ALGEBRAIC STRUCTURES ON GRAPH COHOMOLOGY

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Abstract. We define algebraic structures on graph cohomology and prove that they correspond to algebraic structures on the cohomology of the spaces of imbeddings of S^1 or $\mathbb{R}$ into $\mathbb{R}^n$. As a corollary, we deduce the existence of an infinite number of nontrivial cohomology classes in $\operatorname{Imb}(S^1, \mathbb{R}^n)$ when n is even and greater than 3. Finally, we give a new interpretation of the anomaly term for the Vassiliev invariants in $\mathbb{R}^3$.

1. Introduction

In this paper we consider the spaces $\operatorname{Imb}(S^1, \mathbb{R}^n)$ of imbeddings of the circle into $\mathbb{R}^n$ and the spaces $\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)$ of imbeddings of the real line into $\mathbb{R}^n$ with fixed behavior at infinity, namely, imbeddings that coincide with a fixed imbedded line in $\mathbb{R}^n$ outside a compact subset.

When $n > 3$ one can define, using configuration space integrals, chain maps from certain graph complexes $(\mathcal{D}_n^{k,m}, \delta)$ or $(\mathcal{L}_n^{k,m}, \delta)$ to the de Rham complexes of the spaces of imbeddings:

$$I\colon (\mathcal{D}_n^{k,m}, \delta) \to (\Omega^{(n-3)k+m}(\operatorname{Imb}(S^1, \mathbb{R}^n)), \mathrm{d}), \tag{1.1}$$

$$I\colon (\mathcal{L}_n^{k,m}, \delta) \to (\Omega^{(n-3)k+m}(\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)), \mathrm{d}). \tag{1.2}$$

We will use the same symbol I also for the induced maps in cohomology. Our main interest is to determine which algebraic structures are preserved by the maps I.

On the one side, in fact, the de Rham complexes are differential graded commutative algebras. Moreover, there exist a multiplication $\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n) \times \operatorname{Imb}(\mathbb{R}, \mathbb{R}^n) \to \operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)$ given by attaching the end of the first imbedding to the beginning of the second, and rescaling. This operation is non-associative but gives rise, together with the cup product, to a Hopf algebra structure on $H(\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n))$ for $n > 3$. Indeed, as shown in [4], the multiplication is part of an action of the operad of little 2-cubes on $\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)$.

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On the other side, we show in Section 3 that $(\mathcal{D}_n^{k,m}, \delta)$ and $(\mathcal{L}_n^{k,m}, \delta)$ (whose definitions are recalled in Section 2) can be endowed, respectively, with a differential algebra and a differential Hopf algebra structure. The existence of these algebraic structures implies in particular the existence of infinitely many nontrivial classes of trivalent cocycles of even type (see Section 4), and, as a consequence of the results of [5], of infinitely many nontrivial elements of $H(\operatorname{Imb}(S^1, \mathbb{R}^n))$, for every even $n \geq 4$.

The central result of the paper, contained in Section 6, is that, for $n > 3$, the maps I in cohomology respect all the above algebraic structures.

Finally, in Section 7 we consider the case $n = 3$ and $m = 0$ (the case of Vassiliev invariants) and we give a new interpretation of the so-called "anomaly" term [3, 1] as the obstruction for the Bott–Taubes map from trivalent cocycles to Vassiliev invariants to be a coalgebra map.

As a final remark, we recall that there exist other approaches to study the cohomology of spaces of imbeddings based on graph cohomology [6, 7]. It would be interesting to understand our results in these other contexts.

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2. Graph cohomology

We briefly recall the definition of the *graph complexes* given in [5], Section 4. A graph consists of an *oriented circle* and many *edges* joining vertices. The vertices lying on the circle are called *external vertices*, those lying off the circle are called *internal vertices* and are required to be at least trivalent. We define the order $k \geq 0$ of a graph to be minus its Euler characteristic, and the degree $m \geq 0$ to be the deviation of the graph from being trivalent, namely:

$$k = e - v_i,$$
$$m = 2e - v_e - 3v_i,$$

where e is the number of edges, v_i the number of internal vertices and v_e the number of external vertices of the graph.

The graph complexes $(\mathcal{D}_n^{k,m}, \delta)$ depend only on the parity of n, and we will denote by $\mathcal{D}_o^{k,m}$ and $\mathcal{D}_e^{k,m}$ the real vector spaces generated by decorated graphs of order k and degree m, of odd and even type, respectively.

The type of a graph depends on the decoration which we put on it. By definition, the decoration in $\mathcal{D}_o^{k,m}$ is given by numbering all the vertices (with the convention that we first number the external vertices and then the internal ones) and orienting the edges. Then one takes the quotient by the following relations: a cyclic permutations of the external vertices or a permutation of the internal vertices multiplies the graph by the sign of the permutation, a reversal of an orientation of an edge produces a minus sign. An extra decoration is needed on edges connecting the same external vertex, namely an ordering of the two half-edges forming them. The decoration in $\mathcal{D}_e^{k,m}$ is given by numbering the external vertices and numbering the edges, while the relations are as follows: a cyclic permutations of the external vertices or a permutation of the edges multiplies the graph by the sign of the permutation.

Double lines and internal loops are not allowed, namely, in $\mathcal{D}_o^{k,m}$ and $\mathcal{D}_e^{k,m}$ we quotient by the subspace generated by all the graphs containing two edges joining the same pair of vertices and by all the graphs containing edges whose end-points are the same internal vertex.

By an *arc* we mean a piece of the oriented circle between two consecutive vertices, and by a *regular edge* we mean an edge with at least one internal end-point.

The coboundary operators $\delta_o \colon \mathcal{D}_o^{k,m} \to \mathcal{D}_o^{k,m+1}$ and $\delta_e \colon \mathcal{D}_e^{k,m} \to \mathcal{D}_e^{k,m+1}$ are linear operators, whose action on a graph Γ is given by the signed sum of all the graphs obtained from Γ by contracting, one at a time, all the regular edges and arcs of the graphs. The signs are as follows: if we contract an arc or edge connecting the vertex i with the vertex j, and oriented from i to j, then the sign is $(-1)^j$ if $j > i$ or $(-1)^{i+1}$ if $j < i$. If we contract an edge labeled by α, then the sign is $(-1)^{\alpha+1+v_e}$ where v_e is the number of external vertices of the graph. The decoration on Γ induces a decoration on the contracted graphs as follows: contraction of the edge between the vertex i and the vertex j produces a new vertex labeled by $\min(i,j)$ while the vertex labels greater than $\max(i,j)$ are rescaled by one. Similarly when the edge labeled by α is collapsed, the edge labels greater than α are rescaled by one.

The complexes $(\mathcal{L}_o^{k,m}, \delta_o)$ and $(\mathcal{L}_e^{k,m}, \delta_e)$ are defined in the same manner, except that instead of graphs with an oriented circle we consider graphs with an *oriented line*.

The cohomology groups with respect to the above coboundary operators will be denoted by $H^{k,m}(\mathcal{D}_o)$, $H^{k,m}(\mathcal{D}_e)$, $H^{k,m}(\mathcal{L}_o)$ and $H^{k,m}(\mathcal{L}_e)$. When we do not want to specify the parity of these complexes we simply write $\mathcal{D}^{k,m}$ or $\mathcal{L}^{k,m}$. Moreover we set $\mathcal{D} = \bigoplus_{k,m} \mathcal{D}^{k,m}$, $\mathcal{L} = \bigoplus_{k,m} \mathcal{L}^{k,m}$, $H(\mathcal{D}) = \bigoplus_{k,m} H^{k,m}(\mathcal{D})$ and $H(\mathcal{L}) = \bigoplus_{k,m} H^{k,m}(\mathcal{L})$

3. Algebraic structures on graphs

3.1. **Operations on $\mathcal{D}^{k,m}$.** Suppose that Γ_1 and Γ_2 are two graphs with $v_e(\Gamma_1)$ and $v_e(\Gamma_2)$ external vertices, respectively. The sets V_e^1 and V_e^2 of external vertices of Γ_1 and Γ_2 respectively, are cyclically ordered. A (V_e^1, V_e^2)-shuffle is a permutation of the set $V_e^1 \cup V_e^2$ respecting the cyclic order of V_e^1 and V_e^2. If σ is a (V_e^1, V_e^2)-shuffle, then we can combine Γ_1 and Γ_2 in a single graph $\Gamma_1 \bullet_\sigma \Gamma_2$ with $v_e(\Gamma_1) + v_e(\Gamma_2)$ external vertices by placing the legs of Γ_1 and Γ_2 into distinct external vertices on an oriented circle, according to σ.

We put the labels in $\Gamma_1 \bullet_\sigma \Gamma_2$ as follows: the vertices and edges which come from Γ_1 are numbered in the same way as in Γ_1. Next we number the vertices coming from Γ_2 by adding to the corresponding label of Γ_2 the number of labeled vertices of Γ_1 and the edges by adding to the corresponding label of Γ_2 the number of edges of Γ_1. If a vertex or edge is not labeled in Γ_1 or Γ_2, it remains unlabeled in $\Gamma_1 \bullet_\sigma \Gamma_2$ The multiplication $\Gamma_1 \bullet \Gamma_2$ of two graphs Γ_1 and Γ_2 is then defined as

$$\Gamma_1 \bullet \Gamma_2 = (-1)^{\lambda(\Gamma_1,\Gamma_2)} \sum_\sigma \Gamma_1 \bullet_\sigma \Gamma_2$$

where the sum is taken over all the (V_e^1, V_e^2)-shuffles and

$$\lambda(\Gamma_1, \Gamma_2) = \begin{cases} v_e(\Gamma_2)\, e(\Gamma_1) & \text{for } \Gamma_i \in \mathcal{D}_e \\ 0 & \text{for } \Gamma_i \in \mathcal{D}_o. \end{cases} \tag{3.1}$$

In the above formula, $e(\Gamma_1)$ denotes the number of edges of Γ_1.

Extending this product to $\mathcal{D}$ by linearity, we obtain an associative operation called the *shuffle product*:

$$\bullet\colon \mathcal{D}^{k_1,m_1} \otimes \mathcal{D}^{k_2,m_2} \to \mathcal{D}^{k_1+k_2,m_1+m_2}.$$

We define a new grading $|\cdot|$ of the graphs generating $\mathcal{D}_e^{k,m}$ and $\mathcal{D}_o^{k,m}$, by considering the total number of labels of the graph:

$$|\Gamma| \equiv \begin{cases} e(\Gamma) + v_e(\Gamma) & \text{for } \Gamma_i \in \mathcal{D}_e \\ v(\Gamma) & \text{for } \Gamma_i \in \mathcal{D}_o. \end{cases}$$

where $v(\Gamma)$ is the total number of vertices of Γ. Modulo 2, the new grading is equal to $k + m$ in the even case and to m in the odd case.

With respect to the new grading the integration maps I will be grading-preserving modulo 2. The shuffle product is graded commutative with respect to this new grading. In fact one can easily see that $\Gamma_1 \bullet \Gamma_2 = (-1)^{|\Gamma_1||\Gamma_2|}\Gamma_1 \bullet \Gamma_2$. Leibnitz rule also holds between the coboundary operator δ, which has degree -1 with respect to the new grading, and the shuffle product, which has degree 0. More precisely

Proposition 3.1. $\delta(\Gamma_1 \bullet \Gamma_2) = \delta(\Gamma_1) \bullet \Gamma_2 + (-1)^{|\Gamma_1|}\Gamma_1 \bullet \delta(\Gamma_2)$ *for every* $\Gamma_i \in \mathcal{D}$.

Proof. Let σ be a (V_e^1, V_e^2)-shuffle. When we apply the coboundary operator δ to one of the graphs of $\Gamma_1 \bullet_\sigma \Gamma_2$ we can either collapse an edge of Γ_1, or collapse two external vertices of Γ_1, or collapse an edge of Γ_2, or collapse two external vertices of Γ_2, or collapse an external vertex of Γ_1 with an external vertex of Γ_2. The first and second contribution yield $\delta(\Gamma_1) \bullet_\sigma \Gamma_2$, while the third and fourth yield $\Gamma_1 \bullet_\sigma \delta(\Gamma_2)$. Taking into account the signs we obtain the formula

$$\delta(\Gamma_1 \bullet_\sigma \Gamma_2) = \delta(\Gamma_1) \bullet_\sigma \Gamma_2 + (-1)^{|\Gamma_1|}\Gamma_1 \bullet_\sigma \delta(\Gamma_2) + R_\sigma(\Gamma_1, \Gamma_2)$$

where $R_\sigma(\Gamma_1, \Gamma_2)$ is a linear combination of the graphs obtained from $\Gamma_1 \bullet_\sigma \Gamma_2$ by contracting an external vertex of Γ_1 with an external vertex of Γ_2. Now, suppose that σ brings the ith external vertex of Γ_1 next to the jth external vertex of Γ_2 in $\Gamma_1 \bullet_\sigma \Gamma_2$. Collapsing the arc between these two vertices yields a contribution, called Λ, to $R_\sigma(\Gamma_1, \Gamma_2)$. Then consider the (V_e^1, V_e^2)-shuffle σ^τ obtained by composing σ with a transposition of the two vertices considered above. Collapsing the arc in $\Gamma_1 \bullet_{\sigma^\tau} \Gamma_2$ between these vertices yields the contribution $-\Lambda$ to $R_\sigma(\Gamma_1, \Gamma_2)$. This fact in turn implies that $\sum_\sigma R_\sigma(\Gamma_1, \Gamma_2) = 0$, with the sum taken over all (V_e^1, V_e^2)-shuffles. □

As a consequence, $(\mathcal{D}, \delta, \bullet)$ is a differential graded commutative algebra. It also has a unit $\mathbf{1}$ consisting of the graph without edges. It follows that its cohomology $H(\mathcal{D})$ is a graded commutative algebra with unit.

3.2. **Operations on $\mathcal{L}^{k,m}$.** The product $\bullet$ on $\mathcal{L}^{k,m}$ is defined exactly as in the previous case, i.e., as the shuffle product of two graphs. The unit $\mathbf{1}$ is the graph with no edges. In addition we have a comultiplication Δ mapping a graph Γ to the signed sum of $\Gamma' \otimes \Gamma''$ over all possible ways to cut Γ into two disconnected parts Γ' and Γ'' by removing an internal point of one of the arcs of Γ. By convention, if the oriented line is oriented, say, from left to right, then Γ_1 is the left-most component of $\Gamma \setminus \{\text{pt}\}$. A graph Γ is primitive if it cannot be disconnected in a nontrivial way by removing an internal point of an arc, and in this case

$\Delta\Gamma = \mathbf{1}\otimes\Gamma + \Gamma\otimes\mathbf{1}$. To fix the signs in the general case, first of all we order the primitive subgraphs of Γ with respect to the oriented line, and assume that the labels are compatible with this ordering (i.e., all labels of a primitive subgraph are less than any label of a subsequent one). The decoration induced by Γ on Γ' and Γ'' is determined by rescaling the labels of Γ'' and

$$\Delta\Gamma = \sum(-1)^{\lambda(\Gamma',\Gamma'')}\Gamma'\otimes\Gamma'',$$

where $\lambda(\Gamma',\Gamma'')$ is given in eq. (3.1).

We also define a counit ϵ by $\epsilon(\Gamma) = 0$ if $\Gamma \neq \mathbf{1}$ and $\epsilon(\mathbf{1}) = 1$.

Theorem 3.2. *With the above definitions,* $(\mathcal{L}, \bullet, \Delta, \mathbf{1}, \epsilon, \delta)$ *is a differential graded commutative Hopf algebra with unit.*

Proof. The Leibnitz rule and the graded commutativity of the shuffle product hold just as for $\mathcal{D}$, and one can easily verify the coassociativity of the coproduct and the compatibility between product and coproduct. The coboundary operator δ induces a coboundary operator on $\mathcal{L}\otimes\mathcal{L}$ by setting $\delta(\Gamma'\otimes\Gamma'') = \delta(\Gamma')\otimes\Gamma'' + (-1)^{|\Gamma'|}\Gamma'\otimes\delta(\Gamma'')$ on the generators and extending this definition to $\mathcal{L}\otimes\mathcal{L}$ by linearity. Then one has the following equation

$$\Delta\,\delta = \delta\,\Delta.$$

In fact, it easy to check that, for any graph Γ, the explicit expression for $\Delta(\delta\Gamma)$ contains the same graphs (with the same signs) of the explicit expression for $\delta\Delta(\Gamma)$. Finally, since this bialgebra is $\mathbb{N}$-graded and the only element in degree zero is the unit, the antipode is automatically defined. □

Remark 3.3. If we pass to the dual, then our algebraic structures correspond to the ones considered in [7] (see also [2] for the degree zero in the odd case).

4. Trivalent cocycles of even type

A consequence of the algebra structure on $H^{k,0}(\mathcal{D}_e)$ is the following

Proposition 4.1. *For every* $k \in \mathbb{N}$ *we have* $H^{2k,0}(\mathcal{D}_e) \neq 0$

It is well known that the odd-case analogue of this Proposition holds, thanks to the existence of Vassiliev knot invariants at any order [2].

Proof. Let us denote by the $\Psi = \frac{1}{4}\Phi - \frac{1}{3}\Gamma_1$ the graph cocycle of figure 1. We want to show that, for every $l \geq 2$, the lth power of Ψ is nontrivial. First, we notice that $\delta_e(\Psi) = 0$ and hence, thanks to Proposition 3.1, we also have $\delta_e(\Psi^{\bullet l}) = 0$. Moreover $\Psi^{\bullet l}$ cannot be δ_e-exact since it has

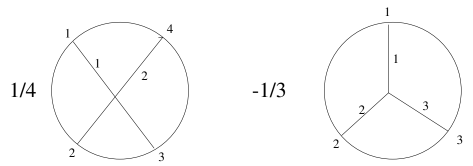

FIGURE 1. Even cocycle of order 2

degree zero and there are no graphs of negative degree. Therefore, it is enough to prove that, for every $l \geq 2$, the cocycle $\Psi^{\bullet l}$ is a linear combination of graphs with at least one coefficient different from zero.

Let us denoted by Γ_l the graph represented in figure 2. This graph has l triples of edges are attached on the external circle in such a way that they do not overlap.

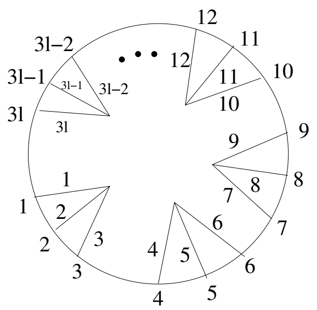

FIGURE 2

We will prove by induction that the coefficient of Γ_l in $\Psi^{\bullet l}$ is different from zero for every $l \geq 2$. The proof of this fact for $l = 2$ goes as follows: in the shuffle product

$$\Psi \bullet \Psi = \frac{1}{16}\ \Phi \bullet \Phi - \frac{1}{6}\ \Phi \bullet \Gamma_1 + \frac{1}{9}\ \Gamma_1 \bullet \Gamma_1$$

there are exactly three contributions to Γ_2, all coming from $\Gamma_1 \bullet \Gamma_1$. Since each of these three contributions has coefficient ± 1, we have that the coefficient of Γ_2 in $\Psi^{\bullet 2}$ is different from zero.

Next, we suppose that the coefficient of Γ_{l-1} in $\Psi^{\bullet(l-1)}$ is different from zero, and we observe that all the contributions to the graph Γ_l in $\Psi^{\bullet l}$ arise from the shuffle product of Γ_{l-1} with Γ_1. An easy computation shows that these contributions are all equal up to an even permutation of labels, and hence they cannot cancel out. $\square$

It is known from [5], Theorem 1.1, that the maps

$$I\colon H^{k,0}(\mathcal{D}) \to H^{(n-3)k}(\,\mathrm{Imb}\,(S^1, \mathbb{R}^n))$$

are injective for every $n > 3$. This means that each of the graph cocycles of the above Proposition produces a nontrivial cohomology class of $\mathrm{Imb}\,(S^1, \mathbb{R}^n)$ for every even $n \geq 4$. Hence we have:

Corollary 4.2. *For any $n > 3$ and for any positive integer k_0, there are nontrivial cohomology classes on $\mathrm{Imb}\,(S^1, \mathbb{R}^n)$ of degree greater than k_0.*

5. Integration map

We now recall how the maps of eqs. (1.1) and (1.2) are constructed (see [3, 5] for further details). We consider the fiber bundle $p\colon C_{q,t}(\mathbb{R}^n) \to \mathrm{Imb}\,(S^1, \mathbb{R}^n)$ whose fiber over a given imbedding γ is the compactified configuration space of $q+t$ points in $\mathbb{R}^n$, the first q of which are constrained on γ.

Let us fix a symmetric volume form ω^{n-1} on S^{n-1}, namely a normalized top form satisfying the additional condition $\alpha^*\omega^{n-1} = (-1)^n\omega^{n-1}$, where α is the antipodal map. A tautological form θ_{ij} is by definition the pull-back to $C_{q,t}(\mathbb{R}^n)$ of ω^{n-1} via the smooth map $\phi_{ij}\colon C_{q,t}(\mathbb{R}^n) \to S^{n-1}$ which, on the interior of $C_{q,t}(\mathbb{R}^n)$, is defined as

$$\phi_{ij}(x_1, \ldots, x_{q+t}) = \frac{(x_i - x_j)}{|x_i - x_j|} \ .$$

For $i = j$ we use instead the map:

$$C_{q,t}(\mathbb{R}^n) \xrightarrow{\pi} C_{q,0}(\mathbb{R}^n) = C_q \times \mathrm{Imb}\,(S^1, \mathbb{R}^n) \xrightarrow{pr_i \times id} S^1 \times \mathrm{Imb}\,(S^1, \mathbb{R}^n) \xrightarrow{D} S^{n-1}$$

where C_q is a component of the compactified configuration space of q points on S^1, π forgets the t points not lying on the imbedding, pr_i is the projection on the ith point and D is the normalized derivative $D(t, \psi) = \dot{\psi}(t)/|\dot{\psi}(t)|$.

For any given graph $\Gamma \in \mathcal{D}^{k,m}$ with q external vertices and t internal vertices, we construct a differential form $\omega(\Gamma)$ on $C_{q,t}(\mathbb{R}^n)$ by associating the tautological form θ_{ij} to the edge connecting the vertices i and j, and taking the wedge product of these forms over all the edges of Γ. Then $I(\Gamma)$ is set to be the integral of $\omega(\Gamma)$ along the fibers of $p\colon C_{q,t}(\mathbb{R}^n) \to \mathrm{Imb}\,(S^1, \mathbb{R}^n)$. The map I extended by linearity to $\mathcal{D}^{k,m}$ takes value in $\Omega^{(n-3)k+m}(\mathrm{Imb}\,(S^1, \mathbb{R}^n))$. Similarly, one can define $I\colon \mathcal{L}^{k,m} \to \Omega^{(n-3)k+m}(\mathrm{Imb}\,(\mathbb{R}, \mathbb{R}^n))$.

Let us now turn to the algebraic structures in the case of imbeddings of S^1 into $\mathbb{R}^n$. We know from [5], Theorem 4.4, that the integration map $I\colon \mathcal{D} \to \Omega(\mathrm{Imb}\,(S^1, \mathbb{R}^n))$ is a chain map for $n > 3$. An easy check shows that the shuffle product correspond exactly to the wedge product of configuration space integrals. Therefore we have:

Proposition 5.1. *The integration map* $I\colon \mathcal{D} \to \Omega(\operatorname{Imb}(S^1, \mathbb{R}^n))$ *is a homomorphism of differential algebras with unit for* $n > 3$.

Next we consider $\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)$.

Lemma 5.2. *The integration map* $I\colon \mathcal{L} \to \Omega(\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n))$ *is a chain a map for* $n > 3$.

Proof. The only difference with respect to the case of imbeddings of the circle is that now one has to consider also faces describing points on the imbedding escaping to infinity, possibly along with external points. A main feature of the compactified configuration spaces is that they split near the codimension-one faces in a product of the configurations collapsing to a certain point (or escaping at the point "infinity") and the configurations which remain far from this collapsing point. Since the imbeddings are fixed outside a compact set, the integration along the faces at infinity yields zero unless the form degree of the integrand is zero on the first component (i.e., the configurations escaping at infinity). This implies that an entire connected component of a graph has to escape to infinity: in fact, whenever exactly one argument of a tautological form goes to infinity, the form degree is entirely carried by the point escaping at infinity as a consequence of

$$\frac{x-y}{|x-y|} \sim \frac{x}{|x|} \qquad \text{for } x \to \infty.$$

If however a connected subgraph Γ yields a zero form after integration on the face at infinity, the relation $(n-1)e = v_e + nv_i - 1$ should hold (where again e is the number of edges, v_e the number of external vertices and v_i the number of internal vertices of Γ). But in a graph whose vertices are at least trivalent $(n-1)e - v_e - nv_i$ is nonnegative. □

The proof of the next Proposition is exactly as for $\operatorname{Imb}(S^1, \mathbb{R}^n)$.

Proposition 5.3. *The integration map* $I\colon \mathcal{L} \to \Omega(\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n))$ *is a homomorphism of differential algebras with unit for* $n > 3$.

6. Hopf algebras

We now want to define a coproduct on $\Omega(\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n))$. In the following we will call *long knots* the elements of $\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)$. We first fix some convention in the definition of $\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)$; namely, we choose a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ such that all the elements of $\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)$ coincide, outside a compact subset, with the reference imbedding $\varsigma(t) = \mathbf{e}_1 t$. Then we observe that two long knots γ_1 and γ_2 can be composed to a new long knot $m(\gamma_1, \gamma_2)$ by

$$m(\gamma_1,\gamma_2)(t) = \begin{cases} \Phi_-(\gamma_1(\phi_-(t))) & t \leq 0, \\ \Phi_+(\gamma_2(\phi_+(t))) & t \geq 0. \end{cases}$$

where $\Phi_\pm(x_1, x_2, \dots, x_n) = (\phi_\pm^{-1}(x_1), x_2, \dots, x_n)$ and $\phi_\pm \colon \mathbb{R}^\pm \to \mathbb{R}$ are any pair of diffeomorphisms, e.g., $\phi_\pm(t) = \tan\left(2\arctan(t) \mp \frac{\pi}{2}\right)$. Roughly speaking we are attaching γ_1 and γ_2 one after the other.

The product m has a unit e given by the linear imbedding ς used to define $\operatorname{Imb}(\mathbb{R}, \mathbb{R}^n)$. The pullback of the product $m\colon \operatorname{Imb}(\mathbb{R},\mathbb{R}^n) \times \operatorname{Imb}(\mathbb{R},\mathbb{R}^n) \to \operatorname{Imb}(\mathbb{R},\mathbb{R}^n)$ defines a non-coassociative coproduct

$$m^* \colon \Omega(\operatorname{Imb}(\mathbb{R},\mathbb{R}^n)) \to \Omega(\operatorname{Imb}(\mathbb{R},\mathbb{R}^n)) \hat{\otimes} \Omega(\operatorname{Imb}(\mathbb{R},\mathbb{R}^n)).$$

where $\hat{\otimes}$ denotes the topological tensor product of Fréchet spaces. The evaluation at e defines a counit η:

$$\eta(\omega) = \begin{cases} \omega(e) & \text{if } \deg \omega = 0 \\ 0 & \text{if } \deg \omega > 0. \end{cases}$$

We observe however that the non-coassociative coproduct m^* gives rise to an associative operation in cohomology (in fact the product m is associative up to a homotopy given by composing the long knot with a suitable diffeomorphism of $\mathbb{R}$). More precisely, we have:

Proposition 6.1. $(H(\operatorname{Imb}(\mathbb{R},\mathbb{R}^n)), \wedge, 1, \Delta, \eta)$ *is a graded commutative and cocommutative Hopf algebra for* $n > 3$.

Proof. As shown in [4], there exists an action of the little 2-cubes operad $\mathcal{LC}_2$ on $\operatorname{Imb}(\mathbb{R},\mathbb{R}^n)$. This means that there are operations on the space $\operatorname{Imb}(\mathbb{R},\mathbb{R}^n)$ corresponding to each element of $\mathcal{LC}_2$. In particular, it turns out that one of these operations is the multiplication m described above. Passing on the cochain level we obtain an action of operad $\Omega(\mathcal{LC}_2)$ on $\Omega(\operatorname{Imb}(\mathbb{R},\mathbb{R}^n))$, which give rise in cohomology to an action of the operad $H(\mathcal{LC}_2)$ on $H(\operatorname{Imb}(\mathbb{R},\mathbb{R}^n))$. Since the operad $H(\mathcal{LC}_2)$ is the linear dual of the Gerstenhaber operad, we have that the coproduct is coassociative and cocommutative. The compatibility between the wedge product and the coproduct is obvious. Finally, the existence of the antipode follows from the fact that $H(\operatorname{Imb}(\mathbb{R},\mathbb{R}^n))$ is $\mathbb{N}$-graded with only one element in degree zero. □

Remark 6.2. A more explicitly proof of the cocommutativity of the coproduct is based on the observation that the composition m of two long knots is commutative up to homotopy. In fact, one can shrink one of the two long knots in a very small region and slide it along the other long knot (see [7, 8] for details).

Our central result is then the following

Theorem 6.3. *The map* $I\colon H(\mathcal{L}) \to H(\operatorname{Imb}(\mathbb{R},\mathbb{R}^n))$ *is a Hopf algebra homomorphism for* $n>3$.

Proof. First, we notice that for $n>3$, the degree of the differential form $I(\Gamma)$ is zero if and only if $\Gamma = \mathbf{1}$. Therefore we have $\eta(I(\Gamma)) = \epsilon(\Gamma)$.

We now have to show that the coproducts are compatible; viz.,

$$\Delta \circ I = (I\otimes I)\circ\Delta. \tag{6.1}$$

To prove this, consider any two cycles Z_1 and Z_2 (of degree k and l) of imbeddings. Let $Z = m_*(Z_1, Z_2)$ be the $(k+l)$-cycle obtained by attaching the two cycles:

$$Z(u_1,\dots,u_k,v_1,\dots,v_l)(t) = \begin{cases} \Phi_-(Z_1(u_1,\dots,u_k)(\phi_-(t))) & t\le 0,\\ \Phi_+(Z_2(v_1,\dots,v_l)(\phi_+(t))) & t\ge 0.\end{cases}$$

Identity (6.1) is equivalent to

$$\int_Z I(\Gamma) = \sum (-1)^{\lambda(\Gamma',\Gamma'')} \int_{Z_1} I(\Gamma')\int_{Z_2} I(\Gamma''), \tag{6.2}$$

for any Z_1 and Z_2 and for any $\Gamma\in H(\mathcal{L})$, where we write $\Delta\Gamma = \sum(-1)^{\lambda(\Gamma',\Gamma'')}\Gamma'\otimes\Gamma''$.

To prove (6.2), let us introduce the $(k+l+1)$-chain $\mathfrak{Z}$ by

$$\mathfrak{Z}(R,u_1,\dots,u_k,v_1,\dots,v_l)(t) = \tag{6.3}$$
$$= \begin{cases} \Phi_-(Z_1(u_1,\dots,u_k)(\phi_-(t+R))) - \varsigma(R) & t\le -R,\\ \varsigma(t) & -R<t<R,\\ \Phi_+(Z_2(v_1,\dots,v_l)(\phi_+(t-R))) + \varsigma(R) & t\ge R,\end{cases}$$

with $R\in[0,+\infty)$. In practice we are moving the support of the cycles Z_1 and Z_2 far apart, and the parameter R measure the distance between the two cycles. Since $\delta\Gamma = 0$ and since I is a chain map (Lemma 5.2), by Stokes' Theorem we get

$$0 = \int_{\partial\mathfrak{Z}} I(\Gamma) = -\int_Z I(\Gamma) + \lim_{R\to+\infty} J_R$$

with

$$J_R(\Gamma) = \int_{\mathfrak{Z}(R,\cdot)} I(\Gamma).$$

Let us write our graph cocycle as $\Gamma = \sum_i c_i\Gamma_i$. Each $I(\Gamma_i)$ can be split into two parts as follows. We fix $R\in(0,+\infty)$ and suppose Γ_i has q external vertices and t internal vertices. Then we define $C^{1,R}_{q,t}(\mathbb{R}^n)$ to be the subbundle of $C_{q,t}(\mathbb{R}^n)$ where the points corresponding to the external vertices of every primitive subgraph of Γ_i lie either all to the left of R or all to the right of $-R$. We also let $C^{2,R}_{q,t}(\mathbb{R}^n)$ be the fiberwise complement of $C^{1,R}_{q,t}(\mathbb{R}^n)$ in $C_{q,t}(\mathbb{R}^n)$. Now we define $I_{\alpha,R}(\Gamma_i)$, $\alpha = 1, 2$, to be the integral of the differential form $\omega(\Gamma_i)$ performed along the fibers of $C^{\alpha,R}_{q,t}(\mathbb{R}^n)$. Finally we set $I_{\alpha,R}(\Gamma) = \sum_i c_i I_{\alpha,R}(\Gamma_i)$ and we have $J_R(\Gamma) = \int_{\mathfrak{Z}(R,\cdot)} I_{1,R}(\Gamma) + \int_{\mathfrak{Z}(R,\cdot)} I_{2,R}(\Gamma)$.

One immediately sees that $\lim_{R\to\infty} \int_{\mathfrak{Z}(R,\cdot)} I_{1,R}(\Gamma)$ is equal to the right-hand side of eq. (6.2), and hence what we have to prove is that

$$\lim_{R\to\infty} \int_{\mathfrak{Z}(R,\cdot)} I_{2,R}(\Gamma_i) = 0. \tag{6.4}$$

We now need a generalization of Lemma 10 of [1]. Let Γ be a connected graph with v_e external vertices and v_i internal vertices, and let $\omega(\Gamma)$ be the product of tautological forms associated to the graph Γ. We denote by g_Γ the integral of $\omega(\Gamma)$ over the internal vertices of Γ, namely the push-forward of $\omega(\Gamma)$ along the map $p\colon C_{v_e,v_i}(\mathbb{R}^n) \to C_{v_e,0}(\mathbb{R}^n)$ that forgets the internal points. Let us write g_Γ in coordinates as $g_\Gamma = g_{\Gamma_I}(x_1, \dots, x_{v_e}, \gamma)\, \mathrm{d}x^I$, where I is a multi-index. Suppose moreover that $\mathbf{x}(T) = (x_1(T), \dots, x_{v_e}(T))$ a sequence in the configuration space with the property that there is a pair of points whose distance diverges as T goes to infinity.

Lemma 6.4. *With the above notations, we have*

$$\lim_{T\to\infty} g_\Gamma|_{\mathbf{x}(T)} = 0.$$

Proof. Case 1. We consider first the case when Γ has no edges whose end-points are both external. Then g_Γ turns out to be bounded whenever its arguments $x_1, \dots, x_{v_e}$ run in a bounded subdomain of $\mathbb{R}^n$ (namely, g_Γ do not diverge if two or more points collapse). We can always suppose that the pair of points whose distance diverges fastest are $x_1(T)$ and $x_{v_e}(T)$. Using the translation invariance of the integral, we also suppose $x_{v_e}(T) = 0$. Therefore we have that $|x_1(T)| \to \infty$ as $T \to \infty$.

We claim that if we rescale all the variables by $1/|x_1|$ we get

$$g_{\Gamma_I}(x_1, \dots, x_{v_e-1}, 0) = \left(\frac{1}{|x_1|}\right)^{\alpha_\Gamma} g_{\Gamma_I}(x_1/|x_1|, \dots, x_{v_e-1}/|x_1|, 0).$$

where $\alpha_\Gamma = (n-1)e - nv_i$. In fact, when performing the integral along the fibers of $p\colon C_{v_e,v_i}(\mathbb{R}^n) \to C_{v_e,0}(\mathbb{R}^n)$, it is convenient to rescale the integration variables by $1/|x_1|$. This yields the contribution $-nv_i$. At this point the tautological forms θ_{ij}, whose degree is $n-1$ and whose number is the same as the number of edges e, are obtained by rescaling the arguments of the functions ϕ_{ij}, and this yields the contribution $(n-1)e$. Moreover, using the fact that $2e - 3v_i - v_e \geq 0$ and $v_e > 0$, we have

$$\alpha_\Gamma = (n-1)e - nv_i \geq (n-1)\frac{3}{2}v_i + \frac{n-1}{2}v_e - nv_i = \frac{n-3}{2}v_i + \frac{n-1}{2}v_e > 0.$$

Now, $x_k/|x_1|$ is bounded for every $k = 2, \ldots, v_e - 1$, and hence the quantity $g_{\Gamma_I}(x_1/|x_1|, \ldots, x_{v_e-1}/|x_1|, 0)$ remains bounded when $|x_1|$ goes to infinity. This proves the first case.

Case 2. When Γ has edges whose end-points are both external, we denote by $\widetilde{\Gamma}$ the same graph with these edges removed. We have then $g_\Gamma = g_{\widetilde{\Gamma}} h_\Gamma$ where h_Γ is the product of the functions associated to the removed edges. If $\widetilde{\Gamma}$ is the empty graph, then there is at least one edge whose end-points go far apart. Since the imbeddings are fixed outside a compact set, asymptotically h_Γ is the pull-back of a volume form via a constant function, i.e., h_Γ vanishes in the limit $T \to \infty$. If $\widetilde{\Gamma}$ is non empty but connected we are done by the argument in Case 1. The last possibility is when $\widetilde{\Gamma}$ is non empty and disconnected. If inside a connected component there is a pair of points whose distance diverges, we are also done by the argument in Case 1, otherwise there is at least one edge whose end-points go far apart, and in this case h_Γ goes to zero. □

We now show that eq. (6.4) holds. Suppose Γ_i has q external vertices and t internal vertices, and consider $C^{2,R}_{q,t}(\mathbb{R}^n)$. For every element of $C^{2,R}_{q,t}(\mathbb{R}^n)$ consider a primitive subgraph Ξ of Γ_i such that the preimage l of its left-most point on the imbedding is less than $-R$, while the preimage r of its right-most point is greater than R. Let $\Xi_0, \ldots, \Xi_k$ be the connected component of Ξ minus the oriented line, ordered following the order of their left-most external vertices. With the symbols l_i and r_i we mean the preimages of the left-most and right-most among the points corresponding to the external vertices of Ξ_i. In particular, $l_0 = l$. If $r_0 > 0$ then the result follows from Lemma 6.4 applied to g_{Ξ_0}. If $r_0 < 0$ then $l_1 < 0$. If $l_1 < -R$ then we repeat our considerations for Ξ/Ξ_0, otherwise we have two possibilities: $r_1 < R$, and the integral of $\omega(\Gamma_i)$ on this portion of $C^{2,R}_{q,t}(\mathbb{R}^n)$ is zero for dimensional reasons, or $r_1 > R$ and we can apply Lemma 6.4 to g_{Ξ_1}.

This concludes the proof of the Theorem. □

7. The "anomaly" term

Let us consider now the case of ordinary framed long knots, i.e., imbeddings of $\mathbb{R}$ into $\mathbb{R}^3$ with a choice of a trivialization of their normal bundle. Following [3], we know that I is not a chain map because of an anomaly term due to contributions from the most hidden faces of the boundaries of the compactified configuration spaces. More precisely, if we restrict to trivalent graphs, the following equation holds

$$\mathrm{d}\tilde{I} = I\delta$$

where

$$\tilde{I}(\Gamma) = I(\Gamma) + c_\Gamma I(\Theta).$$

Here c_Γ are certain (unknown) coefficients while Θ is the graph with one chord only, so $\int_K I(\Theta)$ is the self-linking number $\mathrm{slk}(K)$ of the framed long knot K. As a consequence, the Bott–Taubes map $\tilde{I}$ associates to each cocycle of trivalent graphs an invariant of framed long knots.

Let Γ be a cocycle of trivalent graphs, Z_1 and Z_2 two framed long knots, $Z = m(Z_1, Z_2)$ their product and $\mathfrak{Z}$ the 1-chain of eq. (6.3). Then $\mathrm{d}I(\Gamma) + c_\Gamma \mathrm{d}I(\Theta) = \mathrm{d}\tilde{I}(\Gamma) = I(\delta(\Gamma)) = 0$ and hence, applying Stokes' Theorem, we have

$$\lim_{R\to\infty} \int_{\mathfrak{Z}(R)} I(\Gamma) - \int_Z I(\Gamma) = \int_{\mathfrak{Z}} \mathrm{d}I(\Gamma) = -c_\Gamma(\mathrm{slk}(\mathfrak{Z}(R)) - \mathrm{slk}(Z)) = 0,$$

since the self-linking number of $\mathfrak{Z}(R)$ does not depend on R and it is therefore equal to the self-linking number of Z. As a consequence, using the same arguments of the proof of Theorem 6.3, we deduce that also in three dimensions the map I, when restricted to trivalent graphs, is compatible with the coproducts. On the other hand the Bott–Taubes map $\tilde{I}$ is a map of coalgebras if and only if

$$\int_Z \tilde{I}(\Gamma) = \sum (-1)^{\lambda(\Gamma',\Gamma'')} \int_{Z_1} \tilde{I}(\Gamma') \int_{Z_2} \tilde{I}(\Gamma''). \tag{7.1}$$

The left-hand side of (7.1) is given by

$$\int_Z I(\Gamma) + c_\Gamma \int_Z I(\Theta) = \sum \int_{Z_1} I(\Gamma') \int_{Z_2} I(\Gamma'') + c_\Gamma \left(\mathrm{slk}(Z_1) + \mathrm{slk}(Z_2)\right),$$

while the right-hand side is

$$\sum \int_{Z_1} (I(\Gamma') + c_{\Gamma'} I(\Theta)) \int_{Z_2} (I(\Gamma'') + c_{\Gamma''} I(\Theta)) =$$
$$= \sum \int_{Z_1} I(\Gamma') \int_{Z_2} I(\Gamma'') + c_{\Gamma''}\mathrm{slk}(Z_2) \int_{Z_1} I(\Gamma') + c_{\Gamma'}\mathrm{slk}(Z_1) \int_{Z_2} I(\Gamma'') +$$
$$+ c_{\Gamma'} c_{\Gamma''} \mathrm{slk}(Z_1)\mathrm{slk}(Z_2)$$

In particular we can write eq. (7.1) in the case when Γ is obtained by "attaching" the oriented lines of two graphs Γ_1 and Γ_2. By using the fact of [1] that $c_\Gamma = 0$ if Γ is not primitive, then one easily sees that (7.1) holds if and only if

$$c_{\Gamma_2}\mathrm{slk}(Z_2)\int_{Z_1} I(\Gamma_1) + c_{\Gamma_1}\mathrm{slk}(Z_1)\int_{Z_2} I(\Gamma_2) + c_{\Gamma_1}c_{\Gamma_2}\mathrm{slk}(Z_1)\mathrm{slk}(Z_2) = 0.$$

Hence, if c_{Γ_1} is different from zero, $\tilde{I}$ is not a coalgebra map (e.g. choose Z_1, Z_2 and Γ_2 such that $\mathrm{slk}(Z_2) = 0$, $\mathrm{slk}(Z_1) \neq 0$ and $\int_{Z_2} I(\Gamma_2) \neq 0$). In other words, the anomaly term $c_\Gamma I(\Theta)$ is the obstruction for $\tilde{I}$ to be a coalgebra map.

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