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Calculus of Variations

On the long-time behavior of some mathematical models for nematic liquid crystals

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Abstract A model describing the evolution of a liquid crystal substance in the nematic phase is investigated in terms of two basic state variables: the *velocity field* $\boldsymbol{u}$ and the *director field* $\boldsymbol{d}$, representing the preferred orientation of molecules in a neighborhood of any point in a reference domain. After recalling a known existence result, we investigate the long-time behavior of weak solutions. In particular, we show that any solution trajectory admits a non-empty ω-limit set containing only stationary solutions. Moreover, we give a number of sufficient conditions in order that the ω-limit set contains a single point. Our approach improves and generalizes existing results on the same problem.

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1 Introduction

In this paper we analyze the long-time behavior of weak solutions to the system

$$\boldsymbol{u}_t + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \nu\Delta\boldsymbol{u} = \operatorname{div}\left(-p\mathbb{I} - L(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) - \delta(L\Delta\boldsymbol{d} - \boldsymbol{f}(\boldsymbol{d}))\otimes\boldsymbol{d}\right), \quad (1)$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad (2)$$

$$\boldsymbol{d}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{d} - \delta\boldsymbol{d}\cdot\nabla\boldsymbol{u} - L\Delta\boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d}) = 0, \quad (3)$$

describing the evolutionary behavior of *nematic* liquid crystal flows (we refer to the monographs [5,6] for a detailed presentation of the physical foundations of continuum theories of liquid crystals). Actually, system (1–3) can be seen as a simplification of the original Ericksen–Leslie model [7,13], that still keeps a good level of compliance with experimental results. The model couples the Navier-Stokes Eq. 1 for the macroscopic velocity $\boldsymbol{u}$ (p denoting as usual the pressure), with the incompressibility condition (2) and with the Eq. 3 ruling the behavior of the local orientation vector $\boldsymbol{d}$ of the liquid crystal. Here, the function $\boldsymbol{f}$ represents the gradient w.r.t. $\boldsymbol{d}$ of the configuration energy F of the crystal. We choose F to be a double well potential having minima for $|\boldsymbol{d}| = 1$ and growing at infinity at most as a fourth order polynomial. This provides a standard relaxation of the physical constraint $|\boldsymbol{d}| = 1$, which is very difficult to treat mathematically.

In this paper, the system is complemented with the homogeneous Dirichlet boundary condition for $\boldsymbol{u}$, the no-flux condition for $\boldsymbol{d}$, and with initial conditions. It is settled in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ for $d = 2$ or $d = 3$. No restriction is assumed on the viscosity coefficient ν.

Regarding the parameter δ, we will take $\delta \geq 0$, with the case $\delta > 0$ denoting the presence of a *stretching effect* on the molecules of the crystal. Some of our results, however, hold only for $\delta = 0$. Actually, the situation $\delta > 0$ is more difficult to be treated mathematically since the term $\delta\boldsymbol{d}\cdot\nabla\boldsymbol{u}$ prevents from using maximum principle arguments in (3). For this reason, even if the initial datum $\boldsymbol{d}_0$ satisfies the (relaxed) physical constraint $|\boldsymbol{d}_0| \leq 1$ almost everywhere, the same may not be true for $\boldsymbol{d}(t)$, for positive times, if $\delta > 0$.

A mathematical analysis of system (1–3) has been first addressed in the papers [14] and [15] (in this second work, an even more general model is taken into account). There, the authors consider the case $\delta = 0$ and prove existence of a unique classical solution for $d = 2$, and also in dimension $d = 3$ under the additional assumption that the viscosity ν is sufficiently large. These results have been extended to the case $\delta > 0$ in the paper [19]. Finally, the restriction on the viscosity has been recently dropped in [2], where *weak* solutions are considered and a global existence result for the 3D system (1–3) is proved in that regularity frame. Of course, uniqueness is not known to hold in that regularity setting. A similar result is essentially contained also in the recent paper [3], where analogous estimates are derived but no formal statement of an existence result is provided.

The Dirichlet boundary condition for $\boldsymbol{u}$ and either a nonhomogeneous Dirichlet or the no-flux boundary condition for $\boldsymbol{d}$ are treated there. Moreover, let us quote the recent paper [9], where these results have been extended to a more general system (1–3), where also temperature effects are taken into account. We note, however, that the results of [9] require different boundary conditions for $\boldsymbol{u}$ (namely, the so-called *complete slip* conditions).

The long-time behavior of system (1–3) has been analyzed in the recent work [20], still considering the case $d = 2$ or the case $d = 3$ with the large viscosity ν, and periodic boundary conditions. More precisely, in [20] the authors show existence of a nonempty ω-limit set for any strong bounded solution emanating from smooth initial data. Moreover, by using the

Simon-Łojasiewicz inequality, they prove that, for the nonlinearity $\boldsymbol{f} = (|\boldsymbol{d}|^2 - 1)\boldsymbol{d}$, this ω-limit set contains only one point.

Stability and asymptotic stability properties of this model (actually, with even more complete stretching terms) have also been studied in [3], where the long-time behavior of solutions is analyzed in the case of periodic boundary conditions. More precisely, the authors prove, by means of formal estimates, that weak solutions become eventually smoother for large times, which suffices to have existence of non-empty ω-limit sets.

Finally, in the recent contribution [12], the existence of a smooth global attractor of finite fractal dimension is obtained in two dimensions of space.

Our aim in this paper is to extend the results of [3,20] in the following directions:

(i) we address the case $d = 3$ without the large viscosity assumption considering weak solutions;
(ii) we consider more general C^1 functions $\boldsymbol{f}$;
(iii) we use different boundary conditions and weaker initial data;
(iv) we discuss convergence, as t tends to ∞, of strong solutions in some particular situations.

To get (i), we prove convergence of weak solutions, and, in some situations, we get strong convergence using the fact that weak solutions to the system become eventually smoother for times t larger than some T. This property is well-known for the (uncoupled) three-dimensional N-S system, and we find conditions under which it holds also for the coupled system (1–3). Note that this result is still true for periodic boundary conditions, and so it improves the study done in [20]. In turn, this property (cf. 74–75 below) enables us to obtain properties sufficient to characterize the ω-limit set. Assuming that $\boldsymbol{f}$ is analytic, we apply the generalized Łojasiewicz theorem to get convergence of the variable $\boldsymbol{d}$.

To address question (ii), in particular, to remove the analyticity condition, we make the basic observation that the set of global minimizers of the configuration energy of the crystal coincides with the set of constant unit vectors of $\mathbb{R}^d$. Then, it is easy to prove that any global minimizer $\overline{\boldsymbol{d}}$ satisfies the so-called *normal hyperbolicity* condition. Based on this fact, we can prove that, if the ω-limit set contains a global minimizer, then it coincides with it (i.e., it does not contain any other point). We can also give two precise conditions ensuring the fact that the ω-limit set contains global minimizers, which, unfortunately, require $\delta = 0$. Namely, this happens when either the diffusion coefficient L is large enough, or when the initial energy is very small compared with L (in particular, the initial datum $\boldsymbol{d}_0$ is already close enough to the set of global minimizers in a suitable norm).

The paper is organized as follows. In the next section, we present our assumptions, state the main results, and, for the reader's convenience, we briefly sketch the basic estimates at the core of the existence proof. The proofs of the new results on the long-time behavior are given in Sect. 3.

2 Main results

We let Ω be a smooth, bounded, and connected domain in $\mathbb{R}^d$, $d \in \{2, 3\}$, with the boundary Γ. For simplicity, we also assume $|\Omega| = 1$. We set $H := L^2(\Omega)$, $\boldsymbol{H} := L^2(\Omega)^d$, and denote by $(\cdot, \cdot)$ the scalar product both in H and in $\boldsymbol{H}$ and by $\|\cdot\|$ the related norms. Next, we set $V := H^1(\Omega), \boldsymbol{V} := H^1(\Omega)^d$ and $\boldsymbol{V}_0 := H_0^1(\Omega)^d$. The duality between V' and V, as well as those between $\boldsymbol{V}'$ and $\boldsymbol{V}$ and between $\boldsymbol{V}_0'$ and $\boldsymbol{V}_0$, will be indicated by $\langle \cdot, \cdot \rangle$. Identifying H with H' through the scalar product of H, it is then well known that $V \subset H \subset V'$ with

continuous and dense inclusions. In other words, (V, H, V') constitutes a Hilbert triplet (see, e.g., [16]). Correspondingly, we also have the vectorial analogues $(\boldsymbol{V}, \boldsymbol{H}, \boldsymbol{V}')$ and $(\boldsymbol{V}_0, \boldsymbol{H}, \boldsymbol{V}_0')$. The symbol $\|\cdot\|_X$ will indicate the norm in the generic (real) Banach space X and $\langle\cdot,\cdot\rangle_X$ will stand for the duality between X' and X.

We consider $\boldsymbol{f}$ in the form

$$\boldsymbol{f}(\boldsymbol{d}) = \left(\psi(|\boldsymbol{d}|^2) - 1\right)\boldsymbol{d} = \frac{1}{2}\partial_{\boldsymbol{d}}\left(\widehat{\psi}(|\boldsymbol{d}|^2) - |\boldsymbol{d}|^2\right), \tag{4}$$

where

$$\psi \in C^1\left([0,+\infty); [0,+\infty)\right), \quad \text{with } \psi(0) = 0, \ \ \psi(1) = 1 \ \text{ and } \ \psi'(1) > 0, \tag{5}$$

is an increasing function, and the convex function $\widehat{\psi}$ is defined by

$$\widehat{\psi}' = \psi, \quad \widehat{\psi}(1) = 1. \tag{6}$$

We also assume that there exists a constant $c_\psi > 0$ such that

$$\psi'(r) \le c_\psi \quad \text{for all } \ r \in [0,+\infty). \tag{7}$$

Given $L > 0$, we define the *configuration energy* of the liquid crystal flow as

$$\mathcal{E}(\boldsymbol{d}) := \frac{1}{2}\int_\Omega \left(L|\nabla \boldsymbol{d}|^2 + \widehat{\psi}(|\boldsymbol{d}|^2) - |\boldsymbol{d}|^2\right). \tag{8}$$

The *total energy* is then given by adding to $\mathcal{E}$ the "macroscopic" kinetic energy; namely, we set

$$\mathbb{E}(\boldsymbol{u}, \boldsymbol{d}) := \frac{1}{2}\|\boldsymbol{u}\|^2 + \mathcal{E}(\boldsymbol{d}) = \frac{1}{2}\int_\Omega \left(|\boldsymbol{u}|^2 + L|\nabla \boldsymbol{d}|^2 + \widehat{\psi}(|\boldsymbol{d}|^2) - |\boldsymbol{d}|^2\right). \tag{9}$$

Let us notice that, thanks to the above assumptions (5–7), $\mathcal{E}(\boldsymbol{d}) = 0$ if and only if $\boldsymbol{d}$ is a (constant) unit vector (cf. Lemma 2.13 below for a simple proof).

We will address the following system of PDE's:

$$\boldsymbol{u}_t + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \nu\Delta\boldsymbol{u} = \operatorname{div}\mathbb{S}, \tag{10}$$

$$\mathbb{S} = -p\mathbb{I} - L(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}) - \delta(L\Delta\boldsymbol{d} - \boldsymbol{f}(\boldsymbol{d}))\otimes\boldsymbol{d}, \tag{11}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{12}$$

$$\boldsymbol{d}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{d} - \delta\boldsymbol{d}\cdot\nabla\boldsymbol{u} - L\Delta\boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d}) = 0, \tag{13}$$

where the coefficients ν, L, δ satisfy $\nu, L > 0$ and $\delta \ge 0$. Notice that, by (7), $\boldsymbol{f}(\boldsymbol{d})$ grows at infinity at most as the third power of $|\boldsymbol{d}|$.

The system, supplemented with the boundary and initial conditions

$$\boldsymbol{u} = 0 \quad \text{a.e. on } \ (0,T)\times\Gamma, \tag{14}$$

$$\partial_{\boldsymbol{n}}\boldsymbol{d} = 0 \quad \text{a.e. on } \ (0,T)\times\Gamma, \tag{15}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{d}|_{t=0} = \boldsymbol{d}_0, \quad \text{a.e. in } \ \Omega, \tag{16}$$

will be called Problem (P).

We introduce a precise definition of weak solutions:

Definition 2.1 A weak solution to Problem (P) is a couple $(\boldsymbol{u}, \boldsymbol{d})$ such that

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{V}_0), \tag{17}$$

$$\boldsymbol{d} \in H^1(0, T; L^{3/2}(\Omega)^d) \cap L^\infty(0, T; \boldsymbol{V}) \cap L^2(0, T; H^2(\Omega)^d), \tag{18}$$

for all $T > 0$, $\boldsymbol{u}, \boldsymbol{d}$ satisfy initial and boundary conditions (16, 15), the Eqs. 11–13 are satisfied for a.e. $t \in (0, T)$, and

$$\langle \boldsymbol{u}_t, \boldsymbol{\phi} \rangle - \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\phi} + \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\phi} = - \int_\Omega \mathbb{S} : \nabla \boldsymbol{\phi}, \tag{19}$$

holds for any test function $\boldsymbol{\phi} \in \boldsymbol{W}^{1,3}_{0,\mathrm{div}}(\Omega)$ (i.e., the subspace of $W^{1,3}_0(\Omega)^d$ consisting of divergence-free functions).

Remark 2.2 The regularity of the test function ϕ can be justified thanks to (17, 18 and 13). We have in any case (also if $\delta > 0$)

$$\boldsymbol{u} \otimes \boldsymbol{u}, \quad \nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}, \quad (L\Delta \boldsymbol{d} - \boldsymbol{f}(\boldsymbol{d})) \otimes \boldsymbol{d} \in L^2(0, T; L^{3/2}(\Omega))^{d \times d}, \tag{20}$$

whence their (distributional) divergence belongs to the space $L^2(0, T; W^{-1,3/2}(\Omega))^d$. Note also that the boundary condition (14) is in fact "embedded" into the weak formulation (19).

It is known that Problem (P) admits at least one weak solution $(\boldsymbol{u}, \boldsymbol{d})$. This has been proved in [14] for the case $\delta = 0$ and in [2] for the case $\delta = 1$ (cf. also [3] for the formal computations). Namely, we have

Theorem 2.3 *Let (4–7) hold and let*

$$\boldsymbol{u}_0 \in \boldsymbol{H}, \quad \mathrm{div}\, \boldsymbol{u}_0 = 0, \tag{21}$$

$$\boldsymbol{d}_0 \in \boldsymbol{V}. \tag{22}$$

Then, Problem (P) *possesses a global in time weak solution* $(\boldsymbol{u}, \boldsymbol{d})$*, satisfying, for a.a.* $t > 0$*, the* energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}(\boldsymbol{u}, \boldsymbol{d}) + \| -L\Delta \boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d}) \|^2 + \nu \|\nabla \boldsymbol{u}\|^2 \leq 0. \tag{23}$$

We point out that assumptions (21–22) are equivalent to asking that the initial energy $\mathbb{E}_0 := \mathbb{E}(\boldsymbol{u}_0, \boldsymbol{d}_0)$ is finite.

Remark 2.4 The proof of the above theorem relies on a rather tricky approximation scheme and on refined compactness methods to pass to the limit. It is then worth pointing out that, due to nonuniqueness, our subsequent results on the long-time behavior hold only for those solutions satisfying the energy inequality, in particular for the limit points of the approximate scheme, and not necessarily for all solutions in the regularity frame (17–18). Actually, there may exist "spurious" weak solutions not satisfying the energy inequality (23) which is crucial for investigating the long-time behavior. As a convention, in the sequel we shall restrict the terminology "weak solutions" to those solutions which satisfy (23). Spurious solutions are thus excluded.

As noted above, in the case $\delta = 0$ a maximum principle holds for the $\boldsymbol{d}$-component of any weak solution. For the reader's convenience, we recall the statement and the (simple) proof.

Theorem 2.5 *Let the assumptions of* Theorem 2.3 *hold and let*

$$\delta = 0, \ \textit{and} \ \ |\boldsymbol{d}_0(x)| \le 1 \ \textit{for a.a.} \ x \in \Omega.$$

Then any weak solution $(\boldsymbol{u}, \boldsymbol{d})$ *to* Problem (P) *satisfies*

$$|\boldsymbol{d}(t,x)| \le 1 \quad \textit{for a.a.} \ (t,x) \in (0,\infty) \times \Omega. \tag{24}$$

Proof Testing Eq. 13 by $\boldsymbol{d}$ one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{d}|^2 + \boldsymbol{u}\cdot\nabla|\boldsymbol{d}|^2 - \frac{L}{2}\Delta|\boldsymbol{d}|^2 + L|\nabla\boldsymbol{d}|^2 + \left(\psi(|\boldsymbol{d}|^2) - 1\right)|\boldsymbol{d}|^2 = 0. \tag{25}$$

Then, we notice that, by (5),

$$(\psi(r) - 1)r \ge 0 \quad \forall\, r \ge 1. \tag{26}$$

Thus, (25) represents a parabolic equation for $|\boldsymbol{d}|^2$ (which still satisfies the no-flux b.c.). It is clear that the maximum principle applies, yielding (24). □

Unfortunately, (24) is not known (and not expected) to hold in the case $\delta > 0$.

Although the next result is essentially contained in the paper [20], for completeness it is worth stating and proving existence of (nonempty) ω-limit sets of weak solutions.

Theorem 2.6 *Let the assumptions of* Theorem 2.3 *hold, and let* $(\boldsymbol{u}, \boldsymbol{d})$ *be a weak solution of* Problem (P). *Then, the* ω*-limit set of* $(\boldsymbol{u}, \boldsymbol{d})$ *is nonempty. More precisely, we have*

$$\lim_{t\nearrow+\infty} \boldsymbol{u}(t) = 0 \quad \textit{weakly in} \ \boldsymbol{H}, \tag{27}$$

and any diverging sequence $\{t_n\} \subset [0, +\infty)$ *admits a subsequence, not relabeled, such that*

$$\lim_{n\nearrow+\infty} \boldsymbol{d}(t_n) = \boldsymbol{d}_\infty \quad \textit{weakly in} \ \boldsymbol{V} \quad \textit{and strongly in} \ \boldsymbol{H}, \tag{28}$$

for some $\boldsymbol{d}_\infty \in V$. *Moreover, any such limit point* $\boldsymbol{d}_\infty$ *is a solution of the stationary problem*

$$-L\Delta z + \boldsymbol{f}(z) = 0 \ \textit{ in } \ \Omega, \quad \partial_{\boldsymbol{n}} z = 0 \ \textit{ on } \ \Gamma. \tag{29}$$

Proof Let $\{t_n\} \subset [0, +\infty)$ be a diverging sequence. Then, the energy estimate implies that, at least for a (nonrelabeled) subsequence of n,

$$\boldsymbol{u}(t_n) \to \boldsymbol{u}_\infty \quad \text{weakly in } \boldsymbol{H}, \quad \boldsymbol{d}(t_n) \to \boldsymbol{d}_\infty \quad \text{weakly in } \boldsymbol{V}, \tag{30}$$

for suitable limit functions $\boldsymbol{u}_\infty$ and $\boldsymbol{d}_\infty$. Let us consider the initial and boundary value problem associated to (10–13) on the time interval $[t_n, t_n + 1]$ with "initial values" $\boldsymbol{u}(t_n)$ and $\boldsymbol{d}(t_n)$. It is clear that, setting, $\boldsymbol{u}_n(t) := \boldsymbol{u}(t + t_n)$ and $\boldsymbol{d}_n(t) := \boldsymbol{d}(t + t_n)$, $t \in [0, 1]$, we get a weak solution to the problem on the time interval $[0, 1]$. Then, (23) implies that

$$\nabla\boldsymbol{u}_n \to 0 \quad \text{strongly in } L^2(0,1;\boldsymbol{H}^d), \tag{31}$$

whence, by Poincaré's inequality and (23) again, we have also

$$\boldsymbol{u}_n \to 0 \quad \text{strongly in } L^2(0,1;\boldsymbol{V}_0) \quad \text{and weakly star in } L^\infty(0,1;\boldsymbol{H}). \tag{32}$$

Moreover, we have

$$\boldsymbol{d}_n \to \overline{\boldsymbol{d}} \quad \text{weakly star in } L^\infty(0,1;\boldsymbol{V}) \cap L^2(0,1;H^2(\Omega)^d), \tag{33}$$

for a suitable limit function $\overline{\boldsymbol{d}}$. The growth condition (7) and a comparison argument in (13) then entail

$$\boldsymbol{d}_{n,t} \to \overline{\boldsymbol{d}}_t \quad \text{weakly in } L^2(0,1;L^{3/2}(\Omega)^d). \tag{34}$$

Hence, by the Aubin-Lions lemma, we obtain

$$\boldsymbol{d}_n \to \overline{\boldsymbol{d}} \quad \text{strongly in } L^2(0,1;\boldsymbol{V}). \tag{35}$$

To proceed, we take $\boldsymbol{\phi} \in \boldsymbol{W}^{1,3}_{0,\mathrm{div}}(\Omega)$ and test (10) by $\boldsymbol{\phi}$. Noting that, by (11),

$$\int_\Omega \mathbb{S}_n : \nabla\boldsymbol{\phi} = -L\int_\Omega (\nabla\boldsymbol{d}_n \odot \nabla\boldsymbol{d}_n) : \nabla\boldsymbol{\phi} - \delta\int_\Omega ((L\Delta\boldsymbol{d}_n - \boldsymbol{f}(\boldsymbol{d}_n)) \otimes \boldsymbol{d}_n) : \nabla\boldsymbol{\phi}, \tag{36}$$

and recalling (32, 33), and Remark 2.2, we arrive at

$$\|\boldsymbol{u}_{n,t}\|_{L^2(0,1;\boldsymbol{W}^{-1,3/2}_{\mathrm{div}}(\Omega))} \le c, \tag{37}$$

where $\boldsymbol{W}^{-1,3/2}_{\mathrm{div}}(\Omega)$ denotes the dual space to $\boldsymbol{W}^{1,3}_{0,\mathrm{div}}(\Omega)$ and c denotes a positive constant independent of n.

Thus, from (32, 37), and the Aubin-Lions lemma, we obtain that

$$\boldsymbol{u}_n \to 0 \quad \text{strongly in } C^0([0,1];\boldsymbol{V}_0'), \tag{38}$$

so that, in particular, $\boldsymbol{u}_\infty = 0$, and (32, 38) imply (27). On the other hand, by the energy estimate, we obtain

$$-L\Delta\boldsymbol{d}_n + \boldsymbol{f}(\boldsymbol{d}_n) \to 0 \quad \text{strongly in } L^2(0,1;\boldsymbol{H}), \tag{39}$$

whereas, by (32–35),

$$\boldsymbol{u}_n \cdot \nabla\boldsymbol{d}_n - \delta\boldsymbol{d}_n \cdot \nabla\boldsymbol{u}_n \to 0 \quad \text{weakly in } L^2(0,1;L^{3/2}(\Omega)^d). \tag{40}$$

Thus, comparing terms in (13), we also have that $\boldsymbol{d}_{n,t} \to 0$ in a suitable way. This entails that $\overline{\boldsymbol{d}}$ is constant in time and, therefore, it coincides with $\boldsymbol{d}_\infty$ for all times in $[0,1]$. Moreover, taking the limit in (13), we obtain that $\boldsymbol{d}_\infty$ is a solution to (29), as desired. This completes the proof. □

We now present the main results of this paper, which characterize the ω-limit set of our system as a singleton under a number of different conditions.

Theorem 2.7 *Let the assumptions of* Theorem 2.3 *hold, and, in addition, let* ψ *be* analytic. *Then the ω-limit set of the component $\boldsymbol{d}$ of any weak solution consists of a single point, and we have*

$$\lim_{t\nearrow+\infty} \boldsymbol{d}(t) = \boldsymbol{d}_\infty \quad \textit{strongly in } \boldsymbol{H} \tag{41}$$

for the whole trajectory $\boldsymbol{d}$, where $\boldsymbol{d}_\infty$ is a solution to (29).

Remark 2.8 As usually, when applying the Łojasiewicz inequality, we can also get the rate of convergence of the form

$$\|\boldsymbol{d}(t) - \boldsymbol{d}_\infty\|_{\boldsymbol{H}} \le C(1+t)^{-\frac{\theta}{1-2\theta}},$$

where θ is the Łojasiewicz exponent, and C is a suitably chosen constant depending on the initial energy and on the limit function.

In particular situations, we can also prove a stronger convergence result:

Theorem 2.9 *Under the hypotheses of* Theorem 2.7*, let, in addition,*

$$\delta = 0. \tag{42}$$

Then,

$$\boldsymbol{u}(t) \to 0 \ \textit{strongly in} \ \boldsymbol{V}, \tag{43}$$

$$\lim_{t \nearrow +\infty} \boldsymbol{d}(t) = \boldsymbol{d}_\infty \ \ \textit{strongly in} \ H^2(\Omega)^d. \tag{44}$$

Remark 2.10 The same result was proved in [20] for periodic boundary conditions for $\boldsymbol{u}$ and $\boldsymbol{d}$, large viscosity coefficient, and smooth initial data. Actually, it is easy to check that our argument holds true also in the case of periodic B.C., when $\delta \geq 0$. Hence, the same result of [20] holds without the requirement of large viscosity, and for initial data as in (21, 22). On the other hand, in the case $\delta > 0$ with boundary conditions (14–15), it does not seem possible to repeat the strong estimates required for the proof of (43–44) (some additional boundary terms appear, which is not clear how to control). Hence, extending the statement of Theorem 2.9 to this situation remains an open question.

As in [20, Theorem 1.2] the proofs rely on a suitable version of the *Simon-Łojasiewicz* inequality, proved in [11, Theorem 6]. For the reader's convenience, we report here the statement of a particular case of the (more general) result of [11], in a form suitable for our application:

Theorem 2.11 *Let the energy functional $\mathcal{E}$ be given by (8) with $\hat{\psi}$ analytic. Let $\boldsymbol{p} \in \boldsymbol{V}$ be a critical point of $\mathcal{E}$. Then there exist constants $\theta \in (0, 1/2)$, $\Lambda > 0$ and $\epsilon_1 > 0$ such that the inequality*

$$|\mathcal{E}(\boldsymbol{v}) - \mathcal{E}(\boldsymbol{p})|^{1-\theta} \leq \Lambda \, \| -L\Delta \boldsymbol{v} + \boldsymbol{f}(\boldsymbol{v}) \|_{\boldsymbol{V}'} \tag{45}$$

holds for any $\boldsymbol{v}$ such that

$$\|\boldsymbol{v} - \boldsymbol{p}\|_{\boldsymbol{V}} < \epsilon_1. \tag{46}$$

To apply the preceding Theorem in our situation, we have to show that the inequality (45) holds for $\boldsymbol{v} = \boldsymbol{d}(t)$ in a small $\boldsymbol{H}$-neighbourhood of $\boldsymbol{d}_\infty$:

Lemma 2.12 *Let the energy functional $\mathcal{E}$ be given by (8) with $\hat{\psi}$ analytic. Let $\boldsymbol{d}_\infty \in \boldsymbol{V}$ be a solution of (29). Let K, $P > 0$ be constants. Then there exist $\epsilon > 0$ and $\Lambda > 0$ such that (45) holds for any $\boldsymbol{v}$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} \leq K, \quad \|\boldsymbol{v} - \boldsymbol{d}_\infty\|_{\boldsymbol{H}} \leq \epsilon, \ \textit{and} \ \ |\mathcal{E}(\boldsymbol{v}) - \mathcal{E}(\boldsymbol{d}_\infty)| \leq P. \tag{47}$$

Proof We argue by contradiction. Assume that there is a sequence $\boldsymbol{v}_n$ such that

$$\|\boldsymbol{v}_n\|_{\boldsymbol{V}} \leq K, \ \ \boldsymbol{v}_n \to \boldsymbol{d}_\infty \text{ in } \boldsymbol{H}, \ \ |\mathcal{E}(\boldsymbol{v}_n) - \mathcal{E}(\boldsymbol{d}_\infty)| \leq P$$

and

$$|\mathcal{E}(\boldsymbol{v}_n) - \mathcal{E}(\boldsymbol{d}_\infty)|^{1-\theta} \geq n \| - L\Delta \boldsymbol{v}_n + \boldsymbol{f}(\boldsymbol{v}_n)\|_{\boldsymbol{V}'}, \ \ n = 1, 2, 3, \ldots \tag{48}$$

Then

$$\boldsymbol{f}(\boldsymbol{v}_n) \to \boldsymbol{f}(\boldsymbol{d}_\infty) \text{ in } \boldsymbol{V}', \text{ and } \Delta \boldsymbol{v}_n \to \Delta \boldsymbol{d}_\infty \text{ in } \boldsymbol{V}'.$$

This implies that

$$\nabla \boldsymbol{v}_n \to \nabla \boldsymbol{d}_\infty \text{ in } \boldsymbol{H}, \text{ and, consequently, } \boldsymbol{v}_n \to \boldsymbol{d}_\infty \text{ in } \boldsymbol{V}.$$

Hence, at least for n sufficiently large, (46) holds for $\boldsymbol{v} = \boldsymbol{v}_n$, $\boldsymbol{p} = \boldsymbol{d}_\infty$. Consequently, also (45) is valid. This contradicts (48). □

In the case that $\boldsymbol{f}$ does not satisfy the analyticity condition, we can show that the ω-limit set is a singleton only in particular situations. For this purpose, we first state a simple property:

Lemma 2.13 *Let (5–7) hold. Then, $\overline{\boldsymbol{d}}$ is a global minimizer of $\mathcal{E}$ if and only if $\overline{\boldsymbol{d}}$ is a constant unit vector.*

Proof Thanks to (5–7) the function $r \mapsto \widehat{\psi}(r) - r$ has a minimum at $r = 1$; moreover, $\widehat{\psi}(1) - 1 = 0$. Thus, $\mathcal{E}(\boldsymbol{d})$ is always nonnegative and $\mathcal{E}(\boldsymbol{d}) = 0$ if and only if $\nabla \boldsymbol{d} = 0$ a.e. in Ω and $|\boldsymbol{d}| = 1$ a.e. in Ω, whence the claim follows immediately. □

Our next result is of conditional type and states that, if the ω-limit set of $\boldsymbol{d}(t)$ contains at least one global minimizer $\overline{\boldsymbol{d}}$ of the free energy, then it has to coincide with the set $\{\overline{\boldsymbol{d}}\}$. This is a consequence of the facts that the set of global minimizers of the free energy is a $(d-1)$-dimensional smooth manifold and, on the other hand, the kernel of the linearized operator $z \mapsto -\Delta z + \partial_{\boldsymbol{d}} \boldsymbol{f}(\overline{\boldsymbol{d}})z$ is also a $(d-1)$-dimensional manifold. In other words, the so-called *normal hyperbolicity* condition is satisfied at $\overline{\boldsymbol{d}}$, which implies convergence of the whole trajectory to $\overline{\boldsymbol{d}}$.

Theorem 2.14 *Let the assumptions of* Theorem 2.3 *hold and let us assume that there exist a constant unit vector $\overline{\boldsymbol{d}} \in \mathbb{S}^{d-1}$ and a diverging sequence $\{t_n\}$ such that*

$$\lim_{t_n \nearrow +\infty} \boldsymbol{d}(t_n) = \overline{\boldsymbol{d}} \quad \text{weakly in } \boldsymbol{V}. \tag{49}$$

Then, $\omega - \lim \boldsymbol{d} = \{\overline{\boldsymbol{d}}\}$ and the whole trajectory $\boldsymbol{d}(t)$ converges to $\overline{\boldsymbol{d}}$ strongly in $\boldsymbol{H}$ as $t \nearrow \infty$. If, in addition, (42) holds, then $\boldsymbol{d}(t) \to \overline{\boldsymbol{d}}$ in $H^2(\Omega)^d$.

Remark 2.15 Let us note that the same convergence result for $\boldsymbol{d}$ in $H^2(\Omega)^d$ holds true in case $\delta > 0$ with periodic boundary conditions for $\boldsymbol{u}$ and $\boldsymbol{d}$.

The next results only hold in the case $\delta = 0$. Actually, their proofs rely on the maximum principle proved in Theorem 2.5. In this setting, convergence to a single equilibrium takes place if either the diffusion coefficient L in (13) is large enough, or the "initial energy" $\mathbb{E}_0 := \mathbb{E}(\boldsymbol{u}_0, \boldsymbol{d}_0)$ (cf. 21–22) is small enough (in other words, if the initial datum $\boldsymbol{d}_0$ is sufficiently close to the set of global minimizers). Indeed, we can prove the following two results:

Theorem 2.16 *Let the assumptions of* Theorem 2.5 *hold and, in particular, let $\delta = 0$. Assume that L in (13) satisfies*

$$L > c_\Omega^2, \text{ where } c_\Omega \text{ is the best constant in the Poincaré-Wirtinger inequality.} \tag{50}$$

Then, the ω-limit set of any weak solution starting from $(\boldsymbol{u}_0, \boldsymbol{d}_0)$ consists of a single point $(\boldsymbol{0}, \boldsymbol{d}_\infty)$.

Theorem 2.17 *Let the assumptions of* Theorem 2.5 *hold and, in particular, let* $\delta = 0$. *Assume that there exist* $\kappa > 0$ *and* $\sigma \geq 1$ *such that*

$$\widehat{\psi}(r) - r \geq \kappa(1-r)^{\sigma} \quad \forall\, r \in [0,1]. \tag{51}$$

Then, there exists $\epsilon > 0$ *such that, if* $(\boldsymbol{u}_0, \boldsymbol{d}_0)$ *satisfy* $\mathbb{E}_0 \leq \epsilon$, *then, the* ω*-limit set of any weak solution starting from* $(\boldsymbol{u}_0, \boldsymbol{d}_0)$ *consists of a single point* $(\boldsymbol{0}, \boldsymbol{d}_\infty)$.

3 Proofs

All proofs will be presented in the case $d = 3$, the case $d = 2$ being clearly simpler.

3.1 Proof of Theorem 2.7

Energy estimate. We test (10) by $\boldsymbol{u}$ and (13) by $-L\Delta\boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d})$. Performing standard computations and using, in particular, the incompressibility constraint (12), we readily obtain the energy inequality (23). In particular, we get that the function $t \mapsto \mathbb{E}(\boldsymbol{u}(t), \boldsymbol{d}(t))$ is nonincreasing, whence it tends to some (finite) value $\mathbb{E}_\infty$. Moreover, thanks to (27–28), we get

$$\mathbb{E}_\infty - \mathbb{E}_0 = -\int_0^{+\infty} \mathcal{D}(s)\ \mathrm{d}s \leq 0, \tag{52}$$

where $\mathcal{D}$ denotes the sum of the *dissipative terms*, namely

$$\mathcal{D} := \|-L\Delta\boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d})\|^2 + \nu\|\nabla\boldsymbol{u}\|^2. \tag{53}$$

We deduce from the energy inequality (23 and 5–7) that

$$\boldsymbol{u} \in L^\infty(0,\infty;\boldsymbol{H}) \cap L^2(0,\infty;\boldsymbol{V}_0), \tag{54}$$

$$\boldsymbol{d} \in L^\infty(0,\infty;\boldsymbol{V}), \tag{55}$$

$$-L\Delta\boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d}) \in L^2(0,\infty;\boldsymbol{H}). \tag{56}$$

Relations (54, 55) imply

$$\boldsymbol{u}\cdot\nabla\boldsymbol{d} - \delta\boldsymbol{d}\cdot\nabla\boldsymbol{u} \in L^2(0,\infty;L^{3/2}(\Omega)^d) \tag{57}$$

which, together with (56), yield (cf. 13)

$$\boldsymbol{d}_t \in L^2(0,\infty;L^{3/2}(\Omega)^d). \tag{58}$$

Application of the Łojasiewicz inequality. Our aim is to show that there exists $T > 0$ such that

$$\boldsymbol{d}_t \in L^1(T,\infty;L^{3/2}(\Omega)^d), \tag{59}$$

which implies convergence of $\boldsymbol{d}$ in $L^{3/2}(\Omega)^d$. The pre-compactness of the trajectory in $\boldsymbol{H}$ then concludes the proof of Theorem 2.7.

To this end, we first realize that there exists a constant C such that

$$\|\boldsymbol{u}(t)\|^2 \leq C\|\nabla\boldsymbol{u}(t)\|^{\frac{1}{1-\theta}} \quad \text{for a.a. } t > 0, \tag{60}$$

where $\theta \in (0, \frac{1}{2})$ is the same as in (45). Indeed, if $\|\nabla\boldsymbol{u}\| \leq 1$, then (60) follows by the Poincaré inequality, and if $\|\nabla\boldsymbol{u}\| \geq 1$, the interpolation between $\boldsymbol{V}_0$ and $\boldsymbol{V}_0'$ together with the boundedness of $\boldsymbol{u}$ in $\boldsymbol{V}'$ gives the same estimate.

Now, let $\boldsymbol{d}_\infty$ be an element of the ω-limit set of $\boldsymbol{d}$. Then, integrating (23) from 0 to $+\infty$, we infer that

$$\mathcal{D} \in L^1(0,\infty), \tag{61}$$

and, from Lemma 2.12 and (60), we get

$$\int_t^{+\infty} \mathcal{D}(s)\ \mathrm{d}s = \mathcal{E}(\boldsymbol{d}(t)) - \mathcal{E}(\boldsymbol{d}_\infty) + \frac{1}{2}\|\boldsymbol{u}(t)\|^2 \le C\mathcal{D}(t)^{\frac{1}{2(1-\theta)}}, \tag{62}$$

for all $t > 0$ such that (47) holds. Denoting by $\mathcal{M}$ this set, we obtain that $\mathcal{D}^{1/2} \in L^1(\mathcal{M})$, see [10, Lemma 7.1]. Then also

$$\boldsymbol{u} \in L^1(\mathcal{M}; \boldsymbol{V}_0), \quad -L\Delta\boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d}) \in L^1(\mathcal{M}; \boldsymbol{H}), \tag{63}$$

and so, taking into account the growth of $\boldsymbol{f}$, we get

$$\boldsymbol{u}\cdot\nabla\boldsymbol{d} - \delta\boldsymbol{d}\cdot\nabla\boldsymbol{u} \in L^1(\mathcal{M}, L^{\frac{3}{2}}(\Omega)^d),$$

which implies (cf. 13)

$$\boldsymbol{d}_t \in L^1(\mathcal{M}, L^{\frac{3}{2}}(\Omega)^d). \tag{64}$$

This fact, combined with the pre-compactness of the trajectory of $\boldsymbol{d}$ in $\boldsymbol{H}$ and a simple contradiction argument (see [10]), yields the existence of a large T such that (cf. Lemma 2.12)

$$\|\boldsymbol{d}(t) - \boldsymbol{d}_\infty\| < \epsilon, \quad \forall t \ge T.$$

In other words, the solution $\boldsymbol{d}$ remains in the $\epsilon-$neighbourhood of $\boldsymbol{d}_\infty$ in the space $\boldsymbol{H}$ for $t \ge T$, and Lemma 2.12 applies to $\boldsymbol{d}(t)$ in the whole interval (T,∞). In other words, $\mathcal{M} \supset (T,\infty)$, and (59) holds. This fact, together with pre-compactness of the trajectory yields

$$\boldsymbol{d}(t) \to \boldsymbol{d}_\infty \quad \text{strongly in } \boldsymbol{H}. \tag{65}$$

Theorem 2.7 has been proved.

3.2 Proof of Theorem 2.9

The proof follows the lines of the argument developed in [3, Sect. 3.2]. In particular, we can derive the differential inequality for the dissipative term $\mathcal{D}$ defined in (53) (cf. [3, formula (8)], which is still valid with our boundary conditions in case $\delta = 0$ or with periodic boundary conditions in case $\delta > 0$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} \le C_*\left(\mathcal{D}^3 + 1\right), \tag{66}$$

where the computable constant C_* depends on the parameters of the problem and on the "initial energy" $\mathbb{E}_0$, but is independent of time. We point out that the formal computations in the proof of (66) are valid for the classical solutions, but the result can be justified by a proper approximation.

Next, we show that there is $T_1 > 0$ such that $\mathcal{D} \in L^\infty(T_1,\infty)$. To prove this, we consider the differential inequality

$$y' \le C_*\left(y^3 + 1\right), \quad y(t_0) = 1. \tag{67}$$

Then, there exist (a small) τ (independent of t_0) and (a large) $K > 0$ such that the solution y satisfies

$$\|y\|_{C^0([t_0,t_0+\tau])} \leq K. \tag{68}$$

On the other hand, according to (61), we have

$$\lim_{t\nearrow+\infty} \int_t^{+\infty} \mathcal{D}(s)\ \mathrm{d}s = 0. \tag{69}$$

Thus, for any $\epsilon > 0$ there exists $T > 0$ such that

$$\int_T^{+\infty} \mathcal{D}(s)\ \mathrm{d}s \leq \epsilon. \tag{70}$$

Choosing $\epsilon = \tau/2$ and T correspondingly, we obtain that, for all $t \geq T$, there exists $t_0 \in [t, t+\tau/2]$ such that

$$\mathcal{D}(t_0) \leq \frac{2}{\tau} \int_t^{t+\tau/2} \mathcal{D}(s)\ \mathrm{d}s \leq \frac{2\epsilon}{\tau} = 1. \tag{71}$$

Comparing solutions of (66 and 67) and recalling the choice of t_0, we get from (68) that

$$|\mathcal{D}(s)| \leq K \quad \forall\, s \in [T+\tau/2, +\infty). \tag{72}$$

Setting $T_1 := T + \tau/2$, we deduce from (66) that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}$ is bounded on (T_1, ∞), which together with (61) yields

$$\mathcal{D}(t) \to 0 \ \text{ as } t \to \infty. \tag{73}$$

This implies (using the Poincaré inequality) that,

$$\boldsymbol{u}(t) \to 0, \quad \text{strongly in } \boldsymbol{V}. \tag{74}$$

Taking into account the growth of $\boldsymbol{f}$, we get from (72)

$$\|\boldsymbol{d}\|_{L^\infty(T_1,\infty;H^2(\Omega)^d)} \leq K. \tag{75}$$

To show that $\boldsymbol{d}$ converges to a single point $\boldsymbol{d}_\infty$, we make again use of the Łojasiewicz inequality (45). The same argument applies this time to the strong solution and time $t \geq T_1$. This gives

$$\boldsymbol{d}_t \in L^1(T, \infty; \boldsymbol{H}) \ \text{ for some } T > T_1. \tag{76}$$

It follows that

$$\boldsymbol{d}(t) \to \boldsymbol{d}_\infty \quad \text{strongly in } \boldsymbol{H}. \tag{77}$$

Moreover, by (75) and the growth conditions on $\boldsymbol{f}$,

$$\boldsymbol{f}(\boldsymbol{d}(t)) \to \boldsymbol{f}(\boldsymbol{d}_\infty) \ \text{ strongly in } \boldsymbol{H}, \tag{78}$$

whence, using (73 and 74),

$$\Delta\boldsymbol{d}(t) \to \Delta\boldsymbol{d}_\infty \ \text{ strongly in } \boldsymbol{H}, \tag{79}$$

which concludes the proof.

3.3 Proof of Theorem 2.14

In this section, we show that the energy functional $\mathcal{E}$ satisfies the Łojasiewicz inequality (45) with the exponent $\frac{1}{2}$. Then, arguing as in the proof of Theorem 2.7, we obtain the strong convergence of $\boldsymbol{d}$ in $\boldsymbol{H}$. Moreover, if (42) holds (or we have periodic boundary conditions, cf. Remark 2.10), we have the strong convergence in $H^2(\Omega)^d$ (cf. the proof of Theorem 2.9).

Let us consider the linearized problem associated to (29) at the element $\overline{\boldsymbol{d}}$ of the ω-limit set, i.e.,

$$\mathcal{L}(\overline{\boldsymbol{d}})z := -L\Delta z + \psi(|\overline{\boldsymbol{d}}|^2)z + 2\psi'(|\overline{\boldsymbol{d}}|^2)(\overline{\boldsymbol{d}} \otimes \overline{\boldsymbol{d}})z - z = 0, \quad \partial_{\boldsymbol{n}} z = 0 \text{ on } \Gamma. \tag{80}$$

Let $\overline{\boldsymbol{d}}$ be a global minimizer of $\mathcal{E}$, i.e., a constant unit vector (by Lemma 2.13). We aim to apply the result proved by Simon and reported in [4, Cor. 3.12]. To this end, we introduce the following notation:

$$\begin{aligned} &U \text{ is a } \boldsymbol{V}\text{-neighbourhood of } \overline{\boldsymbol{d}} \in \boldsymbol{V},\\ &\boldsymbol{\mathcal{V}}_0 \text{ is the kernel of } \mathcal{L}(\overline{\boldsymbol{d}}),\\ &S_0 = \{\boldsymbol{d} \in \boldsymbol{V};\ \mathcal{E}'(\boldsymbol{d}) = 0\},\\ &S = \{\boldsymbol{h} \in U;\ \mathcal{E}'(\overline{\boldsymbol{d}} + \boldsymbol{h}) \in \boldsymbol{\mathcal{V}}_0'\}. \end{aligned}$$

In our situation, [4, Cor. 3.12] reads as follows:

Lemma 3.1 *Let $\overline{\boldsymbol{d}} \in S_0$ and assume the following hypotheses:*

(i) *The kernel $\boldsymbol{\mathcal{V}}_0$ of the linearization $\mathcal{L}(\overline{\boldsymbol{d}})$ is a complemented subspace of $\boldsymbol{V}$, i.e., there exists a projection $P \in \mathcal{B}(\boldsymbol{V})$ such that $\boldsymbol{\mathcal{V}}_0 = \mathrm{Rg}P$.*

(ii) *There exists a neighbourhood U of $\overline{\boldsymbol{d}}$ in $\boldsymbol{V}$ such that $\mathcal{E}' \in C^1(U, \boldsymbol{V}')$. Moreover, the range of $\mathcal{L}(\overline{\boldsymbol{d}})$ coincides with $\boldsymbol{\mathcal{V}}_1'$, the space of the elements of $\boldsymbol{V}'$ belonging to the the kernel of the adjoint projection $P' \in \mathcal{B}(\boldsymbol{V}')$.*

(iii) *$(S_0 - \overline{\boldsymbol{d}}) \cap S$ is a neighbourhood of 0 in the critical manifold S.*

Then $\mathcal{E}$ satisfies the Łojasiewicz inequality near $\overline{\boldsymbol{d}}$ with the exponent $\theta = \frac{1}{2}$.

To verify the assumption (i), we test (80) by z and use the condition $\psi(1) = 1$ to obtain

$$L\|\nabla z\|^2 + 2\int_\Omega \psi'(1)|\overline{\boldsymbol{d}} \cdot z|^2 = 0. \tag{81}$$

Hence, taking into account the last condition in (5), we get

$$\nabla z = 0 \quad \text{and} \quad \overline{\boldsymbol{d}} \cdot z = 0 \quad \text{a.e. in } \Omega. \tag{82}$$

Consequently, any solution z to (80), i.e., any element of the kernel, is a constant vector orthogonal to $\overline{\boldsymbol{d}}$ (conversely, it is apparent that any such vector is a solution to (80)). Thus, the kernel of the linearized operator $\mathcal{L}(\overline{\boldsymbol{d}})$ is a $(d-1)$-dimensional plane orthogonal to $\overline{\boldsymbol{d}}$ and containing the origin, which trivially permits to define the projection P.

The first condition in (ii) is obvious since $\boldsymbol{f}$ is C^1 and, by hypotheses, has at most cubic growth. To verify the second condition, we observe that $\boldsymbol{\mathcal{V}}_1'$ is the subspace of $\boldsymbol{V}'$ consisting of the elements that are orthogonal (w.r.t. the duality between $\boldsymbol{V}'$ and $\boldsymbol{V}$) to the plane $\boldsymbol{\mathcal{V}}_0$. Then, computing $\langle \mathcal{L}(\overline{\boldsymbol{d}})z, \boldsymbol{v}_0\rangle$ for generic $z \in \boldsymbol{V}$ and $\boldsymbol{v}_0 \in \boldsymbol{\mathcal{V}}_0$, we obtain (cf. (80) and recall that $\psi(1) = 1$), using (82),

$$\langle \mathcal{L}(\overline{\boldsymbol{d}})z, \boldsymbol{v}_0\rangle = 2\int_\Omega \psi'(|\overline{\boldsymbol{d}}|^2)(\overline{\boldsymbol{d}} \cdot z)(\overline{\boldsymbol{d}} \cdot \boldsymbol{v}_0) = 0, \tag{83}$$

the last equality following from the fact that $\boldsymbol{v}_0 \perp \overline{\boldsymbol{d}}$. Thus, $\mathcal{L}(\overline{\boldsymbol{d}})\boldsymbol{V} \subset \boldsymbol{\mathcal{V}}_1'$. To show the converse inclusion, we choose $\boldsymbol{\zeta} \in \boldsymbol{\mathcal{V}}_1'$ and prove that there exists at least one $\boldsymbol{z} \in \boldsymbol{V}$ such that

$$\mathcal{L}(\overline{\boldsymbol{d}})\boldsymbol{z} = -L\Delta\boldsymbol{z} + 2\psi'(1)(\overline{\boldsymbol{d}} \otimes \overline{\boldsymbol{d}})\boldsymbol{z} = \boldsymbol{\zeta}, \quad \partial_{\boldsymbol{n}}\boldsymbol{z} = 0 \text{ on } \Gamma. \tag{84}$$

This can be seen by approximation. Actually, it is clear that, for any $k \in \mathbb{N}$, there is a solution $\boldsymbol{z}_k$ to

$$-L\Delta\boldsymbol{z}_k + k^{-1}\boldsymbol{z}_k + 2\psi'(1)(\overline{\boldsymbol{d}} \otimes \overline{\boldsymbol{d}})\boldsymbol{z}_k = \boldsymbol{\zeta}, \quad \partial_{\boldsymbol{n}}\boldsymbol{z} = 0 \text{ on } \Gamma. \tag{85}$$

Testing by $\boldsymbol{z}_k$, we have

$$L\|\nabla\boldsymbol{z}_k\|^2 + k^{-1}\|\boldsymbol{z}_k\|^2 + 2\psi'(1)\|\overline{\boldsymbol{d}} \cdot \boldsymbol{z}_k\|^2 = (\boldsymbol{\zeta}, \boldsymbol{z}_k) = (\boldsymbol{\zeta}, \boldsymbol{z}_k - P\boldsymbol{z}_k). \tag{86}$$

Indeed, $\boldsymbol{\zeta} \perp P\boldsymbol{z}_k$ by assumption. Using the Poincaré-Wirtinger inequality it is then apparent that the right-hand side can be estimated. Then, standard methods permit to check that $\boldsymbol{z}_k$ tend to a solution $\boldsymbol{z}$ to (84), as desired.

The third assumption is satisfied because $0 \in S_0 - \overline{\boldsymbol{d}} \subset S$, and both $S_0 - \overline{\boldsymbol{d}}$ and S have the same dimension (see [4, Proposition 3.6]).

Lemma 3.1 then yields that the Łojasiewicz inequality (45) holds near $\overline{\boldsymbol{d}}$ with the exponent $\theta = 1/2$. If $\delta = 0$ or in case of periodic boundary conditions, repeating the computations leading to (75), we obtain the strong convergence of $\boldsymbol{d}$ in $\boldsymbol{H}$, and, proceeding as in (78, 79), the strong convergence in $H^2(\Omega)^d$, which completes the proof of Theorem 2.14.

3.4 Proof of Theorem 2.16

Let $\boldsymbol{d}_\infty$ be an element of the ω-limit set of the $\boldsymbol{d}$-component of some weak solution. Then, by Theorem 2.6, $\boldsymbol{d}_\infty$ solves the stationary problem, which we rewrite as

$$-L\Delta\boldsymbol{d}_\infty + \left(\psi(|\boldsymbol{d}_\infty|^2) - 1\right)\boldsymbol{d}_\infty = 0. \tag{87}$$

Testing (87) by $\boldsymbol{d}_\infty - (\boldsymbol{d}_\infty)_\Omega$, where $(\boldsymbol{d}_\infty)_\Omega = \int_\Omega \boldsymbol{d}_\infty$, we get

$$L\|\nabla\boldsymbol{d}_\infty\|^2 + \int_\Omega \psi(|\boldsymbol{d}_\infty|^2)\boldsymbol{d}_\infty \cdot (\boldsymbol{d}_\infty - (\boldsymbol{d}_\infty)_\Omega) = \|\boldsymbol{d}_\infty - (\boldsymbol{d}_\infty)_\Omega\|^2 \le c_\Omega^2\|\nabla\boldsymbol{d}_\infty\|^2, \tag{88}$$

where c_Ω is the (best) constant in the Poincaré–Wirtinger inequality. Thus, being L is large (precisely, we need $L > c_\Omega^2$), the latter term can be controlled.

In what follows, we denote

$$\Xi(\boldsymbol{d}) = \frac{1}{2}\widehat{\psi}(|\boldsymbol{d}|^2). \tag{89}$$

A direct check (e.g., computing the Hessian matrix) shows that Ξ is convex, and we notice that $\partial_{\boldsymbol{d}}\Xi(\boldsymbol{d}) = \psi(|\boldsymbol{d}|^2)\boldsymbol{d}$. Thus, we have

$$\begin{aligned}\int_\Omega \psi(|\boldsymbol{d}_\infty|^2)\boldsymbol{d}_\infty \cdot (\boldsymbol{d}_\infty - (\boldsymbol{d}_\infty)_\Omega) &= \left(\partial_{\boldsymbol{d}}\Xi(\boldsymbol{d}_\infty), \boldsymbol{d}_\infty - (\boldsymbol{d}_\infty)_\Omega\right)_H \\ &\ge \int_\Omega \left(\Xi(\boldsymbol{d}_\infty) - \Xi((\boldsymbol{d}_\infty)_\Omega)\right) \\ &= \int_\Omega \Xi(\boldsymbol{d}_\infty) - \Xi\left(\int_\Omega \boldsymbol{d}_\infty\right) \ge 0,\end{aligned} \tag{90}$$

where the latter inequality follows from Jensen's inequality.

From (88) to (90), we obtain that $\boldsymbol{d}_\infty$ is a constant vector. Thus, taking into account that ψ is monotone and $\psi(1) = 1$, we readily obtain from Eq. 87 that either $\boldsymbol{d}_\infty = 0$ or $|\boldsymbol{d}_\infty| = 1$. Hence the set of stationary solutions is disconnected and consists of the isolated point and the two-dimensional manifold. Consequently, either Theorem 2.14 applies, or the whole trajectory tends to 0. In both cases, the ω-limit set is a singleton, as desired.

3.5 Proof of Theorem 2.17

Let us first note that, by (52 and 51), we have

$$\epsilon \geq \mathbb{E}_0 \geq \mathbb{E}_\infty = \frac{1}{2}\int_\Omega \left(L|\nabla \boldsymbol{d}_\infty|^2 + \widehat{\psi}(|\boldsymbol{d}_\infty|^2) - |\boldsymbol{d}_\infty|^2\right). \tag{91}$$

Rewriting the stationary problem (87) and testing it by $\boldsymbol{d}_\infty$, we obtain

$$L\|\nabla \boldsymbol{d}_\infty\|^2 + \int_\Omega \left(\psi(|\boldsymbol{d}_\infty|^2)|\boldsymbol{d}_\infty|^2 - |\boldsymbol{d}_\infty|^2\right) = 0. \tag{92}$$

Dividing (92) by 2 and subtracting the result from (91), we obtain

$$\frac{1}{2}\int_\Omega \left(\widehat{\psi}(|\boldsymbol{d}_\infty|^2) - \psi(|\boldsymbol{d}_\infty|^2)|\boldsymbol{d}_\infty|^2\right) \leq \epsilon. \tag{93}$$

On the other hand, thanks to (6, 5 and 51),

$$\frac{1}{2}\int_\Omega \left(\widehat{\psi}(|\boldsymbol{d}_\infty|^2) - \psi(|\boldsymbol{d}_\infty|^2)|\boldsymbol{d}_\infty|^2\right) \geq \frac{1}{2}\int_\Omega \left(\widehat{\psi}(|\boldsymbol{d}_\infty|^2) - |\boldsymbol{d}_\infty|^2\right) \geq \frac{\kappa}{2}\int_\Omega \left||\boldsymbol{d}_\infty|^2 - 1\right|^\sigma, \tag{94}$$

where also the maximum principle (24) has been used. Thus,

$$\left\||\boldsymbol{d}_\infty|^2 - 1\right\|_{L^1(\Omega)} \leq \left\||\boldsymbol{d}_\infty|^2 - 1\right\|_{L^\sigma(\Omega)} \leq \left(\frac{2\epsilon}{\kappa}\right)^{1/\sigma}. \tag{95}$$

To proceed, we notice that, by standard elliptic regularity results applied to (29), there exists a constant $K_0 > 0$ such that, for any solution $\overline{\boldsymbol{d}}$ of (29) it holds

$$\left\|\overline{\boldsymbol{d}}\right\|_{H^2(\Omega)} \leq K_0. \tag{96}$$

Consequently, for some $K > 0$ depending on K_0, we have

$$\left\||\overline{\boldsymbol{d}}|^2 - 1\right\|_{H^2(\Omega)} \leq K. \tag{97}$$

In particular, $\boldsymbol{d}_\infty$ satisfies (97). Thus, by the Gagliardo–Nirenberg interpolation inequality (we refer, for simplicity, to the case $d = 3$, the case $d = 2$ is even better),

$$\begin{aligned}\left\||\boldsymbol{d}_\infty|^2 - 1\right\|_{L^\infty(\Omega)} &\leq \left\||\boldsymbol{d}_\infty|^2 - 1\right\|_{H^2(\Omega)}^{6/7}\left\||\boldsymbol{d}_\infty|^2 - 1\right\|_{L^1(\Omega)}^{1/7} + \left\||\boldsymbol{d}_\infty|^2 - 1\right\|_{L^1(\Omega)} \\ &\leq K^{6/7}\left(\frac{2\epsilon}{\kappa}\right)^{1/7\sigma} + \left(\frac{2\epsilon}{\kappa}\right)^{1/\sigma} < \eta,\end{aligned} \tag{98}$$

where $\eta > 0$ is a small constant to be chosen later and the last inequality holds provided that ϵ is small enough.

To conclude the proof, we set

$$\alpha := \psi(|\boldsymbol{d}_\infty|^2) - 1 \tag{99}$$

and notice that $\alpha \leq 0$ because of (5–6). Moreover, $\boldsymbol{d}_\infty$ can be interpreted as a solution of the linear elliptic system

$$-L\Delta \boldsymbol{d}_\infty + \alpha \boldsymbol{d}_\infty = 0 \ \text{ in } \ \Omega, \quad \partial_{\boldsymbol{n}} \boldsymbol{d}_\infty = 0 \ \text{ on } \ \Gamma. \tag{100}$$

Testing the above equation by 1, we obtain

$$\int_\Omega \alpha \boldsymbol{d}_\infty = 0. \tag{101}$$

Then, multiplying (100) by $\boldsymbol{d}_\infty - (\boldsymbol{d}_\infty)_\Omega$ and using (101), we infer

$$\begin{aligned} L\|\nabla \boldsymbol{d}_\infty\|^2 &= -\int_\Omega \alpha\, |\boldsymbol{d}_\infty - (\boldsymbol{d}_\infty)_\Omega|^2 - (\boldsymbol{d}_\infty)_\Omega \cdot \int_\Omega \alpha\, (\boldsymbol{d}_\infty - (\boldsymbol{d}_\infty)_\Omega) \\ &\leq c_\Omega^2 \|\alpha\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{d}_\infty\|^2 + |(\boldsymbol{d}_\infty)_\Omega|^2 \int_\Omega \alpha. \end{aligned} \tag{102}$$

The latter term is nonpositive, while the first term on the right-hand side can be controlled provided that η is small enough. Indeed,

$$\begin{aligned} \|\alpha\|_{L^\infty(\Omega)} &= \left\| \psi(|\boldsymbol{d}_\infty|^2) - 1 \right\|_{L^\infty(\Omega)} = \left\| \psi(|\boldsymbol{d}_\infty|^2) - \psi(1) \right\|_{L^\infty(\Omega)} \\ &\leq c_\psi \left\| |\boldsymbol{d}_\infty|^2 - 1 \right\|_{L^\infty(\Omega)} \leq c_\psi \eta, \end{aligned} \tag{103}$$

thanks to (98) and (7). If $L > c_\Omega^2 c_\psi \eta$, we see, in the same way as above, that $\boldsymbol{d}_\infty$ is a constant unit vector. Finally, we take ϵ such that (98) holds, and the proof follows again by applying Theorem 2.14.

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References

1. Blesgen, T.: A generalization of the Navier-Stokes equations to two-phase flow. J. Phys. D Appl. Phys. **32**, 1119–1123 (1999)
2. Cavaterra, C., Rocca, E.: On a 3D isothermal model for nematic liquid crystals accounting for stretching terms, preprint arXiv:1107.3947v1, 1–14 (2011)
3. Climent-Ezquerra B.Guillén-Gonzáles, F., Rodríguez-Bellido, M.A.: Stability for nematic liquid crystals with stretching terms. Int. J. Bifurcation Chaos **20**, 2937–2942 (2010)
4. Chill, R.: On the Łojasiewicz-Simon gradient inequality. J. Funct Anal **201**, 572–601 (2003)
5. Chandrasekhar, S.: Liquid Crystals. Cambridge U. Press, Cambridge (1977)
6. Gennes, P.G.de : The Physics of Liquid Crystals. Oxford Univ. Press, London (1974)
7. Ericksen, J.: Liquid crystals with variable degree of orientation. Arch. Ration. Mech. Anal. **113**, 97–120 (1991)
8. Feireisl, E., Rocca, E., Schimperna, G.: On a non-isothermal model for nematic liquid crystals. Nonlinearity **24**, 243–257 (2011)
9. Feireisl, E., Frémond, M., Rocca, E., Schimperna, G.: A new approach to non-isothermal models for nematic liquid crystals, preprint arXiv:1104.1339v1, 1–21 (2011)

10. Feireisl, E., Simondon, F.: Convergence for semilinear degenerate parabolic equations in several space dimensions. J. Dynam. Differ. Equ. **12**, 647–673 (2000)
11. Gajewski, H., Griepentrog, J.A.: A descent method for the free energy of multicomponent systems. Discrete Contin. Dyn. Syst. **15**, 505–528 (2006)
12. Grasselli, M., Wu, H.: Finite-dimensional global attractor for a system modeling the 2D nematic liquid crystal flow. Z. Angew. Math. Phys. **62**, 979–992 (2011)
13. Leslie, F.M.: Some constitutive equations for liquid crystals. Arch. Ration. Mech. Anal. **28**, 265–283 (1968)
14. Lin, F.-H., Liu, C.: Nonparabolic dissipative systems modeling the flow of liquid crystals. Comm. Pure Appl. Math. **48**, 501–537 (1995)
15. Lin, F.-H., Liu, C.: Existence of solutions for the Ericksen-Leslie system. Arch. Ration. Mech. Anal. **154**, 135–156 (2000)
16. Lions, J.-L.: Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires. (French), Dunod, Gauthier-Villars, Paris (1969)
17. Liu, C., Shen, J.: On liquid crystal flows with free-slip boundary conditions. Discrete Contin. Dynam. Syst. **7**, 307–318 (2001)
18. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. **146**(4), 65–96 (1987)
19. Sun, H., Liu, C.: On energetic variational approaches in modeling the nematic liquid crystal flows. Discrete Contin. Dyn. Syst. **23**, 455–475 (2009)
20. Wu, H., Xu, X., Liu, C.: Asymptotic behavior for a nematic liquid crystal model with different Kkinematic transport properties, preprint arXiv:0901.1751v2, 1–26 (2010)