p^\infty}(L, E) := \varinjlim_{K\subset_f L_\alpha \subset L} \mathrm{Sel}_{p^\infty}(L_\alpha, E); \quad S_p(L, E) := \mathrm{Hom}_{\mathrm{cts}}\left(\mathrm{Sel}_{p^\infty}(L, E), \mathbb{Q}_p/\mathbb{Z}_p\right),$$
$$M_p(L, E) := \varprojlim_{K\subset_f L_\alpha \subset L} M_p(L_\alpha, E),$$

where L_α/K runs over the finite subextension of L/K, and the direct (resp., inverse) limit is taken with respect to the restriction (resp., corestriction) maps. Then Galois conjugation induces an action of the Iwasawa algebra $\mathbb{Z}_p(L)$ on $\mathrm{Sel}_{p^\infty}(L, E)$ and $M_p(L, E)$, and it is known that $\mathrm{Sel}_{p^\infty}(L, E)$ (resp., $M_p(L, E)$) is indeed of cofinite (resp., finite) type over $\mathbb{Z}_p(L)$, i.e. $S_p(L, E)$ is a finite $\mathbb{Z}_p(L)$-module. Here we consider on $S_p(L, E)$ the $\mathbb{Z}_p(L)$-modules structure defined by $\phi^\lambda(*) := \phi(\iota(\lambda) \cdot *)$ for every $\lambda \in \mathbb{Z}_p(L)$ and every $\phi \in S_p(L, E)$.

The Cassels-Tate pairing defines, for every finite extension L/K a skew-symmetric bilinear form

$$S_p(L, E)_{\mathrm{tors}} \otimes S_p(L, E)_{\mathrm{tors}} \to \mathbb{Q}_p/\mathbb{Z}_p. \tag{64}$$

Moreover, for every finite extension L/K the canonical p-adic height pairing introduced by Schneider, Perrin-Riou *et. al.* defines a symmetric $\mathbb{Z}_p$-bilinear form:

$$M_p(L, E) \otimes M_p(L, E) \longrightarrow \mathrm{Gal}(\mathcal{K}/L) \otimes \mathbb{Q}, \tag{65}$$

whose definition comes from a study of the Selmer group $\mathrm{Sel}_{p^\infty}(\mathcal{K}, E)$, or equivalently (via Shapiro's lemma, ctf. [**Gre94b**]) studying continuous Galois cohomology of the Galois deformation $\mathrm{Ta}_p(E_{/\mathbb{Q}})[[\mathrm{Gal}(\mathcal{K}/L)]]$ of the p-adic Tate module of $E_{/\mathbb{Q}}$.

When $S_p(\mathcal{K}, E)$ is (as expected) a torsion $\mathbb{Z}_p(\mathcal{K})$-module, we are also interested in the algebraic p-adic L-function

$$\mathbf{L}_p(\mathcal{K}, E) := \mathrm{char}_{\mathbb{Z}_p(\mathcal{K})}\left(S_p(\mathcal{K}, E)\right) \in \mathbb{Z}_p(\mathcal{K})/\mathbb{Z}_p(\mathcal{K})^*,$$

where $\mathrm{char}_{\mathbb{Z}_p(\mathcal{K})}(\ddagger)$ denotes the characteristic ideal of the finite $\mathbb{Z}_p(\mathcal{K})$-module $\ddagger$. Since $\mathbb{Z}_p(\mathcal{K})$ is regular, this is a principal (non-zero) ideal which we identify, up to p-adic units with any of its generator. In some cases we know that $\mathbf{L}_p(\mathcal{K}, E) \in I^r_{\mathcal{K}/K}$, where $r := \mathrm{rank}_{\mathbb{Z}} E(K)$ and that (up to p-adic units) its image in

$I^r_{\mathcal{K}/K}/I^{r+1}_{\mathcal{K}/K}\otimes\mathbb{Q}$ is given by the product of arithmetic invariants of E/K (e.g. Tamagawa factors and the order of the Tate-Shafarevich group) and the discriminant of the pairing (65) for $L=K$ (viewed as an element of $I^r_{\mathcal{K}/K}/I^{r+1}_{\mathcal{K}/K}\otimes\mathbb{Q}$).

In [**MR04**],[**MR02**] the authors proposed that all these structures can be packaged in a single linear-algebraic object, essentially a skew-Hermitian matrix with entries in $\mathbb{Z}_p(\mathcal{K})$. More precisely, let $\mathcal{R}=(\mathcal{R},\mathfrak{m}_{\mathcal{R}})$ be a commutative local ring equipped with an involution $\iota:\mathcal{R}\to\mathcal{R}$. They define a *basic skew-Hermitian* $\mathcal{R}$-module Φ to be a finite, free $\mathcal{R}$-module, equipped with a skew-Hermitian pairing

$$(-,-)_h:\Phi\otimes_{\mathcal{R}}\Phi^\iota\to\mathfrak{m}_{\mathcal{R}},$$

which is non-degenerate in that the adjoint map $h:=\mathrm{adj}\left((-,-)_h\right):\Phi\to\Phi^*:=\mathrm{Hom}_{\mathcal{R}}\left(\Phi^\iota,\mathcal{R}\right)$ is injective. For every ι-stable ideal $I\subset\mathcal{R}$, they define the R/I-modules $S(\Phi,I)$ and $M(\Phi,I)$ by the exact sequence

$$0\to M(\Phi,I)\to\Phi\otimes\mathcal{R}/I\stackrel{h_I}{\to}\Phi^*\otimes\mathcal{R}/I\to S(\Phi,I)\to 0, \tag{66}$$

where $h_I:=h\otimes_{\mathcal{R}}\mathcal{R}/I$. For every such an ideal, they also construct natural skew-Hermitian pairings

$$c_{\Phi,I}:S(\Phi,I)_{\mathrm{tors}}\otimes S(\Phi,I)^\iota_{\mathrm{tors}}\longrightarrow\mathrm{Frac}(\mathcal{R}/I)/(\mathcal{R}/I); \tag{67}$$

$$h_{\Phi,I}:M(\Phi,I)\otimes M(\Phi,I)^\iota\longrightarrow I/I^2, \tag{68}$$

where $\mathrm{Frac}(*)$ denotes the total ring of fractions of $*$, $\dagger_{\mathrm{tors}}:\ker\left(\dagger\to\dagger\otimes_*\mathrm{Frac}(*)\right)$ and 'skew-Hermitian' refers to the involutions induced on quotient modules by ι. These pairings can be 'naively' described as follows: let $x,y\in M(\Phi,I)$ (resp., $\alpha,\beta\in S(\Phi,I)_{\mathrm{tors}}$) be classes modulo I (resp., modulo $\mathrm{Im}(h_I)$) represented by $\widetilde{x},\widetilde{y}\in\Phi$ (resp., $\widetilde{\alpha},\widetilde{\beta}\in\Phi^*\otimes_{\mathcal{R}}\mathcal{R}/I$). Then, writing $(-,-)_{h_I}:=(-,-)_h \bmod I$ we have:

$$c_{\Phi,I}\left(\alpha\otimes\beta\right):=(s_{\widetilde{\alpha}}s_{\widetilde{\beta}})^{-1}\cdot(\gamma_{\widetilde{\alpha}},\gamma_{\widetilde{\beta}})_{h_I} \bmod \mathcal{R}/I;\quad h_{\Phi,I}(x\otimes y):=(\widetilde{x},\widetilde{y})_h \bmod I^2\in I/I^2,$$

where $s_*\in\mathcal{R}/I$ and $\gamma_*\in\Phi\otimes\mathcal{R}/I$ are such that $s_*\cdot *=h_I(\gamma_*)$. See Section **??** for the details.

In *loc. cit.* the authors proposed that under fairly general conditions on (E,K,p) there exists a skew-Hermitian $\mathbb{Z}_p(\mathcal{K})$-module $\Phi=(\Phi,h)$ which organizes the arithmetic of (E,K,p) in the following sense: for every $K\subset L\subset\mathcal{K}$

a) there exist natural isomorphisms of $\mathbb{Z}_p(L)$-modules

$$S(\Phi,I_{\mathcal{K}/L})\cong S_p(L,E);\quad M(\Phi,I_{\mathcal{K}/L})\cong M_p(L,E).$$

b) If L/K is finite the $\mathbb{Z}_p$-bilinear form:

$$M(\Phi,I_{\mathcal{K}/L})\otimes M(\Phi,I_{\mathcal{K}/L})^\iota\stackrel{h_{\Phi,I_{\mathcal{K}/L}}}{\longrightarrow}I_{\mathcal{K}/L}/I^2_{\mathcal{K}/L}\cong\mathrm{Gal}(\mathcal{K}/L)\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(L)\stackrel{\mathrm{id}\otimes\mathrm{pr}_1}{\longrightarrow}\mathrm{Gal}(\mathcal{K}/L)$$

corresponds to the canonical height pairing (65) under the second isomorphisms in *a*). Here $\mathrm{pr}_1:\mathbb{Z}_p(L):=\mathbb{Z}_p[\mathrm{Gal}(L/K)]\to\mathbb{Z}_p$ is defined by $\sum_{g\in\mathrm{Gal}(K/L)}x_g\cdot g\mapsto x_1$ and the isomorphism of $\mathbb{Z}_p(L)$-modules above is characterized by the property: $\tau-1 \bmod I^2_{\mathcal{K}/L}\mapsto\tau\otimes 1$ for every $\tau\in\mathrm{Gal}(\mathcal{K}/L)$.

c) If L/K is finite, the $\mathbb{Z}_p$-bilinear pairing:

$$S(\Phi,I_{\mathcal{K}/L})_{\mathrm{tors}}\otimes S(\Phi,I_{\mathcal{K}/L})_{\mathrm{tors}}\stackrel{c_{\Phi,I_{\mathcal{K}/L}}}{\longrightarrow}\mathrm{Frac}(\mathbb{Z}_p(L))/\mathbb{Z}_p(L)\cong\mathbb{Q}_p/\mathbb{Z}_p\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(L)\stackrel{\mathrm{id}\otimes\mathrm{pr}_1}{\longrightarrow}\mathbb{Q}_p/\mathbb{Z}_p$$

corresponds to the Cassels-Tate pairing (64) under the first isomorphisms in *a*).

In particular $S_p(\mathcal{K},E)$ is a torsion $\mathbb{Z}_p(\mathcal{K})$-module and fixing any $\mathbb{Z}_p(\mathcal{K})$-basis of Φ we have

$$\mathbf{L}_p(\mathcal{K},E)=\det\left(H_\Phi\right)\cdot\mathbb{Z}_p(\mathcal{K}),$$

where H_Φ is the skew-Hermitian matrix describing h. We can easily deduce from this (cfr. [**MR04**]) an algebraic p-adic BSD formula describing the 'leading coefficent' of $\mathbf{L}_p(\mathcal{K},E)$ in terms of the determinant of the p-adic height pairing on $M_p(E,K)$ (as described above).

Using the work [**Nek06**], in [**MR05**] the same authors proved that, under some additional assumptions such an organizing module Φ exists, and is unique up to (noncanonical) isomorphism. More precisely, the complex $\mathbf{\Phi}:=\left(\Phi\stackrel{h}{\to}\Phi^*\right)$ concentrated in degrees 1 and 2 turns out to be essentially Nekovář's Selmer complex $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K,T_p)$ attached to the p-adic Tate module $T_p := \varprojlim E_{p^n}$. For example, the existence of an organizing module can be proved assuming the following Hypothesis on (E,K,p): for every finite prime v of K, write k_v for the residue field of the completion K_v.

H1. $K/\mathbb{Q}$ is an abelian extension;

H2. for every prime $v|p$ of K, $p \nmid \#\left(\widetilde{E}_v(k_v)\right)$, where $\widetilde{E}_v$ is the reduction of $E_{/K_v}$;

H3. for every prime $v|\mathrm{cond}(E)$ of K, $E(K_v)$ has no non-trivial p-torsion.

We remark that these hypothesis can be weakened. Hypothesis H1 is made to ensure that $S_p(\mathcal{K},E)$ is a torsion $\mathbb{Z}_p(\mathcal{K})$-module, as follows by the work of Kato and Rohlrich. Hypothesis H3 can be relaxed assuming only that p does not divide any of the Tamagawa numbers of E/K_v for $v|\mathrm{cond}(E)$. (We assume in this note the stronger condition H3 since this 'trivializes' unramified local conditions at any place $v \nmid p$, simplifying somehow the exposition.)

8.2. Adding the weight variable. Write $N := \mathrm{cond}(E)$ and let $f_E \in S_2(\Gamma_0(N),\mathbb{Z})$ be the newform attached to $E/\mathbb{Q}$ by the modularity theorem. Hida theory attaches to $E/\mathbb{Q}$ a formal q-expansion

$$\mathbf{g} = \sum_{n\geq 1} \mathbf{a}_n(X)\cdot q^n \in R[[q]],$$

where $R = R_E$ is a local domain, finite over the 'diamond algebra' $\Lambda := \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$. The domain R parametrizes the *Hida family* of f_E. For the purposes if this introduction we only state the following 'weak parametrization property': for every even integer κ in a suitable open disk $U \subset \mathbb{Z}_p$ centered at 2, there exists an *arithmetic point* $\psi_\kappa : R \to \mathbb{Z}_p$ such that

$$g_\kappa := \sum_{n\geq 1} \psi_\kappa(\mathbf{a}_n) \in S_\kappa\left(\Gamma_1(Np),\mathbb{Z}_p\right)$$

is the q-expansion of a (classical) normalized eigenform of level $\Gamma_1(Np)$ and nebentype $\omega^{2-\kappa}$, where ω is the Teichmüller character. g_κ is a p-stabilized ordinary newform of tame conductor N, i.e. $a_p(g_\kappa) = \psi_\kappa(\mathbf{a}_p)$ is a p-adic unit, and the conductor of g_κ is N or Np. Moreover, g_2 is obtained by f_E via the process of p-stabilization (Sec. 9.2 for the details). We call $\mathbf{g}$, or the family $\{g_\kappa\}_\kappa$ the *Hida family* of $E/\mathbb{Q}$. Together with Hypothesis H1-H3 above, we consider for the rest of the introduction the following Hypothesis on (E,p):

H4. $E(\overline{\mathbb{Q}})_p$ is an irreducible $\mathbb{F}_p[G_\mathbb{Q}]$-module (where $G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$);

H5. R is a regular local ring.

In Section 9.3 we also recall Hida's construction of a big p-ordinary self-dual Galois R-representation $\mathbf{T}$ interpolating critical twists of the Deligne representations attached to members of the Hida family $\mathbf{g}$. More precisely, $\mathbf{T}$ is a free R-module of rank two, equipped with a continuous, R-linear action of $G_\mathbb{Q}$ which is unramified at every place $v \nmid N\cdot p\cdot\infty$. For every place v of $\overline{\mathbb{Q}}$ dividing p, there exists a short exact sequence of $R[G_v]$-modules

$$0 \to F_v^+(\mathbf{T}) \to \mathbf{T} \to F_v^-(\mathbf{T}) \to 0,$$

with $F_v^\pm(\mathbf{T}) \xrightarrow{\sim} R$ as R-modules. (Here $G_v \subset G_\mathbb{Q}$ is the decomposition group at v). The interpolation property can be stated as follows: let $\kappa \in U$ be an even integer, and consider the base change $\mathbf{T}_{\psi_\kappa} = \mathbf{T}_\kappa := \mathbf{T}\otimes_{R,\psi_\kappa}\mathbb{Z}_p$. Then $\mathbf{T}_\kappa\otimes\mathbb{Q}_p$ is isomorphic to the Tate twist $V_\kappa := V(\widetilde{g_\kappa})(1-\kappa/2)$ of the Deligne representation $V(\widetilde{g_\kappa})$ of the twisted modular form $\widetilde{g_\kappa} := g_\kappa\otimes\omega^{1-\kappa/2}$. We note that $\widetilde{g_\kappa}$ has level $\Gamma_0(Np)$, and that $\det V_\kappa = \mathbb{Q}_p(1)$, expressing a weak form of the self-duality of $\mathbf{T}$ alluded to above. At weight $\kappa = 2$ we can be more explicit. Thanks to assumption H4, $\mathbf{T}_2 \cong T_p := \varprojlim E(\overline{\mathbb{Q}})_{p^n}$ is the p-adic Tate module of $E/\mathbb{Q}$ as a $G_\mathbb{Q}$-modules. Moreover this induces an isomorphism of G_v-modules between $F_v^+(\mathbf{T}_2)$ and the

p-adic Tate module of $\widehat{E}(\overline{\mathfrak{m}})$, where $\widehat{E}_{/\mathbb{Z}_p}$ is the formal group of $E/\mathbb{Q}_p$ and $\overline{\mathfrak{m}}$ is the maximal ideal of the ring of integers of $\overline{\mathbb{Q}_p}$ (see Section 9.3.4).

8.2.1. *Greenberg Selmer groups.* Let $\mathcal{R}$ be a complete local Noetherian ring with finite residue field of characteristic p, and let M be an $\mathcal{R}[\mathfrak{G}_K]$-module, where $\mathfrak{G}_K := \mathrm{Gal}(K_{Np}/K)$ is the Galois group of the maximal algebraic extension K_{Np}/K which is unramified at every prime $v \nmid Np\infty$ of K. Write Σ_K for the set of primes of K lying over a prime factor of Np. Assume that M is *quasi-ordinary at* p, i.e. assume that for every prime $v|p$ of K there exists an $\mathcal{R}[G_v]$-subsmodule $F_v^+(M) \subset M$, where $G_v \subset G_K \twoheadrightarrow \mathfrak{G}_K$ is a fixed decomposition group at v. Assume also that M is *admissible* (or *continuous*) as an $\mathcal{R}[\mathfrak{G}_K]$-module in the sense of [**Nek06**, Chap. 3]. Then the continuous cohomology groups $H^j(\mathfrak{G}_K, *)$, $H^j(K_w, *)$ are defined for every $\mathcal{R}[\mathfrak{G}_K]$ submodule $*$ of M, as well as the cohomology groups $H^j(K_v, F_v^+(M))$ for every place $v|p$ of K. For every $v \in \Sigma_K$ define the *ordinary part* of $H^1(K_v, M)$:

$$H^1_{\mathrm{ord}}(K_v, M) := \begin{cases} \mathrm{Im}\left(H^1(K_v, F_v^+(M)) \to H^1(K_v, M)\right) & \text{if } v|p\ ; \\ \\ \mathrm{Im}\left(H^1(G_v/I_v, M^{I_v}) \to H^1(K_v, M)\right) & \text{if } v \nmid p, \end{cases}$$

where $I_v \subset G_v$ denotes the inertia subgroup and the maps are the natural ones. The *Greenberg Selmer group* attached to the data $(M, K, \{M_v^+\}_{v|p})$ is then defined by:

$$\mathrm{Sel}_{\mathrm{Gr}}(K, M) := \left\{x \in H^1(\mathfrak{G}_K, M) \ : \ \mathrm{res}_v(x) \in H^1_{\mathrm{ord}}(K_v, M) \text{ for every } v \in \Sigma_K\right\}.$$

More generally: let L/K be a finite subextension of K_{Np}/K, let w be a prime of L dividing p and write $\mathfrak{G}_L := \mathrm{Gal}(K_{Np}/L)$. Fixing a decomposition group $G_w \subset G_L \twoheadrightarrow \mathfrak{G}_L$ and letting v be the prime of K lying below w, the 'v-ordinary structure' $F_v^+(M) \subset M$ naturally give rise to a 'w-ordinary structure' $F_w^+(M) \subset M$ on the $\mathcal{R}[G_w]$-module M (see the beginning of Appending B for more details). The Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(L, M) \subset H^1(\mathfrak{G}_L, M)$ is then defined as above. Finally: let F/K be an arbitrary subextension of K_{Np}/K. We then define:

$$\mathrm{Sel}_{\mathrm{Gr}}(F, M) := \varinjlim_{F_\alpha} \mathrm{Sel}_{\mathrm{Gr}}(F_\alpha, M); \quad M_{\mathrm{Gr}}(F, M) := \varprojlim_{F_\alpha} \mathrm{Sel}_{\mathrm{Gr}}(F_\alpha, M),$$

where F_α/K runs over the set of finite subextension of E/K and the direct (resp., inverse) limit is taken with respect to the restriction (resp., corestriction) maps in Galois cohomology.

Let $\kappa \in U$ be an even integer, and let $\mathcal{T} \in \{\mathbf{T}, \mathbf{T}_\kappa\}$. Write $\mathbb{A}_{\mathcal{T}} := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{T}, \mu_{p^\infty})$ for the Kummer dual of $\mathcal{T}$, equipped with the v-ordinary structure $F_v^+(\mathbb{A}_{\mathcal{T}}) := \mathrm{Hom}_{\mathrm{cts}}(F_v^-(\mathcal{T}), \mu_{p^\infty})$ for every prime $v|p$ of K. For every $K \subset L \subset \mathcal{K}$ define the Selmer groups:

$$\mathrm{Sel}_p(L, \mathbf{g}) := \mathrm{Sel}_{\mathrm{Gr}}(L, \mathbb{A}_{\mathbf{T}}); \quad \mathrm{Sel}_p(L, g_\kappa) := \mathrm{Sel}_{\mathrm{Gr}}(L, \mathbb{A}_{\mathbf{T}_\kappa});$$

$$S_p(L, \mathbf{g}) := \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Sel}_p(L, \mathbf{g}), \mathbb{Q}_p/\mathbb{Z}_p\right); \quad S_p(L, g_\kappa) := \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Sel}_p(L, g_\kappa), \mathbb{Q}_p/\mathbb{Z}_p\right),$$

and the modules of L/K-*universal norms*

$$M_p(L, \mathbf{g}) := M_{\mathrm{Gr}}(L, \mathbf{T}); \quad M_p(L, g_\kappa) := M_{\mathrm{Gr}}(L, \mathbf{T}_\kappa).$$

Galois conjugation induces an action of $R(L)$ (resp., $\mathbb{Z}_p(L)$) on $\mathrm{Sel}_p(L, \mathbf{g})$ and $M_p(L, \mathbf{g})$ (resp., $\mathrm{Sel}_p(L, g_\kappa)$ and $M_p(L, g_\kappa)$). We consider on $S_p(L, \mathbf{g})$ (resp., $S_p(L, g_\kappa)$) the $R(L)$-action (resp., $\mathbb{Z}_p(L)$-action) defined by $\phi^\lambda(*) := \phi(\iota(\lambda) \cdot *)$ for every $\lambda \in R(L)$ (resp., $\lambda \in \mathbb{Z}(L)$). We know [**Gre94b**] that $S_p(L, \mathbf{g})$ and $M_p(L, \mathbf{g})$ (resp., $S_p(L, g_\kappa)$ and $M_p(L, g_\kappa)$) are finite $R(L)$-modules (resp., $\mathbb{Z}_p(L)$-modules). Moreover, thanks to the work of Kato and Rohrlich, we know that $S_p(\mathcal{K}, \mathbf{g})$ (resp., $S_p(\mathcal{K}, g_\kappa)$) is a torsion $R(\mathcal{K})$-module (resp., $\mathbb{Z}_p(\mathcal{K})$-module).

8.2.2. *Height and weight pairings.* The Selmer groups introduced above come equipped with the following 'arithmetic-cohomological structures' (see Section 11.4 for the precise definitions): let $K \subset L \subset \mathcal{L} \subset \mathcal{K}$ and let $\kappa \in U$ be an evan integer. We write

$$\widetilde{I}_{\mathcal{L}/L} := \ker\left(R(\mathcal{L}) \twoheadrightarrow R(L)\right); \quad J_{\mathcal{L}/L,\kappa} := \ker\left(R(\mathcal{L}) \twoheadrightarrow R(L) \overset{\overline{\psi_\kappa}}{\twoheadrightarrow} \mathbb{Z}_p(L)\right),$$

where $\overline{\psi_\kappa}$ is the map induced on Iwasawa algebras by the arithmetic map ψ_κ.

- Nekovář's duality for Selmer complexes [**Nek06**, Ch. 11] attaches to every tower $K \subset L \subset \mathcal{L} \subset \mathcal{K}$ and every even integer $\kappa \in U$ skew-Hermitian *canonical p-adic height pairings*

$$h_{\mathcal{L}/L,\mathbf{g}} : M_p(L,\mathbf{g}) \otimes_{R(L)} M_p(L,\mathbf{g})^\iota \to \widetilde{I}_{\mathcal{L}/L}/\widetilde{I}^2_{\mathcal{L}/L};$$

$$h_{\mathcal{L}/L,g_\kappa} : M_p(L,g_\kappa) \otimes_{\mathbb{Z}_p(L)} M_p(L,g_\kappa)^\iota \to I_{\mathcal{L}/L}/I^2_{\mathcal{L}/L},$$

defined in terms of Galois cohomology of the (constant) 'Galois deformations' $\mathbf{T}(\mathcal{L}) := \mathbf{T} \otimes_R R(\mathcal{L})$ and $\mathbf{T}_\kappa(\mathcal{L}) := \mathbf{T}_\kappa \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\mathcal{L})$ of $\mathbf{T}(L) = \mathbf{T}(\mathcal{L})/\widetilde{I}_{\mathcal{L}/L}$ and $\mathbf{T}_\kappa(L) = \mathbf{T}_\kappa(\mathcal{L})/I_{\mathcal{L}/L}$ respectively. These pairings are compatible under specialization at weight κ, i.e. we have a commutative diagram

$$\begin{array}{ccccc} M_p(L,\mathbf{g}) & \times & M_p(L,\mathbf{g})^\iota & \xrightarrow{h_{\mathcal{L}/L,\mathbf{g}}} & \widetilde{I}_{\mathcal{L}/L}/\widetilde{I}^2_{\mathcal{L}} \\ \downarrow{\scriptstyle \psi_{\kappa *}} & & \downarrow{\scriptstyle \psi_{\kappa *}} & & \downarrow{\scriptstyle \psi_\kappa} \\ M_p(L,g_\kappa) & \times & M_p(L,g_\kappa)^\iota & \xrightarrow{h_{\mathcal{L}/L,g_\kappa}} & I_{\mathcal{L}/L}/I^2_{\mathcal{L}/L}, \end{array}$$

where $\psi_{\kappa *}$ is the map induced on cohomology groups by the arithmetic map ψ_κ.
- Nekovář's wide generalization of Cassels-Tate and Flach pairings gives for every $K \subset L \subset \mathcal{K}$ and every even integer $\kappa \in U$ skew-Hermitian parings:

$$c_{L,\mathbf{g}} : S_p(L,\mathbf{g})_{R(L)\text{-tors}} \otimes_{R(L)} S_p(L,\mathbf{g})^\iota_{R(L)\text{-tors}} \to \mathrm{Frac}(R(L))/R(L);$$

$$c_{L,g_\kappa} : S_p(L,g_\kappa)_{\mathbb{Z}_p(L)\text{-tors}} \otimes_{\mathbb{Z}_p(L)} S_p(L,g_\kappa)^\iota_{\mathbb{Z}_p(L)\text{-tors}} \to \mathrm{Frac}(\mathbb{Z}_p(L))/\mathbb{Z}_p(L),$$

defining again studying the Galois cohomology of the modules $\mathbf{T} \otimes_R R(L)$ and $\mathbf{T}_\kappa \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(L)$ respectively.
- Making again use of Nekovář's formalism, we will attach in Section 10.5 to every $\kappa \in U$ and every $K \subset L \subset \mathcal{L} \subset \mathcal{K}$ a lift of $h_{\mathcal{L}/L,g_\kappa}$ to a *canonical pairing*

$$h^{\mathrm{wt}}_{\mathcal{L}/L,g_\kappa} : M_p(L,g_\kappa) \otimes_{\mathbb{Z}_p(L)} M_p(L,g_\kappa)^\iota \longrightarrow J_{\mathcal{L}/L,\kappa}/J^2_{\mathcal{L}/L,\kappa},$$

More precisely, the composition of $h^{\mathrm{wt}}_{\mathcal{L}/L,g_\kappa}$ with the projection induced by $\overline{\psi}_\kappa : J_{\mathcal{L}/L,\kappa} \twoheadrightarrow I_{\mathcal{L}/L}$ equals $h_{\mathcal{L}/L,g_\kappa}$. This pairing is defined studying the Galois cohomology of the 'two-variable' deformation $\mathbf{T}(\mathcal{L})$ of $\mathbf{T}_\kappa(L) = \mathbf{T}(\mathcal{L})/J_{\mathcal{L}/L,\kappa}$. In particular the pairings $h_{L,\kappa} := h_{L/L,\kappa}$ is attached intrinsically to the Hida deformation $\mathbf{T}(L)$ of $\mathbf{T}_\kappa(L)$.

The height and the abstract Cassels-Tate pairings just mentioned are indeed strictly related one another, and are manifestations of Nekovář's wide generalization of Poitou-Tate duality to Selmer complexes.

At weight $\kappa = 2$ we recover the constructions above. More precisely: as proved in [**Gre94b**] there exists natural isomorphisms $S_p(L,E) \cong S_p(L,g_2)$ and $M_p(L,E) \cong M_p(L,g_2)$. Moreover it follows by the results in [**Nek06**, Sec. 11.4] and [**Nek06**, Sec. 10.] that these isomorphisms identify, for every finite subextension L/K of $\mathcal{K}/L$ the $\mathbb{Z}_p$-bilinear forms:

$$M_p(L,g_2) \otimes M_p(L,g_2) \overset{h_{\mathcal{K}/L,g_2}}{\longrightarrow} I_{\mathcal{K}/L}/I^2_{\mathcal{K}/L} \cong \mathrm{Gal}(\mathcal{K}/L) \otimes \mathbb{Z}_p(L) \overset{\mathrm{id}\otimes\mathrm{pr}_1}{\longrightarrow} \mathrm{Gal}(\mathcal{K}/L);$$

$$S_p(L,g_2)_{\mathrm{tors}} \otimes S_p(L,g_2)_{\mathrm{tors}} \overset{c_{L,g_2}}{\longrightarrow} \mathrm{Frac}(\mathbb{Z}_p(L))/\mathbb{Z}_p(L) \cong \mathbb{Q}_p/\mathbb{Z}_p \otimes \mathbb{Z}_p(L) \overset{\mathrm{id}\otimes\mathrm{pr}_1}{\longrightarrow} \mathbb{Q}_p/\mathbb{Z}_p$$

with the $\mathcal{K}/L$ Height pairing (64) and Cassels-Tate pairing (65) respectively.

8.2.3. *Organizing modules.* It is natural to wonder (cfr. [**MR05**, Sec. 1]) if we can lift Mazur-Rubin organizer of the arithmetic of (E,K,p) to a skew-Hermitian $R(\mathcal{K})$-module Φ which organizes the arithmetic of the whole Hida family $\mathbf{g}$ over K. This means that via the 'linear-algebraic' constructions (66), (67) and (68), Φ encodes all the above 'cohomological structures' for varying even weight $\kappa \in U$ and intermediate fields $K \subset L \subset \mathcal{L} \subset \mathcal{K}$.

The following theorem answers positively this question, at least under the running assumptions. Its proof (whose details will be given in Section 11.4) follows easily combining Nekovář's duality formalism for Selmer complexes with the work of Mazur-Rubin.

THEOREM 8.1. *Assume that (E, K, p) satisfies hypothesis H1-H5 above. Then there exists a basic skew-Hermitian $R(\mathcal{K})$-module $\Phi = (\Phi, h)$, free of rank $r_\Phi := \mathrm{rank}_{\mathbb{Z}} E(K) + \dim_{\mathbb{F}_p} Ш(E/K)_p$ over $R(\mathcal{K})$, satisfying the following properties. Write*

$$\mathcal{S} := \mathrm{coker}\left(\Phi \xrightarrow{h} \Phi^*\right) = H^2(\mathbf{\Phi}),$$

and write $H_\Phi \in \mathrm{GL}_{r_\Phi}(\mathfrak{m}_{R(\mathcal{K})})$ for the skew-Hermitian matrix describing the morphism h with respect to a (fixed) basis $\mathbf{u} = \{u_1, \dots, u_{r_\Phi}\}$ of Φ.

1. *There exists a canonical isomorphism of $R(\mathcal{K})$-modules*

$$\mathcal{S} \cong S_p(\mathcal{K}, \mathbf{g}).$$

For every sub-extension L/K of $\mathcal{K}/L$, there exist canonical isomorphisms of $R(L)$-modules

$$(\dagger) \qquad S(\Phi, \widetilde{I}_{\mathcal{K}/L}) \cong S_p(L, \mathbf{g}); \quad M_p(\Phi, \widetilde{I}_{\mathcal{K}/L}) \cong M_p(L, \mathbf{g}).$$

2. *For every even integer $\kappa \in U$ and every intermediate field $K \subset L \subset \mathcal{K}$, there exist canonical isomorphisms of $\mathbb{Z}_p(L)$-modules*

$$(\ddagger) \qquad S(\Phi, J_{\mathcal{K}/L,\kappa}) \cong S_p(L, g_\kappa); \quad M_p(\Phi, J_{\mathcal{K}/K,\kappa}) \cong M_p(L, g_\kappa).$$

3. *For every sub-extension $K \subset L \subset \mathcal{K}$, the second isomorphisms in $(\dagger)$ identifies the pairing (68) for $I = \widetilde{I}_{\mathcal{K}/K}$: $h_{\Phi, \widetilde{I}_{\mathcal{K}/K}}$ with the canonical p-adic height pairing $h_{\mathcal{K}/L, \mathbf{g}}$. Similarly the first isomorphism in $(\dagger)$ identifies the pairing (67) for $\widetilde{I}_{\mathcal{K}/K}$: $c_{\Phi, \widetilde{I}_{\mathcal{K}/L}}$ with the abstract Cassels-Tate pairing $c_{L,\mathbf{g}}$.*

4. *For every intermediate field $K \subset L \subset \mathcal{K}$, the second isomorphisms in $(\ddagger)$ identifies the pairing (68) for $I = J_{\mathcal{K}/K,\kappa}$: $h_{\Phi, J_{\mathcal{K}/K,\kappa}}$ with $h^{\mathrm{wt}}_{\mathcal{K}/K, g_\kappa}$. Similarly the first isomorphism in $(\ddagger)$ identifies the pairing (67) for $I = J_{\mathcal{K}/K,\kappa}$: $c_{\Phi, J_{\mathcal{K}/K,\kappa}}$ with c_{L, g_κ}.*

5. *Write $f_\Phi := \det(H_\Phi)$ and $f_\Phi^\kappa := \overline{\psi}_\kappa(f_\Phi)$ for every even integer $\kappa \in U$. Then*

$$f_\Phi \cdot R(\mathcal{K}) = \mathrm{char}_{R(\mathcal{K})}\left(S_p(\mathcal{K}, \mathbf{g})\right); \quad f_\Phi^\kappa \cdot \mathbb{Z}_p(\mathcal{K}) = \mathrm{char}_{\mathbb{Z}_p(\mathcal{K})}\left(S_p(\mathcal{K}, g_\kappa)\right).$$

Writing $r_\kappa := \mathrm{rank}_{\mathbb{Z}_p}\left(S_p(K, g_\kappa)\right)$, f_Φ satisfies the 'functional equation'

$$\iota(f_\Phi) = (-1)^{r_\kappa} \cdot f_\Phi.$$

6. *(p-adic BSD formula) $f_\Phi \in \left(J_{\mathcal{K}/K,\kappa}\right)^{r_\kappa}$ and, up to p-adic units*

$$f_\Phi \equiv \#\left(S_p(K, g_\kappa)_{\mathrm{tors}}\right) \cdot \det\left(h^{\mathrm{wt}}_{\mathcal{K}/K, g_\kappa}\right) \mod J^{r_\kappa+1}_{\mathcal{K}/K,\kappa}.$$

The complex $\mathbf{\Phi}$, together with its skew-Hermitian structure is isomorphic in the derived category $\mathcal{D} = \mathcal{D}(R(\mathcal{K}))$ of complexes of $R(\mathcal{K})$-modules to Nekovář's Selmer complex $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$, equipped with its skew-Hermitian global cup-product pairing (see Sec. 11.2 for precise definitions).

Notations and assumptions. The following notations will remain fixed through this note.

- $p \geq 5$ is a rational prime,
- $E_{/\mathbb{Q}}$ is an elliptic curve of conductor N_E, with ordinary (i.e. good ordinary or multiplicative) reduction at p.
- $N := N_E \cdot p^{-\mathrm{ord}_p(N_E)}$ is the *tame conductor* of $E_{/\mathbb{Q}}$.
- We fix an embedding $\rho_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, under which we consider $\overline{\mathbb{Q}}$ as a subfield of $\overline{\mathbb{Q}_p}$. This also fixes a decomposition group $G_p := \rho_p^*(G_{\mathbb{Q}_p}) \subset G_{\mathbb{Q}}$.
- $K/\mathbb{Q}$ is a number field.
- $S_f = S_{K,f} := \{v|N \cdot p\}$ is the set of primes of K dividing $N \cdot p$ and $S := S_f \cup \{v|\infty\}$. We also write $S_p := \{v \in S_f : v|p\}$, $S_p^{\mathrm{good}} := \{v \in S_p : E_{/K}$ has good reduction at $v\}$ and $S_p^{\mathrm{split}} := \{v \in S_p : E_{/K}$ has split multiplicative reduction at $v\}$.
- $G_{K,S} := \mathrm{Gal}(K_S/K)$, where $K_S \subset \overline{K} = \overline{\mathbb{Q}}$ is the maximal algebraic extension of K which is unramified at any prime $v \notin S$.
- $K \subset \mathcal{K} \subset K_S$ is the maximal $\mathbb{Z}_p$-power extension of K (inside $\overline{K}$).
- For $v \in S_f$, we fix an embedding $\rho_v : \overline{K} \hookrightarrow \overline{K}_v$, where K_v is the completion of K at v. This also fixes a decomposition group $G_v := \rho_v^*(G_{K_v}) \hookrightarrow G_K \twoheadrightarrow G_{K,S}$. For every $G_{K,S}$-module M, we consider M as a G_{K_v}-module via ρ_v^*.
- For every field F s.t. $\mathrm{char}(F) = 0$ we denote by $\chi_{cy} : G_F \twoheadrightarrow \mathrm{Gal}(F(\mu_{p^\infty})/F) \hookrightarrow \mathbb{Z}_p^*$ the p-adic cyclotomic character and by $\kappa_{cy} : G_F \twoheadrightarrow 1 + p\mathbb{Z}_p$ the composition of χ_{cy} with projection to principal units.
- $\omega : \mathbb{F}_p^* \xrightarrow{\sim} \mu_{p-1} \subset \mathbb{Z}_p^*$ is the Teichmüller lift.
- Given a field F s.t. $\mathrm{char}(F) = 0$ and a $\mathbb{Z}_p[\mathrm{Gal}(\overline{F}/F)]$-module M, $M(1) := M \otimes_{\mathbb{Z}_p} \varprojlim \mu_{p^n}(\overline{F})$ is the Tate twist of M (with diagonal G_F-action).

We will always assume the following

HYPOTHESIS 1. *$E(\overline{\mathbb{Q}})_p$ is an irreducible $\mathbb{F}_p[G_{\mathbb{Q}}]$-module.*

Starting from Section 10, we also assume that (E, K, p) satisfies the following assumptions. We write $R(E, p)$ to denote the branch of Hida's universal p-ordinary Hecke algebra of tame conductor N attached to $E/\mathbb{Q}$. It is the local domain denoted R_g in Sec. 9.2 below.

HYPOTHESIS 2. *$R(E, p)$ is a regular local ring.*

HYPOTHESIS 3. *$E(K)_p = 0$ and $E(K_v)_p = 0$ for every prime $v|N$.*

Hypothesis 1 and 2 are not too restrictive. For example we have the following proposition, which follows by the discussion in [**NP00**, Sec. 4.3.9].

PROPOSITION 1. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication. Write $\mathscr{P}_E$ for the set of primes $p \geq 5$ such that:*

i) $E/\mathbb{Q}_p$ has ordinary reduction;

ii) (E, p) satisfies Hypothesis 1 and 2, with $R(E, p)$ isomorphic to the Iwasawa algebra $\mathbb{Z}_p[[X]]$.

Then $\mathscr{P}_E$ is a set of primes of Dirichlet density one.

9. Hida theory

Let $f = f_E = \sum_{n\geq 1} a_n \cdot q^n \in S_2(\Gamma_0(N_E), \mathbb{Z})$ be the newform attached to $E/\mathbb{Q}$ by the modularity theorem. Since $E/\mathbb{Q}_p$ has ordinary reduction, $a_p = a_p(E) \in \mathbb{Z}_p^*$ is a p-adic unit and Hensel's Lemma gives a factorization:

$$X^2 - a_p X + \mathbf{1}_{N_E}(p) \cdot p = (X - \alpha_p) \cdot (X - \beta_p) \in \mathbb{Z}_p[X],$$

with $\alpha_p \in \mathbb{Z}_p^*$ and $\beta_p \in p\mathbb{Z}_p$ (under ρ_p). Here $\mathbf{1}_{N_E}$ is the trivial Dirichlet character modulo N_E. The *p-stabilization* g of f is the modular form

$$g := \sum_{n\geq 1} a_n q^n - \beta_p \cdot \sum_{n\geq 1} a_n q^{np} \in S_2(\Gamma_0(Np), \mathbb{Z}_p).$$

g is a (normalized) eigenform on $\Gamma_0(Np)$, with Hecke eigenvalue a_ℓ (resp., α_p) for every prime $\ell \neq p$ (resp., for $\ell = p$). In particular its conductor is N_E. We note that, if $E/\mathbb{Q}_p$ has multiplicative reduction then $\alpha_p = a_p$, $\beta_p = 0$ and $g = f$. In any case g is a *p-stabilized ordinary newform* of tame conductor N, with the terminology of Hida.

9.1. Jacobians of modular curves. For $r \geq 0$ let $\Phi_r := \Gamma_1(p^r) \cap \Gamma_0(N)$ and write $X_r := X(\Phi_r)_{/\mathbb{Q}}$ for the modular curve of level Φ_r over $\mathrm{Spec}(\mathbb{Q})$, as defined in [**Roh97**] or [**DDT95**, Chap. I]. Then X_r is a smooth proper model over $\mathbb{Q}$ of the compact Riemann surface $\Phi_r\backslash\mathfrak{H}^* =: X_r^{\mathrm{an}}$ (where $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1_{\mathbb{Q}}$ is the extended upper half-plane), together with a $\mathbb{Q}$-morphism $\boldsymbol{j} : X_r \to \mathbb{P}^1_{\mathbb{Q}}$. The affine scheme $Y_{r/\mathbb{Q}} := \boldsymbol{j}^{-1}(\mathbb{A}^1_{\mathbb{Q}})$ (coarsely, if $r = 0$) represents the functor sending a $\mathbb{Q}$-scheme T to the set $\mathcal{M}_{\Phi_r}(T)$ of T-isomorphism classes of elliptic curves$_{/T}$ with a structure of level Φ_r. In particular: for every subfield $k \subset \mathbb{C}$, we have a bijection $Y_r(\bar{k}) \simeq \mathcal{M}_{\Phi_r}(\bar{k}) := \left\{(A, C, P)_{/\bar{k}}\right\}\Big/\cong$, where A is an elliptic curve over $\bar{k}$, $C \subset A(\bar{k})_N$ is a cyclic subgroup of order N and $P \in A(\bar{k})_{p^r}$ is a point of exact order p^r. The isomorphism $\Phi_r\backslash\mathfrak{H} \xrightarrow{\sim} Y_r(\mathbb{C}) \simeq M_{\Phi_r}(\mathbb{C})$ is defined mapping $\tau \in \mathfrak{H}$ to the class represented by $(\mathbb{C}/\Lambda_\tau, \left\langle N^{-1} \bmod \Lambda_\tau\right\rangle, p^{-r} \bmod \Lambda_\tau)$, where $\Lambda_\tau := \mathbb{Z}+\mathbb{Z}\tau$ (and $\langle\star\rangle$ denotes the group generated by $\star$).

Let $J_{r/\mathbb{Q}} := \mathrm{Jac}(X_r)$ be the Jacobian of X_r and $\mathrm{Ta}_p(J_r) := \varprojlim_n J_r(\overline{\mathbb{Q}})_{p^n}$ its p-adic Tate module. It is known that the natural $G_{\mathbb{Q}}$-action on $\mathrm{Ta}_p(J_r)$ is unramified at every prime $\ell \nmid Np$. We will write $\mathfrak{G} = \mathrm{Gal}(\mathbb{Q}_{Np}/\mathbb{Q})$ for the Galois group of the maximal algebraic extension $\mathbb{Q} \subset \mathbb{Q}_{Np} \subset \overline{\mathbb{Q}}$ unramified at every prime not dividing $Np \cdot \infty$, so that $\mathrm{Ta}_p(J_r)$ is a continuous $\mathbb{Z}_p[\mathfrak{G}]$-module.

Let $\mathfrak{h}_r := \mathfrak{h}(r) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ be the Hecke algebra of level Γ_r over $\mathbb{Z}_p$. Here $\mathfrak{h}(r)$ is the $\mathbb{Z}$-algebra generated by the Hecke operators T_ℓ for ℓ prime and the diamond operators $\langle d\rangle$ for $d \in (\mathbb{Z}/p^r\mathbb{Z})^*$ acting on the space $S_2(\Phi_r, \mathbb{Z}) \subset S_2(\Gamma_1(Np^r), \mathbb{Z})$ of weight-two cuspidal forms with integer Fourier coefficients. We also write $U_\ell = T_\ell$ when $\ell | Np^r$. It is a finite flat $\mathbb{Z}_p$-algebra, so by Hensel's Lemma the natural map $\mathfrak{h}_r \cong \prod_{\mathfrak{m}} \mathfrak{h}_{r,\mathfrak{m}}$ is an isomorphism of rings, where $\mathfrak{m}$ runs through the (finite set of) maximal ideals of $\mathfrak{h}_r$ (and $(-)_{\mathfrak{m}}$ denotes localization). We consider the T_p-decomposition

$$\mathfrak{h}_r = \mathfrak{h}_r^{\mathrm{nil}} \times \mathfrak{h}_r^{\mathrm{ord}},$$

where $\mathfrak{h}_r^{\mathrm{ord}}$ (resp., $\mathfrak{h}_r^{\mathrm{nil}}$) is the product of the $\mathfrak{h}_{r,\mathfrak{m}}$'s with $T_p \notin \mathfrak{m}$ (resp, $T_p \in \mathfrak{m}$). We also write $e_r^{\mathrm{ord}} \in \mathfrak{h}_r$ for the idempotent corresponding to projection onto the *ordinary part* $\mathfrak{h}_r^{\mathrm{ord}}$. More generally, for every $\mathfrak{h}_r$-module M we define its ordinary part $M^{\mathrm{ord}} := e_r^{\mathrm{ord}} \cdot M = M \otimes_{\mathfrak{h}_r} \mathfrak{h}_r^{\mathrm{ord}}$.

We can also represent $\mathfrak{h}(r)$ as a sub-ring of $\mathrm{Corr}_{\mathbb{Q}}(X_r) \subset \mathrm{End}_{\mathbb{Q}}(J_r)$, where $\mathrm{Corr}_{\mathbb{Q}}(X_{/\mathbb{Q}})$ denotes the ring of correspondences on $X \times X$ defined over $\mathbb{Q}$ [**Roh97**, pag. 89]. In particular $\mathrm{Ta}_p(J_r)$ is equipped with a structure of an $\mathfrak{h}_r[\mathfrak{G}]$-module. We can characterize T_ℓ, U_ℓ and $\langle d\rangle$ as endomorphisms of J_r by their action on $Y_r^{\mathrm{an}} := Y_r(\mathbb{C})$ as follows. Identifying $Y_r^{\mathrm{an}} \simeq M_{\Phi_r}(\mathbb{C})$ as above, T_ℓ (resp., U_ℓ) is induced by the map $Y_r^{\mathrm{an}} \to \mathrm{Div}\,(Y_r^{\mathrm{an}})$:

$$(A, Q, P) \mapsto \sum_{\mathcal{L}} (A/\mathcal{L}, Q \bmod \mathcal{L}, P \bmod \mathcal{L})$$

where $\mathcal{L} \subset A_\ell$ runs over all subgroup of order ℓ (resp., such that $\mathcal{L} \cap Q = 0$ and $\mathcal{L} \cap \langle P\rangle = 0$). Finally $\langle d\rangle$ is induced by the automorphism $Y_r^{\mathrm{an}} \xrightarrow{\sim} Y_r^{\mathrm{an}}$ sending (A, Q, P) to $(A, Q, d \cdot P)$. (Using the natural identification $S_2(\Gamma_r,\mathbb{C}) = H^0\left(X_r^{\mathrm{an}}, \Omega^1_{X_r^{\mathrm{an}}}\right)$ between weight-two cusp forms and holomorphic differentials on X_r^{an}, Abel-Jacobi theorem allows us to identify

$$J_r^{\mathrm{an}} := J_r(\mathbb{C}) = S_2(\Phi_r,\mathbb{C})^*/H_1(X_r^{\mathrm{an}},\mathbb{Z}), \tag{69}$$

where $(-)^*$ denotes the $\mathbb{C}$-dual. Then the action already defined on the L.H.S. corresponds to the action of $\mathfrak{h}(r)$ induced by composition on $S_2(\Phi_r,\mathbb{C})^*$, which can be proved to preserve integral homology.)

Let us define

$$\mathfrak{h}_\infty^{\mathrm{ord}} := \varprojlim_{r\geq 1} \mathfrak{h}_r^{\mathrm{ord}}; \quad \mathrm{Ta}_\infty^{\mathrm{ord}} := \varprojlim_{r\geq 1} \mathrm{Ta}_p(J_r)^{\mathrm{ord}}.$$

The first limit is taken with respect to 'restriction of endomorphisms'. The second limit is taken with respect to the morphisms induced by Albanese functoriality by the $\mathbb{Q}$-maps $X_{r+1}\to X_r$ attached to $\Phi_{r+1}\subset\Phi_r$. As we are considering $r\geq 1$ in the limits, the transition maps are compatibles with Hecke action, so that $\mathrm{Ta}_\infty^{\mathrm{ord}}$ is an $\mathfrak{h}_\infty^{\mathrm{ord}}[\mathfrak{G}]$-module. Diamond operators gives morphisms $\mathbb{Z}_p[(\mathbb{Z}_p/p^r\mathbb{Z})^*]\to\mathfrak{h}_r^{\mathrm{ord}}$ inducing on the limit a "diamond morphism"

$$[\ ] : \mathbb{Z}_p[[\mathbb{Z}_p^*]] \longrightarrow \mathfrak{h}_\infty^{\mathrm{ord}}.$$

(We note that this normalization differ by other ones found in literature, e.g. from that of [**Hid86a**] and [**EPW06**], where $[z]\mapsto z^2\cdot\langle z\rangle$ is used.) Writing $\Gamma := 1+p\mathbb{Z}_p$, $[\ ]$ in particular equips $\mathfrak{h}_\infty^{\mathrm{ord}}$ with the structure of an algebra over the *Hida algebra* $\Lambda := \mathbb{Z}_p[[\Gamma]]$. Thanks to the work of Hida [**Hid86b**, Th. 3.1], [**Hid86a**, Th. 3.1] (see also Section 8 of [**Hid86a**]) we know that $\mathfrak{h}_\infty^{\mathrm{ord}}$ is a finite, flat Λ-algebra and $\mathrm{Ta}_\infty^{\mathrm{ord}}$ is a free Λ-module of finite rank. Since Λ ia a local complete Noetherian ring, using again Hensel Lemma we obtain a finite decomposition $\mathfrak{h}_\infty^{\mathrm{ord}} = \prod_{\mathfrak{m}} \mathfrak{h}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$, where $\mathfrak{h}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$ is the localization of $\mathfrak{h}_\infty^{\mathrm{ord}}$ at the maximal ideal $\mathfrak{m}$. Let $\phi_g : \mathfrak{h}_1 \twoheadrightarrow \mathfrak{h}_1^{\mathrm{ord}} \to \mathbb{Z}$ be the morphism of $\mathbb{Z}_p$-algebras attached to g, i.e.: $\phi_g(T_\ell) = a_\ell(g) = a_\ell(E)$ for every prime $\ell\neq p$, $\phi_g(U_p) = a_p(g) = \alpha_p\in\mathbb{Z}_p^*$ and $\phi_g(\langle d\rangle) = 1$ for every $d\in\mathbb{F}_p^*$. We denote by the same symbol the morphism of $\mathbb{Z}_p$-algebras

$$\phi_g : \mathfrak{h}_\infty^{\mathrm{ord}} \twoheadrightarrow \mathfrak{h}_1^{\mathrm{ord}} \xrightarrow{\phi_g} \mathbb{Z}_p$$

induced by ϕ_g, and write $\mathfrak{m}_g\in\mathrm{Spec}(\mathfrak{h}_\infty^{\mathrm{ord}})$ for the maximal ideal s.t. ϕ_g factorizes through $\mathfrak{h}_\infty^{\mathrm{ord}}\twoheadrightarrow\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}$. We then define the $\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}[\mathfrak{G}]$-module

$$\mathrm{Ta}_{\infty,\mathfrak{m}}^{\mathrm{ord}} := \mathrm{Ta}_\infty^{\mathrm{ord}}\otimes_{\mathfrak{h}_\infty^{\mathrm{ord}}}\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}.$$

Under Hypothesis 2 it is known that $\mathrm{Ta}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$ is free of rank two as an $\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}$-module. (Indeed a regular ring is Gorestein, and this implies the freeness of $\mathrm{Ta}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$.) Moreover, as a manifestation of the Eichler-Shimura congruence relation (see also [**Roh97**] or [**DDT95**, Ch. I]) : for every prime $\ell\nmid Np$

$$\mathrm{Trace}\left(\mathrm{Frob}_\ell|\,\mathrm{Ta}_{\infty,\mathfrak{m}}^{\mathrm{ord}}\right) = T_\ell; \quad \det\left(\mathrm{Frob}_\ell|\,\mathrm{Ta}_{\infty,\mathfrak{m}}^{\mathrm{ord}}\right) = \ell\cdot[\ell], \tag{70}$$

where $\mathrm{Frob}_\ell\in\mathfrak{G}$ is an arithmetic Frobenius at ℓ and we have written again T_ℓ for the projection of the ℓ-th Hecke operator on $\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}$.

REMARK 9.1. For every $j\in\mathbb{Z}/(p-1)\mathbb{Z}$ let $\varepsilon_j := \frac{1}{p-1}\sum_{a\in\mathbb{F}_p^*}\omega^{-j}(a)\cdot a\in\mathbb{Z}_p[\mathbb{F}_p^*]\subset\Lambda$, so that every $\mathbb{Z}_p[\mathbb{F}_p^*]$-module M decomposes as $M = \bigoplus_j \varepsilon_j\cdot M$. As $\phi_g(\varepsilon_0) = 1$ and $\phi_g(\varepsilon_j) = 0$ for every $j\neq 0$ we have $\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}} = \varepsilon_0\cdot\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}$, i.e. $\mathbb{F}_p^*$ acts trivially on $\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}$. In particular (70) (combined with the Chebotarev density theorem) gives us:

$$\det_{\mathfrak{h}_{\infty,\mathfrak{m}_g}^{ord}}\left(\mathrm{Ta}_{\infty,\mathfrak{m}}^{\mathrm{ord}}\right) \xrightarrow{\sim} \mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}\otimes\chi_{cy}\cdot[\kappa_{cy}] = \mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}(\chi_{cy}\cdot[\kappa_{cy}]) \tag{71}$$

as $\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}[\mathfrak{G}]$-modules. (We recall that κ_{cy} is the composition of the p-adic cyclotomic character χ_{cy} with projection to principal units on $\mathbb{Z}_p^*$.)

9.2. The domain R. Let $\mathfrak{p}'_g := \ker(\phi_g) \in \mathrm{Spec}(\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g})$. By [**Hid86a**, Cor. 1.4] (see also Sec. 12.7.5 of [**Nek06**]) the localization of $\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}$ at $\mathfrak{p}'_g$ is a discrete valuation ring. In particular $\mathfrak{p}'_g$ contains a unique minimal prime ideal $\mathfrak{p}_{\min}$ such that ϕ_g factorizes through

$$R_g := \mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}/\mathfrak{p}_{\min}.$$

Then R_g is a local domain, finite over $[\ ] : \Lambda \to R$ and whose localization at the prime $\mathfrak{p}_g := \mathfrak{p}'_g/\mathfrak{p}_{\min}$ is a discrete valuation ring. R_g is called the *branch* of $\mathfrak{h}^{\mathrm{ord}}_{\infty}$ attached to g. With the terminology of [**Hid86b**, pag. 253] $\mathscr{R} := \mathrm{Frac}(R)$ is the *primitive component* of $\mathfrak{h}^{\mathrm{ord}}_{\infty} \otimes \mathrm{Frac}(\Lambda)$ corresponding to g. We write from now on

$$R = R_g; \quad \mathfrak{p} := \mathfrak{p}_g.$$

For every positive integer n we write $\mathbf{a}_n \in R$ for the projection in R of the n-th Hecke operator $T_n \in \mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}$. (Here T_n is defined as a polynomial in the T_ℓ's, U_ℓ's and $\langle d\rangle$'s by the usual recipe [**Shi71**, Ch. 3].)

We define an *arithmetic point* of R to be a morphisms of $\mathbb{Z}_p$-algebras $\psi : R \to \overline{\mathbb{Q}_p}$ such that the composition of ψ with $\Gamma := (1+p\mathbb{Z}_p) \subset \mathbb{Z}_p^* \to R$ is of the form $x \mapsto \chi(x)\cdot x^{k-2}$, for some integer $k \geq 2$ and some finite order character $\chi : 1+p\mathbb{Z}_p \to \overline{\mathbb{Q}_p}^*$. Then k is the *weight of* ψ, and χ its *wild character.* We denote by $\mathcal{X}^{\mathrm{arith}} = \mathcal{X}^{\mathrm{arith}}(R)$ the set of arithmetic points of R. Given $\psi \in \mathcal{X}^{\mathrm{arith}}$ we write $\mathcal{O}_\psi := \psi(R)$, ψ^{wild} for the wild character of ψ and

$$r(\psi) := \max\left\{1, \mathrm{ord}_p\left(\mathrm{cond}\left(\psi^{\mathrm{wild}}\right)\right)\right\}$$

(where $\mathrm{cond}(\star)$ denotes the conductor of $\star$, viewed as a character of finite order on $\mathbb{Z}_p^* = \mathbb{F}_p^* \times \Gamma$). We consider every element of R as a function on $\mathcal{X}^{\mathrm{arith}}$ letting $r(\psi) := \psi(r)$ for every $r \in R$ and $\psi \in \mathcal{X}^{\mathrm{arith}}$. An *arithmetic prime* $\mathfrak{q} \in \mathrm{Spec}(R)$ is defined as the kernel of an arithmetic map. Given $\psi \in \mathcal{X}^{\mathrm{arith}}$ we also write $\psi = \psi_{\mathfrak{q}}$ with $\mathfrak{q} := \ker(\psi)$. In the following we will use the terminology arithmetic prime and arithmetic point interchangeably, letting the context explains if we are considering a morphism of its kernel. Moreover, given a local ring $\mathcal{O}$ we will also write $\mathcal{X}^{\mathrm{arith}}(R;\mathcal{O}) = \mathcal{X}^{\mathrm{arith}}(\mathcal{O})$ to denote the set of arithmetic primes such that $\psi(R) = \mathcal{O}$.

Let us consider the formal q-eqxpansion

$$\mathbf{g} := \sum_{n\geq 1} \mathbf{a}_n \cdot q^n \in R[[q]].$$

Hida's control theorem [**Hid86b**, Cor. 1.2] (see also [**Hid86a**, Th. 3.5]) implies that for every arithmetic point $\psi \in \mathcal{X}^{\mathrm{arith}}$ of weight $k \geq 2$

$$\mathbf{g}_\psi := \sum_{n\geq 1} \mathbf{a}_n(\psi)\cdot q^n \in S_k\left(\Phi_{r(\psi)}, \omega^{2-k}\cdot\psi^{\mathrm{wild}}, \mathcal{O}_\psi\right)$$

is the q-expansion of a classical normalized eigenform of weight k, level $\Gamma_0(Np^{r(\psi)})$, character $\omega^{2-k}\cdot\psi^{\mathrm{wild}}$ and Fourier coefficients in $\mathcal{O}_\psi \cap \overline{\mathbb{Q}}$. (Using the definition of $r(\psi)$, here we identify ω and ψ^{wild} with the induced characters:

$$\left(\mathbb{Z}/Np^{r(\psi)}\mathbb{Z}\right)^* \twoheadrightarrow \left(\mathbb{Z}/p^{r(\psi)}\mathbb{Z}\right)^* = \mathbb{F}_p^* \times \Gamma/\Gamma_{r(\psi)} \longrightarrow \overline{\mathbb{Q}_p}^*,$$

where for every $r \geq 1$ we write $\Gamma_r := \Gamma^{p^{r-1}}$.) Moreover it is a *$p$-stabilized ordinary newform (of tame conductor N).* In other words this means: $\mathbf{g}_\psi$ is a common eigenform for all Hecke operators T_ℓ ($\ell \nmid Np$) and U_ℓ ($\ell | Np$), the p-th Fourier coefficient $\mathbf{a}_p(\psi) \in \overline{\mathbb{Z}_p}^*$ and $N|\mathrm{cond}(\mathbf{g}_\psi)$ (i.e. the system of Hecke eigenvalues $\{\mathbf{a}_\ell(\psi) : \ell \neq p\}$ does not arise from any eigenform of level not divided by N).

We note that $\phi_g =: \psi_{\mathfrak{p}} \in \mathcal{X}^{\mathrm{arith}}$ (with $\mathfrak{p} = \mathfrak{p}_g := \ker(\phi_g)$) is an arithmetic point of weight 2 and trivial wild character, and $g = \mathbf{g}_{\phi_g}$ with the notations above. Again by [**Hid86b**, Cor. 1.4] we know that for every $\psi_{\mathfrak{q}} \in \mathcal{X}^{\mathrm{arith}}$, the localization $R_{\mathfrak{q}}$ is a discrete valuation ring, unramified over the localization of Λ at the height-one prime $\mathfrak{q} \cap \Lambda$. In particular for every topological generator $\gamma \in \Gamma$, $\mathfrak{p} \cap \Lambda = (\gamma - 1)$ is the augmentation ideal of Λ. We can use this result to describe R as a ring of $\mathbb{Z}_p$-valued (locally) analytic functions (cfr. [**GS93**, Sec. 2]). More precisely: for every open subset $V \subset \mathbb{Z}_p$ denote by $\mathcal{A}(V)$ the ring of $\mathbb{Z}_p$-valued analytic functions on V. We endow $\mathcal{A}(V)$ with a structure of Λ algebra via the unique ring morphism $\Lambda \to \mathcal{A}(\mathbb{Z}_p)$ whose restriction to $\Gamma \subset \Lambda^*$ is defined mapping $x \in \Gamma$ to the power series $x^{k-2} = \sum_{n=0}^{\infty} \frac{\log_p(x)^n}{n!} \cdot (k-2)^n$. As R is finite over Λ, $R_{\mathfrak{p}}$ is unramified over $\Lambda_{\mathfrak{p}\cap\Lambda}$, and the inclusion of residue fields $\mathbb{Q}_p = \mathrm{Frac}(\Lambda/(\gamma-1)) \subset \mathrm{Frac}(R/\mathfrak{p})$ is an equality, we see that there exists an open neighborhood $2 \in U \subset \mathbb{Z}_p$ and a unique injective morphism of Λ-algebra

$$\mathcal{M}_g : R \hookrightarrow \mathcal{A}(U).$$

In particular $\psi_2(r) := \mathcal{M}_g(r)|_{k=2} = \phi_g(r)$ for every $r \in R$. More generally: for every integer $\kappa \in U$ the map induced by evaluation at κ:

$$\psi_\kappa : R \overset{\mathcal{M}_g}{\longrightarrow} \mathcal{A}(U) \overset{\mathrm{ev}_\kappa}{\longrightarrow} \mathbb{Z}_p \tag{72}$$

is an arithmetic point ψ_κ of weight κ and trivial wild character (cfr. Introduction).

9.2.1. *The 'twisted Hida family'.* Let us consider the *critical character*

$$\theta_R : \mathbb{Z}_p^* \twoheadrightarrow \Gamma \overset{x\mapsto\sqrt{x}}{\longrightarrow} \Gamma \overset{[\ ]}{\longrightarrow} R^*; \quad \theta_R^2 = [\ ].$$

(As $p \neq 2$ Hensel Lemma tells us that Γ is uniquely 2-divisible, so $\sqrt{x} = x^{1/2}$ makes sense for every $x \in \Gamma$. The equality $\theta_R(x)^2 = [x]$ for $x = \omega_x \cdot \gamma_x \in \mathbb{Z}_p^* = \mathbb{F}_p^* \times \Gamma$ follows by the fact (see Rem. 9.1) that $\mathbb{F}_p^*$ acts trivially on R via the structural morphism $[\ ]$, i.e. $[x] = [\gamma_x]$.)

For every *even* integer κ we write $\mathcal{X}_\kappa^{\mathrm{arith}} \subset \mathcal{X}^{\mathrm{arith}}$ for the subset of arithmetic points of weight κ and $\mathcal{X}_{\mathrm{even}}^{\mathrm{arith}} := \bigcup_\kappa \mathcal{X}_\kappa^{\mathrm{arith}}$. Given $\psi \in \mathcal{X}_\kappa^{\mathrm{arith}}$, the map $\vartheta_\psi^{\mathrm{wild}} : \Gamma \to \overline{\mathbb{Q}_p}^*$; $\gamma \mapsto \gamma^{\frac{2-\kappa}{2}} \cdot (\psi \circ \theta_R)(\gamma)$ factorizes through $\Gamma/\Gamma_{r(\psi)}$, and we can define the character

$$\vartheta_\psi := \vartheta_\psi^{\mathrm{wild}} \cdot \omega^{\frac{2-\kappa}{2}} : \mathbb{F}_p^* \times \Gamma/\Gamma_{r(\psi)} \longrightarrow \overline{\mathbb{Q}_p}^* \quad \text{such that} \quad \mathbf{g}_\psi \in S_\kappa\left(\Phi_{r(\psi)}, \vartheta_\psi^2\right).$$

In other words ϑ_ψ is a square root of the character of the p-stabilized newform $\mathbf{g}_\psi$, so that

$$g_\psi := \mathbf{g}_\psi \otimes \vartheta_\psi^{-1} \in S_\kappa\left(\Gamma_0\left(Np^{2\cdot r(\psi)}\right), \mathcal{O}_\psi\right)$$

is an eigenform of level $Np^{2\cdot r(\psi)}$ with *trivial* character. We refer to the family $\{g_\psi\}_{\psi \in \mathcal{X}_{\mathrm{even}}^{\mathrm{arith}}}$ as the *twisted Hida family* attached to $E/\mathbb{Q}$ (or better to g).

9.3. The representation T. Let us write $\mathbb{T} := \mathrm{Ta}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}} \otimes_{\mathfrak{h}_{\infty,\mathfrak{m}_g}^{\mathrm{ord}}} R$. As recalled above, it is a free rank-two R-module, with a continuous R-linear action of $\mathfrak{G}$. With the notations above (70) rephrases as: for every $\ell \nmid Np$ the characteristic polynomial of Frob_ℓ on $\mathbb{T}$ is given by

$$\det\left(1 - X \cdot \mathrm{Frob}_\ell | \mathbb{T}\right) = 1 - \mathbf{a}_p X - \ell[\ell] X^2 \in R[X]. \tag{73}$$

In order to obtain a self-dual representation, we consider the *critical twist*

$$\mathbf{T} := \mathbb{T} \otimes_R \Theta_R^{-1},$$

where the critical character Θ_R is defined as the composition

$$\Theta_R : \mathfrak{G} \twoheadrightarrow \mathrm{Gal}\left(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}\right) \overset{\chi_{cy}}{\longrightarrow} \mathbb{Z}_p^* \overset{\theta_R}{\longrightarrow} R^*.$$

As $\Theta_R^2 = [\kappa_{cy}]$, we see by Rem. 9.1 that $\det_R \mathbf{T} \overset{\sim}{\to} R(1) := R \otimes \chi_{cy}$ is the Tate twist of R.

9.3.1. *Self-duality.* As explained in [**NP00**, Sec. 1.6] (and previously proved by Ohta) the Weil pairings on the Jacobians $\{J_r\}_{r\geq 1}$ define an R-bilinear, skew-symmetric and $\mathfrak{G}$-equivariant map

$$\pi_{\mathfrak{m}_g}:\mathrm{Ta}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}\otimes_{\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}}\mathrm{Ta}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}\longrightarrow\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}\otimes\chi_{cy}\cdot[\kappa_{cy}]=\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}(1)\otimes[\kappa_{cy}],$$

inducing a 'skew-symmetric' isomorphisms of $\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}[\mathfrak{G}]$-modules

$$\mathrm{adj}\left(\pi_{\mathfrak{m}_g}\right):\mathrm{Ta}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}\stackrel{\sim}{\rightarrow}\mathrm{Hom}_{\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}}\left(\mathrm{Ta}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g},\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}(1)\otimes[\kappa_{cy}]\right).$$

(This follows in particular by the discussion in Sec. 1.6.10 of *loc. cit.*, using that under Hypothesis 1 we know by [**MT90**, Théorème 7] that $\mathrm{Hom}_\Lambda\left(\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g},\Lambda\right)\stackrel{\sim}{\rightarrow}\mathfrak{h}^{\mathrm{ord}}_{\infty,\mathfrak{m}_g}$ as Λ-modules.) Taking the quotient $\pi_{\mathfrak{m}_g}\otimes R$ and twisting by Θ_R^{-1} we thus obtain an R-bilinear, skew-symmetric $\mathfrak{G}$-equivariant pairing

$$\pi_R:\mathbf{T}\otimes_R\mathbf{T}\longrightarrow R(1)\otimes[\kappa_{cy}]\otimes\Theta_R^{-2}=R(1),$$

inducing an isomorphism of $R[\mathfrak{G}]$-modules

$$\mathrm{adj}(\pi_R):\mathbf{T}\stackrel{\sim}{\rightarrow}\mathrm{Hom}_R\left(\mathbf{T},R(1)\right).$$

9.3.2. *Ramification at p.* By the work of Mazur-Wiles and Tilouine we know that the restriction of $\mathbb{T}$ to $G_{\mathbb{Q}_p}$ is reducible: let w be a prime of $\overline{\mathbb{Q}}$ dividing p, defined by an embedding $\rho_w:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. Write $I_w\subset G_w\subset G_{\mathbb{Q}}$ for the inertia subgroup of the corrsponding decomposition group $G_w=\rho_w^*(G_{\mathbb{Q}_p})$ at w. There exists a short exact sequence of $R[G_w]$-modules

$$0\rightarrow\mathbb{T}_w^+\rightarrow\mathbb{T}\rightarrow\mathbb{T}_w^-\rightarrow 0, \tag{74}$$

with $\mathbb{T}_w^\pm$ free of rank one over R. I_w acts trivially on $\mathbb{T}_w^-$ and via the character $\chi_{cy}\cdot[\chi_{cy}]$ on $\mathbb{T}_w^+$, and the arithmetic Frobenius $\mathrm{Frob}_w\in G_w/I_w$ acts on $\mathbb{T}_w^-$ via multiplication by $\mathbf{a}_p\in R^*$. We refer the reader to [**NP00**, Prop. 1.5.4] for precise references. Defining $\mathbf{T}_w^\pm:=\mathbb{T}_w^\pm\otimes\Theta_R^{-1}$, we find an exact sequence of $R[G_w]$-modules

$$0\rightarrow\mathbf{T}_w^+\stackrel{i_w^+}{\rightarrow}\mathbf{T}\stackrel{p_w^-}{\rightarrow}\mathbf{T}_w^-\rightarrow 0. \tag{75}$$

We also write $F_w^\pm(\mathbf{T}):=\mathbf{T}_w^\pm$. This exact sequence is 'self-dual' in the following sense. For the dual representation $\mathbf{T}^*(1):=\mathrm{Hom}_R(\mathbf{T},R(1))$, let us define $F_w\left(\mathbf{T}^*(1)\right):=\mathrm{Hom}_R(\mathbf{T}_w^\mp,R(1))$. Since $\pi_R:\mathbf{T}\times\mathbf{T}\rightarrow R(1)$ is skew-symmetric and $\mathbf{T}_w^\pm\stackrel{\sim}{\rightarrow}R$, we see that $\mathrm{adj}(\pi_R)$ induces an isomorphism of short exact sequences of $R[G_w]$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_w^+(\mathbf{T}) & \longrightarrow & \mathbf{T} & \longrightarrow & F_w^-(\mathbf{T}) & \longrightarrow & 0\\ & & \downarrow\sim & & \downarrow\mathrm{adj}(\pi_R) & & \downarrow\sim & & \\ 0 & \longrightarrow & F_w^+(\mathbf{T}^*(1)) & \longrightarrow & \mathbf{T}^*(1) & \longrightarrow & F_w^-(\mathbf{T}^*(1)) & \longrightarrow & 0. \end{array} \tag{76}$$

With the notations and terminology introduced in Section 0.5 we can sum up the discussion of this Section as follows: let $K/\mathbb{Q}$ be a number field and recall that $G_{K,S}:=\mathrm{Gal}(K_{Np}/K)$ denotes the Galois group of the maximal algebraic extension K_{Np}/K which is unramified for every prime $v\nmid Np\infty$ of K. Then π_R is a perfect $R[G_{K,S}]$-duality between $\mathbf{T}\in{}_{R[G_{K,S}]}\text{-Mod}$ and itself, such that for every prime $v|p$ of K:

$$\mathbf{T}_v^+\perp\!\!\!\perp_{\pi_R}\mathbf{T}_v^+, \tag{$v|p$}$$

i.e. $\mathbf{T}_v^+$ is its own π_R-orthogonal complement.

More generally: let $\phi:R\twoheadrightarrow A$ be a surjective morphism of local $\mathbb{Z}_p$-algebra. For every $R[G_{K,S}]$-module M we write $M_\phi:=R\otimes_{R,\phi}A\in{}_{A[G_{K,S}]}\text{-Mod}$. Then, putting $F_v^\pm(\mathbf{T}_\phi):=(\mathbf{T}_v^\pm)_\phi$ and recalling that $\mathbf{T}$, $\mathbf{T}_v^\pm$ are free finite R-modules we obtain:

$$\pi_\phi=\pi_{R,\phi}:=\pi_R\otimes_{R,\phi}A:\mathbf{T}_\phi\otimes_A\mathbf{T}_\phi\longrightarrow A(1)$$

is a perfect $A[G_{K,S}]$-duality between $\mathbf{T}_\psi$ and itself, such that

$$(v|p) \qquad F_v^+(\mathbf{T}_\phi) \perp\!\!\perp_{\pi_\phi} F_v^+(\mathbf{T}_\phi)$$

for every prime $v|p$ of K.

9.3.3. *Specialization at arithmetic primes.* Let $\psi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{O})$ be an arithmetic prime of *even* weight $k \in 2\mathbb{Z}$. Let us write $F := \mathrm{Frac}(\mathcal{O})$ and $\mathbf{V}_\psi := \mathbf{T}_\psi \otimes_{\mathcal{O}} F$ (with the notations of the preceding Section). Letting $\star$ denotes the eigenform $\mathbf{g}_\psi$ of its twist g_ψ, write $\rho_\star : G_{\mathbb{Q}} \to \mathrm{GL}_2(F)$ for the Deligne representation attached to $\star$, and let $V(\star)$ be a representation space for $\rho_{g_{\mathfrak{q}}}$. Using the irreducibility of $V(\star)$, the Eichler-Shimura relations (70) and retracing the definitions, we easily conclude that there exists an isomorphism of $F[G_{\mathbb{Q}}]$-modules

$$(77) \qquad \mathbf{V}_\psi \cong V(\mathbf{g}_\psi) \otimes \left(\theta_\psi^{-1} \cdot \chi_{cy}^{1-k/2}\right) = \left(V(g_\psi) \otimes \theta_\psi^{-1}\right) \otimes \chi_{cy}^{1-k/2} = V(g_\psi)(1-k/2).$$

In other words we see that $\mathbf{T}_\psi$ is an $\mathcal{O}$-stable Galois lattice in the $(1-k/2)$-*critical twist* of the Deligne representation attached to g_ψ. In particular by the results recalled in the preceding Sections we recover the well known fats (proved by Wiles *et. al.*) that $V(g_\kappa)(1-k/2)$ is a (nearly) self-dual and p-ordinary representation.

We can sum up this discussion saying that $\mathbf{T}$ 'parametrize' the family of Deligne representations attached to the elements of the twisted Hida family $\{g_\psi\}_{\psi \in \mathcal{X}^{\mathrm{arith}}_{\mathrm{even}}}$.

9.3.4. *Specialization at ϕ_g.* We now describe more precisely the isomorphism (77) for the arithmetic prime $\phi_g \in \mathcal{X}^{\mathrm{arith}}(R;\mathbb{Z}_p)$ of weight 2 attache to the elliptic curve $E_{/\mathbb{Q}}$. We begin by recalling how a p-ordinary structure is explicitly defined on the representation $T_p = \mathrm{Ta}_p(E_{/\mathbb{Q}}) := \varprojlim_{n\geq 1} E(\overline{\mathbb{Q}})_{p^n}$.

Let $M \in \{E_{p^\infty} = E(\overline{\mathbb{Q}})_{p^\infty}, T_p\}$, and let w be a prime of $\overline{\mathbb{Q}}$ lying over p. We will use the notations of Section 9.3.2. Since $E/\mathbb{Q}$ is ordinary at p, it is well known that we have exact sequences of G_w-modules

$$(78) \qquad 0 \to F_w^+(M) \to M \to F_w^-(M) \to 0,$$

with $F_w^\pm(M)$ co-free (resp., free) of rank one over $\mathbb{Z}_p$. Moreover, the inertia $I_w \subset G_w$ act trivially on $F_w^-(M)$ and via the p-adic cyclotomic character χ_{cy} on $F_w^+(M)$, and an arithmetic Frobenius in G_w/I_w acts on $F_w^-(M)$ via multiplication by the p-adic unit $\alpha_p = \mathbf{a}_p(\phi_g) = a_p(g)$. The filtration (78) can be explicitly described as follows.

Assume first that $E/\mathbb{Q}_p$ has good ordinary reduction, and let $\widetilde{E}/\mathbb{F}_p$ be the reduction of E modulo p. (Here $\mathbb{F}_p$ is the field with p elements.) By [**Sil86**, Ch. VII], the reduction map $E(\overline{\mathbb{Q}}_p)_{p^n} \to \widetilde{E}(\overline{\mathbb{F}}_p)_{p^n}$ is surjective for every $n \geq 1$. We then obtain surjective maps of $G_{\mathbb{Q}_p}$-modules $E(\overline{\mathbb{Q}_p})_{p^\infty} \twoheadrightarrow \widetilde{E}(\overline{\mathbb{F}_p})_{p^\infty}$ and $T_p \twoheadrightarrow \mathrm{T}_p(\widetilde{E})$ (where $\mathrm{T}_p(\widetilde{E})$ is the p-adic Tate module of the $\widetilde{E}/\mathbb{F}_p$). Since by assumption $\widetilde{E}(\overline{\mathbb{F}}_p)_{p^n} \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ for every $n \geq 1$ (i.e. $E_{/\mathbb{Q}_p}$ has ordinary reduction), $F_w^-(E_{p^\infty}) := \widetilde{E}(\overline{\mathbb{F}}_p)_{p^\infty}$ (resp., $F_w^-(T_p) := \mathrm{T}_p(\widetilde{E})$) is co-free (resp., free) of rank one over $\mathbb{Z}_p$. Identifying $E(\overline{\mathbb{Q}}_p)_{p^n}$ with $E(\overline{\mathbb{Q}})_{p^n}$ under the embedding $\rho_w : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing the prime w, we obtain (78) defining $F_w^+(M)$ as the kernel of the projection $M \twoheadrightarrow F_w^-(M)$.

Assuming that $E/\mathbb{Q}_p$ has multiplicative reduction, the Tate parametrization gives us an isomorphism

$$\Phi_{Tate} : \overline{\mathbb{Q}}_p^*/q^{\mathbb{Z}} \xrightarrow{\sim} E(\overline{\mathbb{Q}}_p),$$

where $q = q(E/\mathbb{Q}_p) \in p\mathbb{Z}_p$ is the Tate period of $E/\mathbb{Q}_p$ [**Sil94**, Ch. V]. Let $\chi_{un} : G_{\mathbb{Q}_p} \twoheadrightarrow G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p} \to \{\pm 1\}$ be the unramified quadratic character on $G_{\mathbb{Q}_p}$. Writing $\chi_p = 1$ (resp., $\chi_p := \chi_{un}$) if $a_p(E) = 1$ (resp., $a_p(E) = -1$), i.e. if $E/\mathbb{Q}_p$ has split (resp., non-split) multiplicative reduction, Φ_{Tate} induces short exact sequences of $G_{\mathbb{Q}_p}$-modules (see [**Sil94**, Ch. V])

$$0 \to \mu_{p^n}(\overline{\mathbb{Q}}_p) \otimes \chi_p \to E(\overline{\mathbb{Q}}_p)_{p^n} \to \mathbb{Z}/p^n\mathbb{Z} \otimes \chi_p \to 0$$

for every $n \geq 1$, where $\mathbb{Z}/p^n\mathbb{Z}$ has trivial Galois action. Taking the direct (resp., inverse) limit for $n \to \infty$, and identifying $E(\overline{\mathbb{Q}}_p)_{p^n}$ with $E(\overline{\mathbb{Q}})_{p^n}$ under the embedding $\rho_w : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing the prime w we obtain the exact sequence (78) for $M = E_{p^\infty}$ (resp., $M = T_p(E)$) with $F_w^+(E_{p^\infty}) := \mathbb{Q}_p/\mathbb{Z}_p(1) \otimes \chi_p$ (resp., $F_w^+(T_p) := \mathbb{Z}_p(1) \otimes \chi_p$).

Let $J_1 \twoheadrightarrow E^{\min}_{/\mathbb{Q}}$ be the optimal elliptic curve attached by the Eichler-Shimura construction to the eigenform $f_E \in S_2(\Phi_1, \mathbb{Z})$ [**DDT95**]. Letting R acts on the p-adic Tate module $T_p^{\min} := \mathrm{Ta}_p(E^{\min}_{/\mathbb{Q}})$ via the morphism ϕ_g (so that $\mathbf{a}_p$ acts via the unit root $\alpha_p = a_p(g)$) Eichler-Shimura theory (cfr. [**DDT95**, Ch. 1], [**Hid86a**, Sec. 9], [**Gre94a**]) tells us that the natural projections induce isomorphisms of $R[\mathfrak{G}]$-modules:

$$\mathbf{T}_{\phi_g} := \mathbf{T} \otimes_{\mathbb{Z}_p, \phi_g} \mathbb{Z}_p \cong \mathrm{Ta}_p(J_1)^{\mathrm{ord}} \otimes_{\mathfrak{h}_1^{\mathrm{ord}}, \phi_g} \mathbb{Z}_p \cong T_p^{\min}. \tag{79}$$

By the isogeny theorem we also know that there exists an isogeny over $\mathbb{Q}$ between $E^{\min}$ and E, and thanks to our irreducibility assumption Hypothesis 1 this induces an isomorphism of $\mathbb{Z}_p[G_{\mathbb{Q}}]$-modules on p-adic Tate modules: $T_p \cong T_p^{\min}$. Combined with (79) this gives us an isomorphism of $\mathbb{Z}_p[\mathfrak{G}]$-modules

$$\mathbf{T}_{\phi_g} \cong T_p. \tag{80}$$

Moreover, letting w be a prime of $\overline{\mathbb{Q}}$ lying over p and recalling that $F_w^-(T_p) \cong \mathbb{Z}_p$ as $\mathbb{Z}_p$-modules we easily see that this isomorphism extends to an isomorphism of short exact sequences of $\mathbb{Z}_p[G_w]$-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_w^+(\mathbf{T}_{\phi_g}) & \longrightarrow & \mathbf{T}_{\phi_g} & \longrightarrow & F_w^-(\mathbf{T}_{\phi_g}) & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & F_w^+(T_p) & \longrightarrow & T_p & \longrightarrow & F_w^-(T_p) & \longrightarrow & 0. \end{array}$$

Let $W : T_p \otimes_{\mathbb{Z}_p} T_p \to \mathbb{Z}_p(1)$ be the p-adic Weil pairing (defined as in [**Sil86**, Ch.3]), inducing isomorphisms of $\mathbb{Z}_p[\mathfrak{G}]$-modules $T_p \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p, \mathbb{Z}_p(1))$ and $E_{p^\infty} \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p, \mu_{p^\infty})$. Since W is alternating and $F_w^{\pm}(T_p)$ is free of rank one as a $\mathbb{Z}_p$-module, W induces isomorphisms of $\mathbb{Z}_p[G_w]$-modules $F_w^{\pm}(T_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(F_w^{\mp}(T_p), \mathbb{Z}_p(1))$ and $F_w^{\pm}(E_{p^\infty}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(F_w^{\mp}(T_p), \mu_{p^\infty})$. Moreover, as π_R is also skew-symmetric and $\mathbf{T}_w^{\pm} \cong R$ as R-modules, multiplying eventually π_R by a unit we can (and will from now on) assume that

$$\pi_{\phi_g} = \pi_{R,\phi_g} = W, \tag{81}$$

i.e. that the perfect duality π_R specializes at ϕ_g to the Weil pairing.

10. Selmer complexes in Hida theory

10.1. Selmer complexes. Let $K \subseteq L \subseteq \mathcal{K}$ be a subextension of $\mathcal{K}/K$, and let $\psi \in \mathcal{X}^{\mathrm{arith}}(R)$ be an arithmetic prime of R. Using the notations of Section 0.11 and Section 0.16, let $X = \mathbf{T}(L)$ (resp., $L(\mathbb{A}_{\mathbf{T}})$, $\mathbf{T}_\psi(L)$, $L(\mathbb{A}_{\mathbf{T}_\psi})$), and write $\mathcal{R} = \mathcal{R}_X := R$ (resp., R, $\mathcal{O}_\psi$, $\mathcal{O}_\psi$). Then X is a continuous $\mathcal{R}(L)[G_{K,S}]$-module equipped with a $\mathcal{R}(L)[G_{K_v}]$-submodule $i_v^+ = i_v^+(X) : X_v^+ \hookrightarrow X$ for every $v \in S_p$. (We recall that for every Galois extension F/K, and any local complete $\mathbb{Z}_p$-algebra A we write $A(F) = A(F/K) = A[[\mathrm{Gal}(F/K)]] := \varprojlim_{i \in I} A[\mathrm{Gal}(F_i/K)]$ for the (complete) F/K Iwasawa algebra over A, where $\{F_i/K\}_{i\in I}$ is the set of finite sub-extensions of F/K.)

Let us consider a subset $\Sigma \subset S_f$, containing the set S_p of primes of K dividing p. Using the notations of Appendix A, we consider *Nekovář's Selmer complexes*

$$\widetilde{C}_f^\bullet(G_{K,S}, X; \Delta_\Sigma(X)) \in \mathrm{Kom}(\mathcal{R}(L)),$$

together with the corresponding derived objects

$$\widetilde{\mathbf{R}\Gamma}_f(K, X) := \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta_\Sigma(X)) \in \mathcal{D}^b(\mathcal{R}(L)),$$

with local conditions $\Delta_\Sigma(X) = \{\Delta_{\Sigma,v}(X)\}_{v \in S_f}$ defined by

$$\Delta_{\Sigma,v}(X) := \begin{cases} C^\bullet_{\mathrm{cont}}(K_v, X_v^+) & \text{if } v|p \ ; \\ C^\bullet_{\mathrm{cont}}(G_{K_v}/I_{K_v}, X^{I_{K_v}}) & \text{if } v \nmid p,\ v \in \Sigma \ ; \\ 0 & \text{if } v \notin \Sigma \ . \end{cases}$$

(As usual $I_{K_v} \subset G_{K_v}$ denotes the inertia subgroup at v.) For every $q \geq 0$, we write

$$\widetilde{H}^q_f(K, X) := H^q\left(\widetilde{\mathbf{R}\Gamma}_f(K, X)\right) \in {}_{\mathcal{R}(L)}\mathrm{Mod}$$

for the corresponding *extended Selmer group*, a finite (resp., cofinite) $\mathcal{R}(L)$-module for $X \in \{\mathbf{T}(L), \mathcal{O}_\psi(L)\}$ (resp., $X \in \{L(\mathbf{T}), L(\mathbf{T}_\psi)\}$).

LEMMA 10.1. *Let $S_p \subset \Sigma \subset S_f$. The natural morphism of complexes*

$$\iota_\Sigma : \widetilde{C}_f^\bullet(G_{K,S}, X; \Delta_{S_p}(X)) \longrightarrow \widetilde{C}_f^\bullet(G_{K,S}, X; \Delta_\Sigma(X))$$

is a quasi-isomorphism.

PROOF. We will prove in Lemma 10.7 of Section 10.4 that, under our assumptions $C^\bullet_{\mathrm{cont}}(K_v, Y(L))$ is acyclic for every $v \in S_f - S_p$ and $Y \in \{\mathbf{T}, \mathbf{T}_\psi\}$. Since $L(\mathbb{A}_Y)$ is isomorphic to the Kummer dual of $Y(L)^\iota$ (see Section 0.19), Tate local duality tells us that $C^\bullet_{\mathrm{cont}}(K_v, L(\mathbb{A}_Y))$ is also acyclic for $v \in S_f - S_p$. Then $C^\bullet_{\mathrm{cont}}(K_v, X)$ is acyclic for every $\mathcal{R}[G_{K,S}]$-module X we are considering. Using the inflation maps attached to $G_{K_v} \twoheadrightarrow G_{K_v}/I_{K_v} \cong \widehat{\mathbb{Z}}$ together with the fact that the group $\mathbb{Z}_p$ has p-cohomological dimension 1, we conclude that

$$C^\bullet_{\mathrm{cont}}(G_{K_v}/I_{K_v}, X^{I_{K_v}}) \ \text{ is acyclic for every } v \in S_f - S_p.$$

Then we obtain by construction an isomorphism in $\mathcal{D}(\mathcal{R}(L))$:

$$\mathrm{Cone}\,(\iota_\Sigma) \cong \bigoplus_{v \in \Sigma;\ v \nmid p} C^\bullet_{\mathrm{cont}}(G_{K_v}/I_{K_v}, X^{I_{K_v}}) \cong 0,$$

i.e. ι_Σ is a quasi-isomorphism, as was to be proved. □

In follows in particular that $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta_\Sigma(X))$ does not depend, up to canonical isomorphism, on the choice of Σ. This justify our notation $\widetilde{\mathbf{R}\Gamma}_f(K, X)$. We will write from now on

$$\widetilde{C}_f^\bullet(K, X) = \widetilde{C}_f^\bullet(G_{K,S}, X) := \widetilde{C}_f^\bullet(G_{K,S}, X; \Delta_{S_p}(X))$$

and we will identify $\widetilde{\mathbf{R}\Gamma}_f(K, X)$ with the corresponding derived object $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta_{S_p}(X)) \in \mathcal{D}(\mathcal{R}(L))$.

10.2. Shapiro's Lemma and Iwasawa theory. Let $X \in \{\mathbf{T}, \mathbf{T}_\psi\}$ and let $\mathcal{R} = \mathcal{R}_X \in \{R, \mathcal{O}_\psi\}$ be the corresponding 'coefficient ring'. Let $K \subset L \subset \mathcal{K}$ be a (possibly infinite) subextension of $\mathcal{K}/K$. Using the notations of Section B, we will write

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) := \widetilde{\mathbf{R}\Gamma}_f(K, X(L)) \in \mathcal{D}_{\mathrm{ft}}\left(\mathcal{R}(L)\right);$$

$$\widetilde{H}^*_{f,\mathrm{Iw}}(L/K, X) := H^*\left(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X)\right) \in \left({}_{\mathcal{R}(L)}\mathrm{Mod}\right)_{\mathrm{ft}};$$

$$\widetilde{\mathbf{R}\Gamma}_f(K_S/L, \mathbb{A}_X) := \widetilde{\mathbf{R}\Gamma}_f(K, L(\mathbb{A}_X)) \in \mathcal{D}_{\mathrm{cft}}\left(\mathcal{R}(L)\right);$$

$$\widetilde{H}^*_f(K_S/L, \mathbb{A}_X) := H^*\left(\widetilde{\mathbf{R}\Gamma}_f(K_S/L, \mathbb{A}_X)\right) \in \left({}_{\mathcal{R}(L)}\mathrm{Mod}\right)_{\mathrm{cft}}.$$

For every finite extension $K \subset E \subset L$ let $S_{E,f}$ (resp., $S_{E,p}$) be the set of finite primes of E dividing primes in $S_f = S_{K,f}$ (resp, dividing p), and let $G_{E,S} := \mathrm{Gal}(K_S/E)$ be the Galois group of the maximal algebraic extension of E which is unramified outside $S_E = S_{E,f} \cup \{v|\infty\}$. As in the preceding Section we write $\widetilde{C}^\bullet_f(E, \dagger) := \widetilde{C}^\bullet_f(G_{E,S}, \dagger; \Delta_{S_{E,p}}(\dagger))$, $\widetilde{\mathbf{R}\Gamma}_f(E, \dagger) := \widetilde{\mathbf{R}\Gamma}_f(G_{E,S}, \dagger; \Delta_{S_{E,p}}(\dagger))$ and $\widetilde{H}^*_f(E, \dagger) := \widetilde{H}^*_f(G_{E,S}, \dagger; \Delta_{S_{E,p}}(\dagger))$, for $\dagger \in \{X, X(E), E(X)\}$. As explained in details in Appendix B, Nekovář's generalization of Shapiro's Lemma gives us a natural isomorphism

$$\mathrm{Sh}_{E,f} : \widetilde{\mathbf{R}\Gamma}_f(K, X(E)) \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(E, X) \in \mathcal{D}(\mathcal{R})$$

coming from a quasi-isomorphisms on the underline complexes of $\mathcal{R}$-modules (we recall $X(E) := X \otimes_{\mathcal{R}} \mathcal{R}[\mathrm{Gal}(E/K)]$). This induces in cohomology isomorphisms of $\mathcal{R}(E) := \mathcal{R}[\mathrm{Gal}(E/K)]$-modules (denoyed by the same symbol)

$$\mathrm{Sh}^*_{E,f} : \widetilde{H}^*_f(K, X(E)) \xrightarrow{\sim} \widetilde{H}^*_f(E, X) \in {}_{\mathcal{R}(L)}\mathrm{Mod}. \tag{82}$$

Here the $\mathcal{R}(E)$-action on $\widetilde{H}^*_f(E, X)$ comes for the (generalized) conjugation action of $\mathrm{Gal}(E/K)$ (see again Appendix B for details). Since $X(E) \cong E(X) := \mathrm{Hom}_{\mathcal{R}}\left(\mathcal{R}(E), X\right)$ as $\mathcal{R}(E)[G_{K,S}]$-modules for every finite Galois extension E/K, we obtain similar isomorphisms, denoted again $\mathrm{Sh}_{E,f}$ replacing $\widetilde{\mathbf{R}\Gamma}_f(K, X(E))$ and $\widetilde{H}^*_f(K, X(E))$ with $\widetilde{\mathbf{R}\Gamma}_f(K, E(X))$ and $\widetilde{H}^*_f(K, E(X))$ respectively. (See again *loc.* cit. for the details.)

Given finite subextensions $K \subset E \subset E' \subset \mathcal{K}$, the formalism of [**Nek06**, Ch. 8] (recalled in Appendix B) gives us generalized *restriction* and *corestriction* morphisms in $\mathcal{D}(\mathcal{R})$:

$$\mathrm{res} := \mathrm{res}_{f,E'/E} : \widetilde{\mathbf{R}\Gamma}_f(E, X) \to \widetilde{\mathbf{R}\Gamma}_f(G_{E'}, X); \quad \mathrm{cor} := \mathrm{cor}_{f,E'/E} : \widetilde{\mathbf{R}\Gamma}_f(E', X) \to \widetilde{\mathbf{R}\Gamma}_f(E, X).$$

We use the same notation to denote the corresponding maps induced on cohomology. For every (possibly infinite) subextension $K \subset L \subset \mathcal{K}$ we can then consider the 'naive' L-Iwasawa objects:

$$\widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f,\mathrm{Iw}}(L/K, X) := \varprojlim_{E,\mathrm{cor}} \widetilde{\mathbf{R}\Gamma}_f(E, X) \in \mathcal{D}(\mathcal{R}); \quad \widetilde{H}^{*,\mathrm{naive}}_{f,\mathrm{Iw}}(L/K, X) := \varprojlim_{E,\mathrm{cor}} \widetilde{H}^*_f(E, X) \in {}_{\mathcal{R}(L)}\mathrm{Mod};$$

$$\widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f}(K_S/L, \mathbb{A}_X) := \varinjlim_{E,\mathrm{res}} \widetilde{\mathbf{R}\Gamma}_f(E, \mathbb{A}_X) \in \mathcal{D}(\mathcal{R}); \quad \widetilde{H}^{*,\mathrm{naive}}_{f}(K_S/L, \mathbb{A}_X) := \varinjlim_{E,\mathrm{res}} \widetilde{H}^*_f(E, \mathbb{A}_X) \in {}_{\mathcal{R}(L)}\mathrm{Mod},$$

where the limit is taken over the set of finite subextensions $K \subset E \subset L$. We recall that $\mathrm{Gal}(E/K)$ acts by (generalized) Galois conjugation on $\widetilde{H}^*_f(E, X)$ and such an action can be defined on the complex $\widetilde{C}^\bullet_f(E, X)$ only up to homotopy (see Appendix B for the details). Then conjugation defines a natural structure of $\mathcal{R}(L)$-module on $\widetilde{H}^{*,\mathrm{naive}}_{f,\mathrm{Iw}}(L/K, X)$ and $\widetilde{H}^{*,\mathrm{naive}}_{f}(K_S/L, \mathbb{A}_X)$, but via such a 'naive' definition $\widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f,\mathrm{Iw}}(L/K, X)$ and $\widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f}(K_S/L, \mathbb{A}_X)$ lives a priori only in $\mathcal{D}(\mathcal{R})$. Following ideas of Greenberg [**Gre94b**], Nekovář's solved this problem using his version of Shapiro's Lemma. More precisely: as explained in details in Appendix B, Shapiro's isomorphisms (82) induce natural isomorphisms in $\mathcal{D}(\mathcal{R})$:

$$\mathrm{Sh}_{f,L} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) \cong \widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f,\mathrm{Iw}}(L/K, X); \tag{83}$$

$$\mathrm{Sh}_{f,L} : \widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f,\mathrm{Iw}}(K_S/L, \mathbb{A}_X) \cong \widetilde{\mathbf{R}\Gamma}_f(K_S/L, \mathbb{A}_X). \tag{84}$$

More precisely, these isomorphisms are defined by the inverse (resp., inductive) limit of the Shapiro quasi-isomorphisms $(\mathrm{Sh}_{E,f})_{E/K}$, combined with natural isomorphisms of complexes of $\mathcal{R}$-modules:

$$\widetilde{C}_f^\bullet(K,X(L))\cong\varprojlim_{\mathrm{pr}_*,E}\widetilde{C}_f^\bullet(K,X(E));\quad \widetilde{C}_f^\bullet(K,L(\mathbb{A}_X))\cong\varinjlim_{\mathrm{pr}_*^\star,E}\widetilde{C}_f^\bullet(K,E(\mathbb{A}_X)).$$

Here E/K runs again over the finite subextensions of L/K. The inverse limit is taken with respect to the maps $\mathrm{pr}_*=\left(\mathrm{pr}_{E'/E}\right)_*$ induced on Selmer complexes by the natural projections $\mathrm{pr}:X(E')\twoheadrightarrow X(E)$ defined by restriction of automorphisms for every tower of finite subextensions $L/E'/E/K$. The direct limit is taken with respect to the maps $(\mathrm{pr}^\star)_*:=\left(\mathrm{pr}^\star_{E'/E}\right)_*$ induced on complexes by the duals $\mathrm{pr}^\star:=\mathrm{pr}^\star_{E'/K}:\mathrm{Hom}_{\mathcal{R}}(\dagger,\mathbb{A}_X)(\mathrm{pr}_{E'/E}):E(\mathbb{A}_X)\hookrightarrow E'(\mathbb{A}_X)$ of the projections $\mathrm{pr}:\mathcal{R}(E')\twoheadrightarrow\mathcal{R}(E)$. The first (resp., second) isomorphism comes again from the maps induced on Selmer complexes by the natural projection $X(L):=X\otimes_{\mathcal{R}}\mathcal{R}[[\mathrm{Gal}(L/K)]]\twoheadrightarrow X\otimes_{\mathcal{R}}\mathcal{R}[\mathrm{Gal}(E/K)]:=X(E)$ (resp., comes from the natural maps induced on complexes by the natural maps $E(X)\to\varinjlim_F\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}(F),\mathbb{A}_X)=:L(A_{\mathbb{X}})$). (Of course we implicitly stated that the Shapiro isomorphisms $\mathrm{Sh}_{*,f}$ 'transform' $(\mathrm{pr}_{E'/E})_*$ (resp., $(\mathrm{pr}^\star_{E'/E})_*$) in $\mathrm{cor}_{f,E'/E}$ (resp., $\mathrm{res}_{f,E'/E}$). We refer again to Appendix B for the details.) Then (83) and (84) identify the 'naive' limits with complexes naturally living in $\mathcal{D}(\mathcal{R}(L))$, and we can use this isomorphisms to give $\widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f,\mathrm{Iw}}(L/K,X)$ and $\widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f}(K_S/L,\mathbb{A}_X)$ the required $\mathcal{R}(L)$-structure. Moreover it can be proved (cfr. App. B) that taking cohomology in (83) and (84) we also obtain natural isomorphisms

$$\mathrm{Sh}^*_{f,L}:\widetilde{H}^*_{f,\mathrm{Iw}}(L/K,X)\xrightarrow{\sim}\varprojlim_{E,\mathrm{pr}_*}\widetilde{H}^*_f(K,X(E))\xrightarrow{\sim}\widetilde{H}^{*,\mathrm{naive}}_{f,\mathrm{Iw}}(L/K,X)\in\mathcal{D}(\mathcal{R}(L));\tag{85}$$

$$\mathrm{Sh}^*_{f,L}:\widetilde{H}^{*,\mathrm{naive}}_{f,\mathrm{Iw}}(L/K,\mathbb{A}_X)\xrightarrow{\sim}\varinjlim_{E,(\mathrm{pr}^\star)_*}\widetilde{H}^*_f(K,E(\mathbb{A}_X))\xrightarrow{\sim}\widetilde{H}^*_f(K_S/L,\mathbb{A}_X)\in\mathcal{D}(\mathcal{R}(L)).\tag{86}$$

We will use (83) and (85) (resp., (84) and (86)) to identify $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,X)$ and $\widetilde{H}^*_{f,\mathrm{Iw}}(L/K,X)$ (resp., $\widetilde{\mathbf{R}\Gamma}_f(K_S/L,\mathbb{A}_X)$ and $\widetilde{H}^*_f(K_S/L,\mathbb{A}_X)$) with the corresponding *naive* objects $\widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_{f,\mathrm{Iw}}(L/K,X)$ and $\widetilde{H}^{*,\mathrm{naive}}_{f,\mathrm{Iw}}(L/K,X)$ (resp., $\widetilde{\mathbf{R}\Gamma}^{\mathrm{naive}}_f(K_S/L,\mathbb{A}_X)$ and $\widetilde{H}^{*,\mathrm{naive}}_f(K_S/L,\mathbb{A}_X)$) respectively.

10.3. Control theorems.

10.3.1. *Abstract case.* Let $X\in\{\mathbf{T},\mathbf{T}_\psi\}$, $K\subseteq L\subseteq\mathcal{K}$ and $\mathcal{R}=\mathcal{R}_X$ be as in the preceding Section. Let $\phi:\mathcal{R}(L)\twoheadrightarrow A$ be a surjective morphism of complete local Noetherian rings such that $\mathcal{P}:=\ker(\phi)=(\boldsymbol{x})\in\mathrm{Spec}(\mathcal{R})$ is generated by an $\mathcal{R}(L)$-regular sequence $(\boldsymbol{x})=(x_1,\dots,x_n)\subset\mathfrak{m}_{\mathcal{R}(L)}$ (where $\mathfrak{m}_{\mathcal{R}(L)}$ is the maximal ideal of $\mathcal{R}(L)$), and write $Y_{[\mathcal{P}]}:=X(L)\otimes_{\mathcal{R}(L),\phi}A$. Then $Y_{[\mathcal{P}]}$ is a continuous $A[G_{K,S}]$-module, equipped with $A[G_v]$-submodules $i_v^+(Y_{[\mathcal{P}]}):=i_v^+(X(L))\otimes_{\mathcal{R}(L),\phi}A:\left(Y_{[\mathcal{P}]}\right)_v^+:=X(L)_v^+\otimes_{\mathcal{R}(L),\phi}A\hookrightarrow Y_{[\mathcal{P}]}$ for every $v\in S_p$ (recall that X is a free $\mathcal{R}$-module of finite type, hence $X(L)$ is finite and free over $\mathcal{R}(L)$). Let us consider Selmer complexes

$$\widetilde{C}_f^\bullet(K,Y_{[\mathcal{P}]}):=\widetilde{C}_f^\bullet(G_{K,S},Y_{[\mathcal{P}]};\Delta_{S_p}(Y_{[\mathcal{P}]}))\in\mathrm{K}(A);\quad \widetilde{\mathbf{R}\Gamma}_f(K,Y_{[\mathcal{P}]}):=\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Y_{[\mathcal{P}]};\Delta_{S_p}(Y_{[\mathcal{P}]}))\in\mathcal{D}(A),$$

defined exactly an is Section 10.1 (using the G_v-filtrations $\left(Y_{[\mathcal{P}]}\right)_v^+$ already defined). As usual we write

$$\widetilde{H}^*_f(K,Y_{[\mathcal{P}]}):=H^*\left(\widetilde{\mathbf{R}\Gamma}_f(K,Y_{[\mathcal{P}]})\right)\in\ ({}_A\mathrm{Mod})_{\mathrm{ft}}\,,$$

Gently abusing notations, for every $M\in\mathcal{D}^b(\mathcal{R}(L))$ we will write $M\otimes^{\mathbf{L}}_{\mathcal{R}(L),\phi}A$ to denote both $\phi_*\circ\mathbf{L}\phi^*M\in\mathcal{D}^b(\mathcal{R}(L))$ and $\mathbf{L}\phi^*M\in\mathcal{D}^b(A)$. (The derived category under consideration will make the notation clear.)

Proposition 10.2. *There exists a canonical isomorphism in $\mathcal{D}(A)$:*

$$\widetilde{\mathbf{R}\Gamma}_f(K,X)\otimes^{\mathbf{L}}_{\mathcal{R}(L),\phi}A\cong\widetilde{\mathbf{R}\Gamma}_f(K,Y_{[\mathcal{P}]}).$$

Proof. This is a special case of Lemma 0.4. □

10.3.2. *Galois deformations.*

PROPOSITION 10.3. *Let $K \subset L \subset L' \subset \mathcal{K}$ be a tower of subextensions of $\mathcal{K}/K$. Let $X = \mathbf{T}$ (resp., $X = \mathbf{T}_\psi$ for $\psi \in \mathcal{X}^{\mathrm{arith}}(R)$), and let $\mathcal{R} = R$ (resp, $\mathcal{R} = \mathcal{O}_\psi$).*

a) There exists a canonical isomorphism in $\mathcal{D}(\mathcal{R}(L))$:

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L'/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(L'),\varepsilon_{L'/L}} \mathcal{R}(L) \cong \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X), \tag{87}$$

where $\varepsilon_{L'/L} : \mathcal{R}(L') \twoheadrightarrow \mathcal{R}(L)$ is the projection induced by restriction of automorphisms.

b) Assume that $\mathrm{Gal}(L'/K) \overset{\sim}{\to} \mathbb{Z}_p^{k+1}$ ($k \geq 0$) and $\mathrm{Gal}(L/K) \overset{\sim}{\to} \mathbb{Z}_p^k$, and fix a topological generator $\sigma_{L'/L} \in \mathrm{Gal}(L'/L)$. Then (87) induces short exact sequences of $\mathcal{R}(L)$-modules:

$$0 \to \widetilde{H}^q_{f,\mathrm{Iw}}(L'/K, X)/\left(\sigma_{L'/L} - 1\right) \to \widetilde{H}^q_{f,\mathrm{Iw}}(L/K, X) \overset{i_{\sigma_{L'/L}}}{\to} \widetilde{H}^{q+1}_{f,\mathrm{Iw}}(L'/K, X)[\sigma_{L'/L} - 1] \to 0. \tag{88}$$

PROOF. *a)* is a special case of Proposition 0.13 (i.e. an easy corollary of the preceding Proposition).

b) Under our assumptions $\mathrm{Gal}(L/K)$ is a direct summand of $\mathrm{Gal}(L'/K)$; fix topological generators $\sigma_1, \ldots, \sigma_k, \sigma_{L'/L}$ of $\mathrm{Gal}(L'/K)$, such that $\sigma_1, \ldots, \sigma_k$ is a set of topological generators of $\mathrm{Gal}(L/K)$. We then obtain (non-canonical) isomorphisms $\mathcal{R}(L') \overset{\sim}{\to} (\mathcal{R}[[\sigma_1 - 1, \ldots, \sigma_k - 1]])\,[[\sigma_{L'/L} - 1]] \overset{\sim}{\to} \mathcal{R}(L)[[\sigma_{L'/L} - 1]]$, inducing a short exact sequence on Selmer complexes (cfr. Lemma 0.4):

$$0 \to \widetilde{C}^\bullet_f(G_{K,S}, X(L')) \overset{\sigma_{L'/L}-1}{\to} \widetilde{C}^\bullet_f(G_{K,S}, X(L')) \to \widetilde{C}^\bullet_f(G_{K,S}, X(L)) \to 0.$$

(Indeed the control theorem comes exactly from this short exact sequence of complexes; cfr. the proof Lemma 0.4.) Taking cohomology and recalling the definitions we conclude the proof. □

10.3.3. *Weight deformations.* We recall that we are assuming R regular, so that every height one prime of R (in particular every arithmetic prime) is a principal ideal.

PROPOSITION 10.4. *Let $\psi \in \mathcal{X}^{\mathrm{arith}}(R)$, with $\mathfrak{p}_\psi := \ker(\psi) = (\varpi_\psi)$ and let $K \subset L \subset \mathcal{K}$. Let $\psi(L) : R(L) \twoheadrightarrow \mathcal{O}_\psi(L)$ be the morphisms induced on Iwasawa algebras by ψ. There exists a canonical isomorphism in $\mathcal{D}(\mathcal{O}_\psi(L))$:*

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \otimes^{\mathbf{L}}_{R(L),\psi(L)} \mathcal{O}_\psi(L) \cong \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi).$$

This induces short exact sequences of $\mathcal{O}_\psi(L)$-modules:

$$0 \to \widetilde{H}^q_{f,\mathrm{Iw}}(L/K, \mathbf{T})/\varpi_\psi \to \widetilde{H}^q_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi) \overset{i_{\varpi_\psi}}{\to} \widetilde{H}^{q+1}_{f,\mathrm{Iw}}(L/K, \mathbf{T})[\varpi_\psi] \to 0. \tag{89}$$

PROOF. The first isomorphism follows directly from Prop. 10.2, noting that the kernel of $\psi(L)$ is the principal ideal of the domain $R(L)$ generated by ϖ_ψ,

$$\begin{aligned}\mathbf{T}_\psi(L) := \left(\mathbf{T}_\psi \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi(L)\right) < -1 > &\cong \left(\mathbf{T} \otimes_{R,\psi} \mathcal{O}_\psi \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi(L)\right) < -1 > \cong (\mathbf{T} \otimes_{R,\psi} \mathcal{O}_\psi(L)) < -1 > \\ &\cong (\mathbf{T} \otimes_R R(L)) < -1 > \otimes_{R(L),\psi(L)} \mathcal{O}_\psi(L) =: \mathbf{T}(L) \otimes_{R(L),\psi(L)} \mathcal{O}_\psi(L)\end{aligned}$$

as $\mathcal{O}_\psi(L)[G_{K,S}]$-modules, and similar isomorphisms of $\mathcal{O}_\psi(L)[G_v]$-modules are obtained replacing $\mathbf{T}$ with $\mathbf{T}^+_v$ for every $v \in S_p$. As explained in the proof of Lemma 0.4, the control theorems comes from an exact sequence of complexes of $R(L)$-modules:

$$0 \to \widetilde{C}^\bullet_f(G_{K,S}, \mathbf{T}(L)) \overset{\varpi_\psi}{\to} \widetilde{C}^\bullet_f(G_{K,S}, \mathbf{T}(L)) \to \widetilde{C}^\bullet_f(G_{K,S}, \mathbf{T}_\psi(L)) \to 0.$$

Taking cohomology we obtain the second statement, with i_{ϖ_ψ} the connecting morphism in the associated long exact cohomology sequence. □

10.4. Duality and perfectness. Let $X \in \{\mathbf{T}, \mathbf{T}_\psi\}$, $K \subseteq L \subseteq \mathcal{K}$ and $\mathcal{R} = \mathcal{R}_X$ be as in the preceding Section. Let us write $\pi_X := \pi_R : \mathbf{T} \otimes_R \mathbf{T} \to R(1)$ (resp., $\pi_X := \pi_R \otimes_{R,\psi} \mathcal{O}_\psi : \mathbf{T}_\psi \otimes_{\mathcal{O}_\psi} \mathbf{T}_\psi \to \mathcal{O}_\psi$) if $X = \mathbf{T}$ (resp., $X = \mathbf{T}_\psi$). With the terminology of the Section 0.6 (cfr. Section 0.8) π_X is a (skew-symmetric) perfect duality between X and itself, such that $X \perp_{\pi_X} X$ and $X_v^+ \perp\perp_{\pi_X} X_v^+$ for every $v \in S_p$. Then the constructions of Section 0.17 gives us a skew-Hermitian perfect duality

$$\pi_X(L) : X(L) \otimes_{\mathcal{R}(L)} X(L)^\iota \longrightarrow \mathcal{R}(L)(1),$$

such that $X(L) \perp_{\pi_X(L)} X(L)^\iota$ and $X(L)_v^+ \perp\perp_{\pi_X(L)} (X(L)_v^+)^\iota$ for every $v \in S_p$. Thanks to Nekovář's theory and global class-field theory (see Section 0.7 and Section 0.10) we can attach to $\pi_X(L)$ a global cup-product pairing in $\mathcal{D}(\mathcal{R}(L))$:

$$\cup_{\boldsymbol{\pi_X(L)}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X)^\iota \longrightarrow \mathcal{R}(L)[-3]$$

and an abstract Cassels-Tate pairing:

$$\widetilde{c}_{\pi_X(L),2,2} : \widetilde{H}^2_{f,\mathrm{Iw}}(L/K, X)_{\mathrm{tor}} \otimes_{\mathcal{R}(L)} \widetilde{H}^2_{f,\mathrm{Iw}}(L/K, X)^\iota_{\mathrm{tor}} \longrightarrow \mathscr{R}(L)/\mathcal{R}(L),$$

where $*_{\mathrm{tor}}$ denotes the $\mathcal{R}(L)$-torsion submodule of $*$ and $\mathscr{R}(L) := \mathrm{Frac}(\mathcal{R}(L))$ is the total ring of fractions of $\mathcal{R}(L)$.

Let S be a ring. We recall that a complex $C \in \mathcal{D}^b(S)$ is said *perfect* (resp., perfect of *perfect amplitude* contained in $[a,b]$) if there exists a quasi-isomorphism $P \to C$ with P a bounded complex of projective, finitely generated S-modules (resp., such that $P^j = 0$ if $j > b$ or $j < a$). In this case we write $C \in \mathcal{D}_{\mathrm{parf}}(S)$ (resp., $C \in \mathcal{D}^{[a,b]}_{\mathrm{parf}}(S)$). We have the following fundamental theorem.

THEOREM 10.5. *a) The cup-product* $\cup_{\boldsymbol{\pi_X(L)}}$ *induces by adjunction an isomorphism in* $\mathcal{D}(\mathcal{R}(L))$*:*

$$\alpha_X(L) := \mathrm{adj}(\cup_{\boldsymbol{\pi_X(L)}}) := \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathcal{R}}\left(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X)^\iota, \mathcal{R}(L)\right)[-3].$$

Moreover $\cup_{\boldsymbol{\pi_X(L)}}$ *is skew-Hermitian, i.e. the following diagram commutes in* $\mathcal{D}(\mathcal{R}(L))$*:*

$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X)^\iota & \xrightarrow{\cup_{\boldsymbol{\pi_X(L)}}} & \mathcal{R}(L)[-3] \\ \downarrow{\scriptstyle s_{12}} & & \uparrow{\scriptstyle \iota} \\ \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X)^\iota \otimes^{\mathbf{L}}_{\mathcal{R}(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) & \xrightarrow{-(\cup_{\boldsymbol{\pi_X(L)}})^\iota} & \mathcal{R}(L)^\iota[-3] \end{array} .$$

b) $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) \in \mathcal{D}^{[1,2]}_{\mathrm{parf}}(\mathcal{R}(L))$*.*
c) The abstract Cassels-Tate pairing $\widetilde{c}_{\pi_X(L),2,2}$ *is skew-hermitian.*
d) There exists for every $q \in \mathbb{Z}$ *a canonical isomorphism of* $\mathcal{R}(L)$*-modules*

$$\widetilde{H}^q_f(K_S/L, \mathbb{A}_X) \cong \mathrm{Hom}_{\mathrm{cts}}\left(\widetilde{H}^{3-q}_{f,\mathrm{Iw}}(L/K, X), \mathbb{Q}_p/\mathbb{Z}_p\right)$$

*(*cts = continuous *refers to the discrete topology on* $\mathbb{Q}_p/\mathbb{Z}_p$ *and the* $\mathfrak{m}_{\mathcal{R}(L)}$*-adic topology on* $\widetilde{H}^*_{f,\mathrm{Iw}}(L/K, X)$*).*

In the proof we will use the following Lemmas.

LEMMA 10.6. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring,* $C \in \mathcal{D}_{\mathrm{ft}}(A)$ *and* $\boldsymbol{x} = (x_1, \ldots, x_d) \subset \mathfrak{m}$ *an* A*-regular sequence. If* $H^q\left(C \otimes^{\mathbf{L}}_A A/\boldsymbol{x}\right) = 0$ *then* $H^q(C) = 0$*. In particular:* $C \otimes^{\mathbf{L}}_A A/\boldsymbol{x} \xrightarrow{\sim} 0$ *in* $\mathcal{D}_{\mathrm{ft}}(A/\boldsymbol{x})$ *implies* $C \xrightarrow{\sim} 0$ *in* $\mathcal{D}_{\mathrm{ft}}(A)$*.*

PROOF. We prove the lemma by induction on d.

If $d = 1$ we have $C \otimes^{\mathbf{L}}_A A/\boldsymbol{x} \xrightarrow{\sim} C \otimes_A \left(A \xrightarrow{x_1} A\right) \xrightarrow{\sim} \mathrm{Cone}\left(C \xrightarrow{x_1} C\right)$, where $(A \xrightarrow{x_1} A)$ (concentrated in degrees -1 and 0) is a free resolution of the A-module $A/\boldsymbol{x}$. Taking cohomology this induces injections

$$H^q(C)/x_1 H^q(C) \hookrightarrow H^q\left(C \otimes^{\mathbf{L}}_A A/\boldsymbol{x}\right).$$

By hypothesis $H^q(C \otimes^{\mathbf{L}}_A A/\boldsymbol{x}) = 0$, so $H^q(C) = 0$ by Nakayama's lemma.

Let now $d > 1$, and let $A' := A/(x_1, \ldots, x_{d-1})$. Then $A/\boldsymbol{x} = A'/x_d A'$ and by assumptions x_d is not a zero divisor in A'. We have an isomorphism

$$C \otimes^{\mathbf{L}}_A A/\boldsymbol{x} \xrightarrow{\sim} \left(C \otimes^{\mathbf{L}}_A A'\right) \otimes^{\mathbf{L}}_{A'} A'/x_d.$$

By assumption and the case already proved we conclude $H^q\left(C \otimes^{\mathbf{L}}_A A'\right) = 0$, so by induction $H^q(C) = 0$. □

LEMMA 10.7. $\mathbf{R}\Gamma_{\mathrm{cont}}(K_v, X(L)) \cong 0 \in \mathcal{D}(\mathcal{R}(L))$ for every $v \in S_f - S_p$.

PROOF. Lemma 0.4 (cfr. the proof of Proposition 0.13) gives a canonical isomorphism in $\mathcal{D}(\mathcal{R}(L))$:

$$\mathbf{R}\Gamma_{\mathrm{cont}}(K_v, X(\mathcal{K})) \otimes^{\mathbf{L}}_{\mathcal{R}(\mathcal{K}),\varepsilon_{\mathcal{K}/L}} \mathcal{R}(L) \cong \mathbf{R}\Gamma_{\mathrm{cont}}(K_v, X(L)),$$

where $\varepsilon_{\mathcal{K}/L}$ is the projection $\mathcal{R}(\mathcal{K}) \twoheadrightarrow \mathcal{R}(L)$ induced by restriction of automorphisms. Then to prove the Lemma we can assume $L = \mathcal{K}$. Given an arithmetic map $\psi \in \mathcal{X}^{\mathrm{arith}}(R; \mathcal{O}_\psi)$ the prime ideal $J_{\mathcal{K},\psi} := \ker\left(R(\mathcal{K}) \overset{\psi(\mathcal{K})}{\twoheadrightarrow} \mathcal{O}_\psi(\mathcal{K}) \overset{\varepsilon_{\mathcal{K}/K}}{\twoheadrightarrow} \mathcal{O}_\psi\right) \in \mathrm{Spec}(R(\mathcal{K}))$ is generated by an $R(\mathcal{K})$-regular sequence, so that *loc. cit.* gives us an isomorphism in $\mathcal{D}(\mathcal{O}_\psi)$:

$$\mathbf{R}\Gamma_{\mathrm{cont}}(K_v, \mathbf{T}(\mathcal{K})) \otimes^{\mathbf{L}}_{R(\mathcal{K})} R(\mathcal{K})/J_{\mathcal{K},\psi} \cong \mathbf{R}\Gamma_{\mathrm{cont}}(K_v, \mathbf{T}_\psi). \tag{90}$$

(Indeed, as in the preceding Section we easily obtain $\mathbf{T}(\mathcal{K})/J_{\mathcal{K},\phi} \cong \mathbf{T}_\psi$ as $\mathcal{O}_\phi[G_{K,S}]$-modules.) Then the Lemma will follows once we will prove $\mathbf{R}\Gamma_{\mathrm{cont}}(K_v, \mathbf{T}(\mathcal{K})) \cong 0 \in \mathcal{D}(R(\mathcal{K}))$ for every $v \in S_f - S_p$. But combining the preceding Lemma with (90) for $\psi = \phi_g \in \mathcal{X}^{\mathrm{arith}}(R, \mathbb{Z}_p)$ (the arithmetic prime attached to f_E) and the isomorphism of $\mathbb{Z}_p[G_{K,S}]$-modules $\mathbf{T}_{\phi_g} \cong T_p := \mathrm{Ta}_p(E_{/\mathbb{Q}})$ (proved in Section (9.3.4)) this last assertion will follows by the claim:

$$\mathbf{R}\Gamma_{\mathrm{cont}}(K_v, T_p)) \cong 0 \in \mathcal{D}(\mathbb{Z}_p) \quad \text{for every } v \in S_f - S_p. \tag{91}$$

To prove this we simply use Hypothesis 3 and Tate local duality: in fact $H^0(K_v, T_p) = 0$ (e.g. by [**Sil86**, pag. 184]) and (using the Weil pairing) $H^2(K_v, T_p) = 0$ since it is the Pontrjagin dual of $H^0(K_v, E_{p^\infty}) = E(K_v)_{p^\infty} = 0$ (by of Hyp. 3). Finally, since $v \nmid p$, Tate's formula for the local Euler characteristic [**Mil04**, pag. 31] shows that ($H^j(K_v, T_p) = 0$ for every $j \neq 1$ and) $H^1(K_v, T_p)$ is finite, so

$$0 = H^0(K_v, E_{p^\infty}) = H^0(K_v, E_{p^\infty})_{/\mathrm{div}} \xrightarrow{\sim} H^1(K_v, T_p)_{\mathbb{Z}_p-\mathrm{tors}} = H^1(K_v, T_p).$$

(Here the second equality is again [**Sil86**, pag. 184] and the isomorphism is the connecting morphism attached to the short exact sequence of G_v-modules $0 \to T_p \to T_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to E_{p^\infty} \to 0$ [**Tat76**].) This proves (91) and the Lemma. □

PROOF OF THEOREM 10.5. *a*) Recall that, with the notations of Section 10.1 we have

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X)\text{`} = \text{'}\widetilde{C}^\bullet_f(G_{K,S}, X(L); \Delta_{S_p}(X(L))),$$

where $\Delta_{S_p}(X(L))$ is the set of local conditions attached to $i_v^+ : X_v^+(L) \hookrightarrow X(L)$ (resp., $0 \to X(L)$) for $v \in S_p$ (resp., $v \in S_f - S_p$). It then follows by the exactness of $\pi_X(L)$ and Proposition 0.2 that we have an isomorphism in $\mathcal{D}(\mathcal{R}(L))$:

$$\mathrm{Cone}\,(\alpha_X(L)) \cong \bigoplus_{v \in S_f - S_p} \mathbf{R}\Gamma_{\mathrm{cont}}(K_v, X(\mathcal{K})).$$

Then the first assertion follows by Lemma 10.7. The second assertion is a special case of Lemma 0.5, recalling that π_X is skew-symmetric, so that $\pi_X(L)$ is skew-Hermitian (see Section 0.17).

b) Let $f : S \to T$ be a morphism of rings. Then by construction of left derived functors: $- \otimes^{\mathbf{L}}_{S,f} T$ (or better $\mathbf{L}f^*$) maps $\mathcal{D}^{[a,b]}_{\mathrm{parf}}(S)$ to $\mathcal{D}^{[a,b]}_{\mathrm{parf}}(T)$. Then using the control theorems proved in the preceding sections, it is sufficient to prove the statement for $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$ (cfr. also to the proof of point of Lemma 10.7).

Since $G_{K,S}$ and G_v ($v \in S_f$) have p-cohomological dimension two, and $\mathbf{T}$, $\mathbf{T}_v^\pm$ ($v \in S_f$) are free $\mathcal{R}$-modules (so that $\mathbf{T}(\mathcal{K})$ and $\mathbf{T}(\mathcal{K})_v^+$ ($v \in S_f$) are finite free $R(\mathcal{K})$-modules), it follows by [**Nek06**, Prop. 4.2.9] that:

$$\left\{\mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, \mathbf{T}(\mathcal{K})); \mathbf{R}\Gamma_{\mathrm{cont}}(K_v, X_v^+(\mathcal{K})),\ \mathbf{R}\Gamma_{\mathrm{cont}}(K_v, X(\mathcal{K})),\ v \in S_f\right\} \subset \mathcal{D}^{[0,2]}_{\mathrm{parf}}(R(\mathcal{K}))$$

(Recall that we are considering $X_v^+ = 0$ if $v \in S_f - S_p$). By definition we then obtain

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/L, \mathbf{T})^{\iota} = \text{'}$$

$$\mathrm{Cone}\left(C^{\bullet}_{\mathrm{cont}}(G_{K,S}, \mathbf{T}(\mathcal{K})) \oplus \bigoplus_{v \in S_p} C^{\bullet}_{\mathrm{cont}}(K_v, \mathbf{T}_v^+(\mathcal{K})) \overset{\mathrm{res}_{S_p} - i^+_{S_p}}{\to} \bigoplus_{v \in S_p} C^{\bullet}_{\mathrm{cont}}(K_v, \mathbf{T}(\mathcal{K}) \right) [-1] \in \mathcal{D}^{[0,3]}_{\mathrm{parf}}(\mathcal{R}(L)).$$

We claim that:

$$\widetilde{H}^3_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) = 0. \tag{92}$$

This would conclude the proof. Indeed (92) would imply (see the discussion in [**Nek06**, Sec. 4.2.8])

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) \in \mathcal{D}^{[0,2]}_{\mathrm{parf}}(R(\mathcal{K})).$$

Then we would obtain $\mathbf{R}\mathrm{Hom}_{R(\mathcal{K})}\left(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})^{\iota}, R(\mathcal{K})\right)[-3] \in \mathcal{D}^{[1,3]}_{\mathrm{parf}}(R(\mathcal{K}))$ and using the isomorphism $\alpha_{\mathbf{T}}(\mathcal{K})$ from $a)$ we finally would obtain:

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) \in \mathcal{D}^{[0,2]}_{\mathrm{parf}}(R(\mathcal{K})) \cap \mathcal{D}^{[1,3]}_{\mathrm{parf}}(R(\mathcal{K})) = \mathcal{D}^{[1,2]}_{\mathrm{parf}}(R(\mathcal{K})).$$

(For the last equality see again [**Nek06**, Sec. 4.2.8].)

Since T_p (resp., $F_v^+(T_p)$ for $v \in S_p$) is obtained as the quotient of $\mathbf{T}(\mathcal{K})$ (resp., $\mathbf{T}(\mathcal{K})_v^+$) by the ideal $J_{\mathcal{K},\phi_g}$, which is generated by an $R(\mathcal{K})$-regular sequence (cfr. the proof of Lemma 10.7), Lemma 10.6 implies that to prove (92) it is sufficient to prove that $\widetilde{H}^3_f(K, T_p) = 0$. This follows by Hyp. 3 : $E(K)_{p^\infty} = 0$. Indeed, since E_{p^∞} is the Kummer dual of T_p, using Nekovář's generalized Poitou-Tate duality (precisely Prop. 0.8 and the exact sequence (166)) we see that the Pontrjagin dual of $\widetilde{H}^3_f(K, T_p)$ is a submodule of $H^0(G_{K,S}, E_{p^\infty}) = 0$.

$c)$ (resp., $d)$) is a special case of Lemma 0.7 (resp., Lemma 0.14). □

10.5. p-adic pairings. Fix a $\mathbb{Z}_p^d$-extension $\mathcal{L}/K$. For every subextension $K \subseteq L \subseteq \mathcal{L}$ and every arithmetic point $\psi \in \mathcal{X}^{\mathrm{arith}}(R; \mathcal{O}_\psi)$ we write $J_{L,\psi} = J_{\mathcal{L}/L,\psi} := \ker\left(R(\mathcal{L}) \overset{\varepsilon_{\mathcal{L}/L}}{\twoheadrightarrow} R(L) \overset{\psi(L)}{\twoheadrightarrow} \mathcal{O}_\psi(L)\right) \subset \mathfrak{m}_{R(\mathcal{L})}$; it is an ideal generated by an $R(\mathcal{L})$-regular sequence. More precisely: fix topological generators $\sigma_1, \ldots, \sigma_d$ of $\mathrm{Gal}(\mathcal{L}/K) \overset{\sim}{\to} \mathbb{Z}_p^d$ and let ϖ_ψ be a generator of $\mathfrak{p}_\psi := \ker(\psi)$. Then, identifying $R(\mathcal{L}) \overset{\sim}{\to} R[[X_1, \ldots, X_d]]$ (with $X_j := \sigma_j - 1$), $J_{L,\psi} = (\varpi_\psi, \gamma_1, \ldots, \gamma_d)$, with $\gamma_j = (X_j + 1)^{p^{n_j}} - 1$ for some integer $n_j \geq 0$. As $\mathbf{T}(\mathcal{L})_{[J_{L,\psi}]} := \mathbf{T}(\mathcal{L}) \otimes_{R(\mathcal{L}), \psi(L) \circ \varepsilon_{\mathcal{L}/L}} \mathcal{O}_\psi(L) \overset{\sim}{\to} \mathbf{T}_\psi(L)$ as $\mathcal{O}_\psi(L)[G_{K,S}]$-modules and similarly $(\mathbf{T}(\mathcal{L})_v^+)_{[J_{L,\psi}]} \cong \mathbf{T}_\psi(L)$ as $\mathcal{O}_\psi(L)[G_{K_v}]$-modules for every $v \in S_p$ (cfr. the preceding Sections) the construction of Section C gives us a Bockstein map in $\mathcal{D}(\mathcal{O}_\psi(L))$:

$$\beta^{\mathrm{wt}}_{L,\psi} := \beta_{J_{\mathcal{L}/L,\psi}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi)[1] \otimes_{\mathcal{O}_\psi} J_{L,\psi}/J^2_{L,\psi}.$$

(Referring to Section C for the details, we recall that $\beta^{\mathrm{wt}}_{L,\psi}$ is obtained, via the 'control theorems' of Section 10.3 applying the derived functor $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{L}/K, \mathbf{T}) \otimes^{\mathbf{L}}_{R(\mathcal{L})} -$ to the exact triangle in $\mathcal{D}(R(\mathcal{L}))$: $R(\mathcal{L})/J^2_{L,\psi} \to \mathcal{O}_\psi(L) \to J_{L,\psi}/J^2_{L,\psi}[1]$.) The associated *derived 'height' paring* is then defined as the morphism in $\mathcal{D}(\mathcal{O}_\psi(L))$ (cfr. Section C):

$$\begin{aligned}\widetilde{h}^{\mathrm{wt}}_{\mathcal{L}/L,\psi} := \widetilde{h}_{J_{\mathcal{L}/L,\psi}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi) \otimes^{\mathbf{L}}_{\mathcal{O}_\psi(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi)^{\iota} &\overset{\beta^{\mathrm{wt}}_{L,\psi} \otimes \mathrm{id}}{\longrightarrow} \\ \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi)[1] \otimes^{\mathbf{L}}_{\mathcal{O}_\psi(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi)^{\iota} \otimes_{\mathcal{O}_\psi(L)} J_{L,\psi}/J^2_{L,\psi} & \\ \overset{\cup_{\boldsymbol{\pi_\psi(L)}}[1] \otimes \mathrm{id}}{\longrightarrow} \mathcal{O}_\psi(L)[-2] \otimes_{\mathcal{O}_\psi(L)} J_{L,\psi}/J^2_{L,\psi} = J_{L,\psi}/J^2_{L,\psi}[-2], &\end{aligned}$$

where $\cup_{\boldsymbol{\pi_\psi(L)}} = \cup_{\boldsymbol{\pi_{\mathbf{T}_\psi}(L)}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi) \otimes^{\mathbf{L}}_{\mathcal{O}_\psi(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi)^{\iota} \to \mathcal{O}_\psi(L)[-3]$ is the global cup-product pairing in $\mathcal{D}(\mathcal{O}_\psi(L))$ induced by $\pi_\psi := \pi_R \otimes_{R,\psi} \mathcal{O}_\psi$ defined in Section 10.4 (and we use the canonical isomorphism $(*[1]) \otimes^{\mathbf{L}} \dagger \xrightarrow{\sim} \left(* \otimes^{\mathbf{L}} \dagger\right)[1])$. This pairing induces in cohomology an $\mathcal{O}_\psi(L)$-bilinear form:

$$\widetilde{h}^{\mathrm{wt}}_{\mathcal{L}/L,\psi,1,1} : \widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi) \otimes_{\mathcal{O}_\psi(L)} \widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)^\iota \longrightarrow J_{L,\psi}/J^2_{L,\psi}.$$

If $\mathcal{L} = L$ we simply write $\widetilde{h}^{\mathrm{wt}}_{L,\psi} := \widetilde{h}^{\mathrm{wt}}_{L/L,\psi}$ and $\widetilde{h}^{\mathrm{wt}}_{L,\psi,1,1} := \widetilde{h}^{\mathrm{wt}}_{L/L,\psi,1,1}$. In particular taking $\mathcal{L} = L = K$ (and writing $\mathfrak{p}_\psi := \ker(\psi)$) we obtain a *'derived weight pairing'*:

$$\widetilde{h}^{\mathrm{wt}}_{K,\psi} : \widetilde{\mathbf{R}\Gamma}_f(K,\mathbf{T}_\psi) \otimes^{\mathbf{L}}_{\mathcal{O}_\psi} \widetilde{\mathbf{R}\Gamma}_f(K,\mathbf{T}_\psi) \longrightarrow \mathfrak{p}_\psi/\mathfrak{p}^2_\psi[-2],$$

and the corresponding *weight paring* on cohomology:

$$\widetilde{h}^{\mathrm{wt}}_{K,\psi,1,1} : \widetilde{H}^1_f(K,\mathbf{T}_\psi) \otimes_{\mathcal{O}_\psi} \widetilde{H}^1_f(K,\mathbf{T}_\psi) \longrightarrow \mathfrak{p}_\psi/\mathfrak{p}^2_\psi.$$

In a similar way, replacing in the constructions above $\mathbf{T}$ with $\mathbf{T}_\psi$ and $J_{L,\psi}$ with the $\mathcal{L}/L$-augmentation ideal $I_{L,\psi} = I_{\mathcal{L}/L,\psi} := \ker\left(\varepsilon_{\mathcal{L}/L} : \mathcal{O}_\psi(\mathcal{L}) \twoheadrightarrow \mathcal{O}_\psi(L)\right) \subset \mathfrak{m}_{\mathcal{O}_\psi(\mathcal{L})}$ in $\mathcal{O}_\psi(\mathcal{L})$, the constructions of Section C gives us a Bockstein map:

$$\beta_{L,\psi} := \beta_{I_{\mathcal{L}/L,\psi}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)[1] \otimes_{\mathcal{O}_\psi} I_{L,\psi}/I^2_{L,\psi},$$

(obtained, via Prop. 10.3 applying $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{L}/K,\mathbf{T}_\psi) \otimes^{\mathbf{L}}_{\mathcal{O}_\psi(\mathcal{L})} -$ to the the exact triangle in $\mathcal{D}(\mathcal{O}_\psi(\mathcal{L}))$: $\mathcal{O}_\psi(\mathcal{L})/I^2_{L,\psi} \to \mathcal{O}_\psi(L) \to I_{L,\psi}/I^2_{L,\psi}[1]$) and the associated *'derived canonical height pairing'*

$$\begin{aligned}\widetilde{h}_{\mathcal{L}/L,\psi} := \widetilde{h}_{I_{\mathcal{L}/L,\psi}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi) \otimes^{\mathbf{L}}_{\mathcal{O}_\psi(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)^\iota &\xrightarrow{\beta_{L,\psi}\otimes\mathrm{id}} \\ \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)[1] \otimes^{\mathbf{L}}_{\mathcal{O}_\psi(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)^\iota \otimes_{\mathcal{O}_\psi(L)} I_{L,\psi}/I^2_{L,\psi} & \\ \xrightarrow{\cup_{\pi_\psi(L)}[1]\otimes\mathrm{id}} \mathcal{O}_\psi(L)[-2] \otimes_{\mathcal{O}_\psi(L)} I_{L,\psi}/I^2_{L,\psi} = I_{L,\psi}/I^2_{L,\psi}[-2],&\end{aligned}$$

inducing in cohomology the *canonical height pairing*:

$$\widetilde{h}_{\mathcal{L}/L,\psi,1,1} : \widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi) \otimes_{\mathcal{O}_\psi(L)} \widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)^\iota \longrightarrow I_{L,\psi}/I^2_{L,\psi}.$$

Write $\mathfrak{p}_\psi(L) := J_{L/L,\psi}$ for the kernel of $\psi(L) : R(L) \twoheadrightarrow \mathcal{O}_\psi(L)$, which is a principal ideal generated by ϖ_ψ. We have a canonical decomposition of $\mathcal{O}_\psi(L)$-modules

$$J_{L,\psi}/J^2_{L,\psi} = I_{L,\psi}/I^2_{L,\psi} \oplus \mathfrak{p}_\psi(L)/\mathfrak{p}_\psi(L)^2,$$

induced by the natural projections $\varepsilon_{\mathcal{L}/L} : J_{L,\psi} \twoheadrightarrow \mathfrak{p}_\psi(L)$ and $\psi(\mathcal{L}) : J_{L,\psi} \twoheadrightarrow I_{L,\psi}$. This induces a decomposition

$$\widetilde{h}^{\mathrm{wt}}_{\mathcal{L}/L,\psi,1,1} = \widetilde{\mathbf{h}}_{\mathcal{L}/L,\psi,1,1} \oplus \widetilde{\mathbf{h}}^{\mathrm{wt}}_{L,\psi,1,1}.$$

LEMMA 10.8. *$\widetilde{\mathbf{h}}^{\mathrm{wt}}_{L,\psi,1,1} = \widetilde{h}^{\mathrm{wt}}_{L,\psi,1,1}$ and $\widetilde{\mathbf{h}}_{\mathcal{L}/L,\psi,1,1} = \widetilde{h}_{\mathcal{L}/L,\psi,1,1}$*

PROOF. This is a special case of Lemma 0.15. □

It follows in particular by the Lemma that $\widetilde{\mathbf{h}}_{\mathcal{L}/L,\psi,1,1}$ (resp., $\widetilde{\mathbf{h}}^{\mathrm{wt}}_{L,\psi,1,1}$) depends only on the Galois deformation $\mathbf{T}_\psi(\mathcal{L})$ of $\mathbf{T}_\psi(L) = \mathbf{T}_\psi(\mathcal{L}) \otimes_{\mathcal{O}_\psi(\mathcal{L})} \mathcal{O}_\psi(L)$ (resp, the Hida deformation $\mathbf{T}(L)$ of $\mathbf{T}_\psi(L) = \mathbf{T}(L) \otimes_{R(L),\psi(L)} \mathcal{O}_\psi$).

10.5.1. *Another description of $\widetilde{h}^{\mathrm{wt}}_{L,\psi,1,1}$.* Fix a generator ϖ_ψ of the principal ideal $\mathfrak{p}_\psi := \ker(\psi)$. We recall (cfr. Section 10.4) that the perfect skew-Hermitian pairing $\pi(L) : \mathbf{T}(L) \otimes_{\mathcal{R}(L)} \mathbf{T}(L)^\iota \to R(L)(1)$ induced by the perfect skew-symmetric duality $\pi = \pi_R : \mathbf{T} \otimes_R \mathbf{T} \to R(1)$ induces, via Nekovář's construction described in Section 0.10 skew-Hermitian pairing (cfr. Theorem 10.5) of $R(L)$-modules:

$$\widetilde{c}_{\pi(L),2,2} : \widetilde{H}^2_{f,\mathrm{Iw}}(L/K,\mathbf{T})_{R(L)-\mathrm{tors}} \otimes_{R(L)} \widetilde{H}^2_{f,\mathrm{Iw}}(L/K,\mathbf{T})^\iota_{R(L)-\mathrm{tors}} \longrightarrow \mathscr{R}(L)/R(L),$$

where $\mathscr{R}(L) := \mathrm{Frac}(R(L))$. We have a (non-canonical) isomorphism of $\mathcal{O}_\psi(L)$-modules:

$$\theta_{\varpi_\psi} : \left(\mathscr{R}(L)/R(L)\right)[\varpi_\psi] \xrightarrow{\sim} \mathfrak{p}_\psi(L)/\mathfrak{p}_\psi(L)^2,$$

defined by $\theta_{\varpi_\psi}([r/\varpi_\psi]) := (\varpi_\psi \cdot r) \mod \mathfrak{p}_\psi(L)^2 = \psi(L)(r) \cdot \varpi_\psi \mod \mathfrak{p}_\psi(L)^2$. Recall also the morphism of $\mathcal{O}_\psi(L)$-modules

$$i_{\varpi_\psi} : \widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi) \to \widetilde{H}^2_{f,\mathrm{Iw}}(L/K,\mathbf{T})[\varpi_\psi]$$

defined in (89).

PROPOSITION 10.9. *The following diagram of $\mathcal{O}_\psi(L)$ modules commutes:*

$$\begin{array}{ccc}
\widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)\otimes_{\mathcal{O}_\psi(L)}\widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)^\iota & \xrightarrow{\widetilde{h}^{\mathrm{wt}}_{L,\psi,1,1}} & \mathfrak{p}_\psi(L)/\mathfrak{p}_\psi(L)^2 \\
\downarrow{\scriptstyle i_{\varpi_\psi}\otimes i^\iota_{\varpi_\psi}} & & \uparrow{\scriptstyle \theta_{\varpi_\psi}} \\
\widetilde{H}^2_{f,\mathrm{Iw}}(L/K,\mathbf{T})[\varpi_\psi]\otimes_{\mathcal{O}_\psi(L)}\widetilde{H}^2_{f,\mathrm{Iw}}(L/K,\mathbf{T})^\iota[\varpi_\psi] & \xrightarrow{\widetilde{c}_{\pi(L),2,2}} & \left(\mathscr{R}(L)/R\right)[\varpi_\psi].
\end{array}$$

PROOF. This is a special case of Proposition 0.17. □

COROLLARY 10.10. $\widetilde{h}^{\mathrm{wt}}_{L,\psi,1,1}$ *is skew-Hermitian (with respect to the involution induced on $\mathfrak{p}_\psi(L)/\mathfrak{p}_\psi(L)^2$ by the Iwasawa involution ι on $R(L)$). In particular $\widetilde{h}^{\mathrm{wt}}_{K,\psi,1,1}$ is skew-symmetric.*

PROOF. This follows directly by the preceding Proposition, recalling that $\widetilde{c}_{\pi(L),2,2}$ is skew-Hermitian (Theorem 10.5) and noting that θ_{ϖ_ψ} 'commutes' with the involution ι (as $\iota(\varpi_\psi)=\varpi_\psi$). □

10.5.2. *Another description of $\widetilde{h}_{\mathcal{L}/L,\psi,1,1}$.* Let us assume (for simplicity) in this Section that L/K is a (possibly trivial) $\mathbb{Z}_p$-power extension (referring to Section 0.22 for the general situation). Let as fix an isomorphism: $\mathrm{Gal}(\mathcal{L}/L)\xrightarrow{\sim}\mathbb{Z}_p^k$, and let $\sigma_j\in\mathrm{Gal}(\mathcal{L}/L)$ for $j=1,\ldots,k$ be the topological generators corresponding to the canonical basis of $\mathbb{Z}_p^k$. Write $\mathcal{L}_j/L$ for the $\mathbb{Z}_p$-extension corresponding to σ_j, i.e. $\mathcal{L}_j\subset\mathcal{L}$ is the subfield fixed by the closed subgroup of $\mathrm{Gal}(\mathcal{L}/L)$ corresponding to $\bigoplus_{t\neq j;1\le t\le k}\mathbb{Z}_p\subset\mathbb{Z}_p^k$ under the fixed isomorphism. Then $\mathrm{Gal}(\mathcal{L}_j/L)=\sigma_j^{\mathbb{Z}_p}$ is topologically generated by σ_j.

The construction of Section 0.10, applied to the $\mathcal{O}_\psi(\mathcal{L}_j)$-modules $\mathbf{T}_\psi(\mathcal{L}_j)$ and the perfect skew-Hermitian duality $\pi_\psi(\mathcal{L}_j):\mathbf{T}_\psi(\mathcal{L}_j)\otimes_{\mathcal{O}_\psi(\mathcal{L}_j)}\mathbf{T}_\psi(\mathcal{L}_j)^\iota\to\mathcal{O}_\psi(\mathcal{L}_j)(1)$ (where $\pi_\psi:=\pi\otimes_{R,\psi}\mathcal{O}_\psi$) gives us skew-Hermitian pairings:

$$\widetilde{c}_{\pi_\psi(\mathcal{L}_j),2,2}:\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{L}_j/K,\mathbf{T}_\psi)_{\mathcal{O}_\psi(\mathcal{L}_j)\text{-tors}}\otimes_{\mathcal{O}_\psi(\mathcal{L}_j)}\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{L}_j/K,\mathbf{T}_\psi)^\iota_{\mathcal{O}_\psi(\mathcal{L}_j)\text{-tors}}\longrightarrow\mathscr{O}_\psi(\mathcal{L}_j)/\mathcal{O}_\psi(\mathcal{L}_j),$$

where $\mathscr{O}_\psi(\mathcal{L}_j):=\mathrm{Frac}\left(\mathcal{O}_\psi(\mathcal{L}_j)\right)$. Moreover (88) gives us a morphism of $\mathcal{O}_\psi(L)$-modules:

$$i_{\sigma_j}:\widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)\longrightarrow\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{L}_j/K,\mathbf{T}_\psi)[\sigma_j-1],$$

and we have an isomorphism of $\mathcal{O}_\psi(L)$-modules: $\theta_{\sigma_j}:\left(\mathscr{O}_\psi(\mathcal{L}_j)/\mathcal{O}_\psi(\mathcal{L}_j)\right)[\sigma_j-1]\xrightarrow{\sim}I_{\mathcal{L}_j/L,\psi}/I^2_{\mathcal{L}_j/L,\psi}$, defined by $[\alpha/(\sigma_j-1)]\mapsto\alpha\cdot(\sigma_j-1)\bmod I^2_{\mathcal{L}_j/L,\psi}=\varepsilon_{\mathcal{L}_j/L}(\alpha)\cdot(\sigma_j-1)\bmod I^2_{\mathcal{L}_j/L,\psi}$. Finally, let us note that we have a canonical decomposition:

$$I_{\mathcal{L}/L,\psi}/I^2_{\mathcal{L}/L,\psi}\cong\bigoplus_{j=1}^k I_{\mathcal{L}_j/L,\psi}/I^2_{\mathcal{L}_j/L,\psi},$$

induced on 'augmentation ideals' by the natural projections $\varepsilon_{\mathcal{L}/\mathcal{L}_j}:\mathcal{O}_\psi(\mathcal{L})\twoheadrightarrow\mathcal{O}_\psi(\mathcal{L}_j)$ defined by restriction of automorphisms.

PROPOSITION 10.11. *For every $j=1,\ldots,k$, the following diagram of $\mathcal{O}_\psi(L)$-modules commutes:*

$$\begin{array}{ccc}
\widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)\otimes_{\mathcal{O}_\psi(L)}\widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)^\iota & \xrightarrow{\widetilde{h}_{\mathcal{L}/L,\psi,1,1}} & I_{\mathcal{L}/L,\psi}/I^2_{\mathcal{L}/L,\psi} \\
\Vert & & \downarrow{\scriptstyle \varepsilon_{\mathcal{L}/\mathcal{L}_j}} \\
\widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)\otimes_{\mathcal{O}_\psi(L)}\widetilde{H}^1_{f,\mathrm{Iw}}(L/K,\mathbf{T}_\psi)^\iota & \xrightarrow{\widetilde{h}_{\mathcal{L}_j/K,\psi,1,1}} & I_{\mathcal{L}_j/L,\psi}/I^2_{\mathcal{L}_j/L,\psi} \\
\downarrow{\scriptstyle i_{\sigma_j}\otimes i^\iota_{\sigma_j}} & & \uparrow{\scriptstyle \theta_{\sigma_j}} \\
\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{L}_j/K,\mathbf{T}_\psi)[\sigma_j-1]\otimes_{\mathcal{O}_\psi(L)}\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{L}_j/K,\mathbf{T}_\psi)^\iota[\sigma_j-1] & \xrightarrow{\widetilde{c}_{\pi_\psi(\mathcal{L}_j),2,2}} & \left(\mathscr{O}_\psi(\mathcal{L}_j)/\mathcal{O}_\psi(\mathcal{L}_j)\right)[\sigma_j-1].
\end{array}$$

PROOF. Writing $I_{(j)} \subset \mathcal{O}_\psi(\mathcal{L})$ for the ideal generated by $\{\sigma_i - 1\}_{i \neq j}$ we have $\mathcal{O}_\psi(\mathcal{L}_j) = \mathcal{O}_\psi(\mathcal{L})/I_{(j)}$, so that

$$\begin{aligned}\mathbf{T}_\psi(\mathcal{L}_j) &:= \big(\mathbf{T}_\psi \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi(\mathcal{L}_j)\big) < -1 > \cong \big(\mathbf{T}_\psi \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi(\mathcal{L})/I_{(j)}\big) < -1 > \\ &\cong \big(\mathbf{T}_\psi \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi(\mathcal{L})\big) < -1 > \otimes_{\mathcal{O}_\psi(\mathcal{L}),\varepsilon_{\mathcal{L}/\mathcal{L}_j}} \mathcal{O}_\psi(\mathcal{L}_j) =: \mathbf{T}_\psi(\mathcal{L}) \otimes_{\mathcal{O}_\psi(\mathcal{L}),\varepsilon_{\mathcal{L}/\mathcal{L}_j}} \mathcal{O}_\psi(\mathcal{L}_j).\end{aligned}$$

Moreover $I_{\mathcal{L}_j/L,\psi} \subset \mathcal{O}_\psi(\mathcal{L}_j)$ is the image of $I_{\mathcal{L}/L,\psi} \subset \mathcal{O}_\psi(\mathcal{L})$ under the natural projection $\varepsilon_{\mathcal{L}/\mathcal{L}_j}$. Then the commutativity of the upper square follows for Lemma 0.15, applied with $\mathcal{P} = I_{\mathcal{L}/L,\psi}$, $\boldsymbol{x} = (\sigma_j - 1)$ and $\boldsymbol{y} = (\sigma_i - 1 : i \neq j,\ 1 \leq i \leq k)$.

The commutativity of the lower square follows directly from Prop. 0.17. □

COROLLARY 10.12. *$\widetilde{h}_{\mathcal{L}/L,\psi,1,1}$ is skew-Hermitian (for the the involution induced on $I_{\mathcal{L}/L,\psi}/I^2_{\mathcal{L}/L,\psi}$ by Iwasawa main involution on $\mathcal{O}_\psi(\mathcal{L})$.)*

PROOF. We have $\sigma_j^{\pm 1} \cdot \alpha \equiv \alpha \bmod I_{\mathcal{L}_j/L,\psi}$ and $-\iota(\sigma_j - 1) = \sigma_j^{-1} \cdot (\sigma_j - 1) \equiv (\sigma_j - 1) \mod I^2_{\mathcal{L}_j/L,\psi}$ for every $\alpha \in \mathcal{O}_\psi(\mathcal{L}_j)$, so that

$$\theta_{\sigma_j} \circ \iota \left[\frac{\alpha}{\sigma_j - 1}\right] = -\theta_{\sigma_j} \left[\frac{\sigma_j \cdot \iota(\alpha)}{\sigma_j - 1}\right] = -\theta_{\sigma_j} \left[\frac{\iota(\alpha)}{\sigma_j - 1}\right] = -\left[\iota(\alpha)(\sigma_j - 1)\right] = \left[\iota\big(\alpha(\sigma_j - 1)\big)\right] = \iota \circ \theta_{\sigma_j} \left[\frac{\alpha}{\sigma_j - 1}\right],$$

i.e. θ_{σ_j} 'commutes' with Iwasawa involution ι. It then follows by the preceding Proposition and the properties of Cassels-Tate pairings that each $\widetilde{h}_{\mathcal{L}_j/L,\psi,1,1}$ is skew-Hermitian. Using again the preceding Proposition we conclude. □

REMARK 10.13. We have a canonical isomorphism of $\mathcal{O}_\psi(L)$-modules:

$$\mathrm{Gal}(\mathcal{L}/L) \otimes_{\mathbb{Z}_p} \mathcal{O}_\psi(L) \cong I_{\mathcal{L}/L,\psi}/I^2_{\mathcal{L}/L,\psi},$$

induced by $g \otimes 1 \mapsto g - 1 \bmod I^2_{\mathcal{L}/L,\psi}$ for every $g \in \mathrm{Gal}(\mathcal{L}/L)$. This isomorphism 'transforms' the involution ι on the *R.H.S.* in the involution $-\mathrm{id} \otimes \iota$ on the *L.H.S.* In particular, taking $L = K$ we obtain from the preceding Corollary a *symmetric* height pairing:

$$\widetilde{h}_{\mathcal{L}/K,\psi,1,1} : \widetilde{H}^1_f(K, \mathbf{T}_\psi) \times \widetilde{H}^1_f(K, \mathbf{T}_\psi) \longrightarrow \mathrm{Gal}(\mathcal{L}/K) \otimes_{\mathbb{Z}_p} \mathcal{O}_\psi.$$

10.6. ***R*-adic pairings.** Let $K \subset L \subset \mathcal{L} \subset \mathcal{K}$ be as in Section 10.5. Write $J_{\mathcal{L}/L} := \ker\left(R(\mathcal{L}) \twoheadrightarrow R(L)\right)$ for the $\mathcal{L}/L$-augmentation ideal in $R(\mathcal{L})$. This is a prime ideal generated by an $R(\mathcal{L})$-sequence (cfr. Section 10.5), and the construction of Sec. 0.20 gives us a Bockstein map:

$$\beta_{J_{\mathcal{L}/L}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T})[1] \otimes_{R(L)} J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L},$$

(defined via the the control theorem Prop. 10.3 applying $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{L}/K, \mathbf{T}) \otimes^{\mathbf{L}}_{R(\mathcal{L})} -$ to the exact triangle in $\mathcal{D}(R(\mathcal{L}))$: $R(\mathcal{L})/J^2_{\mathcal{L}/L} \to R(L) \to J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L}[1]$) with associated *R-adic derived pairing*:

$$\begin{aligned}\widetilde{H}_{\mathcal{L}/L} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \otimes^{\mathbf{L}}_{R(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T})^\iota &\stackrel{\beta_{J_{\mathcal{L}/L}} \otimes \mathrm{id}}{\longrightarrow} \\ \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T})[1] &\otimes^{\mathbf{L}}_{R(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, \mathbf{T})^\iota \otimes_{R(L)} J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L} \\ &\stackrel{\cup_{\boldsymbol{\pi_{R(L)}}}[1] \otimes \mathrm{id}}{\longrightarrow} R[-2] \otimes_R J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L} = J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L}[-2],\end{aligned}$$

where $\cup_{\boldsymbol{\pi_{R(L)}}}$ is the global cup-product pairing in $\mathcal{D}(R(L))$ attached to the the perfect skew-symmetric duality $\pi = \pi_R : \mathbf{T} \otimes_R \mathbf{T} \to R(1)$ in Sec. 0.17. As above we write

$$\widetilde{H}_{\mathcal{L}/L,1,1} : \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \otimes_{R(L)} \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T})^\iota \longrightarrow J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L}$$

for the R-bilinear form induced by $\widetilde{H}_{\mathcal{L}/L}$ on $(1,1)$-cohomology.

10.6.1. *Specializations.* Let $\psi \in \mathcal{X}^{\mathrm{arith}}(R, \mathcal{O}_\psi)$, inducing a morphism $\psi(L) : R(\mathcal{L}) \twoheadrightarrow \mathcal{O}_\psi(L) \cong R(\mathcal{L}) \otimes_R R/\varpi_\psi$, such that $I_{\mathcal{L}/L,\psi} = \psi(L)\left(J_{\mathcal{L}/L}\right)$. Let us denote again by the same symbol the induced projection $J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L} \twoheadrightarrow I_{\mathcal{L}/L,\psi}/I^2_{\mathcal{L}/L,\psi}$. Since $\mathbf{T}_\psi(\ddagger) = \mathbf{T}(\ddagger) \otimes_{R,\psi} \mathcal{O}_\psi = \mathbf{T}(\ddagger)/\varpi_\psi$ for every sub-extension $\ddagger/K$ of $\mathcal{K}/L$, a direct application of Lemma 0.16 proves the following Lemma.

LEMMA 10.14. *Let us write $\psi(L)_* : \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \to \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi)$ for the morphisms induced on Selmer complexes by $\psi(L)$. Then the following diagram of $R(L)$-modules commutes:*

$$\begin{array}{ccc} \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \otimes_{R(L)} \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T})^\iota & \xrightarrow{\widetilde{H}_{\mathcal{L}/L,1,1}} & J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L} \\ \downarrow{\scriptstyle \psi(L)_* \otimes \psi(L)^\iota_*} & & \downarrow{\scriptstyle \psi(L)} \\ \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi) \otimes_{R(L)} \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\psi)^\iota & \xrightarrow{\widetilde{h}_{\mathcal{L}/L,\psi,1,1}} & I_{\mathcal{L}/L,\psi}/I^2_{\mathcal{L}/L,\psi}. \end{array}$$

10.6.2. *Relations with Cassels-Tate pairing.* Assume that L/K is a $\mathbb{Z}_p$-power extension, and let $\mathcal{L}_j/L$ be as in Section 10.5.2. Then the analogue of Proposition 10.11 holds for $\widetilde{H}_{\mathcal{L}/L,1,1}$, i.e. using similar notations as in *loc. cit.*: under the canonical decomposition of $R(L)$-modules $J_{\mathcal{L}/L}/J^2_{\mathcal{L}/L} \cong \bigoplus_{j=1}^k J_{\mathcal{L}_j/L}/J^2_{\mathcal{L}_j/L}$ we have

$$\widetilde{H}_{\mathcal{L}/L,1,1} = \bigoplus_{j=1}^k \widetilde{H}_{\mathcal{L}_j/L,1,1}; \quad \widetilde{H}_{\mathcal{L}_j/L,1,1} = \vartheta_{\sigma_j} \circ \widetilde{c}_{\pi(\mathcal{L}_j),2,2} \circ \zeta_{\sigma_j} \otimes \zeta^\iota_{\sigma_j}.$$

Here $\zeta_{\sigma_j} : \widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \to \widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{L}_j/K, \mathbf{T})[\sigma_j - 1]$ is the morphism coming from (88), the isomorphism $\vartheta_{\sigma_j} : \mathscr{R}(\mathcal{L}_j)/R(\mathcal{L}_j)[\sigma_j - 1] \xrightarrow{\sim} J_{\mathcal{L}_j/L}/J^2_{\mathcal{L}_j/L}$ is defined by $\vartheta_{\sigma_j}\left(\left[\frac{r}{\sigma_j - 1}\right]\right) = r \cdot (\sigma_j - 1) \mod J^2_{\mathcal{L}_j/L}$ and

$$\widetilde{c}_{\pi(\mathcal{L}_j),2,2} : \widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{L}_j/K, \mathbf{T})_{R(\mathcal{L}_j)\text{-tors}} \otimes_{R(\mathcal{L}_j)} \widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{L}_j/K, \mathbf{T})^\iota_{R(\mathcal{L}_j)\text{-tors}} \longrightarrow \mathscr{R}(\mathcal{L}_j)/R(\mathcal{L}_j)$$

is the (skew-Herimitian) Cassels-Tate pairing attached to the perfect (skew-Hermitian) duality $\pi(\mathcal{L}_j) : \mathbf{T}(\mathcal{L}_j) \otimes_{R(\mathcal{L}_j)} \mathbf{T}(\mathcal{L}_j)^\iota \to R(\mathcal{L}_j)(1)$ induced by the (skew-symmetric) perfect duality $\pi = \pi_R : \mathbf{T} \otimes_R \mathbf{T} \to R(1)$ (cfr. Section 0.10 and Section 0.17). In particular we conclude that $\widetilde{H}_{\mathcal{L}/K,1,1}$ is *skew-Hermitian.*

11. Organizing modules and p-adic L-functions

In this Section we review the constructions of [**MR05**], and we show how the 'weight variable' may be naturally included in their theory of 'organizing modules'. Following Mazur-Rubin, we then apply the resulting theory to the the study of algebraic p-adic L-function.

Write $\overline{R} := R(\mathcal{K})$ and $\overline{\mathbb{Z}_p} := \mathbb{Z}_p(\mathcal{K})$. Let $\mathcal{P}$ be an ideal of $\overline{R}$ which is stable under the action of Iwasawa involution ι. Then $A := \overline{R}/\mathcal{P}$ is equipped with an involution compatible with $\iota : \overline{R} \to \overline{R}$, which we denote again by ι. Moreover, for every A-module N, we write $N^* := \mathrm{Hom}_A(N^\iota, A)$, where N^ι denotes the A-module with the same underline abelian group as N, but with A-action obtained composing the original action with $\iota : A \to A$. We will frequently consider the case $N = M \otimes_{\overline{R}} A$, for a free $\overline{R}$-module M of finite rank. In this case we have canonical identifications

$$M^\iota \otimes_{\overline{R}} A = N^\iota; \quad M^* \otimes_{\overline{R}} A = N^*,$$

where the first (resp., second) isomorphism is defined by $m \otimes a \mapsto \iota(a) \cdot m$ (resp., $\psi \mapsto \psi \bmod \mathcal{P}$).

11.1. Skew-Hermitian modules (cfr. [**MR05**, Sec. 4]**).** A *skew-Hermitian* $\overline{R}$-module is a pair (Φ, h), where Φ is a free $\overline{R}$-module of finite type and $h : \Phi \hookrightarrow \Phi^*$ is an injective, *skew-Hermitian* morphism of $\overline{R}$-modules. In other words: writing

$$(-,-)_h : \Phi \times \Phi^\iota \longrightarrow \overline{R}$$

for the $\overline{R}$-bilinear form defined by $(x, y)_h := h(x)(y)$, we assume that $(-,-)_h$ is non-degenerate, and satisfies

$$(x, y)_h = -\iota\,(y, x)_h \tag{93}$$

for every $x, y \in \Phi$. We say that (Φ, h) is a *basic skew-Hermitian module* if $(-,-)_h : \Phi \times \Phi^\iota \to \overline{\mathfrak{m}}$ takes values in the maximal ideal $\overline{\mathfrak{m}}$ of $\overline{R}$.

A morphism of skew-Hermitian modules $(\Phi, h) \to (\Psi, \rho)$ is a morphism of $\overline{R}$-modules $\psi : \Phi \to \Psi$ such that the following diagram commutes:

$$\begin{array}{ccc} \Phi & \xrightarrow{h} & \Phi^* \\ {\scriptstyle\psi}\downarrow & & \uparrow{\scriptstyle\psi^*} \\ \Psi & \xrightarrow{\rho} & \Psi^*. \end{array}$$

We fix for the rest of this section a skew-Hermitian module $\Phi = (\Phi, h)$. We also write $\mathcal{S} := \mathcal{S}(\Phi) := \mathrm{coker}\,(h)$, sitting in an exact sequence of $\mathcal{R}$-modules

$$0 \to \Phi \xrightarrow{h} \Phi^* \to \mathcal{S} \to 0,$$

which also give a free resolution of length 1 of $\mathcal{S}$.

Let $\mathcal{P} \subset \overline{R}$ and $A := \overline{R}/\mathcal{P}$ be as above, and write $\Phi_{\mathcal{P}} := \Phi \otimes_{\overline{R}} A$, $h_{\mathcal{P}} := h \otimes A : \Phi_{\mathcal{P}} \to \Phi^* \otimes_{\overline{R}} A = \Phi^*_{\mathcal{P}}$, where $\Phi^*_{\mathcal{P}}$ is an abbreviation for $(\Phi_{\mathcal{P}})^*$. We define

$$M(\mathcal{P}) := \ker\,(h_{\mathcal{P}}) \cong \mathrm{Tor}_1^{\overline{R}}\,(\mathcal{S}, A)\,; \quad \mathcal{S}(\mathcal{P}) := \mathrm{coker}\,(h_{\mathcal{P}}) \cong \mathcal{S} \otimes_{\overline{R}} A,$$

giving rise to an exact sequence of A-modules

$$0 \to M(\mathcal{P}) \to \Phi_{\mathcal{P}} \xrightarrow{h_{\mathcal{P}}} \Phi^*_{\mathcal{P}} \to \mathcal{S}(\mathcal{P}) \to 0. \tag{94}$$

By (93) and this exact sequence we obtain a commutative diagram of A-modules:

$$\begin{array}{ccccccc} M(\mathcal{P})^\iota & \hookrightarrow & \Phi^\iota_{\mathcal{P}} & \xrightarrow{-h^\iota_{\mathcal{P}}} & (\Phi^*_{\mathcal{P}})^\iota & \twoheadrightarrow & \mathcal{S}(\mathcal{P})^\iota \\ & & \downarrow\wr & & \uparrow\wr & & \\ \mathrm{Hom}_A(\mathcal{S}(\mathcal{P}), A) & \hookrightarrow & \mathrm{Hom}_A\,(\Phi^*_{\mathcal{P}}, A) & \xrightarrow{\mathrm{Hom}(h_{\mathcal{P}})} & \mathrm{Hom}_A(\Phi_{\mathcal{P}}, A) & \twoheadrightarrow & \mathrm{Ext}^1_{\overline{R}}\,(\mathcal{S}, A)\,. \end{array} \tag{95}$$

Here $\Phi^\iota_{\mathcal{P}} := (\Phi_{\mathcal{P}})^\iota = \Phi^\iota \otimes_{\overline{R}} A$, the first vertical map is the canonical isomorphism sending $x \in \Phi^\iota_{\mathcal{P}}$ in $\{\mathrm{Hom}_A(\Phi^\iota_{\mathcal{P}}, A) \ni \psi \mapsto \psi(x)\}$, while the second is given by $\mathrm{Hom}(\Phi_{\mathcal{P}}, A) \ni \psi \mapsto \iota \circ \psi \in \mathrm{Hom}(\Phi^\iota_{\mathcal{P}}, A)^\iota$. This gives in particular isomorphisms of A-modules

$$M(\mathcal{P})^\iota \xrightarrow{\sim} \mathrm{Hom}_A(\mathcal{S}(\mathcal{P}), A); \quad \mathcal{S}(\mathcal{P})^\iota \xrightarrow{\sim} \mathrm{Ext}^1_{\overline{R}}(\mathcal{S}, A). \tag{96}$$

As in [**MR05**, Sec. 4] (and with notations which will be explained below), this isomorphisms can be used to construct a skew-Hermitian 'height pairing'

$$\widetilde{h}_{\Phi_{\mathcal{P}},1,1} : M(\mathcal{P}) \otimes_A M(\mathcal{P})^\iota \longrightarrow \mathcal{P}/\mathcal{P}^2,$$

and a skew-Hermitian 'Cassels-Tate pairing':

$$\widetilde{c}_{\Phi_{\mathcal{P}},2,2} : \mathcal{S}(\mathcal{P})_{\mathrm{tors}} \otimes_A \mathcal{S}(\mathcal{P})^\iota_{\mathrm{tors}} \longrightarrow \mathscr{A}/A,$$

where $\mathscr{A} := \mathrm{Frac}(A)$ and $_{\mathrm{tors}}$ refers to the A-torsion. (Here skew-Hermitian is with respect to the involution induces by $\iota : A \to A$, meaning that an analogue of the relation (93) holds.) These pairings can also be defined directly as follows. (See Sec. 4 of *loc. cit.*, or the following Section for a more 'conceptual' definition.)

Let $x \in M(\mathcal{P}) \subset \Phi_{\mathcal{P}}$ and $y \in M(\mathcal{P})^\iota \subset \Phi^\iota_{\mathcal{P}}$, and let $\widetilde{x} \in \Phi$ and $\widetilde{y} \in \Phi^\iota$ be liftings of x and y respectively. By definition, $h_{\mathcal{P}}(x) = 0$ and $h_{\mathcal{P}}(y) = 0$, so that $(\widetilde{x}, z)_h \in \mathcal{P}$ and $(x', y)_h \in \mathcal{P}$ for every $z \in \Phi^\iota$ and $z' \in \Phi$. This implies that the projection of $(\widetilde{x}, \widetilde{y})_h$ in $\mathcal{P}/\mathcal{P}^2$ depends only on x and y. We then define

$$\widetilde{h}_{\Phi_{\mathcal{P}},1,1}(x \otimes y) := (\widetilde{x}, \widetilde{y})_h \mod \mathcal{P}^2.$$

Let now $x \in \mathcal{S}(\mathcal{P})_{\mathrm{tors}}$ and $y \in \mathcal{S}(\mathcal{P})^\iota_{\mathrm{tors}}$, with $a \cdot x = 0$ and $b \cdot y = 0$ respectively, for some $a, b \in A$. Let also $\widetilde{x}^* \in \Phi^*_{\mathcal{P}}$ and $\widetilde{y}^* \in (\Phi^*_{\mathcal{P}})^\iota$ be any lift of x and y respectively (under the projection (94)). By construction, there exist $\widetilde{x} \in \Phi_{\mathcal{P}}$ and $\widetilde{y} \in \Phi^\iota_{\mathcal{P}}$ such that $h_{\mathcal{P}}(\widetilde{x}) = a \cdot \widetilde{x}^*$ and $h^\iota_{\mathcal{P}}(\widetilde{y}) = b \cdot \widetilde{y}^*$. Writing $(-,-)_{h_{\mathcal{P}}} : \Phi_{\mathcal{P}} \times \Phi^\iota_{\mathcal{P}} \to A$ for the (skew-Hermitian) A-bilinear form corresponding to $h_{\mathcal{P}}$, it can be easily checked that the formula

$$\widetilde{c}_{\Phi_{\mathcal{P}},2,2}(x \otimes y) := (ab)^{-1} \cdot (\widetilde{x}, \widetilde{y})_{h_{\mathcal{P}}} \mod A \in \mathscr{A}/A$$

defines a (skew-Hermitian) pairing (i.e. it depends only on x and y).

11.1.1. *Associated complexes and duality.* Let $\Phi = (\Phi, h)$, $\mathcal{P} \in \mathrm{Spec}(\overline{R})$ and $A := \overline{R}/\mathcal{P}$ be as in the preceding Section. We consider the *associated complex* of A-modules

$$\mathbf{\Phi}_{\mathcal{P}} := \left(\Phi_{\mathcal{P}} \xrightarrow{h_{\mathcal{P}}} \Phi^*_{\mathcal{P}}\right),$$

with $\Phi_{\mathcal{P}}$ in degree one. (When $\mathcal{P} = 0$, we write simply $\mathbf{\Phi} := \left(\Phi \xrightarrow{h} \Phi^*\right)$ for $\mathbf{\Phi}_0$.) By construction:

$$M(\mathcal{P}) = H^1(\mathbf{\Phi}_{\mathcal{P}}); \quad \mathcal{S}(\mathcal{P}) = H^2(\mathbf{\Phi}_{\mathcal{P}}).$$

We have

$$\mathbf{\Phi}_{\mathcal{P}} \otimes_A \mathbf{\Phi}^\iota_{\mathcal{P}} = \left(\Phi_{\mathcal{P}} \otimes \Phi^\iota_{\mathcal{P}} \xrightarrow{\partial^2} (\Phi^*_{\mathcal{P}} \otimes \Phi^\iota_{\mathcal{P}}) \oplus (\Phi_{\mathcal{P}} \otimes (\Phi^*_{\mathcal{P}})^\iota) \xrightarrow{\partial^3} \Phi^*_{\mathcal{P}} \otimes (\Phi^*_{\mathcal{P}})^\iota\right),$$

concentrated in degrees $[2,4]$, where

$$\partial^2 = (h_{\mathcal{P}} \otimes \mathrm{id}, -\mathrm{id} \otimes h^\iota_{\mathcal{P}}), \quad \partial^3 = (\mathrm{id} \otimes h^\iota_{\mathcal{P}}) \oplus (h_{\mathcal{P}} \otimes \mathrm{id}).$$

It follows by the definition of skew-Hermitian module that the formula:

$$(x^* \otimes y) \oplus (x \otimes y^*) \mapsto x^*(y) - \iota(y^*(x)),$$

for every $x \in \Phi_{\mathcal{P}}$, $y \in \Phi^\iota_{\mathcal{P}}$, $x^* \in \Phi^*_{\mathcal{P}}$ and $y^* \in (\Phi^*_{\mathcal{P}})^\iota$ defines a morphism of complexes of A-modules

$$\cup_{\Phi_{\mathcal{P}}} : \mathbf{\Phi}_{\mathcal{P}} \otimes_A \mathbf{\Phi}^\iota_{\mathcal{P}} \longrightarrow A[-3].$$

LEMMA 11.1. *The cup-product $\cup_{\Phi_\mathcal{P}}$ induces by adjunction an isomorphism*

$$\alpha_{\mathbf{\Phi}_\mathcal{P}} := \mathrm{adj}\left(\cup_{\Phi_\mathcal{P}}\right) : \mathbf{\Phi}_\mathcal{P} \cong \mathrm{Hom}_A\left(\mathbf{\Phi}_\mathcal{P}^\iota, A\right)[-3] = \mathrm{Hom}_A\left(\mathbf{\Phi}_\mathcal{P}, A\right)^\iota[-3].$$

Moreover $\cup_{\Phi_\mathcal{P}}$ is skew-Hermitian, i.e. the following diagram of complexes of A-modules:

$$\begin{array}{ccc} \mathbf{\Phi}_\mathcal{P} \otimes_A \mathbf{\Phi}_\mathcal{P}^\iota & \xrightarrow{\ \cup_{\Phi_\mathcal{P}}\ } & A[-3] \\ {\scriptstyle s_{12}}\downarrow & & \uparrow{\scriptstyle \iota} \\ \mathbf{\Phi}_\mathcal{P}^\iota \otimes_A \mathbf{\Phi}_\mathcal{P} & \xrightarrow{\ -\left(\cup_{\Phi_\mathcal{P}}\right)^\iota\ } & A^\iota[-3] \end{array}$$

commutes.

PROOF. This follows by an easy computations (cfr. the commutative diagram (95)). □

11.1.2. *'Derived pairings'.* The notations are as in the preceding Sections. We recall that by definition $\mathbf{\Phi}$ is a complex of free $\overline{R}$-modules (of finite type). In particular the functor $\mathbf{\Phi} \otimes_{\overline{R}} -$ (on the homotopy category of complexes of $\overline{R}$-modules) maps quasi-isomorphisms to quasi-isomorphisms, so that it can be derived trivially to a functor defined on $\mathcal{D}^b(\overline{R})$. Applying it to the exact triangle in $\mathcal{D}(\overline{R})$:

$$\mathcal{P}/\mathcal{P}^2 \to \overline{R}/\mathcal{P}^2 \to A \to \mathcal{P}/\mathcal{P}^2[1],$$

we obtain (cfr. Sec. 10.5) a 'Bockstein map' in $\mathcal{D}(\overline{R})$:

$$\beta_\mathcal{P} : \mathbf{\Phi}_\mathcal{P} = \mathbf{\Phi} \otimes_{\overline{R}} A \longrightarrow \mathbf{\Phi} \otimes_{\overline{R}} \mathcal{P}/\mathcal{P}^2[1] \cong \mathbf{\Phi}[1] \otimes_{\overline{R}} \mathcal{P}/\mathcal{P}^2 \cong \mathbf{\Phi}_\mathcal{P}[1] \otimes_A \mathcal{P}/\mathcal{P}^2.$$

We define the 'derived height pairing' as the morphism in $\mathcal{D}(A)$:

$$\widetilde{h}_{\Phi_\mathcal{P}} : \mathbf{\Phi}_\mathcal{P} \otimes_A \mathbf{\Phi}_\mathcal{P}^\iota \stackrel{\beta_\mathcal{P}\otimes\mathrm{id}}{\longrightarrow} \mathbf{\Phi}_\mathcal{P}[1] \otimes_A \mathbf{\Phi}_\mathcal{P}^\iota \otimes_A \mathcal{P}/\mathcal{P}^2 \stackrel{\cup_{\Phi_\mathcal{P}}[1]\otimes\mathrm{id}}{\longrightarrow} \mathcal{P}/\mathcal{P}^2[-2].$$

LEMMA 11.2. *The pairing $H^{1,1}(\widetilde{h}_{\Phi_\mathcal{P}}) : M(\mathcal{P}) \otimes_A M(\mathcal{P})^\iota \to \mathcal{P}/\mathcal{P}^2$ induced by $\widetilde{h}_{\Phi_\mathcal{P}}$ is equal to $\widetilde{h}_{\Phi_\mathcal{P},1,1}$.*

PROOF. By construction, the morphism $\beta_\mathcal{P}$ is represented by the diagram:

$$\mathbf{\Phi}_\mathcal{P} = \mathbf{\Phi} \otimes_{\overline{R}} A \stackrel{\mathrm{id}\otimes\mathrm{pr}}{\longleftarrow} \mathbf{\Phi} \otimes_{\overline{R}} \mathrm{Cone}\left(\mathcal{P}/\mathcal{P}^2 \stackrel{i}{\to} \overline{R}/\mathcal{P}^2\right) \stackrel{\mathrm{id}\otimes -p}{\longrightarrow} \mathbf{\Phi} \otimes_{\overline{R}} \left(\mathcal{P}/\mathcal{P}^2[1]\right) \stackrel{\sim}{\to} \mathbf{\Phi}_\mathcal{P}[1] \otimes_A \mathcal{P}/\mathcal{P}^2,$$

where: p is the natural projection, pr is induces by the projection $\overline{R}/\mathcal{P}^2 \twoheadrightarrow A = \overline{R}/\mathcal{P}$ and the last isomorphism is defined by $x \otimes y \mapsto (-1)^j (x \otimes 1) \otimes y$ for $x \in \mathbf{\Phi}^j$ (and identifying again $\mathbf{\Phi}_\mathcal{P} = \mathbf{\Phi} \otimes_{\overline{R}} A$).

Let us take $\overline{x} = x \otimes 1 \in M(\mathcal{P}) = H^1(\mathbf{\Phi} \otimes A)$, so that $h(x) \otimes 1 = 0$, i.e. $h(x) = \sum_j \alpha_j \cdot x_j^*$ for elements $\alpha_j \in \mathcal{P}$ and $x_j^* \in \Phi^*$. Then $-\sum_j x_j^* \otimes [\alpha_j] + x \otimes 1 \in \Phi^* \otimes \mathcal{P}/\mathcal{P}^2 \oplus \Phi \otimes \overline{R}/\mathcal{P}^2$ is a 1-cocycle lifting $x \otimes 1$ under $\mathrm{id} \otimes \mathrm{pr}$. It follows that

$$H^1\left(\beta_\mathcal{P}\right)(\overline{x}) \text{ is represented by } \sum_j (x_j^* \otimes 1) \otimes [\alpha_j] \in \Phi_\mathcal{P}^* \otimes_A \mathcal{P}/\mathcal{P}^2.$$

By the definitions of $\cup_{\Phi_\mathcal{P}}$ and $\widetilde{h}_{\Phi_\mathcal{P},1,1}$ we thus obtain, for every $\overline{y} = y \otimes 1 \in M(\mathcal{P})^\iota \subset \Phi^\iota \otimes A$:

$$H^{1,1}\left(\widetilde{h}_{\Phi_\mathcal{P}}\right)(\overline{x} \otimes \overline{y}) = \sum_j (x_j^*(y) \otimes 1) \otimes [\alpha_j]^\iota = {}'\left(x, y\right)_h \bmod \mathcal{P}^2 = \widetilde{h}_{\Phi_\mathcal{P},1,1}(\overline{x} \otimes \overline{y}) \in \mathcal{P}/\mathcal{P}^2,$$

concluding the proof. □

11.1.3. *'Cassels-Tate pairings'.* With the notations of the preceding Sections, let $\mathscr{P} := \left(A \stackrel{-i}{\to} \mathscr{A}\right)$ and $\mathbf{R}\Gamma_!(-) := - \otimes_A \mathscr{P}$. The constructions of Section 0.10 gives us a morphism of complexes of A-modules:

$$\widetilde{c}_{\Phi_\mathcal{P}} : \mathbf{R}\Gamma_!\left(\mathbf{\Phi}_\mathcal{P}\right) \otimes_A \mathbf{R}\Gamma_!\left(\mathbf{\Phi}_\mathcal{P}^\iota\right) \to \mathbf{R}\Gamma_!\left(\mathbf{\Phi}_\mathcal{P} \otimes_A \mathbf{\Phi}_\mathcal{P}^\iota\right) \stackrel{\cup_{\Phi_\mathcal{P}}\otimes\mathrm{id}}{\longrightarrow} \mathbf{R}\Gamma_!\left(A\right)[-3],$$

inducing an A-bilinear pairing:

$$H^{2,2}\left(\widetilde{c}_{\Phi_\mathcal{P}}\right) : \mathcal{S}(\mathcal{P})_{\mathrm{tors}} \otimes_A \mathcal{S}(\mathcal{P})_{\mathrm{tors}}^\iota \longrightarrow \mathscr{A}/A.$$

LEMMA 11.3. $H^{2,2}\left(\widetilde{c}_{\Phi_\mathcal{P}}\right) = \widetilde{c}_{\Phi_\mathcal{P},2,2}$.

Proof. We use the notations of Section 0.10. Let $x \in \mathcal{S}(\mathcal{P})_{\mathrm{tors}}$ and $y \in \mathcal{S}(\mathcal{P})^{\iota}_{\mathrm{tors}}$. As in the definition of $\widetilde{c}_{\Phi_{\mathcal{P}},2,2}$ we choose $a \in A$, $\widetilde{x}^* \in \Phi^*_{\mathcal{P}}$ and $\widetilde{x} \in \Phi_{\mathcal{P}}$ such that $x = [\widetilde{x}^*]$ and $h_{\mathcal{P}}(\widetilde{x}) = a\widetilde{x}^*$, and similarly $b \in A$, $\widetilde{y}^* \in (\Phi^*_{\mathcal{P}})^{\iota}$ and $\widetilde{y} \in \Phi^{\iota}_{\mathcal{P}}$ such that $h^{\iota}_{\mathcal{P}}(\widetilde{y}) = b\widetilde{y}^*$. Then

$$X := \widetilde{x}^* + \widetilde{x} \otimes a^{-1} \in \left(\mathbf{R}\Gamma_!(\mathbf{\Phi}_{\mathcal{P}})\right)^2; \quad Y := \widetilde{y}^* + \widetilde{y} \otimes b^{-1} \in \left(\mathbf{R}\Gamma_!(\mathbf{\Phi}^{\iota}_{\mathcal{P}})\right)^2$$

are 2-cocycles, whose cohomology classes lift x and y respectively under the projection in (164). We then compute the composition (165) on $X \otimes Y$, obtaining the 4-cocycle

$$\widetilde{x}^* \otimes \widetilde{y}^* + \widetilde{x}^* \otimes \widetilde{y} \otimes b^{-1} \in \left(\mathbf{R}\Gamma_!\left(\mathbf{\Phi}_{\mathcal{P}} \otimes_A \mathbf{\Phi}^{\iota}_{\mathcal{P}}\right)\right)^4.$$

Applying $\cup_{\Phi_{\mathcal{P}}} \otimes \mathrm{id}$ to this element, and identifying $H^4\left(\mathbf{R}\Gamma_!(A)[-3]\right) \overset{\sim}{\leftarrow} \mathscr{A}/A$ (using again (164)) we obtain:

$$H^{2,2}\left(\widetilde{c}_{\Phi_{\mathcal{P}}}\right)(x \otimes y) = b^{-1} \cdot \widetilde{x}^*(y) \bmod A = (ab)^{-1} \cdot (\widetilde{x}, \widetilde{y})_{h_{\mathcal{P}}} \bmod A = \widetilde{c}_{\Phi_{\mathcal{P}},2,2}(x \otimes y).$$

□

11.2. Organizing modules. Following [**MR05**], we say that a skew-Hermitian $\overline{R}$-module $\Phi = (\Phi, h)$ *organizes the arithmetic* of $\mathbf{g}$ over $\mathcal{K}$ if there exists an isomorphism $\psi : \mathbf{\Phi} \overset{\sim}{\to} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$ in $\mathcal{D}(\overline{R})$, such that the following diagram commutes in $\mathcal{D}(\overline{R})$:

$$\begin{array}{ccc} \mathbf{\Phi} \otimes_{\overline{R}} \mathbf{\Phi}^{\iota} & \xrightarrow{\quad \cup_{\Phi} \quad} & \overline{R}[-3] \\ \downarrow{\scriptstyle \psi\otimes\psi^{\iota}} & & \| \\ \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) \otimes^{\mathbf{L}}_{\overline{R}} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})^{\iota} & \xrightarrow{\quad \cup_{\overline{\pi}_R} \quad} & \overline{R}[-3] \end{array}$$

where $\cup_{\Phi} := \cup_{\Phi_0}$ and $\overline{\pi}_R := \pi_R(\mathcal{K})$ is the perfect duality over $R(\mathcal{K})$ attached to π_R (cft. Section 10.4). Combining the results of [**MR05**, Sections 5-6] with the results recalled in Section 10.4 we obtain the following

Theorem 11.4. *Assume that $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$ is a torsion $\overline{R}$-module. Then there exists a basic skew-Hermitian module (Φ, h) which organizes the arithmetic of $\mathbf{g}$ over $\mathcal{K}$.*

Moreover, if (Ψ, ρ) is another basic skew-Hermitian module organizing the arithmetic of $\mathbf{g}$ over $\mathcal{K}$, there exists a (noncanonical) isomorphism of skew-Hermitian modules $(\Phi, h) \to (\Psi, \rho)$.

Proof. By Theorem 10.5, $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$ can be represented in $\mathcal{D}(\overline{R})$ by a complex $\boldsymbol{P} := \left(P_1 \overset{\partial}{\to} P_2\right)$ concentrated in degrees 1 and 2, where P_j is a projective, hence free $\overline{R}$-module of finite type. Moreover, the isomorphism $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) \overset{\sim}{\to} \mathbf{R}\mathrm{Hom}_{\overline{R}}\left(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})^{\iota}, \overline{R}\right)[-3]$ (from Theorem 10.5) induces an isomorphism in $\mathcal{D}(\overline{R})$ between $\boldsymbol{P}$ and $\mathrm{Hom}_{\overline{R}}(\boldsymbol{P}^{\iota}, \overline{R})[-3] = \left(P^*_2 \overset{\partial^*}{\to} P^*_1\right)$ (where we recall $M^* := \mathrm{Hom}_{\overline{R}}\left(M^{\iota}, \overline{R}\right)$ for an $\overline{R}$-module M, and $\partial^* := \mathrm{Hom}(\partial^{\iota})$). From this we obtain isomorphisms

$$\widetilde{H}^1_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) \overset{\sim}{\to} \ker(\partial) \overset{\sim}{\to} \ker(\partial^*) \overset{\sim}{\to} (\mathrm{coker}(\partial))^* \overset{\sim}{\to} \left(\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})\right)^*.$$

Since $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$ is assumed to be a torsion $\overline{R}$-module, it follows that $\widetilde{H}^1_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) = 0$, i.e. ∂ is injective.

Let us fix an isomorphism $\vartheta : \boldsymbol{P} \overset{\sim}{\to} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$ in $\mathcal{D}(\overline{R})$, and define the morphism in $\mathcal{D}(\overline{R})$:

$$\cup_{\boldsymbol{P}} : \boldsymbol{P} \otimes_{\overline{R}} \boldsymbol{P}^{\iota} \overset{\vartheta\otimes\vartheta^{\iota}}{\longrightarrow} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) \otimes^{\mathbf{L}}_{\overline{R}} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})^{\iota} \overset{\cup_{\overline{\pi}_R}}{\longrightarrow} \overline{R}[-3].$$

(We remark that, since $\boldsymbol{P}$ is a complex of free $\overline{R}$-modules, this pairing is actually a morphism in the homotopy category, i.e. comes from a morphism of complexes of $\overline{R}$-modules.) By *b)* of Theorem 10.5 it is skew-Hermitian, i.e. the following diagram commutes in $\mathcal{D}(\overline{R})$:

(97)
$$\begin{array}{ccc} \boldsymbol{P} \otimes_{\overline{R}} \boldsymbol{P}^{\iota} & \xrightarrow{\cup_{\boldsymbol{P}}} & \overline{R}[-3] \\ \downarrow{\scriptstyle s_{12}} & & \uparrow{\scriptstyle \iota} \\ \boldsymbol{P}^{\iota} \otimes_{\overline{R}} \boldsymbol{P} & \xrightarrow{-(\cup_{\boldsymbol{P}})^{\iota}} & \overline{R}^{\iota}[-3]. \end{array}$$

The commutativity of this diagram can be reformulated as follows: let

$$\alpha_{\boldsymbol{P}} := \mathrm{adj}\,(\cup_{\boldsymbol{P}}) : \boldsymbol{P} \xrightarrow{\sim} \mathrm{Hom}_{\overline{R}}\left(\boldsymbol{P}^{\iota}, \overline{R}\right)[-3] \xrightarrow{\sim} \mathrm{Hom}_{\overline{R}}\left(\boldsymbol{P}, \overline{R}\right)^{\iota}[-3]$$

be the isomorphism in $\mathcal{D}(\overline{R})$ induced by $\cup_{\boldsymbol{P}}$ (the last map induced by $\phi \mapsto \iota \circ \phi$). Applying $\mathrm{Hom}(-, \overline{R})$ to the map $\alpha^{\iota}_{\boldsymbol{P}}[3] : \boldsymbol{P}^{\iota}[3] \to \mathrm{Hom}_{\overline{R}}(\boldsymbol{P}, \overline{R})$ we obtain a morphism

$$\gamma : \mathrm{Hom}_{\overline{R}}\left(\mathrm{Hom}_{\overline{R}}\left(\boldsymbol{P}, \overline{R}\right), \overline{R}\right) \to \mathrm{Hom}_{\overline{R}}\left(\boldsymbol{P}^{\iota}[3], \overline{R}\right) \xrightarrow{\sim} \mathrm{Hom}_{\overline{R}}\left(\boldsymbol{P}^{\iota}, \overline{R}\right)[-3] \xrightarrow{\sim} \mathrm{Hom}_{\overline{R}}\left(\boldsymbol{P}, \overline{R}\right)^{\iota}[-3].$$

(We remark that the first isomorphism 'involves signs', i.e. is not given by the identity.) Moreover, since $\boldsymbol{P}$ is a complex of free $\overline{R}$-modules, the canonical map $\varepsilon : \boldsymbol{P} \xrightarrow{\sim} \mathrm{Hom}_{\overline{R}}\left(\mathrm{Hom}_{\overline{R}}\left(\boldsymbol{P}, \overline{R}\right), \overline{R}\right)$ is an isomorphism (see [**Nek06**, Sec. 1.2.8] for the precise definition). We then obtain a morphism

$$\widetilde{\alpha_{\boldsymbol{P}}} := \gamma \circ \varepsilon : \boldsymbol{P} \longrightarrow \mathrm{Hom}_{\overline{R}}(\boldsymbol{P}, \overline{R})^{\iota}[-3].$$

Then the commutativity of (97) is equivalent to the following identity in $\mathcal{D}(\overline{R})$:

(98)
$$\widetilde{\alpha_{\boldsymbol{P}}} = -\alpha_{\boldsymbol{P}}.$$

In the terminology of [**MR05**, Sec. 6], we can summarize the discussion above as follows. $\boldsymbol{P}$ is a complex of free $\overline{R}$-modules concentrated in degrees 1 and 2, with injective coboundary map $\partial : P_1 \to P_2$, and $\alpha_{\boldsymbol{P}}$ is a 'skew-Hermitian, degree 3, perfect pairing on $\boldsymbol{P}$' in the derived category $\mathcal{D}(\overline{R})$. (This last statement meaning precisely that (98) holds (cfr. Def. 6.1 in *loc. cit*).) Applying Prop. 6.5 of *loc. cit.*, we obtain the following statement: there exists a basic skew-Hermitian module $\Phi = (\Phi, h)$, together with an isomorphism $\varphi : \boldsymbol{\Phi} \xrightarrow{\sim} \boldsymbol{P}$ in $\mathcal{D}(\overline{R})$, such that the following diagram commutes in $\mathcal{D}(\overline{R})$:

$$\begin{array}{ccc} \boldsymbol{\Phi} & \xrightarrow{\alpha_{\boldsymbol{\Phi}}} & \mathrm{Hom}_{\overline{R}}\left(\boldsymbol{\Phi}, \overline{R}\right)^{\iota}[-3] \\ \downarrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi^*} \\ \boldsymbol{P} & \xrightarrow{\alpha_{\boldsymbol{P}}} & \mathrm{Hom}_{\overline{R}}\left(\boldsymbol{P}, \overline{R}\right)^{\iota}[-3], \end{array}$$

where $\varphi^* := \mathrm{Hom}\left(\varphi, \overline{R}\right)^{\iota}[-3]$ and $\alpha_{\boldsymbol{\Phi}}$ is the isomorphism defined in Lemma 11.1. By construction, this is equivalent to the commutativity in $\mathcal{D}(\overline{R})$ of the diagram:

$$\begin{array}{ccc} \boldsymbol{\Phi} \otimes_{\overline{R}} \boldsymbol{\Phi}^{\iota} & \xrightarrow{\cup_{\Phi}} & \overline{R}[-3] \\ \downarrow{\scriptstyle \varphi \otimes \varphi^{\iota}} & & \| \\ \boldsymbol{P} \otimes_{\overline{R}} \boldsymbol{P}^{\iota} & \xrightarrow{\cup_{\boldsymbol{P}}} & \overline{R}[-3]. \end{array}$$

Taking $\psi := \vartheta \circ \varphi : \boldsymbol{\Phi} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$ we see that Φ organizes the arithmetic of $\mathbf{g}$ over $\mathcal{K}$.

The last statement is [**MR05**, Prop. 6.6], concluding the proof. ☐

11.2.1. *Specializations and comparison of pairings.* We assume in this Section that the ideal $\mathcal{P} \subset \overline{R}$ is (invariant under the action of ι and) generated by an $\overline{R}$-regular sequence. As usual $A := \overline{R}/\mathcal{P}$ and we write $T_{\mathcal{P}} := \mathbf{T}(\mathcal{K})/\mathcal{P}$. We write

$$\pi_{\mathcal{P}} : T_{\mathcal{P}} \otimes_A T^{\iota}_{\mathcal{P}} \to A(1)$$

for the perfect duality induced by $\overline{\pi}_R$. We write $\widetilde{\mathbf{R}\Gamma}_f(K, T_{\mathcal{P}})$ for the Selmer Complexes attached to the Greenberg local conditions: $F^+_v(T_{\mathcal{P}}) := \mathbf{T}(\mathcal{K})^+_v \otimes_{\overline{R}} A$ (resp., $F^+_v(T_{\mathcal{P}}) := 0$) for every prime $v \in S_f$ dividing (resp., not dividing) p (cfr. Section 10). We recall that under these assumptions on $\mathcal{P}$, the constructions in Appendix C attaches to the '$\mathcal{P}$-deformation' $\mathbf{T}(\mathcal{K})$ of $T_{\mathcal{P}}$ a Bockstein map:

$$\beta_{\mathcal{P}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) \cong \widetilde{\mathbf{R}\Gamma}_{f}(K, \mathbf{T}(\mathcal{K})) \longrightarrow \widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})[1] \otimes_A \mathcal{P}/\mathcal{P}^2,$$

and the corresponding abstract '$\mathcal{P}$-height-pairing'

$$\widetilde{h}_{\mathcal{P}} := (\cup_{\boldsymbol{\pi}_{\mathcal{P}}}[1] \otimes \mathrm{id}) \circ (\beta_{\mathcal{P}} \otimes \mathrm{id}) : \widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}}) \otimes^{\mathbf{L}}_A \widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})^{\iota} \longrightarrow \mathcal{P}/\mathcal{P}[-2].$$

As usual $\widetilde{h}_{\mathcal{P},1,1} := H^{1,1}(\widetilde{h}_{\mathcal{P}})$ is the map induced on $(1,1)$-cohomology. Moreover, the construction of Section 0.10 attaches to $\mathcal{P}$, or better to $\pi_{\mathcal{P}}$ an abstract derived Cassels-Tate pairing:

$$\widetilde{c}_{\pi_{\mathcal{P}}} : \mathbf{R}\Gamma_! \left(\widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})\right) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma_! \left(\widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})^{\iota}\right) \longrightarrow \mathbf{R}\Gamma_!(A[-3]),$$

and the corresponding Cassels-Tate pairing:

$$\widetilde{c}_{\pi_{\mathcal{P}},2,2} = H^{2,2}(\widetilde{c}_{\pi_{\mathcal{P}}}) : \widetilde{H}^2_f(K, T_{\mathcal{P}})_{A\text{-tors}} \otimes_A \widetilde{H}^2_f(K, T_{\mathcal{P}})_{A\text{-tors}} \longrightarrow \mathscr{A}/A.$$

Proposition 11.5. *Assume that there exists a skew-Hermitian module $\Phi = (\Phi, h)$ which organizes the arithmetic of $\mathbf{g}$ over $\mathcal{K}$, with an isomorphism $\psi : \mathbf{\Phi} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$. Then ψ induces an isomorphism $\psi_{\mathcal{P}} : \mathbf{\Phi}_{\mathcal{P}} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})$ in $\mathcal{D}(A)$ such that the following diagrams:*

$$\begin{array}{ccc}
\mathbf{\Phi}_{\mathcal{P}} \otimes_A \mathbf{\Phi}^{\iota}_{\mathcal{P}} & \xrightarrow{\ \cup_{\Phi_{\mathcal{P}}}\ } & A[-3] \\
\downarrow{\scriptstyle \psi_{\mathcal{P}} \otimes \psi^{\iota}_{\mathcal{P}}} & & \Vert \\
\widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}}) \otimes^{\mathbf{L}}_A \widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})^{\iota} & \xrightarrow{\ \cup_{\boldsymbol{\pi}_{\mathcal{P}}}\ } & A[-3];
\end{array}$$

$$\begin{array}{ccc}
\mathbf{\Phi}_{\mathcal{P}} \otimes_A \mathbf{\Phi}^{\iota}_{\mathcal{P}} & \xrightarrow{\ \widetilde{h}_{\mathbf{\Phi}_{\mathcal{P}}}\ } & \mathcal{P}/\mathcal{P}^2[-2] \\
\downarrow{\scriptstyle \psi_{\mathcal{P}} \otimes \psi^{\iota}_{\mathcal{P}}} & & \Vert \\
\widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}}) \otimes^{\mathbf{L}}_A \widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})^{\iota} & \xrightarrow{\ \widetilde{h}_{\mathcal{P}}\ } & \mathcal{P}/\mathcal{P}^2[-2];
\end{array}$$

$$\begin{array}{ccc}
\mathbf{R}\Gamma_!(\mathbf{\Phi}_{\mathcal{P}}) \otimes_A \mathbf{R}\Gamma_!(\mathbf{\Phi}^{\iota}_{\mathcal{P}}) & \xrightarrow{\ \widetilde{c}_{\Phi_{\mathcal{P}}}\ } & \mathbf{R}\Gamma_!(A)[-3] \\
\downarrow{\scriptstyle \mathbf{R}\Gamma_!(\psi_{\mathcal{P}}) \otimes \mathbf{R}\Gamma_!(\psi^{\iota}_{\mathcal{P}})} & & \Vert \\
\mathbf{R}\Gamma_!(\widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma_! \left(\widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})^{\iota}\right) & \xrightarrow{\ \widetilde{c}_{\pi_{\mathcal{P}}}\ } & \mathbf{R}\Gamma_!(A)[-3]
\end{array}$$

commute in $\mathcal{D}(A)$. Moreover identifying $M(\mathcal{P}) \xrightarrow{\sim} \widetilde{H}^1_f(K, T_{\mathcal{P}})$ (resp., $\mathcal{S}(\mathcal{P}) \xrightarrow{\sim} \widetilde{H}^2_f(K, T_{\mathcal{P}})$) under the isomorphism induced in cohomology by $\psi_{\mathcal{P}}$, we have $\widetilde{h}_{\mathcal{P},1,1} = \widetilde{h}_{\Phi_{\mathcal{P}},1,1}$ (resp., $\widetilde{c}_{\Phi_{\mathcal{P}},2,2} = \widetilde{c}_{\pi_{\mathcal{P}},2,2}$).

Proof. Define $\psi_{\mathcal{P}}$ as the composition

$$\psi_{\mathcal{P}} : \mathbf{\Phi}_{\mathcal{P}} = \mathbf{\Phi} \otimes_{\overline{R}} A \xrightarrow{\psi \otimes \mathrm{id}} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}) \otimes^{\mathbf{L}}_{\overline{R}} A \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}}),$$

where the second isomorphism comes from the control theorems of Sec. 10.3. The commutativity of the first diagram follows by the fact that $\cup_{\Phi_{\mathcal{P}}}$ (resp., $\cup_{\boldsymbol{\pi}_{\mathcal{P}}}$) is obtained applying $- \otimes^{\mathbf{L}}_{\overline{R}} A$ to $\cup_{\Phi}$ (resp., $\cup_{\overline{\boldsymbol{\pi}}_{\boldsymbol{R}}}$). This is clear from the definitions (resp., follows by c) of Lemma 0.4).

Moreover, by construction we obtain a commutative diagram of Bockstein maps:

$$\begin{array}{ccc}
\mathbf{\Phi}_{\mathcal{P}} & \xrightarrow{\ \beta_{\mathcal{P}}\ } & \mathbf{\Phi}_{\mathcal{P}}[1] \otimes_A \mathcal{P}/\mathcal{P}^2 \\
\downarrow{\scriptstyle \psi_{\mathcal{P}}} & & \downarrow{\scriptstyle \psi_{\mathcal{P}}[1] \otimes \mathrm{id}} \\
\widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}}) & \xrightarrow{\ \beta_{\mathcal{P}}\ } & \widetilde{\mathbf{R}\Gamma}_{f}(K, T_{\mathcal{P}})[1] \otimes_A \mathcal{P}/\mathcal{P}^2.
\end{array}$$

Then retracing the definitions of the 'height' and 'Cassels-Tate' pairings, the commutativity of the second and third diagram respectively follows formally by the commutativity of the first. The last assertion follows by Lemmas 11.2 and 11.3. □

11.3. Determinants and p-adic L-functions. Following [**MR04**],[**MR05**] in this Section we show how the existence of organizing modules applies to the study of algebraic p-adic functions. In particular, we can easily deduce 'functional equations' and p-adic Birch and Swinnerton-Dyer formulas (cfr. [**Sch83**],[**PR87**],[**PR92**],[**BD95**]) relating the Taylor expansion of p-adic L-functions to the determinant of the p-adic pairings from Section C.

Let $\mathcal{R}$ be a regular local ring. For every finite, torsion $\mathcal{R}$-module M we write

$$\mathrm{char}_{\mathcal{R}}(M) := \prod_{\mathrm{ht}(\mathcal{P})=1} \mathcal{P}^{\mathrm{length}_{R_{\mathcal{P}}}(M_{\mathcal{P}})},$$

where $(-)_{\mathcal{P}}$ is the localization of $(-)$ at $\mathcal{P}$, and the product is taken over all height-one primes in $\mathrm{Spec}(\mathcal{R})$ [**Bou89**, Ch. 7]. Since every height-one prime of $\mathcal{R}$ is principal, this is a principal ideal. If M is a finite $\mathcal{R}$-module of positive rank, we put $\mathrm{char}_{\mathcal{R}}(M) := 0$.

We fix in this Section an arithmetic prime $\psi \in \mathcal{X}^{\mathrm{arith}}(R;\mathbb{Z}_p)$ with value in $\mathbb{Z}_p$ and with associated p-stabilized modular form g_ψ.

11.3.1. *p-adic L-functions.* Let us define an *algebraic p-adic L-function* $\mathbf{L}_p(\mathcal{K}, g_\psi)$ of $g_\psi/\mathcal{K}$ to be any generator of the characteristic ideal

$$\mathrm{char}_{\mathbb{Z}_p(\mathcal{K})}\left(\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}_\psi)\right).$$

Let $I := \ker\left(\varepsilon_{\mathbb{Z}_p} : \mathbb{Z}_p(\mathcal{K}) \twoheadrightarrow \mathbb{Z}_p\right)$ be the augmentation ideal in $\mathbb{Z}_p(\mathcal{K})$. We say that $\mathbf{L}_p(\mathcal{K}, g_\psi)$ *vanishes to order* $r \in \mathbb{N}$ if $\mathbf{L}_p(\mathcal{K}, g_\psi) \in I^r$. We then define its *$r$-th derivative*

$$\mathbf{L}_p(\mathcal{K}, g_\psi)^{(r)} \in I^r/I^{r+1}$$

to be the projection of $\mathbf{L}_p(\mathcal{K}, g_\psi)$ modulo I^{r+1}.

In the same way, define an *algebraic p-adic L-function* $\mathbf{L}_p(\mathcal{K}, \mathbf{g})$ of $\mathbf{g}/\mathcal{K}$ to be any generator of

$$\mathrm{char}_{R(\mathcal{K})}\left(\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})\right).$$

Let $J = J_\psi := \ker\left(R(\mathcal{K}) \twoheadrightarrow R \overset{\psi}{\twoheadrightarrow} \mathbb{Z}_p\right)$, where the first map is the augmentation map. As above: $\mathbf{L}_p(\mathcal{K}, \mathbf{g})$ *vanishes to order $r \in \mathbb{N}$ at the arithmetic prime ψ* if $\mathbf{L}_p(\mathcal{K}, \mathbf{g}) \in J^r$ and we define its *r-th derivative at ψ*

$$\mathbf{L}_p(\mathcal{K}, \mathbf{g})^{(r)} = \mathbf{L}_p(\mathcal{K}, \mathbf{g})^{(r,\psi)} \in J^r/J^{r+1}$$

as the projection of $\mathbf{L}_p(\mathcal{K}, \mathbf{g})$ modulo J^{r+1}.

Remark 11.6. Let us write $\mathbb{A} := \mathrm{Hom}_{\mathrm{cts}}\left(\mathbf{T}(\mathcal{K}), \mu_{p^\infty}\right)$ for the Kummer dual of $\mathbf{T}(\mathcal{K})$. By the results recalled in Section 0.11, $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$ is the Pontrjagin dual of $\widetilde{H}^1_f(K, \mathbb{A}) := \widetilde{H}^2_f(G_{K,S}, \mathbb{A}; \Delta(\mathbb{A}))$, where $\Delta(\mathbb{A})$ are the 'dual local conditions' to that defining $\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T})$. Then, as costumary in Iwasawa theory, we can define $\mathbf{L}_p(\mathcal{K}, \mathbf{g})$ as the characteristic ideal of the Pontrjagin dual of a suitable discrete big (extended) Selmer group. A similar remark also applies to $\mathbf{L}_p(\mathcal{K}, g_\psi)$.

11.3.2. *Functional equations.*

Proposition 11.7. *There exists p-adic L-functions* $\mathbf{L}_p(\mathcal{K}, \mathbf{g})$ *and* $\mathbf{L}_p(\mathcal{K}, g_\psi)$ *such that:*

$$\mathbf{L}_p(\mathcal{K}, \mathbf{g})^\iota = w(E/K) \cdot \mathbf{L}_p(\mathcal{K}, \mathbf{g}); \quad \mathbf{L}_p(\mathcal{K}, g_\psi)^\iota = w(E/K) \cdot \mathbf{L}_p(\mathcal{K}, g_\psi),$$

where

$$w(E/K) := (-1)^{\mathrm{rank}_{\mathbb{Z}_p} \widetilde{H}^1_f(K, \mathbf{T}_\psi)} = (-1)^{\mathrm{rank}_R \widetilde{H}^1_f(K, \mathbf{T})}.$$

PROOF. Let us begin by proving the first ‘functional equation’.

The statement is trivial if $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})$ has positive rank over $\overline{R}$, so we can assume that $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})$ is a torsion $\overline{R}$-module. Let us fix a skew-Hermitian $\overline{R}$-module $\Phi=(\Phi,h)$ which organizes the arithmetic of $\mathbf{g}/\mathcal{K}$, whose existence in guaranteed by Th. 11.4. By construction we have an isomorphism of $\overline{R}$-modules $\mathcal{S}\xrightarrow{\sim}\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})$, where we recall that $\mathcal{S}:=\mathcal{S}(\Phi)$ has by definition a free $\overline{R}$-resolution

$$0\to\Phi\xrightarrow{h}\Phi^*\to\mathcal{S}\to 0.$$

Choosing an $\overline{R}$-basis $\{u_1,\dots,u_{r_\Phi}\}$ of Φ, this in turn gives us a free $\overline{R}$-presentation:

$$0\to\overline{R}^{r_\Phi}\xrightarrow{{}^{\mathbf{t}}H_\Phi}\overline{R}^{r_\Phi}\to\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})\to 0,$$

where

$$H_\Phi:=\big((u_i,u_j)_h\big)_{1\le i,j\le r_\Phi}\in\mathrm{GL}_{r_\Phi}(\overline{R})$$

is the matrix of the skew-Hermitian form $(-,-)_h$ with respect to the fixed basis. Localizing at $\mathcal{P}$ for every height-one prime $\mathcal{P}\in\mathrm{Spec}(\overline{R})$, it follows by the structure theorem for finite modules over principal ideal domains that $\mathrm{ord}_\mathcal{P}(\det H_\Phi)=\mathrm{ord}_\mathcal{P}\left(\mathrm{char}_{\overline{R}}\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})\right)$ for every height one prime $\mathcal{P}$ of $\overline{R}$. Since R (and hence $\overline{R}$) is regular we have $\overline{R}=\bigcap_{\mathrm{ht}(\mathcal{P})=1}\overline{R}_\mathcal{P}$, so that

$$\mathbf{L}_p(\mathcal{K},\mathbf{g}):=\det(H_\Phi) \tag{99}$$

is a p-adic L-function for $\mathbf{g}/\mathcal{K}$. Since $(-,-)_h$ is a skew-Hermitian pairing, H_Φ is a skew-Hermitian matrix, i.e. ${}^{\mathbf{t}}H_\Phi=-H_\Phi^\iota$ (where $(m_{ij})^\iota=(\iota(m_{ij}))$). It follows that

$$\iota(\det(H_\Phi))=(-1)^{r_\Phi}\cdot\det(H_\Phi). \tag{100}$$

As $T_J:=\mathbf{T}(\mathcal{K})/J\xrightarrow{\sim}\mathbf{T}_\psi$ as Galois-modules, with the notations of Sec. 11.2.1 we have $\widetilde{\mathbf{R}\Gamma}_f(K,\mathbf{T}_\psi)\xrightarrow{\sim}\widetilde{\mathbf{R}\Gamma}_f(K,T_J)$. By Prop. 11.5 we can then identify $\widetilde{H}^1_f(K,\mathbf{T}_\psi)\xrightarrow{\sim}M(J)$ and $\widetilde{H}^2_f(K,\mathbf{T}_\psi)\xrightarrow{\sim}\mathcal{S}(J)$ (under the isomorphism $\mathbf{\Phi}_J\xrightarrow{\sim}\widetilde{\mathbf{R}\Gamma}_f(K,T_J)$). Since ι acts trivially on $\mathbb{Z}_p=\overline{R}/J$, by definition we have an exact sequence

$$0\to M(J)\to\Phi_J\xrightarrow{h_J}\mathrm{Hom}_{\mathbb{Z}_p}(\Phi_J,\mathbb{Z}_p)\to\mathcal{S}(J)\to 0,$$

and the bilinear form $(-,-)_{h_J}$ induced by h_J is skew-symmetric, so that $\mathrm{rank}_{\mathbb{Z}_p}(h_J):=\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Im}(h_J))$ is even. We thus obtain:

$$\mathrm{rank}_{\mathbb{Z}_p}\widetilde{H}^1_f(K,\mathbf{T}_\psi)=\mathrm{rank}_{\mathbb{Z}_p}M(J)=\mathrm{rank}_{\mathbb{Z}_p}\mathcal{S}(J)=r_\Phi-\mathrm{rank}_{\mathbb{Z}_p}(h_J)\equiv r_\Phi \bmod 2.$$

Combining this congruence with (99) and (100) we conclude the proof of the first formula.

To prove the second ‘functional equation’, we can again assume that $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T}_\psi)$ is a torsion $\overline{\mathbb{Z}_p}=\mathbb{Z}_p(\mathcal{K})$-module. It follows by (89) and $c)$ of Theorem 10.5 that we have an isomorphism of $\overline{\mathbb{Z}_p}$-modules

$$\frac{\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})}{\overline{\mathfrak{p}_\psi}\cdot\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})}\xrightarrow{\sim}\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T}_\psi),$$

where $\overline{\mathfrak{p}_\psi}=\varpi_\psi\cdot\overline{R}$ is the kernel of the map induced by ψ on Iwasawa algebras. In particular $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})$ is a torsion $\overline{R}$-module and there exists an organizing module Φ for $\mathbf{g}/\mathcal{K}$. Using the notations above, and again Prop. 11.5 we can identify $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T}_\psi)\xrightarrow{\sim}\mathcal{S}(\overline{\mathfrak{p}_\psi})\xrightarrow{\sim}\mathcal{S}/\overline{\mathfrak{p}_\psi}$. Since $M(\overline{\mathfrak{p}_\psi})=0$ by the torsion assumption and (96) we then obtain a free $\overline{\mathbb{Z}_p}$-presentation:

$$0\to\overline{\mathbb{Z}_p}^{r_\Phi}\xrightarrow{{}^{\mathbf{t}}\widetilde{H}_\Phi}\overline{\mathbb{Z}_p}^{r_\Phi}\to\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T}_\psi)\to 0,$$

where $\widetilde{H}_\Phi\in\mathrm{GL}_{r_\Phi}(\overline{\mathbb{Z}_p})$ is obtained applying $\overline{\psi}$ to H_Φ. It follows that

$$\mathbf{L}_p(\mathcal{K},g_\psi):=\det\left(\widetilde{H}_\Phi\right)=\overline{\psi}\left(\mathbf{L}_p(\mathcal{K},\mathbf{g})\right)$$

is a p-adic L-function for $g_\psi/\mathcal{K}$. Since $\overline{\psi}$ commutes with ι, we obtain the second formula from the first.

Finally, we note that

$$\mathrm{rank}_R \widetilde{H}^1_f(K,\mathbf{T}) \equiv r_\Phi \equiv \mathrm{rank}_{\mathbb{Z}_p} \widetilde{H}^1_f(K,\mathbf{T}_\psi) \bmod 2.$$

The second equality has already been observed above. For the first, we apply exactly the same argument replacing J, $M(J)$ and $\mathcal{S}(J)$ with the augmentation ideal $\overline{I} := \ker\left(\overline{R} \twoheadrightarrow R\right)$, $M(\overline{I}) \xrightarrow{\sim} \widetilde{H}^1_f(K,\mathbf{T})$ and $\mathcal{S}(\overline{I}) \xrightarrow{\sim} \widetilde{H}^2_f(K,\mathbf{T})$ respectively. (Indeed $\mathbf{T}(\mathcal{K})/\overline{I} \xrightarrow{\sim} \mathbf{T}$ as Galois modules and ι induces the identity on $R = \overline{R}/\overline{I}$.) □

11.3.3. *A p-adic BSD formula.* Recall our p-adic pairing

$$\widetilde{h}^{\mathrm{wt}}_{\mathcal{K}/K,\psi,1,1} : \widetilde{H}^1_f(K,\mathbf{T}_\psi) \otimes_{\mathbb{Z}_p} \widetilde{H}^1_f(K,\mathbf{T}_\psi) \longrightarrow J_\psi/J_\psi^2.$$

Let us define the determinant of $\widetilde{h}^{\mathrm{wt}}_{\mathcal{K}/K,\psi,1,1}$ by the formula

$$\det\left(\widetilde{h}^{\mathrm{wt}}_{\mathcal{K}/K,\psi,1,1}\right) := \det\left(\left(\widetilde{h}^{\mathrm{wt}}_{\mathcal{K}/K,\psi,1,1}(u_i \otimes u_j)\right)_{1\le i,j\le \widetilde{r}_\psi}\right) \in \left(J_\psi^{\widetilde{r}_\psi}/J_\psi^{\widetilde{r}_\psi+1}\right)/\mathbb{Z}_p^*,$$

where $u_1,\dots,u_{\widetilde{r}_\psi}$ is any $\mathbb{Z}_p$-basis of (the free $\mathbb{Z}_p$-module) $\widetilde{H}^1_f(K,\mathbf{T}_\psi)$. (Here we use the natural 'multiplication' $\left(J/J^2\right)^l \to J^l/J^{l+1}$ to consider the determinant as an element of $J^{\widetilde{r}_\psi}/J^{\widetilde{r}_\psi+1}$, and $[\alpha]=[\beta] \in (-)/\mathbb{Z}_p^*$ if and only if $\alpha = u\cdot\beta \in (-)$ for some $u \in \mathbb{Z}_p^*$.) In a similar way we define the determinant

$$\det\left(\widetilde{h}_{\mathcal{K}/K,\psi,1,1}\right) \in \left(I^{\widetilde{r}_\psi}/I^{\widetilde{r}_\psi+1}\right)/\mathbb{Z}_p^*$$

of the bilinear form

$$\widetilde{h}_{\mathcal{K}/K,\psi,1,1} : \widetilde{H}^1_f(K,\mathbf{T}_\psi) \otimes_{\mathbb{Z}_p} \widetilde{H}^1_f(K,\mathbf{T}_\psi) \to I/I^2,$$

where I is the augmentation ideal of $\overline{\mathbb{Z}_p}$. By Lemma (10.8) we have:

$$\det\left(\widetilde{h}_{\mathcal{K},\psi,1,1}\right) = \overline{\psi}\left(\det\left(\widetilde{h}^{\mathrm{wt}}_{\mathcal{K},\psi,1,1}\right)\right). \tag{101}$$

THEOREM 11.8. *Assume that $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K,\mathbf{T})$ is a torsion $R(\mathcal{K})$-module, and let $\widetilde{r}_\psi := \mathrm{rank}_{\mathbb{Z}_p}\widetilde{H}^1_f(K,\mathbf{T}_\psi)$. Then $\mathbf{L}_p(\mathcal{K},\mathbf{g})$ vanishes to order $\widetilde{r}_\psi$ at ψ and we have*

$$\mathbf{L}_p(\mathcal{K},\mathbf{g})^{(\widetilde{r},\psi)} = \#\left(\widetilde{H}^2_f(K,\mathbf{T}_\psi)_{\mathrm{tors}}\right)\cdot\det\left(\widetilde{h}^{\mathrm{wt}}_{\mathcal{K}/K,\psi,1,1}\right) \in \left(J_\psi^{\widetilde{r}_\psi}/J^{\widetilde{r}_\psi+1}\right)/\mathbb{Z}_p^*.$$

PROOF. We write in the proof $\widetilde{r} := \widetilde{r}_\psi$ and similarly $J := J_\psi$. Let us fix an organizing module $\Phi = (\Phi, h)$ of the arithmetic of $\mathbf{g}/\mathcal{K}$. By Prop. 11.5 we have $\widetilde{H}^1_f(K,\mathbf{T}_\psi) \xrightarrow{\sim} M(J)$ and $\widetilde{H}^2_f(K,\mathbf{T}_\psi) \xrightarrow{\sim} \mathcal{S}(J)$, where by construction

$$0 \to M(J) \to \Phi_J \xrightarrow{h_J} \Phi_J^* \to \mathcal{S}(J) \to 0$$

is an exact sequence. Let us fix an $\overline{R}$-basis $u_1,\dots,u_{r_\Phi}$ of Φ such that the projection $\widetilde{u}_1,\dots,\widetilde{u}_{\widetilde{r}}$ of the first $\widetilde{r}$ elements in $\Phi_J = \Phi/J\Phi$ form a $\mathbb{Z}_p$-basis of $M(J)$; this is possible since $\mathbb{Z}_p$ is a principal ideal domain. As in the proof of Prop. 11.7, and using the same notations, we can take

$$\mathbf{L}_p(\mathcal{K}/K,\mathbf{g}) := \det\left(H_\Phi\right) = \sum_{\sigma\in S_{r_\Phi}} \epsilon(\sigma)\cdot(u_1,u_{\sigma(1)})_h\cdots(u_{r_\Phi},u_{\sigma(r_\Phi)})_h, \tag{102}$$

where S_n is the permutation group on n-elements. By definition $0 = h_J(\widetilde{u_j})(\widetilde{\star}) = (\widetilde{u}_j,\widetilde{\star})_{h_J} := (u_j,\star)_h \bmod J$ for every $\star \in \Phi^*$ and $j \le \widetilde{r}$, so that the sum in (102) belongs to $J^{\widetilde{r}}$. Moreover, assume that $\sigma \in S_n$ satisfies $\sigma(t) \le \widetilde{r}$ for some $t \ge \widetilde{r}+1$. Then $(u_t,u_{\sigma(t)})_h = -\iota\left(\left(u_{\sigma(t)},u_t\right)_h\right) \in J$, so that taking (102) modulo $J^{\widetilde{r}+1}$ we can disregard the contribution of these σ's. In other words:

$$\mathbf{L}_p(\mathcal{K},\mathbf{g}) \equiv A\cdot B \bmod J^{\widetilde{r}+1}, \tag{103}$$

where, putting $\hat{r} := r_\Phi - \widetilde{r}$ and $v_j := u_{j+\widetilde{r}}$ for $1 \le j \le \hat{r}$ we write

$$A := \sum_{\sigma\in S_{\widetilde{r}}} \epsilon(\sigma)\cdot(u_1,u_{\sigma(1)})_h\cdots(u_{\widetilde{r}},u_{\sigma(\widetilde{r})})_h; \quad B := \sum_{\sigma\in S_{\hat{r}}} \epsilon(\sigma)\cdot(v_1,v_{\sigma(1)})_h\cdots(v_{\hat{r}},v_{\sigma(\hat{r})})_h.$$

Using the exact sequence above we find (as in the proof of Prop. 11.7): there exists a p-adic unit $u \in \mathbb{Z}_p^*$ such that

$$B \equiv u \cdot \# \left(\widetilde{H}^2_f(K, \mathbf{T}_\psi) \right) \mod J. \tag{104}$$

Finally, as $(u_i, u_j)_h \mod J^2 = \widetilde{h}_{\Phi_J,1,1}(\widetilde{u}_i \otimes \widetilde{u}_j)$ by definition, we have:

$$A = \det\left((u_i, u_j)_h\right) = \det\left(\widetilde{h}_{\Phi_J,1,1}(\widetilde{u}_i \otimes \widetilde{u}_j)\right) = \det\left(\widetilde{h}^{\mathrm{wt}}_{\mathcal{K}/K,\psi,1,1}\right) \in \left(J^{\widetilde{r}}/J^{\widetilde{r}+1}\right)/\mathbb{Z}_p^*, \tag{105}$$

where the last equality follows by the last statement of Prop. 11.5. Combining (103), (104) and (105) we obtain the formula in the statement. □

Remark 11.9. In the preceding proof we used the fact that $\mathbb{Z}_p$ is a principal ideal domain. Indeed all the results above and the following Corollary are valid, *mutatis mutandis* for an arithmetic prime $\psi \in \mathcal{X}^{\mathrm{arith}}(R; \mathcal{O})$ such that $\psi(R) = \mathcal{O}$ is a discrete valuation ring.

Corollary 11.10. *Assume that $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathcal{K}/K, \mathbf{T}_\psi)$ is a torsion $\mathbb{Z}_p(\mathcal{K})$-module. Then $\mathbf{L}_p(\mathcal{K}, E)$ vanishes to order $\widetilde{r}_\psi$ and*

$$\mathbf{L}_p(\mathcal{K}, g_\psi)^{(\widetilde{r}_\psi)} = \# \left(\widetilde{H}^2_f(K, \mathbf{T})_{\mathrm{tors}} \right) \cdot \det\left(\widetilde{h}_{\mathcal{K}/K,\psi,1,1} \right) \in \left(I^{\widetilde{r}_\psi}/I^{\widetilde{r}_\psi+1} \right)/\mathbb{Z}_p^*.$$

Proof. As in the proof of Prop. 11.7 we have $\overline{\psi}\left(\mathbf{L}_p(\mathcal{K}, \mathbf{g})\right) = \mathbf{L}_p(\mathcal{K}, g_\psi)$ under our assumptions. Then $\mathbf{L}_p(\mathcal{K}, g_\psi)$ vanishes to order $\widetilde{r}_\psi$ and by (101) we obtain the statement applying $\overline{\psi}$ to the formula displayed in in the preceding Theorem. □

11.4. Proof of Theorem 8.1. In this Section the hypothesis and notations are those used in the Introduction. Recall the family of primes $\{\psi_\kappa\}_\kappa$ defined by (72) and indexed by even integers κ. We will write $\mathbf{T}_\kappa := \mathbf{T}_{\psi_\kappa}$. We claim that for every algebraic extension $K \subset L \subset \mathcal{K}$ and every even integer $\kappa \in U$ there exists natural isomorphisms of $R(L)$-modules:

$$\widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \cong M_p(L, \mathbf{g}); \quad \widetilde{H}^1_f(K_S/L, \mathbb{A}_\mathbf{T}) \cong \mathrm{Sel}_p(L, \mathbf{g}), \tag{106}$$

and natural isomorphisms of $\mathbb{Z}_p(L)$-modules:

$$\widetilde{H}^1_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\kappa) \cong M_p(L, g_\kappa); \quad \widetilde{H}^1_f(K_S/L, \mathbb{A}_{\mathbf{T}_\kappa}) \cong \mathrm{Sel}_p(L, g_\kappa), \tag{107}$$

Before giving the proof, we show how Theorem 8.1 follows from this.

First of all: Theorem 10.5 tells us that taking the Pontrjaging duals of (106) and (107) we obtain isomorphisms of $R(L)$ and $\mathbb{Z}_p(L)$-modules respectively:

$$\widetilde{H}^2_{f,\mathrm{Iw}}(L/K, \mathbf{T}) \cong S_p(L, \mathbf{g}); \quad \widetilde{H}^2_{f,\mathrm{Iw}}(L/K, \mathbf{T}_\kappa) \cong S_p(L, g_\kappa). \tag{108}$$

The 'arithmetic pairings' mentioned in the Introduction are then defined, via the isomorphisms (106), (107) and (108) as the 'height' and 'Cassels-Tate' pairings defined on the corresponding Selmer complexs in Section 10:

$$h_{\mathcal{L}/L,\mathbf{g}} := \widetilde{H}_{\mathcal{L}/L,1,1}; \quad h_{\mathcal{L}/L,g_\kappa} := \widetilde{h}_{\mathcal{L}/L,\psi_\kappa,1,1};$$
$$c_{L,\mathbf{g}} := \widetilde{c}_{\pi_R(L),2,2}; \quad c_{L,\mathbf{g}_\kappa} := \widetilde{c}_{\pi_\psi(L),2,2};$$
$$h^{\mathrm{wt}}_{\mathcal{L}/L,g_\kappa} := \widetilde{h}^{\mathrm{wt}}_{\mathcal{L}/L,\psi_\kappa,1,1},$$

where $\pi_\kappa := \pi_R \otimes_{R,\psi_\kappa} \mathbb{Z}_p : \mathbf{T}_\kappa \otimes_{\mathbb{Z}_p} \mathbf{T}_\kappa \to \mathbb{Z}_p(1)$. Moreover, under our assumption H1 we know by the work of Kato and Rohrlich (see , e.g. [**Gre97**, Theorem 1.5]) that $\widetilde{H}^2_{f,\mathrm{Iw}}(K^{\mathrm{cycl}}_\infty/K, T_p) \cong S_p(L, g_2)$ is a (finite) torsion $\mathbb{Z}_p(K^{\mathrm{cycl}}_\infty)$-module, where $K^{\mathrm{cycl}}_\infty/K$ is the cyclotomic $\mathbb{Z}_p$-extension of K. It then follows from the Control Theorems proved in Section 14.2 that

$$\widetilde{H}^2_f(\mathcal{K}/K, \mathbf{T}) \cong S_p(\mathcal{K}, \mathbf{g}) \text{ is a (finte) torsion } R(\mathcal{K})\text{-module.}$$

Theorem 8.1 follows immediately from Theorem 11.4, Proposition 11.5, Prop. 11.7 and Theorem 11.8.

Coming to the proof of (106) and (107): using Iwasawa theory (cfr. Appendix 12) we can assume that L/K is a finite sub-extension of $\mathcal{K}/K$; in particular $G_{L/K} := \mathrm{Gal}(L/K)$, $G_{L_w/K_v} := \mathrm{Gal}(L_w/K_v)$

and G_{k_w/k_v} are finite p-group for every prime $w|v$ of L (where k_* is the residue field at $*$). We note that, writing $\mathcal{T}$ for $\mathbf{T}$ or $\mathbf{T}_\kappa$ we have:

$$\mathbf{R}\Gamma_{\mathrm{cont}}(L_w,\mathcal{T}) \text{ and } \mathbf{R}\Gamma_{\mathrm{cont}}(L_w,\mathbb{A}_{\mathcal{T}}) \text{ are acyclic complexes for every prime } w\nmid p \text{ of } L. \tag{109}$$

Indeed, using local Tate duality the statement for $\mathbf{R}\Gamma_{\mathrm{cont}}(L_w,\mathbb{A}_{\mathcal{T}})$ follows from that for $\mathbf{R}\Gamma_{\mathrm{cont}}(L_w,\mathcal{T})$. Moreover, as in the proof of Theorem 10.5, we can assume (using the control Theorems) that $\mathcal{T}=\mathbf{T}_2\cong T_p$ is the p-adic Tate module of $E_{/\mathbb{Q}}$. In this case we have, for every prime w of L dividing $N:=\mathrm{cond}(E)$:

$$0\overset{\mathrm{H3}}{=}E(K_v)_{p^\infty}=H^0(G_{L_w/K_v},E(L_w)_{p^\infty}) \implies E(L_w)_{p^\infty}=0,$$

where $v|N$ is the prime of K lying below w. (The implication above follows easily from the facts that G_{L_w/K_v} is a finite p-groups acting on a finite p-groups [**Ser79**, Lemma 2 pag. 138].) Exactly as in the proof of Theorem 10.5 we conclude that $\mathbf{R}\Gamma_{\mathrm{cont}}(L_w,T_p)$ is acyclic, proving also (109).

We now prove that, for every prime $w|p$ of L:

$$H^0(L_w,F_w^-(\mathcal{T}))=0. \tag{110}$$

For the proof of this assertion when $\mathcal{T}=\mathbf{T}_\kappa$ we refer to [**NP00**, Sec. (3.1.5)], recalling that E is assumed to have good ordinary reduction at p. Then the statement for $\mathcal{T}=\mathbf{T}$ follows applying Nakayama's Lemma to the injection $H^0(L_v,F_v^-(\mathbf{T}))\otimes_{R,\psi_\kappa}\mathbb{Z}_p\hookrightarrow H^0(L_w,\mathbf{T}_\kappa)=0$.

Analogously: for every prime $w|p$ of L we have

$$H^0(L_w,F_w^-(\mathbb{A}_{\mathcal{T}}))=0. \tag{111}$$

Indeed, for $\mathbf{T}=\mathbf{T}_2=T_p$ we have $F_w^-(T_p)\cong\widetilde{E}_w[p^\infty]=$ the p^∞-torsion of the reduction of $E_{/L_w}$ (see Section 9.3.4). Letting $v|p$ be the prime of K lying below w, by assumption H2 we have $H^0(G_{k_w/k_v},\widetilde{E}_w[p^\infty])\cong\widetilde{E}_v(k_v)_{p^\infty}=0$, so that as above we conclude $H^0(L_w,F_v^-(\mathbb{A}_{T_p}))=\widetilde{E}_w(k_w)_{p^\infty}=0$. Applying cohomology to the exact sequence of G_{L_w}-modules $0\to F_w^-(\mathbb{A}_{\mathbf{T}_\kappa})\to F_w^-(\mathbb{A}_{\mathbf{T}})\overset{\times\varpi_\kappa}{\to}F_w^-(\mathbb{A}_{\mathbf{T}})\to 0$ (where ϖ_κ is a generator of $\ker(\psi_\kappa)$) we also obtain $(\star_\kappa)$ $H^0(L_w,F_w^-(\mathbb{A}_{\mathbf{T}}))[\varpi_\kappa]\cong H^0(L_w,F_w^-(\mathbb{A}_{T_p}))$. Combining $(\star)_2$ with what already observed we conclude $H^0(L_w,F_w^-(\mathbb{A}_{\mathbf{T}}))[\varpi_2]=0$, so $H^0(L_w,F_w^-(\mathbb{A}_{\mathbf{T}}))=0$ by Nakayama's Lemma (as $H^0(L_w,F_w^-(\mathbb{A}_{\mathbf{T}}))$ is an $\mathcal{R}$-module of cofinite type). Using again $(\star)_\kappa$ we conclude the proof of (111) for $\mathcal{T}\in\{\mathbf{T},\mathbf{T}_\kappa\}$.

Combining (109), (110) and (111) with the exact sequences (158) and (166) we immediately conclude: for every finite subextension L/K of $\mathcal{K}/K$ the natural maps induces isomorphisms

$$\widetilde{H}_f^1(L,\mathbf{T})\cong M_p(L,\mathbf{g});\quad \widetilde{H}_f^1(L,\mathbb{A}_{\mathbf{T}})\cong \mathrm{Sel}_p(L,\mathbf{g});$$

$$\widetilde{H}_f^1(L,\mathbf{T}_\kappa)\cong M_p(L,g_\kappa);\quad \widetilde{H}_f^1(L,\mathbb{A}_{\mathbf{T}_\kappa})\cong \mathrm{Sel}_p(L,g_\kappa).$$

As explained above this concludes the proof of (106) and (107).

12. Cyclotomic Iwasawa theory

In this Section we analyze more closely two-variable cyclotomic Iwasawa theory, i.e. we take $K = \mathbb{Q}$ and $\mathcal{K} := \mathbb{Q}_\infty \subset \mathbb{Q}(\mu_{p^\infty})$ the cyclotomic $\mathbb{Z}_p$-extension. One of our principal aims is to formulate a two-variable main conjecture (see Sec. 12.5) 'explaining' and motivating the p-adic Birch and Swinnerton-Dyer conjecture proposed in **[Ven12]**.

In addition to Hypothesis 1, 2 and 3, we asume that p is not an *anomalous prime* if $E/\mathbb{Q}_p$ has good reduction. In this framework our Hypotheses are the following.

Fix an elliptic curve $E/\mathbb{Q}$ of conductor N_E and a rational prime p. Let $f_E = \sum_{n\geq 1} a_n(E)\cdot q^n \in S_2(\Gamma_0(N_E),\mathbb{Z})$ be the newform attached to $E/\mathbb{Q}$ by the modularity theorem. We assume that (E,p) satisfies the following conditions:

(**Irr**) $E(\overline{\mathbb{Q}})_p$ is an irreducible $\mathbb{F}_p[G_\mathbb{Q}]$-module;

(**Fro**) Either $p\|N_E$ or $p\nmid N_E$ and $a_p(E)\not\equiv 0,1 \bmod p$;

(**Tam**) $p\nmid 6\cdot\prod_{\ell|N} E(\mathbb{Q}_\ell)_{\mathrm{tors}}$, where $N := N_E/p^{\mathrm{ord}_p(N_E)}$;

(**Reg**) $R = R_g$ is a regular local ring, where $g\in S_2(\Gamma_0(Np),\mathbb{Z}_p)$ is the p-stabilization of f_E.

From Proposition 1 we easily obtain the following statement, showing that our theory is non-vacuous.

Proposition 12.1. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication. The set of rational primes p such that (E,p) satisfies the above conditions has Dirichlet density one.*

We quote from the beginning the following result, which follows combining deep results of Kato and Rohrlich with the results of the preceding Section.

Theorem 12.2. *There exists a skew-Hermitian $R(\mathbb{Q}_\infty)$-module which organizes the arithmetic of the Hida family $\mathbf{g}/\mathbb{Q}_\infty$. Moreover $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,\mathbf{T}_\psi)$ is a torsion $\mathcal{O}_\psi(\mathbb{Q}_\infty)$-module for every $\psi\in\mathcal{X}^{\mathrm{arith}}(R,\mathcal{O}_\psi)$.*

Proof. The work of Kato and Rohrlich (see, e.g. Theorem 1.5 in Section 1 of **[Gre97]**) implies that the p-primary Selmer group $Sel(\mathbb{Q}_\infty,E_{p^\infty})$ of $E/\mathbb{Q}_\infty$ is $\mathbb{Z}_p(\mathbb{Q}_\infty)$-cotorsion. Then the comparison results between Selmer complexes and Greenberg Selmer groups tell us that $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,T_p)$ is a torsion $\mathbb{Z}_p(\mathbb{Q}_\infty)$. Using the Control Theorems proved in Section 10.3 and Theorem 10.5, we conclude that $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,\mathbf{T})$ is a torsion $R(\mathbb{Q}_\infty)$-module, and that $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,\mathbf{T}_\psi)$ is a torsion $\mathcal{O}_\psi(\mathbb{Q}_\infty)$-module for every arithmetic prime $\psi\in\mathcal{X}^{\mathrm{arith}}(R,\mathcal{O}_\psi)$. The existence of an organizing module for the arithmetic of $\mathbf{g}/\mathbb{Q}_\infty$ follows by Theorem 11.4. □

12.1. Mellin transforms. In order to compare the computations above with the conjecture proposed in **[Ven12]**, we will apply the 'Mellin transform' to the constructions of the preceding Sections.

For every open p-adic neighborhood U of $2\in\mathbb{Z}_p$, let $\mathcal{A}(U)\subset\mathbb{Z}_p[[k-2]]$ be the ring of those power series in $(k-2)$ converging for $k\in U$. We can endow $\mathcal{A}(U)$ with a structure of $\Lambda := \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$-algebra, via the unique embedding $\Lambda\hookrightarrow\mathcal{A}(U)$ sending the group-like element $\gamma\in 1+p\mathbb{Z}_p$ to the analytic function on U: $k\mapsto\gamma^{k-2} := \exp_p\big((k-2)\cdot\log_p(\gamma)\big)$. As recalled in Section 9.2 (cf. **[GS93]**) there exists an open neighborhood $2\in U = U_E\subset\mathbb{Z}_p$, together with a unique morphism of Λ-algebras $\mathcal{M}_2 = \mathcal{M}_g : R\hookrightarrow\mathcal{A}(U)$ such that

$$\mathcal{M}_2(r)|_{k=2} = \phi_g(r)$$

for every $r\in R$. (Recall that $\phi_g = \psi_\mathfrak{p}\in\mathcal{X}^{\mathrm{arith}}(R;\mathbb{Z}_p)$ is the $\mathbb{Z}_p$-valued arithmetic point defined by $\phi_g(\boldsymbol{a}_\ell) = a_\ell(g)$ for every prime ℓ.)

The (p-adic) cyclotomic character induces a canonical isomorphism $\chi_{cy} : \mathcal{G} := \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \xrightarrow{\sim} 1+p\mathbb{Z}_p$. Writing $\mathcal{A}$ for the ring of $\mathbb{Z}_p$-valued p-adic analytic functions on $\mathbb{Z}_p$, we consider the morphism of $\mathbb{Z}_p$-algebras

$$\mathcal{M}_1 : \overline{\mathbb{Z}_p} := \mathbb{Z}_p(\mathbb{Q}_\infty)\hookrightarrow\mathcal{A}$$

induced by the character $\mathcal{G}\hookrightarrow\mathcal{A}$ sending $g\in\mathcal{G}$ to the analytic function on $\mathbb{Z}_p$: $s\mapsto\chi_{cy}(g)^{s-1}$.

Finally, let $\mathcal{A}(U,\mathbb{Z}_p) \subset \mathbb{Z}_p[[k-2,s-1]]$ be the ring of formal power series in $(k-2,s-1)$ which converges for $k \in U$ and $s \in \mathbb{Z}_p$. Then there exists a unique morphism of $\mathbb{Z}_p$-algebras

$$\mathcal{M}_{2,1} : \overline{R} := R(\mathbb{Q}_\infty) \longrightarrow \mathcal{A}(U,\mathbb{Z}_p)$$

such that $\mathcal{M}_{2,1}|_R = \mathcal{M}_2$ and $\mathcal{M}_{2,1}|_{\overline{\mathbb{Z}_p}} = \mathcal{M}_1$.

12.2. Algebraic p-adic L-functions. We write

$$\mathbf{L}_p(E,s) := \mathcal{M}_1\left(\mathrm{char}_{\mathbb{Z}_p(\mathbb{Q}_\infty)}\left(\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,T_p)\right)\right) \in \mathcal{A}$$

for the Mellin transform of any (algebraic) p-adic L-function of $E/\mathbb{Q}_\infty$. (With this notation, s is the 'cyclotomic variable'.) Then $\mathbf{L}_p(E,s)$ is determined only up to multiplication by units in $\overline{\mathbb{Z}_p}$. We will also write $\mathbf{L}_p(\mathbb{Q}_\infty,E)$ fo the characteristic ideal of $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,T_p)$. The 'functional equation' satisfied by $\mathbf{L}_p(\mathbb{Q}_\infty,E)$ translates into the following:

PROPOSITION 12.3. *(Functional equation) There exists* $\mathbf{L}_p(E,s)$ *such that*

$$\mathbf{L}_p(E,s) = w(E/\mathbb{Q}) \cdot \mathbf{L}_p(E,2-s).$$

PROOF. For $g \in \mathcal{G}$ we have $\mathcal{M}_1(\iota(g))(s) = \mathcal{M}_1(g^{-1})(s) = \chi_{cy}(g)^{1-s} = \chi_{cy}(g)^{(2-s)-1} = \mathcal{M}_1(g)(2-s)$. Since $\overline{\mathbb{Z}_p} = \mathbb{Z}_p[[\sigma_{cy}-1]]$ for every topological generator $\sigma_{cy} \in \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, it follows that $((\mathcal{M}_1 \circ \iota)(x))(s) = (\mathcal{M}_1(x))(2-s)$ for every $x \in \overline{\mathbb{Z}_p}$. Then the Proposition is a reformulation of the functional equation for $\mathbf{L}_p(\mathbb{Q}_\infty,E)$ displayed in Prop. 11.7. □

REMARK 12.4. Let $E^\dagger(\mathbb{Q})$ be the extended Mordell-Weil group of $E_{/\mathbb{Q}}$. Assuming the finiteness of $Ш(E/\mathbb{Q})_{p^\infty}$, the results of Section 4 of Part 1 give us

$$\widetilde{r} := \mathrm{rank}_{\mathbb{Z}} E^\dagger(\mathbb{Q}) = \mathrm{rank}_{\mathbb{Z}_p} E(\mathbb{Q}) + \delta_p,$$

where $\delta_p := 1$ (resp, $\delta_p := 0$) if $E/\mathbb{Q}_p$ has (resp., has not) split multiplicative reduction. In particular, via the Birch and Swinnerton-Dyer conjecture, the sign $w(E/\mathbb{Q})$ in the functional equation satisfied by $\mathbf{L}_p(E,s)$ should be different from the sign in the functional equation of the complex Hasse-Weil L-function $L(E/\mathbb{Q},s)$ if and only if we are in the 'exceptional case', i.e. $\delta_p = 1$.

In a similar way, we write

$$\mathbf{L}_p(\mathbf{g},k,s) := \mathcal{M}_{2,1}\left(\mathrm{char}_{R(\mathbb{Q}_\infty)}\left(\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,\mathbf{T})\right)\right) \in \mathcal{A}(U,\mathbb{Z}_p)/\overline{R}^*$$

for the Mellin transform of any algebraic p-adic L-function of $\mathbf{g}/\mathbb{Q}_\infty$. We will also write $\mathbf{L}_p(\mathbb{Q}_\infty,\mathbf{g})$ for the characteristic ideal of $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,\mathbf{T})$.

PROPOSITION 12.5. *Let* $e_{\mathrm{gen}} := \mathrm{rank}_R \widetilde{H}^1_f(\mathbb{Q},\mathbf{T})$. *We have:*
1. $\mathbf{L}_p(\mathbf{g},2,s) = \mathbf{L}_p(E,s)$.
2. $\mathbf{L}_p(\mathbf{g},k,s) \in (s-1)^{e_{\mathrm{gen}}} \cdot \mathcal{A}(U,\mathbb{Z}_p)$.
3. *(Functional equation) There exists* $\mathbf{L}_p(\mathbf{g},k,s)$ *such that:*

$$\mathbf{L}_p(\mathbf{g},k,s) = (-1)^{e_{\mathrm{gen}}} \cdot \mathbf{L}_p(\mathbf{g},k,2-s).$$

PROOF. 1. Since $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,T_p)$ is a torsion $\overline{\mathbb{Z}_p}$-module by Theorem 12.2, we see as in the proof of Prop. 11.7 that $\overline{\psi}_{\mathfrak{p}}(\mathbf{L}_p(\mathbb{Q}_\infty,\mathbf{g})) = \mathbf{L}_p(\mathbb{Q}_\infty,E)$. As $\left(\mathcal{M}_1 \circ \overline{\psi}_{\mathfrak{p}}\right)(*) = (\mathcal{M}_{2,1}(*))|_{(k,s)=(2,s)}$, recalling the definitions we obtain the statement.

2. Let $\overline{I}$ be the augmentation ideal of $\overline{R}$. By Section 10.3 and Theorem 10.5 we have an isomorphism of R-modules

$$\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,\mathbf{T})/\overline{I} \xrightarrow{\sim} \widetilde{H}^2_f(\mathbb{Q},\mathbf{T}).$$

Moreover (again as in the proof of Prop. 11.7) $\widetilde{H}^1_f(\mathbb{Q},\mathbf{T})$ and $\widetilde{H}^2_f(\mathbb{Q},\mathbf{T})$ have the same R-rank, so that

$$\mathrm{rank}_R\left(\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty,\mathbf{T})/\overline{I}\right) = e_{\mathrm{gen}}.$$

Localizing $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty, \mathbf{T})$ at the height-one prime $\overline{I}$ (and using the structure theorem for finite torsion modules over PID's) we deduce easily from this: $\mathrm{length}_{\overline{R}_{\overline{I}}}\left(\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty, \mathbf{T})_{\overline{I}}\right) \geq e_{\mathrm{gen}}$, so that

$$\mathbf{L}_p(\mathbb{Q}_\infty, \mathbf{g}) \in \overline{I}^{e_{\mathrm{gen}}}.$$

Applying the Mellin transform $\mathcal{M}_{2,1}$ we conclude $\mathbf{L}_p(\mathbf{g}, k, s) \in (s-1)^{e_{\mathrm{gen}}}$, as claimed. (Indeed, $\overline{I} = (\sigma_{cy} - 1) \cdot \overline{R}$ for every topological generator $\sigma_{cy} \in \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, and $(\mathcal{M}_{2,1}(\sigma_{cy}-1))(k,s)$ is by definition the analytic function $\sum_{n\geq 1} \frac{\log_p(\chi_{cy}(\sigma_{cy}))^n}{n!} \cdot (s-1)^n \in (s-1) \cdot \mathcal{A}(U, \mathbb{Z}_p)$.)

3. As $\overline{R} = R[[\sigma_{cy} - 1]]$ (with σ_{cy} as above), as in the proof of Prop. 12.3 we see that $((\mathcal{M}_{2,1} \circ \iota)(x))(k,s) = (\mathcal{M}_{2,1}(x))(k, 2-s)$ for every $x \in \overline{R}$. Then the statement is a reformulation of the functional equation for $\mathbf{L}_p(\mathbb{Q}_\infty, \mathbf{g})$ proved in Prop. 11.7. □

Remark 12.6. As a direct consequence of Greenberg Conjecture **[Gre94a]**,**[NP00]** we expect that $e_{\mathrm{gen}} = 1$ or 0, depending on the *sign* of the Hida family $\mathbf{g}$. See Section 12.5 below for more details.

We are also interested in two other algebraic p-adic L-functions. The first one is the Mellin transform of the characteristic ideal of the R-torsion submodule of $\widetilde{H}^2_f(\mathbb{Q}, \mathbf{T})$, *i.e.*

$$\mathbf{L}_p(\mathbf{g}, k) := \mathcal{M}_2\left(\mathrm{char}_R\left(\widetilde{H}^2_f(\mathbb{Q}, \mathbf{T})_{\mathrm{tors}}\right)\right) \in \mathcal{A}(U)/R^*$$

The second one is the 'generic restriction' of $\mathbf{L}_p(\mathbf{g}, k, s)$ to the line $s = 1$, defined by:

$$\mathbf{L}_p^{\mathrm{gen}}(\mathbf{g}, k) := \left.\frac{\mathbf{L}_p(\mathbf{g}, k, s)}{(s-1)^{e_{\mathrm{gen}}}}\right|_{s=1} \in \mathcal{A}(U)/R^*.$$

(See Theorem 12.16 for the precise relation between these two closely related p-adic L-functions.)

Remark 12.7. It follows from d) of Theorem 10.5 and standard structure theorems for symplectic finite modules over principal ideal domains that the characteristic ideal of $\widetilde{H}^2_f(K, \mathbf{T})_{\mathrm{tors}}$ is (represented by) a square in R. In particular

$$\mathbf{L}_p(\mathbf{g}, k) = \left(\mathbf{L}_p^{\frac{1}{2}}(\mathbf{g}, k)\right)^2 \in \mathcal{A}(U)/R^*$$

for a suitable 'square-root (algebraic) p-adic L-function' $\mathbf{L}_p^{\frac{1}{2}}(\mathbf{g}, k)$.

12.3. Pairings and regulators. Let $\mathcal{J} \subset \mathcal{A}(U, \mathbb{Z}_p)$ be the ideal generated by $(k-2)$ and $(s-1)$, and let as usual $J := \ker\left(\overline{R} \overset{\varepsilon_R}{\to} R \overset{\phi_g}{\to} \mathbb{Z}_p\right)$. The Mellin transform induces a morphism $\mathcal{M}_{2,1} : J/J^2 \to \mathcal{J}/\mathcal{J}^2$. For $* = k, s$, let us write $\partial_*(f(k,s)) := \frac{\partial}{\partial *} f(k,s)\big|_{(k,s)=(2,1)}$ (for every $f(k,s) \in \mathcal{A}(U, \mathbb{Z}_p)$). Then ∂_* induces a morphism of $\mathbb{Z}_p$-modules $\partial_* : \mathcal{J}/\mathcal{J}^2 \to \mathbb{Z}_p$.

The isomorphism $\mathbf{T}_{\phi_g} \cong \mathrm{Ta}_p(E_{/\mathbb{Q}}) =: T_p$ of Section 9.3.4 allows us to identify $\widetilde{H}^1_f(K, \mathbf{T}_{\phi_g})$ with $\widetilde{H}^1_f(K, T_p)$, where the latter is defined using the ordinary structure on T_p (recalled in Section 9.3.4 too). We can then consider the p-adic pairing

$$\widetilde{h}^{\mathrm{wt}}_{\mathbb{Q},1,1} = \widetilde{h}^{\mathrm{wt}}_{\mathbb{Q},\phi_g,1,1} : \widetilde{H}^1_f(\mathbb{Q}, T_p) \otimes_{\mathbb{Z}_p} \widetilde{H}^1_f(\mathbb{Q}_p, T_p) \longrightarrow \mathfrak{p}/\mathfrak{p}^2$$

attached in Section 10.5.1 to the Hida deformation $\mathbf{T}$ of $T_p := \mathrm{Ta}_p(E_{/\mathbb{Q}}) \cong \mathbf{T}_{\phi_g}$ at the arithmetic prime $\mathfrak{p} := \phi_g$. The *p-adic weight pairing* attached to $E/\mathbb{Q}$:

$$\langle -, - \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} : \widetilde{H}^1_f(\mathbb{Q}, T_p) \times \widetilde{H}^1_f(\mathbb{Q}, T_p) \longrightarrow \mathbb{Z}_p$$

is defined by the formula

$$\langle x, y \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} := \left(\partial_k \circ \mathcal{M}_2 \circ \widetilde{h}^{\mathrm{wt}}_{\mathbb{Q},1,1}\right)(x \otimes y),$$

for every $x, y \in \widetilde{H}^1_f(K, T_p)$. We also have a 'height pairing'

$$\widetilde{h}_{\mathbb{Q}_\infty/\mathbb{Q},1,1} = \widetilde{h}_{\mathbb{Q}_\infty/\mathbb{Q},\phi_g,1,1} : \widetilde{H}^1_f(\mathbb{Q}, T_p) \otimes_{\mathbb{Z}_p} \widetilde{H}^1_f(\mathbb{Q}_p, T_p) \longrightarrow I/I^2$$

attached to the Galois deformation $T_p(\mathbb{Q}_\infty)$ of T_p at the augmentation ideal $I \subset \mathbb{Z}_p(\mathbb{Q}_\infty)$. Define the *cyclotomic p-adic height pairing* attached to $E/\mathbb{Q}$:

$$\langle -,-\rangle^{\mathrm{MTT}}_{\mathbb{Q},p} : \widetilde{H}^1_f(\mathbb{Q},T_p)\times \widetilde{H}^1_f(\mathbb{Q},T_p)\longrightarrow \mathbb{Z}_p$$

by the formula

$$\langle x,y\rangle^{\mathrm{MTT}}_{\mathbb{Q},p} := \left(\partial_s\circ \mathcal{M}_1\circ \widetilde{h}_{\mathbb{Q}_\infty/\mathbb{Q},1,1}\right)(x\otimes y).$$

(Here Nek and MTT are abbreviations for Nekovář and Mazur-Tate-Teitelbaum respectively.) The following Proposition follows by the discussion in Sections 10.5.1 and 10.5.2.

PROPOSITION 12.8. *$\langle -,-\rangle^{\mathrm{Nek}}_{\mathbb{Q},p}$ (resp., $\langle -,-\rangle^{\mathrm{MTT}}_{\mathbb{Q},p}$) is a skew-symmetric (resp., symmetric) $\mathbb{Q}_p$-bilinear form.*

Let now $\mathcal{I} := (s-1)\cdot \mathcal{A}(U,\mathbb{Z}_p)$ and $\overline{I} := \ker(\varepsilon_R)$ the augmentation ideal in $\overline{R}$, so that we have a morphism $\mathcal{M}_{2,1} : \overline{I}/\overline{I}^2 \to \mathcal{I}/\mathcal{I}^2$. The *cyclotomic $\mathcal{A}(U)$-adic height pairing* attached to $\mathbf{T}$:

$$\langle -,-\rangle^{\mathrm{cycl}}_{\mathbf{T}} : \widetilde{H}^1_f(\mathbb{Q},\mathbf{T})\times \widetilde{H}^1_f(\mathbb{Q},\mathbf{T})\longrightarrow \mathcal{A}(U)$$

is defined by the formula

$$\langle x,y\rangle^{\mathrm{cycl}}_{\mathbf{T}} := \frac{\partial}{\partial s}\mathcal{M}_{2,1}\left(\widetilde{H}_{\mathbb{Q}_\infty/\mathbb{Q},1,1}(x\otimes y)\right)\bigg|_{s=1}.$$

Here the $\overline{I}/\overline{I}^2$-pairing $\widetilde{H}_{\mathbb{Q}_\infty/\mathbb{Q},1,1}$ is defined in Section 10.6.

PROPOSITION 12.9. *$\langle -,-\rangle^{\mathrm{cycl}}_{\mathbf{T}}$ is a symmetric R-bilinear form, satisfying (with the notations of Lemma* 10.14*)*

$$\langle x,y\rangle^{\mathrm{cycl}}_{\mathbf{T}}\Big|_{k=2} = \langle \phi_{g*}(x),\phi_{g*}(y)\rangle^{\mathrm{MTT}}_{\mathbb{Q},p}$$

for every $x,y\in \widetilde{H}^1_f(\mathbb{Q},\mathbf{T})$.

PROOF. This follows by the discussion in Sec. 10.6. □

In order to compute derivatives of the generic p-adic L-function $\mathbf{L}^{\mathrm{gen}}_p(\mathbf{g},k)$, we now also introduce a 'derived regulator'. Let $\widetilde{H}^1_{f,\infty}\subset \widetilde{H}^1_f(\mathbb{Q},T_p)$ be the (left=right) radical of (the restriction of) $\langle -,-\rangle^{\mathrm{Nek}}_{\mathbb{Q},p}$. Let $\{P_1,\ldots,P_{r_\infty}\}$ be a $\mathbb{Z}_p$-basis of $\widetilde{H}^1_{f,\infty}$ (which is a free $\mathbb{Z}_p$-module, as $\widetilde{H}^1_f(\mathbb{Q},T_p)$ is free by, e.g. Theorem 10.5), where $r_\infty := \mathrm{rank}_{\mathbb{Z}_p}\widetilde{H}^1_{f,\infty}$. Since $\widetilde{H}^1_{f,\infty}$ is p-adically saturated in $\widetilde{H}^1_f(\mathbb{Q},T_p)$, we can complete $\{P_j\}$ to a $\mathbb{Z}_p$-basis $\{P_1,\ldots,P_{r_\infty},Q_1,\ldots,Q_t\}$ of $\widetilde{H}^1_f(\mathbb{Q},T_p)$ (with $r_\infty+t=\widetilde{r}$). Define

$$\mathcal{R}_{\mathbb{Q},p} := \mathcal{R}^\infty_{\mathbb{Q},p}\cdot \mathcal{R}^{\mathrm{Nek}}_{\mathbb{Q},p},$$

with 'partial regulators' defined by:

$$\mathcal{R}^\infty_{\mathbb{Q},p} := \det\left(\left(\langle P_i,P_j\rangle^{\mathrm{MTT}}_{\mathbb{Q},p}\right)_{1\le i,j\le r_\infty}\right);\quad \mathcal{R}^{\mathrm{Nek}}_{\mathbb{Q},p} := \det\left(\left(\langle Q_i,Q_j\rangle^{\mathrm{NEK}}_{\mathbb{Q},p}\right)_{1\le i,j\le t}\right).$$

These are well-defined elements of $\mathbb{Q}_p/\mathbb{Z}_p^*$.

REMARK 12.10. In Section 4 of Part 1 we defined an embedding $i^\dagger_E : E^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p \hookrightarrow \widetilde{H}^1_f(K,T_p)\otimes\mathbb{Q}_p$ and a p-adic weight pairing on $E^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p$, denote again by $\langle -,-\rangle^{\mathrm{Nek}}_{\mathbb{Q},p}$. Indeed retracing the definitions it follows easily from Prop. 10.9 that the this pairing is precisely the 'restriction' of $\langle -,-\rangle^{\mathrm{Nek}}_{\mathbb{Q},p}\otimes\mathbb{Q}_p$ (as defined above) to the extended Mordell-Weil group $E^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p$.

REMARK 12.11. With the notations of the preceding Remark, if follows easily from the results of [**Nek06**, sec. 11.4] that the 'restriction' of $\langle -,-\rangle^{\mathrm{MTT}}_{\mathbb{Q},p}\otimes\mathbb{Q}_p$ to $E^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p$ is, up to sign, the bilinear form denoted by the same symbol in Sec. 5 of Part 1. The latter is essentially the p-adic height pairing on $E^\dagger(\mathbb{Q})$ appearing in the formulation of the p-adic Birch and Swinnerton-Dyer conjectures proposed in [**MTT86**] (explaining our notation).

REMARK 12.12. Assume that $\text{Ш}(E/\mathbb{Q})_{p^\infty}$ is finite, so that $i_E^\dagger : E^\dagger(\mathbb{Q}) \otimes \mathbb{Q}_p \cong \widetilde{H}^1_f(\mathbb{Q}, T_p) \otimes \mathbb{Q}_p$ is an isomorphism. Moreover, under the assumptions of this Section, it is easily verified that this isomorphism identifies the $\mathbb{Z}_p$-lattices $E^\dagger(\mathbb{Q}) \otimes \mathbb{Z}_p$ and $\widetilde{H}^1_f(\mathbb{Q}, T_p)$. Using $\mathbb{Z}_p$-basis of $\widetilde{H}^1_f(\mathbb{Q}, T_p)$ coming from $\mathbb{Z}$-basis of $E^\dagger(\mathbb{Q})/\text{tors}$, we can define a regulator $\mathcal{R}_{\mathbb{Q},p}$ belonging to $\mathbb{Q}_p$ (and not only to $\mathbb{Q}_p/\mathbb{Z}_p^*$). The preceding Remarks then imply that this regulator is precisely that appearing in the p-adic Birch and Swinnerton-Dyer conjecture proposed in Sec. 6 of Part 1. (See Section 5 of *loc. cit.* for more details.)

12.4. p-adic BSD formulas.

THEOREM 12.13. *Let* $\widetilde{r} := \text{rank}_{\mathbb{Z}_p} \widetilde{H}^1_f(\mathbb{Q}, T_p)$. *Then* $\mathbf{L}_p(\mathbf{g}, k, s) \in \mathcal{J}^{\widetilde{r}}$ *and there exists a p-adic unit* $u \in \mathbb{Z}_p^*$ *such that:*

$$\mathbf{L}_p(\mathbf{g}, k, s) \equiv u \cdot \# \left((\text{Ш}(E/\mathbb{Q})_{p^\infty})_{/\text{div}} \right) \cdot \det \left(\langle -, - \rangle^{\text{Nek}}_{\mathbb{Q},p} \cdot (k-2) + \langle -, - \rangle^{\text{MTT}}_{\mathbb{Q},p} \cdot (s-1) \right) \mod \mathcal{J}^{\widetilde{r}+1},$$

where the determinant is computed with respect to any $\mathbb{Z}_p$*-basis of* $\widetilde{H}^1_f(\mathbb{Q}, T_p)$.

PROOF. Using the notations of Section 10.5, it follows directly by the definitions and the discussion in Sections 10.5.1 and 10.5.2 that

$$\mathcal{M}_{2,1} \left(\widetilde{h}^{\text{wt}}_{\mathbb{Q}_\infty/\mathbb{Q},1,1}(x \otimes y) \right) \equiv \langle x, y \rangle^{\text{Nek}}_{\mathbb{Q},p} \cdot (k-2) + \langle x, y \rangle^{\text{MTT}}_{\mathbb{Q},p} \cdot (s-1) \mod \mathcal{J}^2.$$

Then the statement follows immediately by Theorem 11.8, together with the equality

$$\# \left(\widetilde{H}^2_f(\mathbb{Q}, T_p)_{\text{tors}} \right) = \left((\text{Ш}(E/\mathbb{Q})_{p^\infty})_{/\text{div}} \right).$$

Indeed Nekovář's generalized Poitou-Tate duality (Sec. 0.11) and the existence of the Weil pairing (Section 9.3.4) imply that $\widetilde{H}^2_f(\mathbb{Q}, T_p)_{\text{tors}}$ is the Pontrjagin dual of $\widetilde{H}^1_f(\mathbb{Q}, E_{p^\infty})_{/\text{div}}$, so that these two groups have the same order. Moreover it follows immediately from [**Nek06**, Sec. 9.3.7] (or easily from (166) and Section 9.3.4) that, under our assumptions: $\widetilde{H}^1_f(\mathbb{Q}, E_{p^\infty})_{/\text{div}} \xrightarrow{\sim} \text{Sel}_{p^\infty}(\mathbb{Q}, E_{p^\infty})_{/\text{div}}$. By construction this last group is isomorphic to $(\text{Ш}(E/\mathbb{Q})_{p^\infty})_{/\text{div}}$. (More precisely: using global duality it can be easily proved that actually we have an isomorphism: $\widetilde{H}^2_f(\mathbb{Q}, T_p)_{\text{tors}} \xrightarrow{\sim} (\text{Ш}(E/\mathbb{Q})_{p^\infty})_{/\text{div}}$.) □

COROLLARY 12.14. *With the notations of the preceding Theorem,* $\text{ord}_{s=1} \mathbf{L}_p(E, s) \geq \widetilde{r}$, *and we have*

$$\left. \frac{\mathbf{L}_p(E, s)}{(s-1)^{\widetilde{r}}} \right|_{s=1} = \# \left((\text{Ш}(E/\mathbb{Q})_{p^\infty})_{/\text{div}} \right) \cdot \det \left(\langle -, - \rangle^{\text{MTT}}_{\mathbb{Q},p} \right) \in \mathbb{Q}_p/\mathbb{Z}_p^*,$$

where the determinant is computed with respect to any $\mathbb{Z}_p$*-basis of* $\widetilde{H}^1_f(\mathbb{Q}, T_p)$.

PROOF. By Prop. 12.5 we know that $\mathbf{L}_p(E, s)$ is obtained evaluating $\mathbf{L}_p(\mathbf{g}, k, s)$ at $k = 2$. Evaluating the R.H.S. of the p-adic BSD formula of the preceding Theorem we obtain the statement. □

COROLLARY 12.15. *With the notations above, let* $\widetilde{r}_{\text{gen}} := \widetilde{r} - e_{\text{gen}}$. *Then:*

1. $\text{ord}_{k=2} \mathbf{L}^{\text{gen}}_p(\mathbf{g}, k) \geq \widetilde{r}^{\text{gen}}$;
2. *if* $r_\infty = e_{\text{gen}}$, *then we have an equality*

$$\left. \frac{\mathbf{L}^{\text{gen}}_p(\mathbf{g}, k)}{(k-2)^{\widetilde{r}_{\text{gen}}}} \right|_{k=2} = \# \left((\text{Ш}(E/\mathbb{Q})_{p^\infty})_{/\text{div}} \right) \cdot \mathcal{R}_{\mathbb{Q},p} \in \mathbb{Q}_p/\mathbb{Z}_p^*.$$

PROOF. Combining the preceding Theorem with *b*) of Prop. 12.5, we know that $\mathbf{L}_p(\mathbf{g}, k, s)$ lies in the intersection $\mathcal{J}^{\widetilde{r}} \cap (s-1)^{e_{\text{gen}}} \cdot \mathcal{A}(U, \mathbb{Z}_p)$. Recalling the definition of $\mathbf{L}^{\text{gen}}_p(\mathbf{g}, k)$, 1. follows immediately. Under the assumption $e_{\text{gen}} = r_\infty$, and using Proposition 12.8, the formula in the statement follows by the Theorem and a simple computation. □

It is interesting to know the relation between the derivative $\mathbf{L}^{\text{gen}}_p(\mathbf{g}, k)$ of $\mathbf{L}_p(\mathbf{g}, k, s)$ along the cyclotomic direction and $\mathbf{L}_p(\mathbf{g}, k)$. This relation is made explicit in the following Theorem, which is also 'included' in [**Nek06**, Th. 11.7.11] . We note the analogy with [**PR87**, Théorèm 1], in which the anticyclotomic variable plays the role of the weight variable, and the module of universal norms plays the role of $\widetilde{H}^1_f(\mathbb{Q}, \mathbf{T})$. (See also [**MR05**, Th. 10.2].)

THEOREM 12.16. *We have*

$$\mathbf{L}_p^{\mathrm{gen}}(\mathbf{g},k) = \mathbf{L}_p(\mathbf{g},k)\cdot \det\left(\langle -,-\rangle_{\mathbf{T}}^{\mathrm{cycl}}\right) \in \mathcal{A}(U)/R^*,$$

where the determinant is computed on any R-basis of the free R-module $\widetilde{H}^1_f(\mathbb{Q},\mathbf{T})$.

PROOF. We can prove the Theorem with a similar argument to that used in the proof of Theorem 11.8 (with some complication coming from the fact the R is not a principal ideal domain). Alternatively, we can use the following argument, which is essentially the one used in [**PR87**], [**Pla97**] and [**Nek06**, Th. 11.7.11] to prove similar statements.

First of all, we note that $\widetilde{H}^1_f(\mathbb{Q},\mathbf{T})$ is a free R-module (of rank e_{gen}). In fact we know (e.g. by *c*) of Theorem 10.5) that $\widetilde{H}^1_f(\mathbb{Q},\mathbf{T})$ has no R-torsion. By the structure theorem for finite modules over 2-dimensional normal local rings [**Bou89**, Ch. 7], there exists a short exact sequence of R-modules

$$0\to \widetilde{H}^1_f(\mathbb{Q},\mathbf{T})\to R^n\to A\to 0;\quad \#A<\infty,\quad n\geq 0.$$

Using the exact sequence (89), the associated exact sequence of $\mathfrak{p}=\varpi\cdot R$ 'torsion and cotorsion' gives us an injection:

$$A[\varpi]\subset \widetilde{H}^1_f(\mathbb{Q},T_p)\xrightarrow{\sim}\mathbb{Z}_p^{\widetilde{r}}$$

so that $A[\varpi]=0$. The tautological exact sequence $A[\varpi]\subset A\xrightarrow{\varpi} A\twoheadrightarrow A/\varpi$ gives also $A/\varpi=0$, so finally $A=0$ by Nakayama's lemma, as claimed.

Let us fix an organizing $\overline{R}$-module $\Phi=(\Phi,h)$ for $\mathbf{g}/\mathbb{Q}_\infty$, and a topological generator of $\sigma_{cy}\in \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, so that we can identify $\overline{R}\xrightarrow{\sim} R[[X]]$ and $\overline{I}=(X)$ with $X:=\sigma_{cy}-1$. Let us consider the following composition:

$$\vartheta: M(X)\xrightarrow{\sim}\mathcal{S}[X]\xrightarrow{\gamma}\mathcal{S}/X\xrightarrow{\sim}\mathcal{S}(X)\xrightarrow{\delta}\mathrm{Hom}_R\left(M(X),R\right), \tag{112}$$

where (with the notations introduced in Section 11.1) $\mathcal{S}(X):=\mathcal{S}(\overline{I})$, $M(X):=M(\overline{I})$ and $\mathcal{S}[X]$ denotes the X-torsion in $\mathcal{S}$. The morphisms are defined as follows: the first isomorphism comes from the connecting morphism attached to the following snake diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi & \xrightarrow{X} & \Phi & \longrightarrow & \Phi_X:=\Phi/X & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h_X} & & \\
0 & \longrightarrow & \Phi^* & \xrightarrow{X} & \Phi^* & \longrightarrow & \mathrm{Hom}_R\left(\Phi_X,R\right) & \longrightarrow & 0,
\end{array}$$

where $h_X:=h\otimes_{\overline{R}}\overline{R}/X=h\otimes_{\overline{R}}R$. The morphism γ is the natural map $\mathcal{S}[X]\hookrightarrow\mathcal{S}\twoheadrightarrow\mathcal{S}/X$. The second isomorphism comes from the definitions. Finally

$$\delta:\mathcal{S}(X)\xrightarrow{\mathrm{can}}\mathrm{Hom}_R\left(\mathrm{Hom}_R\left(\mathcal{S}(X),R\right),R\right)\xrightarrow{\sim}\mathrm{Hom}_R\left(M(X),R\right),$$

where the isomorphism is the R-dual of the first isomorphism in (96). Let us write

$$[-,-]_\vartheta: M(X)\otimes_R M(X)\longrightarrow R$$

for the bilinear form attached to (112). It follows immediately from the definitions that

$$\widetilde{h}_{\Phi_X,1,1}(m\otimes n)=[m,n]_\vartheta\cdot X \bmod \overline{I}^2\in\overline{I}/\overline{I}^2$$

for every $m,n\in M(X)$. Using Proposition 11.5 to identify $M(X)$ with $\widetilde{H}^1_f(\mathbb{Q},\mathbf{T})$ and correspondingly $\widetilde{h}_{\Phi_X,1,1}$ with $\widetilde{H}_{\mathbb{Q}_\infty,1,1}$, we conclude:

$$\det\left(\widetilde{H}_{\mathbb{Q}_\infty,1,1}\right)=\det\left([-,-]_\vartheta\right)\cdot X^{e_{\mathrm{gen}}}=\mathrm{char}_R\left(\mathrm{coker}\left(\vartheta\right)\right)\cdot X^{e_{\mathrm{gen}}}\in\left(\overline{I}^{e_{\mathrm{gen}}}/\overline{I}^{e_{\mathrm{gen}}+1}\right)/R^*, \tag{113}$$

where the determinants are computed on any R-basis of $\widetilde{H}^1_f(\mathbb{Q},\mathbf{T})\xrightarrow{\sim}M(X)$.

Before concluding the proof we recall some standard facts from the theory of finitely generated $R[[X]]$-modules. Let $f: M \to N$ be a morphism of finite R-modules, such that $\ker(f)$ and $\mathrm{coker}(f)$ are torsion R-modules. We write

$$\mathcal{H}(f) := \frac{\mathrm{char}_R\,(\mathrm{coker}(f))}{\mathrm{char}_R(\ker(f))} \in \mathrm{Frac}(R)/R^*.$$

If $f: M \to N$ and $g: N \to O$ are morphisms of the above type, we have

$$\mathcal{H}(g \circ f) = \mathcal{H}(f) \cdot \mathcal{H}(g). \tag{114}$$

Moreover, for every finite torsion $\overline{R} = R[[X]]$-module T with characteristic ideal $f_T(X) := \mathrm{char}_R(T)$, write $\gamma_T : T[X] \hookrightarrow T \twoheadrightarrow T/X$. Then we have:

1. $\mathrm{ord}_{X=0} f_T(X) \geq r_X := \mathrm{rank}_R\,(T/X)$;
2. The following properties are equivalent: *i)* $r_X = \mathrm{ord}_{X=0} f_T(X)$; *ii)* $\ker(\gamma_T)$ and $\mathrm{coker}\,(\gamma_T)$ are torsion R-modules; *iii)* the localization $T \otimes_{\overline{R}} \overline{R}_{\overline{I}}$ is a semi-simple $\overline{R}_{\overline{I}}$ module;
3. if the conditions in 2. are satisfied, then we have

$$\left.\frac{f_T(X)}{X^{r_X}}\right|_{X=0} \equiv \mathcal{H}(\gamma_T) \bmod R^*.$$

All this properties (which can be proved exactly as in the 'classical' case of finite torsion modules over $\mathbb{Z}_p[[X]]$) are proved in details in [**Pla97**, Sec. 4].

We can now easily conclude the proof. Assume first that $\mathcal{S}$ is semi-simple at the augmentation ideal $\overline{I}$, so that the kernel and cokernel of γ are finite, torsion R-modules. Since (by definition) $\ker(\delta)$ is pseudo-isomorphic to $\mathcal{S}(X)_{\mathrm{tors}} \stackrel{\sim}{\to} \widetilde{H}^2_f(\mathbb{Q}, \mathbf{T})_{\mathrm{tors}}$, using 3. above, (113), (114) and the freeness of $\widetilde{H}^1_f(\mathbb{Q}, \mathbf{T})$ we obtain: the characteristic ideal $\mathbf{L}_p(\mathbb{Q}_\infty, \mathbf{g})$ of $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty, \mathbf{T}) \stackrel{\sim}{\to} \mathcal{S}$ has the form:

$$\mathbf{L}_p(\mathbb{Q}_\infty, \mathbf{g}) \equiv \mathbf{L}_p^{\mathrm{gen}} \cdot X^{e_{\mathrm{gen}}} \bmod \overline{I}^{e_{\mathrm{gen}}+1}; \quad \mathbf{L}_p^{\mathrm{gen}} \in R - \{0\},$$

and we have the equality:

$$\det\left(\widetilde{H}_{\mathbb{Q}_\infty,1,1}\right) \cdot \mathrm{char}_R\left(\widetilde{H}^2_f(\mathbb{Q}, \mathbf{T})_{\mathrm{tors}}\right) = \mathbf{L}_p^{\mathrm{gen}} \cdot X^{e_{\mathrm{gen}}} \in \left(\overline{I}^{e_{\mathrm{gen}}}/\overline{I}^{e_{\mathrm{gen}}+1}\right)/R^*. \tag{115}$$

Retracing the definitions: $\mathcal{M}_2\left(\mathbf{L}_p^{\mathrm{gen}}\right) \cdot \log_p\left(\chi_{cy}(\sigma_{cy})\right)^{e_{\mathrm{gen}}} = \mathbf{L}_p^{\mathrm{gen}}(\mathbf{g}, k)$ and, up to multiplications by elements in R^*

$$\log_p(\chi_{cy}(\sigma_{cy}))^{e^{\mathrm{gen}}} \cdot \text{`}\mathcal{M}_{2,1}\text{'}\left(\det\left(\widetilde{H}_{\mathbb{Q}_\infty,1,1}\right)\right) = \det\left(\langle -,-\rangle_{\mathbf{T}}^{\mathrm{cycl}}\right) \in \mathcal{I}^{e_{\mathrm{gen}}}/\mathcal{I}^{e_{\mathrm{gen}}+1}.$$

It follows that applying $\mathcal{M}_{2,1}$ to the equality (115) we conclude the proof (when $\mathcal{S}$ is semi-simple at $\overline{I}$).

Finally, let us assume that $\mathcal{S}$ is not semi-simple at $\overline{I}$. Then $\mathbf{L}_p(\mathbb{Q}_\infty, \mathbf{g}) \in \left(X^{e_{\mathrm{gen}}+1}\right)$, so that $\mathbf{L}_p(\mathbf{g}, k, s)$ belongs to $(s-1)^{e_{\mathrm{gen}}+1} \cdot \mathcal{A}(U, \mathbb{Z}_p)$ and $\mathbf{L}_p^{\mathrm{gen}}(\mathbf{g}, k) \equiv 0$ vanishes identically. On the other hand the non-semi-simplicity of $\mathcal{S}$ at $\overline{I}$ implies that $\ker(\gamma) = \mathcal{S}[X] \cap X \cdot \mathcal{S}$ has positive rank over R (i.e. is non-zero in our case), so that $[-,-]_\vartheta$ has a non-trivial left (and right) radical. It follows by (113) that $\det\left(\widetilde{H}_{\mathbb{Q}_\infty,1,1}\right)$, and so $\det\left(\langle -,-\rangle_{\mathbf{T}}^{\mathrm{cycl}}\right)$ also vanishes. □

Corollary 12.17. *The following properties are equivalent:*

1. $\mathbf{L}_p^{\mathrm{gen}}(\mathbf{g}, k) \not\equiv 0$*;*
2. $\langle -,-\rangle_{\mathbf{T}}^{\mathrm{cycl}}$ *is non-degenerate;*
3. $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty, \mathbf{T})$ *is semi-simple at the augmentation ideal* $\overline{I}$*.*

Proof. This follows immediately by the preceding proof. □

Corollary 12.18. *Let* $\mathcal{U}^{\mathrm{wt}} := \mathrm{Im}\left(\widetilde{H}^1_f(\mathbb{Q}, \mathbf{T}) \to \widetilde{H}^1_f(\mathbb{Q}, T_p)\right) \subset \widetilde{H}^1_{f,\infty}$*, and assume that the following conditions are satisfied:*

i) $r_\infty = e_{\mathrm{gen}}$*;*

ii) the restriction $\left.\langle -,-\rangle_{\mathbb{Q},p}^{\mathrm{MTT}}\right|_{\mathcal{U}^{\mathrm{wt}} \times \mathcal{U}^{\mathrm{wt}}}$ *is non-degenerate.*

Then $\mathbf{L}_p(\mathbf{g},k)$ *has order of vanishing* $\widetilde{r}_{\rm gen} := \widetilde{r} - e_{\rm gen}$ *at* $k=2$ *and we have an equality:*

$$\frac{\mathbf{L}_p(\mathbf{g},k)}{(k-2)^{\widetilde{r}_{\rm gen}}}\bigg|_{k=2} = \left(\left[\widetilde{H}^1_{f,\infty} : \mathcal{U}^{\rm wt}\right]\right)^{-2} \cdot \#\left((\text{Ш}(E/\mathbb{Q})_{p^\infty})_{/\rm div}\right) \cdot \mathcal{R}^{\rm Nek}_{\mathbb{Q},p} \in \mathbb{Q}_p^*/\mathbb{Z}_p^*.$$

PROOF. Since $\widetilde{H}^1_f(\mathbb{Q},\mathbf{T})$ is a free R-module, by (89) the assumptions $r_\infty = e_{\rm gen}$ means that $\mathcal{U}^{\rm wt}$ has finite index in the radical $\widetilde{H}^1_{f,\infty}$ of $\langle -,-\rangle^{\rm Nek}_{\mathbb{Q},p}\Big|_{\widetilde{H}^1_f(T_p)\times\widetilde{H}^1_f(T_p)}$. Recalling the definitions, Prop. 12.9 and our assumptions give

$$\left[\widetilde{H}^1_{f,\infty} : \mathcal{U}^{\rm wt}\right]^2 \cdot \mathcal{R}^\infty_{\mathbb{Q},p} = \det\left(\langle -,-\rangle^{\rm cycl}_{\mathbf{T}}\right)\Big|_{k=2} \in \mathbb{Q}_p^*/\mathbb{Z}_p^*. \tag{116}$$

In particular $\mathcal{R}_{\mathbb{Q},p} \neq 0$ and combining Cor. 12.15 and Th. 12.16 we see that

$$\mathrm{ord}_{k=2}\mathbf{L}_p^{\rm gen}(\mathbf{g},k) = \widetilde{r}_{\rm gen} = \mathrm{ord}_{k=2}\mathbf{L}_p(\mathbf{g},k).$$

Finally, the formula in the statement follows taking the $\widetilde{r}_{\rm gen}$-th derivative of the equality in Th. 12.16, using Cor. 12.15 and (116). □

REMARK 12.19. Combining Prop. 10.9, the exact sequence (89) and the non-degenerancy of the localization at $\mathfrak{p}$ of the Cassels-Tate pairing we see easily that the following conditions are equivalent:

1. $r_\infty = e_{\rm gen}$;
2. $\widetilde{H}^2_f(\mathbb{Q},\mathbf{T}) \otimes_R R_\mathfrak{p}$ is a semi-simple $R_\mathfrak{p}$-module.

By the discussion in [**Ven12**], we expect that this conditions should always be satisfied, i.e. that $\widetilde{h}^{\rm wt}_{\mathbb{Q},1,1}$ is 'as non-degenerate as possible'. If $\widetilde{H}^2_f(\mathbb{Q},\mathbf{T})$ turns out to be non-semisimple at $\mathfrak{p}$, we have to consider derived regulators attached to higher '$\mathfrak{p}$-graded quotients' of $\widetilde{H}^1_f(\mathbb{Q},T_p)$ [**Nek06**, Sec. 12.7] in order to obtain generalizations of Cor. 12.15 and Cor. 12.18 (cfr. [**BD95**]).

REMARK 12.20. The preceding Remark and Rem. 12.6 lead us to expect that $r_\infty = e_{\rm gen} \in \{0,1\}$. Moreover Schneider conjecture on the non-degenerancy of the cyclotomic p-adic height [**Sch82**],[**MTT86**] suggests that $\mathcal{R}^\infty_{\mathbb{Q},p}$, and so $\mathcal{R}_{\mathbb{Q},p}$ should be non-zero. By the preceding proof, this would imply that $\det\left(\langle -,-\rangle^{\rm cycl}_{\mathbf{T}}\right)$ does not vanish at $k=2$, so in particular, by Theorem 12.16: $\mathbf{L}_p^{\rm gen}(\mathbf{g},k) \not\equiv 0$ has order of vanishing $\widetilde{r}^{\rm gen}$ at $k=2$. (For a discussion of this topic and some examples, we refer the reader to [**Ven12**, Sec. 5] and Section 12.6 below.*)*

12.5. Relations with the Mazur-Kitagawa p-adic L-function. We now consider the analytic side of the matter, i.e. the Mazur-Kitagawa p-adic L-function of R.

12.5.1. *Analytic p-adic L-functions.* Under our Hypothesis (**Irr**), Sec. 3.4 of [**EPW06**] (working on ideas of Mazur and Kitagawa [**Kit94**]) attaches to R an element

$$L_p^{\rm MK}(\mathbf{g}) \in \overline{R}/R^*$$

interpolating the Mazur-Tate-Teitelbaum p-adic L-functions attached to the elements of $\mathbf{g}$. Here we write $L_p^{\rm MK}(\mathbf{g})$ to denote the projection in $\overline{R}$ of any L-function denoted $L(\mathfrak{m},N,1) \in \mathfrak{h}^{\rm ord}_{\infty,\mathfrak{m}}[[\mathcal{G}]]$ in *loc. cit.* (where $\mathfrak{m}$ is as in Section 9.2 and 1 in the argument of the p-adic L-function stands for the trivial character).

More precisely, $L_p^{\rm MK}(\mathbf{g})$ satisfies the following interpolation property. Given an arithmetic map $\psi = \psi_\mathfrak{q} \in \mathcal{X}^{\rm arith}(R)$ let us write $\mathcal{O}_\mathfrak{q} := \psi_\mathfrak{q}(R)$ and $\overline{\psi}_\mathfrak{q} : \overline{R} \to \mathcal{O}_\mathfrak{q}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]] =: \overline{\mathcal{O}_\mathfrak{q}}$ for the morphism induced by the arithmetic map $\psi_\mathfrak{q}$. Then there exists $\alpha \in R^*$ such that

$$\overline{\psi}_\mathfrak{q}\left(L_p^{\rm MK}(\mathbf{g})\right) = \psi_\mathfrak{q}(\alpha) \cdot L^{\rm MTT}_{p,\Omega_\mathfrak{q}}(\mathbf{g}_\mathfrak{q}) \in \overline{\mathcal{O}_\mathfrak{q}} \tag{117}$$

for every arithmetic map $\psi_\mathfrak{q} \in \mathcal{X}^{\rm arith}(R)$. Here $\Omega_\mathfrak{q} = \Omega^+_{\mathbf{g}_\mathfrak{q}} \in \mathbb{C}$ is a certain (fixed) 'canonical' Shimura period for $\mathbf{g}_\mathfrak{q}$ (see [**EPW06**, Sec. 3.1]) and $L^{\rm MTT}_{p,\Omega_\mathfrak{q}}(\mathbf{g}_\mathfrak{q})$ is the Mazur-Tate-Teitelbaum p-adic L-function attached in [**MTT86**] to $\mathbf{g}_\mathfrak{q}$ (and the unique 'allowable p-root' $\mathbf{a}_p(\mathfrak{q}) = a_p(\mathbf{g}_\mathfrak{q})$), normalized with respect to $\Omega_\mathfrak{q}$.

The power series $L^{\mathrm{MTT}}_{p,\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}})$ is characterized by the following interpolation property. For every even Dirichlet character ψ of conductor c_ψ, and every $0 < s_0 < \mathrm{weight}(\mathfrak{q})$, define

$$L^{\mathrm{alg}}_{\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}}, \psi, s_0) := \frac{c_\psi^{s_0-1} \cdot (s_0-1)! \cdot \tau(\psi)}{(2\pi i)^{s_0-1} \cdot \Omega_{\mathfrak{q}}} \cdot L(\mathbf{g}_{\mathfrak{q}}, \overline{\psi}, s_0) \in \mathrm{Frac}\left(\mathcal{O}_{\mathfrak{q}}\right),$$

where $\tau(\psi)$ denotes the Gauss sum and $L(\mathbf{g}_{\mathfrak{q}}, \overline{\psi}, s)$ is the Hecke complex L-function of $\mathbf{g}_{\mathfrak{q}}$ twisted by $\overline{\psi}$. Let us identify $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \overset{\sim}{\to} 1 + p\mathbb{Z}_p$ under the p-adic cyclotomic character χ_{cy}. Let $u \in 1 + p\mathbb{Z}_p$ be a topological generator, so that $\overline{\mathcal{O}_{\mathfrak{q}}} = \mathcal{O}_{\mathfrak{q}}[[X]]$ with $X := u - 1$, and write $L^{\mathrm{MTT}}_{p,\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}}, X) := L^{\mathrm{MTT}}_{p,\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}})$. Then: for every Dirichlet character $\eta : \mathbb{Z}_p^* \twoheadrightarrow 1 + p\mathbb{Z}_p^* \to \overline{\mathbb{Q}}_p^*$ of conductor p^m $(m \geq 0)$ and every integer $0 < s_0 < \mathrm{weight}(\mathfrak{q})$:

$$\begin{aligned} \eta\chi_{cy}^{s_0-1}\left(L^{\mathrm{MTT}}_{p,\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}})\right) &:= L^{\mathrm{MTT}}_{p,\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}}, \eta(u) \cdot u^{s_0-1} - 1) \\ &= \mathbf{a}_p(\mathfrak{q})^{-m} \cdot \left(1 - \frac{\eta\omega^{1-s_0}(p) \cdot p^{s_0-1}}{\mathbf{a}_p(\mathfrak{q})}\right) \cdot L^{\mathrm{alg}}_{\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}}, \eta\omega^{1-s_0}, s_0), \end{aligned} \tag{118}$$

where ω is the Teichmüller character. (The Weierstrass preparation theorem immediately implies that $L^{\mathrm{MTT}}_{p,\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}})$ is determined by these values.) From now on we write simply

$$L_p(\mathbf{g}_{\mathfrak{q}}) := L^{\mathrm{MTT}}_{p,\Omega_{\mathfrak{q}}}(\mathbf{g}_{\mathfrak{q}}).$$

As follows by the results in [**GV00**, Sec. 3] (again under Hyp. 1) we can choose $\Omega_{\mathfrak{p}} = \Omega_E$ as the real Nerón period of $E/\mathbb{Q}$, i.e. the complex period appearing in the classical Birch and Swinnerton-Dyer conjecture. (We recall that $g = \mathbf{g}_{\mathfrak{p}}$ is the p-stabilization of f_E.) Here we insist to make this choice for $\Omega_{\mathfrak{p}}$, and to normalize $L_p^{\mathrm{MK}}(\mathbf{g})$ in such a way that

$$\overline{\psi}_{\mathfrak{p}}\left(L_p^{\mathrm{MK}}(\mathbf{g})\right) = L_p(E) := L^{\mathrm{MTT}}_{p,\Omega_E}(\mathbf{g}_{\mathfrak{p}}). \tag{119}$$

Then $L_p^{\mathrm{MK}}(\mathbf{g})$ is a well-defined element of $\overline{R}$ up to multiplication by a unit $\alpha \equiv 1 \bmod \mathfrak{p}$ of R.

12.5.2. *Two-variable main conjecture.* Let us write

$$L_p(\mathbf{g}, k, s) := \mathcal{M}_{2,1}\left(L_p^{\mathrm{MK}}(\mathbf{g})\right) \in \mathcal{A}(U, \mathbb{Z}_p)$$

as the Mellin transform of any $L_p^{\mathrm{MK}}(\mathbf{g})$. By the discussion above, this is a well-defined element of $\mathcal{A}(U, \mathbb{Z}_p)$ up to multiplication by a unit $\alpha \in R^*$ such that $\alpha(2) = \psi_{\mathfrak{p}}(\alpha) = 1$.

CONJECTURE 12.21. $L_p(\mathbf{g}, k, s + k/2 - 1) = \mathbf{L}_p(\mathbf{g}, k, s) \in \mathcal{A}(U, \mathbb{Z}_p)/\overline{R}^*$.

In the preceding Conjecture, the 'twist' $s \mapsto s + \frac{k}{2} - 1$ in the cyclotomic variable takes care of the twist $\mathbf{T} \mapsto \mathbf{T} := \mathbf{T} \otimes_R \Theta_R^{-1}$ in the Hida deformation, since (as discussed in Sec. 9.3.3) this produces a twist by $\mathbb{Z}_p(1 - k/2)$ in the specialization of $\mathbf{T}$ at an arithmetic point of weight k. We can also reformulate the conjecture as follows. Let us write again $\Theta_R : \overline{R} \to R$ for the morphism of R-algebras induced by Θ_R. We can lift Θ_R to a morphism $\overline{\Theta}_R : \overline{R} \to \overline{R}$ such that $\varepsilon_R \circ \overline{\Theta}_R = \Theta_R$, defining

$$\overline{\Theta}_R\left(\sum_{j=0}^{\infty} r_j \cdot (\sigma_{cy} - 1)^j\right) := \sum_{j=0}^{\infty} r_j \cdot (\Theta_R(\sigma_{cy}) \cdot \sigma_{cy} - 1)^j$$

(where as usual $\sigma_{cy} \in \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ is a topological generator). We can reformulate the preceding conjecture as follows.

CONJECTURE 12.22. $\overline{\Theta}_R\left(L_p^{\mathrm{MK}}(\mathbf{g})\right)$ *generates the characteristic ideal of* $\widetilde{H}^2_{f,\mathrm{Iw}}(\mathbb{Q}_\infty, \mathbf{T})$.

12.5.3. *Functional equations.* As shown in [**How07**, Prop. 2.3.6], given $\mathfrak{q} \in \mathcal{X}^{\text{arith}}(R)$, the functional equation studied in [**MTT86**] reads in our case:

(120) $$\chi\left(L_p(\mathbf{g}_\mathfrak{q})\right) = w(\mathbf{g}) \cdot \chi^{-1}\Theta_\mathfrak{q}(\langle N \rangle) \cdot \chi^{-1}\Theta_\mathfrak{q}^2\left(L_p(\mathbf{g}_\mathfrak{q})\right)$$

for every continuous character $\chi : \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \to \overline{\mathbb{Q}}_p^*$, where $\langle N \rangle \in 1 + p\mathbb{Z}_p$ is the projection of N to principal units and

$$\Theta_\mathfrak{q} : G_\mathbb{Q} \overset{\Theta_R}{\to} R^* \overset{\psi_\mathfrak{q}}{\to} \overline{\mathbb{Q}}_p^*.$$

(Recall that Θ_R factorizes through $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, since $\Delta = (\mathbb{Z}/Np\mathbb{Z})^*$ acts trivially on R via the diamond morphism.) Here $w(\mathbf{g}) \in \pm$, which is independent on the arithmetic point $\mathfrak{q}$, is the *sign of the Hida family* alluded to in Rem. 12.6. It equals minus the eigenvalue of the Atkin-Lehner operator w_N acting on f_E. Writing $\mathrm{sign}(E/\mathbb{Q})$ to be the sign in the functional equation satisfied by the Hecke L-series $L(f_E, s) = L(E/\mathbb{Q}, s)$ at $s = 1$, we also have:

$$w(\mathbf{g}) = (-1)^{m_p(E)} \cdot \mathrm{sign}(E/\mathbb{Q}); \quad m_p(E) := \begin{cases} 1 & \text{if} \quad a_p(E) = 1; \\ 0 & \quad \text{otherwise.} \end{cases}$$

In other words, $w(\mathbf{g})$ differs by $\mathrm{sign}(E/\mathbb{Q})$ if and only if $E/\mathbb{Q}$ has split multiplicative reduction.

Let $\kappa \in U \cap \mathbb{Z}^{\geq 2}$ be an even integer, and define the arithmetic point of weight κ and trivial character

$$\mathfrak{q}_\kappa := \ker\left(\psi_\kappa : R \overset{\mathcal{M}_2}{\to} \mathcal{A}(U) \overset{f \mapsto f(\kappa)}{\to} \mathbb{Z}_p\right) \in \mathcal{X}^{\text{arith}}(R); \quad g_\kappa := \mathbf{g}_{\mathfrak{q}_\kappa}.$$

It follows by (117) that $L_p(\mathbf{g}, \kappa, s) = \alpha(\kappa) \cdot \chi_{cy}^{s-1}\left(L_p(g_\kappa)\right)$. As $\Theta_{\mathfrak{q}_\kappa}(g) = \chi_{cy}(g)^{\kappa/2-1}$, taking $\mathfrak{q} = \mathfrak{q}_\kappa$ and $\chi = \chi_{cy}^{s-1}$ ($s \in \mathbb{Z}_p$) in equation (120) we then obtain

$$L_p(\mathbf{g}, \kappa, s) = w(\mathbf{g}) \cdot \langle N \rangle^{\kappa/2-s} \cdot L_p(\mathbf{g}, \kappa, \kappa - s).$$

It follows that, writing $\Lambda_p(\mathbf{g}, k, s) := \langle N \rangle^{s/2} \cdot L_p(\mathbf{g}, k, s)$, we have a functional equation

(121) $$\Lambda_p(\mathbf{g}, k, s) \equiv w(\mathbf{g}) \cdot \Lambda_p(\mathbf{g}, k, k - s).$$

By the description of $w(\mathbf{g})$ given above, we see that the classical Birch and Swinnerton-Dyer conjecture predicts: $w(\mathbf{g}) \overset{?}{=} (-1)^{\mathrm{rank}_{\mathbb{Z}_p} E^\dagger(\mathbb{Q})}$. In any case, recalling that $\mathrm{rank}_R \widetilde{H}^1_f(\mathbb{Q}, \mathbf{T})$ has the same parity as $\mathrm{rank}_{\mathbb{Z}_p} \widetilde{H}^1_f(\mathbb{Q}, T_p)$ (by Prop. 11.7), the parity conjecture proved by Nekovář in Section 12 of [**Nek06**] gives:

$$w(\mathbf{g}) = (-1)^{e_{\text{gen}}}.$$

Then (121) is consistent, via Conjecture 12.21 with the functional equation (Prop. 12.5) satisfied by $\mathbf{L}_p(\mathbf{g}, k, s)$.

12.5.4. *Weight-variable main conjecture.* By (121) we see that $w(\mathbf{g}) = -1$ implies that $L_p(\mathbf{g}, k, s)$ vanishes on the central critical line $s = k/2$, so that it is divisible by $(s - k/2)$. We define the generic restriction of $L_p(\mathbf{g}, k, s)$ to the central critical line as the analytic function on U:

$$L_p^{\text{gen}}(\mathbf{g}, k) := \left.\frac{L_p(\mathbf{g}, k, s)}{(s - k/2)^{e(\mathbf{g})}}\right|_{s=k/2} \in \mathcal{A}(U); \quad e(\mathbf{g}) := \begin{cases} 1 & \text{if} \quad w(\mathbf{g}) = -1; \\ 0 & \text{if} \quad w(\mathbf{g}) = +1. \end{cases}$$

Then Greenberg Conjecture [**Gre94a**] and the conjectural equality of the order of vanishing of $L(g_\kappa, k)$ and $L_p(g_\kappa, k)$ at $s = \kappa/2$ for $\kappa > 2$ (with the notations introduced above) coming from Bloch-Kato conjectures predict that $L_p^{\text{gen}}(\mathbf{g}, k)$ is not identically zero. Moreover, since $\widetilde{H}^1_f(\mathbb{Q}, \mathbf{T})$ 'interpolates' the Bloch-Kato Selmer groups attached to lattices in the Deligne representations $\{V_\mathfrak{q}\}_{\mathfrak{q} \in \mathcal{X}^{\text{arith}}(R)}$ of Sec. 9.3.3, again by Bloch-Kato conjecture we expect the equality:

$$e(\mathbf{g}) \overset{?}{=} e_{\text{gen}}.$$

(We refer the reader to [**NP00**] for more details.) As a direct consequence of Conjecture 12.21 we are then lead to the following

CONJECTURE 12.23. $L_p^{\mathrm{gen}}(\mathbf{g},k)=\mathbf{L}_p^{\mathrm{gen}}(\mathbf{g},k)\in\mathcal{A}(U)/R^*$.

12.5.5. *Relations with p-adic BSD conjectures.* Combining Cor. 12.15, Rem. 12.19, and the Tate-Shafarevich conjecture, the preceding Conjecture leads us to expect the following equality:

$$\left.\frac{L_p^{\mathrm{gen}}(\mathbf{g},k)}{(k-2)^{\widetilde{r}_{\mathrm{gen}}}}\right|_{k=2}\overset{?}{\equiv}\#\left(\text{Ш}(E/\mathbb{Q})\right)\cdot\mathcal{R}_{\mathbb{Q},p}\bmod\mathbb{Z}_p^*,\tag{122}$$

where $\widetilde{r}_{\mathrm{gen}}:=\mathrm{rank}_{\mathbb{Z}}E^\dagger(\mathbb{Q})-e(\mathbf{g})$.

We note that, under our normalization (119), the leading coefficient of $L_p^{\mathrm{gen}}(\mathbf{g},k)$ at $k=2$ is a well-defined element of $\mathbb{Q}_p$ (i.e. does not depend on the choice of $L_p^{\mathrm{MK}}(\mathbf{g})\in\overline{R}/R^*$). Moreover, assuming the finiteness of $\text{Ш}(E/\mathbb{Q})$, we can as well define $\mathcal{R}_{\mathbb{Q},p}$ as an element of $\mathbb{Q}_p$ (see Rem. 12.12). In [**Ven12**] the following more precise conjecture is proposed (cfr. Rem. 12.12 and Rem. 12.20):

1. $\mathrm{ord}_{k=2}L_p^{\mathrm{gen}}(\mathbf{g},k)\overset{?}{=}\widetilde{r}_{\mathrm{gen}}$;
2. the leading coefficient of $L_p^{\mathrm{gen}}(\mathbf{g},k)$ at $k=2$ is given by:

$$\left.\frac{L_p^{\mathrm{gen}}(\mathbf{g},k)}{(k-2)^{\widetilde{r}_{\mathrm{gen}}}}\right|_{k=2}\overset{?}{\doteq}\mathcal{E}_p\cdot\mathbf{BSD}(E/\mathbb{Q})\cdot\mathcal{R}_{\mathbb{Q},p}\in\mathbb{Q}_p^*.\tag{123}$$

Here $\mathbf{BSD}(E/\mathbb{Q})$ is the p-part of the algebraic factor appearing in the R.H.S. of the classical Birch and Swinnerton-Dyer conjecture, i.e.

$$\mathbf{BSD}(E/\mathbb{Q}):=\frac{\#\left(\text{Ш}(E/\mathbb{Q})\right)\cdot\prod_{\ell\neq\infty}c_\ell(E/\mathbb{Q})}{\#\left(E(\mathbb{Q})_{\mathrm{tors}}\right)^2},$$

where $c_\ell(E/\mathbb{Q})=[E(\mathbb{Q}_\ell):E_0(\mathbb{Q}_\ell)]$ is the Tamagawa factor of $E/\mathbb{Q}_\ell$ (see [**Sil86**, Ch. VII]). Moreover the 'Euler factor' $\mathcal{E}_p$ satisfies (see [**Ven12**, Sec. 6.1])

$$\mathcal{E}_p\doteq(1-\alpha_p)^2\ \left(\text{resp., }1,\ \mathrm{ord}_p(q_E)^{-1}=c_p(E/\mathbb{Q}_p)^{-1}\right)$$

if $E/\mathbb{Q}_p$ has good (resp., non-split multiplicative, split multiplicative) reduction, where $\doteq$ denotes equality up to some power of 2 (and $\alpha_p=\mathbf{a}_p(\mathfrak{p})$ is the p-adic unit defined in Sec. 1). Using Hyp. (**Irr**) (resp., Hyp. (**Fro**), Hyp. (**Tam**)) we see that $E(\mathbb{Q})_{\mathrm{tor}}$ (resp., $(1-\alpha_p)$, $c_\ell(E/\mathbb{Q}_\ell)$ for $\ell\neq p$) is a p-adic unit. As $c_p(E/\mathbb{Q})\leq 4$ if $E/\mathbb{Q}_p$ has not split multiplicative reduction [**Sil86**, pag. 359] (and $p\geq 5$ by assumption), it follows

$$\#\left(\text{Ш}(E/\mathbb{Q})\right)\equiv\mathcal{E}_p\cdot\mathbf{BSD}(E/\mathbb{Q})\bmod\mathbb{Z}_p^*,$$

so that (122) and (123) are consistent with each others.

12.6. Exceptional-zero formulas. We assume in this section that $E/\mathbb{Q}$ is exceptional at p, i.e. that $E/\mathbb{Q}_p$ has split multiplicative reduction. Let $q_E\in p\mathbb{Z}_p$ be the Tate period of $E/\mathbb{Q}_p$, and let

$$\log_E:E(\mathbb{Q}_p)\overset{\Phi_{Tate}^{-1}}{\to}\mathbb{Q}_p^*\overset{\log_{q_E}}{\to}\mathbb{Q}_p$$

be the formal group logarithm on $E/\mathbb{Q}_p$. In light of the main conjectures of the preceding Sections, the following theorem is an algebraic manifestation of the 'exceptional-zero formulas' proved by Greenberg and Stevens in [**GS93**] and by Bertolini and Darmon in [**BD07**].

THEOREM 12.24. *Assume that $E/\mathbb{Q}_p$ has split multiplicative reduction and that $\text{Ш}(E/\mathbb{Q})_{p^\infty}$ is finite.*

1. *If $E(\mathbb{Q})$ is finite, $\mathrm{ord}_{s=1}\mathbf{L}_p(E,s)=1$ and we have equalities in $\mathbb{Q}_p^*/\mathbb{Z}_p^*$:*

$$\mathbf{L}_p^{\mathrm{gen}}(\mathbf{g},2)=\#\left(\text{Ш}(E/\mathbb{Q})_{p^\infty}\right)\cdot\log_p(q_E)=\frac{d}{ds}\mathbf{L}_p(E,s)_{s=1}.$$

2. *If $\mathrm{rank}_{\mathbb{Z}}E(\mathbb{Q})=1$, then $\mathbf{L}_p^{\mathrm{gen}}(\mathbf{g},k)\equiv\mathbf{L}_p(\mathbf{g},k)$ has order of vanishing 2 at $k=2$, and we have an equality in $\mathbb{Q}_p^*/\mathbb{Z}_p^*$:*

$$\frac{d^2}{dk^2}\mathbf{L}_p(\mathbf{g},k)_{k=2}=2\cdot\#\left(\text{Ш}(E/\mathbb{Q})_{p^\infty}\right)\cdot\log_E(\mathbf{P})^2,$$

where $\mathbf{P}$ is any generator of $E(\mathbb{Q})$ modulo torsion.

PROOF. As $Ш(E/\mathbb{Q})_{p^\infty}$ is finite, we have an isomorphism $i_E^\dagger : E^\dagger(\mathbb{Q}) \otimes \mathbb{Z}_p \xrightarrow{\sim} \widetilde{H}_f^1(\mathbb{Q}, T_p)$, under which we identify these $\mathbb{Z}_p$-modules (see Remarks 12.10, 12.11 and 12.12). Let $q_E \in p\mathbb{Z}_p$ be the p-adic Tate period attached to $E_{/\mathbb{Q}_p}$ (see Section 9.3.4).

If $E(\mathbb{Q})$ is finite, then $\widetilde{H}_f^1(\mathbb{Q}, T_p) = q_E^{\mathbb{Z}} \otimes \mathbb{Z}_p$. Moreover, as $\mathrm{rank}_R \widetilde{H}_f^1(\mathbb{Q}, \mathbf{T}) \equiv \mathrm{rank}_{\mathbb{Z}_p} \widetilde{H}_f^1(\mathbb{Q}, T_p) \bmod 2$ by Prop. 11.7 and $\widetilde{H}_f^1(\mathbb{Q}, \mathbf{T})$ is a free R-module, Corollary 10.4 gives us:

$$\widetilde{H}_f^1(\mathbb{Q}, \mathbf{T}) \xrightarrow{\sim} R; \quad \widetilde{H}_f^1(\mathbb{Q}, T_p) = \widetilde{H}_{f,\infty}^1; \quad \mathscr{U}^{\mathrm{wt}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \widetilde{H}_f^1(\mathbb{Q}, V_p)$$

(with $\mathscr{U}^{\mathrm{wt}}$ is as in Cor. 12.18). In particular we have $\widetilde{r} = r_\infty = e_{\mathrm{gen}} = 1$ and $\widetilde{r}_{\mathrm{gen}} = 0$ with the notations above. By [**Nek06**, Theorem 11.3.9] (or Lemma 10.7 in Part 2) we we obtain:

$$\mathcal{R}_{\mathbb{Q},p}^\infty = \mathcal{R}_{\mathbb{Q},p} = \det\left(\langle -, - \rangle_{\mathbb{Q},p}^{\mathrm{MTT}}\right) = \langle q_E, q_E \rangle_{\mathbb{Q},p}^{\mathrm{MTT}} = \log_p(q_E).$$

Finally, thanks to the proof of Manin conjecture given in [**BSDGP96**], we know that $\log_p(q_E) \neq 0$. Combined with Cor. 12.14 and Cor. 12.15, this proves 1.

Let us now assume that $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1$, and let us fix a generator $\mathbf{P}$ of $E(\mathbb{Q})/\mathrm{tor}$. Then, writing $\widetilde{\mathbf{P}} = (\mathbf{P}, y_{\mathbf{P}}) \in E^\dagger(\mathbb{Q})$ for a lift of $\mathbf{P}$, $\widetilde{H}_f^1(\mathbb{Q}, T_p) = \mathbb{Z}_p \cdot q_E \oplus \mathbb{Z}_p \cdot \widetilde{\mathbf{P}}$ (recall that by assumption $E(K)_p = 0$). Combining Remark 12.10 with the computations carried out in Section 4.4 of Part 1 we have:

$$\left\langle q_E, \widetilde{\mathbf{P}} \right\rangle_{\mathbb{Q},p}^{\mathrm{Nek}} = \frac{1}{2} \cdot \log_{q_E}(y_{\mathbf{P}}) = \frac{1}{2} \cdot \log_E(\mathbf{P}) \in \mathbb{Q}_p^*.$$

In particular $\widetilde{H}_{f,\infty}^1 = 0 = \mathscr{U}^{\mathrm{wt}}$, so that $r_\infty = e_{\mathrm{gen}} = 0$, and $\widetilde{r} = 2 = \widetilde{r}_{\mathrm{gen}}$. Moreover the skew-symmetricity of $\langle -, - \rangle_{\mathbb{Q},p}^{\mathrm{Nek}}$ gives:

$$\mathcal{R}_{\mathbb{Q},p} = \mathcal{R}_{\mathbb{Q},p}^{\mathrm{Nek}} = \det \begin{pmatrix} 0 & \frac{1}{2} \cdot \log_E(\mathbf{P}) \\ -\frac{1}{2} \cdot \log_E(\mathbf{P}) & 0 \end{pmatrix} = \log_E(\mathbf{P})^2 \in \mathbb{Q}_p^*/\mathbb{Z}_p^*.$$

Together with Cor. 12.18 this proves 2., and with it the proposition. □

12.6.1. **Example : $\mathbf{X_0(11)}$ at $\boldsymbol{p = 11}$.** We close this section giving the simplest 'exceptional example'.

Let $p = 11$ and let us consider the elliptic curve

$$X_0(11) : y^2 + y = x^3 - x^2 - 10x - 20,$$

which is the curve denoted $11A1$ in Cremona's tables. $X_0(11)$ has split multiplicative reduction at $p = 11$ and $E(\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}/5\mathbb{Z}$, so it satisfies Hypothesis 3. Moreover there exists no $\mathbb{Q}$-rational 11-isogeny defined on $X_0(11)$, so that Hypothesys 1 is satisfied. Finally, Hypothesis 2 is satisfied with $R = \mathbb{Z}_{11}[[1 + p\mathbb{Z}_{11}]]$, since we know that $S_2(\Gamma_1(11), \mathbb{C})$ is one-dimensional, generated by $f_{X_0(11)} = g$ (see [**Hid86a**, Cor. 1.3]).

Write q for the 11-adic Tate period of $X_0(11)_{/\mathbb{Q}_{11}}$ and $\mathbf{g}$ for the Hida family attached to $X_0(11)$ at $p = 11$, such that $\mathbf{g}_{\mathfrak{p}} = f_{X_0(11)}$. The computations in [**MTT86**, Sec. 13, Ch. II] tell us: $q = 11^5 \cdot u$ and $\mathscr{L}_{11}(E) := \log_{11}(q)/\mathrm{ord}_{11}(q) = 11 \cdot v$, with $u, v \in \mathbb{Z}_{11}^*$. Moreover by the work of Kolyvagin it is known that $Sel(\mathbb{Q}, X_0(11)_{11^\infty}) = 0$, so that the preceding theorem gives:

$$\mathbf{L}_{11}^{\mathrm{gen}}(\mathbf{g}, 2) = \log_{11}(q) = 11 \cdot u; \quad u \in \mathbb{Z}_{11}^*. \tag{124}$$

Fix a topological generator γ of $1 + 11\mathbb{Z}_{11}$, and a topological generator $\sigma_{cy} \in \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ such that $\chi_{cy}(\sigma_{cy}) = \gamma$. Then we have $\overline{R} = \mathbb{Z}_{11}[[X, Y]]$, where $X := [\gamma] - 1 \in R$ is a generator of $\mathfrak{p}$ and $Y := \sigma_{cy} - 1$ generates the augmentation ideal $\overline{I}$ of $\overline{R}$. Let us write

$$\mathbf{L}_{11}(\mathbb{Q}_\infty, \mathbf{g}) \equiv Y \cdot \mathbf{L}_{11}^{\mathrm{gen}} \bmod \overline{I}^2; \quad \mathbf{L}_{11}^{\mathrm{gen}} \in R.$$

Then (recalling the definitions) (124) gives us $\log_p(\gamma) \cdot \psi_{\mathfrak{p}}(\mathbf{L}_{11}^{\mathrm{gen}}) = 11 \cdot u$, i.e.

$$\mathbf{L}_{11}^{\mathrm{gen}} \in R^*.$$

It follows $\mathbf{L}_{11}(\mathbb{Q}_\infty, \mathbf{g}) = Y \in \overline{R}/\overline{R}^*$, so that

$$\mathbf{L}_{11}(\mathbf{g}, k, s) \equiv \left(\gamma^{s-1} - 1\right) \bmod \overline{R}^*; \quad \mathbf{L}_{11}(X_0(11), s) \equiv \left(\gamma^{s-1} - 1\right) \bmod \overline{\mathbb{Z}_{11}}^*.$$

Moreover, using Th. 12.16 (and the proof of Cor. 12.18): $\mathbf{L}_{11}^{\mathrm{gen}}(\mathbf{g}, k) \equiv 11 \equiv \mathbf{L}_{11}(\mathbf{g}, k) \bmod R^*$.

On the analytic side of the matter, the Mazur-Kitagawa 11-adic L-function attached to $X_0(11)$ is computed in [**EPW06**, Sec. 5.3], where it is shown that

$$L_{11}^{\mathrm{MK}}(\mathbf{g}) \equiv \sigma_{cy} - \Theta_R(\sigma_{cy}) \bmod \overline{R}^*.$$

(Note that [**EPW06**] uses a different normalization for the diamond morphism, so that their $\langle\gamma\rangle_p^{1/2} \cdot \gamma^{-1}$ is our $\Theta(\sigma_{cy})$‘ $= [\gamma]^{1/2}$’.) We immediately deduce:

$$L_{11}(\mathbf{g}, k, s) \equiv \left(\gamma^{s-k/2} - 1\right) \bmod \overline{R}^*; \quad L_{11}(\mathbf{g}, k, s + k/2 - 1) \equiv \mathbf{L}_{11}(\mathbf{g}, k, s) \bmod \overline{R}^*,$$

i.e. Conjecture 12.21 holds for $(E_{/\mathbb{Q}}, p) = (X_0(11)_{/\mathbb{Q}}, 11)$.

Part 3

A note on Kato zeta elements and exceptional zero formulas

Introduction

We fix in this note an elliptic curve $A_{/\mathbb{Q}}$ having an odd prime p of *split* multiplicative reduction. For every integer n we write $\mathbf{Q}_n \subset \mathbb{Q}(\boldsymbol{\mu}_{p^{n+1}})$ for the sub-field of degree p^n over $\mathbb{Q}$ and $\mathbf{Q}_\infty := \bigcup_{n\in\mathbb{N}} \mathbf{Q}_n$ for the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let $\Phi_\infty = \bigcup_{n\in\mathbb{N}} \Phi_n$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$, so that Φ_n is the completion of $\mathbf{Q}_n$ at (the unique prime above) p. We will identify the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(\mathbf{Q}_\infty/\mathbb{Q})]]$ with $\mathbb{Z}_p[[\mathrm{Gal}(\Phi_\infty/\mathbb{Q}_p)]]$. Let T be a finite $\mathbb{Z}_p$-module equipped with a continuous linear action of $G = G_\mathbb{Q}$ (resp., $G_{\mathbb{Q}_p}$), and write $F_n = \mathbf{Q}_n$ (resp., Φ_n) for every $0 \le n \le \infty$. Then we write $H^1_{\mathrm{Iw}}(F_\infty, T\otimes\mathbb{Q}) := \left(\varprojlim_{n\in\mathbb{N}} H^1(F_n,T)\right)\otimes\mathbb{Q}$, the inverse limit being taken with respect to the corestriction maps.

Tate's theory **[Tat95]** gives a p-adic analytic isomorphism: $\Phi_{\mathrm{Tate}} : \left(\mathbb{G}_m/q_A^{\mathbb{Z}}\right)_{/\mathbb{Q}_p} \cong A_{/\mathbb{Q}_p}$, where $q_A \in p\mathbb{Z}_p$ is the *Tate period* of $A_{/\mathbb{Q}_p}$. Since q_A has positive valuation, identifying $A(\overline{\mathbb{Q}})_{p^n} \cong \left(\overline{\mathbb{Q}}_p^*/q_A^{\mathbb{Z}}\right)_{p^n}$ as $G_{\mathbb{Q}_p}$-modules via Φ_{Tate}, we obtain a surjective morphism of $G_{\mathbb{Q}_p}$-modules $\pi_{q_A} : A(\overline{\mathbb{Q}})_{p^n} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$, defined by $\pi_{q_A}(x \bmod q_A^{\mathbb{Z}}) := \frac{p^n\cdot \mathrm{ord}_p(x)}{\mathrm{ord}_p(q_A)} \bmod p^n$. This induces on p-adic Tate modules a surjective morphism of $\mathbb{Z}_p[G_{\mathbb{Q}_p}]$-modules: $\mathrm{Ta}_p(A) \twoheadrightarrow \mathbb{Z}_p$. Composed with restriction from $G_\mathbb{Q}$ to $G_{\mathbb{Q}_p}$ this induces *residue maps*:

$$\partial_{p,n} : H^1(\mathbf{Q}_n, V_p(A)) \to H^1(\Phi_n,\mathbb{Q}_p); \quad \partial_{p,\infty} := \lim_{n\to\infty} \partial_{p,n} : H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A)) \to H^1_{\mathrm{Iw}}(\Phi_\infty,\mathbb{Q}_p),$$

where $V_p(A) := \mathrm{Ta}_p(A)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Writing I_Λ for the augmentation ideal of Λ, the work of Coleman gives a morphism of Λ-modules $\mathscr{C}_\infty : H^1_{\mathrm{Iw}}(\Phi_\infty,\mathbb{Q}_p) \to I_\Lambda\otimes\mathbb{Q}$, 'interpolating' the Bloch-Kato dual exponential maps attached to the trivial representations $\mathbb{Q}_p$ of G_{Φ_n}. (See Section 13 for the details.) Composed with $\partial_{p,\infty}$ this gives the *Coleman map*:

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A)) \to I_\Lambda\otimes\mathbb{Q}; \quad \boldsymbol{u} \mapsto \mathscr{L}_{\boldsymbol{u}},$$

Following Rubin **[Rub94]**,**[Rub98]** we write $L_p(\boldsymbol{u},s)$ for the p-adic Mellin transform of the measure $\mathscr{L}_{\boldsymbol{u}}$: for every $s\in\mathbb{Z}_p$

$$L_p(\boldsymbol{u},s) := \chi_{\mathrm{cycl}}^{s-1}\left(\mathscr{L}_{\boldsymbol{u}}\right),$$

where $\chi_{\mathrm{cycl}} : \mathrm{Gal}(\mathbf{Q}_\infty/\mathbb{Q}) \cong 1+p\mathbb{Z}_p$ is the p-adic cyclotomic character. It is an (locally) analytic p-adic function. We think of $L_p(\boldsymbol{u},s)$ as a p-adic L-function: this is possible thanks to the work of Kato (recalled below), allowing us to construct the Mazur-Tate-Teitelbaum p-adic L-function of $A_{/\mathbb{Q}}$ by this recipe.

12.7. Exceptional zero formulas I: abstract case. Let us fix for the rest of this Section a non-zero 'universal norm' $\boldsymbol{u} = \lim_{n\to\infty} u_n \in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))$, for elements $u_n \in H^1(\mathbf{Q}_n, V_p(A))$.

By construction $L_p(\boldsymbol{u},1) = 0$, so that we are interested in the value of its first derivative at $s=1$. The following Proposition, whose proof is postponed to Section 17, gives an explicit description of this derivative. We recall that the *L-invariant* of $A_{/\mathbb{Q}_p}$ **[MTT86]** is defined by

$$\mathscr{L}_p(A) := \frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)},$$

where $\log_p$ is Iwasawa's p-adic logarithm. We decompose $\partial_p := \partial_{p,0}$ as

$$\partial_p = \partial_p^{\log}\oplus\partial_p^{\mathrm{ord}} : H^1(\mathbb{Q}, V_p(A)) \to \mathrm{Hom}_{\mathrm{continuous}}\left(G^{\mathrm{ab}}_{\mathbb{Q}_p},\mathbb{Q}_p\right) \cong \mathbb{Q}_p^{\log}\oplus\mathbb{Q}_p^{\mathrm{ord}}.$$

Here $G^{\mathrm{ab}}_{\mathbb{Q}_p} = \mathrm{Gal}(\mathbb{Q}_p(\boldsymbol{\mu}_{p^\infty})/\mathbb{Q}_p)\times G_{\mathbb{F}_p}$ is the Galois group of the maximal abelian extension of $\mathbb{Q}_p$, and $\mathbb{Q}_p^\dagger$ is a copy of $\mathbb{Q}_p$. The isomorphism is defined by $\psi \mapsto (\log_p(1+p)^{-1}\cdot\psi(\gamma_0), \psi(\mathrm{Frob}_p))$, where $\gamma_0 \in \mathrm{Gal}(\mathbb{Q}_p(\boldsymbol{\mu}_{p^\infty})/\mathbb{Q}_p)$ is such that $\chi_{\mathrm{cycl}}(\gamma_0) = 1+p$ and $\mathrm{Frob}_p \in G_{\mathbb{F}_p}$ is the Frobenius element.

Proposition 12.25. $\frac{d}{ds}L_p(\boldsymbol{u},s)_{s=1} = \left(1-p^{-1}\right)^{-1}\cdot\mathscr{L}_p(A)\cdot\partial_p^{\log}(u_0)$.

Let $H^1_f(\mathbb{Q},V_p(A)) \supset A(\mathbb{Q})\otimes\mathbb{Q}_p$ be the compact Selmer group with $\mathbb{Q}_p$-coefficients, defined by the exact sequence:

$$0 \to H^1_f(\mathbb{Q},V_p(A)) \to H^1(\mathbb{Q},V_p(A)) \stackrel{\prod_\ell \mathrm{res}_\ell}{\longrightarrow} \prod_{\ell\ \mathrm{prime}} \frac{H^1(\mathbb{Q}_\ell, V_p(A))}{A(\mathbb{Q}_\ell)\otimes\mathbb{Q}_p}.$$

(Here we identify $A(\mathbb{Q}_\ell)\otimes\mathbb{Q}_p$ with a submodule of $H^1(\mathbb{Q}_\ell, V_p(A))$ via the Kummer map.) It is easily seen that $\ker(\partial_p) = \ker\left(\partial_p^{\log}\right)$ (see Section 17.1), so that combining the preceding Proposition with the description of $H^1_f(\mathbb{Q}, V_p(A))$ given in [**Gre97**] (cfr. the proof of Lemma 14.6) we obtain the following:

COROLLARY 12.26. $\frac{d}{ds}L_p(\boldsymbol{u}, s)_{s=1} = 0$ *if and only if* $u_0 \in H^1_f(\mathbb{Q}, V_p(A))$.

12.7.1. We assume for the rest of this Section the following:

Hypothesis: $0 \neq u_0 \in H^1_f(\mathbb{Q}, V_p(A))$.

REMARK 12.27. The Hypothesis $u_0 \neq 0$ is made only to avoid trivial cases. Indeed $H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))$ has no non-trivial $I_\Lambda\otimes\mathbb{Q}$-torsion, and $H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))/I_\Lambda$ injects into $H^1(\mathbb{Q}, V_p(A))$ under $\boldsymbol{u}\mapsto u_0$, so that we can eventually divide $\boldsymbol{u}$ by a power of a generator of I_Λ to get $u_0 \neq 0$.

By the Corollary $\frac{d}{ds}L_p(\boldsymbol{u}, s)_{s=1} = 0$. In this case we can express the second derivative of $L_p(\boldsymbol{u}, s)$ at $s = 1$ in terms of the canonical p-adic cyclotomic height of u_0. We begin by recalling some constructions from (among others) [**MTT86**], [**Nek93**], [**Nek06**], referring to Section 14 for the details.

Write $\widetilde{H}^1_f(\mathbb{Q}, V_p(A))$ for *Nekovář's extended Selmer group* attached to the p-ordinary representation $V_p(A)$ [**Nek06**, Ch. 6], sitting in a short exact sequence

$$0 \to q_A\cdot\mathbb{Q}_p \to \widetilde{H}^1_f(\mathbb{Q}, V_p(A)) \to H^1_f(\mathbb{Q}, V_p(A)) \to 0 \tag{125}$$

and admitting a natural splitting $\widetilde{\sigma} : H^1_f(\mathbb{Q}, V_p(A)) \hookrightarrow \widetilde{H}^1_f(\mathbb{Q}, V_p(A))$. (Like (non-strict) Greenberg Selmer groups [**Gre91**], Nekovář's extended Selmer modules 'capture algebraically' trivial-zeros of p-adic L-functions in the sense of [**MTT86**].) To clarify its structure further: let $A^\dagger(\mathbb{Q})$ be the extended Mordell-Weil group of $A_{/\mathbb{Q}}$ [**MTT86**],[**BD96**], giving rise to an exact sequence

$$0 \to q_A\cdot\mathbb{Q}_p \to A^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p \to A(\mathbb{Q})\otimes\mathbb{Q}_p \to 0 \tag{126}$$

admitting a canonical section $\sigma : A(\mathbb{Q})\otimes\mathbb{Q}_p \hookrightarrow A^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p$. Then there exists a natural injective morphism of short exact sequences $i_A^\dagger : (126) \hookrightarrow (125)$ respecting the natural sections σ and $\widetilde{\sigma}$. Combined with Kummer theory this gives: $\widetilde{H}^1_f(\mathbb{Q}, V_p(A))$ is isomorphic to $A^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p$ provided that the p-part of the Tate-Shafarevich group Ш$(A/\mathbb{Q})$ is finite. We will identify $H^1_f(\mathbb{Q}, V_p(A))$ and $A^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p$ as submodules of $\widetilde{H}^1_f(\mathbb{Q}, V_p(A))$ via $\widetilde{\sigma}$ and $i_A^\dagger$ respectively.

Section 11 of [**Nek06**] constructs a *canonical extended (cyclotomic) p-adic height*

$$\langle -, -\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} : \widetilde{H}^1_f(\mathbb{Q}, V_p(A)) \times \widetilde{H}^1_f(\mathbb{Q}, V_p(A)) \longrightarrow \mathbb{Q}_p.$$

It is a symmetric $\mathbb{Q}_p$-bilinear pairing, satisfying

$$\langle q_A, q_A\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} = \log_p(q_A); \quad \langle q_A, P\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} = \log_A(P)$$

for every $P \in H^1_f(\mathbb{Q}, V_p(A))$. Here (with an abuse of notation) we write

$$\log_A : H^1_f(\mathbb{Q}, V_p(A)) \stackrel{\mathrm{res}_p}{\longrightarrow} A(\mathbb{Q}_p)\otimes\mathbb{Q}_p \stackrel{\log_A}{\longrightarrow} \mathbb{Q}_p,$$

where $\log_A : A(\mathbb{Q}_p)\otimes\mathbb{Q}_p \to \mathbb{Q}_p$ is the formal group logarithm on $A_{/\mathbb{Q}_p}$, defined via Tate's uniformization as the $\mathbb{Q}_p$-linear extension of

$$\log_{q_A}\circ\Phi^{-1}_{\mathrm{Tate}} : A(\mathbb{Q}_p) \cong \mathbb{Q}_p^*/q_A^{\mathbb{Z}} \to \mathbb{Q}_p,$$

with $\log_{q_A} := \log_p - \mathscr{L}_p(A)\cdot\mathrm{ord}_p$ is the branch of the p-adic logarithm vanishing at $q_A \in p\mathbb{Z}_p$. The *p-adic (cyclotomic) regulator* of $A_{/\mathbb{Q}}$ is then defined by

$$\mathcal{R}^{\mathrm{Nek}}_{\mathbb{Q},p} := \det\left(\left(\langle P_i, P_j\rangle^{\mathrm{Nek}}_{\mathbb{Q},p}\right)_{1\leq i,j\leq\widetilde{r}}\right) \in \mathbb{Q}_p,$$

where $\{P_j\}_{j=1}^{\widetilde{r}}$ is any basis of $A^\dagger(\mathbb{Q})/\mathrm{Torsion}$. We also consider the *Schneider* $\log_p$*-height*

$$\langle -, -\rangle^{\mathrm{Sch}}_{\mathbb{Q},p} : H^1_f(\mathbb{Q}, V_p(A)) \times H^1_f(\mathbb{Q}, V_p(A)) \longrightarrow \mathbb{Q}_p,$$

defined by the formula

$$(127)\qquad \langle P,Q\rangle^{\mathrm{Sch}}_{\mathbb{Q},p} := \frac{1}{\log_p(q_A)}\cdot\det\begin{pmatrix}\langle q_A,q_A\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} & \langle q_A,P\rangle^{\mathrm{Nek}}_{\mathbb{Q},p}\\ \langle Q,q_A\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} & \langle P,Q\rangle^{\mathrm{Nek}}_{\mathbb{Q},p}\end{pmatrix} = \langle P,Q\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} - \frac{\log_A(P)\cdot\log_A(Q)}{\log_p(q_A)}.$$

(We recall that $\log_p(q_A)$ is known to be non-zero by [**BSDGP96**].) As suggested by the notation, it is proved in [**Nek06**, Sec. 11] (see also [**Nek93**, Sec. 7-8]) that the restriction of $\langle -,-\rangle^{\mathrm{Sch}}_{\mathbb{Q},p}$ to $A(\mathbb{Q})\times A(\mathbb{Q})$ is the 'canonical p-adic height pairing' constructed in [**Sch82**] (see also [**PR92**]). (We note that the definition of the Schneider $\log_p$-height given here differs from that given in [**MTT86**, pag. 34], as the latter contains the extra factor $\mathrm{ord}_p(q_A)$ in the denominator appearing on the R.H.S. of (127). See also [**Nek93**, Sec. 7.14] for a discussion of this point.) With this notations we have:

Proposition 12.28. $\log_A(u_0)\cdot\frac{d^2}{ds^2}L_p(\boldsymbol{u},s)_{s=1} = -2\cdot\left(1-p^{-1}\right)^{-1}\cdot\mathscr{L}_p(A)\cdot\langle u_0,u_0\rangle^{\mathrm{Sch}}_{\mathbb{Q},p}$.

The proof of this Proposition will be given in Section 17. Retracing the definitions above we easily obtain the following:

Corollary 12.29. *Assume that* $\dim(A(\mathbb{Q})\otimes\mathbb{Q})=1$ *and that* $u_0\in A(\mathbb{Q})\otimes\mathbb{Q}_p$. *Then*

$$\frac{1}{2}\cdot\frac{d^2}{ds^2}L_p(\boldsymbol{u},s)_{s=1} = \frac{-1}{(1-p^{-1})\cdot\mathrm{ord}_p(q_A)}\cdot\frac{\log_A(u_0)}{\log_A(\mathbb{P})^2}\cdot\mathcal{R}^{\mathrm{Nek}}_{\mathbb{Q},p},$$

where $\mathbb{P}$ *is any generator of* $A(\mathbb{Q})$ *modulo its torsion subgroup.*

We note that Gross-Zagier-Kolyvagin's theorem implies that the assumptions of the preceding Corollary are satisfied if the complex Hasse-Weil L-function of $A_{/\mathbb{Q}}$ has order of vanishing 1 at $s=1$.

12.8. Exceptional zero formulas II: Kato's zeta elements. As explained more precisely in Section 16 below: Kato has constructed an Euler system for the Tate module of $A_{/\mathbb{Q}}$, giving in particular an element $\boldsymbol{\zeta}^{\mathrm{Kato}}_\infty\in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty,V_p(A))$ s.t. $\mathscr{L}_{\boldsymbol{\zeta}^{\mathrm{Kato}}_\infty}\in I_\Lambda\otimes\mathbb{Q}$ is essentially the Mazur-Tate-Teitelbaum (cyclotomic) p-adic measure $\mathcal{L}_p(A)$, 'interpolating' the algebraic part $\frac{L(A,\chi,1)}{\Omega_A}$ of the special values of the complex Hasse-Weil L-functions of $A_{/\mathbb{Q}}$ twisted by finite-order characters $\chi:\mathrm{Gal}(\mathbf{Q}_\infty/\mathbb{Q})\to\overline{\mathbb{Q}}^*$. Moreover, thanks to work of Rohrlich we know that $\mathcal{L}_p(A)\neq 0$, so that $\boldsymbol{\zeta}^{\mathrm{Kato}}_\infty\neq 0$. Then we can define the *order of vanishing* $\rho_{\mathrm{Kato}}\geq 0$ of $\boldsymbol{\zeta}^{\mathrm{Kato}}_\infty$ by: $\boldsymbol{\zeta}^{\mathrm{Kato}}_\infty\in I_\Lambda^{\rho_{\mathrm{Kato}}}\otimes\mathbb{Q}\backslash I_\Lambda^{\rho_{\mathrm{Kato}}+1}\otimes\mathbb{Q}$. Let us fix a topological generator $\gamma_0\in\mathrm{Gal}(\mathbf{Q}_\infty/\mathbb{Q})$ and let us write $\varpi:=\gamma_0-1$ for the corresponding generator of I_Λ. We define (cfr. Remark 12.27)

$$\boldsymbol{z}^{\mathrm{Kato}}_{\infty,\varpi} = \lim_{n\to\infty}\boldsymbol{z}^{\mathrm{Kato}}_{n,\varpi} := \varpi^{-\rho_{\mathrm{Kato}}}\cdot\boldsymbol{\zeta}^{\mathrm{Kato}}_\infty;\quad \boldsymbol{z}^{\mathrm{Kato}}_0 := \log_p(\gamma_0)^{\rho_{\mathrm{Kato}}}\cdot\boldsymbol{z}^{\mathrm{Kato}}_{0,\varpi}\neq 0.$$

Let us write $L_p(A,s)=\chi^{s-1}_{\mathrm{cycl}}(\mathcal{L}_p(A))$ for the (cyclotomic) p-adic L-function of $A_{/\mathbb{Q}}$. Thanks to the work of Kato we can then 'specialize' the results of the preceding Section to $L_p(A,s)$, obtaining the following 'p-adic Gross-Zagier formulas', whose proofs are explained in details in Section 17. We refer the reader to the articles of Bertolini and Darmon, e.g. [**BD07**], [**BD98**] for analogues and deeper results in the anticyclotomic setting.

The following Theorem is the well-known Mazur-Tate-Teitelbaum exceptional zero formula, proved by Greenberg and Stevens in [**GS93**].

Theorem 12.30. (Kato's work + Prop. 12.25) $\frac{d}{ds}L_p(A,s)_{s=1}=\mathscr{L}_p(A)\cdot\frac{L(A/\mathbb{Q},1)}{\Omega_A}$.

For every prime ℓ dividing the conductor N of $A_{/\mathbb{Q}}$ we write $E_\ell(X):=1-a_\ell(A)\cdot X\in\mathbb{Z}[X]$, where $a_\ell(A)\in\{0,\pm1\}$ is the usual 'solution-count number' attached to $A_{/\mathbb{Q}_\ell}$ [**Sil86**, Appendix C16]. Let us write $\mathcal{E}_N:=\prod_{\ell|N}E_\ell\left(\ell^{-1}\right)^{-1}$.

Theorem 12.31. (Kato's work + Prop. 12.28) *Assume that* $\boldsymbol{z}^{\mathrm{Kato}}_0\in H^1_f(\mathbb{Q},V_p(A))$. *Then*

1. $L_p(A,s)$ *vanishes to order at least* $\rho_{\mathrm{Kato}}+2$ *at* $s=1$*;*

2. $\log_A\left(\boldsymbol{z}_0^{\mathrm{Kato}}\right)\cdot \left.\frac{L_p(A,s)}{(s-1)^{2+\rho_{\mathrm{Kato}}}}\right|_{s=1} = -\mathcal{E}_N\cdot\mathscr{L}_p(A)\cdot\left\langle \boldsymbol{z}_0^{\mathrm{Kato}},\boldsymbol{z}_0^{\mathrm{Kato}}\right\rangle_{\mathbb{Q},p}^{\mathrm{Sch}}.$

As in the preceding Section, the following Corollary follows easily from the preceding Theorem.

Corollary 12.32. (Kato's work + Cor. 12.29) *Assume* $\dim\left(A(\mathbb{Q})\otimes\mathbb{Q}\right)=1$ *and* $\boldsymbol{z}_0^{\mathrm{Kato}}\in A(\mathbb{Q})\otimes\mathbb{Q}_p$ *(e.g.* $\mathrm{ord}_{s=1}L(A,s)=1$*). Then*

$$\left.\frac{L_p(A,s)}{(s-1)^{2+\rho_{\mathrm{Kato}}}}\right|_{s=1} = \frac{-\mathcal{E}_N}{\mathrm{ord}_p(q_A)}\cdot\frac{\log_A(\boldsymbol{z}_0^{\mathrm{kato}})}{\log_A(\mathbb{P})^2}\cdot\mathcal{R}_{\mathbb{Q},p}^{\mathrm{Nek}},$$

where $\mathbb{P}$ *is any generator of* $A(\mathbb{Q})$ *modulo its torsion subgroup.*

13. The Coleman map

Fix a generator $(\zeta_{p^m})_{m\in\mathbb{N}}$ of $\mathbb{Z}_p(1):=\varprojlim\boldsymbol{\mu}_{p^m}(\overline{\mathbb{Q}}_p)$. For every $n\in\mathbb{N}$, Φ_n is the unique subfield of $\mathbb{Q}_p(\zeta_{p^{n+1}})$ of degree p^n over $\mathbb{Q}_p$ and $\Phi_\infty:=\bigcup_{n\in\mathbb{N}}\Phi_n$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$. For every $n\leq\infty$ we write $\mathbf{G}_n:=\mathrm{Gal}(\Phi_n/\mathbb{Q}_p)$, $\boldsymbol{\Lambda}_n:=\mathbb{Z}_p[[\mathbf{G}_n]]$ for the completed (if $n=\infty$) group ring of $\mathbf{G}_n$ over $\mathbb{Z}_p$ and $\mathbf{I}_n:=\ker\left(\varepsilon_n:\boldsymbol{\Lambda}_n\twoheadrightarrow\mathbb{Z}_p\right)$ for its augmentation ideal. We also write $\mathcal{O}_n:=\mathcal{O}_{\Phi_n}$ for the ring of integers of Φ_n and $\mathfrak{m}_n:=\max\left(\mathcal{O}_n\right)$ for its maximal ideal.

Given a finite $\mathbb{Z}_p$-module T, equipped with a continuous $\mathbb{Z}_p$-linear action of $G_{\mathbb{Q}_p}$, we write $H^q_{\mathrm{Iw}}\left(\Phi_\infty,T\right):=\varprojlim_{n\in\mathbb{N}}H^q(\Phi_n,T)$, the limit being taken with respect to the corestriction maps.

For every finite extension $L/\mathbb{Q}_p$, we write $\exp_L^*:H^1(L,\mathbb{Q}_p)\to L$ for the Bloch-Kato dual exponential map of the trivial G_L-representation $\mathbb{Q}_p$. Writing $\mathcal{O}_L$ for the ring integers of L and

$$\exp_L: L\overset{\exp_p}{\longrightarrow}\mathcal{O}_L^*\otimes\mathbb{Q}_p\to\left(\varprojlim L^*/L^{*p^n}\right)\otimes\mathbb{Q}_p\overset{\text{Kummer}}{\cong}H^1(L,\mathbb{Q}_p(1)),$$

the dual exponential is characterized by the 'commutativity' of the following diagram:

(128)
$$\begin{array}{ccccccc}
L & \times & L & \xrightarrow{\times} & L & \xrightarrow{\mathrm{Trace}_{L/\mathbb{Q}_p}} & \mathbb{Q}_p\\
\uparrow{\scriptstyle\exp_L^*} & & \downarrow{\scriptstyle\exp_L} & & & & \|\\
H^1(L,\mathbb{Q}_p) & \times & H^1(L,\mathbb{Q}_p(1)) & \xrightarrow{\cup} & H^2(L,\mathbb{Q}_p(1)) & \xrightarrow{\mathrm{inv}_L} & \mathbb{Q}_p
\end{array}$$

We will write $\langle -,-\rangle_L:=\mathrm{inv}_L\circ\cup: H^1(L,\mathbb{Q}_p)\times H^1(L,\mathbb{Q}_p(1))\to\mathbb{Q}_p$ for the (perfect) local Tate pairing and we will identify from now on the p-adic completion $L^*\widehat{\otimes}\mathbb{Z}_p$ of L^* with $H^1(L,\mathbb{Z}_p(1))$ via the Kummer isomorphism. Moreover, for every $n\in\mathbb{N}$ we abbreviate $\exp_n^*:=\exp_{\Phi_n}^*$.

Remark 13.1. As proved by Kato [**Kat93**, Chapter II] we have $H^0(L,B_{\mathrm{dR}}^+)\cong H^1(L,B_{\mathrm{dR}}^+)$, the isomorphism being defined by "cupping" with $\log_p\circ\chi_{\mathrm{cycl}}\in H^1(L,\mathbb{Q}_p)$, and

$$\exp_L^*: H^1(L,\mathbb{Q}_p)\to H^1(L,B_{\mathrm{dR}}^+)\cong H^0(L,B_{\mathrm{dR}}^+)=L.$$

Here B_{dR}^+ is the valuation ring of Fontaine's field of periods B_{dR} and the first map is induced by inclusion.

13.1. Statements. The following result, based on work of Coleman [**Col79**], will be the key for relating Kato's zeta elements $\boldsymbol{\zeta}_\infty^{\mathrm{kato}}$ to the p-adic L-function. Its statement is essentially [**Rub98**, Proposition A.2] and its proof will be recalled in the following Section.

Proposition 13.2. (Cfr. [**Rub98**, Appendix]) *There exists a unique morphism of* $\boldsymbol{\Lambda}_\infty$*-modules*

$$\mathscr{C}_\infty: H^1_{\mathrm{Iw}}\left(\Phi_\infty,\mathbb{Z}_p\right)\longrightarrow\mathbf{I}_\infty$$

such that for every $\boldsymbol{\psi}=\lim_{n\to\infty}\psi_n\in H^1_{\mathrm{Iw}}\left(\Phi_\infty,\mathbb{Z}_p\right)$ *and every non-trivial character* χ *of* $\mathbf{G}_n$

$$\chi\left(\mathscr{C}_\infty(\boldsymbol{\psi})\right)=\tau(\chi)\cdot\sum_{\gamma\in\mathbf{G}_n}\chi^{-1}(\gamma)\cdot\exp_n^*(\psi_n^\gamma).$$

(Here $\tau(\chi):=\sum_{\alpha\in(\mathbb{Z}/p^m\mathbb{Z})^*}\chi(\alpha)\cdot\zeta_{p^m}^\alpha$ *is the Gaussian sum of* χ*, where* $p^m\leq p^{n+1}$ *is the conductor of* χ*.)*

Since $\mathscr{C}_\infty$ takes values in $\mathbf{I}_\infty$, its *special value* $\varepsilon_\infty \circ \mathscr{C}_\infty : H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p) \xrightarrow{0} \mathbb{Z}_p$ is the zero map. This leads us to consider its *derivative at* $\mathbf{I}_\infty$:

$$\mathscr{C}'_\infty : H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p) \xrightarrow{\mathscr{C}_\infty} \mathbf{I}_\infty \xrightarrow{\text{proj.}} \mathbf{I}_\infty/\mathbf{I}^2_\infty.$$

The following description of $\mathscr{C}'_\infty$ (whose proof will be given in Section 13.3) is fundamental for the proof of the results of the introduction. Let us fix a topological generator $\gamma_0 \in \mathrm{Gal}(\Phi_\infty/\mathbb{Q}_p)$. We write $\varpi := \gamma_0 - 1 \in \mathbf{I}_\infty$ and $\log_p(\varpi) := \log_p(\chi_{\mathrm{cycl}}(\gamma_0))$.

PROPOSITION 13.3. *Let* $\beta_{p,\varpi} := \log_p(\varpi) \cdot (1 - p^{-1}) \in \mathbb{Z}^*_p$. *For every* $\boldsymbol{\psi} = \lim \psi_n \in H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p)$

$$\mathscr{C}'_\infty(\boldsymbol{\psi}) = \psi_0(\mathrm{Frob}_p) \cdot \left\{ \frac{\varpi}{\beta_{p,\varpi}} \right\} \in \mathbf{I}_\infty/\mathbf{I}^2_\infty,$$

where $\mathrm{Frob}_p = \mathrm{rec}_{\mathbb{Q}_p}(p) \in G_{\mathbb{F}_p} \subset G^{\mathrm{ab}}_{\mathbb{Q}_p}$ *is the arithmetic Frobenius (and* $\{*\} := * \bmod \mathbf{I}^2_\infty$*).*

13.2. Rubin's description. Since fundamental for the methods of this note, in this section we recall Rubin's explicit description of $\mathscr{C}_\infty$ [**Rub98**],[**Rub94**]. As in *loc. cit.* we define for every $n \in \mathbb{N}$:

$$x_n := p + \mathrm{Trace}_{\mathbb{Q}_p(\boldsymbol{\mu}_{p^{n+1}})/\Phi_n} \left(\sum_{k=0}^{n} \frac{\zeta_{p^{n+1-k}} - 1}{p^k} \right) \in \Phi_n.$$

LEMMA 13.4. 1. $x_0 = 0$ *and* $\mathrm{Trace}_{\Phi_{n+m}/\Phi_n}(x_{n+m}) = x_n$ *for every* $m, n \in \mathbb{N}$.
2. *For every non-trivial character* χ *of* $\mathbf{G}_n$*:*

$$\chi\Big(\sum_{\gamma \in \mathbf{G}_n} x_n^\gamma \cdot \gamma \Big) = \tau(\chi).$$

PROOF. 1. follows by a simple computation, while 2. is easily proved using standard properties of Gaussian sum [**Lan90**, Ch. 3, Th. 1.1]. □

The following key Lemma is due to Coleman:

LEMMA 13.5. (Coleman) *There exists a (unique) principal unit* $g(X) \in 1 + (p, X) \cdot \mathbb{Z}_p[[X]]$ *s.t.:*
1. $\log_p(g(0)) = p$*;*
2. $\mathfrak{col}_n := g(\zeta_{p^{n+1}} - 1) \in 1 + \mathfrak{m}_n$ *and* $\log_p(\mathfrak{col}_n) = x_n$ *for every* $n \in \mathbb{N}$*;*
3. $\mathrm{Norm}_{\Phi_{n+m}/\Phi_n}(\mathfrak{col}_{n+m}) = \mathfrak{col}_n$ *for every* $n, m \in \mathbb{N}$.

PROOF. (Cfr. [**Rub00**, Appendix D]) Let us consider the power series

$$f(X) := X - \frac{1}{p-1} \sum_{\mu \in \boldsymbol{\mu}_{p-1} \subset \mathbb{Z}^*_p} \frac{[\mu](X)}{\mu} \in X^2 \cdot \mathbb{Z}_p[[X]]; \quad \Xi_f(X) := \sum_{k=0}^{\infty} \frac{(f \circ [p^k])(X)}{p^k} \in \mathbb{Q}_p[[X]],$$

where $[a](X) := (1+X)^a - 1 \in X \cdot \mathbb{Z}_p[[X]]$ for every $a \in \mathbb{Z}_p$. (We refer to [**Col79**, Sec. 5] for the proof of the convergence of Ξ_f.) Since $f \in X^2 \cdot \mathbb{Z}_p[[X]]$, applying Theorem 24 of [**Col79**] (with $\mathfrak{F} = \mathbb{G}_{m/\mathbb{Z}_p}$, $a = \frac{p}{p-1}$, $b = 0$ and f as above with the notations of *loc. cit.*) we conclude that there exists a unique power series $g^o(X) \in 1 + (p, T) \cdot \mathbb{Z}_p[[X]]$ such that

$$\log(g^o(X)) = \frac{p}{p-1} + \Xi_f(X) = \frac{p}{p-1} + \sum_{k=0}^{\infty} \Big(\frac{(X+1)^{p^k} - 1}{p^k} - \underbrace{\frac{1}{p-1} \cdot \sum_{\mu \in \boldsymbol{\mu}_{p-1}} \frac{(X+1)^{\mu \cdot p^k} - 1}{\mu \cdot p^k}}_{\vartheta_k(X)} \Big). \tag{129}$$

Let us write $\mathfrak{T}$ for the operator $h(X) \mapsto \sum_{\delta \in \boldsymbol{\mu}_{p-1}} (h \circ [\delta])(X)$. We note that

$$p^k \cdot \mathfrak{T}(\vartheta_k(X)) = \sum_{\mu} \mu^{-1} \sum_{\delta} \big([\delta \cdot \mu \cdot p^k](X) \big) = \mathfrak{T}\big([p^k](X)\big) \cdot \sum_{\mu} \mu = 0.$$

Then taking $g(X) := \prod_{\mu\in\boldsymbol{\mu}_{p-1}} (g^o \circ [\mu])(X)$ we obtain:

$$\log(g(X)) = \mathfrak{T}(\log(g^o(X))) = p + \sum_{k=0}^{\infty} \sum_{\mu\in\boldsymbol{\mu}_{p-1}} \frac{(1+X)^{\mu\cdot p^k} - 1}{p^k}.$$

Since by construction $\mathfrak{col}_n := g(\zeta_{p^{n+1}} - 1) = \mathrm{Norm}_{\mathbb{Q}_p(\mu_{p^{n+1}})/\Phi_n}\big(g^o(\zeta_{p^{n+1}} - 1)\big)$ and $g^o(X)$ is a principal unit, we conclude that $\mathfrak{col}_n \in 1 + \mathfrak{m}_n$. Evaluating at $X = \zeta_{p^n} - 1$ and recalling the definition of x_n we deduce 1. and 2. Finally: since $\mathbb{Q}_p(\boldsymbol{\mu}_{p^{n+1}})$ (resp, $\mathbb{Q}_p(\boldsymbol{\mu}_\ell)$ for a prime $\ell \neq p$) is totally ramified (resp., unramified), the torsion submodule of $\mathbb{Q}_p\left(\boldsymbol{\mu}_{p^{n+1}}\right)^*$ equals $\boldsymbol{\mu}_{p-1} \times \boldsymbol{\mu}_{p^{n+1}}$, and since $\Phi_n \cap \mathbb{Q}_p\left(\boldsymbol{\mu}_p\right) = \mathbb{Q}_p$ we have $(\Phi_n^*)_{\mathrm{tors}} = \boldsymbol{\mu}_{p-1}$. This implies that $\log_p$ is injective on $1 + \mathfrak{m}_n$ (recalling: $p \neq 2$). As $\{\log_p(\mathfrak{col}_n)\}_{n\in\mathbb{N}}$ is a trace-compatible system by Lemma 13.4 and 2., this proves 3. ☐

Proof of Proposition 13.2. Let $n \in \mathbb{N}$. For every $\psi_n \in H^1(\Phi_n, \mathbb{Z}_p)$ define

$$\mathscr{C}_n(\psi_n) := \left(\sum_{\gamma\in\mathbf{G}_n} x_n^\gamma \cdot \gamma\right) \cdot \left(\sum_{\gamma\in\mathbf{G}_n} \exp_n^*(\psi_n^\gamma) \cdot \gamma^{-1}\right) \in \mathbb{Q}_p[\mathbf{G}_n] \tag{130}$$

Combining Lemma 13.5 and equation (128) we can rewrite

$$\begin{aligned} \mathscr{C}_n(\psi_n) &:= \sum_{\gamma\in\mathbf{G}_n} \mathrm{Trace}_{\Phi_n/\mathbb{Q}_p}\Big(x_n^\gamma \cdot \exp_n^*(\psi_n)\Big) \cdot \gamma \\ &= \sum_{\gamma\in\mathbf{G}_n} \mathrm{Trace}_{\Phi_n/\mathbb{Q}_p}\Big(\log_p(\mathfrak{col}_n^\gamma) \cdot \exp_n^*(\psi_n)\Big) \cdot \gamma \\ &= \sum_{\gamma\in\mathbf{G}_n} \langle \psi_n, \mathfrak{col}_n^\gamma \rangle_{\Phi_n} \cdot \gamma. \end{aligned} \tag{131}$$

Since the local Tate pairing $\langle -, - \rangle_{\Phi_n}$ maps $\Phi_n^* \widehat{\otimes} \mathbb{Z}_p \times H^1(\Phi_n, \mathbb{Z}_p)$ to $\mathbb{Z}_p$, we conclude that $\mathscr{C}_n$ in fact defines a map

$$\mathscr{C}_n : H^1(\Phi_n, \mathbb{Z}_p) \longrightarrow \mathbf{\Lambda}_n := \mathbb{Z}_p[\mathbf{G}_n].$$

As $\langle -, - \rangle_{\Phi_n}$ is $\mathbf{G}_n$-equivariant (with respect to the conjugation action on cohomology and the trivial action on $\mathbb{Q}_p$) it follows immediately from (131) that $\mathscr{C}_n$ is a morphism of $\mathbf{\Lambda}_n$-modules. Moreover, using the 'projection formulas' $\left\langle \mathrm{Norm}_{\Phi_{n+k}/\Phi_n}(\dagger), \ddagger \right\rangle_{\Phi_{n+k}} = \langle \dagger, \ddagger \rangle_{\Phi_n}$ **[Ser67]**, (131) easily implies that the following diagram commutes for every $m \geq n \in \mathbb{N}$:

$$\begin{array}{ccc} H^1(\Phi_m, \mathbb{Z}_p) & \xrightarrow{\mathscr{C}_m} & \mathbf{\Lambda}_m \\ \downarrow{\scriptstyle \mathrm{Norm}_{\Phi_m/\Phi_n}} & & \downarrow{\scriptstyle \mathbf{G}_m \twoheadrightarrow \mathbf{G}_n} \\ H^1(\Phi_n, \mathbb{Z}_p) & \xrightarrow{\mathscr{C}_n} & \mathbf{\Lambda}_n. \end{array}$$

We then obtain on the limit the desired Coleman map of $\mathbf{\Lambda}_\infty$-module:

$$\mathscr{C}_\infty = \lim_{n\to\infty} \mathscr{C}_n : H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p) \longrightarrow \mathbf{\Lambda}_\infty.$$

The fact that $\mathscr{C}_\infty$ maps to the augmentation ideal $\mathbf{I}_\infty$ and has the characterizing interpolation property follows by (130) and Lemma 13.4. ☐

13.3. Derivative of the Coleman map. In this Section we prove Proposition 13.3, giving a simple explicit formula for the derivative of $\mathscr{C}_\infty$.

The local Tate pairings $\langle -, - \rangle_{\Phi_n}$ for $n \in \mathbb{N}$ combine to give a $\mathbf{\Lambda}_\infty$-bilinear pairing

$$\langle -, - \rangle_{\Phi_\infty} : H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p) \times H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1))^\iota \longrightarrow \mathbf{\Lambda}_\infty,$$

defined by the following formula:

$$\langle \boldsymbol{\psi}, \boldsymbol{u} \rangle_{\Phi_\infty} = \lim_{n\to\infty} \sum_{\gamma \in \mathbf{G}_n} \langle \psi_n, u_n^\gamma \rangle_{\Phi_n} \cdot \gamma$$

for every $\boldsymbol{\psi} = \lim_{n\to\infty} \psi_n \in \varprojlim H^1(\Phi_n, \mathbb{Z}_p)$ and every $\boldsymbol{u} = \lim_{n\to\infty} u_n \in \varprojlim H^1(\Phi_n, \mathbb{Z}_p(1))$. Here $\iota : \mathbf{\Lambda}_\infty \to \mathbf{\Lambda}_\infty$ denotes Iwasawa involution induced by $g \mapsto g^{-1}$ on group-like elements; for every $\mathbf{\Lambda}_\infty$-module M we write M^ι for the $\mathbb{Z}_p$-module M, with $\mathbf{\Lambda}_\infty$-action obtained twisting the original action by ι.

Identifying as usual $H^1(\Phi_n, \mathbb{Z}_p(1)) = \Phi_n^* \widehat{\otimes} \mathbb{Z}_p$ by Kummer theory, Lemma 13.5 allows us to define:

$$\mathfrak{col} := \lim_{n\to\infty} \left(\mathfrak{col}_n \widehat{\otimes} 1\right) \in H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)).$$

Then the proof of Prop. 13.2 (specifically equation (131)) gives us the following:

LEMMA 13.6. $\mathscr{C}_\infty = \langle -, \mathfrak{col} \rangle_{\Phi_\infty} : H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p) \longrightarrow \mathbf{I}_\infty$.

Recall the fixed topological generator $\gamma_0 \in \mathbf{G}_\infty$ and the corresponding generator $\varpi := \gamma_0 - 1 \in \mathbf{I}_\infty$

LEMMA 13.7. *There exists a unique* $\mathfrak{d}_\varpi \in H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1))$ *such that* $\mathfrak{col} = \varpi \cdot \mathfrak{d}_\varpi$.

PROOF. The exact sequence of Galois modules $0 \to \mathbf{\Lambda}_\infty(1) \overset{\varpi}{\to} \mathbf{\Lambda}_\infty(1) \overset{\varepsilon_\infty}{\to} \mathbb{Z}_p(1) \to 0$, together with Shapiro's Lemma gives us a long cohomology exact sequence of $\mathbf{\Lambda}_\infty$-modules:

$$\cdots \overset{\delta}{\to} H^q_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)) \overset{\varpi}{\to} H^q_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)) \overset{\varepsilon_{\infty *}}{\to} H^q(\mathbb{Q}_p, \mathbb{Z}_p(1)) \overset{\delta}{\to} H^{q+1}_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)) \overset{\varpi}{\to} \cdots$$

(see Remark 14.4 below). We deduce that $H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p)$ has no ϖ-torsion and that we have an injective morphism of $\mathbb{Z}_p$-modules

$$H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)) \Big/ \varpi \cdot H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)) \hookrightarrow H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) = \mathbb{Q}_p^* \widehat{\otimes} \mathbb{Z}_p. \tag{132}$$

The unicity of $\mathfrak{d}$ is then clear. Moreover we know by Lemma 13.5 that $\mathfrak{col}_0 = 1$, so $\mathfrak{col}$ is in the kernel of (132) i.e.: $\mathfrak{col}$ belongs to $\varpi \cdot H^1(\Phi_\infty, \mathbb{Z}_p(1))$ as claimed. □

Let us write $p^{\mathbb{Z}_p}$ for the p-adic completion of $p^{\mathbb{Z}} \subset \mathbb{Q}_p^*$, so that we identify $H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong p^{\mathbb{Z}_p} \oplus (1 + p\mathbb{Z}_p)$. By local class field theory **[Ser67]** the reciprocity map gives an isomorphism $\mathrm{rec}_{\mathbb{Q}_p} : \mathbb{Q}_p^* \widehat{\otimes} \mathbb{Q}_p \xrightarrow{\sim} G^{\mathrm{ab}}_{\mathbb{Q}_p} \widehat{\otimes} \mathbb{Q}_p$, inducing an isomorphism

$$H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \Big/ \bigcap_{n\in\mathbb{N}} \mathrm{Norm}_{\Phi_n/\mathbb{Q}_p}\left(H^1(\Phi_n, \mathbb{Z}_p(1))\right) \cong \mathbf{G}_\infty \overset{\log_p(\chi_{\mathrm{cycl}})}{\cong} \mathbb{Z}_p.$$

In particular we have $p^{\mathbb{Z}_p} = \bigcap_{n\in\mathbb{N}} \mathrm{Norm}_{\Phi_n/\mathbb{Q}_p}\left(H^1(\Phi_n, \mathbb{Z}_p(1))\right)$.

The preceding two lemmas reduce the computation of $\mathscr{C}'_\infty$ to the computation of the 'universal norm' $\varepsilon_{\infty *}(\mathfrak{d}_\varpi) \in p^{\mathbb{Z}_p}$. This can be done using again the work of Coleman **[Col79]**, and precisely the so called Coleman isomorphism which we now briefly recall. Let $\mathbb{Z}_{p,n} := \mathbb{Z}_p[\zeta_{p^{n+1}}]$ and let $\mathcal{V}_\infty := \varprojlim_{n\in\mathbb{N}} (\mathbb{Z}_{p,n})^*$ (limit with respect to the norm maps). The Coleman isomorphism gives an isomorphism of $\widetilde{\mathbf{G}}_\infty := \mathrm{Gal}(\mathbb{Q}_p(\boldsymbol{\mu}_{p^\infty})/\mathbb{Q}_p)$-modules **[CS06**, Cor. 2.3.7]:

$$\mathcal{V}_\infty \cong \{h \in \mathbb{Z}_p[[X]] : \mathcal{N}(h) = h\}\,; \quad \mathbf{v} = \lim_{n\to\infty} v_n \mapsto F_{\mathbf{v}},$$

where $\mathcal{N}$ is Coleman norm operator **[Col79**, Sec. Theorem 11]. (We will need only the existence of the morphism $\mathcal{V}_\infty \to \mathbb{Z}_p[[X]]$.) The action of $g \in \widetilde{\mathbf{G}}_\infty$ on $h \in \mathbb{Z}_p[[X]]$ is given by $h^g(X) := h\left((X+1)^{\chi_{\mathrm{cycl}}(g)} - 1\right)$ and the power series $F_{\mathbf{v}}$ is characterized (via Weierstrass preparation) by: $F_{\mathbf{v}}(\zeta_{p^{n+1}} - 1) = v_n$ for every $n \in \mathbb{N}$. Let us consider the composition

$$\mathcal{V}_\infty \to \varprojlim_{n\in\mathbb{N}} \mathcal{O}_n^* \overset{\mathrm{Kummer}}{\to} H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)), \tag{133}$$

where the first map is defined by $(v_n)_{n\in\mathbb{N}} \mapsto \left(\mathrm{Norm}_{\mathbb{Q}_p(\boldsymbol{\mu}_{p^{n+1}})/\Phi_n}(v_n)\right)_{n\in\mathbb{N}}$. Since $\mathrm{Norm}_{\mathbb{Q}_p(\boldsymbol{\mu}_p)/\mathbb{Q}_p}(v_0) = 1$ for every $\mathbf{v} \in \mathcal{V}_\infty$ by the discussion above, the argument of the preceding proof implies that the image of the above map is contained in $\mathbf{I}_\infty \cdot H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)) \cong H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)) \otimes_{\mathbf{\Lambda}_\infty} \mathbf{I}_\infty$. Then composing (133) with the projection $\mathbf{I}_\infty \to \mathbf{I}_\infty/\mathbf{I}^2_\infty$ induces a morphism of $\mathbf{\Lambda}_\infty$-modules

$$\mathfrak{N}_\varpi : \mathcal{V}_\infty \longrightarrow H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)) \otimes_{\mathbf{\Lambda}_\infty} \mathbf{I}_\infty/\mathbf{I}^2_\infty \overset{\varepsilon_{\infty *}\otimes \mathrm{id}}{\longrightarrow} p^{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbf{I}_\infty/\mathbf{I}^2_\infty \cong p^{\mathbb{Z}_p},$$

where the last map (dependent on the choice of ϖ) is defined by $* \otimes \{\varpi\} \mapsto *$. This morphism is easily described by the following:

LEMMA 13.8. $\mathfrak{N}_\varpi(\mathbf{v}) = p^{\frac{\log_p(F_{\mathbf{v}}(0))}{\log_p(\varpi)}} \in p^{\mathbb{Z}_p}$.

PROOF. For every $n \in \mathbb{N}$ write $\nu_n = \mathrm{Norm}_{\mathbb{Q}_p(\zeta_{p^{n+1}})/\Phi_n}(\zeta_{p^{n+1}} - 1)$. Then ν_n is a local parameter in $\mathcal{O}_n$ and we can identify by Kummer theory:

$$H^1(\Phi_n, \mathbb{Z}_p(1)) \cong \Phi_n^* \widehat{\otimes} \mathbb{Z}_p = \nu_n^{\mathbb{Z}_p} \oplus (1 + \mathfrak{m}_n)\,, \tag{134}$$

where $\nu_n^{\mathbb{Z}_p} \cong \mathbb{Z}_p$ is the p-adic completion on $\nu_n^{\mathbb{Z}} \subset \Phi_n^*$. Letting $\mathbf{v} = \lim v_n$ we can thus write

$$\lim_{n\to\infty} \Big(\mathrm{Norm}_{\mathbb{Q}_p(\zeta_{p^{n+1}})/\Phi_n}(v_n)\Big) = \varpi \cdot \lim_{n\to\infty} (\nu_n^{z_n} \oplus \xi_n) \in H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1)), \tag{135}$$

for some $z_n \in \mathbb{Z}_p$ and $\xi_n \in 1 + \mathfrak{m}_n$. We note that $\mathrm{Norm}_{\mathbb{Q}_p(\zeta_{p^{n+1}})/\mathbb{Q}_p(\zeta_{p^n})}(\zeta_{p^{n+1}} - 1) = \zeta_{p^n} - 1$ for every $n \in \mathbb{N}$ (since $X^p - \zeta_{p^n}$ is the minimal polynomial of $\zeta_{p^{n+1}}$ over $\mathbb{Q}_p(\zeta_{p^n})$ by total ramification). Then corestriction respects the decompositions (176), so that $z_\varpi := z_n$ is independent on n and $\{\xi_n\}_{n\in\mathbb{N}}$ is norm compatible. Define $\boldsymbol{\beta} := \lim \xi_n \in H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Z}_p(1))$. As $\xi_0 = 1$ and $\nu_0 := \mathrm{Norm}_{\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p}(\zeta_p - 1) = p$, we have by the preceding proof $(135) \equiv \varpi \cdot \lim_{n\to\infty} (\nu_n^{z_\varpi}) \mod (\varpi^2)$, so that by definition:

$$\mathfrak{N}_\varpi(\mathbf{v}) = p^{z_\varpi}. \tag{136}$$

To compute z_ϖ we first note:

$$\wp_\varpi \text{“} = \varpi \cdot \lim_{n\to\infty} (\nu_n^{z_\varpi}) \text{”} := \lim_{n\to\infty} \left(\prod_{\mu \in \boldsymbol{\mu}_{p-1}} \frac{\zeta_{p^{n+1}}^{\mu\cdot\gamma_0} - 1}{\zeta_{p^{n+1}}^{\mu} - 1} \right)^{z_\varpi} \in \mathcal{V}_\infty,$$

and its associated Coleman power series is given by:

$$F_{\wp_\varpi}(X) = \prod_{\mu \in \boldsymbol{\mu}_{p-1}} \left(\frac{(X+1)^{\mu\cdot\chi_{\mathrm{cycl}}(\gamma_0)} - 1}{(X+1)^{\mu} - 1} \right)^{z_\varpi}.$$

In a similar way, writing $\mathbf{v}^0 := \lim \Big(\mathrm{Norm}_{\mathbb{Q}_p(\zeta_{p^{n+1}})/\Phi_n}(v_n)\Big)$, we have:

$$F_{\mathbf{v}^0}(X) = \prod_{\mu \in \boldsymbol{\mu}_{p-1}} (F_{\mathbf{v}} \circ [\mu])(X); \quad F_{\varpi\cdot\boldsymbol{\beta}}(X) = (F_{\boldsymbol{\beta}} \circ [\chi_{\mathrm{cycl}}(\gamma_0)])(X) \big/ F_{\boldsymbol{\beta}}(X).$$

Since $(f \circ [a])(0) = f(0)$ for every $a \in \mathbb{Z}_p$ and $f \in \mathbb{Z}_p[[X]]$, we finally obtain from (135):

$$F_{\mathbf{v}}(0)^{p-1} = F_{\mathbf{v}^0}(X)\Big|_{X=0} = F_{\wp_\varpi}(X) \cdot F_{\varpi\cdot\boldsymbol{\beta}}(X)\Big|_{X=0} = \chi_{\mathrm{cycl}}(\gamma_0)^{(p-1)\cdot z_\varpi}.$$

Applying $\log_p$ to this equation we obtain: $\log_p(F_{\mathbf{v}}(0)) = \log_p(\varpi) \cdot z_\varpi$, which combined with (136) concludes the proof. □

COROLLARY 13.9. $\varepsilon_{\infty *}(\mathfrak{d}_\varpi) = p^{\frac{p}{p-1}\cdot\frac{1}{\log_p(\varpi)}} \in p^{\mathbb{Z}_p}$.

PROOF. Let $g \in 1 + (p, X)\mathbb{Z}_p[[X]]$ be the power series defined in Lemma 13.5, so that $g = F_{\mathfrak{col}}$. Since $\mathfrak{col}_n \in \Phi_n^*$ for every $n \in \mathbb{N}$, looking at the definitions we have

$$p^{p-1} \cdot \varepsilon_{\infty *}(\mathfrak{d}_\varpi) = \mathfrak{N}_\varpi(\mathfrak{col}).$$

As $\log_p(g(0)) = p$ by 1. of Lemma 13.5, applying the preceding Lemma we obtain the statement. □

We can now finish the proof of Proposition 13.3.

Proof of Proposition 13.3. Combining Lemma 13.6, Lemma 13.7 and Corollary 13.9 we have:

$$\mathscr{C}'_\infty(\boldsymbol{\psi}) = \left\{\langle \boldsymbol{\psi}, \mathfrak{col}\rangle_{\Phi_\infty}\right\} = \langle \boldsymbol{\psi}, \mathfrak{d}_\varpi\rangle_{\Phi_\infty}\cdot\{\varpi\} = \varepsilon_\infty\left(\langle \boldsymbol{\psi}, \mathfrak{d}_\varpi\rangle_{\Phi_\infty}\right)\cdot\{\varpi\}$$
$$= \langle \psi_0, \varepsilon_{\infty *}(\mathfrak{d}_\varpi)\rangle_{\mathbb{Q}_p}\cdot\{\varpi\} = \langle \psi_0, p\rangle_{\mathbb{Q}_p}\cdot\left\{\frac{\varpi}{\beta_{p,\varpi}}\right\}.$$

We conclude the proof using again local class field theory [**Ser67**]: $\langle \chi, p\rangle_{\mathbb{Q}_p} = \chi\left(\mathrm{rec}_{\mathbb{Q}_p}(p)\right) = \chi(\mathrm{Frob}_p)$ for every $\chi \in \mathrm{Hom}_{\mathrm{cont}}(G_{\mathbb{Q}_p}, \mathbb{Z}_p)$.

□

14. Nekovář's extended height

In this section we briefly sketch Nekovář's construction of the extended p-adic cyclotomic height on the Nekovář-Selmer complex attached to the ordinary representation $\mathrm{Ta}_p(A)$. The reference for the material in this section is [**Nek06**], in particular Chapter 11.

14.1. Selmer complexes. Fix a positive integer N divisible by p. Let $L/\mathbb{Q}$ be a number field unramified at every finite place $v \nmid N$. We write $\mathfrak{G}_L := \mathrm{Gal}(\mathbb{Q}_N/L)$, where $\mathbb{Q}_N \subset \overline{\mathbb{Q}}$ is the maximal algebraic extension of $\mathbb{Q}$ which is unramified at every finite prime $\ell \nmid N$. We fix for every prime $v|\ell|N$ of L an embedding $\rho_v : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, which also fix a morphism $\rho_v^* : G_{L_v} \hookrightarrow G_L \twoheadrightarrow \mathfrak{G}_L$, where $L_v := \rho_v(L)\cdot\mathbb{Q}_\ell$ is the completion of L at v and $\rho_v^*(\sigma) = \left(\rho_v^{-1}\circ\sigma\circ\rho_v\right)\big|_{\mathbb{Q}_N}$. Every G_L-module will be considered as a G_{L_v}-module via ρ_v^*, for every $v|\ell|N$. Let $R = (R, \mathfrak{m}_R, \mathbb{F}_R)$ be a complete local Noetherian ring with finite residue field $\mathbb{F}_R$ of characteristic p. Let $\mathscr{S} \subset R$ be a multiplicative system (containing the identity) and let $\mathcal{R} := \mathscr{S}^{-1}R$. By an *$\mathcal{R}$-adic representation of* $\mathfrak{G}_L$ we mean a localization $\mathcal{X} := X \otimes_R \mathcal{R}$ of a finite R-module X, equipped with a continuous R-linear action of $\mathfrak{G}_L$ for the $\mathfrak{m}_R$-adic topology on X. For $G \in \{\mathfrak{G}_L, G_{L_v}\}$ we write

$$C^\bullet_{\mathrm{cont}}(G, \mathcal{X}) := C^\bullet_{\mathrm{cont}}(G, X)\otimes_R \mathcal{R} \quad \left(\text{resp., } \mathbf{R}\Gamma_{\mathrm{cont}}(G,\mathcal{X}),\ H^q(G,\mathcal{X})\right)$$

for the complex of *continuous $\mathcal{X}$-valued cochains on G* (resp., its image in the derived category $\mathcal{D}(\mathcal{R})$ on complexes of $\mathcal{R}$-modules, its cohomology) [**Nek06**, Chapter 3]. (More precisely: $C^\bullet_{\mathrm{cont}}(G, X)$ is the complex of continuous non-homogeneous cochains of G with values on X. This is defined exactly as in the classical (discrete) case, e.g. [**NSW00**], but with the term continuous referring to the $\mathfrak{m}_R$-adic topology on X and the profinite topology on G.) We also write $C^\bullet_{\mathrm{cont}}(L_v, -) := C^\bullet_{\mathrm{cont}}(G_{L_v}, -)$, $\mathbf{R}\Gamma_{\mathrm{cont}}(L_v, -) := \mathbf{R}\Gamma_{\mathrm{cont}}(G_{L_v}, -)$ and $H^q(L_v, -) := H^q(G_{L_v}, -)$.

Let $\mathcal{X}$ be an $\mathcal{R}$-adic representation of $\mathfrak{G}_L$. We fix for every prime $v|N$ an $R[G_{L_v}]$-submodule $X_v^+ \subset X$. Letting $\mathcal{X}_v^+ := X_v^+ \otimes_R \mathcal{R}$, the *(Greenberg) local conditions* at v is the morphism of complexes of $\mathcal{R}$-modules:

$$i_v^+ : C^\bullet(L_v, \mathcal{X}_v^+) \longrightarrow C^\bullet_{\mathrm{cont}}(L_v, \mathcal{X})$$

induced by the inclusion $X_v^+ \subset X$. The *Nekovář-Selmer complex* of $\mathcal{X} = \left(\mathcal{X}, \{\mathcal{X}_v^+\}_{v|N}\right)$ over L is then defined as the complex:

$$\widetilde{C}^\bullet_f(\mathfrak{G}_L, \mathcal{X}) := \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(\mathfrak{G}_L, \mathcal{X}) \oplus \bigoplus_{v|N} C^\bullet_{\mathrm{cont}}(L_v, \mathcal{X}_v^+) \xrightarrow{\mathrm{res}_N - i_N^+} \bigoplus_{v|N} C^\bullet_{\mathrm{cont}}(L_v, \mathcal{X})\right)[-1].$$

We write $i_N^+ := \oplus_{n|N} i_v^+$ and $\mathrm{res}_N = \oplus_{v|N}\mathrm{res}_v$, where $\mathrm{res}_v : C^\bullet_{\mathrm{cont}}(\mathfrak{G}_L, \mathcal{X}) \to C^\bullet_{\mathrm{cont}}(L_v, \mathcal{X})$ is the 'restriction' morphism of complexes induced by the morphism of pairs: $(\rho_v^*, \mathrm{id}) : (\mathfrak{G}_L, \mathcal{X}) \to (G_{L_v}, \mathcal{X})$. Let

$$\widetilde{\mathbf{R}\Gamma}_f(\mathfrak{G}_L, \mathcal{X}) \in \mathcal{D}(\mathcal{R}); \quad \widetilde{H}^q_f(\mathfrak{G}_L, \mathcal{X}) \in {}_{\mathcal{R}}\mathrm{Mod}$$

be the image of $\widetilde{C}^\bullet_f(\mathfrak{G}_L, \mathcal{X})$ in the derived category and its cohomology respectively. Under our assumptions the usual finiteness theorems for Galois cohomology of discrete modules imply that $\widetilde{H}^q_f(\mathfrak{G}_L, \mathcal{X})$ is a finite $\mathcal{R}$-module [**Nek06**, Sec. 4.2].

Lemma 14.1. *There exists an exact sequence of finite $\mathcal{R}$-modules:*

$$\cdots \to \bigoplus_{v|N} H^{q-1}(L_v, \mathcal{X}_v^-) \to \widetilde{H}^q_f(\mathfrak{G}_L, \mathcal{X}) \to H^q(\mathfrak{G}_L, \mathcal{X}) \to \bigoplus_{v|N} H^q(L_v, \mathcal{X}_v^-) \to \cdots, \tag{137}$$

where $\mathcal{X}_v^- := \mathcal{X}/\mathcal{X}_v^+ \in \mathcal{R}[G_{L_v}]$-Mod.

Proof. Writing $\mathrm{U}_v^-(\mathcal{X}) := \mathrm{Cone}\big(C_f^\bullet(L_v, \mathcal{X}) \stackrel{-i_v^+}{\to} C^\bullet_{\mathrm{cont}}(L_v, \mathcal{X})\big)$ we have an exact triangle in $\mathcal{D}(\mathcal{R})$:

$$\bigoplus_{v|N} \mathrm{U}_v^-(\mathcal{X})[-1] \to \widetilde{\mathbf{R}\Gamma}_f(\mathfrak{G}_L, \mathcal{X}) \to \mathbf{R}\Gamma_{\mathrm{cont}}(\mathfrak{G}_L, \mathcal{X}). \tag{138}$$

Since $C^\bullet_{\mathrm{cont}}(L_v, -)$ maps short exact sequences of $\mathcal{R}$-adic representations in short exact sequences of complexes [**Nek06**, Prop. 3.4.2], for every place $v|N$ the natural projection induces an isomorphism in the derived category:

$$\mathrm{U}_v^-(\mathcal{X}) \cong \mathbf{R}\Gamma_{\mathrm{cont}}(L_v, \mathcal{X}_v^-) \in \mathcal{D}(\mathcal{R}).$$

Taking the long cohomology exact sequence attached to (185) we obtain (137). □

14.1.1. *Galois deformations.* Let $\mathbb{L}/L$ be a Galois extension contained in $\mathbb{Q}_N/L$. Write $\Gamma_{\mathbb{L}} := \mathrm{Gal}(\mathbb{L}/L)$ and $\mathcal{R}_{\mathbb{L}} := \mathcal{R}[[\Gamma_{\mathbb{L}}]] = R[[\Gamma_{\mathbb{L}}]] \otimes_R \mathcal{R}$ for the completed group algebra of $\Gamma_{\mathbb{L}}$ over $\mathcal{R}$. For every $M \in \mathcal{R}_{\mathbb{L}}[\mathfrak{G}_L]$-Mod and every $n \in \mathbb{Z}$, $M < n > \in \mathcal{R}_{\mathbb{L}}[\mathfrak{G}_L]$-Mod denotes the $\mathcal{R}_{\mathbb{L}}$-module M, with $\mathfrak{G}_L$-action obtained multiplying the original action by $\chi_{\mathbb{L}}^n$, where

$$\chi_{\mathbb{L}} : \mathfrak{G}_L \twoheadrightarrow \Gamma_{\mathbb{L}} \subset \mathcal{R}_{\mathbb{L}}^*$$

denotes the tautological $\mathfrak{G}_L$-representation. For every $\mathcal{R}[\mathfrak{G}_L]$-module T we denote by

$$T_{\mathbb{L}} := (T \otimes_{\mathcal{R}} \mathcal{R}_{\mathbb{L}}) < -1 > \in \mathcal{R}_{\mathbb{L}}[\mathfrak{G}_L]\text{-Mod}$$

its $\mathbb{L}$-*deformation.* Writing $\varepsilon_{\mathbb{L}} : \mathcal{R}_{\mathbb{L}} \twoheadrightarrow \mathcal{R}$ for the augmentation map, we have $\varepsilon_{\mathbb{L}}^*(T_{\mathbb{L}}) \cong T$ as $\mathcal{R}[\mathfrak{G}_L]$-modules, where $\varepsilon_{\mathbb{L}}^*(-) := - \otimes_{\mathcal{R}_{\mathbb{L}}, \varepsilon_{\mathbb{L}}} \mathcal{R} : \mathcal{R}_{\mathbb{L}}$-Mod $\to \mathcal{R}$-Mod.

Let $\mathcal{X} = \big(\mathcal{X}, \{\mathcal{X}_v^+\}_{v|N}\big)$ be an $\mathcal{R}$-adic representation of $\mathfrak{G}_L$. Then $\mathcal{X}_{\mathbb{L}} = \big(\mathcal{X}_{\mathbb{L}}, \{(\mathcal{X}_v^+)_{\mathbb{L}}\}_{v|N}\big)$ is an $\mathcal{R}_{\mathbb{L}}$-adic representation. In the following proposition $\mathbf{L}\varepsilon_{\mathbb{L}}^* : \mathcal{D}(\mathcal{R}_{\mathbb{L}}) \to \mathcal{D}(\mathcal{R})$ denotes the left derived functor of the functor $\varepsilon_{\mathbb{L}}^* : C^\bullet \mapsto C^\bullet \otimes_{\mathcal{R}_{\mathbb{L}}, \varepsilon_{\mathbb{L}}} \mathcal{R}$ on complexes of $\mathcal{R}_{\mathbb{L}}$-modules.

Proposition 14.2. (Control Theorem) *Assume that* $\Gamma_{\mathbb{L}} \cong \mathbb{Z}_p^g$ *for some* $g \geq 1$. *There exists a canonical isomorphism in* $\mathcal{D}(\mathcal{R})$*:*

$$\mathbf{L}\varepsilon_{\mathbb{L}}^* \Big(\widetilde{\mathbf{R}\Gamma}_f(\mathfrak{G}_L, \mathcal{X}_{\mathbb{L}})\Big) \cong \widetilde{\mathbf{R}\Gamma}_f(\mathfrak{G}_L, \mathcal{X}).$$

Proof. The proof proceeds by induction on the dimension $g \geq 1$ of $\Gamma_{\mathbb{L}}$. We sketch the proof for $g = 1$ (the only case needed for the results of this note), referring to [**Nek06**, Prop. 8.10.1] for the general case.

Fix a topological generator $\gamma_{\mathbb{L}} \in \Gamma_{\mathbb{L}} \stackrel{\sim}{\to} \mathbb{Z}_p$ and write $\varpi_{\mathbb{L}} := \gamma_{\mathbb{L}} - 1$ for the corresponding generator of $\ker(\varepsilon_{\mathbb{L}})$, so that we have an isomorphisms $\mathcal{R}_{\mathbb{L}} \cong \mathcal{R}[[\varpi_{\mathbb{L}}]]$ and $\mathcal{R}_{\mathbb{L}}/(\varpi_{\mathbb{L}}) \cong \mathcal{R}$. Let $M \in \{\mathcal{X}, \mathcal{X}_v^+\}$ and $G \in \{\mathfrak{G}_L, G_{L_v}\}$. Tensoring $\mathcal{R} = \mathcal{R}_{\mathbb{L}}/(\varpi_{\mathbb{L}})$ with M we obtain short exact sequences of $\mathcal{R}_{\mathbb{L}}[G]$-modules:

$$0 \to M_{\mathbb{L}} \stackrel{\varpi_{\mathbb{L}}}{\to} M_{\mathbb{L}} \to M \to 0.$$

Applying $C^\bullet_{\mathrm{cont}}(G, -)$ we obtain a short exact sequences of complexes of $\mathcal{R}_{\mathbb{L}}$-modules [**Nek06**, Prop. 3.4.2]

$$0 \to C^\bullet_{\mathrm{cont}}(G, M_{\mathbb{L}}) \stackrel{\varpi_{\mathbb{L}}}{\to} C^\bullet_{\mathrm{cont}}(G, M_{\mathbb{L}}) \to C^\bullet_{\mathrm{cont}}(G, M) \to 0,$$

compatibles under res_v and i_v^+. Combining these exact sequences gives a similar short exact sequence for $\widetilde{C}_f^\bullet(\mathfrak{G}_L, -)$, i.e. a quasi isomorphism of complexes of $\mathcal{R}_{\mathbb{L}}$-modules:

$$\widetilde{C}_f^\bullet(\mathfrak{G}_L, \mathcal{X}_{\mathbb{L}}) \otimes_{\mathcal{R}_{\mathbb{L}}} \big[\mathcal{R}_{\mathbb{L}} \stackrel{\varpi_{\mathbb{L}}}{\to} \mathcal{R}_{\mathbb{L}}\big] \stackrel{\mathrm{qis}}{\longrightarrow} \widetilde{C}_f^\bullet(\mathfrak{G}_L, \mathcal{X}), \tag{139}$$

where $\big[\mathcal{R}_{\mathbb{L}} \stackrel{\varpi_{\mathbb{L}}}{\to} \mathcal{R}_{\mathbb{L}}\big] \stackrel{\mathrm{qis}}{\to} \mathcal{R}$ is concentrated in degrees -1 and 0, i.e. is an $\mathcal{R}_{\mathbb{L}}$-projective resolution of $\mathcal{R}$. This in turns defines the the canonical isomorphism in the statement [**Har66**, Ch. II, Cor. 5.11]. □

14.1.2. *Shapiro's lemma and Iwasawa modules.* Let $\mathbb{L} = \bigcup_{\alpha\in\mathscr{A}} L_\alpha$ be an abelian extension of L contained in $\mathbb{Q}_N$, for finite abelian extensions L_α/L. For every $\alpha \in \mathscr{A}$ we write $\mathfrak{G}_\alpha := \mathfrak{G}_{L_\alpha}$. For $\alpha, \beta \in \mathscr{A}$ s.t. $L_\alpha \subset L_\beta$ Section 8 of [**Nek06**] constructs canonical *corestriction* morphisms:

$$\mathrm{cor}_\alpha^\beta : \widetilde{H}_f^q(\mathfrak{G}_\beta, \mathcal{X}) \to \widetilde{H}_f^q(\mathfrak{G}_\alpha, \mathcal{X}),$$

together with a *homotopy action* of $\Gamma_\alpha = \mathrm{Gal}(L_\alpha/L)$ on $\widetilde{H}_f^q(\mathfrak{G}_\alpha, \mathcal{X})$ and a canonical *Shapiro's isomorphism* of $\mathcal{R}_\alpha := \mathcal{R}[\Gamma_\alpha]$-modules

$$\mathrm{Sh}_\alpha : \widetilde{H}_f^q(\mathfrak{G}_L, \mathcal{X}_\alpha) \xrightarrow{\sim} \widetilde{H}_f^q(\mathfrak{G}_\alpha, \mathcal{X}).$$

Here $\mathcal{X}_\alpha := \mathcal{X}_{L_\alpha}$. All these constructions commute with the usual ones under the natural morphism $\widetilde{H}_f^q(\mathfrak{G}_\alpha, \mathcal{X}) \to H^q(\mathfrak{G}_\alpha, \mathcal{X})$. Moreover $\mathrm{cor}_\alpha^\beta$ corresponds, via the Shapiro's isomorphisms $\mathrm{Sh}_\star$, to the map $\mathrm{pr}_* : \widetilde{H}_f^q(\mathfrak{G}_L, \mathcal{X}_\beta) \to \widetilde{H}_f^q(\mathfrak{G}_L, \mathcal{X}_\alpha)$ induced by the natural projection $\Gamma_\beta \twoheadrightarrow \Gamma_\alpha$. This allows us to define the $\mathcal{R}_\mathbb{L}$-module:

$$\widetilde{H}_{f,\mathrm{Iw}}^q(\mathbb{L}, \mathcal{X}) := \left(\varprojlim_{\mathrm{cor};\ \alpha\in\mathscr{A}} \widetilde{H}_f^q(\mathfrak{G}_\alpha, X) \right) \otimes_R \mathcal{R}.$$

We recall that $\mathcal{R} := \mathscr{S}^{-1}R$ and $\big(\mathcal{X}, \{\mathcal{X}_v^+\}_{v|N}\big)$ is obtained as the $\mathscr{S}$-localization of the R-adic representation $\big(X, \{X_v^+\}_{v|N}\big)$ of $\mathfrak{G}_L$. Since $X_\mathbb{L} := (X \otimes_R R[[\Gamma_\mathbb{L}]]) < -1 >= \varprojlim X_\alpha$ as $R_\mathbb{L}[\mathfrak{G}_L]$-modules and $\mathcal{X}_\mathbb{L} = X_\mathbb{L} \otimes_R \mathcal{R}$, the isomorphisms Sh_α combines to give an isomorphism of $\mathcal{R}_\mathbb{L}$-modules [**Nek06**, Prop. 8.8.6]

$$\mathrm{Sh} : \widetilde{H}_f^q(\mathfrak{G}_L, \mathcal{X}_\mathbb{L}) \cong \varprojlim_{\mathrm{pr}_*;\ \alpha\in\mathscr{A}} \widetilde{H}_f^q(\mathfrak{G}_L, \mathcal{X}_\alpha) \cong \widetilde{H}_{f,\mathrm{Iw}}^q(\mathbb{L}, \mathcal{X}), \tag{140}$$

where the first isomorphism is induced by the natural projections $\mathcal{X}_\mathbb{L} \twoheadrightarrow \mathcal{X}_\alpha$. In what follows, we will identify $\widetilde{H}_f^1(\mathfrak{G}_L, \mathcal{X}_\mathbb{L}) = \widetilde{H}_{f,\mathrm{Iw}}^q(\mathbb{L}, \mathcal{X})$ under Sh.

Corollary 14.3. *Assume that $\Gamma_\mathbb{L} \cong \mathbb{Z}_p$, and let $\varpi := \gamma_\mathbb{L} - 1$ for a topological generator $\gamma_\mathbb{L} \in \Gamma_\mathbb{L}$. There exist short exact sequences of $\mathcal{R}$-modules*

$$0 \to \widetilde{H}_{f,\mathrm{Iw}}^q(\mathbb{L}, \mathcal{X})/\varpi \to \widetilde{H}_f^q(\mathfrak{G}_L, \mathcal{X}) \stackrel{\beta_\varpi^q}{\to} \widetilde{H}_{f,\mathrm{Iw}}^{q+1}(\mathbb{L}, \mathcal{X})[\varpi] \to 0$$

Proof. Apply cohomology to the control theorem Prop. 14.2. (See in particular (139) in its proof.) □

Remark 14.4. Let E be a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$, let $\mathbb{E}/E$ by a $\mathbb{Z}_p^g$-extension (i.e. $\Gamma_\mathbb{E} \xrightarrow{\sim} \mathbb{Z}_p^g$) and let T be a continuous $R[G_E]$-module, finite over R. Then the analogues of Prop. 14.2, (140) (and Lemma 14.3 if $g = 1$) obviously hold for the continuous cohomology of T and $T_\mathbb{E}$. (In particular this justifies the argument used in the proof of Lemma 13.7.) As above we will identify

$$H^q(E, T_\mathbb{E}) \cong \varprojlim_{\mathrm{cor}} H^q(E_\alpha, T) =: H_{\mathrm{Iw}}^q(\mathbb{E}, T)$$

via the Shapiro's isomorphism. If $E \subset \mathbb{Q}_N$ is a global field and T is an $R[\mathfrak{G}_E]$-module, a similar canonical isomorphism holds for the cohomology module $H^1(\mathfrak{G}_E, T_\mathbb{E})$ with restricted ramification.

14.1.3. *Class field theory.* Let $\mathcal{R}(1) := \mathcal{R} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. We consider $\mathcal{X}_{\mathrm{full}} = (\mathcal{R}(1), \{0\}_{v|N})$ and we write

$$C_{\mathrm{c,cont}}^\bullet(\mathfrak{G}_L, \mathcal{R}(1)) := \widetilde{C}_f^\bullet(\mathfrak{G}_L, \mathcal{X}_{\mathrm{full}}); \quad H_\mathrm{c}^q(\mathfrak{G}_L, \mathcal{R}(1)) := \widetilde{H}_f^q(\mathfrak{G}_L, \mathcal{X}_{\mathrm{full}}).$$

Proposition 14.5. *The invariant maps of local class field theory induce an isomorphism of $\mathcal{R}$-modules:*

$$\underline{\mathrm{inv}}_{L,N} : H_\mathrm{c}^3(\mathfrak{G}_L, \mathcal{R}) \cong \mathcal{R}.$$

Proof. Lemma 14.1 gives an exact sequence of $\mathcal{R}$-modules:

$$H^2(\mathfrak{G}_L, \mathcal{R}(1)) \stackrel{\mathrm{res}_N}{\to} \bigoplus_{v|N} H^2(L_v, \mathcal{R}(1)) \to H_\mathrm{c}^3(\mathfrak{G}_L, \mathcal{R}(1)) \to 0,$$

where the zero on the right follows by $\mathrm{cd}_p(\mathfrak{G}_L) = 2$. On the other hand, the fundament exact sequence of global class field theory tells us that (res_N is injective and that) $\sum_{v|N} \mathrm{inv}_v$ gives an isomorphism

$\operatorname{coker}(\operatorname{res}_N) \xrightarrow{\sim} \mathcal{R}$. (Here $\operatorname{inv}_v : H^2(L_v, \mathcal{R}(1)) \cong \mathcal{R}$ is obtained (taking limit) from the invariant maps of local class field theory.) □

14.1.4. *Global cup-product pairings.* Let $\mathcal{X} = \big(\mathcal{X}, \{\mathcal{X}_v^+\}_{v|N}\big)$ and $\mathcal{Y} = \big(\mathcal{Y}, \{\mathcal{Y}_v^+\}_{v|N}\big)$ be $\mathcal{R}$-adic representations of $\mathfrak{G}_L$. We assume that there exists a $\mathfrak{G}_L$-equivariant morphism of $\mathcal{R}$-modules

$$\pi : \mathcal{X} \otimes_{\mathcal{R}} \mathcal{Y} \longrightarrow \mathcal{R}(1),$$

such $\mathcal{X}_v^+$ is π-*orthogonal* to $\mathcal{Y}_v^+$ for every $v|N$. This means that $\pi(x_v^+ \otimes y_v^+) = 0$ for every $x_v^+ \in \mathcal{X}_v^+$, $y_v^+ \in \mathcal{Y}_v^+$ and every place $v|N$ of L. For $G \in \{\mathfrak{G}_L, G_{L_v}\}$ let $\cup_\pi(G)$ denotes the G-cup-product pairing attached to π [**Nek06**, Sec. 3.4.5]:

$$\cup_\pi(G) : C^\bullet_{\mathrm{cont}}(G, \mathcal{X}) \otimes_{\mathcal{R}} C^\bullet_{\mathrm{cont}}(G, \mathcal{Y}) \xrightarrow{\cup} C^\bullet_{\mathrm{cont}}(G, \mathcal{X} \otimes_{\mathcal{R}} \mathcal{Y}) \xrightarrow{\pi_*} C^\bullet_{\mathrm{cont}}(G, \mathcal{R}(1)).$$

We will denote simply by $\cup_\pi$ both $\cup_\pi(\mathfrak{G}_L)$ and $\bigoplus_{v|N} \cup_\pi(G_{L_v})$. We write $x_f = (x, x^+, \alpha)$ to denote an n-cochain of $\widetilde{C}^\bullet_f(\mathfrak{G}_L, \mathcal{X})$, where $x \in C^n_{\mathrm{cont}}(\mathfrak{G}_L, \mathcal{X})$, $x^+ \in \bigoplus_{v|N} C^n_{\mathrm{cont}}(L_v, \mathcal{X}_v^+)$ and $\alpha \in \bigoplus_{v|N} C^{n-1}_{\mathrm{cont}}(L_v, \mathcal{X})$. Similarly we denote by $y_f = (y, y^+, \beta)$ a generic m-cochain of $\widetilde{C}^\bullet_f(\mathfrak{G}_L, \mathcal{Y})$. Then a simple computation [**Nek06**, Prop. 1.3.2] proves that the formula

$$x_f \cup_\pi^{\mathrm{PT}} y_f := \big(x \cup_\pi y, \alpha \cup_\pi i_N^+(y^+) + (-1)^n \operatorname{res}_N(x) \cup_\pi \beta\big)$$

defines a morphism of complexes of $\mathcal{R}$-modules:

$$\cup_\pi^{\mathrm{PT}} : \widetilde{C}^\bullet_f(\mathfrak{G}_L, \mathcal{X}) \otimes_{\mathcal{R}} \widetilde{C}^\bullet_f(\mathfrak{G}_L, \mathcal{Y}) \longrightarrow C^\bullet_{\mathrm{c,cont}}(\mathfrak{G}_L, \mathcal{R}(1)).$$

Taking the pairing induced by $\cup_\pi^{\mathrm{PT}}$ in $(2,1)$-cohomology and using Prop. 0.4 we define

$$\langle -, - \rangle_{L,\pi}^{\mathrm{PT}} : \widetilde{H}^2_f(\mathfrak{G}_L, \mathcal{X}) \otimes_{\mathcal{R}} \widetilde{H}^1_f(\mathfrak{G}_L, \mathcal{Y}) \longrightarrow H^3_{\mathrm{c}}(\mathfrak{G}_L, \mathcal{R}(1)) \cong \mathcal{R}.$$

(Here PT stands for Poitou-Tate. In fact [**Nek06**] uses the cup-product pairing $\cup_\pi^{\mathrm{PT}}$ to give a wide generalization of classical Poitou-Tate duality. Se especially Sections 5 and 6 of *loc. cit.*)

14.2. The extended p-Selmer group of $A_{/\mathbb{Q}}$. In this section we recall the relation between the Selmer complex attached to $\mathrm{Ta}_p(A)$ and usual Kummer theory for $A_{/\mathbb{Q}}$. First of all, we have to define a p-ordinary structure on $\mathrm{Ta}_p(A)$.

Tate's p-adic analytic uniformization [**Tat95**], [**Sil94**, Chapter V] gives an isomorphism of $G_{\mathbb{Q}_p}$-modules

$$\Phi_{\mathrm{Tate}} : \overline{\mathbb{Q}}_p^*/q_A^{\mathbb{Z}} \xrightarrow{\sim} A(\overline{\mathbb{Q}}_p).$$

We identify $A(\overline{\mathbb{Q}})_{p^n} \cong A(\overline{\mathbb{Q}}_p)_{p^n}$ under the isomorphism induced by the fixed embedding ρ_p. As the Tate period $q_A \in p\mathbb{Z}_p$, we obtain short exact sequences of $G_{\mathbb{Q}_p}$-modules $0 \to \boldsymbol{\mu}_{p^n}(\overline{\mathbb{Q}}_p) \xrightarrow{\Phi_{\mathrm{Tate}}} A(\overline{\mathbb{Q}})_{p^n} \xrightarrow{\pi_{q_A}} \mathbb{Z}/p^n \to 0$, where G_p acts trivially on $\mathbb{Z}/p^n$ and $\pi_{q_A} = \pi_{q_A,n}$ is defined in Section 1. Taking the inverse limit for $n \to \infty$ and extending scalars to $\mathbb{Q}_p$ we obtain a short exact sequence of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules:

$$0 \to \mathbb{Q}_p(1) \xrightarrow{\Phi_{\mathrm{Tate}}} V_p(A) \xrightarrow{\pi_{q_A}} \mathbb{Q}_p \to 0, \tag{141}$$

defining a p-ordinary structure on $V_p(A) := \mathrm{Ta}_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

With the notations of the preceding Section let $N = N_A$ be the conductor of $A_{/\mathbb{Q}}$, so that by assumption p divides N exactly. Given $L \subset \mathbb{Q}_N$, let $\Phi_{\mathrm{Tate}} : V_p(A)_v^+ := \mathbb{Q}_p(1) \hookrightarrow V_p(A)$ (resp., $V_p(A)_v^+ := 0$) for every place v of L dividing p (resp., dividing N/p). Then $V_p(A) = \big(V_p(A), \{V_p(A)_v^+\}_{v|N}\big)$ is a $\mathbb{Q}_p$-adic representation of $\mathfrak{G}_L$ [**Sil86**, Ch. VII] and we write

$$\widetilde{\mathbf{R}\Gamma}_f(L, V_p(A)) := \widetilde{\mathbf{R}\Gamma}_f(\mathfrak{G}_L, V_p(A)); \quad \widetilde{H}^q_f(L, V_p(A)) := \widetilde{H}^q_f(\mathfrak{G}_L, V_p(A)).$$

(Since we are working with $\mathbb{Q}_p$-coefficient, and as suggested by the notations, $\widetilde{\mathbf{R}\Gamma}_f(L, V_p(A))$ does not depend on any choice. In other words, replacing N_A by an integer N' divisible by every prime of bad reduction of $A_{/\mathbb{Q}}$, and $V_p(A)^+_v$ by any $\mathbb{Q}_p[G_{L_v}]$-submodule of $V_p(A)$ for $v \nmid p$, the new $\widetilde{\mathbf{R}\Gamma}_f(\mathrm{Gal}(\mathbb{Q}_{N'}/L), V_p(A))$ is canonically isomorphism to $\widetilde{\mathbf{R}\Gamma}_f(L, V_p(A))$ in the derived category.) Let

$$H^1_f(L, V_p(A)) := \ker\left(H^1(L, \mathrm{Ta}_p(A)) \overset{\prod_v \mathrm{res}_v}{\longrightarrow} \prod_v \frac{H^1(L_v, \mathrm{Ta}_p(A))}{A(L_v)\widehat{\otimes}\mathbb{Z}_p}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \supset A(L) \otimes \mathbb{Q}_p$$

be the classical compact Selmer group with $\mathbb{Q}_p$-coefficients arising from Kummer theory for $A_{/L}$.

LEMMA 14.6. *We have a natural short exact sequence of $\mathbb{Q}_p$-modules*

$$0 \to \bigoplus_{v|p} \mathbb{Q}_p \to \widetilde{H}^1_f(L, V_p(A)) \to H^1_f(L, V_p(A)) \to 0 \tag{142}$$

PROOF. It follows by a theorem of Lutz [**Sil86**, Ch. VII, Prop. 6.3] and local Tate duality that $\mathbf{R}\Gamma_{\mathrm{cont}}(G_\ell, V_p(A))$ is acyclic for every prime $\ell \neq p$. (In particular $A(\mathbb{Q}_\ell) \otimes \mathbb{Q}_p = 0$ for $\ell \neq p$.) Together with: $V_p(A)^-_v \cong \mathbb{Q}_p$ as $\mathbb{Q}_p[G_{L_v}]$-modules for every $v|p$ and $H^0(\mathbb{Q}, V_p(A)) = 0$ this allows us to extract from (137) the exact sequence

$$0 \to \bigoplus_{v|p} H^0(L_v, \mathbb{Q}_p) \to \widetilde{H}^1_f(L, V_p(A)) \overset{\pi}{\to} H^1(\mathfrak{G}_L, V_p(A)) \to \bigoplus_{v|p} \frac{H^1(L_v, V_p(A))}{\Phi_{\mathrm{Tate}*}\left(H^1(L_v, \mathbb{Q}_p(1))\right)}. \tag{143}$$

Using the Tate parimetrization (and Kummer theory) it is easy to show that the image of the local Kummer map $A(L_v) \otimes \mathbb{Q}_p \hookrightarrow H^1(L_v, V_p(A))$ equals the image of the map $\Phi_{\mathrm{Tate}*} : H^1(L_v, \mathbb{Q}_p(1)) \to H^1(L_v, V_p(A))$. Then the image of π in (143) is precisely $H^1_f(L, V_p(A))$, as was to be shown. □

14.2.1. *Self-duality.* The Weil pairing [**Sil86**, Ch. III] defines a perfect, alternating and $G_{\mathbb{Q}}$-equivariant morphism of $\mathbb{Q}_p$-modules

$$\mathcal{W} : V_p(A) \otimes_{\mathbb{Q}} V_p(A) \longrightarrow \mathbb{Q}_p(1).$$

Since $\mathcal{W}$ is alternating $\mathcal{W} \circ \left(\Phi_{\mathrm{Tate}} \otimes \Phi_{\mathrm{Tate}}\right)$ is the zero map, so that by construction $V_p(A)^+_v$ is $\mathcal{W}$-orthogonal to itself for every place $v|N$ of L. Then the constructions of Section 14.1.4 give in particular a $\mathbb{Q}_p$-bilinear form:

$$\langle -, - \rangle^{\mathrm{PT}}_{L,\mathcal{W}} : \widetilde{H}^1_f(L, V_p(A)) \otimes \widetilde{H}^2_f(L, V_p(A)) \longrightarrow \mathbb{Q}_p. \tag{144}$$

We note that two different normalizations are usually used to define $\mathcal{W}$, and the resulting pairings differ by the sign. Here we take the normalization such that [**Tat95**, pag. 328]

$$\mathcal{W}(\Phi_{\mathrm{Tate}}(x) \otimes y) = x \times \pi_{q_A}(y) \tag{145}$$

for every $x \in \mathbb{Q}_p(1)$ and $y \in V_p(A)$ (see (141) for the notations).

14.2.2. *The extended Mordell-Weil group.* We now explain how to generalize Kummer theory for $A_{/L}$, giving an embedding of the extended Mordell-Weil group of $A_{/L}$ in $\widetilde{H}^1_f(L, V_p(A))$. For simplicity of notations we limit ourself to the case $L = \mathbb{Q}$. We recall [**MTT86**] that the *extended Mordell-Weil group* of $A_{/\mathbb{Q}}$ is defined by

$$A^\dagger(\mathbb{Q}) := \{(P, y) \in A(\mathbb{Q}) \times \mathbb{Q}_p^* : \Phi_{\mathrm{Tate}}(y) = P\}.$$

In other words an element of $A^\dagger(\mathbb{Q})$ consists of a $\mathbb{Q}$-rational point on $A_{/\mathbb{Q}}$, together with a distinguished lift under the p-adic Tate parametrization. By construction we have a short exact sequence

$$0 \to \mathbb{Z} \overset{i}{\to} A^\dagger(\mathbb{Q}) \to A(\mathbb{Q}) \to 0,$$

where $i(1) := (0, q_A)$ (and the right map is the natural projection). After extending scalars to $\mathbb{Q}$, this sequence has a natural splitting:

$$\sigma : A(\mathbb{Q}) \otimes \mathbb{Q} \to A^\dagger(\mathbb{Q}) \otimes \mathbb{Q}; \quad \sigma(P) := \frac{1}{\mathrm{ord}_p(q_A)}\left(\mathrm{ord}_p(q_A) \cdot P, y_P^{\mathrm{ord}_p(q_A)} \cdot q_A^{-\mathrm{ord}_p(y_P)}\right)$$

where y_P is any lift of P under Φ_{Tate}. (Note that $y_P^{\mathrm{ord}_p(q_A)} \cdot q_A^{-\mathrm{ord}_p(y_P)}$ is the unique lift of $\mathrm{ord}_p(q_A) \cdot P$ lying in $\mathbb{Z}_p^*$.) We write

$$A^\dagger(\mathbb{Q}) \otimes \mathbb{Q}_p \overset{\sigma}{=} \mathbb{Q}_p \oplus (A(\mathbb{Q}) \otimes \mathbb{Q}_p)$$

for the decomposition induced by σ. We also have a natural splitting of (142):

$$\widetilde{\sigma} : H^1_f(\mathbb{Q}, V_p(A)) \to \widetilde{H}^1_f(\mathbb{Q}, V_p(A)); \quad \widetilde{\sigma}([\xi]) = [\xi, \xi_p^+, (\xi_\ell^o)_{\ell|N}],$$

where we impose that the 1-cocycle $\xi_p^+ \in C^1_{\mathrm{cont}}(G_p, \mathbb{Q}_p(1))$ and $(\xi^o_\ell)_{\ell|N} \in \bigoplus_{\ell|N} V_p(A)$ satisfy the following two conditions:

1. $\delta(\xi^o_\ell) = -\mathrm{res}_\ell(\xi)$ for every $\ell \neq p$, and $\delta(\xi^o_p) = \Phi_{\mathrm{Tate}*}(\xi^+_p) - \mathrm{res}_p(\xi)$;
2. the cohomology class represented by ξ^+_p lies in $\mathbb{Z}_p^* \widehat{\otimes} \mathbb{Q}_p \subset H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$.

Since $V_p(A)^{G_\ell} = 0$ condition 1. implies that ξ^o_ℓ is uniquely determined for $\ell \neq p$. As the kernel of the 'cohomological Tate map' $\Phi_{\mathrm{Tate}*} : H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \to H^1(\mathbb{Q}_p, V_p(A))$ is generated by $q_A \widehat{\otimes} 1 \in \mathbb{Q}_p^* \widehat{\otimes} \mathbb{Q}_p$ and $q_A \in p\mathbb{Z}_p$, conditions 2. and 1. imply that (ξ^+_p, ξ^o_p) is uniquely determined up to elements of the form $(\delta(\eta^+_p), \Phi_{\mathrm{Tate}}(\eta^+_p))$, for $\eta^+_p \in \mathbb{Q}_p(1)$. Since by definition $d_{\widetilde{C}^\bullet_f}(0, \eta^+_p, 0) = (0, \delta(\eta^+_p), \Phi_{\mathrm{Tate}}(\eta_{p^+}))$ this shows that the cohomology class $[\xi, \xi^+_p, (\xi^o_\ell)_\ell]$ is uniquely determined by 1. and 2. The same argument also shows that $\widetilde{\sigma}$ respects coboundaries: "$\widetilde{\sigma}([\delta\psi]) = \left[d_{\widetilde{C}^\bullet_f}(\psi, 0, 0)\right]$". Then $\widetilde{\sigma}$ is 'well defined' and clearly defines a section of (142). We write as above

$$\mathbb{Q}_p \oplus H^1_f(\mathbb{Q}, V_p(A)) \overset{\widetilde{\sigma}}{=} \widetilde{H}^1_f(\mathbb{Q}, V_p(A))$$

for the decomposition attached to $\widetilde{\sigma}$. We can thus define a natural embedding:

$$i^\dagger_A : A^\dagger(\mathbb{Q}) \otimes \mathbb{Q}_p \overset{\sigma}{=} \mathbb{Q}_p \oplus \, (A(\mathbb{Q}) \otimes \mathbb{Q}_p) \overset{\mathrm{id}\oplus\mathrm{Kummer}}{\longrightarrow} \mathbb{Q}_p \oplus H^1_f(\mathbb{Q}, V_p(A)) \overset{\widetilde{\sigma}}{=} \widetilde{H}^1_f(\mathbb{Q}, V_p(A)),$$

which is an isomorphism provided that the p-part of the Tate-Shafarevich group Ш$(A/\mathbb{Q})$ of $A_{/\mathbb{Q}}$ is finite. In what follows we will identify $A^\dagger(\mathbb{Q}) \otimes \mathbb{Q}_p$ and $H^1_f(\mathbb{Q}, V_p(A))$ as sub-modules of $\widetilde{H}^1_f(\mathbb{Q}, V_p(A))$ under $i^\dagger_A$ and $\widetilde{\sigma}$ respectively.

14.3. The extended p-adic height pairing. Recall that $\mathbf{Q}_\infty = \bigcup_{n\in\mathbb{N}} \mathbf{Q}_n$ denotes the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let us fix a topological generator $\gamma_0 \in \Gamma_{\mathbf{Q}_\infty}$ and let $\varpi := \gamma_0 - 1$ be the corresponding generator of I_Λ. (Recall that $\Lambda := \mathbb{Z}_p[[\Gamma_{\mathbf{Q}_\infty}]]$ and $I_\Lambda := \ker(\varepsilon_{\mathbf{Q}_\infty})$ is its augmentation ideal.) We write $\ell_\varpi := \log_p(\chi_{\mathrm{cycl}}(\gamma_0))$, where χ_{cycl} denotes as usual the p-adic cyclotomic character.

Define the *Bockstein map* for $A/\mathbf{Q}_\infty$ by the following composition:

$$\beta^1_{\mathrm{cycl}} : \widetilde{H}^1_f(\mathbb{Q}, V_p(A)) \overset{\widetilde{\beta}^1_\varpi}{\longrightarrow} \widetilde{H}^2_{f,\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A)) \overset{\mathrm{pr}_0}{\longrightarrow} \widetilde{H}^2_f(\mathbb{Q}, V_p(A)) \overset{\times \ell_\varpi}{\longrightarrow} \widetilde{H}^2_f(\mathbb{Q}, V_p(A)).$$

Here $\widetilde{\beta}^1_\varpi$ is defined in Corollary 14.3 and pr_0 denotes the natural projection. Multiplication by ℓ_ϖ serves the purpose of removing the dependence on the choice of γ_0, so that β^1_{cycl} is a canonical morphism. Combining this morphism with (149) we can define *Nekovář's extended p-adic height paring*:

$$\langle -, - \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} : \widetilde{H}^1_f(\mathbb{Q}, V_p(A)) \times \widetilde{H}^1_f(\mathbb{Q}, V_p(A)) \longrightarrow \mathbb{Q}_p$$

by the formula:

$$\langle P, Q \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} := -\left\langle \beta^1_{\mathrm{cycl}}(P), Q \right\rangle^{\mathrm{PT}}_{\mathbb{Q},\mathcal{W}}.$$

It is a *symmetric* $\mathbb{Q}_p$-bilinear form [**Nek06**, Cor. 11.2.2]. In particular $\langle -, - \rangle^{\mathrm{Nek}}_{\mathbb{Q},p}$ gives a $\mathbb{Q}_p$-bilinear symmetric form on both the extended Mordell-Weil group $A^\dagger(\mathbb{Q}) \otimes \mathbb{Q}_p$ and on the classical Selmer group $H^1_f(\mathbb{Q}, V_p(A))$.

The following Lemma follows (as a special case) by the computation in [**Nek06**, Sec. 11.4]. We give a proof for the convenience of the reader.

LEMMA 14.7. *For every $y_f = [(y, y^+_p, \beta)] \in \widetilde{H}^1_f(\mathbb{Q}, V_p(E))$:*

$$\langle q_A, y_f \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} = \log_p([y^+_p]),$$

where $[y^+_p] \in H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p^* \widehat{\otimes} \mathbb{Q}_p$ is the cohomology class represented by y^+_p.

PROOF. Recall that $\widetilde{\beta}^1_{\varpi}([z_f])$, for a 1-cocycle $z_f = (z, z_p^+, \alpha) \in \widetilde{C}^1_f(\mathfrak{G}_N, V_p(A))$, is obtained by the following recipe (see the proof of Prop. 14.2): write $\mathfrak{G}_N := \mathrm{Gal}(\mathbb{Q}_N/\mathbb{Q})$ and let $\widetilde{z}_f = (\widetilde{z}, \widetilde{z}_p^+, \widetilde{\alpha}) \in \widetilde{C}^1_f(\mathfrak{G}_N, V_p(A)_{\mathbf{Q}_\infty})$ be any 1-cochain which lifts z_f under the morphism $\varepsilon_{\mathbf{Q}_\infty *} : \widetilde{C}^\bullet_f(\mathfrak{G}_N, V_p(A)_{\mathbf{Q}_\infty}) \to \widetilde{C}^\bullet_f(\mathfrak{G}_N, V_p(A))$ induced by the 'augmentation map' $V_p(A)_{\mathbf{Q}_\infty} \twoheadrightarrow V_p(A)$. Then the differentials $d_{\widetilde{C}^\bullet_f}(\widetilde{z}_f) = \varpi \cdot \widetilde{\mathfrak{z}}_f$ for a (unique) 2-cocycle $\widetilde{\mathfrak{z}}_f \in \widetilde{C}^2_f(\mathfrak{G}_N, V_p(A)_{\mathbf{Q}_\infty})$. Writing $\mathfrak{z}_f := \varepsilon_{\mathbf{Q}_\infty *}(\widetilde{\mathfrak{z}}_f)$:

$$\widetilde{\beta}^1_{\varpi}([z_f]) = [\,\widetilde{\mathfrak{z}}_f]; \quad \beta^1_{\mathrm{cycl}}([z_f]) = \ell_\varpi \cdot [\mathfrak{z}_f].$$

Let us fix a basis $\{\boldsymbol{\zeta}, \boldsymbol{q}\}$ of the $\mathbb{Q}_p$-module $V_p(A)$, where $\boldsymbol{\zeta} = (\zeta_{p^n})_{n\in\mathbb{N}}$ is a generator of $\mathbb{Z}_p(1)$ and $\boldsymbol{q} = (q_A^{1/p^n})_{n\in\mathbb{N}}$ is a compatible system of p^n-th roots of q_A in $\overline{\mathbb{Q}}_p$. (Here we identify $A(\overline{\mathbb{Q}})_{p^n}$ with the $G_{\mathbb{Q}_p}$-module $\left\{\zeta^i_{p^n} \cdot q_A^{j/p^n} : (i,j) \in (\mathbb{Z}/p^n\mathbb{Z})^2\right\}/q_A^{\mathbb{Z}}$ under Φ_{Tate}.) Then the action of $G_{\mathbb{Q}_p}$ on $V_p(A)$ with respect to this basis can be written:

$$G_{\mathbb{Q}_p} \ni g \mapsto \begin{pmatrix} \chi_{\mathrm{cycl}}(g) & \gamma_{\boldsymbol{q}}(g) \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

Here $\gamma_{\boldsymbol{q}} \in C^1_{\mathrm{cont}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ is a 1-cocycle such that $\Phi_{\mathrm{Tate}*}(\gamma_{\boldsymbol{q}}) = \delta(\boldsymbol{q})$, so that by definition $q_A = (0, q_A) \in A^\dagger(\mathbb{Q})$ is identified under $i_A^\dagger$ with the $[q_f] := [(0, \gamma_{\boldsymbol{q}}, \boldsymbol{q})] \in \widetilde{H}^1_f(\mathbb{Q}, V_p(A))$. Recalling that $V_p(A)_{\mathbf{Q}_\infty} := \left(V_p(A) \otimes_{\mathbb{Z}_p} \Lambda\right) \otimes \chi^{-1}_{\mathbf{Q}_\infty}$ and taking

$$\widetilde{q}_f := \left(0, \gamma_{\boldsymbol{q}} \otimes \chi^{-1}_{\mathbf{Q}_\infty}\Big|_{G_{\mathbb{Q}_p}}, \boldsymbol{q} \otimes 1\right) \in \widetilde{C}^1_f(\mathfrak{G}_N, V_p(A)_{\mathbf{Q}_\infty})$$

as a 1-cochain lifting q_f under $\varepsilon_{\mathbf{Q}_\infty *}$, we easily compute (by the discussion above):

$$\beta^1_{\mathrm{cycl}}([q_f]) = \ell_\varpi \cdot [(0, ?, \boldsymbol{q} \cdot \vartheta_\varpi)].$$

Here ? is a 2-cocycle in $C^2_{\mathrm{cont}}(\mathbb{Q}_p, \mathbb{Q}_p)$ (whose explicit description is not relevant here) and $\vartheta_\varpi \in H^1(\mathbb{Q}_p, \mathbb{Q}_p)$ is the 'derivative' of $\chi_{\mathbf{Q}_\infty}$ with respect to ϖ: $\chi_{\mathbf{Q}_\infty}(g) - 1 \equiv \vartheta_\varpi(g) \cdot \varpi \bmod \varpi^2 \cdot \Lambda$ for every $g \in G_{\mathbb{Q}_p}$ (viewed inside $G_{\mathbb{Q}}$ under the fixed embedding ρ_p^*). Combining (145) and the explicit definition of $\cup^{\mathrm{PT}}_{\mathcal{W}}$ (resp., $\underline{\mathrm{inv}}_N$) given in Section 14.1.4 (resp., Section 0.4) we compute:

$$\begin{aligned} (146) \qquad -\ell_\varpi^{-1} \cdot \langle q_A, y_f \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} &= \ell_\varpi^{-1} \cdot \left\langle \beta^1_{\mathrm{cycl}}([q_f]), y_f \right\rangle^{\mathrm{PT}}_{\mathbb{Q},\mathcal{W}} = \underline{\mathrm{inv}}_N\Big(\left[(0, ?, \boldsymbol{q} \cdot \vartheta_\varpi) \cup^{\mathrm{PT}}_{\mathcal{W}} (y, y_p^+, \beta)\right]\Big) \\ &= \underline{\mathrm{inv}}_N\Big(\left[0, (\boldsymbol{q} \cdot \vartheta_\varpi) \cup_{\mathcal{W}} \Phi_{\mathrm{Tate}}(y_p^+)\right]\Big) = \mathrm{inv}_p\Big(\left[\vartheta_\varpi \cup y_p^+\right]\Big) = \vartheta_\varpi\big(\mathrm{rec}_{\mathbb{Q}_p}([y_p^+])\big). \end{aligned}$$

Here $\mathrm{rec}_{\mathbb{Q}_p} : \mathbb{Q}_p^* \widehat{\otimes} \mathbb{Q}_p \cong G^{\mathrm{ab}}_{\mathbb{Q}_p} \widehat{\otimes} \mathbb{Q}_p$ is the local reciprocity map, normalized so that $\mathrm{rec}_{\mathbb{Q}_p}(p) = \mathrm{Frob}_p$. (For the last equality in (146) see **[Ser67]**.) Since $G^{\mathrm{ab}}_{\mathbb{Q}_p} \cong G_{\mathbb{F}_p} \times \mathrm{Gal}\left(\mathbb{Q}_p(\mu_{p^\infty}/\mathbb{Q}_p)\right)$, $\mathrm{Frob}_p \in G_{\mathbb{F}_p}$ and $\vartheta_\varpi(\widetilde{\gamma}_0) = 1$ for every $\widetilde{\gamma}_0 \in G_{\mathbb{Q}_p}$ s.t. $\rho_p^*(\widetilde{\gamma}_0)\big|_{\mathbf{Q}_\infty} = \gamma_0$, we find:

$$\vartheta_\varpi = \frac{\log_p(\chi_{\mathrm{cycl}})}{\ell_\varpi} \in H^1(\mathbb{Q}_p, \mathbb{Q}_p).$$

Then writing

$$[y_p^+] = (p^z \oplus u) \otimes p^{-n} \in \mathbb{Q}_p^* \widehat{\otimes} \mathbb{Q}_p = \left(p^{\mathbb{Z}_p} \oplus (1 + p\mathbb{Z}_p)\right) \otimes \mathbb{Q}_p$$

we obtain by (146)

$$\langle q_A, y_f \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} = -\log_p\left(\chi_{\mathrm{cycl}} \circ \mathrm{rec}_{\mathbb{Q}_p}(u \otimes p^{-n})\right) = p^{-n} \cdot \log_p(u) = \log_p\left([y_p^+]\right),$$

the second equality by Lubin-Tate theory **[Ser67]**. □

15. An exceptional Rubin's style formula

Let $\varpi := \gamma_0 - 1 \in I_\Lambda$, for a fixed topological generator $\gamma_0 \in \Gamma_{\mathbf{Q}_\infty}$. We identify $\Gamma_{\mathbf{Q}_\infty} \cong \Gamma_{\Phi_\infty}$, $\Lambda \cong \mathbf{\Lambda}$ under the (fixed) embedding $\rho_p^* : G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$. Let

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))^o := \big\{ \boldsymbol{x} \in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A)) : \partial_{p,\infty}(\boldsymbol{x}) \in \varpi \cdot H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p) \big\} \\ = \big\{ \boldsymbol{x} \in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A)) : \mathrm{pr}_0(\boldsymbol{x}) \in H^1_f(\mathbb{Q}, V_p(A)) \big\}, \tag{147}$$

where pr_0 denotes the natural projection $H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p) \to H^1(\mathbb{Q}_p, \mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cont}}(G^{\mathrm{ab}}_{\mathbb{Q}_p}, \mathbb{Q}_p)$. The equality above follows by Remark 14.4 (telling us that the induced map $\mathrm{pr}_0 : H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p)/\varpi \hookrightarrow H^1(\mathbb{Q}_p, \mathbb{Q}_p)$ is injective) and the proof of Lemma 14.6 (showing that $H^1_f(\mathbb{Q}, V_p(A)) = \ker(\partial_{p,0})$).

Given $\boldsymbol{x} \in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))$ such that $\partial_{p,\infty}(\boldsymbol{x}) = \varpi \cdot \boldsymbol{y}$ for some $\boldsymbol{y} \in H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p)$, we write

$$\mathrm{Der}_p(\boldsymbol{x}) := \log_p(\varpi) \cdot \mathrm{pr}_0(\boldsymbol{y})(\mathrm{Frob}_p) \in \mathbb{Q}_p, \tag{148}$$

where as usual $\mathrm{Frob}_p \in G_{\mathbb{F}_p} \subset G^{\mathrm{ab}}_{\mathbb{Q}_p}$ is the arithmetic Frobenius. As suggested by the notation $\mathrm{Der}_p(\boldsymbol{x})$ does not depend on the choice of $\boldsymbol{y}$, i.e. we have the following:

LEMMA 15.1. *Formula* (148) *defines a morphism* $\mathrm{Der}_p : H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))^o \to \mathbb{Q}_p$.

PROOF. The cohomology class $\boldsymbol{y}$ is unique up to the addiction of an element of the ϖ-torsion submodule $H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p)[\varpi]$. Remark 14.4 gives an isomorphism

$$\beta^0_\varpi : \mathbb{Q}_p = H^0(\mathbb{Q}_p, \mathbb{Q}_p) \cong H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p)[\varpi],$$

and a similar (and simpler) argument to that used in the proof of Lemma 14.7 easily proves that

$$\mathrm{Im}\left(H^0(\mathbb{Q}_p, \mathbb{Q}_p) \stackrel{\beta^0_\varpi}{\longrightarrow} H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p) \stackrel{\mathrm{pr}_0}{\longrightarrow} H^1(\mathbb{Q}_p, \mathbb{Q}_p) \right) = \mathbb{Q}_p \cdot \log_p(\chi_{\mathrm{cycl}}) \subset \mathrm{Hom}_{\mathrm{cont}}(G^{\mathrm{ab}}_{\mathbb{Q}_p}, \mathbb{Q}_p).$$

As χ_{cycl} vanishes on the Frobenius Frob_p the Lemma follows. □

The following proposition is an 'exceptional-case' analogue of the main result (i.e. Th. 3.2) of [**Rub94**].

PROPOSITION 15.2. *Let* $\boldsymbol{x} \in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))^o$ *and write* $\boldsymbol{x}_0 := \mathrm{pr}_0(\boldsymbol{x})$. *Then*

$$\log_A(\boldsymbol{x}_0) \cdot \mathrm{Der}_p(\boldsymbol{x}) = \frac{-1}{\mathrm{ord}_p(q_A)} \cdot \det \begin{pmatrix} \langle q_A, q_A \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} & \langle q_A, \boldsymbol{x}_0 \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} \\ \langle \boldsymbol{x}_0, q_A \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} & \langle \boldsymbol{x}_0, \boldsymbol{x}_0 \rangle^{\mathrm{Nek}}_{\mathbb{Q},p} \end{pmatrix}.$$

PROOF. Let $\mathfrak{x}_0 \in C^1_{\mathrm{cont}}(\mathbb{Q}, V_p(A))$ be a 1-cocycle representing $\boldsymbol{x}_0$, and recall (cfr. Section 14.2.2) that we identify $\boldsymbol{x}_0 \in H^1_f(\mathbb{Q}, V_p(A))$ with $\widetilde{\sigma}(\boldsymbol{x}_0) =: \big[(\mathfrak{x}_0, \mathfrak{x}^+_{0,p}, \gamma(\mathfrak{x}_0))\big] \in \widetilde{H}^1_f(\mathbb{Q}, V_p(A))$. Write $V_\infty(A) := V_p(A)_{\mathbf{Q}_\infty}$ for the cyclotomic deformation of $V_p(A)$, and $\mathfrak{G}_n := \mathrm{Gal}(\mathbb{Q}_N/\mathbf{Q}_n)$ for every $n \in \mathbb{N}$ (with $N := \mathrm{cond}(A/\mathbb{Q})$). As $H^1(\mathbf{Q}_{n,v}, V_p(A)) = 0$ for every place $v|\ell \neq p$ of $\mathbf{Q}_n$ and every $n \in \mathbb{N}$ (see the proof of Proposition 14.6), we can use Shapiro's Lemma to indetify:

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A)) \cong \varprojlim_{\mathrm{cor}} H^1(\mathfrak{G}_n, V_p(A)) \cong H^1(\mathfrak{G}, V_\infty(A)).$$

as Λ-modules, with $\mathfrak{G} := \mathfrak{G}_0$ (see Remark 14.4). Let us choose a 1-cocycle

$$\widetilde{\mathfrak{x}} \in C^1_{\mathrm{cont}}(\mathfrak{G}, V_\infty(A)); \quad [\widetilde{\mathfrak{x}}] = \boldsymbol{x} \in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))$$

representing $\boldsymbol{x}$. We also choose cochains

$$\widetilde{\mathfrak{x}}^+_{0,p} \in C^1_{\mathrm{cont}}(\mathbb{Q}_p, V_\infty(A)^+_p); \quad \widetilde{\gamma}(\mathfrak{x}_0) = (\widetilde{\gamma}_\ell(\mathfrak{x}_0))_{\ell|N} \in \bigoplus_{\ell|N} C^0_{\mathrm{cont}}(\mathbb{Q}_\ell, V_\infty(A))$$

lifting $\mathfrak{x}^+_{0,p} \in C^1_{\mathrm{cont}}(\mathbb{Q}_p, V_p(A)^+_p)$ and $\gamma(\mathfrak{x}_0) \in \bigoplus_{\ell|N} C^0_{\mathrm{cont}}(\mathbb{Q}_\ell, V_p(A))$ respectively under the 'augmentation map' $\varepsilon_{\mathbf{Q}_\infty *}$. (Here $V_\infty(A)^+_p := \big(V_p(A)^+_p\big)_{\mathbf{Q}_\infty}$ is the cyclotomic deformation of $V_p(A)^+_p = \mathbb{Q}_p(1) \stackrel{\Phi_{\mathrm{Tate}}}{\to} V_p(A)$.)

Then by construction we have:

$$d_{\widetilde{C}^\bullet_f}\left(\widetilde{\mathfrak{x}}, \widetilde{\mathfrak{x}}^+_{0,p}, \widetilde{\gamma}(\mathfrak{x}_0)\right) = \Big(0, \delta\left(\widetilde{\mathfrak{x}}^+_{0,p}\right), \left(-\mathrm{res}_\ell\left(\widetilde{\mathfrak{x}}_0\right) - \delta\left(\widetilde{\gamma}_\ell(\mathfrak{x}_0)\right)\right)_{\ell\neq p} \oplus \Big(\widetilde{\Phi}_{\mathrm{Tate}*}\left(\widetilde{\mathfrak{x}}^+_{0,p}\right) - \mathrm{res}_p\left(\widetilde{\mathfrak{x}}\right) - \delta\left(\widetilde{\gamma}_p(\mathfrak{x}_0)\right)\Big)\Big)$$

$$= \varpi \cdot \Big(0, \widetilde{\mathfrak{y}}^+_p, \left(\gamma_\ell(\widetilde{\mathfrak{y}}_f)\right)_{\ell|N}\Big) = \varpi \cdot \widetilde{\mathfrak{y}}_f \tag{149}$$

for a 2-cocycle $\widetilde{\mathfrak{y}}_f := \Big(0, \widetilde{\mathfrak{y}}^+_p, \left(\gamma_\ell(\widetilde{\mathfrak{y}}_f)\right)_{\ell|N}\Big) \in \widetilde{C}^2_f(\mathfrak{G}, V_\infty(A))$, where $\widetilde{\Phi}_{\mathrm{Tate}*}$ is the map induced on cochains by $\Phi_{\mathrm{Tate}} \otimes \Lambda : \mathbb{Q}_p(1) \otimes \Lambda \to V_p(A) \otimes \Lambda = V_\infty(A)$. Writing $\mathfrak{y}_f := \varepsilon_{\mathbf{Q}_\infty*}(\widetilde{\mathfrak{y}}_f) =: \Big(0, \mathfrak{y}^+_p, \left(\gamma_\ell(\mathfrak{y}_f)\right)_{\ell|N}\Big)$, we have again by construction (cfr. the proof of Proposition 14.7): $\beta^1_{\mathrm{cycl}}(\boldsymbol{x}_0) = \ell_\varpi \cdot \mathfrak{y}_f$. Retracing the definitions of Sections 0.4, 14.1.4 and 14.3 we obtain: for every $z_f = [(z, z^+_p, (z_\ell)_{\ell|N})] \in \widetilde{H}^1_f(\mathbb{Q}, V_p(A))$

$$\ell^{-1}_\varpi \cdot \langle \boldsymbol{x}_0, z_f\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} = -\langle [\mathfrak{y}_f], z_f\rangle^{\mathrm{PT}}_{\mathbb{Q},\mathcal{W}} = -\mathrm{inv}_p\left(\left[\gamma_p(\mathfrak{y}_f) \cup_{\mathcal{W}} \Phi_{\mathrm{Tate}*}(z^+_p)\right]\right)$$

$$= -\left\langle \gamma^-_p(\mathfrak{y}_f), \left[z^+_p\right]\right\rangle_{\mathbb{Q}_p} = -\gamma^-_p(\mathfrak{y}_f)\left(\mathrm{rec}_{\mathbb{Q}_p}\left(\left[z^+_p\right]\right)\right), \tag{150}$$

where we have written $\gamma^-_p(\mathfrak{y}_f) := \pi_{q_A*}(\gamma_p(\mathfrak{y}_f))$ (cfr. equation (141)) and we have used equation (145) (resp., local class field theory **[Ser67]**) for the third (resp., last) equality. (As above $\mathrm{rec}_{\mathbb{Q}_p} : \mathbb{Q}^*_p \widehat{\otimes} \mathbb{Q}_p \cong G^{\mathrm{ab}}_{\mathbb{Q}_p} \widehat{\otimes} \mathbb{Q}_p$ normalized as in **[Ser67]**.)

Let us write

$$\ell_\varpi \cdot \gamma^-_p(\mathfrak{y}_f) = A_p \cdot \mathrm{Log}_{q_A} + B_p \cdot \psi^{\mathrm{un}}_{\mathbb{Q}_p} \in \mathrm{Hom}_{\mathrm{cont}}(G^{\mathrm{ab}}_{\mathbb{Q}_p}, \mathbb{Q}_p); \quad A_p, B_p \in \mathbb{Q}_p,$$

where

$$\psi^{\mathrm{un}}_{\mathbb{Q}_p} : G^{\mathrm{un}}_{\mathbb{Q}_p} \longrightarrow \mathbb{Q}_p; \quad \mathrm{Log}_{q_A} = -\Big(\log_p(\chi_{\mathrm{cycl}}) + \mathscr{L}_p(A) \cdot \psi^{\mathrm{un}}_{\mathbb{Q}_p}\Big)$$
$$\mathrm{Frob}_p \mapsto 1$$

(so that Log_{q_A} and $\psi^{\mathrm{un}}_{\mathbb{Q}_p}$ form a $\mathbb{Q}_p$-basis of $\mathrm{Hom}_{\mathrm{cont}}(G^{\mathrm{ab}}_{\mathbb{Q}_p}, \mathbb{Q}_p)$). Recall that $\left[\mathfrak{x}^+_{0,p}\right] \in \mathbb{Z}^*_p \widehat{\otimes} \mathbb{Q}_p \subset H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ by construction, so that $\log_A(\boldsymbol{x}_0) = \log_p\left(\left[\mathfrak{x}^+_{0,p}\right]\right)$. Combining Lubin-Tate theory **[Ser67]**, Lemma 14.7 and equation (150) we obtain:

$$\langle \boldsymbol{x}_0, \boldsymbol{x}_0\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} = -\ell_\varpi \cdot \gamma^-_p(\mathfrak{y}_f)\left(\left[\mathfrak{x}^+_{0,p}\right]\right) = -A_p \cdot \mathrm{Log}_{q_A}\left(\mathrm{rec}_{\mathbb{Q}_p}\left(\left[\mathfrak{x}^+_{0,p}\right]\right)\right) = -A_p \cdot \log_A(\boldsymbol{x}_0);$$

$$\log_A(\boldsymbol{x}_0) = \langle q_A, \boldsymbol{x}_0\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} = \langle \boldsymbol{x}_0, q_A\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} = -\ell_\varpi \cdot \gamma^-_p(\mathfrak{y}_f)\left(\mathrm{rec}_{\mathbb{Q}_p}(q_A)\right) = -B_p \cdot \mathrm{ord}_p(q_A).$$

These equations, combined with Lemma 14.7 allows us to rewrite the determinant:

$$\det\begin{pmatrix} \langle q_A, q_A\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} & \langle q_A, \boldsymbol{x}_0\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} \\ \langle \boldsymbol{x}_0, q_A\rangle^{\mathrm{Nek}}_{\mathbb{Q},p} & \langle \boldsymbol{x}_0, \boldsymbol{x}_0\rangle^{\mathrm{Nek}}_{\mathbb{Q},p}\end{pmatrix} = -B_p \cdot \mathrm{ord}_p(q_A)^2 \cdot \Big(B_p - A_p \cdot \mathscr{L}_p(A)\Big) \tag{151}$$

$$= \Big(\log_A(\boldsymbol{x}_0) \cdot \mathrm{ord}_p(q_A)\Big) \cdot \Big(\ell_\varpi \cdot \gamma^-_p(\mathfrak{y}_f)(\mathrm{Frob}_p)\Big).$$

On the other hand: identify as usual $H^1\Big(\mathbb{Q}_p, (\mathbb{Q}_p)_{\mathbf{Q}_\infty}\Big) = H^1\Big(\mathbb{Q}_p, (\mathbb{Q}_p)_{\Phi_\infty}\Big) \cong H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p)$ via Shapiro's Lemma, and write $\gamma^-_p(\widetilde{\mathfrak{y}}_f) := \widetilde{\pi}_{q_A*}(\gamma_p(\widetilde{\mathfrak{y}}_f)) \in C^1_{\mathrm{cont}}\Big(\mathbb{Q}_p, (\mathbb{Q}_p)_{\mathbf{Q}_\infty}\Big)$ for the image of $\gamma_p(\widetilde{\mathfrak{y}}_f)$ under the map induced by $\pi_{q_A} \otimes \Lambda : V_\infty(A) = V_p(A) \otimes \Lambda \twoheadrightarrow \mathbb{Q}_p \otimes \Lambda$. Then equation (149) (together with the naturality of the Shapiro's isomorphism) gives: $\gamma^-_p(\widetilde{\mathfrak{y}}_f)$ is a 1-cocycle and

$$\partial_{p,\infty}(\boldsymbol{x}) = \Big(\widetilde{\pi}_{q_A*} \circ \mathrm{res}_p\left([\widetilde{\mathfrak{x}}]\right)\Big) = -\varpi \cdot \left[\gamma^-_p(\widetilde{\mathfrak{y}}_f)\right]. \tag{152}$$

Since by definition $\gamma_p(\mathfrak{y}_f) = \varepsilon_{\mathbf{Q}_\infty*}(\gamma_p(\widetilde{\mathfrak{y}}_f))$, we have $\gamma^-_p(\mathfrak{y}_f) = \mathrm{pr}_0\left(\left[\gamma^-_p(\widetilde{\mathfrak{y}}_f)\right]\right)$ and combining (152) with Lemma 15.1 we obtain:

$$\mathrm{Der}_p(\boldsymbol{x}) = -\ell_\varpi \cdot \gamma^-_p(\mathfrak{y}_f)(\mathrm{Frob}_p).$$

Together with (151) this proves the formula in the statement. □

Remark 15.3. It seems to the author that 'Nekovář's Rubin-style formula' [**Nek06**, Prop. 11.5.11] is incorrect as stated, and then can not be applied in order to obtain (more quickly) Prop. 15.2. As an example: let $\widetilde{q}_f$ be as in the proof of Lemma 14.7 and take "$x_{\mathrm{Iw},f} := \widetilde{q}_f$" (using again the notations of [**Nek06**, Prop. 11.5.11]). Then *loc. cit.* would give '$\langle q_A, z_f\rangle_{\mathbb{Q},p}^{\mathrm{Nek}} = 0$' for every $z_f \in \widetilde{H}^1_f(\mathbb{Q}, V_p(A))$, which is clearly not the case by Lemm 14.7. (We think that the failure of *loc. cit.* is caused by the presence of ϖ-torsion in $H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q})$. In the example above we have indeed $d_{\widetilde{C}^\bullet_f}(\widetilde{q}_f) = (0, ?, \widehat{c}_p)$ and $\widetilde{\pi}_{q_A*}(\widehat{c}_p) = \varpi \cdot \widetilde{\vartheta}_\varpi$, where the 1-cocyle $\widetilde{\vartheta}_\varpi := \left\{G_{\mathbb{Q}_p} \ni g \mapsto \left(\chi^{-1}_{\mathbf{Q}_\infty}(g) - 1\right)/\varpi\right\}$ represents a cohomology class lying in $H^1_{\mathrm{Iw}}\left(\mathbb{Q}_p, (\mathbb{Q}_p)_{\mathbf{Q}_\infty}\right)[\varpi]$.)

16. Kato's Euler zeta elements

In this Section we recall some of the main properties of the Euler system for $\mathrm{Ta}_p(A)$ constructed by Kato in [**Kat04**]. We refer the reader to [**Rub00**, Section 3.5] for more references, details and applications.

16.1. Dual exponentials. Let B_{dR} be Fontain's (topological) field of periods, and write $B^+_{\mathrm{dR}} = \mathrm{Fil}^0\left(B^+_{\mathrm{dR}}\right)$ for its ring of integers. Let us write $\mathrm{R} := \left\{\left(x^{(n)}\right) \in \prod_{n\in\mathbb{N}} \mathcal{O}_{\mathbb{C}_p} : \left(x^{(n+1)}\right)^p = x^{(n)},\ \forall n\right\}$ for the 'perfection of $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$', where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. Then we have a natural $G_{\mathbb{Q}_p}$-equivariant injective map $\mathbb{Z}_p(1) \hookrightarrow \mathrm{R}^*$ and a (continuous) morphism of groups $\mathrm{Log} : \mathrm{Frac}(\mathrm{R})^* \to B^+_{\mathrm{dR}}$ which makes commutative the following diagram:

$$\begin{array}{ccc} \mathrm{Frac}(\mathrm{R})^* & \xrightarrow{\ \mathrm{Log}\ } & B^+_{\mathrm{dR}} \\ \downarrow & & \downarrow \\ \mathbb{C}_p^* & \xrightarrow{\ \log_p\ } & \mathbb{C}_p. \end{array}$$

The left (resp., right) vertical arrow is $\left(x^{(n)}\right)_{n\in\mathbb{N}} \mapsto x^{(0)}$ (resp., projection to the residue field), and $\log_p$ is the branch of the p-adic logarithm vanishing on p. Fix a sequence $\boldsymbol{q}_A = \left(q_A, q_A^{p^{-1}}, \ldots, q_A^{p^{-n}}, \ldots\right) \in \mathrm{R}$ of p^n-th root of the Tate period and a generator $\boldsymbol{\zeta}$ of $\mathbb{Z}_p(1)$, so that we can identify $\{\boldsymbol{q}_A, \boldsymbol{\zeta}\}$ with a $\mathbb{Q}_p$-basis of $V_p(A)$ via Tate's theory (141). It is not difficult to show that $\boldsymbol{q}_A \mapsto \mathrm{Log}(\boldsymbol{q}_A)$ and $\boldsymbol{\zeta} \mapsto \mathrm{Log}(\boldsymbol{\zeta})$ 'realizes' $V_p(A)$ inside B^+_{dR}, i.e. gives an injective $\mathbb{Q}_p$-linear and $G_{\mathbb{Q}_p}$-equivariant morphism $V_p(A) \hookrightarrow B^+_{\mathrm{dR}}$. Then we easily obtain: for every finite extension $L/\mathbb{Q}_p$

$$H^0\left(L, V_p(A) \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}}\right) = L \cdot \boldsymbol{\xi}_A,$$

where

$$\boldsymbol{\xi}_A := \boldsymbol{q}_A \otimes 1 - \boldsymbol{\zeta} \otimes \left[\mathrm{Log}(\boldsymbol{\zeta})^{-1} \cdot \left(\mathrm{Log}(\boldsymbol{q}_A) - \log_p(q_A)\right)\right].$$

(As $\mathrm{Log}(\boldsymbol{\zeta})$ is a uniformizer in B^+_{dR}, $\boldsymbol{\xi}_A$ indeed lies in $V_p(A) \otimes B^+_{\mathrm{dR}}$ by the diagram above.) As proved in [**Kat93**, Ch. II], the cup-product with $\log_p \circ \chi_{\mathrm{cycl}} \in H^1(L, \mathbb{Q}_p)$ gives an isomorphism $H^0(L, V_p(A) \otimes B^+_{\mathrm{dR}}) \cong H^1(L, V_p(A) \otimes B^+_{\mathrm{dR}})$. The *Bloch-Kato dual exponential map* can then be defined by the composition:

$$\exp^*_{A,L} : H^1(L, V_p(A)) \xrightarrow{i_*} H^1(L, V_p(A) \otimes B^+_{\mathrm{dR}}) \cong \left(V_p(A) \otimes B^+_{\mathrm{dR}}\right)^{G_L} = L \cdot \boldsymbol{\xi}_A \cong L,$$

where the last map sends $\boldsymbol{\xi}_A$ to 1. Since $\pi_{q_A} \otimes \mathrm{id} : V_p(A) \otimes B^+_{\mathrm{dR}} \to \mathbb{Q}_p \otimes B^+_{\mathrm{dR}} = B^+_{\mathrm{dR}}$ maps $\boldsymbol{\xi}_A$ to 1, by construction the following diagram:

$$\begin{array}{ccc} H^1\left(L, V_p(A)\right) & \xrightarrow{\ \exp^*_{A,L}\ } & L \\ \downarrow{\scriptstyle \pi_{q_A*}} & & \| \\ H^1(L, \mathbb{Q}_p) & \xrightarrow{\ \exp^*_L\ } & L \end{array} \tag{153}$$

commutes for every finite extension $L/\mathbb{Q}_p$. For every $n \in \mathbb{N}$ we will write $\exp^*_{A,n} := \exp^*_{A,\Phi_n}$.

16.2. Kato's zeta elements. Let $L(A,s)=L(A/\mathbb{Q},s)$ be the Hasse-Weil complex L-function of $A_{/\mathbb{Q}}$, which is defined for $\Re(s)>\frac{3}{2}$ by the Euler product:

$$L(A,s):=\prod_{\ell\nmid N}\left(1-a_\ell(A)\ell^{-s}+\ell^{1-2s}\right)^{-1}\cdot\prod_{\ell|N}\left(1-a_\ell(A)\ell^{-s}\right)^{-1}=\prod_{\ell}E_\ell(\ell^{-s})^{-1}.$$

Here $N:=\mathrm{cond}(A/\mathbb{Q})$. For every prime $\ell\nmid N$ the Euler factor $E_\ell(X)\in\mathbb{Z}[X]$ is the characteristic polynomial of an arithmetic Frobenius $\mathrm{Frob}_q\in G_{\mathbb{Q}_\ell}$ acting on $V_p(A)$, for every prime $q\nmid N\ell$. For a prime $\ell|N$ we have $E_\ell(X)=1-X$ (resp., $E_\ell(X)=1+X$) if $A_{/\mathbb{Q}_\ell}$ has split (resp., non-split) multiplicative reduction, and $E_\ell(X)=1$ if $A_{/\mathbb{Q}_\ell}$ has additive reduction [**Sil86**, Ch. V]. Given a finite order character $\chi:G_{\mathbb{Q}}\to\mathbb{C}^*$ of conductor f_χ and an integer M we write

$$L_{\{M\}}(A,\chi,s):=\prod_{\ell\nmid f_\chi\cdot M}E_\ell\left(\chi\left(\mathrm{Frob}_\ell\right)\cdot\ell^{-s}\right)^{-1},$$

where $\mathrm{Frob}_\ell\in G_{\mathbb{Q}_\ell}\hookrightarrow G_{\mathbb{Q}}$ is an arithmetic Frobenius at ℓ. Thanks to the modularity theorem proved by Wiles et al. and the work of Hecke we know that $L_{\{M\}}(A,\chi,s)$ can be analytically continued to the whole complex plane. If $M=1$ we simply write $L(A,\chi,s):=L_{\{1\}}(A,\chi,s)$.

The following deep results is due to Kato. Its statement is taken from [**Rub00**, Th. 3.5.3], to which we refer for precise references. We will write $\Omega_A=\Omega_A^+\in\mathbb{R}^*$ for the real (or twice the real) period attached to a global minimal Weierstrass model $A_{/\mathbb{Z}}$ of $A_{/\mathbb{Q}}$ [**Sil86**, Appendix C.16].

THEOREM 16.1. (Kato) *There exist* $\boldsymbol{\zeta}_\infty^{\mathrm{Kato}}=\lim_{n\to\infty}\boldsymbol{\zeta}_n^{\mathrm{Kato}}\in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty,V_p(A))$ *satisfying the following interpolation property: for every* $n\in\mathbb{N}$ *and every character* χ *of* $\mathrm{Gal}(\mathbf{Q}_n/\mathbb{Q})$

$$\sum_{\gamma\in\mathrm{Gal}(\mathbf{Q}_n/\mathbb{Q})}\chi(\gamma)\cdot\exp^*_{A,n}\left(\mathrm{res}_p\big(\gamma\big(\boldsymbol{\zeta}_n^{\mathrm{Kato}}\big)\big)\right)=\frac{L_{\{N\}}(A,\chi,1)}{\Omega_A},$$

where N *is the conductor of* $A_{/\mathbb{Q}}$.

REMARK 16.2. As $\psi(-1)=1$ for every Dirichlet character ψ mod p^{n+1} factoring throught $\mathrm{Gal}(\mathbf{Q}_n/\mathbb{Q})$, a theorem of Shimura [**Shi77b**] gives

$$\frac{L_{\{M\}}(A,\psi,1)}{\Omega_A}\in\mathbb{Q}(\psi)$$

for every such character and every integer M, where $\mathbb{Q}(\psi)/\mathbb{Q}$ is the algebraic extension obtained by adding to $\mathbb{Q}$ the values of ψ. The equalities of the preceding Theorem take place in $\overline{\mathbb{Q}}_p$ (identifying $\overline{\mathbb{Q}}$ as a subfield of $\overline{\mathbb{Q}}_p$ under our fixed embedding).

16.3. Zeta elements and the p-adic L-function. The *(cyclotomic) p-adic L-function* of $A_{/\mathbb{Q}}$ is defined to be a 'measure on $\Gamma_{\mathbf{Q}_\infty}$': $\mathcal{L}_p(A)\in\Lambda\otimes\mathbb{Q}$ satisfying the following interpolation property: for every non-trivial character $\chi:\Gamma_{\mathbf{Q}_\infty}\to\overline{\mathbb{Q}}^*$ of finite order

$$\chi\big(\mathcal{L}_p(A)\big)=\tau(\chi)\cdot\frac{L(A,\chi^{-1},1)}{\Omega_A}. \tag{154}$$

Under our assumptions the existence of $\mathcal{L}_p(A)$ is proved (in greater generality) in [**MTT86**, Ch. I], while its uniqueness follows by the Weierstrass preparation theorem. Given an integer M we write

$$\mathcal{L}_p(A,M):=\prod_{\ell|M,\ \ell\neq p}E_\ell\left(\ell^{-1}\cdot\ \mathrm{Frob}_\ell^{-1}|_{\mathbf{Q}_\infty}\right)\cdot\mathcal{L}_p(A)$$

We denote by $L_p(A,M,s)$ the p-adic Mellin transform of $\mathcal{L}_p(A,M)$, defined for every $s\in\mathbb{Z}_p$ by

$$L_p(A,M,s):=\chi_{\mathrm{cycl}}^{s-1}\big(\mathcal{L}_p(A,M)\big).$$

We will simply write $L_p(A,s)$ for $L_p(A,1,s)$.

THEOREM 16.3. *With the notations of Theorem* 16.1*:* $L_p(\boldsymbol{\zeta}_\infty^{\mathrm{Kato}},s)=L_p(A,N,s)$.

PROOF. Let us write for simplicity $\mathscr{C}_\infty^{\mathrm{Kato}} := \mathscr{C}_\infty\left(\partial_{p,\infty}\left(\boldsymbol{\zeta}_\infty^{\mathrm{Kato}}\right)\right)$, and let us identify $\mathrm{Gal}(\mathbf{Q}_n/\mathbb{Q}) \cong \mathbf{G}_n = \mathrm{Gal}(\Phi_n/\mathbb{Q}_p)$ under a fixed embedding $\rho_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (for every $0 \leq n \leq \infty$). For every character $\chi : \Gamma_{\mathbf{Q}_\infty} \twoheadrightarrow \mathbf{G}_n \to \overline{\mathbb{Q}}_p^*$ of conductor $p^k > 1$:

$$\chi\left(\mathscr{C}_\infty^{\mathrm{Kato}}\right) = \chi\left(\mathscr{C}_\infty \circ \partial_{p,\infty}\left(\boldsymbol{\zeta}_\infty^{\mathrm{Kato}}\right)\right) \overset{\text{Prop. 13.2}}{=} \tau(\chi) \cdot \sum_{\gamma \in \mathbf{G}_n} \chi^{-1}(\gamma) \cdot \exp_n^*\left(\gamma \circ \partial_{p,n}\left(\boldsymbol{\zeta}_n^{\mathrm{Kato}}\right)\right)$$

$$\overset{(153)}{=} \tau(\chi) \cdot \sum_{\gamma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbb{Q})} \chi^{-1}(\gamma) \cdot \exp_{A,n}^*\left(\mathrm{res}_p \circ \gamma\left(\boldsymbol{\zeta}_n^{\mathrm{Kato}}\right)\right) \overset{\text{Theorem 16.1}}{=} \tau(\chi) \cdot \frac{L_{\{N\}}(A, \chi^{-1}, 1)}{\Omega_A}$$

$$= \tau(\chi) \cdot \frac{L_{\{N/p\}}(A, \chi^{-1}, 1)}{\Omega_A} \overset{(154)}{=} \chi\big(\mathcal{L}_p(A, N)\big).$$

By the Weierstrass preparation theorem: $\mathscr{C}_\infty^{\mathrm{Kato}} = \mathcal{L}_p(A, N)$. Applying the Mellin transform to this equality of measures we obtain the statement. □

Thanks to the work of Rohrlich (see again [**Rub00**, Sec. 3.5] for references) we know that $\mathcal{L}_p(A)$ is non-zero, i.e. the special value $L(A, \psi, 1)$ is non-zero for all but finitely many cyclotomic finite-order characters ψ. As a corollary of the preceding Theorem we then obtain:

COROLLARY 16.4. (Rohrlich) $\boldsymbol{\zeta}_\infty^{\mathrm{Kato}} \neq 0$.

17. Proofs

In this section we conclude the proofs of the results stated in Section 3.

Let us fix $\boldsymbol{u} = \lim_{n\to\infty} u_n \in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))$. We recall that the p-adic L-function attached to $\boldsymbol{u}$ is defined by

$$L_p(\boldsymbol{u}, s) := \chi_{\mathrm{cycl}}^{s-1}\Big(\mathscr{C}_\infty \circ \partial_{p,\infty}(\boldsymbol{u})\Big),$$

where $\mathscr{C}_\infty : H^1_{\mathrm{Iw}}(\Phi_\infty, V_p(A)) \to \mathbf{I}_\infty \otimes \mathbb{Q} \cong I_\Lambda \otimes \mathbb{Q}$ is the ($\mathbb{Q}$-linear extension of the) Coleman map defined in Section 13.

17.1. Proposition 12.25 **and Theorem** 12.30**.** By Proposition 13.3 we have

$$\begin{aligned}\frac{d}{ds}L_p(\boldsymbol{u}, s)_{s=1} &= \frac{d}{ds}\left[\chi_{\mathrm{cycl}}^{s-1}\left(\frac{\partial_p(u_0)\,(\mathrm{Frob_p})}{\ell_\varpi \cdot (1-p^{-1})} \cdot \varpi + \varpi^2 \cdot \star\right)\right]_{s=1} \\ &= \frac{d}{ds}\left(\left(\ell_\varpi(s-1) - (\ell_\varpi(s-1))^2/2 + \cdots\right) \cdot \frac{\partial_p(u_0)\,(\mathrm{Frob_p})}{\ell_\varpi \cdot (1-p^{-1})} + \cdots\right)_{s=1} \qquad (155) \\ &= \left(1-p^{-1}\right)^{-1} \cdot \partial_p(u_0)\,(\mathrm{Frob}_p).\end{aligned}$$

Looking at the long $G_{\mathbb{Q}_p}$-exact cohomology sequence attached to (141) we have

$$\mathrm{Im}\Big(H^1(\mathbb{Q}_p, V_p(A)) \overset{\pi_{q_A*}}{\to} H^1(\mathbb{Q}_p, \mathbb{Q}_p)\Big) \overset{\text{Kummer Theory}}{=} \ker\Big(H^1(\mathbb{Q}_p, \mathbb{Q}_p) \overset{*\cup q_A}{\to} H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))\Big)$$

$$\overset{\text{Class Field Theory}}{=} \Big\{\psi \in \mathrm{Hom_{cont}}(G_{\mathbb{Q}_p}^{\mathrm{ab}}, \mathbb{Q}_p) : \psi\left(\mathrm{rec}_{\mathbb{Q}_p}(q_A)\right) = 0\Big\} = \mathbb{Q}_p \cdot \mathrm{Log}_{q_A},$$

where $\mathrm{Log}_{q_A} := \log_p(\chi_{\mathrm{cycl}}) + \mathscr{L}_p(A) \cdot \psi_{\mathbb{Q}_p}^{\mathrm{un}}$ is as in the proof of Prop. 15.2. Then $\partial_p(u_0) = \wp(u_0) \cdot \mathrm{Log}_{q_A}$ for some $\wp(u_0) \in \mathbb{Q}_p$, so that we rewrite (155) as:

$$\frac{d}{ds}L_p(\boldsymbol{u}, s)_{s=1} = \left(1-p^{-1}\right)^{-1} \cdot \wp(\boldsymbol{u}) \cdot \mathrm{Log}_{q_A}(\mathrm{Frob}_p) = \left(1-p^{-1}\right)^{-1} \cdot \mathscr{L}_p(A) \cdot \wp(u_0). \qquad (156)$$

On the other, as $\exp_p(\mathbb{Q}_p) = \mathbb{Z}_p^* \otimes \mathbb{Q}$, the defining property of the dual exponential map $\exp_{\mathbb{Q}_p}^*$ (128) gives $\exp_{\mathbb{Q}_p}^*(\psi_{\mathbb{Q}_p}^{\mathrm{un}}) = 0$ and $\exp_{\mathbb{Q}_p}^*\left(\log_p(\chi_{\mathrm{cycl}})\right) = 1$. We then finally obtain

$$\exp_{\mathbb{Q}_p}^*(u_0) = \wp(u_0) \cdot \exp_{\mathbb{Q}_p}^*(\mathrm{Log_{q_A}}) = \wp(u_0) = \partial_p^{\log}(u_0),$$

which combined with (156) concludes the proof of Proposition 12.25.

Let us now take $\boldsymbol{u} = \zeta_\infty^{\mathrm{Kato}}$: then

$$\begin{aligned} \frac{d}{ds}L_p(A,s)_{s=1} &= \prod_{\ell|N;\ l\neq p} E_\ell\left(\ell^{-1}\right)^{-1} \cdot \frac{d}{ds}L_p(A,N,s)_{s=1} \\ \text{(Th. 16.3)} \quad &= \prod_{\ell|N;\ l\neq p} E_\ell\left(\ell^{-1}\right)^{-1} \cdot \frac{d}{ds}L_p(\boldsymbol{\zeta}_\infty^{\mathrm{Kato}},s)_{s=1} \\ \text{(Prop. 12.25)} \quad &= \prod_{\ell|N} E_\ell\left(\ell^{-1}\right)^{-1} \cdot \mathscr{L}_p(A) \cdot \left(\exp^*_{\mathbb{Q}_p} \circ \partial_p\right)\left(\boldsymbol{\zeta}_0^{\mathrm{Kato}}\right) \\ \text{(Th. 16.1 + (153))} \quad &= \prod_{\ell|N} E_\ell\left(\ell^{-1}\right)^{-1} \cdot \mathscr{L}_p(A) \cdot \frac{L_{\{N\}}(A,1)}{\Omega_A} = \mathscr{L}_p(A) \cdot \frac{L(A,1)}{\Omega_A}, \end{aligned}$$

concluding the proof of Theorem 12.30.

17.2. Proposition 12.28 **and Theorem** 12.31**.** Let $\boldsymbol{u} = \lim u_n \in H^1_{\mathrm{Iw}}(\mathbf{Q}_\infty, V_p(A))$ s.t. $0 \neq u_0 \in H^1_f(\mathbb{Q}, V_p(A))$. Corollary 12.26 implies: $L_p(\boldsymbol{u}, s)$ vanishes to order at least 2 at $s = 1$. More precisely (cfr. equation (147))

$$\partial_{p,\infty}(\boldsymbol{u}) = \varpi \cdot \boldsymbol{u}'_p,$$

for some $\boldsymbol{u}'_p = \lim_{n\to\infty} u'_{p,n} \in H^1_{\mathrm{Iw}}(\Phi_\infty, \mathbb{Q}_p)$. As $\chi_{\mathrm{cycl}}^{s-1}(\varpi) = (s-1)\cdot \ell_\varpi - \frac{1}{2}(s-1)^2 \cdot \ell_\varpi^2 + \cdots$ and $\mathscr{C}_\infty$ is a morphism of $\Lambda \cong \mathbf{\Lambda}_\infty$-modules, applying Proposition 13.3 (and Lemma 15.1) we obtain:

$$\begin{aligned} \frac{1}{2} \cdot \frac{d^2}{ds^2} L_p(\boldsymbol{u}, s)_{s=1} = \ell_\varpi \cdot \frac{d}{ds}\chi_{\mathrm{cycl}}^{s-1}\Big(\mathscr{C}_\infty\left(\boldsymbol{u}'_p\right)\Big)\Big|_{s=1} \\ = \ell_\varpi \cdot \left(1 - p^{-1}\right)^{-1} \cdot u'_{p,0}\left(\mathrm{Frob}_p\right) = \left(1-p^{-1}\right)^{-1} \cdot \mathrm{Der}_p(\boldsymbol{u}). \end{aligned}$$

(Note that $\mathscr{C}_\infty(\boldsymbol{u}'_p)$, and then $L_p(\boldsymbol{u}, s)$ depends only on $\boldsymbol{u}$, even if $\boldsymbol{u}'_p$ is well defined only up to ϖ-torsion elements.) Combined with proposition (15.2) this formula concludes the proof of Proposition 12.28.

Taking $\boldsymbol{u} = \boldsymbol{z}_{\infty,\varpi}^{\mathrm{Kato}}$ (with the notations of Section 3), using the results of Section 16 and retracing the definitions we easily see that Proposition 12.28 'specializes' to Theorem 12.31.

APPENDIX A

A short course in Nekovář's theory

Notations. Let $\mathcal{R} = (\mathcal{R}, \mathfrak{m})$ be a complete local Noetherian ring with finite residue field $\mathcal{R}/\mathfrak{m}$ of characteristic $p \geq 3$. We write $\mathcal{D}(\mathcal{R})$ for the derived category of complexes of $\mathcal{R}$-modules [**Har66**, Ch. 1].

Let $K/\mathbb{Q}$ be a number field, and let S_f be a finite set of finite primes of K, containing every prime dividing p. We denote by $K_S \subset \overline{K}$ the maximal algebraic extension of K which is unramified outside $S := S_f \cup \{v|\infty\}$. We fix, for every $v \in S_f$, an embedding $\rho_v : \overline{K} \hookrightarrow \overline{K_v}$, and we write $\rho_v^* : G_{K_v} = \mathrm{Gal}(\overline{K_v}/K_v) \hookrightarrow G_K$ for the induced map and $G_v := \rho_v^*(G_{K_v})$ for the corresponding decomposition group. (Here K_v is the completion of K at the prime v.) We write $S_p := \{v \in S_f : v|p\}$.

Let $G \in \{G_{K,S}, G_v, G_{K_v}\}$. For every *admissible* $\mathcal{R}[G]$-module M (in the sense of [**Nek06**, Ch. 3]), we can consider the complex $C^\bullet_{\mathrm{cont}}(G, M)$ of (non-homogeneous) continuous cochains. We write

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G, M) \in \mathcal{D}(\mathcal{R}); \quad H^*(G, M) := H^*\left(C^\bullet_{\mathrm{cont}}(G, M)\right)$$

for the image of $C^\bullet_{\mathrm{cont}}(G, M)$ in the derived category and its cohomology. We also use the notations: $C^\bullet_{\mathrm{cont}}(K_v, M) := C^\bullet_{\mathrm{cont}}(G_{K_v}, M)$, $\mathbf{R}\Gamma_{\mathrm{cont}}(K_v, M) := \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K_v}, M)$ and $H^*(K_v, M) := H^*(G_{K_v}, M)$.

When M is an $\mathcal{R}[G]$-module of finite (resp, co-finite) type over $\mathcal{R}$, then M is admissible precisely when G acts continuously with respect to the $\mathfrak{m}$-adic (resp., discrete) topology on M, and $C^\bullet_{\mathrm{cont}}(G, M)$ is the usual continuous cochain complex. (For example, if M is of finite type over $\mathcal{R}$ we have

$$C^\bullet_{\mathrm{cont}}(G, M) = \varprojlim{}_n C^\bullet_{\mathrm{cont}}(G, M/\mathfrak{m}^n M),$$

considering of course on each $M/\mathfrak{m}^n M$ the discrete topology.)

We consider any admissible $\mathcal{R}[G_{K,S}]$-module M as an admissible $\mathcal{R}[G_{K_v}]$-module ($v \in S_f$) via ρ_v^*. We have natural isomorphism of complexes

$$C^\bullet_{\mathrm{cont}}(G_v, M) \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(K_v, M),$$

i.e. that induced by the 'morphisms of pairs' $(\rho_v^*, \mathrm{id}_M) : (G_v, M) \to (G_{K_v}, M)$, under which we will always identify these complexes. 'Restricting cocycles' from $G_{K,S}$ to G_v via the natural map $G_v \subset G_K \twoheadrightarrow G_{K,S}$ then induces a *restriction map*

$$res_v : C^\bullet_{\mathrm{cont}}(G_{K,S}, M) \to C^\bullet_{\mathrm{cont}}(K_v, M).$$

We also write $res_v : H^q(G_{K,S}, M) \to H^q(K_v, M)$ for the induced map and $res_{S_f} := \oplus_{v \in S_f} res_v$.

0.3. Generalities. Let X be an admissible $\mathcal{R}[G_{K,S}]$-module. A *local condition* $\Delta_v(X)$ for X at $v \in S_f$ is a complex of $\mathcal{R}$-modules $U_v^+(X)$, together with a morphism of complexes of $\mathcal{R}$-modules

$$i_v^+ = i_v^+(X) : U_v^+(X) \longrightarrow C^\bullet_{\mathrm{cont}}(K_v, X).$$

(We usually write $\Delta_v(X) = U_v^+(X)$ when the morphism i_v^+ is clear.)

Given local conditions $\Delta(X) = \{\Delta_v(X)\}_{v \in S_f}$ the associated *Nekovář's Selmer complex* is defined by [**Nek06**, Ch. 6]:

$$\widetilde{C}^\bullet_f(G_{K,S}, X; \Delta(X)) := \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,S}, X) \oplus \bigoplus_{v \in S_f} U_v^+(X) \xrightarrow{res_{S_f} - i^+_{S_f}} \bigoplus_{v \in S_f} C^\bullet_{\mathrm{cont}}(K_v, X)\right)[-1].$$

Here $i^+_{S_f} = \oplus_{v \in S_f} i_v^+$. We denote by $(x_n, (x^+_{n,v}), (x_{n-1,v}))$ (or more simply by (x_n, x_n^+, x_{n-1})) an element of $\widetilde{C}^n_f(G_{K,S}, X; \Delta(X))$.

For $* = \emptyset, \mathrm{ft}, \mathrm{cf}$, let $\mathcal{D}_*(\mathcal{R})$ be the derived category of complexes of $\mathcal{R}$-modules with cohomology of type $*$ over R. Here ft and cf means of finite and co-finite type respectively. Let

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta(X)) \in \mathcal{D}(\mathcal{R})$$

be the image of $\widetilde{C}_f^\bullet(G_{K,S}, X, \Delta(X))$ in $\mathcal{D}(\mathcal{R})$, and

$$\widetilde{H}_f^q(G_{K,S}, X; \Delta(X)) := H^q\left(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta(X))\right).$$

When X is of $*$-type over $\mathcal{R}$ and $U_v^+(X) \in \mathcal{D}_*(\mathcal{R})$, we have $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta(X)) \in \mathcal{D}_*(\mathcal{R})$.

0.4. Class field theory. Let M be an $\mathcal{R}$-module. With trivial $G_{K,S}$-action, it is an admissible $\mathcal{R}[G_{K,S}]$-module, and so is its Tate twist $M(1) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$ (with $\mathbb{Z}_p(1) := \varprojlim_{n\geq 1} \mu_{p^n}(\overline{K})$). We consider the complex

$$\mathcal{C}(K, M) := \tau_{\geq 3}\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,S}, M(1)) \overset{res_{S_f}}{\to} \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(K_v, M(1))\right)[-1],$$

where $\tau_{\geq 3}[\cdots X_2 \overset{\delta}{\to} X_3 \to \cdots] := [0 \to X_3/\mathrm{Im}(\delta) \to X_4 \to \cdots]$ denotes the 'good truncation' in degree three. We note that $\mathcal{C}(K, M) = \tau_{\geq 3}\widetilde{C}_f^\bullet(G_{K,S}, M(1); \Delta_c)$, where $\Delta_{c,v} : 0 \to C^\bullet_{\mathrm{cont}}(K_v, M(1))$ is the 'full local condition' for every $v \in S_f$.

Let us denote by $\mathbb{R}\mathcal{C}(K, M)$ the image of $\mathcal{C}(K, M)$ in the derived category $\mathcal{D}(\mathcal{R})$. It follows by global class field theory that there is an isomorphism in $\mathcal{D}(\mathcal{R})$

$$\underline{\mathrm{inv}}_{S_f}(M) : \mathbb{R}\mathcal{C}(K, M) \overset{\sim}{\to} M[-3],$$

which is functorial in M. We can describe this isomorphism as follows.

For every $v \in S_f$ let $\mathrm{inv}_v = \mathrm{inv}_v(M) : H^2(K_v, M(1)) \overset{\sim}{\to} M$ be the isomorphism defined (taking limits) by the invariant maps $H^2(K_v, \mathbb{Z}/p^n\mathbb{Z}(1)) \overset{\sim}{\to} \mathbb{Z}/p^n\mathbb{Z}$ of local class field theory. Let $m \in M = (M[-3])^3$: by the fundamental exact sequence of global classfield theory, there exists 2-cocycles $(y_v)_{v\in S_f} \in C^\bullet_{\mathrm{cont}}(K_v, M(1))$ such that $m = \sum_{v\in S_f} \mathrm{inv}_v([y_v])$. Moreover if $(y'_v)_{v\in S_f}$ is another sequence with this property, then $([y_v - y'_v])_v = res_{S_f}([x]) \in \bigoplus_{v\in S_f} H^2(K_v, M(1))$ for a 2-cocycle $x \in C^\bullet_{\mathrm{cont}}(G_{K,S}, M(1))$. This implies that $\left[(0, (y_v)_{v\in S_f})\right] = \left[(0, (y'_v)_{v\in S_f})\right] \in \mathcal{C}(K, M)$. We then obtain a morphism of complexes

$$r_M : M[-3] \to \mathcal{C}(K, M),$$

defined in degree three by $m \mapsto \left[(0, (y_v)_{v\in S_f})\right]$. Since $G_{K,S}$ and G_{K_v} have cohomological dimension 2, it follows easily by the definition that r_M is a quasi-isomorphism, so that it induces an isomorphism $M[-3] \overset{\sim}{\to} \mathbb{R}\mathcal{C}(K, M)$ in $\mathcal{D}(\mathcal{R})$. $\underline{\mathrm{inv}}_{S_f}(M)$ is defined as the inverse of this isomorphism. For more details see [**Nek06**, Ch. 5].

We will write again $\underline{\mathrm{inv}}_{S_f}(M) : H^3(\mathcal{C}(K, M)) \overset{\sim}{\to} M$ to denote the isomorphism induced in cohomology.

0.5. Greenberg local conditions. We will consider from now on modules and local conditions of the following (elementary) type.

Let X a free $\mathcal{R}$-module of finite type, with a continuous, $\mathcal{R}$-linear action of $G_{K,S}$. We assume that there exists for every $v \in S_f$ a short exact sequence of $\mathcal{R}[G_v]$-modules

$$0 \to X_v^+ \overset{i_v^+}{\to} X \overset{p_v^-}{\to} X_v^- \to 0,$$

with $X_v^\pm$ free as $\mathcal{R}$-modules. We then define $\Delta_v(X) = C^\bullet_{\mathrm{cont}}(K_v, X_v^+)$, with $i_v^+(X) : C^\bullet_{\mathrm{cont}}(K_v, X_v^+) \to C^\bullet_{\mathrm{cont}}(K_v, X)$ defined as the morphism induced by i_v^+ for every $v \in S_f$ and $\Delta(X) := \{\Delta_v(X)\}_{v\in S_f}$. We will write from now on simply X' = '$\{X; i_v^+ : X_v^+ \hookrightarrow X,\ v \in S_f\}$ to denote the $\mathcal{R}[G_{K,S}]$-module X, together with the choice of $\mathcal{R}[G_v]$-submodules $i_v^+ : X_v^+ \hookrightarrow X$. We will also write

$$\widetilde{C}_f^\bullet(G_{K,S}, X) := \widetilde{C}_f^\bullet(G_{K,S}, X; \Delta(X)) \in \mathrm{K}(\mathcal{R});$$

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) := \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X; \Delta(X)) \in \mathcal{D}^b_{\mathrm{ft}}(\mathcal{R});$$

$$\widetilde{H}_f^*(G_{K,S}, X) := \widetilde{H}_f^*(G_{K,S}, X; \Delta(X)) \in ({}_{\mathcal{R}}\mathrm{Mod})_{\mathrm{ft}}\,.$$

It follows by the definitions [**Nek06**, §(6.1.3)] that we have an exact triangle in $\mathcal{D}(\mathcal{R})$:

$$\bigoplus_{v \in S_f} \mathbf{R}\Gamma_{\mathrm{cont}}(K_v, X_v^-)[-1] \to \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \to \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X). \tag{157}$$

Taking cohomology we obtain a long exact sequence of $\mathcal{R}$-modules

$$\cdots \to \bigoplus_{v \in S_f} H^{q-1}(K_v, X_v^-) \to \widetilde{H}_f^q(G_{K,S}, X) \to H^q(G_{K,S}, X) \to \bigoplus_{v \in S_f} H^q(K_v, X_v^-) \to \cdots, \tag{158}$$

where the last map is obtained composing res_{S_f} with the (sum of) the maps induced by p_v^-.

0.6. Orthogonal local conditions. Let us fix X and Y as in the preceding Section. Let

$$\pi : X \otimes_{\mathcal{R}} Y \to \mathcal{R}(1)$$

be a morphism of $\mathcal{R}$-modules, inducing a *perfect duality* between X and Y, i.e. such that

$$\mathrm{adj}(\pi) : X \overset{\sim}{\to} \mathrm{Hom}_{\mathcal{R}}(Y, \mathcal{R}(1))$$

is an isomorphism of $\mathcal{R}[G_{K,S}]$-modules (where $\mathrm{adj}(\pi)(x) : y \mapsto \pi(x \otimes y)$). We say that X_v^+ is π-orthogonal to Y_v^+, and write $X_v^+ \perp_\pi Y_v^+$ if the following composition is the zero map:

$$X_v^+ \otimes_{\mathcal{R}} Y_v^+ \overset{i_v^+ \otimes i_v^+}{\to} X \otimes_{\mathcal{R}} Y \overset{\pi}{\to} \mathcal{R}(1).$$

We write $X \perp_\pi Y$ if $X_v^+ \perp_\pi Y_v^+$ for every $v \in S_f$.

If $X_v^+ \perp_\pi Y_v^+$ then $\mathrm{adj}\,(\pi)$ induces morphisms of short exact sequences of $\mathcal{R}[G_v]$-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_v^+ & \longrightarrow & X & \longrightarrow & X_v^- & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{adj}(\pi)}\,\wr & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(Y_v^-, \mathcal{R}(1)) & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(Y, \mathcal{R}(1)) & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(Y_v^+, \mathcal{R}(1)) & \longrightarrow & 0, \end{array} \tag{159}$$

for every $v \in S_f$. We write $W_v = W_v(\pi) := \ker(\beta)$, sitting in an exact sequence of $\mathcal{R}[G_v]$-modules

$$0 \to W_v(\pi) \to X_v^- \to \mathrm{Hom}_{\mathcal{R}}\left(Y_v^+, \mathcal{R}(1)\right) \to 0.$$

Moreover we say that X_v^+ is the *(π-)orthogonal complement* of Y_v^+, written

$$X_v^+ \perp\!\perp_\pi Y_v^+$$

if $W_v(\pi) = 0$ (or equivalently, if (159) is an isomorphism of short exact sequences of $\mathcal{R}[G_v]$-modules).

0.7. Global cup-products. Let X, Y and π be as above. For $G \in \{G_{K,S}, G_v\}$, the morphism π induces a cup-product pairing

$$\cup_\pi : C^\bullet_{\mathrm{cont}}(G, X) \otimes_{\mathcal{R}} C^\bullet_{\mathrm{cont}}(G, Y) \to C^\bullet_{\mathrm{cont}}(G, \mathcal{R}(1)).$$

For $x = (x_v) \in \bigoplus_{v \in S_f} C^\bullet_{\mathrm{cont}}(K_v, X)$ and $y = (y_v) \in \bigoplus_{v \in S_f} C^\bullet_{\mathrm{cont}}(K_v, Y)$, we write again $x \cup_\pi y := \oplus_v x_v \cup_\pi y_v$.

LEMMA 0.1. *Assume that $X \perp_\pi Y$ and let $r, s \in \mathcal{R}$.*

a) The formula

$$\begin{aligned} &(x_n, x_n^+, x_{n-1}) \cup_{\boldsymbol{\pi}, \boldsymbol{r}} (y_m, y_m^+, y_{m-1}) := \\ &\qquad \tau_{\geq 3}\Big(x_n \cup_\pi y_m,\ x_{n-1} \cup_\pi \left(r \cdot res_{S_f}(y_m) + (1-r) \cdot i^+_{S_f}(y_m^+)\right) \\ &\qquad\qquad + (-1)^n \left((1-r) \cdot res_{S_f}(x_n) + r \cdot i^+_{S_f}(x_n^+)\right) \cup_\pi y_{m-1}\Big) \end{aligned} \tag{160}$$

defines a morphism of complexes of $\mathcal{R}$-modules

$$\cup_{\boldsymbol{\pi}, \boldsymbol{r}} : \widetilde{C}_f^\bullet(G_{K,S}, X) \otimes_{\mathcal{R}} \widetilde{C}_f^\bullet(G_{K,S}, Y) \to \mathcal{C}(K, R).$$

b) The formula

$$(x_n, x_n^+, x_{n-1}) \otimes (y_m, y_m^+, y_{m-1}) \mapsto \tau_{\geq 3}\left(0, (-1)^n (r-s) \cdot x_{n-1} \cup_\pi y_{m-1}\right)$$

defines a homotopy between $\cup_{\pi,r}$ *and* $\cup_{\pi,s}$.

PROOF. A simple computation [**Nek06**, Prop. 1.3.2]. □

Under the condition of the preceding Lemma, $\cup_{\boldsymbol{\pi},\boldsymbol{r}}$ and $\underline{\mathrm{inv}}_{S_f}(\mathcal{R})$ induce a morphism in $\mathcal{D}(\mathcal{R})$

$$\cup_{\boldsymbol{\pi}} : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X;) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y) \to \mathbb{R}\mathcal{C}(K, R) \xrightarrow{\sim} \mathcal{R}[-3],$$

which is independent on the choice of $r \in \mathcal{R}$. By adjunction we obtain a morphism

$$\boldsymbol{\gamma_\pi} := \mathrm{adj}\,(\cup_{\boldsymbol{\pi}}) : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathcal{R}}\left(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y), \mathcal{R}\right)[-3].$$

(Here $- \otimes^{\mathbf{L}}_{\mathcal{R}} -$ and $\mathbf{R}\mathrm{Hom}_{\mathcal{R}}(-,-)$ are the derived functors attached to the total tensor product $- \otimes_{\mathcal{R}} -$ and $\mathrm{Hom}^{\bullet}_{\mathcal{R}}(-,-)$ on the (homotopy) category of complexes of $\mathcal{R}$ modules [**Nek06**, Ch. 2], [**Har66**, Ch. 2], and adj refers to the isomorphism in [**Har66**, Ch. 2, Prop. 5.15].)

PROPOSITION 0.2. *Assume that $\mathcal{R}$ is a Gorenstein ring (i.e. it is isomorphic in $\mathcal{D}(\mathcal{R})$ to a bounded complex of injective $\mathcal{R}$-modules). We have an exact triangle in $\mathcal{D}(\mathcal{R})$:*

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \xrightarrow{\boldsymbol{\gamma_\pi}} \mathbf{R}\mathrm{Hom}_{\mathcal{R}}\left(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y), \mathcal{R}\right)[-3] \longrightarrow \bigoplus_{v \in S_f} \mathbf{R}\Gamma_{\mathrm{cont}}\left(K_v, W_v(\pi)\right).$$

PROOF. Under our assumptions, this is a special case of [**Nek06**, Prop. 6.7.7]. □

0.8. Specializations. Let us take X, Y and π as in the preceding Section. We also assume $X \perp_\pi Y$. Let $f : \mathcal{R} \twoheadrightarrow \widetilde{\mathcal{R}}$ be a surjective morphism of rings, and let $I := \ker(f)$. For every $\mathcal{R}$-module M we write $\widetilde{M} := M_{\mathcal{R},f}\widetilde{\mathcal{R}} \cong M/I \cdot M$.

For $Z \in \{X, Y\}$, $\widetilde{Z}$ is a free $\widetilde{\mathcal{R}}$-module, with a continuos $\widetilde{R}$-linear action of $G_{K,S}$. Moreover, for every $v \in S_f$ we have short exact sequences of $\widetilde{\mathcal{R}}[G_v]$-modules

$$0 \to \widetilde{Z}_v^+ \to \widetilde{Z} \to \widetilde{Z}_v^- \to 0,$$

where $\widetilde{Z}_v^{\pm} := \widetilde{Z_v^{\pm}}$ are free $\widetilde{\mathcal{R}}$-modules. The morphism π induces a morphism of $\widetilde{\mathcal{R}}[G_{K,S}]$-modules

$$\widetilde{\pi} := \pi \otimes_{\mathcal{R},f} \widetilde{\mathcal{R}} : \widetilde{X} \otimes_{\widetilde{\mathcal{R}}} \widetilde{Y} \longrightarrow \widetilde{\mathcal{R}}(1),$$

which is a perfect duality between $\widetilde{X}$ and $\widetilde{Y}$ over $\widetilde{\mathcal{R}}$, such that $\widetilde{X} \perp_{\widetilde{\pi}} \widetilde{Y}$. In other words: $\widetilde{X}$, $\widetilde{X}_v^{\pm}$, $\widetilde{Y}$, $\widetilde{Y}_v^{\pm}$ and $\widetilde{\pi}$ are again data of the type discussed in the preceding Sections.

We will write $\mathbf{L}f^* : \mathcal{D}^b(\mathcal{R}) \to \mathcal{D}^b(\widetilde{\mathcal{R}})$ for the left derived functor of the base change functor $- \otimes_{\mathcal{R},f} \widetilde{\mathcal{R}}$ on the category of complexes of $\mathcal{R}$-modules. (Here $\mathcal{D}^b(*)$ denoted the full triangulated subcategory of $\mathcal{D}(*)$ whose object are those complexes of $*$-module which are isomorphic in $\mathcal{D}(*)$ to a bounded complex of $*$-modules.)

REMARK 0.3. Write f_* for the exact forgetful functor from $\widetilde{\mathcal{R}}$-modules to $\mathcal{R}$-modules, so that we have a natural isomorphism $f_* \circ \mathbf{L}f^* = - \otimes^{\mathbf{L}}_{\mathcal{R},f} \widetilde{\mathcal{R}}$. Since $f^* \circ f_*$ is isomorphic to the identity functor and $\mathbf{R}f_* = f_* : \mathcal{D}^b(\widetilde{\mathcal{R}}) \to \mathcal{D}^b(\mathcal{R})$ and $\mathbf{L}f^* : \mathcal{D}^b(\mathcal{R}) \to \mathcal{D}^b(\widetilde{\mathcal{R}})$ are adjoint functors [**Har66**, pag. 111], we easily obtain natural isomorphisms for $M \in \mathcal{D}^b(\mathcal{R})$ and $N \in \mathcal{D}^b(\widetilde{\mathcal{R}})$:

$$\mathrm{Hom}_{\mathcal{D}(\widetilde{\mathcal{R}})}\left(\mathbf{L}f^*(M), N\right) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{R})}\left(M \otimes^{\mathbf{L}}_{\mathcal{R},f} \widetilde{\mathcal{R}}, f_*(N)\right);$$

$$\mathrm{Hom}_{\mathcal{D}(\widetilde{\mathcal{R}})}\left(N, \mathbf{L}f^*(M)\right) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{R})}\left(f_*(N), M \otimes^{\mathbf{L}}_{\mathcal{R},f} \widetilde{\mathcal{R}}\right).$$

In other words: a morphism (resp., isomorphism) in $\mathcal{D}(\widetilde{\mathcal{R}})$ between $\mathbf{L}f^*M$ and N 'is the same' as a morphism (resp., isomorphism) in $\mathcal{D}(\mathcal{R})$ between $M \otimes^{\mathbf{L}}_{\mathcal{R},f} \widetilde{\mathcal{R}}$ and f_*N.

LEMMA 0.4. *Assume that* $I = \boldsymbol{x} := (x_1, \ldots, x_d)$ *is generated by an* $\mathcal{R}$*-regular sequence* $\boldsymbol{x}$.
a) For $T \in \{Z, Z_v^{\pm}\}$ *and* $G = G_{K,S}$ *or* G_v, *there exists canonical isomorphisms in* $\mathcal{D}(\widetilde{\mathcal{R}})$:

$$\mathbf{L}f^* \left(\mathbf{R}\Gamma_{\mathrm{cont}}(G, T)\right) \cong \mathbf{R}\Gamma_{\mathrm{cont}}(G, \widetilde{T}).$$

b) There exists canonical isomorphisms in $\mathcal{D}(\widetilde{\mathcal{R}})$

$$\mathbf{L}f^* \left(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Z; \Delta(Z))\right) \cong \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \widetilde{Z}; \Delta(\widetilde{Z})).$$

c) The isomorphisms in b) induce a commutative diagram in $\mathcal{D}(\widetilde{\mathcal{R}})$:

$$\begin{array}{ccc}
\mathbf{L}f^* \left(\widetilde{\mathbf{R}\Gamma}_f(X) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathbf{R}\Gamma}_f(Y)\right) & \xrightarrow{\mathbf{L}f^*(\cup_\pi)} & \mathbf{L}f^* \left(\mathcal{R}[-3]\right) \\
\sim\downarrow & & \sim\downarrow \\
\widetilde{\mathbf{R}\Gamma}_f(\widetilde{X}) \otimes^{\mathbf{L}}_{\widetilde{\mathcal{R}}} \widetilde{\mathbf{R}\Gamma}_f(\widetilde{Y}) & \xrightarrow{\cup_{\widetilde{\pi}}} & \widetilde{\mathcal{R}}[-3]
\end{array}$$

(where $\widetilde{\mathbf{R}\Gamma}_f(-) := \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, -; \Delta(-))$*).*

PROOF. *a)* We prove the statement by induction on d.

Let $d = 1$ and write $x = x_1$, which by assumption is a non-zerodivisor in $\mathcal{R}$. By [**Nek06**, Prop. 3.4.2] (and [**Nek06**, Prop. 3.5.10]), the tautological exact sequence of $\mathcal{R}[G]$-modules $0 \to T \xrightarrow{x} T \to \widetilde{T} \to 0$ gives rise to a short exact sequence of complexes of $\mathcal{R}$-modules

$$0 \to C^\bullet_{\mathrm{cont}}(G, T) \xrightarrow{x} C^\bullet_{\mathrm{cont}}(G, T) \to C^\bullet_{\mathrm{cont}}(G, \widetilde{T}) \to 0, \tag{161}$$

so that we have isomorphisms in the derived category $\mathcal{D}(\mathcal{R})$:

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G, T) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathcal{R}} \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G, T) \otimes_{\mathcal{R}} \mathscr{P}_{\widetilde{\mathcal{R}}} \xrightarrow{\sim} \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G, T) \xrightarrow{x} C^\bullet_{\mathrm{cont}}(G, T)\right) \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G, \widetilde{T}),$$

where $\mathscr{P}_{\widetilde{\mathcal{R}}} := \left(\mathcal{R} \xrightarrow{x} \mathcal{R}\right)$, concentrated in degrees -1 and 0 is a free resolution of the $\mathcal{R}$-module $\widetilde{\mathcal{R}}$. (The first isomorphism follows by the definition of the derived functor $- \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathcal{R}}$.)

Assume now that $d \geq 2$, and write: $x := x_d$, $\boldsymbol{x}' := (x_1, \ldots, x_{d-1})$, $\widetilde{\mathcal{R}}' := \mathcal{R}/\boldsymbol{x}'$ and $\widetilde{T}' := T \otimes_{\mathcal{R}} \widetilde{\mathcal{R}}'$. Then x is a non-zero-divisor in $\widetilde{\mathcal{R}}'$, $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}'/x$ and we have a short exact sequence of $\widetilde{\mathcal{R}}'$-modules:

$$0 \to \widetilde{T}' \xrightarrow{x} \widetilde{T}' \to \widetilde{T} \to 0.$$

Using induction and what already proved we obtain isomorphisms:

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G, \widetilde{T}) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathcal{R}} \xrightarrow{\sim} \left(\mathbf{R}\Gamma_{\mathrm{cont}}(G, T) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathcal{R}}'\right) \otimes^{\mathbf{L}}_{\widetilde{\mathcal{R}}'} \widetilde{\mathcal{R}} \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G, \widetilde{T}') \otimes^{\mathbf{L}}_{\widetilde{\mathcal{R}}'} \widetilde{\mathcal{R}} \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cont}}(G, \widetilde{T}),$$

which is easily seen to depend only on the prime I (i.e. not on the choice of the $\mathcal{R}$-regular sequence generating it). Using the discussion in Remark 0.3, this in turn defines the isomorphism in the statement.

b) The same argument used in the proof of *a)* applies. In fact, assume $d = 1$. The exact sequences (161) are 'compatible' with respect to $res_{S_f} : C^\bullet_{\mathrm{cont}}(G_{K,S},) \to C^\bullet_{\mathrm{cont}}(K_v, -)$ and $i_v^+ : C^\bullet_{\mathrm{cont}}(K_v, (-)_v^+) \to C^\bullet_{\mathrm{cont}}(K_v, -)$, so that they induce a short exact sequence of complexes of $\mathcal{R}$-modules:

$$0 \to \widetilde{C}^\bullet_f(G_{K,S}, Z; \Delta(Z)) \xrightarrow{x} \widetilde{C}^\bullet_f(G_{K,S}, Z; \Delta(Z)) \to \widetilde{C}^\bullet_f(G_{K,S}, \widetilde{Z}; \Delta(\widetilde{Z})) \to 0,$$

which can be rewritten as an isomorphism:

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Z; \Delta(Z)) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathcal{R}} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \widetilde{Z}; \Delta(\widetilde{Z})).$$

For $d \geq 2$, the induction argument proceeds exactly as in the proof of *a)*.

c) This follows from the definitions and the commutativity of the following diagram of complexes of $\mathcal{R}$-modules, where $r \in \mathcal{R}$, $\widetilde{r} = r \bmod I$, $\widetilde{C}^{\bullet}_f(-) := \widetilde{C}^{\bullet}_f(G_{K,S}, -; \Delta(-))$, $\mathcal{C}(-) := \mathcal{C}(K, -)$ (and the morphism r_- is defined in Sec. 0.4) :

$$\begin{array}{ccccc} \widetilde{C}^{\bullet}_f(X) \otimes_{\mathcal{R}} \widetilde{C}^{\bullet}_f(Y) & \xrightarrow{\cup_{\pi,r}} & \mathcal{C}(\mathcal{R}) & \xleftarrow{r_{\mathcal{R}}} & \mathcal{R}[-3] \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{C}^{\bullet}_f(\widetilde{X}) \otimes_{\widetilde{\mathcal{R}}} \widetilde{C}^{\bullet}_f(\widetilde{Y}) & \xrightarrow{\cup_{\widetilde{\pi},\widetilde{r}}} & \mathcal{C}(\widetilde{\mathcal{R}}) & \xleftarrow{r_{\widetilde{\mathcal{R}}}} & \widetilde{\mathcal{R}}[-3]. \end{array}$$

Here the vertical maps are those induces by 'reduction modulo I'. The commutativity of the left-hand (resp., right-hand) square follows by the definitions of the cup-product in Lemma 0.1 (resp., by the functoriality of the invariant maps of local classfield theory). □

0.9. Hermitian case. In this section we assume that $\mathcal{R}$ is equiped with an involution ι, i.e. with a ring isomorphism $\iota : \mathcal{R} \to \mathcal{R}$ such that $\iota^2 = \mathrm{id}$. For every $\mathcal{R}$-module M, we denote by M^{ι} the $\mathcal{R}$-module with the same underline abelian group of M, but with $\mathcal{R}$-action obtained composing the original action with ι. This defines a functor $M \mapsto M^{\iota}$ on the category of $\mathcal{R}$-modules (with $f^{\iota} := f$ for a morphism $M \to N$). For a complex of $\mathcal{R}$-modules X, X^{ι} is defined by $(X^{\iota})^n = (X^n)^{\iota}$. Again this define a functor on the category of complexes of $\mathcal{R}$-module, which derive trivially to a functor on $\mathcal{D}(\mathcal{R})$. For every admissible $\mathcal{R}[G]$-module M ($G \in \{G_{K,S}, G_v\}$), we have $C^{\bullet}_{\mathrm{cont}}(G, M^{\iota}) = C^{\bullet}_{\mathrm{cont}}(G, M)^{\iota}$ as complexes of $\mathcal{R}$-modules.

Let X be as in Sec. 0.5. We write $Y := X^{\iota}$ and $Y_v^{\pm} := (X_v^{\pm})^{\iota}$ ($v \in S_f$), which are again data of the type considered in Sec. 0.5. We assume that there exists a perfect duality $\pi : X \otimes_{\mathcal{R}} Y \to \mathcal{R}(1)$ between X and Y, such that $X \perp_{\pi} Y$. We also assume that there exists $c = \pm 1$ such that

$$\pi' := \pi \circ s_{12} = c \cdot \iota \circ \pi^{\iota} : Y \otimes_{\mathcal{R}} X \longrightarrow \mathcal{R}(1). \tag{162}$$

We note that π' is a perfect duality between Y and X, such that $Y \perp_{\pi'} X$.

Let us write $\widetilde{C}^{\bullet}_f(-) := \widetilde{C}^{\bullet}_f(G_{K,S}, -; \Delta(-))$ and $\mathcal{C} := \mathcal{C}(K, \mathcal{R})$. As above, we have $\widetilde{C}^{\bullet}_f(Y) = \widetilde{C}^{\bullet}_f(X)^{\iota}$, so that Lemma 0.1 gives us a cup product pairing ($r \in \mathcal{R}$)

$$\cup_{\boldsymbol{\pi},\boldsymbol{r}} : \widetilde{C}^{\bullet}_f(X) \otimes_{\mathcal{R}} \widetilde{C}^{\bullet}_f(X)^{\iota} \longrightarrow \mathcal{C},$$

and a corresponding product in the derived category:

$$\cup_{\boldsymbol{\pi}} : \widetilde{\mathbf{R}\Gamma}_f(X) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathbf{R}\Gamma}_f(X)^{\iota} \longrightarrow \mathcal{R}[-3].$$

In the same way π' induces:

$$\cup_{\boldsymbol{\pi'},\boldsymbol{r}} : \widetilde{C}^{\bullet}_f(X)^{\iota} \otimes_{\mathcal{R}} \widetilde{C}^{\bullet}_f(X) \longrightarrow \mathcal{C}; \quad \cup_{\boldsymbol{\pi'}} : \widetilde{\mathbf{R}\Gamma}_f(X)^{\iota} \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow \mathcal{R}[-3].$$

It follows immediately from (162) and Lemma 0.1 that we have

$$\cup_{\boldsymbol{\pi'},\boldsymbol{r}} = c \cdot \left(\iota \circ (\cup_{\boldsymbol{\pi},\boldsymbol{r}})^{\iota}\right).$$

By the functoriality of the isomorphism $\underline{\mathrm{inv}}_{S_f}(-)$ we also obtain the formula:

$$\cup_{\boldsymbol{\pi'}} = c \cdot \left(\iota \circ (\cup_{\boldsymbol{\pi}})^{\iota}\right). \tag{163}$$

Moreover, as in [**Nek06**, Sec. 6.5], the existence of 'transposition operators' for Greenberg local conditions (see also Sec. 6.7 of *loc. cit.*) implies that the following diagram of complexes of $\mathcal{R}$-modules:

$$\begin{array}{ccc} \widetilde{C}^{\bullet}_f(X) \otimes_{\mathcal{R}} \widetilde{C}^{\bullet}_f(X)^{\iota} & \xrightarrow{\cup_{\boldsymbol{\pi},\boldsymbol{r}}} & \mathcal{C} \\ \downarrow{\scriptstyle s_{12}} & & \| \\ \widetilde{C}^{\bullet}_f(X)^{\iota} \otimes_{\mathcal{R}} \widetilde{C}^{\bullet}_f(X) & \xrightarrow{\cup_{\boldsymbol{\pi'},\boldsymbol{1-r}}} & \mathcal{C} \end{array}$$

commutes up to homotopy (with $s_{12}(x \otimes y) = (-1)^{ij} y \otimes x$ for $x \in \widetilde{C}_f^\bullet(X)^i$ and $y \in \widetilde{C}_f^\bullet(X)^j$). In particular (using b) of Lemma 0.1) we obtain the identity in $\mathcal{D}(\mathcal{R})$:

$$\cup_{\boldsymbol{\pi}'} \circ s_{12} = \cup_{\boldsymbol{\pi}}.$$

Combined with (163), this proves the following:

LEMMA 0.5. *We have a commutative diagram in $\mathcal{D}(\mathcal{R})$:*

$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_f(X) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathbf{R}\Gamma}_f(X)^\iota & \xrightarrow{\cup_{\boldsymbol{\pi}}} & \mathcal{R}[-3] \\ \downarrow{\scriptstyle s_{12}} & & \uparrow{\scriptstyle \iota} \\ \widetilde{\mathbf{R}\Gamma}_f(X)^\iota \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathbf{R}\Gamma}_f(X) & \xrightarrow{c\cdot(\cup_{\boldsymbol{\pi}})^\iota} & \mathcal{R}^\iota[-3]. \end{array}$$

In other words: $\cup_{\boldsymbol{\pi}}$ is symmetric-Hermitian (resp., skew-Hermitian) if $c = +1$ (resp., $c = -1$).

In particular: assume that $\iota = \mathrm{id}$. Then $\cup_{\boldsymbol{\pi}}$ is symmetric (resp., skew-symmetric) if $c = +1$ (resp., $c = -1$).

0.10. Cassels-Tate parings. The notations and hypothesis are as in the preceding Section, and we write $\mathscr{R} := \mathrm{Frac}(\mathcal{R})$ for its total ring of fraction. For every complex of $\mathcal{R}$-modules M, let $\mathbf{R}\Gamma_!(M) := M \otimes_{\mathcal{R}} \left(\mathcal{R} \stackrel{-i}{\to} \mathscr{R}\right)$ and $H^*_!(M) := H^*\left(\mathbf{R}\Gamma_!(M)\right)$, where $\left(\mathcal{R} \stackrel{-i}{\to} \mathscr{R}\right) =: \mathscr{P}$ is concentrated in degrees 0 and 1. (The functor $M \mapsto \mathbf{R}\Gamma_!(M)$ derives trivially to a functor $\mathcal{D}(\mathcal{R}) \to \mathcal{D}(\mathcal{R})$.) We note that $\mathbf{R}\Gamma_!(M) \stackrel{\sim}{\to} \mathrm{Cone}\left(M \stackrel{-i}{\to} M \otimes_{\mathcal{R}} \mathscr{R}\right)[-1]$ as complexes of $\mathcal{R}$-modules. In particular, if $M \in \mathcal{D}_{\mathrm{ft}}(\mathcal{R})$ we obtain for every $q \in \mathbb{Z}$ a short exact sequence of $\mathcal{R}$-modules:

$$0 \to H^{q-1}(M) \otimes_{\mathcal{R}} \mathscr{R}/\mathcal{R} \to H^q_!(M) \to H^q(M)_{\mathrm{tors}} \to 0, \tag{164}$$

where $_{\mathrm{tors}}$ refers to the $\mathcal{R}$-torsion.

We have a quasi-isomorphism

$$v : \mathbf{R}\Gamma_!(\mathscr{P}) = \left(\mathcal{R} \stackrel{(-i,-i)}{\to} \mathscr{R} \oplus \mathscr{R} \stackrel{\mathrm{id}\oplus -\mathrm{id}}{\to} \mathscr{R}\right) \to \mathscr{P},$$

defined by the identity in degree 0 and by the projection to the first component in degree 1. For every complexes of $\mathcal{R}$-modules M, N we can consider the composition

$$(M \otimes_{\mathcal{R}} \mathscr{P}) \otimes_{\mathcal{R}} (N \otimes_{\mathcal{R}} \mathscr{P}) \stackrel{s_{23}}{\to} (M \otimes_{\mathcal{R}} N) \otimes_{\mathcal{R}} (\mathscr{P} \otimes_{\mathcal{R}} \mathscr{P}) \stackrel{\mathrm{id}\otimes v}{\to} (M \otimes_{\mathcal{R}} N) \otimes_{\mathcal{R}} \mathscr{P}. \tag{165}$$

For complexes M, N which are cohomologically bounded above, this construction induces a functorial cup-product pairing in $\mathcal{D}(\mathcal{R})$:

$$\cup_! = \cup_{!,M,N} : \mathbf{R}\Gamma_!(M) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathbf{R}\Gamma_!(N) \to \mathbf{R}\Gamma_!(M \otimes^{\mathbf{L}}_{\mathcal{R}} N).$$

(Here $s_{23}((x \otimes y) \otimes (x' \otimes y')) = (-1)^{ij}(x \otimes x') \otimes (y \otimes y')$ for x' (reps., y) of degree j (resp., i).)

LEMMA 0.6. *Assume that $M, N \in \mathcal{D}^-(\mathcal{R})$ are cohomologically bounded above. Then we have a commutative diagram in $\mathcal{D}(\mathcal{R})$:*

$$\begin{array}{ccc} \mathbf{R}\Gamma_!(M) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathbf{R}\Gamma_!(N) & \xrightarrow{\cup_{!,M,N}} & \mathbf{R}\Gamma_!(M \otimes^{\mathbf{L}}_{\mathcal{R}} N) \\ \downarrow{\scriptstyle s_{12}} & & \downarrow{\scriptstyle \mathbf{R}\Gamma_!(s_{12})} \\ \mathbf{R}\Gamma_!(N) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathbf{R}\Gamma_!(M) & \xrightarrow{\cup_{!,N,M}} & \mathbf{R}\Gamma_!(N \otimes^{\mathbf{L}}_{\mathcal{R}} M). \end{array}$$

PROOF. Multiplication $h : (\mathscr{P} \otimes_{\mathcal{R}} \mathscr{P})^3 := \mathscr{R} \otimes \mathcal{R} \to \mathscr{R} = \mathscr{P}^1; \mu \otimes \nu \mapsto \mu \cdot \nu$ defines a homotopy: $h : v \rightsquigarrow v \circ s_{12}$. Then, for every complexes P, Q of $\mathcal{R}$-modules the following diagram:

$$\begin{array}{ccccc} (P \otimes_{\mathcal{R}} \mathscr{P}) \otimes_{\mathcal{R}} (Q \otimes_{\mathcal{R}} \mathscr{P}) & \xrightarrow{s_{23}} & (P \otimes_{\mathcal{R}} Q) \otimes_{\mathcal{R}} (\mathscr{P} \otimes_{\mathcal{R}} \mathscr{P}) & \xrightarrow{\mathrm{id}\otimes v} & (P \otimes_{\mathcal{R}} Q) \otimes_{\mathcal{R}} \mathscr{P} \\ \downarrow{\scriptstyle s_{12}} & & \downarrow{\scriptstyle s_{12}\otimes s_{12}} & & \downarrow{\scriptstyle s_{12}\otimes \mathrm{id}} \\ (Q \otimes_{\mathcal{R}} \mathscr{P}) \otimes_{\mathcal{R}} (P \otimes_{\mathcal{R}} \mathscr{P}) & \xrightarrow{s_{23}} & (Q \otimes_{\mathcal{R}} P) \otimes_{\mathcal{R}} (\mathscr{P} \otimes_{\mathcal{R}} \mathscr{P}) & \xrightarrow{\mathrm{id}\otimes v} & (Q \otimes_{\mathcal{R}} P) \otimes_{\mathcal{R}} \mathscr{P} \end{array}$$

commutes up to homotopy (i.e. in the homotopy category). Recalling the definition of $\cup_{!,*,\dagger}$ this implies the statement of the Lemma. ☐

Let us abbreviate $\widetilde{\mathbf{R}\Gamma}_f(-) := \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, -)$ and $\widetilde{H}^*_f(-) := \widetilde{H}^*_f(G_{K,S}, -)$. Taking $M := \widetilde{\mathbf{R}\Gamma}_f(X)$ and $N := \widetilde{\mathbf{R}\Gamma}_f(X)^\iota$ above, we obtain a morphism in $\mathcal{D}(\mathcal{R})$:

$$\widetilde{c}_\pi : \mathbf{R}\Gamma_!\left(\widetilde{\mathbf{R}\Gamma}_f(X)\right) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathbf{R}\Gamma_!\left(\widetilde{\mathbf{R}\Gamma}_f(X)^\iota\right) \stackrel{\cup_!}{\to} \mathbf{R}\Gamma_!\left(\widetilde{\mathbf{R}\Gamma}_f(X) \otimes^{\mathbf{L}}_{\mathcal{R}} \widetilde{\mathbf{R}\Gamma}_f(X)^\iota\right) \stackrel{\mathbf{R}\Gamma_!(\cup_{\boldsymbol{\pi}})}{\longrightarrow} \mathbf{R}\Gamma_!\left(\mathcal{R}\right)[-3].$$

This morphism induces a pairing in cohomology:

$$\widetilde{c}'_{\pi,2,2} : H^2_!\left(\widetilde{\mathbf{R}\Gamma}_f(X)\right) \otimes_{\mathcal{R}} H^2_!\left(\widetilde{\mathbf{R}\Gamma}_f(X)^\iota\right) \longrightarrow H^1_!\left(\mathcal{R}\right) \stackrel{\sim}{\to} \mathscr{R}/\mathcal{R},$$

the last isomorphism coming from (164) for $M = \mathcal{R}$. Moreover, every term in (164) is a torsion $\mathcal{R}$-module, and the first term is $\mathcal{R}$-divisible. We then see that $\widetilde{c}'_{\pi,2,2}$ factorizes through an $\mathcal{R}$-bilinear form:

$$\widetilde{c}_{\pi,2,2} : \widetilde{H}^2_f(X)_{\mathrm{tors}} \otimes_{\mathcal{R}} \widetilde{H}^2_f(X)^\iota_{\mathrm{tors}} \longrightarrow \mathscr{R}/\mathcal{R},$$

called the *(abstract) Cassels-Tate pairing* on X attached to π.

PROPOSITION 0.7. *We have a commutative diagram of $\mathcal{R}$-modules*

$$\begin{array}{ccc} \widetilde{H}^2_f(X)_{\mathrm{tors}} \otimes_{\mathcal{R}} \widetilde{H}^2_f(X)^\iota_{\mathrm{tors}} & \xrightarrow{\widetilde{c}_{\pi,2,2}} & \mathscr{R}/\mathcal{R} \\ \downarrow{\scriptstyle s_{12}} & & \uparrow{\scriptstyle \iota} \\ \widetilde{H}^2_f(X)^\iota_{\mathrm{tors}} \otimes_{\mathcal{R}} \widetilde{H}^2_f(X)_{\mathrm{tors}} & \xrightarrow{c\cdot(\widetilde{c}_{\pi,2,2})^\iota} & (\mathscr{R}/\mathcal{R})^\iota . \end{array}$$

PROOF. Let us write as above $\cup_! := \cup_{!,\widetilde{\mathbf{R}\Gamma}_f(X),\widetilde{\mathbf{R}\Gamma}_f(X)^\iota}$. We have:

$$\begin{aligned} \widetilde{c}_\pi^{\,\iota} \circ s_{12} &:= \mathbf{R}\Gamma_!(\cup_{\boldsymbol{\pi}}{}^\iota) \circ \cup_!{}^\iota \circ s_{12} \\ \text{(by Lemma 0.6)} \quad &= \mathbf{R}\Gamma_!(\cup_{\boldsymbol{\pi}}{}^\iota \circ s_{12}) \circ \cup_! \\ \text{(by Lemma 0.5)} \quad &= c \cdot \mathbf{R}\Gamma_!(\iota \circ \cup_{\boldsymbol{\pi}}) \circ \cup_! \\ &= c \cdot \mathbf{R}\Gamma_!(\iota) \circ \mathbf{R}\Gamma_!(\cup_{\boldsymbol{\pi}}) \circ \cup_! =: c \cdot \mathbf{R}\Gamma_!(\iota) \circ \widetilde{c}_\pi. \end{aligned}$$

Retracing the definitions this easily implies the commutativity of the diagram in the statement. ☐

0.11. Poitou-Tate duality. Let X be as in Sec. 0.5. For every (continuous) $\mathcal{R}[G_{K,S}]$-module M of finite type over $\mathcal{R}$, we write

$$\mathbb{A}_M := \mathrm{Hom}_{\mathrm{cont}}\left(M, \mu_{p^\infty}\right)$$

for the Kummer dual of M. (Here $\mu_{p^\infty} := \mu_{p^\infty}(\overline{K})$ (as $G_{K,S}$-module) and $_{\mathrm{cont}}$ refers to the $\mathfrak{m}$-adic (resp., discrete) topology on M (resp., μ_{p^∞}).) Then $\mathbb{A}_M$, with the discrete topology has a continuous, $\mathcal{R}$-linear action of $G_{K,S}$; in particular it is admissible.

Defining $\mathbb{A}^\pm_{X,v} := (\mathbb{A}_X)^\pm_v := \mathbb{A}_{X^\mp_v}$ (for $v \in S_f$) as the Kummer dual of $X^\mp_v$, Pontrjagin duality gives us short exact sequences of $\mathcal{R}[G_v]$-modules

$$0 \to (\mathbb{A}_X)^+_v \to \mathbb{A}_X \to (\mathbb{A}_X)^-_v \to 0.$$

Defining $\Delta_v(\mathbb{A}_X) := C^\bullet_{\mathrm{cont}}(K_v, (\mathbb{A}_X)^+_v)$ we can consider the complexes

$$\widetilde{C}^\bullet_f(G_{K,S}, \mathbb{A}_X) := \widetilde{C}^\bullet_f(G_{K,S}, \mathbb{A}_X; \Delta(\mathbb{A}_X)) \in \mathrm{K}(\mathcal{R}); \quad \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \mathbb{A}_X) := \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \mathbb{A}_X; \Delta(\mathbb{A}_X)) \in \mathcal{D}_{\mathrm{cf}}(\mathcal{R}),$$

and the corresponding extended Selmer groups $\widetilde{H}^*_f(G_{K,S},\mathbb{A}_X) := \widetilde{H}^*_f(G_{K,S},\mathbb{A}_X;\Delta(\mathbb{A}_X))$. The representation with 'Greenberg local conditions' $\mathbb{A}_X = \left\{\mathbb{A}_X, \left\{\mathbb{A}^+_{X,v}\right\}_{v\in S_f}\right\}$ is the *Kummer dual representation* of $X = \left\{X, \{X^+_v\}_{v\in S_f}\right\}$. As in Sec. 0.5, we have a long exact cohomology sequence:

$$\cdots \to \bigoplus_{v\in S_f} H^{q-1}(K_v,(\mathbb{A}_X)^-_v) \to \widetilde{H}^q_f(G_{K,S},\mathbb{A}_X) \to H^q(G_{K,S},\mathbb{A}_X) \to \bigoplus_{v\in S_f} H^q(K_v,(\mathbb{A}_X)^-_v) \to \cdots. \tag{166}$$

PROPOSITION 0.8. *For every $q \in \mathbb{Z}$ we have isomorphisms of $\mathcal{R}$-modules:*

$$\widetilde{H}^q_f(G_{K,S},X) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}_p}\left(\widetilde{H}^{3-q}_f(G_{K,S},\mathbb{A}_X), \mathbb{Q}_p/\mathbb{Z}_p\right),$$

PROOF. This is a special case of [**Nek06**, Prop. 6.7.7] (see also *loc. cit.*, Sec. 2.9.1 and Sec. 6.3.5). $\square$

REMARK 0.9. The isomorphisms of the preceding proposition come from cup-product pairings

$$\cup_{\mathbf{ev}} : \widetilde{C}^q_f(G_{K,S},X;\Delta(X)) \times \widetilde{C}^{3-q}_f(G_{K,S},\mathbb{A}_X;\Delta(\mathbb{A}_X)) \to \widetilde{C}^3_f(G_{K,S},\mu_{p^\infty};\Delta_c(\mu_{p^\infty}))$$

satisfying (the usual relation $d\,(x\cup y) = dx\cup y + (-1)^q\cdot x\cup dy$ and) and $(r\cdot x)\cup y = x\cup(r\cdot y)$, together with the isomorphism $\widetilde{H}^3_f(G_{K,S},\mu_{p^\infty};\Delta_c(\mu_{p^\infty})) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$ coming from Section 0.4. (Here $\Delta_c(\mu_{p^\infty})$ is the full local condition $0 \to C^\bullet_{\text{cont}}(G_{K,S},\mu_{p^\infty})$ at every $v\in S_f$). The cup-product $\cup_{\mathbf{ev}}$ is defined by the same formula displayed in Lemma 0.1, once we replace the cup-products induced on complexes by π by those induced using Kummer duality $\text{ev} : X \times \mathbb{A}_X \to \mu_{p^\infty}$.

REMARK 0.10. Let X be a finite $G_{K,S}$-modules of p-power order, equipped with the "empty local conditions" $\Delta_\emptyset(X)_v : C^\bullet_{\text{cont}}(K_v,X)$ at every $v\in S_f$. Using the preceding proposition, (166) becomes the well-known Poitou-Tate 9-terms exact sequence [**Mil04**, Ch. I]. In fact the proofs of Prop. 0.8 and Prop. 0.2 use this 'special case' in an essential way.

APPENDIX B

Iwasawa theory

Let $\mathcal{K}/K$ be a $\mathbb{Z}_p^d$-extension for some $d \geq 1$ (i.e. a Galois extension with Galois group isomorphic to the additive group $\mathbb{Z}_p^d$). We are interested in the variation of Selmer complexes with respect to finite subextensions $K \subset L \subset \mathcal{K}$. In particular, given a discrete (resp., compact) $\mathcal{R}[G_{K,S}]$-module A (resp., M), we are interested in the Selmer complex of $A/\mathcal{K}$

$$\widetilde{\mathbf{R}\Gamma}_f(K_S/\mathcal{K}, A; \Delta(A)) := \varinjlim_{\mathrm{res},L} \widetilde{\mathbf{R}\Gamma}_f(\mathrm{Gal}(K_S/L), A; \Delta_L(A))$$

(resp., in the Complex of '$\mathcal{K}/K$-universal norms'

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, M; \Delta(M)) := \varprojlim_{\mathrm{cores},L} \widetilde{\mathbf{R}\Gamma}_f(\mathrm{Gal}(K_S, L), M; \Delta_L(M))),$$

where $\{\Delta_L(*)\}_L$ are 'compatible' Greenberg local conditions (and the restriction and corestriction morphisms will be defined below). The key fact is that Shapiro's lemma allows us to describe these 'Iwasawa complexes' over $\mathcal{K}$ in terms of complexes over K of the Galois deformations $\mathrm{Hom}_{\mathcal{R},\mathrm{cts}}(\mathcal{R}[[\mathrm{Gal}(\mathcal{K}/K)]], A)$ and $M \otimes_{\mathcal{R}} \mathcal{R}[[\mathrm{Gal}(\mathcal{K}/K)]]$ respectively. In other words, working with cohomology over general coefficient rings allows us to include Iwasawa theory in the theory of Galois deformations. This point of view is well explained in Greenberg [**Gre94b**] (see especially Prop. 3) and [**Nek06**, Sec. 0.11-0.13]. This Section is a summary of some of the results in [**Nek06**, Ch. 8] (which of course works in much greater generality).

Notations. We use the notations of Section A. Let L/K be a finite Galois subextension of K_S/K. Write $S_{L,f}$ for the set of finite primes of L dividing primes in $S_f = S_{K,f}$, and $S_L := S_{L,f} \cup \{v|\infty\}$. Then $G_{L,S_L} := \mathrm{Gal}(K_S/L) = \mathrm{Gal}(L_{S_L}/L)$ is the Galois group of the maximal algebraic extension L_{S_L}/L which is unramified outside S_L. For every $v \in S_f$ we write $w_0|v$ for the prime of L defined by $\rho_v : \overline{K} \hookrightarrow \overline{K_v}$ and $G_{L,w_0} := G_v \cap G_L$ for the corresponding decomposition group. For every other prime $w|v$ we fix $\sigma_w \in G_K$ such that $\sigma_v(w_0) = w$, i.e. w is induced by the embedding $\rho_w := \rho_v \circ \sigma_w^{-1}$. (This amounts to fixing representatives of the double coset space $G_L \backslash G_K / G_v$, i.e.

$$G_K = \coprod_{w|v} G_L \sigma_w G_v.)$$

We also write $G_{K,w} := \sigma_w \cdot G_v \cdot \sigma_w^{-1}$ and $G_{L,w} := \sigma_w \cdot G_{L,w_0} \cdot \sigma_w^{-1}$, which are decomposition groups at w for K and L respectively.

We consider an $\mathcal{R}$-module X, which is assumed to be either of finite type or of co-finite type. We assume that X is equipped with a continuous $\mathcal{R}$-linear action of $G_{K,S}$ (with respect to the $\mathfrak{m}$-adic or the discrete topology respectively), and with Greenberg local conditions for every $v \in S_f$, i.e. with fixed $\mathcal{R}[G_v]$-submodules $i_v^+ : X_v^+ \hookrightarrow X$. For every $w|v \in S_f$, we define the $\mathcal{R}[G_{K,w}]$-modules $X_w^+ := \sigma_w(X_v^+)$ (viewing $X_v^+ \subset X$ under i_v^+), so that we obtain an inclusion of $\mathcal{R}[G_{K,w}]$-modules

$$i_w^+ := \sigma_w \circ i_v^+ \circ \sigma_w^{-1} : X_w^+ \hookrightarrow X.$$

For every finite sub-extension $K \subset L \subset \mathcal{K}$, we will consider local conditions $\Delta(X) = \Delta_L(X) := \{\Delta_w(X)\}_{w \in S_{L,f}}$ defined as usual by $i_w^+ : C^\bullet_{\mathrm{cont}}(L_w, X_w^+) \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G_{L,w}, X_w^+) \to C^\bullet_{\mathrm{cont}}(G_{L,w}, X) \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(L_w, X)$. (As in the preceding Section, we identify $C^\bullet_{\mathrm{cont}}(L_w, *)$ and $C^\bullet_{\mathrm{cont}}(G_{L,w}, *)$ under the isomorphism induced by ρ_w^*.) We will also omit $\Delta(X)$ from the notations, i.e. we write

$$\widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L},X) := \widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L},X;\Delta_L(X));$$

we also use similar notations $\widetilde{C}^{\bullet}_f(G_{L,S_L},X)$ and $\widetilde{H}^{*}_f(G_{L,S_L},X)$.

0.12. Shapiro's lemma. Let $K\subset L\subset K_S$ be a finite Galois extension and let $\mathcal{R}(L):=\mathcal{R}[\mathrm{Gal}(L/K)]$. We consider the $\mathcal{R}[G_{K,S}]$-modules

$$L(X) := \mathrm{Hom}_{\mathcal{R}}\left(\mathcal{R}(L),X\right)\cong X\otimes_{\mathcal{R}}\mathcal{R}(L) =: X(L), \tag{167}$$

where the isomorphism is defined by $f\mapsto \sum_{\sigma\in\mathrm{Gal}(L/K)} f(\sigma)\otimes\sigma$. The Galois action on $L(X)$ (resp., $X(L)$) is given by $f^g(\sigma):=g\cdot f(g^{-1}\sigma)$ (resp., $\left(\sum_\sigma x_\sigma\otimes\sigma\right)^g := \sum_\sigma g(x_\sigma)\otimes g\sigma$) for every $g\in G_{K,S}$. The functors $X\mapsto L(X)$ and $X\mapsto X(L)$ commute with direct and inverse limits; in particular if X is of finite type over $\mathcal{R}$ we have $X(L)=\varprojlim_n\left[\,(X/\mathfrak{m}^nX)\,(L)\right]$.

We note that if A is a discrete $\mathcal{R}[G_{K,S}]$-module, then $A(L)$ is isomorphic as a $G_{K,S}$-module to the induced module

$$\mathrm{Ind}^{G_{K,S}}_{G_{L,S_L}}(A) := \left\{f: G_{K,S}\overset{\text{loc. const.}}{\longrightarrow} A:\ f(\gamma\cdot g)=\gamma\cdot f(g),\quad \forall\gamma\in G_{L,S_L}\right\}$$

with $G_{K,S}$-action given by $f^\tau(g):=f(g\cdot\tau)$, and the isomorphism defined sending a G_{L,S_L}-equivariant map $f: G_{K,S}\to A$ to the element $\sum_{\sigma\in\mathrm{Gal}(L/K)}\left(\widetilde{\sigma}\cdot f(\widetilde{\sigma}^{-1})\right)\otimes\sigma$, where $\widetilde{\sigma}|_L=\sigma$. Moreover Shapiro's lemma asserts the morphism of pairs $\left(G_{L,S_L}\subset G_{K,S},\mathrm{Ind}^{G_{K,S}}_{G_{L,S_L}}(A)\to A;\ f\mapsto f(1)\right)$ gives a quasi-isomorphism

$$\mathrm{sh}: C^{\bullet}_{\mathrm{cont}}(G_{K,S},\mathrm{Ind}^{G_{K,S}}_{G_{L,S_L}}(A))\longrightarrow C^{\bullet}_{\mathrm{cont}}(G_{L,S_L},A).$$

The preceding isomorphism (167) then induces quasi-isomorphisms of complexes of $\mathcal{R}$-modules

$$\mathrm{sh}: C^{\bullet}_{\mathrm{cont}}(G_{K,S},X(L))\longrightarrow C^{\bullet}_{\mathrm{cont}}(G_{L,S_L},X);\quad \mathrm{sh}: C^{\bullet}_{\mathrm{cont}}(G_{K,S},L(X))\longrightarrow C^{\bullet}_{\mathrm{cont}}(G_{L,S_L},X)$$

induced by $(\subset,\mathrm{pr}_1):(G_{K,S},X(L))\to(G_{L,S_L},X)$ and $(\subset, f\mapsto f(\mathrm{id}_L)):(G_{K,S},L(X))\to(G_{L,S_L},X)$ respectively, where $\mathrm{pr}_1\left(\sum_\sigma x_\sigma\otimes\sigma\right)=x_{\mathrm{id}_L}$. This follows immediately if X is of co-finite type (hence discrete), while follows by a limit argument if X is of finite type over $\mathcal{R}$. (More precisely we apply Shapiro's lemma to any of the discrete modules $(X/\mathfrak{m}^nX)\,(L)$ and use the fact recalled above that $X\mapsto X(L)$ commutes with inverse limits.)

Let $v\in S_f$ and let us consider for every $w|v$ the $G_{K,w}$-module

$$X(L_w) := X\otimes_{\mathcal{R}}\mathcal{R}[G_{K,w}/G_{L,w}].$$

(If X is discrete then $X(L_w)\xrightarrow{\sim}\mathrm{Ind}^{G_{K,w}}_{G_{L,w}}(X)$.) We consider also $X(L_w)$ as a $G_{K,v}=G_v$-module via the morphism $\mathrm{Ad}(\sigma_w): G_v\to G_{K,w};\ \gamma\mapsto\sigma_w\gamma\sigma_w^{-1}$. We can easily check that the following defines an isomorphism of G_v-modules:

$$\omega_v=\bigoplus_{w|v}\omega_w : X(L)\xrightarrow{\ \sim\ }\bigoplus_{w|v}X(L_w);\qquad \sum_{\sigma\in\mathrm{Gal}(L/K)} x_\sigma\otimes\sigma\mapsto\left\{\sum_{\gamma\in G_{K,w}/G_{L,w}}\sigma_w\left(x_{\sigma_w^{-1}\cdot\gamma}\right)\otimes\gamma\right\}_{w|v}. \tag{168}$$

(We have $\gamma=\sigma_w\gamma_w\sigma_w^{-1}\in G_{K,w}/G_{L,w}=\sigma_w\cdot G_v/G_{L,w_0}\cdot\sigma_w^{-1}$. Then $\sigma_w^{-1}\cdot\gamma$ is by definition the element $\gamma_w\cdot\sigma_w^{-1}\in\mathrm{Gal}(L/K)=\coprod_{w|v}\left(G_v/G_{L,w_0}\right)\cdot\sigma_w^{-1}$.) As above Shapiro's Lemma induces a quasi isomorphism of complexes of $\mathcal{R}$-modules

$$\mathrm{sh}_w: C^{\bullet}_{\mathrm{cont}}(K_v,X(L_w))\xrightarrow{\sim} C^{\bullet}_{\mathrm{cont}}(G_{K,w},X(L_w))\overset{\mathrm{qis}}{\to} C^{\bullet}_{\mathrm{cont}}(L_w,X),$$

where the first isomorphism is induces by the isomorphism of pairs $(\mathrm{Ad}(\sigma_w^{-1}), \mathrm{id}) : (G_v, X(L_w)) \to (G_{K,w}, X(L_w))$ (recalling the definition of the action of G_v on $X(L_w)$). In particular we obtain a quasi-isomorphism of complexes

$$\mathrm{sh}_{S_f} : \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(K_v, X(L)) \xrightarrow{\bigoplus_v (\omega_v)_*} \bigoplus_{v\in S_f}\bigoplus_{w|v} C^\bullet_{\mathrm{cont}}(K_v, X(L_w)) \xrightarrow{\bigoplus_{v,w}\mathrm{sh}_w} \bigoplus_{w\in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X).$$

It follows by the definitions that for every $w|v$ the following digram of complexes of $\mathcal{R}$-modules commutes:

$$\begin{array}{ccccc} C^\bullet_{\mathrm{cont}}(G_{K,S}, X(L)) & \xrightarrow{(\sigma_w^{-1})_*} & C^\bullet_{\mathrm{cont}}(G_{K,S}, X(L)) & \xrightarrow{\mathrm{sh}} & C^\bullet_{\mathrm{cont}}(G_{L,S_L}, X) \\ \downarrow{\scriptstyle \mathrm{res}_v} & & & & \downarrow{\scriptstyle \mathrm{res}_w} \\ C^\bullet_{\mathrm{cont}}(K_v, X(L)) & \xrightarrow{(\omega_w)_*} & C^\bullet_{\mathrm{cont}}(K_v, X(L_w)) & \xrightarrow{\mathrm{sh}_w} & C^\bullet_{\mathrm{cont}}(L_w, X). \end{array}$$

Here σ_* for $\sigma \in G_{K,S}$ denotes Galois conjugation, i.e. the isomorphism of complexes induces on cochains by $(\mathrm{Ad}(\sigma^{-1}), \sigma) : (G_{K,S}, X) \to (G_{K,S}, X)$. (We recall that for a profinite group G and a continuous G-module M, there exist homotopies $h_\sigma = h_\sigma(G, X)$ between the the identity morphism on $C^\bullet_{\mathrm{cont}}(G, M)$ and σ_*, which are functorial in both G and M.) Fixing (functorial) homotopies $h_\sigma : \mathrm{id} \rightsquigarrow \sigma_*$, the preceding diagram tells us that the diagram:

$$\begin{array}{ccc} C^\bullet_{\mathrm{cont}}(G_{K,S}, X(L)) & \xrightarrow{\mathrm{res}_{S_f}} & \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(K_v, X(L)) \\ \downarrow{\scriptstyle \mathrm{sh}} & & \downarrow{\scriptstyle \mathrm{sh}_{S_f}} \\ C^\bullet_{\mathrm{cont}}(G_{L,S_L}, X) & \xrightarrow{\mathrm{res}_{S_{L,f}}} & \bigoplus_{w\in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X) \end{array}$$

is commutative up to the homotopy $\boldsymbol{s}_X := \bigoplus_{v\in S_f}\bigoplus_{w|v} \mathrm{res}_w \circ \mathrm{sh} \circ h_{\sigma_w^{-1}} : \mathrm{res}_{S_{L,f}} \circ \mathrm{sh} \rightsquigarrow \mathrm{sh}_{S_f} \circ \mathrm{res}_{S_f}$.

Let us define for every $v \in S_f$ the sub-$\mathcal{R}(L)[G_v]$-modules $i_v^+(L) := i_v^+ \otimes \mathrm{id} : X(L)_v^+ := X_v^+(L) \hookrightarrow X(L)$. We also define for every $w|v \in S_f$ the $G_{K,w}$-module

$$X_w^+(L_w) := X_w^+ \otimes_{\mathcal{R}} \mathcal{R}[G_{K,w}/G_{L,w}],$$

(which is isomorphic to $\mathrm{Ind}_{G_{L,w}}^{G_{K,w}}(X_w^+)$ if X_w^+ is a discrete $G_{K,w}$-module). $X_w^+(L_w)$ is also a $G_{K,v} = G_v$-module via $\mathrm{Ad}(\sigma_w) : G_v \to G_{K,w}$. Formula (168) again defines an isomorphism of G_v-modules

$$\omega_v^+ : X(L)_v^+ \xrightarrow{\sim} \bigoplus_{w|v} X_w^+(L_w)$$

such that the following diagram

$$\begin{array}{ccc} X(L)_v^+ & \xrightarrow{\omega_v^+} & \bigoplus_{w|v} X_w^+(L_w) \\ \downarrow{\scriptstyle i_v^+(L)} & & \downarrow{\scriptstyle \bigoplus_{w|v} i_w^+(L_w)} \\ X(L) & \xrightarrow{\omega_v} & \bigoplus_{w|v} X(L_w) \end{array}$$

is commutative. (We have written $i_w^+(L_w) := i_w^+ \otimes \mathrm{id}$.) Shapiro's Lemma gives as above a quasi-isomorphism $\mathrm{sh}_w^+ : C^\bullet_{\mathrm{cont}}(K_v, X_w^+(L_w)) \xrightarrow{\mathrm{qis}} C^\bullet_{\mathrm{cont}}(L_w, X_w^+)$ (for $w|v \in S_f$) such that $i_w^+ \circ \mathrm{sh}_w^+ = \mathrm{sh}_w \circ i_w^+(L_w)$. We then obtain a quasi-isomorphism

$$\mathrm{sh}_{S_f}^+ := \left(\mathrm{sh}_w^+ \circ (\omega_v^+)_*\right)_{w|v\in S_f} : \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(K_v, X(L)_v^+) \xrightarrow{\mathrm{qis}} \bigoplus_{w\in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X_w^+),$$

sitting in a commutative diagram of complexes of $\mathcal{R}$-modules

$$\begin{array}{ccc} \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(K_v, X(L)^+_v) & \xrightarrow{\mathrm{sh}^+_{S_f}} & \bigoplus_{w\in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X^+_w) \\ \downarrow{\scriptstyle i^+_{S_f}} & & \downarrow{\scriptstyle i^+_{S_{L,f}}} \\ \bigoplus_{v\in S_f} C^\bullet_{\mathrm{cont}}(K_v, X(L)) & \xrightarrow{\mathrm{sh}_{S_f}} & \bigoplus_{v\in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X), \end{array}$$

where $i_{S_{L,f}} := \bigoplus_{w\in S_{L,f}} i^+_w$. Define $X^-_w := X/X^+_w$ and $X(L)^-_v := X(L)/X(L)^+_v$ for every $w|v \in S_f$; we then have completely analogous results replacing $+$ with $-$.

Let us define again local conditions $\Delta(X(L)) = \{\Delta(X(L))_v\}_{v\in S_f}$ using the submodules $X(L)^+_v$ (i.e. define $\Delta(X(L))_v$ as the morphism of complexes $i^+_v(L) : C^\bullet_{\mathrm{cont}}(K_v, X(L)^+_v) \to C^\bullet_{\mathrm{cont}}(K_v, X(L))$) and let us write

$$\widetilde{C}^\bullet_f(G_{K,S}, X(L)) := \widetilde{C}^\bullet_f(G_{K,S}, X(L); \Delta(X(L))); \quad \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X(L)) := \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X(L); \Delta(X(L)))$$

for the corresponding Selmer complex. Via 'functoriality of cones' [**Nek06**, Sec. 1.1.6] the constructions above allow us to define a quasi isomorphism of complexes of $\mathcal{R}$-modules

$$\begin{aligned} \mathrm{sh}_f := \mathrm{sh}_{f,E/K} : \widetilde{C}^\bullet_f(G_{K,S}, X(L)) &\xrightarrow{\mathrm{qis}} \widetilde{C}^\bullet_f(G_{L,S_L}, X) \\ (x_n, x^+_n, x_{n-1}) &\mapsto \Big(\mathrm{sh}(x_n), \mathrm{sh}^+_{S_f}(x^+_n), \mathrm{sh}_{S_f}(x_{n-1}) + \boldsymbol{s}_X(x_n)\Big). \end{aligned} \tag{169}$$

Moreover, as proved in [**Nek06**, Sec. 8.1.7.2] the image of sh_f in the homotopy category of complexes of $\mathcal{R}$-modules is canonical (i.e. the homotopy class of sh_f does not depend on the choice of the homotopies h_σ for $\sigma \in G_K$). In particular we obtain a canonical isomorphism in $\mathcal{D}(\mathcal{R})$:

$$\mathrm{sh}_f : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X(L)) \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L}, X). \tag{170}$$

Moreover we see easily that this induces an isomorphism of exact triangles in $\mathcal{D}(\mathcal{R})$ (see (157)):

$$\begin{array}{ccccc} \bigoplus_{v\in S_f} \mathbf{R}\Gamma_{\mathrm{cont}}(K_v, X(L)^-_v)[-1] & \longrightarrow & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X(L)) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{cont}}(G_{K,S}, X(L)) \\ \downarrow{\scriptstyle \mathrm{sh}^-_{S_f}} & & \downarrow{\scriptstyle \mathrm{sh}_f} & & \downarrow{\scriptstyle \mathrm{sh}} \\ \bigoplus_{w\in S_f} \mathbf{R}\Gamma_{\mathrm{cont}}(L_w, X^-_w)[-1] & \longrightarrow & \widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L}, X) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{cont}}(G_{L,S_L}, X). \end{array} \tag{171}$$

Using the isomorphism (167), and defining $L(X)^+_v := \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}(L), X^+_v) \xrightarrow{\sim} X(L)^+_v$ we can replace $X(L)$ with $L(X)$ in (169), (170) and (171).

0.13. Conjugation. With the notations introduced above, let $\gamma \in \mathrm{Gal}(L/K)$ and $x = \sum_\sigma x_\sigma \otimes \sigma \in \mathcal{R}(L)$. The formula $\gamma \star x := \sum_\sigma x_\sigma \otimes \sigma\gamma^{-1}$ defines an action of $\mathrm{Gal}(L/K)$ on $X(L)$ commuting with the $G_{K,S}$-action. This equips $X(L)$ with the structure of an $\mathcal{R}(L)$-module, such that $X(L)^+_v$ is an $\mathcal{R}(L)$-submodule for every $v \in S_f$. Then $\widetilde{C}^\bullet_f(G_{K,S}, X(L))$ is a complex of $\mathcal{R}(L)$-modules.

The isomorphism (170) induces an isomorphism $\gamma_{*,f} : \widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L}, X) \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L}, X)$, for every $\gamma \in \mathrm{Gal}(L/K)$. More precisely: given $g \in G_K$ we can define a morphism of complexes

$$\begin{aligned} g_{*,f} : \widetilde{C}^\bullet_f(G_{L,S_L}, X) &\longrightarrow \widetilde{C}^\bullet_f(G_{L,S_L}, X) \\ \Big(x_n, \big(x^+_{n,w}\big)_w, \big(x_{n-1,w}\big)_w\Big) &\mapsto \Big(g_*(x_n), \big(g^+_{*,w}(x^+_{n,w})\big)_w, \big(g_{*,w}(x_{n-1,w})\big)_w + \boldsymbol{g}_X(x_n)\Big), \end{aligned} \tag{172}$$

whose homotopy class is canonical and such that the following diagram of complexes:

$$\begin{array}{ccc} \widetilde{C}^\bullet_f(G_{K,S}, X(L)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{L,S_L}, X) \\ \downarrow{\scriptstyle (g|_L)_*} & & \downarrow{\scriptstyle g_{*,f}} \\ \widetilde{C}^\bullet_f(G_{K,S}, X(L)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{L,S_L}, X) \end{array} \tag{173}$$

commutes up to homotopy. We now explain the notations used above. First of all, for every $g \in G_K$

$$g_* := \big(\mathrm{Ad}(g^{-1}), g\big)_* : C^\bullet_{\mathrm{cont}}(G_{L,S_L}, X) \to C^\bullet_{\mathrm{cont}}(G_{L,S_L}, X)$$

is the morphism of complexes induced by the morphism of pairs $(\mathrm{Ad}(g^{-1}), g) : (G_{L,S_L}, X) \to (G_{L,S_L}, X)$. For every $w \in S_{L,S_L}$, let us write $g \cdot \sigma_w = \alpha_w \cdot \sigma_{g(w)} \cdot \gamma_w$, with $\alpha_w \in G_L$ and $\gamma_w \in G_v = G_{K,v}$. (Here v is the prime of K lying below w, and $g(w)$ refers to the action of $G_K \twoheadrightarrow \mathrm{Gal}(L/K)$ on the set $G_L \backslash G_K / G_v \simeq \{w|v\}$ of primes of L dividing v.) Then $g_{*,w}$ denotes the composition

$$C^\bullet_{\mathrm{cont}}(L_w, X) \overset{\left(\mathrm{Ad}(\sigma_w),\sigma_w^{-1}\right)_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(L_{w_0}, X) \overset{\left(\mathrm{Ad}(\gamma_w^{-1}),\gamma_w\right)_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(L_{w_0}, X) \overset{\left(\mathrm{Ad}(\sigma_{g(w)}^{-1}),\sigma_{g(w)}\right)_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(L_{g(w)}, X),$$

and similarly $g^+_{*,w}$ is defined by the composition

$$C^\bullet_{\mathrm{cont}}(L_w, X^+_w) \overset{\left(\mathrm{Ad}(\sigma_w),\sigma_w^{-1}\right)_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(L_{w_0}, X^+_{w_0}) \overset{\left(\mathrm{Ad}(\gamma_w^{-1}),\gamma_w\right)_*}{\longrightarrow}$$
$$\longrightarrow C^\bullet_{\mathrm{cont}}(L_{w_0}, X^+_{w_0}) \overset{\left(\mathrm{Ad}(\sigma_{g(w)}^{-1}),\sigma_{g(w)}\right)_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(L_{g(w)}, X^+_{g(w)}).$$

Then $i^+_{S_{L,f}} \circ \left(g^+_{*,w}\right)_{w \in S_{L,S_L}} = (g_{*,w})_{w \in S_{L,f}} \circ i^+_{S_{L,f}}$ and it follows directly that:

$$g_{*,w} \circ \mathrm{res}_w = \left(\alpha_w^{-1}\right)_* \circ g_* \circ \mathrm{res}_{g(w)}.$$

Recalling that the conjugation morphism on $C^\bullet_{\mathrm{cont}}(G_{L,S_L}, X)$ attached to every element of G_L is homotopic to the identity, we obtain a homotopy

$$\boldsymbol{g}_X := \bigoplus_{w \in S_{L,f}} \left(\mathrm{res}_{g(w)} \circ h_{\alpha_w} \circ g_*\right) : \mathrm{res}_{S_{L,f}} \circ g_* \rightsquigarrow (g_{*,w})_{w \in S_{L,f}} \circ \mathrm{res}_{S_{L,f}}$$

(where as above $h_\star$ is a (bi-functorial) homotopy between the identity and $(\star)_*$ for every $\star \in G_K$). Again by functoriality of cones [**Nek06**, Sec. 1.1.6] we see that (172) induces a quasi-isomorphism of complexes. For a proof of the commutativity of (173) in the homotopy category see [**Nek06**, Sec. 8.1.7.3].

We then obtain for every $g \in G_K$ a commutative diagram of isomorphisms in $\mathcal{D}(\mathcal{R})$

(174)
$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X(L)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L}, X) \\ \downarrow{\scriptstyle (g|_L)_*} & & \downarrow{\scriptstyle g_{*,f}} \\ \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X(L)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L}, X). \end{array}$$

In particular we see that G_L acts trivially on $\widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L}, X)$, so that we have a natural $\mathcal{R}$-linear action of $\mathrm{Gal}(L/K)$ on cohomology, i.e.

(175)
$$\widetilde{H}^*_f(G_{L,S_L}, X) \in \left({}_{\mathcal{R}(L)}\mathrm{Mod}\right)_\dagger,$$

where $\dagger = \mathrm{ft}$ (resp., $\dagger = \mathrm{cf}$) if X is of finite (resp., co-finite) type over $\mathcal{R}$.

Using the isomorphism (167) we can replace $X(L)$ with $L(X)$ in the preceding discussion (where the action of $\gamma \in \mathrm{Gal}(L/K)$ on $f \in L(X)$ is given by $(\gamma \star f)(\sigma) = f(\sigma \cdot \gamma)$ for every $\sigma \in \mathrm{Gal}(L/K)$.)

0.14. Restriction. Let $E/L/K$ be a tower of finite Galois sub-extensions of K_S. We have a *restriction morphism* of complexes of $\mathcal{R}$-modules

(176)
$$\mathrm{res}_{E/L,f} : \widetilde{C}^\bullet_f(G_{L,S_L}, X) \longrightarrow \widetilde{C}^\bullet_f(G_{E,S_E}, X)$$
$$(x_n, x_n^+, x_{n-1}) \mapsto \Big(\mathrm{res}_{E/L}(x_n), r^+_{S_L}\left(x_n^+\right), r_{S_L}\left(x_{n-1}\right) + \boldsymbol{r}_X(x_n)\Big),$$

canonical up to homotopy, and such that the following diagram is commutative in the homotopy category:

$$\begin{array}{ccccc} \widetilde{C}^\bullet_f(G_{K,S}, L(X)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{L,S_L}, X) & \xleftarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K,S}, X(L)) \\ \downarrow (i_{E/L})_* & & \downarrow \mathrm{res}_{L/K,f} & & \downarrow (j_{E/L})_* \\ \widetilde{C}^\bullet_f(G_{K,S}, E(X)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{E,S_E}, X) & \xleftarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K,S}, X(E)), \end{array} \tag{177}$$

where $i_{E/L}$ (resp., $j_{E/L}$) is defined sending $f : \mathrm{Gal}(L/K) \to X$ to the function $\{\sigma \in \mathrm{Gal}(E/K) \mapsto f(\sigma|_L)\}$ (resp, sending $x \otimes \gamma \in X \otimes \mathrm{Gal}(L/K)$ to $\sum_{\sigma|_L=\gamma} x \otimes \sigma \in X \otimes \mathrm{Gal}(E/K)$). In particular we obtain a canonical morphism $\mathrm{res}_{E/L,f} : \widetilde{\mathbf{R}\Gamma}_f(G_{L,S_L}, X) \to \widetilde{\mathbf{R}\Gamma}_f(G_{E,S_E}, X)$.

In (176) $\mathrm{res}_{E/L} : C^\bullet_{\mathrm{cont}}(G_{L,S_L}, X) \to C^\bullet_{\mathrm{cont}}(G_{E,S_E}, X)$ is the restriction morphism attached to the inclusion $G_{E,S_E} \subset G_{L,S_L}$. Given $w' \in S_{E,f}$ with $w'|w \in S_{L,f}$, we have $\sigma_{w'} = \beta_{w'} \cdot \sigma_w \cdot \alpha_{w'}$, with $\beta_{w'} \in G_L$ and $\alpha_{w'} \in G_{K,v}$. Then

$$r_{S_L} := \left(r_{w'|w}\right) : \bigoplus_{w \in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X) \to \bigoplus_{w' \in S_{E,f}} C^\bullet_{\mathrm{cont}}(E_{w'}, X),$$

where $r_{w'|w}$ denotes the composition

$$\begin{aligned} C^\bullet_{\mathrm{cont}}(L_w, X) &\overset{\left(\mathrm{Ad}(\sigma_w),\sigma_w^{-1}\right)_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(G_{L,w_0}, X) \overset{\left(\mathrm{Ad}(\alpha_{w'}),\alpha_{w'}^{-1}\right)_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(G_{L,w_0}, X) \\ &\overset{(\subset,\mathrm{id})_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(G_{E,w'_0}, X) \overset{\left(\mathrm{Ad}(\sigma_{w'}^{-1}),\sigma_{w'}\right)_*}{\longrightarrow} C^\bullet_{\mathrm{cont}}(G_{E,w'}, X) = C^\bullet_{\mathrm{cont}}(E_{w'}, X) \end{aligned}$$

(where w'_0 is the prime of E induced by ρ_v). The morphism

$$r^+_{S_L} := \left(r^+_{w'|w}\right) : \bigoplus_{w \in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X^+_w) \to \bigoplus_{w' \in S_{E,f}} C^\bullet_{\mathrm{cont}}(E_{w'}, X^+_{w'})$$

is defined in a similar way. We have $i^+_{S_{E,f}} \circ r^+_{S_L} = r_{S_L} \circ i^+_{S_{L,f}}$ and a homotopy

$$\boldsymbol{r}_X := \bigoplus_{w' \in S_{E,f}} \left(\mathrm{res}_{w'} \circ \mathrm{res}_{E/L} \circ h_{\beta_{w'}}\right) : \mathrm{res}_{S_{E,f}} \circ \mathrm{res}_{E/L} \rightsquigarrow r_{S_L} \circ \mathrm{res}_{S_{L,f}},$$

where as usual we have fixed bi-functorial homotopies $h_g : \mathrm{id}_{C^\bullet_{\mathrm{cont}}(G_{L,S},X)} \rightsquigarrow g_*$ for $g \in G_L$. For a proof of the independence of $\mathrm{res}_{E/L,f}$ (up to homotopy) on this choice and the commutativity of (177) in the homotopy category see again [**Nek06**, Sec. 8.1].

0.15. Norm. Let H be an open normal subgroup of a profinite group G, and let us fix a section $H\backslash G \to G$; $\gamma \mapsto \overline{\gamma}$ of the natural projection. Then for every discrete G-module M the formula

$$\mathrm{cor}^H_G(\psi)(g_1, \ldots, g_n) := \sum_{\gamma \in H\backslash G} \overline{\gamma}^{-1} \cdot \psi\left(\overline{\gamma} \cdot g_1 \cdot \overline{\gamma g_1}^{-1}, \ldots, \overline{\gamma g_1 \cdots g_{n-1}} \cdot g_n \cdot \overline{\gamma g_1 \cdots g_n}^{-1}\right)$$

defines a morphism of complexes $\mathrm{cor}^H_G : C^\bullet_{\mathrm{cont}}(H, M) \to C^\bullet_{\mathrm{cont}}(G, M)$, whose homotopy class does not depend on the choice of the section $\gamma \mapsto \overline{\gamma}$. (The cohomological functor $\{H^q(H, *) \to H^q(G, *)\}_{q \geq 0}$ induced by cor^H_G is that defined via universality of the 'fixed module functor' on the category of discrete G-modules by the natural transformation $H^0(H, *) \Rightarrow H^0(G, *)$ defined by the norm map $M^H \to M^G$; $m \mapsto \sum_{\gamma \in H\backslash G} \gamma^{-1}(m)$ [**NSW00**, Ch. I-II].) Since cor^H_G is functorial in M, this definition extends to admissible G-modules.

Let $E/L/K$ be finite Galois extensions contained in K_S. Given a section $i : G_{E,S_E}\backslash G_{L,S_L} \to G_{L,S_L}$ we obtain a morphism

$$\mathrm{cor}_{E/L} : C^\bullet_{\mathrm{cont}}(G_{E,S_E}, X) \longrightarrow C^\bullet_{\mathrm{cont}}(G_{L,S_L}, X).$$

Fixing a section $i_{w_0} : G_{E,w_0'}\backslash G_{L,w_0} \to G_{L,w_0}$ (where as usual $w_0'|w_0|v$ are the primes induced by ρ_v), a construction completely analogous to that of the preceding section gives us morphisms

$$c_{S_E} : \bigoplus_{w'\in S_{E,f}} C^\bullet_{\mathrm{cont}}(E_{w'}, X) \to \bigoplus_{w\in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X);$$
$$c^+_{S_E} : \bigoplus_{w'\in S_{E,f}} C^\bullet_{\mathrm{cont}}(E_{w'}, X^+_{w'}) \to \bigoplus_{w\in S_{L,f}} C^\bullet_{\mathrm{cont}}(L_w, X^+_w),$$

such that $c_{S_E} \circ i^+_{S_{E,f}} = i^+_{S_{L,f}} \circ c^+_{S_E}$ and $\boldsymbol{c}_X : \mathrm{res}_{S_{L,f}} \circ \mathrm{cor}_{E/L} \rightsquigarrow c_{S_E} \circ \mathrm{res}_{S_{E,f}}$. (The homotophy $\boldsymbol{c}_X$ depends on the sections i, i_{w_0} and on the choice of the usual homotopies h_g for $g \in G_L$.) As in the preceding Sections, we obtain a *corestriction (or norm) morphism*

$$\mathrm{cor}_f := \mathrm{cor}_{f,E/L} : \widetilde{C}^\bullet_f(G_{E,S_E}, X) \longrightarrow \widetilde{C}^\bullet_f(G_{L,S_L}, X),$$

whose homotophy class can be shown to be independent on any choices. The relation with Shapiro's Lemma is given by the following diagram, commutative up to homotopy:

$$\begin{array}{ccccc} \widetilde{C}^\bullet_f(G_{K,S}, X(E)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{E,S_E}, X) & \xleftarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K,S}, E(X)) \\ \downarrow{\scriptstyle \mathrm{pr}_*} & & \downarrow{\scriptstyle \mathrm{cor}_{E/L,f}} & & \downarrow{\scriptstyle (j^*_{E/L})_*} \\ \widetilde{C}^\bullet_f(G_{K,S}, X(L)) & \xrightarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{L,S_L}, X) & \xleftarrow{\mathrm{sh}_f} & \widetilde{C}^\bullet_f(G_{K,S}, L(X)). \end{array} \tag{178}$$

(Here pr is the natural projection and $j^*_{E/L}$ is the map induced by $j_{E/L} : \mathrm{Gal}(L/K) \ni \sigma \mapsto \sum_{\widetilde{\sigma}|_L=\sigma} \widetilde{\sigma} \in \mathcal{R}[\mathrm{Gal}(E/K)]$.) For the details we refer the reader once again to [**Nek06**, Ch. 8].

0.16. $\mathbb{Z}_p$-power extensions. Let $\mathcal{K}/K$ be the maximal $\mathbb{Z}_p$-power extension of K, i.e. the composition of all $\mathbb{Z}_p$-extensions of K inside $\overline{K}$. Since every $\mathbb{Z}_p$-extension is unramified at every finite prime not dividing p, we have $\mathcal{K} \subset K_S$. For every (possibly infinite) subfield $K \subset L \subset \mathcal{K}$ we write

$$\mathcal{R}(L) := \mathcal{R}[[\mathrm{Gal}(L/K)]] = \varprojlim \mathcal{R}[\mathrm{Gal}(E/K)]$$

for the completed group algebra of $\mathrm{Gal}(L/K)$ over $\mathcal{R}$, where E/L runs over all finite sub-extensions of L/K. Inversion $g \mapsto g^{-1}$ on group-like elements induces an involution $\iota : \mathcal{R}(L) \to \mathcal{R}(L)$ (i.e. an $\mathcal{R}$-linear ring isomorphism such that $\iota^2 = \iota$). Given an $\mathcal{R}(L)[G_{K,S}]$-module M we write M^ι for the $\mathcal{R}[G_{K,S}]$-module M, with $\mathcal{R}(L)$-action defined by $r|_{M^\iota} \cdot m := \iota(r) \cdot m$ for every $r \in \mathcal{R}(L)$ and $m \in M$. For every $n \in \mathbb{Z}$ we also write $M < n >$ for the $\mathcal{R}(L)$-module M with $G_{K,S}$-action defined by $g|_{M<n>} \cdot m := \chi_L(g)^n \cdot g(x)$ for every $g \in G_{K,S}$ and $m \in M$, where χ_L denotes the tautological representation:

$$\chi_L : G_{K,S} \twoheadrightarrow \mathrm{Gal}(L/K) \hookrightarrow \mathcal{R}(L)^*.$$

For every subfield $K \subset L \subset \mathcal{K}$ let us define the $\mathcal{R}(L)[G_{K,S}]$-modules

$$X(L) := \varprojlim X(E); \quad L(X) := \varinjlim E(X), \tag{179}$$

where E/K runs over all finite sub-extensions of L/K and the inverse (resp., direct) limit is taken with respect to projection $\mathcal{R}[\mathrm{Gal}(E'/K)] \twoheadrightarrow \mathcal{R}[\mathrm{Gal}(E/K)]$ (resp., the morphisms $i_{E'/E} : E(X) \to E'(X)$ (177)) for $K \subset E \subset E' \subset L$.

0.16.1. Let us consider in this Section an $\mathcal{R}[G_{K,S}]$-module $T = X$, *free of finite type* over $\mathcal{R}$. Since the natural map $\ddagger \otimes_{\mathcal{R}} \mathcal{R}(L) := \ddagger \otimes_{\mathcal{R}} \varprojlim (\mathcal{R}[\mathrm{Gal}(E/K)]) \to \varprojlim (\ddagger \otimes_{\mathcal{R}} \mathcal{R}[\mathrm{Gal}(E/K)])$ (where $K \subset E \subset L$ runs over the finite Galois subextensions of L/K) is an isomorphism for every $\mathcal{R}$-module of finite type, and since by definition $z \in \mathcal{R}(E)$ (resp., $g \in G_{K,S}$) acts on $T(E) := T \otimes_{\mathcal{R}} \mathcal{R}[\mathrm{Gal}(E/K)]$ via multiplication by $\mathrm{id} \otimes \iota(z)$ (resp., via $g \otimes g|_E$) (recall L/K is abelian), we obtain a canonical isomorphism of $\mathcal{R}(L)[G_{K,S}]$-modules:

$$T(L) \cong (T \otimes_{\mathcal{R}} \mathcal{R}(L)^\iota) < -1 > \stackrel{\mathrm{id}\otimes\iota}{\longrightarrow} (T \otimes_{\mathcal{R}} \mathcal{R}(L)) < -1 >, \tag{180}$$

(where we consider on $\mathcal{R}(L)$ the trivial $G_{K,S}$-action). In a similar way we obtain canonical isomorphisms of $\mathcal{R}(L)[G_v]$-modules, for $v \in S_f$:

$$T(L)_v^{\pm} := T_v^{\pm}(L) \cong \left(T_v^{\pm} \otimes_{\mathcal{R}} \mathcal{R}(L)\right) < -1 > .$$

We will from now on identify $T(L)_*^{\dagger}$ with $\left(T_*^{\dagger} \otimes_{\mathcal{R}} \mathcal{R}(L)\right) < -1 >$ under these isomorphisms. We can then define the Selmer complex of $\mathcal{R}(L)$-modules:

$$\widetilde{C}^{\bullet}_{f,\mathrm{Iw}}(L/K,T) := \widetilde{C}^{\bullet}_f(G_{K,S}, T(L); \Delta(T(L)))$$

using the local conditions $\Delta(T(L)) = \{\Delta_v(T(L))\}$ induced on complexes by the embeddings $i_v^+ \otimes \mathcal{R}(L) : T(L)_v^+ \hookrightarrow T(L)$ for every $v \in S_f$. We also write

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K,T) \in \mathcal{D}(\mathcal{R}(L))_{\mathrm{ft}}; \quad \widetilde{H}^*_{f,\mathrm{Iw}}(L/K,T) \in \left({}_{\mathcal{R}(L)}\mathrm{Mod}\right)_{\mathrm{ft}}$$

for the image of $\widetilde{C}^{\bullet}_{f,\mathrm{Iw}}(L/K,T)$ in the derived category and its cohomology respectively. The following is the main result of this appendix. For the (not difficult) details of its proof we refer to [**Nek06**, Prop. 8.8.6].

PROPOSITION 0.11. 1. *The canonical morphism of complexes of $\mathcal{R}$-modules:*

$$\widetilde{C}^{\bullet}_{f,\mathrm{Iw}}(L/K,T) \xrightarrow{\sim} \varprojlim_{E/K;\ \mathrm{pr}_*} \widetilde{C}^{\bullet}_f(G_{K,S}, T(E))$$

is an isomorphism. As usual E/K runs over the finite sub-extensions of L/K and the limit is taken with respect to the maps induced on Selmer complexes by the projections $\mathrm{pr} : T_*^{\dagger}(E') \twoheadrightarrow T_*^{\dagger}(E)$ *for* $E \subset E'$.

2. *The isomorphism in* 1. *and the Shapiro's morphisms of complexes* (169) *induce on cohomology isomorphisms of $\mathcal{R}(L)$-modules:*

$$\mathrm{sh}_{f,L/K} : \widetilde{H}^*_{f,\mathrm{Iw}}(L/K,T) \xrightarrow{\sim} \varprojlim_{E/K;\ \mathrm{pr}_*} \widetilde{H}^*_f(G_{K,S}, T(E)) \xrightarrow{\sim} \varprojlim_{E/K;\ \mathrm{cor}_f} \widetilde{H}^*_f(G_{E,S_E}, T).$$

Here the second isomorphism is defined by $\varprojlim_{E/K} \left(\mathrm{sh}_{f,E/K}\right)_*$ (see diagram (178)*) and the structure of $\mathcal{R}(L)$-module on the* R.H.S. *is induced on the limit by Galois conjugation (see* (174) *and* (175)*).*

0.16.2. Let us fix in this Section an $\mathcal{R}[G_{K,S}]$-module $X = A$, of *cofinite type* over $\mathcal{R}$. Let us consider the Selmer complex of $\mathcal{R}(L)$-modules:

$$\widetilde{C}^{\bullet}_f(K_S/L, A) := \widetilde{C}^{\bullet}_f\left(G_{K,S}, L(X); \Delta(L(X))\right),$$

where $\Delta(L(X)) := \{\Delta_v(L(X))\}_{v \in S_f}$ is defined by the morphisms induced on complexes by the embedding of $\mathcal{R}(L)[G_v]$-modules:

$$\varprojlim \left(i_v^+\right)_* : L(X)_v^+ := \varprojlim_{E/K} \mathrm{Hom}_{\mathcal{R}}\left(\mathcal{R}(E), A_v^+\right) \hookrightarrow \varprojlim_{E/K} \mathrm{Hom}_{\mathcal{R}}\left(\mathcal{R}(E), A\right),$$

where as usual E/K runs over the finite sub-extensions of L/K. We write

$$\widetilde{\mathbf{R}\Gamma}_f(K_S/L, A) \in \left(\mathcal{D}(\mathcal{R}(L))\right)_{\mathrm{cft}}; \quad \widetilde{H}^*_f(K_S/L, A) \in \left({}_{\mathcal{R}(L)}\mathrm{Mod}\right)_{\mathrm{cft}}$$

for the image of $\widetilde{C}^{\bullet}_f(K_S/L, A)$ in the derived category and its cohomology respectively. We have the following 'discrete' analogous of Prop. 0.11, whose proof is also much easier (as, contrary to $\varprojlim$, the functor $\varinjlim$ is exact).

PROPOSITION 0.12. *The natural morphism of complexes of $\mathcal{R}$-modules:*

$$\varinjlim_{E/K} \widetilde{C}^{\bullet}_f(G_{K,S}, E(A)) \xrightarrow{\sim} \widetilde{C}^{\bullet}_f(K_S/L, A)$$

is an isomorphism. This isomorphisms, together with the Shapiro morphisms of complexes (169) *induces isomorphisms of $\mathcal{R}(L)$-modules (cfr. diagram* (177)*):*

$$\mathrm{sh}_{f,K_S/L} : \varinjlim_{E/K;\ \mathrm{res}_f} \widetilde{H}^*_f(G_{E,S_E}, A) \xrightarrow{\sim} \varinjlim_{E/K} \widetilde{H}^*_f(G_{K,S}, E(A)) \xrightarrow{\sim} \widetilde{H}^*_f(K_S/L, A),$$

where the structure of $\mathcal{R}(L)$*-module on the* L.H.S. *is induced on the limit by the conjugation action of* $\mathrm{Gal}(E/K)$ *on* $\widetilde{H}^*_f(G_{E,S_E}, A)$ *defined in* (174) *and* (175).

0.17. Global cup-products. We consider in this section $X = \{X, \{X_v^+\}_{v\in S_f}\}$, $Y = \{Y, \{Y_v^+\}_{v\in S_f}\}$ and $\pi : X \otimes_{\mathcal{R}} Y \to \mathcal{R}(1)$ as in Section 0.6. We also assume that π is a perfect duality and that $X_v^+ \perp_\pi Y_v^+$ for every $v \in S_f$. Let us also consider a (possibly infinite) sub-extension L/K of $\mathcal{K}/K$.

It is easily seen that under our assumptions the morphism of $\mathcal{R}(L)[G_{K,S}]$-modules:

$$\begin{aligned}\pi(L) : X(L) \otimes_{\mathcal{R}(L)} Y(L)^\iota &\cong (X \otimes_{\mathcal{R}} \mathcal{R}(L)) < -1 > \otimes_{\mathcal{R}(L)} ((Y \otimes_{\mathcal{R}} \mathcal{R}(L)) < -1 >)^\iota \\ &\overset{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\iota}{\longrightarrow} (X \otimes_{\mathcal{R}} \mathcal{R}(L)) < -1 > \otimes_{\mathcal{R}(L)} (Y \otimes_{\mathcal{R}} \mathcal{R}(L)) < 1 > \\ &\cong (X \otimes Y) \otimes_{\mathcal{R}} \mathcal{R}(L) \overset{\pi\otimes\mathrm{id}}{\longrightarrow} \mathcal{R}(1) \otimes_{\mathcal{R}} \mathcal{R}(L) \cong \mathcal{R}(L)(1)\end{aligned}$$

defines a perfect duality between the $\mathcal{R}(L)$-modules $X(L)$ and $Y(L)^\iota$, such that $X(L)_v^+ \perp_{\pi(L)} (Y(L)_v^+)^\iota$ for every $v \in S_f$. In particular the constructions of Section 0.7 give us a global cup-product in $\mathcal{D}(\mathcal{R}(L))$:

$$\cup_{\boldsymbol{\pi(L)}} : \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(L)} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, Y)^\iota \longrightarrow \mathcal{R}(L)[-3].$$

We note that, if $X = Y$ and $\pi \circ s_{12} = \pm\pi$, then $\pi(L)^\iota \circ s_{12} = \pm\iota \circ \pi(L)$, where $\iota : \mathcal{R}(L) \cong \mathcal{R}(L)$ is the $\mathcal{R}$-linear involution induced by inversion on $\mathrm{Gal}(L/K)$. In other words: if π is symmetric (resp., skew-symmetric) then $\pi(L)$ is symmetric-Hermitian (resp., skew-Hermitian).

0.18. Control Theorems. Let X be as in the preceding section. Let us consider a tower of extensions $K \subset L \subset L' \subset \mathcal{K}$ and let us write $\varepsilon_{L'/L} : \mathcal{R}(L') \twoheadrightarrow \mathcal{R}(L)$ be the augmentation map induced by restriction of automorphisms $\mathrm{Gal}(L'/K) \twoheadrightarrow \mathrm{Gal}(L/K)$.

PROPOSITION 0.13. *There exists a canonical isomorphism in* $\mathcal{D}(\mathcal{R}(L))$*:*

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L'/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(L'),\varepsilon_{L'/L}} \mathcal{R}(L) \cong \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L/K, X).$$

PROOF. Fix an isomorphism $\mathcal{G} := \mathrm{Gal}(\mathcal{K}/K) \overset{\sim}{\to} \mathbb{Z}_p^d$ $(d \geq 1)$ and corresponding topological generators $\sigma_1, \ldots, \sigma_d \in \mathcal{G}$, so that $\mathcal{R}(\mathcal{K}) \overset{\sim}{\to} \mathcal{R}[[X_1, \ldots, X_d]]$ with variables $X_j := \sigma_j - 1$. For every subextension E/K of $\mathcal{K}/K$ the closed subgroup $\mathrm{Gal}(\mathcal{K}/E)$ is topologically generated by $\boldsymbol{x}_E := \left\{\sigma_j^{p^{n_j}} = (X_j + 1)^{p^{n_j}} - 1\right\}_{j=1}^d$, for positive integers $n_j \geq 0$, so that the kernel of $\varepsilon_{\mathcal{K}/E}$ is identified with the ideal generated by $\boldsymbol{x}_E \subset \mathfrak{m}_{\mathcal{R}(\mathcal{K})}$. This is clearly an $\mathcal{R}(\mathcal{K})$-regular sequence, so that Lemma (0.4) gives us a canonical isomorphism:

$$\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(\mathcal{K}),\varepsilon_{\mathcal{K}/E}} \mathcal{R}(E) \overset{\sim}{\to} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(E/K, X) \in \mathcal{D}(\mathcal{R}(E)).$$

Then using twice what already proved we obtain canonical isomorphisms:

$$\begin{aligned}\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L'/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(L'),\varepsilon_{L'/L}} \mathcal{R}(L) &\overset{\sim}{\to} \left(\widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(\mathcal{K}),\varepsilon_{\mathcal{K}/L'}} \mathcal{R}(L')\right) \otimes^{\mathbf{L}}_{\mathcal{R}(L'),\varepsilon_{L'/L}} \mathcal{R}(L) \\ &\overset{\sim}{\to} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(\mathcal{K}/K, X) \otimes^{\mathbf{L}}_{\mathcal{R}(\mathcal{K}),\varepsilon_{\mathcal{K}/L}} \mathcal{R}(L) \overset{\sim}{\to} \widetilde{\mathbf{R}\Gamma}_{f,\mathrm{Iw}}(L, X).\end{aligned}$$

□

0.19. Pontrjagin duality. Let $T = \{T, \{T_v^+\}_{v\in S_f}\}$ be an $\mathcal{R}[G_{K,S}]$-modules, free of finite type over $\mathcal{R}$, and let L/K be an arbitrary subextension of $\mathcal{K}/K$. Let us write $\mathbb{A}_T = \left\{\mathbb{A}_T, \{\mathbb{A}_{T,v}^+\}_{v\in S_f}\right\}$ for the Kummer dual $\mathbb{A}_T := \mathrm{Hom}_{\mathrm{cont}}(T, \mathbb{Q}_p/\mathbb{Z}_p(1))$ of T, with 'local condition' $\mathbb{A}_{T,v}^+ := \mathrm{Hom}_{\mathrm{cont}}(T_v^-, \mathbb{Q}_p/\mathbb{Z}_p(1)) \hookrightarrow \mathbb{A}_T$.

PROPOSITION 0.14. *There exists canonical isomorphisms of* $\mathcal{R}(L)$*-modules*

$$\widetilde{H}^*_{f,\mathrm{Iw}}(L/K, T) \cong \mathrm{Hom}_{\mathbb{Z}_p}\left(\widetilde{H}^{*-3}_f(K_S/L, \mathbb{A}_T)^\iota, \mathbb{Q}_p/\mathbb{Z}_p\right).$$

The rest of this Section will be devoted to the proof of this proposition. We begin by recalling some facts from Matlis duality referring again to **[Nek06]** for the details: let $\mathcal{S} = (\mathcal{S}, \mathfrak{m}_{\mathcal{S}})$ be a complete Noetherian local ring, with *finite* residue field $\Bbbk_{\mathcal{S}} := \mathcal{S}/\mathfrak{m}_{\mathcal{S}}$ of characteristic p. For every compact or discrete $\mathcal{S}$-module V we write $\mathscr{P}(V) := \mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ (considering the discrete topology on $\mathbb{Q}_p/\mathbb{Z}_p$) and $\mathscr{M}_{\mathcal{S}}(V) := \mathrm{Hom}_{\mathcal{S}}(V, \mathscr{P}(\mathcal{S}))$. Then the following two fact are known:

M1 For every $\mathcal{S}$-module V which is either of finite or co-finite type, the canonical map:

$$V \xrightarrow{\sim} \mathscr{M}_{\mathcal{S}}\left(\mathscr{M}_{\mathcal{S}}(V)\right)$$

of V into its bi-$\mathscr{P}(\mathcal{S})$-dual is an isomorphism of $\mathcal{S}$-modules.

M2 For every $\mathcal{S}$-module V which is either of finite or co-finite type, the canonical map:

$$\mathscr{M}_{\mathcal{S}}(V) \xrightarrow{\sim} \mathscr{P}(V)$$

induced by evaluation at the identity of $\mathcal{S}$ is an isomorphism of $\mathcal{S}$-modules.

In other words **M1** tells us that $\mathscr{P}(\mathcal{S})$ is a *dualizing functor* on the category of $\mathcal{S}$-modules of finite (resp., co-finite) type, so by Matlis duality the Pontrjagin dual $\mathscr{P}(\mathcal{S})$ of $\mathcal{S}$ is an injective hull of $\Bbbk_{\mathcal{S}}$. Indeed **M1** follows by **M2** and Prontrjagin duality. See Sections 2.2, 2.3 and 2.9 of **[Nek06]** for details and precise references.

We now show how the Proposition will follows from **M1**, **M2** and the following claim:

C Let $G \in \{G_{K,S}, G_v\}$. For every discrete $\mathcal{R}(L)[G]$-module A of co-finite type over $\mathcal{R}$, there exists a canonical isomorphism of $\mathcal{R}(L)[G]$-modules:

$$\mathscr{M}_{\mathcal{R}(L)}\left(L\left(A\right)\right) \cong \left\{\left[\mathscr{M}_{\mathcal{R}}(A)\right](L)\right\}^{\iota}.$$

This induces an isomorphisms of functors:

$$\mathscr{M}_{\mathcal{R}(L)} \circ L(*) \Longrightarrow \iota \circ \{*(L)\} \circ \mathscr{M}_{\mathcal{R}} : \left({}_{\mathcal{R}[G]}\mathrm{Mod}\right)_{\mathcal{R}-\mathrm{cft}} \to_{\mathcal{R}(L)[G]} \mathrm{Mod}$$

(where $\dagger -$ ft stands for: of finite type over $\dagger$).

Indeed we then obtain canonical isomorphisms of $\mathcal{R}(L)[G_{K,S}]$-modules:

$$\begin{aligned}
\mathbb{A}_{L(\mathbb{A}_T)} := \mathscr{P}(L(\mathscr{P}(T)(1)))(1) &= \left(\mathscr{P} \circ L(*) \circ \mathscr{P}\right)(T) \\
(\text{using } \mathbf{M2}) \quad &\cong \left(\mathscr{M}_{\mathcal{R}(L)} \circ L(*) \circ \mathscr{M}_{\mathcal{R}}\right)(T) \\
(\text{using } \mathbf{C}) \quad &\cong \left(\iota \circ *(L) \circ \mathscr{M}_{\mathcal{R}} \circ \mathscr{M}_{\mathcal{R}}\right)(T) \\
(\text{using } \mathbf{M1}) \quad &\cong \left(\iota \circ *(L)\right)(T) = T(L)^{\iota},
\end{aligned}$$

expressing a perfect Kummer duality of $\mathcal{R}(L)[G_{K,S}]$-modules:

$$T(L)^{\iota} \otimes_{\mathcal{R}(L)} L(\mathbb{A}_T) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p(1). \tag{181}$$

In a similar way, recalling that $\mathbb{A}^{\pm}_{T,v}$ is defined as the Kummer dual $\mathbb{A}_{T_v^{\mp}}$ of $T_v^{\mp}$ and $L(\mathbb{A}_T)^{\pm}_v := L\left(\mathbb{A}^{\pm}_{T,v}\right)$, we obtain canonical isomorphism of $\mathcal{R}(L)[G_v]$-modules, for every $v \in S_f$:

$$\mathbb{A}_{L(\mathbb{A}_T)^{\pm}_v} := \mathbb{A}_{L\left(\mathbb{A}^{\pm}_{T,v}\right)} := \mathbb{A}_{L\left(\mathbb{A}_{T_v^{\mp}}\right)} \cong \left(T(L)^{\mp}_v\right)^{\iota},$$

expressing a perfect Kummer dualities of $\mathcal{R}(L)[G_v]$-modules:

$$\left(T(L)^{\pm}_v\right)^{\iota} \otimes_{\mathcal{R}(L)} L(\mathbb{A}_T)^{\mp}_v \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p(1). \tag{182}$$

Moreover, again by **C**, (181) and (182) are compatibles (in a suitable sense) under the natural maps 'i_v^+' and 'p_v^-' induced by $i_v^+ : T_v^+ \hookrightarrow T$ and $p_v^- : T \twoheadrightarrow T_v^-$ for $v \in S_f$. In other worlds, with the terminology of Section 0.11: $L(\mathbb{A}_T) = \left\{L(\mathbb{A}_T), (L(\mathbb{A}_T)^+_v)_{v \in S_f}\right\}$ is the Kummer dual of $T(L)^{\iota} = \left\{T(L)^{\iota}, (T(L)^+_v)_{v \in S_f}\right\}$. Then Proposition 0.14 turns out to be a special case of Proposition 0.8, working now with $\mathcal{R}(L)$-coefficients instead of $\mathcal{R}$-coefficients.

Proof of **C**. Let us fix an $\mathcal{R}[G]$-module A, of cofinite type over $\mathcal{R}$. Let us write $T := \mathscr{M}_{\mathcal{R}}(A)$ be the 'Matlis dual' of A, which by assumption (cfr. **M2**) is a finite $\mathcal{R}$-module, and let $I_{\mathcal{R}} := \mathscr{P}(\mathcal{R})$, $I_{\mathcal{R}(L)} := \mathscr{P}(\mathcal{R}(L))$. We will consider on I_* the trivial $G_{K,S}$-action. Letting E/K runs over the finite subextensions of L/K, we have canonical isomorphisms of $\mathcal{R}(L)$-modules:

$$\begin{aligned}
L(I_{\mathcal{R}}) &:= \varinjlim_{E} \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}[\operatorname{Gal}(E/K)], \varinjlim_{n} \operatorname{Hom}_{\mathbb{Z}_p}\left(\mathcal{R}/\mathfrak{m}_{\mathcal{R}}^n, \mathbb{Q}_p/\mathbb{Z}_p\right)\right)\\
(\mathcal{R}[\operatorname{Gal}(E/K)] \text{ is finite, free over } \mathcal{R}) &= \varinjlim_{E,n} \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}[\operatorname{Gal}(E/K)], \operatorname{Hom}_{\mathbb{Z}_p}\left(\mathcal{R}/\mathfrak{m}_{\mathcal{R}}^n, \mathbb{Q}_p/\mathbb{Z}_p\right)\right)\\
&= \varinjlim_{E,n} \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}[\operatorname{Gal}(E/K)] \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{m}_{\mathcal{R}}^n, \mathbb{Q}_p/\mathbb{Z}_p\right)\\
&= \varinjlim_{E,n} \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}/\mathfrak{m}_{\mathcal{R}}^n[\operatorname{Gal}(E/K)], \mathbb{Q}_p/\mathbb{Z}_p\right)\\
&= \operatorname{Hom}_{\mathcal{R}}\left(\varprojlim_{E,n} \mathcal{R}/\mathfrak{m}_{\mathcal{R}}^n[\operatorname{Gal}(E/K)], \mathbb{Q}_p/\mathbb{Z}_p\right)\\
&= \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}(L), \mathbb{Q}_p/\mathbb{Z}_p\right) =: I_{\mathcal{R}(L)}.
\end{aligned}$$

Moreover, recalling the definitions: $g \in G_{K,S}$ acts on $f = (f_E)_E \in L(I_{\mathcal{R}})$ by $f^g(\dagger) = f(g^{-1}(\dagger)) = f(\chi_L(g)^{-1} \cdot \dagger) = \chi_L(g)^{-1} \cdot f(\dagger)$. In other words we have proved that there exist a canonical isomorphism of $\mathcal{R}(L)[G_{K,S}]$-modules:

$$L(I_{\mathcal{R}}) \cong I_{\mathcal{R}(L)} < -1 > . \tag{183}$$

Using this isomorphism we obtain a canonical isomorphism of $\mathcal{R}[G]$-modules:

$$\begin{aligned}
L(A) &:= \varinjlim_{E} \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}[\operatorname{Gal}(E/K)], A\right)\\
(\text{by } \mathbf{M1}) &\cong \varinjlim_{E} \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}[\operatorname{Gal}(E/K)], \mathscr{M}_{\mathcal{R}} \circ \mathscr{M}_{\mathcal{R}}(A)\right)\\
&:= \varinjlim_{E} \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}[\operatorname{Gal}(E/K)], \operatorname{Hom}_{\mathcal{R}}(T, I_{\mathcal{R}})\right)\\
&= \varinjlim_{E} \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}[\operatorname{Gal}(E/K)] \otimes_{\mathcal{R}} T, I_{\mathcal{R}}\right)\\
&= \varinjlim_{E} \operatorname{Hom}_{\mathcal{R}}\left(T, \operatorname{Hom}_{\mathcal{R}}\left(\mathcal{R}[\operatorname{Gal}(E/K)], I_{\mathcal{R}}\right)\right)\\
(\text{since } T \text{ is finite over } \mathcal{R}) &\cong \operatorname{Hom}_{\mathcal{R}}\left(T, \varinjlim_{E} \operatorname{Hom}_{\mathcal{R}}(\mathcal{R}[\operatorname{Gal}(E/K)], I_{\mathcal{R}})\right)\\
&=: \operatorname{Hom}_{\mathcal{R}}\left(T, L(I_{\mathcal{R}})\right)\\
(\text{by } (183)) &\cong \operatorname{Hom}_{\mathcal{R}}\left(T, I_{\mathcal{R}(L)}\right) < -1 >\\
&= \operatorname{Hom}_{\mathcal{R}(L)}\left(T \otimes_{\mathcal{R}} \mathcal{R}(L), I_{\mathcal{R}(L)}\right) < -1 >\\
&= \operatorname{Hom}_{\mathcal{R}(L)}\left((T \otimes_{\mathcal{R}} \mathcal{R}(L)) < 1 >, I_{\mathcal{R}(L)}\right)\\
(\text{via the adjoint of } \mathrm{id} \otimes \iota, \text{ cfr. } (180)) &= \operatorname{Hom}_{\mathcal{R}(L)}\left(T(L)^{\iota}, I_{\mathcal{R}(L)}\right) =: \mathscr{M}_{\mathcal{R}(L)}\left(T(L)^{\iota}\right).
\end{aligned}$$

Applying $\mathscr{M}_{\mathcal{R}(L)}$ to this isomorphism and recalling **M1** we finally obtain a canonical (and functorial) isomorphism of $\mathcal{R}(L)[G_{K,S}]$:

$$\left(\mathscr{M}_{\mathcal{R}(L)} \circ L(*)\right)(A) \cong T(L)^{\iota} := \left\{\left[\mathscr{M}_{\mathcal{R}}(A)\right](L)\right\}^{\iota} = \left(\iota \circ \{*(L)\} \circ \mathscr{M}_{\mathcal{R}}\right)(A).$$

This concludes the proof of Claim **C**, and with it the proof of Proposition 0.14. □

APPENDIX C

Abstract height pairings

The notations and hypothesis are as in Section A.

Let $\mathcal{R} = (\mathcal{R}, \mathfrak{m}_{\mathcal{R}}; \iota : \mathcal{R} \to \mathcal{R})$ and $X^{\text{'}} = \text{'}\{X; X_v^+,\ v \in S_f\}$ be as in Section 0.9. Fix an ideal $\mathcal{P} \subsetneq \mathcal{R}$ generated by an $\mathcal{R}$-regular sequence, and invariant under the involution $\iota : \mathcal{R} \to \mathcal{R}$. Let $\mathcal{A} = \mathcal{A}_{[\mathcal{P}]} := \mathcal{R}/\mathcal{P}$, $Y = Y_{[\mathcal{P}]} := X \otimes_{\mathcal{R}} \mathcal{A}$ and $Y_v^+ = \left(Y_{[\mathcal{P}]}\right)_v^+ := X_v^+ \otimes_{\mathcal{R}} \mathcal{A}$ $(v \in S_f)$. We denote again by $\iota : \mathcal{A} \to \mathcal{A}$ the involution on $\mathcal{A}$ induced by $\iota : \mathcal{R} \to \mathcal{R}$. Then the perfect duality of $\mathcal{R}[G_{K,S}]$-module $\pi := \pi_X : X \otimes_{\mathcal{R}} X^{\iota} \to \mathcal{R}(1)$ induces a perfect duality of $\mathcal{A}[G_{K,S}]$-modules:

$$\pi_{\mathcal{P}} := \pi \otimes \mathcal{A} : Y \otimes_{\mathcal{A}} Y^{\iota} \longrightarrow \mathcal{A}(1); \quad \text{s.t.} \quad Y \perp_{\pi_{\mathcal{P}}} Y^{\iota}.$$

Then the construction of Section 0.7 and Section 0.9 defines a skew-Hermitian pairing in $\mathcal{D}(\mathcal{A})$:

$$\cup_{\boldsymbol{\mathcal{P}}} := \cup_{\boldsymbol{\pi_{\mathcal{P}}}} : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y) \otimes^{\mathbf{L}}_{\mathcal{A}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y)^{\iota} \longrightarrow \mathcal{A}[-3].$$

0.20. Bockstein maps. As $\mathcal{P} = (x_1, \dots, x_d)$ is generated by an $\mathcal{R}$-regular sequence, we can prove easily that $\mathcal{P}/\mathcal{P}^2$ is a free $\mathcal{A}$-module (of rank d), generated by the residue classes of the generators x_j. The projection $\mathcal{R}/\mathcal{P}^2 \twoheadrightarrow \mathcal{A}$ induces an exact triangle in $\mathcal{D}(\mathcal{R})$

$$\mathcal{P}/\mathcal{P}^2 \to \mathcal{R}/\mathcal{P}^2 \to \mathcal{A} \xrightarrow{\partial_{\mathcal{P}}} \mathcal{P}/\mathcal{P}^2[1]. \tag{184}$$

Lemma 0.4 gives canonical isomorphisms in $\mathcal{D}(\mathcal{R})$ (cfr. Remark 0.3):

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{A} \cong \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y);$$

$$\begin{aligned} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{P}/\mathcal{P}^2[1] &\cong \left(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{A}\right) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2[1] \\ \text{(Lemma 0.4)} \qquad &\cong \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2[1] \\ &\cong \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y) \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2[1] \\ &\cong \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y)[1] \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2, \end{aligned}$$

where the third isonmorphism follows by the flatness of $\mathcal{P}/\mathcal{P}^2$ over $\mathcal{A}$. It follows that applying the derived functor $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} -$ to the exact triangle (184) induces an exact triangle in $\mathcal{D}(\mathcal{R})$:

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{R}/\mathcal{P}^2 \to \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y) \xrightarrow{\beta_{\mathcal{P}}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y)[1] \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2.$$

We call $\beta_{\mathcal{P}}\text{'} = \text{'}\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} \partial_{\mathcal{P}}$ the *Bockstein map* attached to $\mathcal{P}$ (cfr. [**Nek06**, Sec. 11.1.4]). If we want to emphasize the Galois module we are considering we will write $\beta_{X,\mathcal{P}}$ for $\beta_{\mathcal{P}}$.

0.21. Associated height pairings. We define the *height pairing* attached to $\mathcal{P}$ as the morphism in $\mathcal{D}(\mathcal{A}_{[\mathcal{P}]} = \mathcal{A})$ (emphasizing the dependence on $\mathcal{P}$ in the notations):

$$\begin{aligned} \widetilde{h}_{\mathcal{P}} : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes^{\mathbf{L}}_{\mathcal{A}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]})^{\iota} &\xrightarrow{\beta_{\mathcal{P}} \otimes \mathrm{id}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]})[1] \otimes^{\mathbf{L}}_{\mathcal{A}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]})^{\iota} \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2 \\ \cong \left(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes^{\mathbf{L}}_{\mathcal{A}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]})^{\iota}\right)[1] \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2 &\xrightarrow{\cup_{\boldsymbol{\mathcal{P}}}[1] \otimes \mathrm{id}} \mathcal{A}[-2] \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2 \cong \mathcal{P}/\mathcal{P}^2[-2]. \end{aligned}$$

Given integers $i, j \in \mathbb{Z}$ s.t. $i + j = 2$ we will write

$$\widetilde{h}_{\mathcal{P},i,j} := H^{i,j}(\widetilde{h}_{\mathcal{P}}) : \widetilde{H}^i_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes_{\mathcal{A}} \widetilde{H}^j_f(G_{K,S}, Y_{[\mathcal{P}]})^{\iota} \longrightarrow \mathcal{P}/\mathcal{P}^2$$

for the pairing induced by $\widetilde{h}_{\mathcal{P}}$ in (i,j)-cohomology. As above we will also write $\widetilde{h}_{X,\mathcal{P}}$ and $\widetilde{h}_{X,\mathcal{P},i,j}$ in order to emphasize which Galois module we are considering.

0.22. Decomposition of pairings. Assume that $\mathcal{P} = (\boldsymbol{x}, \boldsymbol{y})$ is generated by $\boldsymbol{x} = (x_1, \dots, x_r)$ and $\boldsymbol{y} = (y_1, \dots, y_s)$. Let us write $\mathcal{A}_{\boldsymbol{y}} := \mathcal{R}/\boldsymbol{y}$, $Y_{[\boldsymbol{y}]} := Y \otimes_{\mathcal{R}} \mathcal{A}_{\boldsymbol{y}}$ and $\mathcal{P}_{\boldsymbol{y}} \subset \mathcal{A}_{\boldsymbol{y}}$ for the ideal generated by the projections of $x_1, \dots, x_r$ modulo $\boldsymbol{y}$. Since we are assuming $\mathcal{R}$ (local and) Noetherian, $\mathcal{P}_{\boldsymbol{y}}$ is generated by an $\mathcal{A}_{\boldsymbol{y}}$-regular sequence. As $\mathcal{A} := \mathcal{A}_{[\mathcal{P}]} = \mathcal{A}_{\boldsymbol{y}}/\mathcal{P}_{\boldsymbol{y}}$, we have $Y_{[\boldsymbol{y}]} \otimes_{\mathcal{A}_{\boldsymbol{y}}} \mathcal{A} = Y_{[\mathcal{P}]}$ and using again the ‘control Theorem’ (Lemma 0.4) the general construction explained above produces an exact triangle in $\mathcal{D}(\mathcal{A}_{\boldsymbol{y}})$:

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\boldsymbol{y}]}) \otimes^{\mathbf{L}}_{\mathcal{A}_{\boldsymbol{y}}} \mathcal{A}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}} \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]}) \xrightarrow{\beta_{\mathcal{P}_{\boldsymbol{y}}}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]})[1] \otimes_{\mathcal{A}} \mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}}.$$

We have abbreviated $\beta_{\mathcal{P}_{\boldsymbol{y}}} := \beta_{Y_{[\boldsymbol{y}]},\mathcal{P}_{\boldsymbol{y}}}$ ‘ = ’$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\boldsymbol{y}]}) \otimes^{\mathbf{L}}_{\mathcal{A}_{[\boldsymbol{y}]}} \partial_{\mathcal{P}_{\boldsymbol{y}}}$, where $\partial_{\mathcal{P}_{\boldsymbol{y}}}$ is the ‘connecting morphism’ in the exact triangle $\mathcal{A}_{[\boldsymbol{y}]}/\mathcal{P}^2_{\boldsymbol{y}} \to \mathcal{A} \to \mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}}[1]$. Exactly as above we can define associated height pairings:

$$\widetilde{h}_{\mathcal{P}_{\boldsymbol{y}}} : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes^{\mathbf{L}}_{\mathcal{A}_{\boldsymbol{y}}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]})^{\iota} \longrightarrow \mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}}[-2];$$

$$(i+j=2) \qquad \widetilde{h}_{\mathcal{P}_{\boldsymbol{y}},i,j} := H^{i,j}(\widetilde{h}_{\mathcal{P}_{\boldsymbol{y}}}) : \widetilde{H}^i_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes_{\mathcal{A}} \widetilde{H}^j_f(G_{K,S}, Y_{[\mathcal{P}]})^{\iota} \longrightarrow \mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}}.$$

(We note that we are now considering $\mathcal{A}_{\boldsymbol{y}}$ as our coefficient ring instead of $\mathcal{R}$, so the notations could be a little confusing. Anyway we still continue to use the more compact notations $\widetilde{h}_{\mathcal{P}_{\boldsymbol{y}}}$ and $\widetilde{h}_{\mathcal{P}_{\boldsymbol{y}},i,j}$ instead of the more precise ones $\widetilde{h}_{Y_{[\boldsymbol{y}]},\mathcal{P}_{\boldsymbol{y}}}$ and $\widetilde{h}_{Y_{[\boldsymbol{y}]},\mathcal{P}_{\boldsymbol{y}},i,j}$). Let us write $\mathrm{pr}_{\boldsymbol{y}} : \mathcal{P}/\mathcal{P} \twoheadrightarrow \mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}}$ for the map induced by the projection $\mathcal{R} \twoheadrightarrow \mathcal{A}_{\boldsymbol{y}}$.

LEMMA 0.15. $\widetilde{h}_{\mathcal{P}_{\boldsymbol{y}}} = \mathrm{pr}_{\boldsymbol{y}}[-2] \circ \widetilde{h}_{\mathcal{P}} \in \mathrm{Mor}_{\mathcal{D}(\mathcal{A})}\Big(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes^{\mathbf{L}}_{\mathcal{A}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]}), \mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}}[-2]\Big)$. *In particular, for every* $i+j=2$*:*

$$\widetilde{h}_{\mathcal{P}_{\boldsymbol{y}},i,j} = \mathrm{pr}_{\boldsymbol{y}} \circ \widetilde{h}_{\mathcal{P},i,j}.$$

PROOF. Recall $\mathcal{A} = \mathcal{A}_{[\mathcal{P}]} := \mathcal{R}/\mathcal{P}$ and $Y = Y_{[\mathcal{P}]} := X \otimes_{\mathcal{R}} \mathcal{A}$. Applying the functor $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} -$ to the morphism of exact triangles in $\mathcal{D}(\mathcal{R})$:

$$\begin{array}{ccccccc}
\mathcal{P}/\mathcal{P}^2 & \longrightarrow & \mathcal{R}/\mathcal{P}^2 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{P}/\mathcal{P}^2[1] \\
\downarrow{\scriptstyle \mathrm{pr}_{\boldsymbol{y}}} & & \downarrow & & \| & & \downarrow{\scriptstyle \mathrm{pr}_{\boldsymbol{y}}[1]} \\
\mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}} & \longrightarrow & \mathcal{A}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}}[1],
\end{array}$$

using the canonical isomorphism:

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} \ddagger \cong \Big(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{A}_{\boldsymbol{y}}\Big) \otimes^{\mathbf{L}}_{\mathcal{A}_{\boldsymbol{y}}} \ddagger \overset{\text{Lemma 0.4}}{\cong} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\boldsymbol{y}]}) \otimes^{\mathbf{L}}_{\mathcal{A}_{\boldsymbol{y}}} \ddagger$$

valid for every cohomologically bounded above complex of $\mathcal{A}_{\boldsymbol{y}}$-modules $\ddagger$, and recalling the constructions above we easily find a commutative diagram of Bockstein maps:

$$\begin{array}{ccc}
\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y) & \xrightarrow{\beta_{\mathcal{P}}} & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y)[1] \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2 \\
& {\scriptstyle \beta_{\mathcal{P}_{\boldsymbol{y}}}} \searrow & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{pr}_{\boldsymbol{y}}} \\
& & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y)[1] \otimes_{\mathcal{A}} \mathcal{P}_{\boldsymbol{y}}/\mathcal{P}^2_{\boldsymbol{y}}.
\end{array}$$

Recalling the definition of $\widetilde{h}_*$, we conclude the proof. □

We note that the preceding lemma implies that the ‘height pairing’ $\widetilde{h}_{\mathcal{P}_{\boldsymbol{y}}}$ (and then $\widetilde{h}_{\mathcal{P}_{\boldsymbol{y}},i,j}$ for $i+j=2$) depends only on the quotient $Y_{\boldsymbol{y}} := X/\boldsymbol{y} \cdot X$ as a $G_{K,S}$-module. Moreover, letting $\{z_t\}_t$ be an $\mathcal{R}$-regular sequence generating $\mathcal{P}$, we see that to compute $\widetilde{h}_{\mathcal{P},i,j}$ it is sufficient to compute $\widetilde{h}_{(z_t),i,j}$ for every principal ideal $(z_t) \subset \mathcal{R}$. In Section 0.24 we will give a useful description of the pairings $\widetilde{h}_{(z),i,j}$ attached to a non-zero divisor $z \in \mathcal{R}$ using the formalism of abstract Cassels-Tate pairings.

0.23. Behaviour under specializations. Using again the notations of the preceding sections, let us consider a second ideal $I \subset \mathcal{R}$ generated by an $\mathcal{R}$-regular sequence, and invariant under the involution ι on $\mathcal{R}$. As in Section 0.8 we will write $\widetilde{M} := M \otimes_{\mathcal{R}} \mathcal{R}/I$, for every $M \in {}_{\mathcal{R}}\mathrm{Mod}$. We also assume that the ideal $\widetilde{\mathcal{P}} \subset \widetilde{\mathcal{R}}$ is generated by an $\widetilde{\mathcal{R}}$-regular sequence. We will denote again by ι the involution induced by $\iota : \mathcal{R} \xrightarrow{\sim} \mathcal{R}$ on any quotient module.

Since $\widetilde{Y} := \widetilde{Y_{[\mathcal{P}]}} := (X \otimes_{\mathcal{R}} \mathcal{R}/\mathcal{P}) \otimes_{\mathcal{R}} \mathcal{R}/I$ is canonically isomprphic to $(X \otimes_{\mathcal{R}} \mathcal{R}/I) \otimes_{\widetilde{\mathcal{R}}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} \mathcal{R}/\mathcal{P} =: \widetilde{X} \otimes_{\widetilde{\mathcal{R}}} \widetilde{\mathcal{R}}/\widetilde{\mathcal{P}}$, and similarly $\widetilde{\mathcal{A}} = \widetilde{\mathcal{R}}/\widetilde{\mathcal{P}}$, the constructions of the preceding sections (applied this time to the data $\widetilde{R}$, $\widetilde{\mathcal{P}}$ and $\widetilde{X}$) give a Bockstein map in $\mathcal{D}(\widetilde{\mathcal{A}})$:

$$\beta_{\widetilde{\mathcal{P}}} = \beta_{\widetilde{X},\widetilde{\mathcal{P}}} : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \widetilde{Y}) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \widetilde{Y})[1] \otimes \widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2,$$

with associated derived pairing in $\mathcal{D}(\widetilde{\mathcal{A}})$:

$$\widetilde{h}_{\widetilde{\mathcal{P}}} = \widetilde{h}_{\widetilde{X},\widetilde{\mathcal{P}}} : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \widetilde{Y}) \otimes^{\mathbf{L}}_{\widetilde{\mathcal{A}}} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \widetilde{Y})^{\iota} \longrightarrow \widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2[-2].$$

As usual we write $\widetilde{h}_{\widetilde{\mathcal{P}},i,j} = H^{i,j}\left(\widetilde{h}_{\widetilde{\mathcal{P}}}\right)$ for the morphism induced in (i,j)-cohomology.

Lemma 0.16. *Let $\boldsymbol{\pi_I} : \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y) \to \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, \widetilde{Y})$ be the morphism of complexes induced by the natural projection $\pi_I : \mathcal{R} \twoheadrightarrow \widetilde{\mathcal{R}}$, and let $\boldsymbol{\pi_{I,*}} := H^*(\boldsymbol{\pi_I})$ be the morphism induced on cohomology. For every integers $i + j = 2$ the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{H}^i_f(G_{K,S}, Y) \otimes_{\mathcal{A}} \widetilde{H}^j_f(G_{K,S}, Y)^{\iota} & \xrightarrow{\widetilde{h}_{\mathcal{P},i,j}} & \mathcal{P}/\mathcal{P}^2 \\ \downarrow{\scriptstyle \boldsymbol{\pi_{I,i}} \otimes \boldsymbol{\pi_{I,j}}} & & \downarrow{\scriptstyle \pi_I} \\ \widetilde{H}^i_f(G_{K,S}, \widetilde{Y}) \otimes_{\widetilde{\mathcal{A}}} \widetilde{H}^j_f(G_{K,S}, \widetilde{Y})^{\iota} & \xrightarrow{\widetilde{h}_{\widetilde{\mathcal{P}},i,j}} & \widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2. \end{array}$$

Proof. Let us consider the natural morphism of exact triangles in $\mathcal{D}(\mathcal{R})$:

$$\begin{array}{ccccc} \mathcal{R}/\mathcal{P}^2 & \longrightarrow & \mathcal{A} & \xrightarrow{\partial_{\mathcal{P}}} & \mathcal{P}/\mathcal{P}^2[1] \\ \downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_I[1]} \\ \widetilde{\mathcal{R}}/\widetilde{\mathcal{P}}^2 & \longrightarrow & \widetilde{\mathcal{A}} & \xrightarrow{\partial_{\widetilde{\mathcal{P}}}} & \widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2[1]. \end{array}$$

Applying $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\otimes^{\mathbf{L}}_{\mathcal{R}} -$ to this morphism, using Lemma 0.4 (and its proof) and recalling the definitions we obtain a commutative diagram in $\mathcal{D}(\mathcal{R})$:

$$\begin{array}{ccc}
\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Y) & \xrightarrow{\beta_{\mathcal{P}}} & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Y)[1]\otimes_{\mathcal{A}}\mathcal{P}/\mathcal{P}^2 \\
\uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} \\
\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\otimes^{\mathbf{L}}_{\mathcal{R}}\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\partial_{\mathcal{P}}} & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\otimes^{\mathbf{L}}_{\mathcal{R}}\mathcal{P}/\mathcal{P}^2[1] \\
\downarrow{\scriptstyle\mathrm{id}\otimes\pi_I} & & \downarrow{\scriptstyle\mathrm{id}\otimes\pi_I[1]} \\
\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\otimes^{\mathbf{L}}_{\mathcal{R}}\widetilde{\mathcal{A}} & \xrightarrow{\mathrm{id}\otimes\partial_{\widetilde{\mathcal{P}}}} & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\otimes^{\mathbf{L}}_{\mathcal{R}}\widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2[1] \\
\| & & \| \\
\left(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\otimes^{\mathbf{L}}_{\mathcal{R}}\widetilde{\mathcal{R}}\right)\otimes^{\mathbf{L}}_{\widetilde{\mathcal{R}}}\widetilde{\mathcal{A}} & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\partial_{\widetilde{\mathcal{P}}}} & \left(\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\otimes^{\mathbf{L}}_{\mathcal{R}}\widetilde{\mathcal{R}}\right)\otimes^{\mathbf{L}}_{\widetilde{\mathcal{R}}}\widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2[1] \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},\widetilde{X})\otimes^{\mathbf{L}}_{\widetilde{\mathcal{R}}}\widetilde{\mathcal{A}} & \xrightarrow{\mathrm{id}\otimes\partial_{\widetilde{\mathcal{P}}}} & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S},\widetilde{X})\otimes^{\mathbf{L}}_{\widetilde{\mathcal{R}}}\widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2[1] \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},\widetilde{Y}) & \xrightarrow{\beta_{\widetilde{\mathcal{P}}}} & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S},\widetilde{Y})[1]\otimes_{\widetilde{\mathcal{A}}}\widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2,
\end{array}$$

where every isomorphism denoted by a *tilde* comes from Lemma 0.4 (or better its proof). Retracing the proof of Lemma 0.4 we see easily that this commutative diagram induces in cohomology a commutative diagram of $\mathcal{A}$-modules:

(185)
$$\begin{array}{ccc}
\widetilde{H}^i_f(G_{K,S},Y) & \xrightarrow{H^i(\beta_{\mathcal{P}})} & \widetilde{H}^{i+1}_f(G_{K,S},Y)\otimes_{\mathcal{A}}\mathcal{P}/\mathcal{P}^2 \\
\downarrow{\scriptstyle\boldsymbol{\pi_{I,i}}} & & \downarrow{\scriptstyle\boldsymbol{\pi_{I,i}}\otimes\pi_I} \\
\widetilde{H}^i_f(G_{K,S},\widetilde{Y}) & \xrightarrow{H^i(\beta_{\widetilde{\mathcal{P}}})} & \widetilde{H}^{i+1}_f(G_{K,S},\widetilde{Y})\otimes_{\widetilde{\mathcal{A}}}\widetilde{\mathcal{P}}/\widetilde{\mathcal{P}}^2.
\end{array}$$

Let us write $\cup_{\boldsymbol{*,s,t}} := H^{s,t}(\cup_*)$ for the map induced in (s,t)-cohomology by the cup product pairing $\cup_*$ associated to the ideal $*=\mathcal{P}$ (reps., $*=\widetilde{\mathcal{P}}$) of $\mathcal{R}$ (resp., $\widetilde{\mathcal{R}}$). Then, for $s+t=3$ we know again by c) of Lemma 0.4 that the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{H}^s_f(G_{K,S},Y)\otimes_{\mathcal{A}}\widetilde{H}^t_f(G_{K,S},Y)^{\iota} & \xrightarrow{\cup_{\boldsymbol{\mathcal{P},s,t}}} & \mathcal{A} \\
\downarrow{\scriptstyle\boldsymbol{\pi_{I,s}}\otimes\boldsymbol{\pi_{I,t}}} & & \downarrow{\scriptstyle\pi_I} \\
\widetilde{H}^s_f(G_{K,S},\widetilde{Y})\otimes_{\widetilde{\mathcal{A}}}\widetilde{H}^t_f(G_{K,S},\widetilde{Y})^{\iota} & \xrightarrow{\cup_{\boldsymbol{\widetilde{\mathcal{P}},s,t}}} & \widetilde{\mathcal{A}}.
\end{array}$$

Combined with diagram (185) and the definitions we easily conclude the proof of the Lemma. ☐

0.24. Relation with Cassels-Tate pairings. We assume in this section that $\mathcal{P}$ is a principal ideal, generated by a non-zero divisor $\mathcal{R}\ni\varpi\nmid 0$.

We recall by Section 0.10 that Nekovář's abstract Cassels-Tate pairing defines a skew-Hermitian pairing:

$$\widetilde{c}_{\pi,2,2}:\widetilde{H}^2_f(G_{K,S},X)_{\mathcal{R}-\mathrm{tors}}\otimes_{\mathcal{R}}\widetilde{H}^2_f(G_{K,S},X)^{\iota}_{\mathcal{R}-\mathrm{tors}}\longrightarrow\mathscr{R}/\mathcal{R},$$

where $\mathscr{R}=\mathrm{Frac}(\mathcal{R})$ is the total ring of fractions of $\mathcal{R}$. Moreover we have seen in the proof of Lemma 0.4 that the natural projection induces an exact triangle in $\mathcal{D}(\mathcal{R})$:

$$\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\overset{\varpi}{\to}\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},X)\to\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Y_{[\mathcal{P}]}).$$

The long exact cohomology sequence attached to this triangle gives us in particular an exact sequence of $\mathcal{A} = \mathcal{A}_{[\mathcal{P}]}$-modules:

$$0 \to \widetilde{H}^1_f(G_{K,S}, X) \otimes_{\mathcal{R}} \mathcal{A} \to \widetilde{H}^1_f(G_{K,S}, Y_{[\mathcal{P}]}) \stackrel{i_{\varpi}}{\to} \widetilde{H}^2_f(G_{K,S}, X)[\varpi] \to 0,$$

where $\star[\varpi]$ denotes the ϖ-torsion submodule of $\star$. We can then define an 'height pairing':

$$\begin{aligned} \widetilde{h}'_{\mathcal{P},1,1} : \widetilde{H}^1_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes_{\mathcal{A}} \widetilde{H}^1_f(G_{K,S}, Y_{[\mathcal{P}]})^{\iota} &\stackrel{i_{\varpi}\otimes i^{\iota}_{\varpi}}{\to} \\ \to \widetilde{H}^2_f(G_{K,S}, X)[\varpi] \otimes_{\mathcal{A}} \widetilde{H}^2_f(G_{K,S}, X)^{\iota}[\varpi] &\stackrel{\widetilde{c}_{\pi,2,2}}{\longrightarrow} (\mathscr{R}/\mathcal{R})\,[\varpi] \stackrel{\xi_{\varpi}}{\to} \mathcal{P}/\mathcal{P}^2. \end{aligned}$$

The non-canonical isomorphism ξ_{ϖ} is defined sending $\frac{a}{\varpi} \bmod \mathcal{R} \in \mathscr{R}/\mathcal{R}$ to $(a \cdot \varpi) \bmod \mathcal{P}^2 \in \mathcal{P}/\mathcal{P}^2$. It is easily seen that $\widetilde{h}'_{\mathcal{P},1,1}$ depends only on $\mathcal{P}$ and not on the choice of the generator ϖ. It comes as no surprise that we have defined nothing new.

Proposition 0.17. $\widetilde{h}'_{\mathcal{P},1,1} = \widetilde{h}_{\mathcal{P},1,1}$.

Proof. Write $\mathscr{P}(\mathcal{A}) := (\mathcal{R} \stackrel{\varpi}{\to} \mathcal{R})$, concentrated in degrees -1 and 0. The projection $p_{\mathcal{P}}$ (resp., the map $p'_{\mathcal{P}}$ defined by $\mathcal{R} \ni a \mapsto (a \cdot \varpi) \bmod \mathcal{P} \in \mathcal{P}/\mathcal{P}^2$) induces quasi isomorphism of complexes of $\mathcal{R}$-modules (again denoted by the same symbol) $p_{\mathcal{P}} : \mathscr{P}(\mathcal{A}) \stackrel{\text{qis}}{\to} \mathcal{A}$ and $p'_{\mathcal{P}} : \mathscr{P}(\mathcal{A}) \stackrel{\text{qis}}{\to} \mathcal{P}/\mathcal{P}^2$. Let $\widetilde{\partial_{\mathcal{P}}} : \mathscr{P}(\mathcal{A}) \to \mathscr{P}(\mathcal{A})[1]$ be the morphism of complexes defined in degree -1 by the identity. Let us consider the following diagram in $\mathcal{D}(\mathcal{R})$:

$$\begin{array}{ccccc} & & \stackrel{\rho}{\frown} & & \\ \mathscr{P}(\mathcal{A}) & \xrightarrow{p_{\mathcal{P}}} & \mathcal{A} & \xleftarrow{\widetilde{p_{\mathcal{P}}}} & \operatorname{Cone}\left(\mathcal{P}/\mathcal{P}^2 \stackrel{i}{\to} \mathcal{R}/\mathcal{P}^2\right) \\ \downarrow{\scriptstyle \widetilde{\partial_{\mathcal{P}}}} & & \downarrow{\scriptstyle -\partial_{\mathcal{P}}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathscr{P}(\mathcal{A})[1] & \xrightarrow{p'_{\mathcal{P}}[1]} & \mathcal{P}/\mathcal{P}^2[1] & = & \mathcal{P}/\mathcal{P}^2[1], \end{array}$$

where ρ is defined in degree -1 (resp., 0) by $\mathcal{R} \ni r \mapsto [\varpi \cdot r] \in \mathcal{P}/\mathcal{P}^2$ (resp., the natural projection), and $\widetilde{p_{\mathcal{P}}}$ is defined in degree 0 by the projection $p_{\mathcal{P}}$. By definition, the right-hand square is commutative. Moreover we have $\mathrm{pr} \circ \rho = p'_{\mathcal{P}}[1] \circ \widetilde{\partial_{\mathcal{P}}}$ and $\widetilde{p_{\mathcal{P}}} \circ \rho = p_{\mathcal{P}}$. It follows that the left-hand square also commutes, i.e. $-\widetilde{\partial_{\mathcal{P}}}$ is a lift of $\partial_{\mathcal{P}}$ to $\mathcal{R}$-free resolutions of $\mathcal{A}$ and $\mathcal{P}/\mathcal{P}^2[1]$. Applying $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} -$ and using Prop. 10.2, we obtain by construction a commutative diagram in $\mathcal{D}(\mathcal{R})$:

(186)
$$\begin{array}{ccc} \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{A} & \xrightarrow{\mathrm{id}\otimes\partial_{\mathcal{P}}} & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X)[1] \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{P}/\mathcal{P}^2 \\ \uparrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} \\ \widetilde{C}^{\bullet}_f(G_{K,S}, X) \otimes_{\mathcal{R}} \mathscr{P}(\mathcal{A}) & \xrightarrow{\widetilde{\beta_{\mathcal{P}}}} & \widetilde{C}^{\bullet}_f(G_{K,S}, X)[1] \otimes_{\mathcal{R}} \mathscr{P}(\mathcal{A}) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]}) & \xrightarrow{\beta_{\mathcal{P}}} & \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, Y_{[\mathcal{P}]})[1] \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2, \end{array}$$

where $\widetilde{\beta_{\mathcal{P}}}$ is defined as the composition of $-\mathrm{id} \otimes \widetilde{\partial_{\mathcal{P}}}$ with the natural isomorphism $\dagger \otimes (\ddagger[1]) \stackrel{\sim}{\to} (\dagger[1]) \otimes \ddagger$. For the vertical arrows: the first isomorphism (as the commutativity of the upper square) arises from the definition of the derived tensor product, and the second is the natural projection. Indeed the composition of the vertical arrows are precisely the isomorphisms used to prove the control Theorems: $\widetilde{\mathbf{R}\Gamma}_f(X) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{A} \stackrel{\sim}{\to} \widetilde{\mathbf{R}\Gamma}_f(Y_{[\mathcal{P}]})$ and $\widetilde{\mathbf{R}\Gamma}_f(Y) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{P}/\mathcal{P}^2 \cong \left(\widetilde{\mathbf{R}\Gamma}_f(X) \otimes^{\mathbf{L}}_{\mathcal{R}} \mathcal{A}\right) \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2 \cong \widetilde{\mathbf{R}\Gamma}_f(Y_{[\mathcal{P}]}) \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2$ (see the proof of Lemma 0.4).

Given $[x] \in \widetilde{H}^1_f(G_{K,S}, Y_{[\mathcal{P}]})$ we write $x(\mathcal{P}) \in \widetilde{C}^2_f(G_{K,S}, Y_{[\mathcal{P}]})$ for any 2-cocycle representing the image of $[x]$ under the composition

$$\widetilde{H}^1_f(G_{K,S}, Y_{[\mathcal{P}]}) \stackrel{i_\varpi}{\to} \widetilde{H}^2_f(G_{K,S}, X) \stackrel{\mathrm{mod}\ \varpi}{\to} \widetilde{H}^2_f(G_{K,S}, Y_{[\mathcal{P}]}).$$

As i_ϖ is the connecting morphism attached to the short exact sequence of complexes $\widetilde{C}^\bullet_f(X) \stackrel{\varpi}{\hookrightarrow} \widetilde{C}^\bullet_f(X) \twoheadrightarrow \widetilde{C}^\bullet_f(Y_{[\mathcal{P}]})$, writing $\widetilde{x} \in \widetilde{C}^1_f(G_{K,S}, X)$ for any lift of x under 'reduction modulo ϖ', we have $x(\mathcal{P}) = \overline{x} \bmod \varpi$, where $\overline{x} \in \widetilde{C}^2_f(G_{K,S}, X)$ is the 2-cocycle s.t. $\varpi \cdot \overline{x} = d_{\widetilde{C}^\bullet_f}\widetilde{x}$. In other words referring to diagram (186):

$$\widetilde{x} - \overline{x} \in Z^1\left(\widetilde{C}^\bullet_f(G_{K,S}, X) \otimes_{\mathcal{R}} \mathscr{P}(\mathcal{A})\right)$$

is a 1-cocycle 'lifting' x. Applying $H^1(\widetilde{\beta_{\mathcal{P}}})$ to the cohomology class represented by $\widetilde{x} - \overline{x}$ and using again the commutative diagram (186) we thus obtain:

$$H^1(\beta_{\mathcal{P}}) : \widetilde{H}^1_f(G_{K,S}, Y_{[\mathcal{P}]}) \longrightarrow \widetilde{H}^2_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes_{\mathcal{A}} \mathcal{P}/\mathcal{P}^2$$
$$[x] \mapsto [x(\mathcal{P})] \otimes [\varpi].$$

Recalling the definitions, this formula gives us: for every $[x] \in \widetilde{H}^1_f(G_{K,S}, Y_{[\mathcal{P}]})$ and $[y] \in \widetilde{H}^1_f(G_{K,S}, Y_{[\mathcal{P}]})^\iota$

$$\widetilde{h}_{\mathcal{P},1,1}\left([x] \otimes [y]\right) = \underline{\mathrm{inv}}_{S_f}(\mathcal{A})\left([x(\mathcal{P}) \cup_{\boldsymbol{\mathcal{P}},\mathbf{0}} y]\right) \cdot [\varpi] \in \mathcal{P}/\mathcal{P}^2. \tag{187}$$

Here, recalling the fixed perfect duality $\pi : X \otimes X^\iota \to \mathcal{R}(1)$ (resp., $\pi_{\mathcal{P}} := \pi \otimes_{\mathcal{R}} \mathcal{A}$) such that $X \perp_\pi X^\iota$ (resp., $Y_{[\mathcal{P}]} \perp_{\pi_{\mathcal{P}}} Y^\iota_{[\mathcal{P}]}$) we have written $\cup_{\boldsymbol{\mathcal{P}},\mathbf{0}} := \cup_{\boldsymbol{\pi_{\mathcal{P}}},\mathbf{0}} : \widetilde{C}^\bullet_f(G_{K,S}, Y_{[\mathcal{P}]}) \otimes_{\mathcal{A}} \widetilde{C}^\bullet_f(G_{K,S}, Y_{[\mathcal{P}]}) \to \mathcal{C}(K, \mathcal{A})$ for the cup-product pairing attached to $\pi_{\mathcal{P}}$ (and defined in Lemma 0.1), and $\underline{\mathrm{inv}}_{S_f}(\mathcal{A})$ is the isomorphism $H^3(\mathcal{C}(K, \mathcal{A})) \stackrel{\sim}{\to} \mathcal{A}$ defined in Section 0.4 using local class field theory.

We now compute $\widetilde{h}'_{\mathcal{P},1,1}([x] \otimes [y])$, for $[x] \otimes [y] \in \widetilde{H}^1_f(Y_{[\mathcal{P}]}) \otimes_A \widetilde{H}^1_f(Y_{[\mathcal{P}]})^\iota$. We will use the notations of Section 0.10. Let $\widetilde{x} \in \widetilde{C}^1_f(G_{K,S}, X)$ and $\overline{x} \in \widetilde{C}^2_f(G_{K,S}, X)$ be as above, and define similarly $\widetilde{y}$ and $\overline{y}$. Recall by the discussion above that $\overline{x}$ (resp., $\overline{y}$) is a 2-cocycle in $\widetilde{C}^\bullet_f(G_{K,S}, X)$ (resp., $\widetilde{C}^\bullet_f(G_{K,S}, X)^\iota$) whose cohomology class represents $i_\varpi([x])$ (resp., $i^\iota_\varpi([y])$). Then

$$\overline{x}' := (\overline{x}, \widetilde{x} \otimes \varpi^{-1}) \quad \in \left(\mathbf{R}\Gamma_!\left(\widetilde{C}^\bullet_f(G_{K,S}, X)\right)\right)^2$$
$$(\text{resp.,} \quad \overline{y}' := (\overline{y}, \widetilde{y} \otimes \varpi^{-1}) \in \left(\mathbf{R}\Gamma_!\left(\widetilde{C}^\bullet_f(G_{K,S}, X)^\iota\right)\right)^2 \quad)$$

is a 2-cocycle whose image in $H^2_!\left(\widetilde{C}^\bullet_f(G_{K,S}, X)\right)$ (resp., $H^2_!\left(\widetilde{C}^\bullet_f(G_{K,S}, X)^\iota\right)$) lifts $i_\varpi([x])$ (resp., $i^\iota_\varpi([y])$) under the projection in (164). We compute the composition (165) (for $M := \widetilde{C}^\bullet_f(X)$, $N := \widetilde{C}^\bullet_f(X)^\iota$) on the 4-cocycle $\overline{x}' \otimes \overline{y}'$, obtaing the 4-cocycle:

$$(\overline{x} \otimes \overline{y}, (\overline{x} \otimes \widetilde{y}) \otimes \varpi^{-1}) \in \mathbf{R}\Gamma_!\left(\widetilde{C}^\bullet_f(G_{K,S}, X) \otimes_{\mathcal{R}} \widetilde{C}^\bullet_f(G_{K,S}, X)^\iota\right)^4.$$

Let us now apply $\mathbf{R}\Gamma_!(\cup_{\boldsymbol{\pi},\mathbf{0}}) = \cup_{\boldsymbol{\pi},\mathbf{0}} \otimes \mathrm{id}$, obtaining the 4-cocycle:

$$(\ddagger) := \left(\overline{x} \cup_{\boldsymbol{\pi},\mathbf{0}} \overline{y}, (\overline{x} \cup_{\boldsymbol{\pi},\mathbf{0}} \widetilde{y}) \otimes \varpi^{-1}\right) \in \left(\mathbf{R}\Gamma_!\left(\mathcal{C}(K, \mathcal{R})\right)\right)^4.$$

As $H^4(\mathcal{C}(K, \mathcal{R})) = 0$, there exists $\mathcal{T} \in \mathcal{C}(K, \mathcal{R})^3$ s.t. $d_{\mathcal{C}(K,\mathcal{R})}\mathcal{T} = \overline{x} \cup_{\boldsymbol{\pi},\mathbf{0}} \overline{y}$, so that the cohomology class of $(\ddagger)$ is equal to that of the 4-cocycle

$$(\dagger) := \left(0, (\overline{x} \cup_{\boldsymbol{\pi},\mathbf{0}} \widetilde{y} - \varpi\mathcal{T}) \otimes \varpi^{-1}\right) \in \left(\mathbf{R}\Gamma_!\left(\mathcal{C}(K, \mathcal{R})\right)\right)^4.$$

We have an isomorphism $\gamma : H^4_!\left(\mathcal{C}(K, \mathcal{R})\right) \stackrel{\sim}{\to} H^1_!\left(\mathcal{R}\right) \stackrel{\sim}{\to} \mathscr{R}/\mathcal{R}$, where the first isomorphism is induced by the isomorphism in $\mathcal{D}(\mathcal{R})$: $\mathbf{R}\Gamma_!\left(\underline{\mathrm{inv}}_{S_f}(\mathcal{R})\right) : \mathbf{R}\Gamma_!\left(\mathcal{C}(K, \mathcal{R})\right) \stackrel{\sim}{\to} \mathbf{R}\Gamma_!\left(\mathcal{R}[-3]\right)$, and the second comes from (164) (with $M = \mathcal{R}[-3]$). It follows immediately by the definitions that the image of $i_\varpi([x]) \otimes i^\iota_\varpi([y])$

under $\widetilde{c}_{\pi,2,2}$ is obtained applying γ to the cohomology class of the cocycle (†). In other words we have the formula:

$$\left(\widetilde{c}_{\pi,2,2}\circ(i_{\varpi}\otimes i^{\iota}_{\varpi})\right)([x]\otimes[y])=\gamma([(\dagger)]):=\underline{\mathrm{inv}}_{S_f}(\mathcal{R})\left([\overline{x}\cup_{\boldsymbol{\pi},\mathbf{0}}\widetilde{y}-\varpi\mathcal{T}]\right)\cdot[\varpi^{-1}]\in\mathscr{R}/\mathcal{R}.$$

Since $\overline{x}$ mod $\varpi=x(\mathcal{P})$ (by the discussion above), $\widetilde{y}$ mod $\varpi=y$ (by construction), and $\pi\otimes_{\mathcal{R}}\mathcal{A}=:\pi_{\mathcal{P}}$, by the functoriality of $\underline{\mathrm{inv}}_{S_f}(-)$ and the definition of the global cup-products in Lemma 0.1, we have

$$\left(\underline{\mathrm{inv}}_{S_f}(\mathcal{R})\left([\overline{x}\cup_{\boldsymbol{\pi},\mathbf{0}}\widetilde{y}-\varpi\mathcal{T}]\right)\right)\ \mathrm{mod}\ \mathcal{P}=\underline{\mathrm{inv}}_{S_f}(\mathcal{A})\left([x(\mathcal{P})\cup_{\boldsymbol{\mathcal{P}},\mathbf{0}}y]\right),$$

so that we finally obtain: $\left(\widetilde{c}_{\pi,2,2}\circ(i_{\varpi}\otimes i^{\iota}_{\varpi})\right)([x]\otimes[y])=\underline{\mathrm{inv}}_{S_f}(\mathcal{A})\left([x(\mathcal{P})\cup_{\boldsymbol{\mathcal{P}},\mathbf{0}}y]\right)\cdot[\varpi^{-1}]$, i.e.:

$$\widetilde{h}'_{\mathcal{P},1,1}([x]\otimes[y])=\left(\underline{\mathrm{inv}}_{S_f}(\mathcal{A})\left([x(\mathcal{P})\cup_{\boldsymbol{\mathcal{P}},\mathbf{0}}y]\right)\cdot\varpi\right)\ \mathrm{mod}\ \mathcal{P}^2\in\mathcal{P}/\mathcal{P}^2.$$

Together with (187) this concludes the proof. □

0.25. More general height pairings. Though we have considered in this Appendix a representation X equipped with a perfect duality $\pi: X\otimes_{\mathcal{R}}X^{\iota}\to\mathcal{R}(1)$ (with our main applications in mind) the constructions and results of Sections 0.20-0.23 naturally generalize in the following setting.

Let $\mathcal{R}=(\mathcal{R},\mathfrak{m}_{\mathcal{R}})$ be as in Section A, and let $\mathcal{P}$ be a an ideal of $\mathcal{R}$ generated by an $\mathcal{R}$-regular sequence. Let $X=\{X;X_v^+,\ v\in S_f\}$ be as in Section 0.5. As above we write $\mathcal{A}=\mathcal{A}_{[\mathcal{P}]}=\mathcal{R}/\mathcal{P}$, $Y:=Y_{[\mathcal{P}]}=X\otimes_{\mathcal{R}}\mathcal{A}$ and $Y_v^+:=\left(Y_{[\mathcal{P}]}\right)_v^+:=X_v^+\otimes_{\mathcal{R}}\mathcal{A}$ for every $v\in S_f$. These data is all that is needed to define the Bockstein map:

$$\beta_{\mathcal{P}}:=\beta_{X,\mathcal{P}}:\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Y)\longrightarrow\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Y)[1]\otimes_{\mathcal{A}}\mathcal{P}/\mathcal{P}^2.$$

Let $Z=\{Z;Z_v^+,\ v\in S_f\}$ be a representation 'with Greenberg local condition' as in Section 0.5, but this time considering $\mathcal{A}$ as our Noetherian 'coefficient ring'. Assuming that that there exists a perfect duality $\pi: Y\otimes_{\mathcal{A}}Z\to\mathcal{A}(1)$ such that $Y\perp_{\pi}Z$ we can define 'height pairings'

$$\widetilde{h}_{X,\pi}:\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Y)\otimes^{\mathbf{L}}_{\mathcal{A}}\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Z)\longrightarrow\mathcal{P}/\mathcal{P}^2[-2];$$

$$(i+j=2)\qquad\widetilde{h}_{X,\pi,i,j}:\widetilde{H}^i_f(G_{K,S},Y)\otimes^{\mathbf{L}}_{\mathcal{A}}\widetilde{H}^j_f(G_{K,S},Z)\longrightarrow\mathcal{P}/\mathcal{P}^2$$

exactly as in Section 0.21, replacing $\cup_{\boldsymbol{\mathcal{P}}}$ with the global cup-product pairing

$$\cup_{\boldsymbol{\pi}}:\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Y)\otimes^{\mathbf{L}}_{\mathcal{A}}\widetilde{\mathbf{R}\Gamma}_f(G_{K,S},Z)\longrightarrow\mathcal{A}[-3]$$

induced by the duality π and defined in Section 0.7.

We leave to the interested reader to formulate analogues/generalizations of the results of Sections 0.20-0.23 to this more general setting.

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