

A REMARK ON COMPLETELY REDUCIBLE NEAR-RINGS

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A characterization of the completely reducible (zerosymmetric right) near-rings which are the direct sum of their socle of type 2 and of their 2-radical is given.

1. Completely reducible near-rings

Throughout this note N indicates a zerosymmetric right near-ring; terminology and notation are those of [5]. In particular a non-zero left ideal L of N will be called *minimal* if it does not contain proper left ideals of N ; *simple* if the N -group L has no proper ideals; *of type 2* if the N -group L is monogenic and has no proper N -subgroups.

It is well known that if the near-ring N is *completely reducible* - that is, it is the sum of its minimal left ideals - then every minimal left ideal of N is a simple one, so that N is actually the direct sum of (a few of) its simple left ideals.

Assume that N has at least one left ideal of type 2; then the *socle of type 2* - that is the sum $\zeta_2(N)$ of all left ideals of type 2 of N - is a non-zero left ideal of N and therefore is a direct summand of N (see [1], Lemma 1.3). Moreover the following result holds.

PROPOSITION 1. *A completely reducible near-ring N is the direct sum of its socle of type 2 and of a left ideal which is contained in the*

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annihilator of the N -group $\zeta_2(N)$.

Proof. The near-ring N can be written as the direct sum of $\zeta_2(N)$ and of a left ideal which is the direct sum of simple left ideals L_h ($h \in H$) of N which are not of type 2 : $N = \zeta_2(N) \oplus \left\{ \bigoplus_{h \in H} L_h \right\}$.

Now, for every $h \in H$, L_h is contained in the annihilator $(0 : \zeta_2(N))$ of $\zeta_2(N)$.

Indeed let L be a simple left ideal and x an element of $\zeta_2(N)$ such that the N -subgroup Lx is different from zero; then Lx is simple (being N -isomorphic to L). On the other side, Lx is contained in $\zeta_2(N)$, since $\zeta_2(N)$ is a left ideal. Thus Lx is N -isomorphic to a left ideal of type 2 (see [1], Proposition 2.2) and L too must be of type 2. \square

Looking next to the annihilator $(0 : \zeta_2(N))$, it can be remarked that it is the intersection of the annihilators of the left ideals L of type 2 of N ; in fact the inclusion $(0 : \zeta_2(N)) \subseteq \bigcap (0 : L)$ holds trivially, and the other is a consequence of $\zeta_2(N)$ being the direct sum of (a few of) the left ideals of type 2 and of the distributive property of direct sums.

Hence $(0 : \zeta_2(N))$ contains the intersection of the annihilators of all the N -groups of type 2, that is the 2-radical $J_2(N)$ of N .

These assertions (which are true for every zerosymmetric near-ring) can be strengthened when N is completely reducible, because every group of type 2 over such a near-ring is N -isomorphic to a left ideal of type 2 of N (see [5], Corollary 3.11). Therefore

PROPOSITION 2. *If N is a completely reducible near-ring, the annihilator of $\zeta_2(N)$ is the 2-radical of N .*

Then Proposition 1 can be rewritten as follows

PROPOSITION 3. *A completely reducible near-ring N is the direct sum of its socle of type 2 and of a left ideal which is contained in the*

2-radical $J_2(N)$ of N .

2. ζ_2 -decomposable near-rings

The last result leads to study the intersection between $J_2(N)$ and $\zeta_2(N)$, in order to establish conditions under which N is the direct sum of $\zeta_2(N)$ and $J_2(N)$. It seems useful to examine the question in a slightly more general context; call ζ_2 -decomposable a near-ring N which has its socle of type 2 as a direct summand, and recall that $J_2(N)$ is contained in $(0 : \zeta_2(N))$.

Generally - as shown by the Counterexample 4 of Section 3 - if N is ζ_2 -decomposable but not completely reducible, $J_2(N)$ does not coincide with $(0 : \zeta_2(N))$; however they coincide restrictedly to $\zeta_2(N)$. In order to see this, let us prove

LEMMA 4. *Let N be a ζ_2 -decomposable near-ring. For each left ideal C of N such that $J_2(N) \subseteq C \subseteq (0 : \zeta_2(N))$, the intersection $\zeta_2(N) \cap C$ is the direct sum of all the left ideals of N which are nilpotent and of type 2.*

Proof. Let $\zeta_2(N) \cap C = M$. All the nilpotent left ideals of N are contained in $J_2(N)$ and therefore in C ; among them, those of type 2 are necessarily contained in $\zeta_2(N)$; hence all the left ideals of N which are nilpotent and of type 2 are contained in M . So it will be enough to show that there are nilpotent left ideals of type 2 of N whose sum is the whole M .

Now, the N -group M is the sum of its simple ideals L_i since it is an ideal of the completely reducible N -group $\zeta_2(N)$ (see [5], Proposition 2.48). For each L_i it results $L_i^2 = (0)$ because

$$L_i \subseteq M \subseteq C \subseteq (0 : \zeta_2(N)) \subseteq (0 : L_i).$$

Furthermore, each L_i is a left ideal of N (since M is a direct summand of the N -group $\zeta_2(N)$ and therefore of N) and is of type 2, as it is a simple left ideal of N contained in $\zeta_2(N)$ (see [1], Proposition 2.2). \square

Lemma 4 can be read as follows: *in a ζ_2 -decomposable near-ring N the left ideal $M = \zeta_2(N) \cap (0 : \zeta_2(N))$ coincides with $\zeta_2(N) \cap J_2(N)$. Besides, M is nilpotent of class 2 and is zero if and only if N has no nilpotent left ideal of type 2.*

From this statement the announced characterization follows.

PROPOSITION 5. *A completely reducible near-ring N is the direct sum of $\zeta_2(N)$ and $J_2(N)$ if and only if N has no nilpotent left ideal of type 2.*

3. Examples

There are several classes of near-rings satisfying the condition expressed by Proposition 5.

EXAMPLE 1. Let N be a completely reducible near-ring with right identity; then it is easily seen that N is the direct sum of a finite number of simple left ideals, so that the intersection of all the maximal left ideals of N is zero (see [5], Theorem 2.50). Such an intersection coincides with $J_{\frac{1}{2}}(N)$, for in a near-ring with right identity the maximal left ideals coincide with the O -modular ones. On the other side $J_{\frac{1}{2}}(N)$ contains every left nil ideal of N (see [5], Theorem 5.37) and therefore N has no nilpotent left ideal of type 2. This proves

PROPOSITION 6. *A completely reducible near-ring N with right identity is the direct sum of $\zeta_2(N)$ and $J_2(N)$.*

EXAMPLE 2. Let N be a distributive near-ring completely reducible as a left N -group. Then $J_2(N) = J_1(N) = J_0(N)$ is a nilpotent ideal which is the direct sum of the annihilator $A(N)$ of N and of the sum of the nilpotent left ideals of type 2 of N (see [2], Theorem 6.1). Therefore

PROPOSITION 7. *A completely reducible distributive near-ring N is the direct sum of $\zeta_2(N)$ and $J_2(N)$ if (and only if) $J_2(N)$ is the annihilator of N .*

Observe that if the completely reducible near-ring N is distributive, then $J_2(N)$ is nilpotent. This is also true of the 2-radical of a near-ring sum of its left ideals which are N -simple as N -groups (see [4], Theorem 2), but if N is a general completely reducible near-ring, $J_2(N)$ may be non nilpotent. This can be seen in the near-rings of Example 1 or in the following

EXAMPLE 3. Let N be the (external) direct sum of a field F and of the near-ring L built over the symmetric group S_3 denoted by (1) in [5], p. 410.

Since the near-ring L has no proper left ideal, the only left ideals of $N = F \oplus L$ are F (which is the socle of type 2 of N) and L , which is the 2-radical of N and non nilpotent because it contains idempotent elements.

Finally we show that there exist (ζ_2 -decomposable) near-rings N such that $J_2(N)$ is properly contained in $(0 : \zeta_2(N))$.

EXAMPLE 4. Consider the dihedral group of order 12, $D_{12} = \{a, b \mid 6a = 2b = 0, a+b = b+5a\}$ and let N be the distributive and commutative near-ring built over D_{12} denoted by N_4 in [4]. The only ideal of type 2 of N is $A = \{0, 3a\}$, so that $\zeta_2(N) = A$ and $(0 : \zeta_2(N)) = \{0, 2a, 4a, b, b+2a, b+4a\}$. Moreover $N = \zeta_2(N) \oplus (0 : \zeta_2(N))$; hence N is ζ_2 -decomposable.

However $J_2(N) = \{0, 2a, 4a\}$.

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