A. Lanteri, M. Palleschi and A. J. Sommese
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DOUBLE COVERS OF $\mathbf{P}^n$ AS VERY AMPLE DIVISORS

ANTONIO LANTERI, MARINO PALLESCHI
AND ANDREW J. SOMMESE

Introduction

The classical subject of surfaces containing a hyperelliptic curve (here a double cover of $\mathbf{P}^1$) among their hyperplane sections was settled some years ago by the third author and Van de Ven [SV] (see also [Se], [Io]). This paper is devoted to answering the following general question arising very naturally from that problem.

Let A be a smooth complex projective n-fold $n \geq 2$ and let $\pi : A \to \mathbf{P}^n$ be a double cover. Classify all the smooth $(n+1)$-folds X containing A as a very ample divisor.

Of course two obvious pairs (X, L), where $L = \mathcal{O}_X(A)$, satisfying the above conditions are $(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(2))$ and $(\mathbf{Q}^{n+1}, \mathcal{O}_{\mathbf{Q}^{n+1}}(1))$, where $\mathbf{Q}^n \subset \mathbf{P}^{n+1}$ denotes a smooth quadric hypersurface. The first result we prove is that these two pairs are the only ones occuring for $n \geq 3$ (Theorem (1.5)). This fact relies on topological restrictions imposed to X by A, combined with arguments and results on hyperplane sections.

Surprisingly, the situation becomes far richer for $n = 2$. In this case $K_A = \pi^*\mathcal{O}_{\mathbf{P}^2}(a)$ with $a \geq -2$. For $a = 0$, X is a Fano 3-fold of principal series. Fano threefolds were classified by Iskovskih [I] in case $b_2(X) = 1$ and by Mori and Mukai [MM1], [MM2] when $b_2(X) \geq 2$. For $a \neq 0$ the set-up is more interesting since in this case $\pi^*\mathcal{O}_{\mathbf{P}^2}(1)$ lifts to a line bundle $\mathcal{H} \in \mathrm{Pic}(X)$, whose nefness properties allow us in many instances to apply the results from adjunction theory [S2], [S3], [BS]. For $a = -2$ we get the pairs $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))$, $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$ above and scrolls over $\mathbf{P}^1$. For $a > 0$ we prove that (X, L) is a conic bundle over a smooth surface (Theorem (2.2)), the conic bundle map being given by $\mathcal{H}$.

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The heart of the paper is the study of case $a = -1$, i.e. when A is a Del Pezzo double plane. This is a special case of particular interest in the problem of classifying 3-folds having a Del Pezzo surface among their hyperplane sections. This study will be carried out in a future paper. For $a = -1$ we prove that (X, L) is one of the following pairs (Theorem (3.2))

1) a quadric fibration over $\mathbf{P}^1$;
2) a scroll over a smooth surface;
3) the blow-up at one point of its reduction $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$;
4) the blow-up at two points of its reduction $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$, the two points not lying on a line of $\mathbf{Q}^3$;
5) $(\mathbf{P}^1 \times \mathbf{P}^2, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^2}(2,2))$; or
6) the blow-up at one point of its reduction $(\mathbf{P}(\mathcal{E}), 2\xi - p^*\mathcal{O}_{\mathbf{P}^1}(1))$, where $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2)$, ξ stands for the tautological bundle of $\mathcal{E}$ and $p : \mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$ is the bundle projection.

Pairs in 5), 6) come from Veronese bundles, a very special class of polarized threefolds (X', L') studied in [BS], such that $\phi : X' \to C$ is a $\mathbf{P}^2$-bundle over a smooth curve C and $2K_{X'} + 3L' = \phi^*H$, for an ample line bundle H on C. In our case $C = \mathbf{P}^1$ and a detailed study of the vector bundles on $\mathbf{P}^1$ giving rise to X' leads to the above cases.

We also provide more details about cases 1) and 2) above. Quadric fibrations as in 1), which turn out to be rational, are described as divisors inside $\mathbf{P}^3$-bundles over $\mathbf{P}^1$ and we get a maximal list of 4 possible cases (Theorem (4.4)). In particular for the sectional genus of (X, L) we have $g(X, L) \leq 5$. The proof uses a combinatorial argument to translate the smoothness of the general fibre of X into a bound for $g(X, L)$.

As to scrolls over a surface S as in 2) we prove that they admit a second structure of conic bundles over $\mathbf{P}^2$, given by the map associated with $|\mathcal{H}|$. We describe X as a divisor inside the product $S \times \mathbf{P}^2$, having proved that the conic bundle map has no divisorial fibres and that S can only be either $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, or the Segre-Hirzebruch surface $\mathbf{F}_1$. This further description leads to 4 effective cases (Theorem (5.7)).

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0. Background material

We work over the complex number field. A projective k-fold is an irreducible smooth projective scheme of dimension k. Vector bundles are holomorphic vector bundles. We use standard notation from algebraic geometry. We also adopt some current abuses. Everywhere we do not distinguish between line bundles and invertible sheaves. We freely shift from the multiplicative to the additive notation for line bundles; multiplicative notation with "." omitted is reserved for the intersection product in the Chow rings.

Let V be a projective k-fold and let $\mathcal{L}$ be a line bundle on V. We let $\mathcal{L}^r = c_1(\mathcal{L})^r$. $\mathcal{L}_W$ will denote the restriction of $\mathcal{L}$ to a subvariety W of V. K_V will stand for the canonical bundle of V.

If p_X, p_Y are the projections of a product $X \times Y$ onto the factors, we set $\mathcal{O}_{X\times Y}(m, n) = p_X{}^*\mathcal{O}_X(m) + p_Y{}^*\mathcal{O}_Y(n)$.

A line bundle $\mathcal{L}$ on V is said to be numerically effective (nef, for short) if $\mathcal{L}C \geq 0$ for all curves C on V. In addition $\mathcal{L}$ is said to be big if $\mathcal{L}^k > 0$. We say that $\mathcal{L}$ is spanned if it is spanned by $\Gamma(V, \mathcal{L})$.

For an ample line bundle $\mathcal{L}$ on V, the sectional genus $g(V, \mathcal{L})$ of $(V, \mathcal{L})$ is defined by

$$2g(V, \mathcal{L}) - 2 = (K_V + (k-1)\mathcal{L})\mathcal{L}^{k-1}.$$

If $\mathcal{L}$ is also spanned, then $g(V, \mathcal{L})$ is simply the geometric genus of the smooth curve obtained by intersecting $k-1$ general elements of the complete linear system $|\mathcal{L}|$. We also set $d(V, \mathcal{L}) = \mathcal{L}^k$.

(0.1) Special Varieties.

We denote by $\mathbf{Q}^k$ a smooth quadric hypersurface of $\mathbf{P}^{k+1}$. Let V be a projective k-fold. V is said to be Fano (Del Pezzo for $k=2$) if $-K_V$ is ample. If $\mathcal{L}$ is an ample line bundle on V, we say that $(V, \mathcal{L})$ is a Del Pezzo (respectively Mukai) pair if $-K_V = (k-1)\mathcal{L}$ (respectively $-K_V = (k-2)\mathcal{L}$). Let $\mathcal{L}$ be an ample line bundle on V. We say that $(V, \mathcal{L})$ is a scroll (respectively a quadric bundle, respectively a Del Pezzo fibration) over a normal variety W of dimension h, if there exists a surjective morphism with connected fibres $p : V \to W$ and an ample line bundle H on W such that $K_V + (k-h+1)\mathcal{L} = p^*H$ (respectively $K_V + (k-h)\mathcal{L} = p^*H$, respectively $K_V + (k-h-1)\mathcal{L} = p^*H$). In particular, if

$(V, \mathscr{L})$ is a scroll over either a curve or a surface W with $k-h>0$, then W is smooth and V is a $\mathbf{P}^{k-h}$-bundle over W and $\mathscr{L}_f = \mathscr{O}_{\mathbf{P}^{k-h}}(1)$ for every fibre f of p [S2, (3.3)].

In some situations we will use the following result.

(0.2) LEMMA *Let $\mathscr{L}$ be an ample and spanned line bundle on a projective 3-fold V and assume that $(V, \mathscr{L})$ is a scroll over a smooth surface S. Let $p: V \to S$ be the scroll projection. Then any smooth element $Y \in |\mathscr{L}|$ contains some fibres of p, which are (-1)-lines of $(Y, \mathscr{L}_Y)$.*

Proof. Consider the rank-2 vector bundle $\mathscr{E} = p_*\mathscr{L}$. Then $V = \mathbf{P}(\mathscr{E})$, the tautological bundle being $\mathscr{L}$, which is ample. Hence $\mathscr{E}$ is ample and then $c_2(\mathscr{E}) > 0$. Since Y is defined by a general section $s \in \Gamma(S, \mathscr{E})$, it contains the fibres of p corresponding to the zero set of s, which consists of $c_2(\mathscr{E})$ points. Now let f be a fibre of p contained in Y; as $p_{|Y}: Y \to S$ contracts f to a smooth point of S, we have that f is a (-1)-curve inside Y. Finally, since $(K_X + 2\mathscr{L})f = 0$, $(X, \mathscr{L})$ being a scroll, we get $\mathscr{L}_Y f = 1$ by adjunction. □

(0.3) Reductions [S2, (0.5)].

Let $\mathscr{L}$ be an ample and spanned line bundle on a projective k-fold V. We say that a pair $(V', \mathscr{L}')$, consisting of a projective k-fold V' and an ample line bundle $\mathscr{L}'$, is a reduction of $(V, \mathscr{L})$ if

(0.3.1) there exists a morphism $\rho: V \to V'$ expressing V as V' blown-up at a finite set B,

(0.3.2) $\mathscr{L} = \rho^*\mathscr{L}' - [\rho^{-1}(B)]$ (equivalently $K_V + (k-1)\mathscr{L} = \rho^*(K_{V'} + (k-1)\mathscr{L}')$).

Recall that if $K_V + (k-1)\mathscr{L}$ is nef and big, then there exists a reduction $(V', \mathscr{L}')$ of $(V, \mathscr{L})$ and $K_{V'} + (k-1)\mathscr{L}'$ is ample [S2, (4.5)]. Note that in this case such a reduction is unique up to isomorphism and that the positive dimensional fibres of ρ are precisely the linear $\mathbf{P}^{k-1} \subset V$ with normal bundle $\mathscr{O}_{\mathbf{P}^{k-1}}(-1)$. Furthermore ρ induces a bijection between the smooth elements of $|\mathscr{L}|$ and the smooth divisors of $|\mathscr{L}'|$, passing through B.

In particular, in the special case of threefolds, we need to recall the following fact (e.g. see [SV, (0.3.3)]).

(0.3.3) Let $(V', \mathscr{L}')$ be the reduction of $(V, \mathscr{L})$, let $\rho: V \to V'$ be the reduction morphism. Let S be any smooth element of $|\mathscr{L}|$ and let $S' = \rho(S)$. Then

$(S', \mathscr{L}'_{S'})$ is the reduction of $(S, \mathscr{L}_S)$. In particular, if ρ contracts t (-1)-planes of $(V, \mathscr{L})$, then

$$K_{S'}^2 = K_S^2 + t \geq K_S^2.$$

For all the results of adjunction theory we will need for pairs $(V, \mathscr{L})$ with $\mathscr{L}$ very ample, we refer to [SV], [S3] and especially [BS].

Now we prove some very ampleness results which we will need in Section 3.

(0.4) THEOREM. i) *Let A and B be two disjoint linear subspaces of $\mathbf{P}^n$. Let $\pi : Y \to \mathbf{P}^n$ be the blow-up of $A \cup B$ and let $L = \pi^* \mathcal{O}_{\mathbf{P}^n}(\lambda) - \pi^{-1}(A) - \pi^{-1}(B)$. Then*

i) *L is very ample, if $\lambda \geq 3$.*

ii) *L is very ample outside the proper transform of $\langle A, B \rangle$, the linear span of A and B, if $\lambda = 2$.*

Proof. First let $s = n - 1 - \dim A$, and consider the morphisms

$$\begin{array}{ccc} & \mathbf{P}_A^n & \\ {\scriptstyle p_A} \swarrow & & \searrow {\scriptstyle \sigma_A} \\ \mathbf{P}^s & & \mathbf{P}^n \end{array},$$

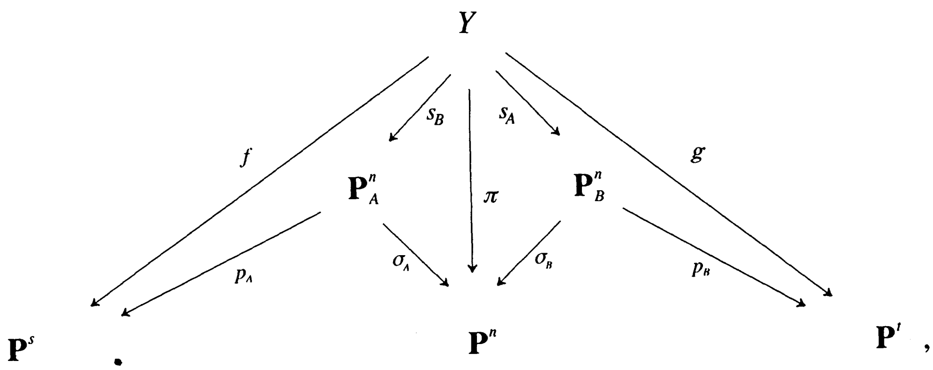
where $\sigma_A : \mathbf{P}^n{}_A \to \mathbf{P}^n$ is the blowing-up of $\mathbf{P}^n$ along A and $p_A : \mathbf{P}^n{}_A \to \mathbf{P}^s$ is the $\mathbf{P}^{\dim A + 1}$-bundle morphism associated to the projection from A. Note that

(0.4.1) $$\sigma_A^* \mathcal{O}_{\mathbf{P}^n}(1) - \sigma_{A^{-1}}(A) = p_A^* \mathcal{O}_{\mathbf{P}^s}(1).$$

Note also that

(0.4.2) the map $(p_A, \sigma_A) : \mathbf{P}^n{}_A \to \mathbf{P}^s \times \mathbf{P}^n$ is an embedding.

Now let $t = n - 1 - \dim B$, define $\mathbf{P}^n{}_B$, σ_B, p_B similarly, and consider the following commutative diagram



where $s_B : Y \to \mathbf{P}^n{}_A$ ($s_A : Y \to \mathbf{P}^n{}_B$ respectively) stands for the blowing-up at $\sigma_A^{-1}(B)$ ($\sigma_B^{-1}(A)$ respectively). Since

$$L = \pi^*\mathcal{O}_{\mathbf{P}^n}(\lambda - 2) + (\pi^*\mathcal{O}_{\mathbf{P}^n}(1) - \pi^{-1}(A)) + (\pi^*\mathcal{O}_{\mathbf{P}^n}(1) - \pi^{-1}(B)),$$

we get from (0.4.1) and the above diagram

$$L = \pi^*\mathcal{O}_{\mathbf{P}^n}(\lambda - 2) + f^*\mathcal{O}_{\mathbf{P}^s}(1) + g^*\mathcal{O}_{\mathbf{P}^t}(1).$$

So to prove i) it is enough to show that the map

$$(f, \pi, g) : Y \to \mathbf{P}^s \times \mathbf{P}^n \times \mathbf{P}^t$$

is an embedding. To see this note that the map

$$(s_A, s_B) : Y \to \mathbf{P}^n{}_A \times \mathbf{P}^n{}_B$$

is an embedding since A and B are disjoint. Then recalling (0.4.2) we see that the map

$$(f, \pi, \pi, g) : Y \to \mathbf{P}^s \times \mathbf{P}^n \times \mathbf{P}^n \times \mathbf{P}^t$$

is an embedding. Let $\Delta : \mathbf{P}^n \to \mathbf{P}^n \times \mathbf{P}^n$ be the diagonal map. Then looking at the commutative diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle (f,\pi,g)}\swarrow & & \searrow{\scriptstyle (f,\pi,\pi,g)} \\ \mathbf{P}^s \times \mathbf{P}^n \times \mathbf{P}^t & \xrightarrow[(\mathrm{id}_{\mathbf{P}^s},\, \Delta,\, \mathrm{id}_{\mathbf{P}^t})]{} & \mathbf{P}^s \times \mathbf{P}^n \times \mathbf{P}^n \times \mathbf{P}^t \end{array}$$

we conclude that (f, π, g) is an embedding. To prove ii) simply note that the map

$$(f, g) : Y \to \mathbf{P}^s \times \mathbf{P}^t$$

is an embedding outside $\pi^{-1}(\langle A, B \rangle)$. In fact the rational map $\mathbf{P}^n \dashrightarrow \mathbf{P}^s \times \mathbf{P}^t$ given by the projections from A, B gives an embedding when restricted to $\mathbf{P}^n \setminus \langle A, B \rangle$. □

(0.5) *Remarks.* 1) The same argument proving (0.4, i) also shows that if $\pi : Y \to \mathbf{P}^n$ is the blowing-up along a single linear subspace A, then the line bundle $L = \pi^* \mathcal{O}_{\mathbf{P}^n}(\lambda) - \pi^{-1}(A)$ is very ample if $\lambda \geq 2$.

2) More generally, the same argument actually shows that given r mutually disjoint linear subspaces, the statement of Theorem (0.4), with $\lambda \geq r + 1$, is true.

Theorem (0.4) also gives the following corollary, which we will use in Section 3.

(0.6) Corollary. *Let $\mathbf{Q}^n \subset \mathbf{P}^{n+1}$ be a smooth hyperquadric and let $x, y \in \mathbf{Q}^n$ be two points not lying on a line contained in $\mathbf{Q}^n$. Let $\rho : Z \to \mathbf{Q}^n$ be the blowing up at x and y and let $E_x = \rho^{-1}(x)$, $E_y = \rho^{-1}(y)$. Then the line bundle $\rho^* \mathcal{O}_{\mathbf{Q}^n}(2) - E_x - E_y$ is very ample on Z.*

Proof. Let $\pi : Y \to \mathbf{P}^{n+1}$ be the blowing up of $\mathbf{P}^{n+1}$ at x and y, let $\mathcal{E}_x = \pi^{-1}(x)$, $\mathcal{E}_y = \pi^{-1}(y)$ and consider the commutative diagram

$$\begin{array}{ccc} Z & \subset & Y \\ \rho \downarrow & & \downarrow \pi \\ \mathbf{Q}^n & \subset & \mathbf{P}^{n+1}. \end{array}$$

Since the line through x and y is transverse to $\mathbf{Q}^n$, its proper transform via π does not intersect Z, hence $\rho^* \mathcal{O}_{\mathbf{Q}^n}(2) - E_x - E_y$, which is the restriction to Z of

$\pi^*\mathcal{O}_{\mathbf{P}^n}(2) - \mathscr{E}_x - \mathscr{E}_y$, is very ample by (0.4, ii). □

The following easy vanishing result will be useful in Section 5.

(0.7) LEMMA. *Let L be an ample line bundle on a smooth 3-fold X. Assume that there exists a smooth surface $S \in |L|$. If E is a nef line bundle on X such that $E^2L > 0$, then*

$$H^2(X, K_X + E) = 0.$$

Proof. Look at the exact cohomology sequence of

$$0 \to K_X + E \to K_X + E + L \to K_S + E_S \to 0.$$

Our assumptions imply that E_S is nef and big, hence $h^1(K_S + E_S) = 0$; furthermore $h^2(K_S + E + L) = 0$, since $E + L$ is ample and so we are done. □

1. High dimension results

In all this paper A will stand for a smooth complex projective n-fold. Let $\pi : A \to \mathbf{P}^n$ be a finite morphism of degree $d \geq 2$ and assume that A is contained in a smooth complex projective $(n+1)$-fold X as a very ample divisor.

In this section we shall also assume that $n > d$.

(1.1) Since $n > d$, π induces an isomorphism $H_2(A, \mathbf{Z}) \cong H_2(\mathbf{P}^n, \mathbf{Z}) = \mathbf{Z}$ and $h^1(\mathcal{O}_A) = 0$ [La, (3.2)]. As a consequence, we have $\mathrm{Pic}(A) = \mathbf{Z}$, generated by $\pi^*\mathcal{O}_{\mathbf{P}^n}(1)$.

(1.2) By the Lefschetz theorem $\pi^*\mathcal{O}_{\mathbf{P}^n}(1)$ extends uniquely to the ample generator of $\mathrm{Pic}(X) = \mathbf{Z}$, which we call $\mathscr{H}$. Then $\mathscr{H}_A = \pi^*\mathcal{O}_{\mathbf{P}^n}(1)$. Moreover, if $L = [A]$ then $L = a\mathscr{H}$, for some integer $a \geq 1$.

(1.3) LEMMA. *Assume that $\pi : A \to \mathbf{P}^n$ is associated with a proper sublinear system of $|\mathscr{H}_A|$, namely $h^0(\mathscr{H}_A) > n + 1$. Let d be prime. Then the morphism $q : A \to \mathbf{P}^{n+t}$ $(t \geq 1)$ associated with $|\mathscr{H}_A|$ is birational and its image $q(A)$ is a variety of degree d in $\mathbf{P}^{n+t}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{q} & q(A) \subset \mathbf{P}^{n+t} \\ \pi \downarrow & \swarrow \rho & \\ \mathbf{P}^n & & \end{array}$$

where ρ is the restriction to $q(A)$ of the projection of $\mathbf{P}^{n+t}$ onto $\mathbf{P}^n$ from a $\mathbf{P}^{t-1}$ not intersecting $q(A)$. Obviously $\deg \rho = \deg q(A)$ and so

$$d = \deg \pi = \deg q \deg \rho = \deg q \deg q(A).$$

Note that $\deg q(A) \neq 1$, $q(A)$ being non-degenerate. Since d is prime, it thus follows that $\deg q(A) = d$ and $\deg q = 1$. □

(1.4) Let things be as in (1.3), i.e. $h^0(\mathcal{H}_A) > n + 1$ and d prime. By slicing down $\mathbf{P}^n$ with general hyperplanes up to getting a line we obtain a ladder as follows:

$$\begin{array}{ccccc} \mathbf{P}^{1+t} & & \mathbf{P}^{2+t} & & \mathbf{P}^{n+t} \\ \uparrow & & \uparrow & & \uparrow q \\ D & \subset & S & \subset \cdots \subset & A \\ \downarrow & & \downarrow & & \downarrow \pi \\ \mathbf{P}^1 & & \mathbf{P}^2 & & \mathbf{P}^n, \end{array}$$

where D is a smooth curve and S a smooth surface. By (1.3) the morphism q_D is birational and since it is associated with a divisor of $|\mathcal{H}_D|$, which has degree d, we conclude that $q(D)$ is a curve of degree d.

We are now able to characterize (X, L) in the case $d = 2 (< n)$.

(1.5) THEOREM. *Let A be a double cover of $\mathbf{P}^n$ contained in X as a very ample divisor and let $L = [A]$. If $n > 2$, then (X, L) is either $(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(2))$, or $(\mathbf{Q}^{n+1}, \mathcal{O}_{\mathbf{Q}^{n+1}}(1))$ where $\mathbf{Q}^{n+1}$ is a smooth quadric hypersurface of $\mathbf{P}^{n+2}$.*

Proof. Assume that $h^0(\mathcal{H}_A) = n + 1 + t$, $t > 0$. We thus get a diagram as in (1.4) and a smooth curve D such that $q(D)$ is a curve of degree 2. Therefore $g(D) = 0$. As a consequence, $(S, \mathcal{H}_S)$ is either i) $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(e))$, $e = 1,2$, or ii) a rational scroll [S1, (1.5.2)] Case i) cannot occur as $d = 2$, while in case ii), going backwards over the ladder in (1.4) we conclude (e.g. see [LP]) that $(A, \mathcal{H}_A)$ is either $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1))$ or a scroll. In the latter case however we would have $H^2(A, \mathbf{Z}) = \mathbf{Z}^2$, contradicting (1.1). In the former case (X, L) is either $(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(2))$, or $(\mathbf{Q}^{n+1}, \mathcal{O}_{\mathbf{Q}^{n+1}}(1))$ as expected.

To complete the proof we assume that

$$h^0(\mathcal{H}_A) = n + 1. \tag{1.5.1}$$

From the exact sequence

$$0 \to (1-a)\mathcal{H} \to \mathcal{H} \to \mathcal{H}_A \to 0,$$

by the Kodaira vanishing theorem (and by the Lefschetz theorem and (1.1) when $a = 1$), we get

$$h^0(\mathcal{H}) = \begin{cases} h^0(\mathcal{H}_A), \text{ if } a \geq 2 \\ h^0(\mathcal{H}_A) + 1, \text{ if } a = 1. \end{cases} \tag{1.5.2}$$

Claim. *If $h^0(\mathcal{H}_A) = n + 1$, then $a \geq 2$.*

By contradiction, assume $a = 1$ and $h^0(\mathcal{H}_A) = n + 1$. Then $h^0(\mathcal{H}) = n + 2$, by (1.5.2) and so π extends to a map $P : X \to \mathbf{P}^{n+1}$, which is associated to $|\mathcal{H}|$. Since in this case $\mathcal{H} = L$, we have that P is an embedding. On the other hand $P_{|A} = \pi$ has degree 2, a contradiction. This proves the claim.

By the claim and (1.5.2) we have $h^0(\mathcal{H}) = h^0(\mathcal{H}_A) = n + 1$. Hence π extends to a meromorphic map

$$P : X \dashrightarrow \mathbf{P}^n,$$

whose indeterminacy locus is finite since A is very ample. Let C be a general fibre of the projection onto $\mathbf{P}^n$ of the graph of P. By identifying C with its image in X, C is the smooth intersection of n general elements of $|\mathcal{H}|$. Therefore

$$2 = CA = aC\mathcal{H}, \tag{1.5.3}$$

and so $a = 2$, $C\mathcal{H} = 1$. Moreover, recalling that $\mathrm{Pic}(X) = \mathbf{Z}$, by (1.2), we have $K_X = r\mathcal{H}$, for some integer r; then by adjunction we get

$$2g(C) - 2 = \deg K_C = (K_X + n\mathcal{H})C = r + n. \tag{1.5.4}$$

Furthermore, since A is very ample, (1.5.3) says that $g(C) = 0$. Therefore (1.5.4) gives $r = -n - 2 = -(\dim X + 1)$, hence $X = \mathbf{P}^{n+1}$ and $L = \mathcal{O}_{\mathbf{P}^{n+1}}(2)$. In this case it would be $(A, \mathcal{H}_A) = (\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1))$, hence $h^0(\mathcal{H}_A) = n + 2$, contradicting (1.5.1). □

2. Double covers of $\mathbf{P}^2$

In this section A will be a smooth complex projective surface, $\pi : A \to \mathbf{P}^2$ a finite morphism of degree 2, X a smooth 3-fold containing A as a very ample divisor; we set $L = [A]$. Recall that $K_A = \pi^* \mathcal{O}_{\mathbf{P}^2}(a)$, where the branch locus of π has degree $2(a+3)$. Of course

(2.0.1) $$a \geq -2.$$

If $C \in | \pi^* \mathcal{O}_{\mathbf{P}^2}(1) |$ is a general element, by the Riemann-Hurwitz formula we get that C has genus

(2.0.2) $$g(C) = a + 2.$$

(2.1) LEMMA. *Let $a \neq 0$. Then there exists a line bundle $\mathcal{H} \in \mathrm{Pic}(X)$ such that its restriction to A is $\mathcal{H}_A = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$.*

Proof. This follows from the adjunction formula $(K_X + L)_A = K_A$, the fact that $K_A = \pi^* \mathcal{O}_{\mathbf{P}^2}(a)$ $(a \neq 0)$, and the Lefschetz theorem claiming that the cokernel of the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(A)$ is torsion free. □

In view of (2.0.1) we divide our analysis according to the possible values of a. We first deal with case $a > 0$.

(2.2) THEOREM. *If $a > 0$, then (X, L) is a conic bundle over a smooth surface.*

Proof. Since $K_A = (K_X + L)_A$ is ample, it follows that

(2.2.1) $$A \text{ cannot contain } (-1)\text{-curves.}$$

CLAIM. *$K_X + L$ is nef.*

To see this start assuming, by contradiction, that $K_X + 2L$ is not nef. Then (X, L) is as in [S3, (0.2, c)]. However, pairs $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$ and scrolls over a curve are ruled out, since our A is neither $\mathbf{P}^2$ nor a $\mathbf{P}^1$-bundle, having an ample canonical bundle.

Now assume that $K_X + 2L$ is nef but not big. Then (X, L) is as in [S3, (0.3)]. First assume that (X, L) is a Del Pezzo pair, then from $K_X = -2L$ and by adjunction, A cannot have an ample canonical bundle. Secondly assume that (X, L) is a quadric fibration over a curve B; then (A, L_A) is a conic bundle over

B. This case is ruled out for the same reason as above. Finally assume that (X, L) is a scroll over a smooth surface S. In this case A would contain some (-1)-curve by (0.2), contradicting (2.2.1).

Therefore $K_X + 2L$ is nef and big. So (X, L) admits a reduction, but (X, L) coincides with its reduction since otherwise the reduction morphism would contract some (-1)-curves of A, contradicting again (2.2.1). Then, by looking at the list of the exceptions to the nefness of $K_X + L$ one easily sees that in no case K_A could be ample (see [S3, (0.4)] and [BS, sec. 1]). This proves the claim.

It follows from the claim that $\mathscr{H}$ itself is nef, hence $\mathscr{H}^3 \geq 0$. If $\mathscr{H}^3 = 0$ then $K_X + L$ is not big and then, since $\dim X = 3$, (X, L) would be either a Mukai pair, a Del Pezzo fibration over a curve or a conic bundle over a surface [S3, (0.4)]. The first two cases are ruled out since the corresponding K_A is not ample. So, (X, L) will be proved to be a conic bundle once we have shown that $\mathscr{H}$ is not big.

By contradiction assume that $\mathscr{H}^3 \geq 1$. By the Hodge index theorem [BSS, (0.15)], we have

$$(\mathscr{H}^2)(\mathscr{H}L^2) \leq (\mathscr{H}^2 L)^2.$$

By setting $\delta = \mathscr{H}L^2 = \deg L_C$ and recalling that $\mathscr{H}^2 L = (\mathscr{H}_A)^2 = 2$ we thus get $\mathscr{H}^3 \delta \leq 4$, so that the possible values for $(\mathscr{H}^3, \delta)$ are

$$\begin{array}{ll} \mathscr{H}^3 = 1 & \delta = 1,2,3,4 \\ \mathscr{H}^3 = 2 & \delta = 1,2 \\ \mathscr{H}^3 = 3,4 & \delta = 1. \end{array} \tag{2.2.2}$$

Since L_C is a very ample line bundle of degree δ on C it must be

$$g(C) \leq (\delta - 1)(\delta - 2)/2. \tag{2.2.3}$$

It is easy to check that (2.0.2), (2.2.2), (2.2.3) are compatible only if

$$\mathscr{H}^3 = 1,\ \delta = 4,\ a = 1.$$

On the other hand using again the Hodge index theorem we get

$$2(L^3) = (L^3)(\mathscr{H}^2 L) \leq (\mathscr{H}L^2)^2 = \delta^2 = 16;$$

hence $L^3 \leq 8$. Note also that L^3 is even since the genus formula for $G \in |L_A|$ gives:

$$2g(G) - 2 = (K_A + L_A)G = a(\mathscr{H}L^2) + L^3 = L^3 + \delta = L^3 + 4.$$

Furthermore $L^3 \neq 2,4$, since otherwise K_A would not be ample; hence $L^3 = 6$ or 8. In both cases the above formula shows that G cannot be a plane curve, hence $|L_A|$ cannot embed A in $\mathbf{P}^3$; therefore $h^0(L_A) \geq 5$. On the other hand the Castelnuovo inequality rules out case $L^3 = 6$, while for $L^3 = 8$ it implies $h^0(L_A) = 5$. Then A would be embedded by $|L_A|$ in $\mathbf{P}^4$, however its characters do not satisfy the numerical formula holding for these surfaces [H, p. 434].

Therefore (X, L) is a conic bundle over a surface. The smoothness of the base follows from [Be]. □

Case $a = -2$ is very easy. In fact we have

(2.3) THEOREM. *If* $a = -2$ *then* (X, L) *is either* $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2))$, $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$, *or a scroll over* $\mathbf{P}^1$.

Proof. Since $K_A = \pi^*\mathcal{O}_{\mathbf{P}^2}(-2)$, A is a Del Pezzo surface with canonical bundle in $2\mathrm{Pic}(A)$, hence A is a quadric surface. Then the assertion follows from [Ba, Th. 5]. □

As to case $a = 0$, the fact that K_A is trivial prevents us from proving that the line bundle $\pi^*\mathcal{O}_{\mathbf{P}^2}(1)$ lifts to some element of $\mathrm{Pic}(X)$. So we only have the following result

(2.4) PROPOSITION. *Let* $a = 0$. *Then* X *is a Fano* 3*-fold of the principal series,* $L = -K_X$, *and* $|-K_X|$ *contains a smooth surface endowed with an irreducible curve of genus* 2.

It only remains to deal with case $a = -1$, which is equivalent to assuming that A is a Del Pezzo surface doubly covering $\mathbf{P}^2$. This will be done in the next three sections.

A class of pairs (X, L) fitting into all the above results is given by following

(2.5) PROPOSITION. *Let* E *be any rank* 2 *spanned holomorphic vector bundle over* $\mathbf{P}^2$, *let* $p : X = \mathbf{P}(E) \to \mathbf{P}^2$ *be the corresponding* $\mathbf{P}^1$*-bundle and let* $\mathcal{L} = \mathcal{O}_{\mathbf{P}(E)}(m) + p^*\mathcal{O}_{\mathbf{P}^2}(n)$, *with suitable* $m, n \in \mathbf{Z}$. *Let* A *be a smooth element of* $|\mathcal{L}|$, *and let* $\pi = p_{|A}$ *be the* $\mathbf{P}^1$*-bundle projection restricted to* A. *If* $m > 1$ *and* $n \geq 0$, *then* π *is a finite morphism of degree* m.

Proof. The finiteness of π is equivalent to A not containing any fibre of p. This is equivalent to the section of the bundle $p_*\mathscr{L}$ corresponding to A not being zero at any point. On the other hand $p_*\mathscr{L} = p_*\mathscr{O}_{\mathbf{P}(E)}(m) \otimes \mathscr{O}_{\mathbf{P}^2}(n) = E^{(m)} \otimes \mathscr{O}_{\mathbf{P}^2}(n)$, where $E^{(m)}$ denotes the m-th symmetric power of E. Since $m \geq 2$, the rank of $E^{(m)} \otimes \mathscr{O}_{\mathbf{P}^2}(n)$ is at least 3. Therefore, since $E^{(m)} \otimes \mathscr{O}_{\mathbf{P}^2}(n)$ is spanned a general section is nowhere zero. □

As a consequence of (2.5) we get a class of 3-folds containing double covers of $\mathbf{P}^2$ as very ample divisors. For instance, by taking $E = \mathscr{O}_{\mathbf{P}^2}(e) \oplus \mathscr{O}_{\mathbf{P}^2}$, with $e \geq 0$, we have that the following facts are equivalent [Bar, §3]: i) $\mathscr{L}$ is ample, ii) $\mathscr{L}$ is very ample, iii) $m > 0$ and $n > (m-1)e$. Hence, for $m = 2, n > e, A$ is a smooth double cover of $\mathbf{P}^2$ contained in X as a very ample divisor; moreover by the adjunction formula we get

$$a = n + e - 3.$$

Note that for $a \geq 0$, (X, L) is a conic bundle over $\mathbf{P}^2$, as in (2.2), (2.4); this happens for any $e \geq 0$, provided that $n \geq e + 3$. For $a = -1$ we have $e + 2 = n > e$, so that $e = 0$, $n = 2$ and then (X, L) is exactly the pair occurring in case (3.2.5) of the next section. Finally for $a = -2$ we get $e = 0$, $n = 1$ so that $(X, L) = (\mathbf{P}^2 \times \mathbf{P}^1, \mathscr{O}_{\mathbf{P}^2 \times \mathbf{P}^1}(1,2))$ is a scroll over $\mathbf{P}^1$ as in (2.3).

3. The Del Pezzo double plane

From now on we will deal with case $a = -1$. In this section we prove a general result, while the next two are devoted to special subcases. Note that for $a = -1$, $\pi : A \to \mathbf{P}^2$ is the so-called Del Pezzo double plane. In particular, A is a Del Pezzo surface since

(3.0.1) $$-K_A = \pi^*\mathscr{O}_{\mathbf{P}^2}(1)$$

is ample. Moreover

(3.0.2) $$K_A^2 = 2.$$

Before stating the main result of this section it is convenient to recall the following fact.

(3.1) *Remark.* Let S be a Del Pezzo surface. Every smooth surface S' dominated by S via a birational morphism is a Del Pezzo surface too.

This follows immediately from the Nakai-Moishezon criterion.

(3.2) THEOREM. *Let* (X, L) *be as in Section* 2 *and assume* $a = -1$. *Then either*

(3.2.1) (X, L) *is a quadric fibration over* $\mathbf{P}^1$ (*for more information see Section* 4),

(3.2.2) (X, L) *is a scroll over a surface* (*for a precise description see Section* 5),

(3.2.3) (X, L) *admits* $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$ *as a reduction, the reduction morphism* $X \to \mathbf{P}^3$ *being the blow-up at a single point,*

(3.2.4) (X, L) *admits* $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$ *as a reduction, the reduction morphism* $X \to \mathbf{Q}^3$ *being the blow-up at two points not lying on a line of* $\mathbf{Q}^3$,

(3.2.5) $(X, L) = (\mathbf{P}^2 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^1}(2,2))$, *or*

(3.2.6) (X, L) *admits* (X', L') *as a reduction, where* X' *is the blow-up* $\sigma : X' \to \mathbf{P}^3$ *along a line* ℓ, $L' = \sigma^*\mathcal{O}_{\mathbf{P}^3}(2) + F$, F *being the proper transform of a plane through* ℓ, *and the reduction morphism* $X \to X'$ *is the blow-up at a single point not lying on* $\sigma^{-1}(\ell)$.

The proof of (3.2) takes the rest of this section. The first step is following

(3.3) LEMMA. *Let* (X, L) *be as in* (3.2). *Then* $K_X + 2L$ *is nef and big unless in cases* (3.2.1), (3.2.2).

Proof. As a first thing assume that $K_X + 2L$ is not nef. Then by [SV], (X, L) is either $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$, or a scroll over a smooth curve C. In the last case A, which is a smooth element of $|L|$, is a $\mathbf{P}^1$-bundle over C and then, since A is rational, it follows that $C = \mathbf{P}^1$. Anyway in all the above cases we have $K_A^2 = 9$ or 8, which contradicts (3.0.2). Therefore $K_X + 2L$ is nef. Assume that it is not big. Then, according to adjunction theory [S3, (0.3)], either (X, L) is a quadric bundle over a smooth curve C and $C = \mathbf{P}^1$ since A, which is rational, is a conic bundle over C (case (3.2.1)), or (X, L) is as in (3.2.2), or $K_X = -2L$. But the last case cannot occur since in that case, by adjunction and (3.0.2), (X, L) would be $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(1))$, hence A would be isomorphic to a quadric surface, a contradiction. □

In view of (3.3) we can proceed assuming that $K_X + 2L$ is nef and big. Let (X', L') be the reduction of (X, L) and let $\rho : X \to X'$ be the corresponding reduction morphism.

(3.4) *Remark.* $K_{X'} + L'$ is not nef.

Proof. Let $S' = \rho(A)$. Then S' is a smooth element of $|L'|$ and by adjunction $(K_{X'} + L')_{S'} = K_{S'}$. If $K_{X'} + L'$ were nef, then so would $K_{S'}$ be. On the other hand $-K_{S'}$ has to be ample in view of (3.1), a contradiction. □

(3.5) By adjunction theory [S3, (0.4)] and [BS, (1.2)] it follows from (3.4) that (X', L') is one of the following pairs:

(3.5.1) $(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$,
(3.5.2) $(\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$,
(3.5.3) X' is a $\mathbf{P}^2$-bundle over a smooth curve C and $2K_{X'} + 3L' = \phi^*H$, where $\phi: X' \to C$ is the bundle projection and H is an ample line bundle on C.

Let $S' = \rho(A)$ as before; then according to (0.3.3), ρ is the blowing-up of X' at t points, where

$$t = K_{S'}^2 - K_A^2 = K_{S'}^2 - 2.$$

We deal with the above cases separately.

In case (3.5.1) we have $K_{S'}^2 = 3$, hence $\rho: X \to \mathbf{P}^3$ is the blowing-up at a single point x and $L = \rho^*\mathcal{O}_{\mathbf{P}^3}(3) - \rho^{-1}(x)$.

In case (3.5.2) we have $K_{S'}^2 = 4$, hence $\rho: X \to \mathbf{Q}^3$ is the blowing-up at two points x, y and $L = \rho^*\mathcal{O}_{\mathbf{Q}^3}(2) - \rho^{-1}(x) - \rho^{-1}(y)$. Note that x, y cannot lie on a line $\ell \subset \mathbf{Q}^3$; otherwise we would get $L\rho^{-1}(\ell) = 0$, contradicting the ampleness of L.

The very ampleness of L in the above cases follows from (0.5.1), (0.6). In both cases the general element of $|L|$ is a Del Pezzo surface with $K_A^2 = 2$; in fact it is either a cubic surface blown-up at a point or a complete intersection of two quadrics blown-up at two general points.

As a last thing assume that (X', L') is as in (3.5.3). First of all note that $C = \mathbf{P}^1$. Actually S', which is a rational surface, being dominated by A, inherits a conic bundle structure on C from the $\mathbf{P}^2$-bundle structure of X'.

Let $X' = \mathbf{P}(\mathcal{E})$ and let F be a fibre of $\phi: X' \to \mathbf{P}^1$. Since $L'_F = \mathcal{O}_{\mathbf{P}^2}(2)$ according to (3.5.3), we have $(K_{X'} + 2L')_F = \mathcal{O}_{\mathbf{P}^2}(1)$ and so we can assume that $\mathcal{E} = \phi_*(K_{X'} + 2L')$. For shortness let ξ be the tautological bundle of $\mathcal{E}$; then

(3.5.4) $$\xi = K_{X'} + 2L'.$$

Note that

$$2\xi = 2(K_{X'} + 2L') = 2K_{X'} + 3L' + L' = \phi^* H + L'$$

is the sum of a nef and an ample line bundle, hence ξ is ample and so is $\mathscr{E}$. Therefore

(3.5.5) $$\mathscr{E} = \oplus_{i=1,\dots,3} \mathscr{O}_{\mathbf{P}^1}(a_i), \text{ where } a_i > 0 \ (i = 1,2,3).$$

By the canonical bundle formula

(3.5.6) $$K_{X'} = -3\xi + (\alpha - 2)F, \text{ where } \alpha = c_1(\mathscr{E}) = a_1 + a_2 + a_3.$$

From (3.5.4), (3.5.6) we see that

$$L' = 2\xi + (1 - (\alpha/2))F.$$

In particular, recalling also (3.5.5), we have that

(3.5.7) $$\alpha \text{ is even and } \geq 4.$$

On the other hand, recalling the basic relation for the tautological bundle ξ

$$\xi^3 - \phi^* \alpha \xi^2 = 0,$$

adjunction formula gives

$$K_{S'}^2 = (K_{X'} + L')^2 L' = (-\xi + ((\alpha/2) - 1)F)^2 (2\xi + (1 - (\alpha/2))F) = 5 - (\alpha/2).$$

So recalling (0.3.3) and (3.0.2) we get

(3.5.8) $$t = K_{S'}^2 - K_A^2 = 3 - (\alpha/2).$$

Since $t \geq 0$, by combining (3.5.7) with (3.5.8) we get only the following possibilities:

(a) $\alpha = 6$, with $t = 0$;

(b) $\alpha = 4$, with $t = 1$.

(3.6) Proposition. *Cases* (a) *and* (b) *lead respectively to* (3.2.5) *and* (3.2.6).

Proof. In case (a) note that $L' = 2(\xi - F)$. This shows that $\xi - F$ is ample, which means that $\mathscr{E} \otimes \mathscr{O}_{\mathbf{P}^1}(-1)$ is ample. Therefore, recalling (3.5.5) we get $a_i - 1 > 0$ $(i = 1,2,3)$. Hence $\mathscr{E} = \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(2) = \mathscr{O}_{\mathbf{P}^1}(2) \otimes \mathscr{E}'$, where $\mathscr{E}'$ is the trivial bundle. This shows that $X' = \mathbf{P}^1 \times \mathbf{P}^2$. Let $\xi' = \mathscr{O}_{\mathbf{P}^1 \times \mathbf{P}^2}(0,1)$ be the tautological bundle of $\mathscr{E}'$; then $\xi = \xi' + 2F$ and so $L' = 2(\xi - F) = 2(\xi' + F) = \mathscr{O}_{\mathbf{P}^1 \times \mathbf{P}^2}(2,2)$. Finally $(X, L) = (X', L')$, since $t = 0$. Conversely, note that such an L is very ample and by adjunction

$$K_A = (K_X + L)_A = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(0, -1)_A,$$

so that (3.0.2) is fulfilled and the projection of X onto the $\mathbf{P}^2$ factor exhibits A as a Del Pezzo double plane.

As to case (b) of course we have $\mathscr{E} = \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2) = \mathcal{O}_{\mathbf{P}^1}(1) \otimes \mathscr{E}'$, where $\mathscr{E}' = \mathcal{O}_{\mathbf{P}^1}{}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. Let ξ' be the tautological bundle of $\mathscr{E}'$. Then $\xi' = \xi - F$. Since $\mathscr{E}'$ is spanned with $h^0(\mathscr{E}') = 4$, ξ' defines a morphism $\sigma: X' \to \mathbf{P}^3$. Moreover the basic relation for the tautological bundle ξ' gives $\xi'^3 = 1$, hence σ is generically 1 to 1. Now look at the surface $Y = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}{}^{\oplus 2}) \subset X'$. Since we have the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^1}(1) \to \mathscr{E}' \to \mathcal{O}_{\mathbf{P}^1}{}^{\oplus 2} \to 0,$$

with $\mathcal{O}_{\mathbf{P}^1}{}^{\oplus 2}$ a quotient of $\mathscr{E}'$, we conclude that ξ'_Y is the tautological bundle of $Y = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}{}^{\oplus 2})$, and thus that σ maps the horizontal fibres of Y to points, with $\sigma(Y)$ a line ℓ. Since ξ' is spanned, it follows that if C is an irreducible curve on X' contracted by σ to a point, then $\xi'_C = \mathcal{O}_C$. From this it follows that $C \subset Y$ and therefore that C is a horizontal fibre of Y. This shows that X' is $\mathbf{P}^3$ blown-up along a line ℓ via σ. Since $\xi' = \sigma^*\mathcal{O}_{\mathbf{P}^3}(1)$ we get

$$L' = 2\xi - F = 2\xi' + F = \sigma^*\mathcal{O}_{\mathbf{P}^3}(2) + F.$$

Moreover since $t = 1$, X is the blown-up of X' at a single point x. It remains to show that $\sigma(x) \notin \ell$. By contradiction, assume that $\sigma(x) \in \ell$. Let ℓ' be the proper transform of $\sigma^{-1}(\sigma(x))$ under the blow-up map $\rho: X \to X'$. Then

$$L\ell' = (\rho^*(\sigma^*\mathcal{O}_{\mathbf{P}^3}(2) + F) - E)\ell' = (\rho^*F - E)\ell' = 1 - 1 = 0,$$

contradicting the ampleness of L. Conversely let (X, L) be as in (3.2.6), let $\rho: X \to X'$ be the reduction morphism and let E be the corresponding exceptional divisor. Then the morphism $\beta = \rho \circ \sigma: X \to \mathbf{P}^3$ exhibits X as $\mathbf{P}^3$ blown-up along a line ℓ and a point $x \notin \ell$. Since $F = \sigma^*\mathcal{O}_{\mathbf{P}^3}(1) - \sigma^{-1}(\ell)$, we have

(3.6.1) $$L = \rho^*(\sigma^*\mathcal{O}_{\mathbf{P}^3}(2) + F) - E = \beta^*\mathcal{O}_{\mathbf{P}^3}(3) - \beta^{-1}(x) - \beta^{-1}(\ell)$$

and then L is very ample in view of (0.4, i). Let A be a general element of $|L|$. It is clear from (3.6.1) that A corresponds to a smooth cubic surface $S \subset \mathbf{P}^3$ containing ℓ and x ; hence A is isomorphic to S blown-up at x and then it is a Del Pezzo surface with $K_A^2 = 2$. □

4. Quadric fibrations over $\mathbf{P}^1$

In this section we provide more details on pairs (X, L) occurring in case (3.2.1) of the above section. Let $p : X \to \mathbf{P}^1$ be the morphism expressing X as a quadric fibration over $\mathbf{P}^1$. Then $K_X + 2L = p^*H$, for some ample line bundle H on $\mathbf{P}^1$. Note that all fibres of p are irreducible. Actually were there a fibre $\mathscr{F} = P + Q$, from

$$\mathcal{O}_P = [\mathscr{F}]_P = [P]_P + [Q]_P$$

since $[Q]_P = \mathcal{O}_P(1)$, we would get $[P]_P = \mathcal{O}_P(-1)$, so that P could be contracted, but this gives a contradiction since PQ, which is a line of Q, cannot be contracted.

Let $\mathscr{E} = p_*L$. For every fibre $\mathscr{F}$ of p we have $h^0(L_{\mathscr{F}}) = 4$, since $|L|$ embeds $\mathscr{F}$ as a quadric of $\mathbf{P}^3$. This implies that $\mathscr{E}$ is a rank-4 vector bundle on $\mathbf{P}^1$. Moreover $\mathscr{E}$ is spanned. To see this let $x \in \mathbf{P}^1$, let $\mathscr{F} = p^{-1}(x)$ and consider the obvious diagram

$$\begin{array}{ccc} \Gamma(L) & \to & \Gamma(L_{\mathscr{F}}) \\ \downarrow & & \downarrow \\ \Gamma(\mathscr{E}) & \to & \Gamma(\mathscr{E}_x) \end{array}$$

where the vertical arrows are isomorphisms. Since $|L|$ embeds $\mathscr{F}$ as a quadric of $\mathbf{P}^3$ the restriction homomorphism $\Gamma(L) \to \Gamma(L_{\mathscr{F}})$ is clearly surjective and then so is also the homomorphism $\Gamma(\mathscr{E}) \to \Gamma(\mathscr{E}_x)$.

Since $\mathscr{E}$ is a spanned rank-4 bundle on $\mathbf{P}^1$ we have

$$\mathscr{E} = \oplus_{i=0,\dots,3} \mathcal{O}_{\mathbf{P}^1}(a_i), \text{ with } a_i \geq 0.$$

We let $\delta = \sum a_i$. Consider the projective bundle $P = \mathbf{P}(\mathscr{E})$, let $p^{\sim} : P \to \mathbf{P}^1$ be the projection and let ξ be the tautological line bundle of $\mathscr{E}$ on P. Then, from the relation $\xi^4 - \xi^3 p^{\sim *} c_1(\mathscr{E}) = 0$ we get

$$\xi^4 = \delta. \tag{4.0.1}$$

Moreover since $\mathscr{E} = p_*L$ and X embeds fibrewise inside P, we have that

$$\xi_X = L \quad \text{and} \quad X \in |2\xi - bF|,$$

where F stands for a fibre of $p^{\sim}$

(4.1) LEMMA. i) *The integers* δ *and* b *are related as follows*: $2\delta = 3b + 6$ (*in particular* b *is even*).

ii) *We have* $b \geq 0$.

Proof. By the canonical bundle formula we know that

$$K_P = -4\xi + p^{-*}\mathcal{O}_{\mathbf{P}^1}(\delta - 2)$$

and then, by adjunction,

$$K_X = -2L + p^*\mathcal{O}_{\mathbf{P}^1}(\delta - b - 2). \tag{4.1.1}$$

Now recalling the relation $K_A^2 = 2$ we get by adjunction and (4.0.1)

$$2 = K_A^2 = (K_X + L)^2 L = (-\xi + (\delta - b - 2)F)^2 X\xi = (-\xi + (\delta - b - 2)F)^2(2\xi - bF)\xi = -2\delta + 3b + 8.$$

This proves i). To prove ii) note that by (4.1.1) we have

$$p^*H = K_X + 2L = p^*\mathcal{O}_{\mathbf{P}^1}(\delta - b - 2).$$

Thus the ampleness of H implies that $\delta - b - 2 = \deg H \geq 1$, hence $\delta \geq b + 3$. Then the assertion follows by combining i) with this inequality. □

We can also compute the numerical invariants of (X, L) in terms of b.

(4.2) *Remark.* We have

1) $$d = d(X, L) = 2b + 6,$$

2) $$g = g(X, L) = (b/2) + 2.$$

Proof. We have, recalling (4.0.1),

$$d = L^3 = \xi^3 X = \xi^3(2\xi - bF) = 2\xi^4 - b\xi^3 F = 2\delta - b,$$

hence (4.1; i) gives 1). Genus formula, taking into account (4.1; i) and (4.0.1), gives 2). □

(4.3) LEMMA. *We have* $\delta \geq 2b$.

Proof. Let $x \in \mathbf{P}^1$ and consider the fibre $\mathcal{E}_x = \bigoplus_{i=0,\dots,3}\mathcal{O}_{\mathbf{P}^1}(a_i)_x$. Denoting by z_i, $i = 0, \dots, 3$ a generator of the stalk $\mathcal{O}_{\mathbf{P}^1}(a_i)_x$ we can look at z_0, z_1, z_2, z_3 as a set of homogeneous coordinates in the fibre $F_x = \mathbf{P}(\mathcal{E}_x)$. So the quadric $\mathcal{F}_x$ cut out by X on F_x is represented by a second degree homogeneous equation in the z_i's.

Note that products $z_i z_j$ are elements of the second symmetric power

$$\mathscr{E}_x^{(2)} = [\oplus_{i \le j} \mathscr{O}_{\mathbf{P}^1}(a_i + a_j)]_x.$$

Since $\Gamma(2\xi) \cong \Gamma(\mathscr{E}^{(2)})$, we have

$$\Gamma(2\xi - bF) \cong \Gamma(\oplus_{i \le j} \mathscr{O}_{\mathbf{P}^1}(a_i + a_j - b)).$$

On the other hand $X \in |2\xi - bF|$, so that

($*$) every summand $z_i z_j$ appearing in the equation of $\mathscr{F}_x$ can be looked at as the restriction to x of a section of $\mathscr{O}_{\mathbf{P}^1}(a_i + a_j - b)$.

Now we use ($*$) and the smoothness of $\mathscr{F}_x$ for a general $x \in \mathbf{P}^1$ to prove the desired inequality by a combinatorial argument. Let $\phi(z_0, z_1, z_2, z_3) = 0$ be the equation of $\mathscr{F}_x$, for $x \in \mathbf{P}^1$ a general point.

a) If ϕ contains terms $z_i z_j$ and $z_h z_k$ with i,j,h,k all different (a complete set of crossed terms, for short), then the assertion is true.

Actually, in view of ($*$) we have both $a_i + a_j - b \ge 0$ and $a_h + a_k - b \ge 0$ and summing them up we get $\delta - 2b \ge 0$.

b) If ϕ contains three squares, then the assertion is true.

Assume that ϕ contains the squares z_0^2, z_1^2, z_2^2; then $2a_j - b \ge 0$ for $j = 0,1,2$ by ($*$). On the other hand, since $\mathscr{F}_x$ is smooth ϕ must also contain a term $z_3 z_i$. If $i = 3$ then ($*$) implies $2a_3 - b \ge 0$ and summing up all the inequalities we get $\delta - 2b \ge 0$. Assume that $i \ne 3$, e.g. let $i = 0$. Then ($*$) gives $a_0 + a_3 - b \ge 0$. On the other hand the above inequalities for $j = 1,2$ give $a_1 + a_2 - b \ge 0$ and summing up we get $\delta - 2b \ge 0$.

Note that the latter argument also proves

c) If ϕ contains two squares and a crossed term involving the two remaining coordinates, then the assertion is true.

d) If ϕ contains two squares, then the assertion is true.

Actually assume that ϕ contains the squares z_0^2 and z_1^2. Since $\mathscr{F}_x$ is smooth ϕ has to contain some crossed term involving z_2 and z_3. In view of c) we can assume that there are two distinct such terms involving z_2 and z_3 respectively. E.g. assume that ϕ contains $z_2 z_0$; then, in view of a), b) and c) we can assume that ϕ contains just 4 terms, the fourth being the crossed term $z_3 z_0$. But then, taking derivatives of

ϕ, we immediately see that $\mathscr{F}_x$ would be singular, a contradiction.

e) If ϕ contains just one square, then the assertion is true.

Actually let ϕ contain z_0^2, then ϕ must contain crossed terms involving z_1, z_2, z_3, no more than 3 in view of the above. Assume that there are three such distinct crossed terms; if no one of them contains z_0, by reasoning as above one gets $\delta - 2b \geq 0$. So we can assume that one of them involves z_0, say z_1z_0; then an easy check shows that the second one can only be either z_2z_0 or z_2z_1 and the third one z_3z_0 or z_3z_1 accordingly. In both cases, taking derivatives of ϕ with respect to z_2 and z_3, one sees that $\mathscr{F}_x$ would be singular, contradiction. So there can be just two crossed terms involving z_1, z_2 and z_3. Since $\mathscr{F}_x$ is irreducible, at least one of them cannot contain z_0. Let z_1z_2 be this one; then the remaining one must be either z_3z_1, z_3z_2, z_0z_1, or z_0z_2 but in all cases, taking derivatives of ϕ one would conclude that $\mathscr{F}_x$ is singular, contradiction.

f) final step.

In view of a) and e) we can assume that ϕ contains neither squares, nor a complete set of crossed terms. So let z_0z_1 be a term of ϕ. Since ϕ has to involve all coordinates and the term z_2z_3 cannot occur by a), there are two distinct crossed terms containing z_2 and z_3. In view of the symmetry between z_0 and z_1 we can thus assume that the term involving z_2 is z_2z_0 and then, due to the above, the third one can only be z_3z_0. But in this case $\mathscr{F}_x$ would be even reducible, a contradiction. This concludes the proof. □

As a consequence of (4.1) and (4.3) we have that b is even and $0 \leq b \leq 6$, So, recalling also (4.2) we get the following

(4.4) THEOREM. *Let (X, L) be a quadric fibration as in* (3.2.1). *Then X embeds fibrewise in a projective bundle $\mathbf{P}(\mathscr{E})$, where $\mathscr{E} = \bigoplus_{i=0,\dots,3} \mathscr{O}_{\mathbf{P}^1}(a_i)$, with $a_i \geq 0$, and*

$$X \in |2\xi - bF|, \quad L = \xi_X$$

where ξ is the tautological bundle of $\mathscr{E}$ and F is a fibre of $\mathbf{P}(\mathscr{E})$. Moreover the possible values of the invariants b, $\delta = \sum a_i$, d, g are those listed in the following table

b	δ	d	g
0	3	6	2
2	6	10	3
4	9	14	4
6	12	18	5.

5. The further structure of the scrolls in (3.2.2)

Let (X, L) be a scroll as in case (3.2.2) of Section 3. Then $K_X + 2L$ is nef but not big, this giving rise to a morphism $p_X : X \to S$ exhibiting X as scroll over a smooth surface S; furthermore

(5.0.1) $$K_X + 2L = p_X{}^*H \text{ for an ample } H \in \mathrm{Pic}(S).$$

Note that

(5.0.2) $$H \text{ is spanned,}$$

since so is $K_X + 2L$ [SV, (0.1)]. As before $K_X + L = -\mathscr{H} \in \mathrm{Pic}(X)$ and recall that the double cover morphism $\pi : A \to \mathbf{P}^2$ is associated with a (not necessarily proper) base point free sublinear system of $|\mathscr{H}_A| = |\pi^*\mathcal{O}_{\mathbf{P}^2}(1)|$. A fortiori

(5.0.3) $$\mathscr{H}_A \text{ is spanned by its global sections.}$$

(5.1) Lemma. *$\mathscr{H}$ is a spanned line bundle and $\mathscr{H}^3 = 0$.*

Proof. The proof is divided into four parts.

Step.1. $\mathscr{H}$ is nef and $|\mathscr{H}|_{|A} = |\mathscr{H}_A|$.
Let $E = K_X + 2L$ and look at the exact sequence

$$0 \to -E \to \mathscr{H} \to \mathscr{H}_A \to 0.$$

Note that $E^2L = (K_X + 2L)^2L > 0$ since the Kodaira dimension of $K_X + 2L$ is 2 and recall that E is nef. Therefore (0.7) and Serre duality give $h^1(-E) = 0$. As a consequence the restriction homomorphism

(5.1.1) $$H^0(X, \mathscr{H}) \to H^0(A, \mathscr{H}_A)$$

induced in cohomology by the above sequence is surjective, i.e. $|\mathscr{H}|_{|A} = |\mathscr{H}_A|$. As to the nefness of $\mathscr{H}$, if there were a curve $\mathscr{C}$ in X such that $\mathscr{H}\mathscr{C} < 0$ the base locus Z of $|\mathscr{H}|$ would contain $\mathscr{C}$. Therefore Z would have a nonempty intersection with A, this producing base points for $|\mathscr{H}|_{|A} = |\mathscr{H}_A|$ and contradicting (5.0.3).

Step 2. $\mathscr{H}^3 = 0$.
Argue by contradiction and so, since $\mathscr{H}$ is nef, assume that $\mathscr{H}^3 > 0$. Let $\delta = \mathscr{H}L^2$. Note that $\delta = \deg L_C$, where $C \in |\mathscr{H}_A|$ is an elliptic curve in view of (2.0.2), hence $\delta \geq 3$, since L is very ample. In addition

$$\mathscr{H}^2 L = (\mathscr{H}_A)^2 = K_A^2 = 2.$$

Then, the Hodge index theorem [BBS, (0.15)] gives the inequality

$$\mathscr{H}^3 \delta \leq 4.$$

This implies $\mathscr{H}^3 = 1$ and $\delta = 3$ or 4. By applying once more the Hodge index theorem we get

(5.1.2) $$2L^3 \leq \delta^2 = 9 \text{ or } 16,$$

and so

(5.1.3) $$L^3 \leq \begin{cases} 4, & \text{if } \delta = 3, \\ 8, & \text{if } \delta = 4. \end{cases}$$

Now, the genus formula, applied to the very ample line bundle L_A gives

(5.1.4) $$2g(L_A) - 2 = L_A^2 + L_A K_A = L^3 - \mathscr{H} L^2 = L^3 - \delta,$$

and so L^3 and δ have the same parity. Thus (5.1.3) gives the following possibilities:

$$L^3 = \begin{cases} 1{,}3, & \text{if } \delta = 3, \\ 2{,}4{,}6{,}8, & \text{if } \delta = 4. \end{cases}$$

Assume that $L^3 = 8$. Then (5.1.2) reads as the equality

$$16 = L_A^2 \mathscr{H}_A^2 = (L_A \mathscr{H}_A)^2 = 16\,;$$

and then, by the Hodge index theorem we immediately see that $L_A = 2\mathscr{H}_A$, which in turn gives $L = 2\mathscr{H}$, hence $K_X = -(L + \mathscr{H}) = -3\mathscr{H}$. But then X would be a Fano 3-fold of index 3; this implies that $X = \mathbf{Q}^3$, $\mathscr{H} = \mathscr{O}_{\mathbf{Q}^3}(1)$, $L = \mathscr{O}_{\mathbf{Q}^3}(2)$. In particular A would be isomorphic to a complete intersection of type (2,2), contradicting the fact that $K_A^2 = 2$. So case $L^3 = 8$ cannot occur.

Now come back to the remaining cases. By recalling that A is a Del Pezzo surface, the Kodaira vanishing theorem immediately shows that

$$h^i(L_A) = h^i(K_A + (L_A - K_A)) = 0 \text{ for } i = 1{,}2.$$

Hence the Riemann-Roch theorem gives

(5.1.5) $$h^0(L_A) = 1 + (L_A^2 - L_A K_A)/2 = 1 + (L^3 + \delta)/2.$$

This immediately shows that case $\delta = 3$ and case $\delta = 4$ with $L^3 = 2$ cannot occur. Otherwise $h^0(L_A) \leq 4$ and so $|L_A|$ would embed A into $\mathbf{P}^3$, a contradiction, since the only Del Pezzo surfaces in $\mathbf{P}^3$ are those of degree ≤ 3, which are not isomorphic to A. On the other hand case $\delta = 4$ with $L^3 = 4$ cannot occur either. Actually, in this case, (5.1.5) would give $h^0(L_A) = 5$ and so $|L_A|$ would embed A into $\mathbf{P}^4$, while the double point formula for surfaces in $\mathbf{P}^4$ [H, p. 434] implies

$$-16 = L^3(L^3 - 10) + 12\chi(\mathcal{O}_A) = 2K_A^2 + 5L_AK_A = -5\delta,$$

a contradiction. Finally consider case $\delta = 4$ with $L^3 = 6$. In this case (5.1.4) gives $g(L_A) = 2$; hence (A, L_A) is a rational surface of sectional genus 2. In view of the classification theory this implies that (A, L_A) is a rational conic bundle and then (X, L) is a quadric bundle over $\mathbf{P}^1$. In this case however note that condition $(K_X + 2L)^2L > 0$ is not fulfilled, contradicting (5.0.1).

Step 3. $h^0(\mathcal{H}_A) = 3$.
Let R be the ramification divisor of $\pi: A \to \mathbf{P}^2$, recall that the branch divisor $\pi(R)$ is a smooth plane quartic curve and consider the exact sequences

$$\begin{array}{ccccccccc} 0 \to \mathcal{H}_A - R = \pi^*\mathcal{O}_{\mathbf{P}^2}(-1) & \to & \mathcal{H}_A = \pi^*\mathcal{O}_{\mathbf{P}^2}(1) & \xrightarrow{r} & \mathcal{H}_R \to 0 \\ 0 \quad \to \quad \mathcal{O}_{\mathbf{P}^2}(-3) & \to & \mathcal{O}_{\mathbf{P}^2}(1) & \xrightarrow[s]{} & \mathcal{O}_{\pi(R)}(1) \to 0. \end{array}$$

Let $\alpha': H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) \to H^0(A, \mathcal{H}_A)$, $\beta': H^0(\pi(R), \mathcal{O}_{\pi(R)}(1)) \to H^0(R, \mathcal{H}_R)$, r', s', be the homomorphisms induced by the isomorphisms α, $\pi^*\mathcal{O}_{\mathbf{P}^2}(1) \cong \mathcal{H}_A$, $\beta: \mathcal{O}_{\pi(R)}(1) \cong \mathcal{H}_R$ and by r, s at the H^0-cohomology level. Also s' is an isomorphism, since $h^i(\mathcal{O}_{\mathbf{P}^2}(-3)) = 0$ for $i = 0,1$. This shows that $\beta' \circ s' = r' \circ \alpha'$ is an isomorphism, so that r' is surjective. Therefore we shall have

$$h^0(\mathcal{H}_A) = h^0(\mathcal{H}_R)$$

once we have shown that $h^0(\mathcal{H}_A - R) = 0$. This follows immediately from

$$(\mathcal{H}_A - R)\,\mathcal{H}_A = \pi^*\mathcal{O}_{\mathbf{P}^2}(-1)\pi^*\mathcal{O}_{\mathbf{P}^2}(1) < 0.$$

Finally, $h^0(\mathcal{H}_R) = h^0(\mathcal{O}_{\pi(R)}(1)) = 3$ since $\pi_{|R}$ is an isomorphism between the ramification divisor and the branch locus.

Step 4. $\mathcal{H}$ is spanned.
In view of step 3, $|\mathcal{H}_A|$ is generated by three independent divisors, which by step 1 can be extended to three divisors H_i, $i = 1,2,3$, of $|\mathcal{H}|$. Their intersection Γ cannot meet A otherwise $\Gamma \cap A$ would be the base locus of the linear system generated by $H_{i|A}$, $i = 1,2,3$, which is nothing but $|\mathcal{H}_A|$ by step 3 and so (5.0.3)

would be contradicted. Since A is (very) ample, $|\mathscr{H}|$ thus contains three divisors meeting at a possibly empty finite set of points Z. Now step 2 says that $Z = \emptyset$ and this is enough to conclude. □

We are now able to show that X admits a conic bundle structure besides the scroll one of (X, L).

(5.2) THEOREM. *$(X, L + 2\mathscr{H})$ is a conic bundle over $\mathbf{P}^2$ via the morphism q_X associated with $|\mathscr{H}|$.*

Proof. First of all by Lemma (5.1) $|\mathscr{H}|$ defines a morphism Φ whose image is $\mathbf{P}^2$. This will follow once we have shown that $h^0(\mathscr{H}) = 3$. To see this recall that the restriction morphism

$$H^0(X, \mathscr{H}) \to H^0(A, \mathscr{H}_A)$$

is surjective, as proved in (5.1) step 1. Furthermore $h^0(\mathscr{H} - A) = 0$, as $(\mathscr{H} - A)f = -Af < 0$, where f is a fibre of the morphism associated with $|\mathscr{H}|$. All this gives $h^0(\mathscr{H}) = h^0(\mathscr{H}_A)$ and so step 3 in (5.1) is enough to conclude. Consider the Stein factorization of the morphism Φ onto $\mathbf{P}^2$ and get the diagram

$$\begin{array}{ccc} & \mathbf{X} & \\ {}^{q_X}\swarrow & & \searrow{}^{\Phi} \\ Z & \xrightarrow[\varphi]{} & \mathbf{P}^2 \end{array}$$

where φ is finite and Z a normal surface. For the general fibre F of q_X we have $0 = F\mathscr{H} = -(K_X + L)F = -\deg K_F - LF$ and so $\deg K_F = -LF < 0$, which yields

$$\deg K_F = -2, \; F = \mathbf{P}^1, \; LF = 2.$$

So if G is a connected component of a fibre of Φ, we have

(5.2.1) $$LG = LF = 2.$$

Assume that $\deg \varphi \geq 2$ and let f be a non-connected fibre of Φ. Since $\Phi_{|A} = \pi : A \to \mathbf{P}^2$ is a $2:1$ morphism and A intersects every connected component of f, we get $Af = 2$ and f has exactly two connected components G_1 and G_2. We thus have $G_iA = 1$, contradicting (5.2.1). Thus $\deg \varphi = 1$. Since Z is normal, Zariski's main theorem claims that φ is an isomorphism. So $Z = \mathbf{P}^2$, $\Phi = q_X$ and $\mathscr{H} = q_X{}^*M$ for some ample line bundle M on $\mathbf{P}^2$. Finally letting $\mathscr{L} = L + 2\mathscr{H}$ we have

$$K_X + \mathscr{L} = K_X + L + 2\mathscr{H} = \mathscr{H} = q_X{}^*M,$$

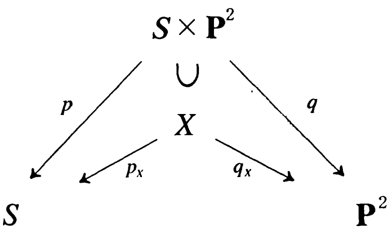
and this shows that $(X, \mathscr{L})$ is a conic bundle over $\mathbf{P}^2$ in the sense of (0.1). $\square$

In order to further describe these pairs (X, L) endowed with a double structure, we need the following

(5.3) LEMMA. *The morphism* $X \xrightarrow{(p_X, q_X)} S \times \mathbf{P}^2$ *is an embedding.*

Proof. First of all note that if ℓ is a fibre of p_X, then $q_X(\ell)$ is a line in $\mathbf{P}^2$. To see this recall that $L\ell = 1$, p_X being the scroll projection, and so $\mathscr{H}\ell = -\deg K_\ell - L\ell = 1$. The projection formula will give the result. To see that (p_X, q_X) is an injection, choose $x, y \in X$. If $p_X(x) \neq p_X(y)$ there is nothing to prove, otherwise there exists a fibre ℓ of p_X containing them. The above remark says that $q_X(x) \neq q_X(y)$. Finally (p_X, q_X) is an immersion. Let τ_x be a tangent vector to X at x. If $dp_X(\tau_x) \neq 0$, there is nothing to show, otherwise τ_x is tangent to a fibre ℓ of p_X. Since $q_{X|\ell}: \ell \to \mathbf{P}^1$ is an embedding, we have $dq_X(\tau_x) \neq 0$. $\square$

From now on let p, q stand for the projection of $M = S \times \mathbf{P}^2$ onto its factors. Note that $p_{|X} = p_X$, $q_{|X} = q_X$, so that we have the following commutative diagram



(5.4) *Remark.* L extends to a line bundle $\mathscr{L}$ on $S \times \mathbf{P}^2$, moreover

$$\mathscr{L} = p^*[D'] + q^*\mathcal{O}_{\mathbf{P}^2}(1) \text{ and } [X] = p^*[D] + q^*\mathcal{O}_{\mathbf{P}^2}(1)$$

for suitable effective divisors D', D on S.

Proof. First of all $K_X + 2L = p_X{}^*H$ in view of (5.0.1) and $\mathscr{H} = -K_X - L = q_X{}^*\mathcal{O}_{\mathbf{P}^2}(1)$, so that

$$L = K_X + 2L + \mathscr{H} = (p^*H + q^*\mathcal{O}_{\mathbf{P}^2}(1))_X.$$

Moreover, $[X] = p^*[D] + q^*\pi^*\mathcal{O}_{\mathbf{P}^2}(x)$ where D is a divisor on S and x an integer. For every point s on S, $p_X{}^{-1}(s) = p^{-1}(s)X$ is a line with respect to L since (X, L) is a scroll via p_X. Therefore

$$1 = \mathscr{L}Xp^{-1}(s).$$

Let D_1 and D_2 be two effective divisors on S meeting transversely at $t = D_1D_2$ distinct points. Then

$$t = \mathscr{L}Xp^*(D_1)p^*(D_2) = (p^*(D_1)p^*(D_2))x\,(q^*\mathcal{O}_{\mathbf{P}^2}(1))^2 = tx. \qquad \square$$

(5.5) *Remark.* Define $C_y = p(q_X{}^{-1}(y))$ for a general $y \in \mathbf{P}^2$ and note that C_y is smooth, p being an isomorphism between $q_X{}^{-1}(y)$ and C_y. Moreover, if q_X has no 2-dimensional fibres, we have $h^0(C_y) \geq 3$ due to the injection $\mathbf{P}^2 \to |C_y|$ obtained by sending $y \in \mathbf{P}^2$ to C_y.

(5.6) Examples. The following special cases are very important as will be clear from (5.7).

(5.6.1) Assume that $S = \mathbf{P}^2$. By (5.4) we have

$$[X] = \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(a, 1) \text{ and } \mathscr{L} = \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(b, 1),$$

with a, b integers, $a \geq 0$, X being an effective divisor of $\mathbf{P}^2 \times \mathbf{P}^2$. If F stands for a fibre of the conic bundle map q_X, we have $2 = LF = \mathscr{L}XF = ab$ and so the only possibilities for (X, L) are either

(α) $\quad [X] = \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,1), \quad L = \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(2,1)_X$; or

(β) $\quad [X] = \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(2,1), \quad L = \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,1)_X$.

(5.6.2) The conic bundle map q_X cannot have 2-dimensional fibres.

By contradiction, let F be a 2-dimensional fibre of q_X. Then the general smooth surface $A \in |L|$ meets F along a curve C (compare [BS, sec. 2]). Moreover, since $(K_X + L)_F$ is trivial, we get by adjunction that $(K_A)_C$ is trivial and thus $-K_A$ cannot be ample, contradicting (3.0.1).

(5.6.3) Assume that $S = \mathbf{P}^1 \times \mathbf{P}^1$ and that $C_y{}^2 = 2$, where $C_y = p(q_X{}^{-1}(y))$ for a general $y \in \mathbf{P}^2$ as in (5.5).

In view of (5.4) we can write

$$[X] = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2}(a,b,1), \quad \mathcal{L} = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2}(c,d,1).$$

Letting $h = q^*\mathcal{O}_{\mathbf{P}^2}(1)$, we have

$$Xh^2 = q_X^{-1}(y) = p^*C_y h^2,$$

which yields $[C_y] = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(a,b)$. Assumption $C_y^{\,2} = 2$ implies $a = b = 1$, as $a \geq 0$, X being effective. From now on choose as C_y a curve reducible in $\ell_1 \cup \ell_2$, where ℓ_1, ℓ_2 correspond to two effective generators of $\mathrm{Pic}(\mathbf{P}^1 \times \mathbf{P}^1)$. Line bundle $\mathcal{H} = -K_X - L$ extends to $-(K_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2} + X + \mathcal{L})$ and so

$$\mathcal{H} = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2}(-c+1, -d+1, 1)_X.$$

Since $\mathcal{H}_f = \mathcal{O}_f$ where $f = q_X^{-1}(y)$, $\mathcal{H}$ must be trivial on the two components of f corresponding to ℓ_1, ℓ_2, namely $p^*\ell_i h^2$, $i = 1,2$. All this implies $d = c = 1$. Therefore the only possibility for (X, L) is

$$(\gamma) \qquad [X] = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2}(1,1,1), \quad L = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^2}(1,1,1)_X.$$

(5.6.4) Assume that $S = \mathbf{F}_1$ (the 1st Segre-Hirzebruch surface) and that $C_y^{\,2} = 1$, where $C_y = p(q_X^{-1}(y))$ for a general $y \in \mathbf{P}^2$ as in (5.5).

Let r be a fibre of the ruling of $\mathbf{F}_1$ and let C_0 be the fundamental section, i.e. the unique exceptional curve. Choosing

$$e = C_0 + r, \quad r$$

as generators for $\mathrm{Pic}(\mathbf{F}_1)$, we have that

$$E = p^*e, \quad R = p^*r, \quad h = q^*\mathcal{O}_{\mathbf{P}^2}(1)$$

are generators for $\mathrm{Pic}(\mathbf{F}_1 \times \mathbf{P}^2)$, so that we can write, by (5.4),

$$[X] = aE + bR + h, \quad \mathcal{L} = cE + dR + h.$$

As in the above example we have

$$X h^2 = q_X^{-1}(y) = p^*C_y h^2,$$

which yields $[C_y] = ae + br$ on $\mathbf{F}_1$. Assumption $C_y^{\,2} = 1$ implies $a = 1$, $b = 0$, $C_y = e$. From now on choose as C_y a curve reducible in $C_0 \cup r$. Line bundle $\mathcal{H} = -K_X - L$ extends to $-(K_{\mathbf{F}_1\times\mathbf{P}^2} + X + \mathcal{L})$ and so

$$\mathcal{H} = -((c-1)E + (d-1)R - h)_X.$$

Since $\mathcal{H}_f = \mathcal{O}_f$ where $f = q_X^{-1}(y)$, $\mathcal{H}$ must be trivial on the two components of f corresponding to C_0 and r, namely $p^*C_0h^2$, Rh^2. All this implies $d = c = 1$. Therefore the only possibility for (X, L) is

(δ) $$[X] = E + h, \quad L = (E + R + h)_X.$$

(5.6.5) Note that in all the above cases, (α), (β), (γ), (δ) the line bundle corresponding to $[X]$ is spanned. Hence the associated linear system does contain a smooth threefold. Moreover L is very ample since it is the restriction to X of a line bundle $\mathscr{L}$ which is immediately seen to be very ample on $S \times \mathbf{P}^2$. Finally $|L|$ contains a smooth element which is in fact a Del Pezzo double plane and the degree 2 morphism is induced by q. Actually in all cases an immediate check gives $-K_A = q_{|A}{}^*\mathcal{O}_{\mathbf{P}^2}(1)$ and so A is a Del Pezzo surface with $K_A{}^2 = 2$.

The main result of this section is the following

(5.7) THEOREM. *Scrolls as in* (3.2.2) *are exactly the pairs* (α), (β), (γ), (δ) *described in* (5.6).

Proof. In view of (5.6.2) the conic bundle map q_X has only 1-dimensional fibres. Let $C_y = p(q_X^{-1}(y))$ for $y \in \mathbf{P}^2$ as in (5.5) and recall that p_X maps isomorphically $q_X{}^{-1}(y)$ onto its image C_y, so we have that C_y is isomorphic to $\mathbf{P}^1$.

The line bundle $[C_y]$ on S associated with C_y is spanned. To see this let s be any point on S. Since $p_X{}^{-1}(s)$ has dimension 1, $\Gamma = q_X p_X^{-1}(s)$ has dimension ≤ 1; we can thus find a point $x \in \mathbf{P}^2 \setminus \Gamma$. Therefore $s \notin p_X q_X^{-1}(x)$, but $p_X q_X^{-1}(x) \in |C_y|$.

Moreover $[C_y]$ is big. By contradiction assume that $C_y{}^2 = 0$. Since C_y is rational, S is a ruled surface, C_y is a fibre and so $h^0(C_y) \leq 2$, which contradicts (5.5).

Recall that $K_X + 2L = p_X{}^*H$ for an ample and spanned line bundle H on S by (5.0.1) and (5.0.2). Now the projection formula gives

(5.7.1) $$HC_y = H p_X(q_X^{-1}(y)) = (K_X + 2L) q_X^{-1}(y) = -2 + 4 = 2$$

and so the Hodge index theorem yields

(5.7.2) $$H^2 C_y^2 \leq 4.$$

Since H is spanned, if $H^2 = 1$ we have $(S, H) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$ and (5.6.1) leads to cases (α), (β). If $H^2 = 2$, either $(S, H) = (\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,1))$ or $\tau : S \to \mathbf{P}^2$ is a double cover and $H = \tau^*\mathcal{O}_{\mathbf{P}^2}(1)$. In the former case by (5.7.2) and the fact that on $\mathbf{P}^1 \times \mathbf{P}^1$ the self intersection of any divisor is even, we have $C_y^2 = 2$ and so (5.6.3) leads to (γ). In the latter case, let $2c$ be the degree of the branch locus of τ. Since A dominates S via p_X, S is a Del Pezzo surface by (3.1). Then since $-K_S = \tau^*\mathcal{O}_{\mathbf{P}^2}(3 - c)$ has to be ample we have $1 \leq c \leq 2$. For $c = 1$ we fall into the above case, while for $c = 2$, S is isomorphic to A, but this contradicts (0.2).

We can thus assume $H^2 \geq 3$, which by (5.7.2) gives

(5.7.3) $$C_y^2 = 1.$$

If C_y is ample, since it is also spanned, (5.7.3) implies $(S, [C_y]) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$ and (5.6.1) leads to cases (α), (β). To finish with, assume that C_y is not ample and $C_y^2 = 1$. So there exists an irreducible reduced curve E such that $C_yE = 0$; note that $E \neq C_y$, C_y being big. From the spannedness of $[C_y]$ we deduce there exists an effective divisor C' on S, linearly equivalent to C_y, with

$$C' = E + R, \quad R > 0.$$

We have $2 = HC' = EH + RH$ in view of (5.7.1) and so the ampleness of H says that $HR = 1$ and R is irreducible and reduced. Since C' is the image via p_X of a reducible fibre of the conic bundle map q_X, we also have $ER = 1$ and so, by (5.7.3), $1 = C'^2 = C'(E + R) = C'R = 1 + R^2$. We thus deduce $R^2 = 0$, $HR = 1$ and $R = \mathbf{P}^1$, R being isomorphic via p_X to a component of a reducible fibre of the conic bundle map. This says that (S, H) is a rational scroll. On the other hand S is a Del Pezzo surface by (3.1), S being dominated by A. Hence either $S = \mathbf{P}^1 \times \mathbf{P}^1$ or $S = \mathbf{F}_1$. The first case cannot happen in view of (5.7.3) since the self intersection $C_y^2 = 1$ cannot occur on $\mathbf{P}^1 \times \mathbf{P}^1$. In the second case (5.6.4) leads to (δ). □

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Antonio Lanteri and Marino Palleschi
Dipartimento di Matematica "F. Enriques"- Università
Via C. Saldini, 50
I-20133 Milano, Italy
LANTERI@VMIMAT. MAT. UNIMI. IT
APELLES@VMIMAT. MAT. UNIMI. IT

Andrew J. Sommese
Department of Mathematics
University of Notre Dame
Notre Dame, INDIANA 46556, U. S. A.
sommese@hobbes. math. nd. edu