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THRESHOLD RESUMMED SPECTRA IN $B \to X_u l \nu$ DECAYS IN NLO (I)

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Abstract

We evaluate threshold resummed spectra in $B \to X_u l \nu$ decays in next-to-leading order. We present results for the distribution in the hadronic variables E_X and m_X^2/E_X^2, for the distribution in E_X and for the distribution in E_X and E_l, where E_X and m_X are the total energy and the invariant mass of the final hadronic state X_u respectively and E_l is the energy of the charged lepton. We explicitly show that all these spectra (where there is no integration over the hadronic energy) can be directly related to the photon spectrum in $B \to X_s \gamma$ via short-distance coefficient functions.

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1 Introduction and summary of the results

A long-standing problem in particle physics is the understanding of strong interactions at low energies. While at very low energies, of the order of the hadronic scale $\Lambda \approx 300$ MeV, perturbative QCD is of no use and alternative methods have been developed in decades (such as quark models, chiral lagrangians, lattice QCD, etc.), at intermediate energies, of the order of a few GeV, perturbative computations can be combined with non-perturbative models to predict a variety of cross sections and decay rates. Among these moderate hard scale phenomena is beauty physics, which is indeed characterized by a hard scale of a few GeV. The measured decay spectra often receive large contributions at the endpoints — in the case of the hadron energy spectrum, in the middle of the domain — from long-distance effects related to soft interactions between the heavy quark and the light degrees of freedom.

The main non perturbative effect is the well-known Fermi motion, which classically can be described as a small vibration of the heavy quark inside the B meson because of the momentum exchange with the valence quark; in the quantum theory it is also the virtuality of the heavy quark that matters. This effect is important in the end-point region, because it produces some smearing of the partonic spectra.

These long distance effects manifest themselves in perturbation theory in the form of series of large infrared logarithms, coming from an "incomplete" cancellation of infrared divergencies in real and virtual diagrams. The probability for instance for a light quark produced in a process with a hard scale Q to evolve into a jet with an invariant mass smaller than m is written in leading order as [1]:

$$\begin{aligned} J(m) &= 1 + A_1\,\alpha_S \int_0^1 \frac{d\omega}{\omega} \int_0^1 \frac{d\theta^2}{\theta^2}\, \Theta\left(\frac{m^2}{Q^2} - \omega\,\theta^2\right) - A_1\,\alpha_S \int_0^1 \frac{d\omega}{\omega} \int_0^1 \frac{d\theta^2}{\theta^2} \\ &= 1 - \frac{A_1}{2}\,\alpha_S \log^2\left(\frac{Q^2}{m^2}\right), \end{aligned} \tag{1}$$

where ω is the energy of a gluon emitted by the light quark normalized to the hard scale, θ is its emission angle and A_1 is a positive constant (see sec. 3). The first integral on the r.h.s. is the real contribution while the second integral is the virtual one. Both integrals are separately divergent for $\omega = 0$ — soft singularity — as well as for $\theta = 0$ — collinear singularity, but their sum is finite. "Complete" real-virtual cancellation occurs only for $m = Q$, i.e. in the completely inclusive evolution of the quark line, while for $m < Q$ there is a left-over double logarithm because of the smaller integration region of the real diagrams. Multiple gluon emission occurs in higher orders of perturbation theory; it can be described as a classical branching process and gives rise to the double logarithmic series, i.e. to powers of the last term $\alpha_S \log^2\left(Q^2/m^2\right)$ on the r.h.s. of eq. (1) [1, 2].

We may say that perturbation theory "signals" long-distance effects in a specific way — even though a quantitative description of the latter has to include also some truly non-perturbative component. A theoretical study of the universality of these long-distance effects can therefore be done *inside* perturbation theory, by comparing the logarithmic structure of different distributions. In other words, if these long-distance effects are universal, this has certainly to show up in perturbation theory: things have to work in perturbation theory first. The aim of this work is to study the relation of long-distance effects between different distributions by means of resummed perturbation theory.

In general, let us consider the semi-inclusive decays

$$B \rightarrow X_q + (\text{non QCD partons}), \tag{2}$$

where X_q is any hadronic final state coming from the fragmentation of the light quark $q = u, d, s$ and the non QCD partons are typically a photon, a lepton-neutrino pair, a lepton-antilepton pair, etc. This system of particle(s), with total four-momentum q_μ, constitutes a "probe" for the hadronic process, as in the case of deep-inelastic-scattering (DIS) of leptons off hadrons. Without any generality loss, we can work in the b rest frame, where $p_b^\mu = m_b v^\mu$, with m_b being the beauty mass and $v^\mu = (1; 0, 0, 0)$ being the classical 4-velocity. The hadronic subprocess in (2) is characterized by the following three scales:

$$m_b, \qquad E_X \quad \text{and} \quad m_X \qquad\qquad\qquad (m_b \geq E_X), \tag{3}$$

where m_X and E_X are the invariant mass and the total energy of the final hadronic state X_q, respectively. We are interested in the so-called threshold region, which can be defined in all generality as the one having

$$m_X \ll E_X. \tag{4}$$

The region (4) is sometimes called radiation-inhibited, because the emitted radiation naturally produces final states with an invariant mass of the order of the hard scale: $m_X \sim O(E_X)$. It is also called semi-inclusive because experimentally, to satisfy the constraint (4), most hadronic final states have to be discarded.

The processes we are going to consider are the well-known radiative decay with a real photon in the final state,

$$B \to X_s + \gamma \tag{5}$$

and the semi-leptonic decay, [4]

$$B \to X_u + l + \nu. \tag{7}$$

In perturbative QCD, the hadronic subprocess in (2) consists of a heavy quark decaying into a light quark which evolves later into a jet of soft and collinear partons because of infrared divergencies. In leading order, one only considers the emission of soft gluons at small angle by the light quark (see eq. (1)); the final state X_q consists of a jet with the leading (i.e. most energetic) quark q originating the jet itself. In next-to-leading order one has to take into account two different single-logarithmic effects: (a) hard emission at small angle by the light quark q and (b) soft emission at large angle by the heavy quark. Because of (a), the final state consists of a jet with many hard partons and, in general, the leading parton is no longer the quark q which originated the jet itself. Because of (b), the final state does not contain only an isolated jet, but also soft partons in any space direction. The main result of [3] is that the large threshold logarithms appearing in (2) are conveniently organized as a series of the form:

$$\begin{aligned} & \sum_{n=1}^{\infty}\sum_{k=1}^{2n} c_{nk}\, \alpha^n(Q)\, \log^k \frac{Q^2}{m_X^2} \\ = \quad & c_{12}\, \alpha(Q)\, \log^2 \frac{Q^2}{m_X^2} + c_{11}\, \alpha(Q)\, \log \frac{Q^2}{m_X^2} + c_{24}\, \alpha^2(Q)\, \log^4 \frac{Q^2}{m_X^2} + c_{23}\, \alpha^2(Q)\, \log^3 \frac{Q^2}{m_X^2} + \cdots\cdots, \end{aligned} \tag{8}$$

where $\alpha(Q) = \alpha_S(Q)$ is the QCD coupling and the hard scale Q is determined by the final hadronic energy E_X [5]:

$$Q = 2E_X. \tag{9}$$

These large logarithms are factorized into a universal QCD form factor. Let us summarize the derivation of (8) and (9). We take the infinite mass limit for the beauty quark while keeping the hadronic energy and the hadronic mass fixed [6]:

$$m_b \to \infty, \qquad \text{with} \qquad E_X \ \text{and} \ m_X \to \text{const.} \tag{10}$$

[4]The results for the semileptonic decay are easily extended to the radiative decay with the photon converting into a lepton pair,

$$B \to X_s + l + \bar{l}. \tag{6}$$

[5]The factor two is inserted in such a way that the hard scale coincides with m_b in the radiative decay (see later). The essential point however is that Q is proportional to E_X via a proportionality constant of order one, whose precise value is irrelevant.

[6]This limit has not to be confused with that one relevant for the shape function, also called structure function of the heavy flavors, which is $E_X \to \infty$, $m_X \to \infty$ with $m_X^2/E_X \to$ const (the latter implies $m_b \to \infty$, but the converse is not true). The shape function describes soft interactions only and therefore does not factorize the whole logarithmic structure, missing the large logarithms coming from hard collinear emission off the light quark .

This takes us into an effective theory in which the beauty quark is replaced by a static quark, as recoil effects are neglected in the limit (10). If we write the beauty quark momentum as $p_b = m_b v + k$, where k is a soft momentum, the infinite mass limit of the propagator is easily obtained as:

$$S_F(p) \, = \, \left(\frac{1+\hat{v}}{2} \, + \, \frac{\hat{k}}{2m} \right) \frac{1}{v \cdot k \, + \, k^2/(2m) \, + \, i\epsilon} \quad \rightarrow \quad \frac{1 \, + \, \hat{v}}{2} \, \frac{1}{v \cdot k \, + \, i\epsilon} \qquad \text{(static limit)}, \tag{11}$$

where $\hat{a} \equiv \gamma_\mu a^\mu$. As discussed above, the beauty quark contributes to the QCD form factor via large logarithms coming from soft emissions, which are correctly described by a static quark. Since the light quark propagator is not touched by the limit (10), we conclude that all soft and/or collinear emissions are correctly described by this limit. Since the heavy flavor mass has disappeared with the limit (10), the only remaining scales in the hadronic subprocess are m_X and E_X. Only one adimensional quantity can be constructed out of them, for example the ratio E_X/m_X, which is therefore the only possible argument for the large logarithms, in agreement with (8). Furthermore, the hard scale Q is given by the greatest scale in the game, i.e. by the hadronic energy E_X, in agreement with (9).

The argument given above, however, is not rigorous: let us refine it. The limit (10) is indeed singular in quantum field theory: one cannot remove degrees of freedom without paying some price. Let us consider for simplicity's sake the semileptonic decay (7), even though the conclusions are general. The vector and axial-vector currents responsible for the $b \to u$ transition are conserved or partially conserved in QCD, implying that the $O(\alpha)$ virtual corrections are ultraviolet finite. These corrections contain however terms of the form

$$\gamma_0 \, \alpha \, \log \frac{m_b}{E_X} \qquad\qquad \text{(ordinary QCD)}, \tag{12}$$

which diverge in the limit (10) (γ_0 is a constant). If one takes the limit (10) *ab initio*, i.e. before integrating the loop, some divergence is expected in the loop integrals, as it is indeed the case. Technically, that occurs because the static propagator is of the form $1/(k_0 + i\epsilon)$ (see eq. (11)) and, unlike the ordinary propagator, has no damping for $|\vec{k}| \to \infty$. It can be shown that the $b \to u$ vector and axial-vector currents are no more conserved or partially conserved in the static theory. Therefore, unlike the QCD case, the $O(\alpha)$ virtual corrections are ultraviolet divergent in the static theory and produce, after renormalization, terms corresponding to (12) of the form

$$\gamma_0 \, \alpha \, \log \frac{\mu}{E_X} \qquad\qquad \text{(effective theory)}, \tag{13}$$

in which basically the heavy flavor mass m_b is replaced by the renormalization point μ — the coefficient γ_0 being the same. The hadronic subprocess in the static theory therefore has amplitudes depending on the physical scales E_X and m_X as well as on the renormalization scale μ. If we neglect terms suppressed by inverse powers of the beauty mass $\sim 1/m_b^n$, we have that the physical scale m_b is replaced by the renormalization point μ in the effective theory: $m_b \to \mu$. The effective currents $\tilde{J}_\nu$ and the coupling constant α are renormalized at the scale μ: $\tilde{J}_\nu = \tilde{J}_\nu(\mu)$ and $\alpha = \alpha(\mu)$. The effective amplitudes contain terms of the form $\alpha^n \log^k \mu/E_X$ $(k \le n)$, which are large logarithms for $\mu \gg E_X$ or $\mu \ll E_X$. To have convergence of the perturbative series, the large logarithms above must be resummed by taking $\mu = O(E_X)$, i.e. $\mu = k \, E_X$ with $k = O(1)$. This implies that the effective currents and the coupling are evaluated at a scale of the order of the hadronic energy: $\tilde{J}_\nu = \tilde{J}_\nu(k \, E_X)$ and $\alpha = \alpha(k \, E_X)$. We have therefore proved that the hard scale Q is fixed by the final hadronic energy E_X and not by the beauty mass m_b:

$$Q \, = \, \mu \, = \, k \, E_X \qquad\qquad \text{with } k \, = \, O(1). \tag{14}$$

Let us go back to the general process (2). Kinematics gives:

$$2E_X \, = \, m_b \left(1 \, - \, \frac{q^2}{m_b^2} \, + \, \frac{m_X^2}{m_b^2} \right). \tag{15}$$

The simplest processes are those with a light-like probe, i.e. with $q^2 = 0$, where

$$2E_X \, = \, m_b \left(1 \, + \, \frac{m_X^2}{m_b^2} \right) \, \simeq \, m_b. \tag{16}$$

This case corresponds to the radiative decay (5). In this case, the final hadronic energy is always large and of the order of the heavy-flavor mass:

$$Q \approx m_b \qquad \text{(radiative decay)}. \tag{17}$$

On the other hand, in the semi-leptonic decay (7) [7], the lepton pair can have a large invariant mass,

$$q^2 \sim O\left(m_b^2\right), \tag{18}$$

implying a substantial reduction of the hard scale:

$$Q \ll m_b. \tag{19}$$

This fact is one of the complications in the threshold resummation of the semileptonic decay spectra: while in the radiative decay (5), the hard scale Q is always large in the threshold region, and of the order of m_b, this is no longer true in the semileptonic decay. The hadronic subprocesses have in general different hard scales in the two decays. If one integrates over q^2, for example because of undetected neutrino momentum, there is a mixing of hadronic contributions with different hard scales in the semileptonic case. However, it turns out by explicit computation that the contributions from a large q^2, i.e. with a small hard scale in the hadronic subprocess, are rather suppressed (see sec. 4).

At fixed Q, the large logarithms in (2) can be factorized into a QCD form factor, which is universal in the sense that it depends only on the hadronic subprocess. The differences between, let us say, the radiative decay (5) and the semileptonic decay (7) only enter in the specific form of a short-distance coefficient function multiplying the QCD form factor (and in the form of a remainder function collecting non factorized, small contributions, see next section).

The discussion above can be summarized as follows. The hard scale $Q = 2\,E_X$ in (2) appears in the argument in the infrared logarithms as well as in the argument of the running coupling. In the radiative decay, because of kinematics, the hard scale is always large and of the order of the beauty mass: $Q \approx m_b$, while in the semileptonic case kinematical configurations are possible with $Q \approx m_b$ as well as with $Q \ll m_b$. The main complication in semileptonic decays is that by performing kinematical integrations (for example over the neutrino energy), one may integrate over the hard scale of the hadronic subprocess. While in radiative decays the hard scale is fixed, in the semileptonic decays there can be a mixing of different hadronic subprocesses. A non-trivial picture of some semileptonic decay spectra emerges: there are long-distance effects which cannot be extracted by the radiative decay, related to a small final hadronic energy, but their effect turns out to be small at the end because of a kinematical suppression of the states with a small hard scale. The decay spectra in (7) can therefore be divided into two classes:

1. distributions in which the hadronic energy E_X is not integrated over. These distributions can be related via short-distance coefficients to the photon spectrum in the radiative decay (5). In particular, the structure of the threshold logarithms is the same as in decay (5). In this paper we restrict ourselves to these simpler distributions;

2. distributions in which the hadronic energy is integrated over and therefore all the hadronic energies contribute. These are for instance the hadron mass distribution or the charged lepton energy distribution. In all these cases, the structure of the threshold logarithms is different from that one in (5) and by far more complicated. The analysis of some of these distributions, which present novel features with respect to $B \to X_s \gamma$, is given in [5].

Let us make a simple analogy with e^+e^- annihilation into hadrons. In the center-of-mass (c.o.m.) frame, the final state consists of a $q\overline{q}$ pair, which are emitted back to back with a high virtuality and evolve later into two jets:

$$e^+ + e^- \to q + \overline{q} \to J_q + J_{\overline{q}}. \tag{20}$$

[7] The same is also true for the radiative decay with the photon converting into a lepton pair (6).

Roughly speaking, the final state X_q in (2), consisting in a single jet, is "half" of that in (20), consisting of the two jets J_q and $J_{\overline{q}}$. Deviations from this independent fragmentation picture arise in next-to-leading order because of large-angle soft emission by the heavy quark in (2), which has no analogue in (20). The structure of e^+e^- hadronic final states is conveniently analyzed by means of so-called shape variables, one of the most studied being the heavy jet mass m_H^2, defined as

$$m_H^2 \, = \, \max\left\{m_R^2, m_L^2\right\}, \tag{21}$$

where m_R and m_L are the invariant masses of the particles in the right and left hemispheres of the event respectively. The hemispheres are defined cutting the space with a plane orthogonal to the thrust axis $\vec{n}$, the latter defined as the direction maximizing

$$\sum_i |\vec{p}_i \cdot \hat{n}|, \tag{22}$$

i.e. basically the sum of length of longitudinal momenta. The sum extends over all hadrons — partons in the perturbative computation. For $m_H \ll Q$, where Q is the hard scale to be identified here with the c.o.m. energy, hard emission at large angle by the $q\overline{q}$ pair cannot occur and the final state consists of two narrow jets around the original $q\overline{q}$ direction, which can be identified with $\vec{n}$. The $O(\alpha)$ computation gives large logarithms of similar form to those in (8) [4]:

$$\alpha(Q)\,\log^2\frac{Q^2}{m_H^2} \quad \text{and} \quad \alpha(Q)\,\log\frac{Q^2}{m_H^2}. \tag{23}$$

There is not a simple relation between, let us say, the heavy jet mass distribution at the Z^0 peak,

$$\frac{d\sigma}{dm_H^2}\,(Q \, = \, m_Z) \tag{24}$$

and the integral of this quantity over Q from a small energy $\epsilon \sim m_H$ up to m_Z with some weight function $\phi(Q)$:

$$\frac{d\hat{\sigma}}{dm_H^2} \, = \, \int_\epsilon^{m_Z} dQ\,\phi(Q)\,\frac{d\sigma}{dm_H^2}\,(Q)\,. \tag{25}$$

Radiative B decays (5) and semileptonic spectra (7) in class 1. are the analog of the former distribution (24), while semileptonic spectra in class 2. are the analog of the latter case (25). The analog of the suppression in the semileptonic spectra 2. of the contributions from large q^2 is the suppression of the weight function $\phi(Q)$ for $Q \ll m_Z$.

Many properties of the distributions we are going to derive in this work can be understood with a qualitative discussion on the hadron energy spectrum,

$$\frac{d\,\Gamma}{d\,E_X}, \tag{26}$$

which exhibits a remarkable phenomenon related to the occurrence of infrared singularities inside the physical domain, instead than at the boundary as it is usually the case. This phenomenon has been studied in the framework of jet physics and is known as the "Sudakov shoulder" [6, 7, 8, 3]. Let us discuss it in the present case in physical terms. In lowest order, the semileptonic decay (7) involves three massless partons in the final state:

$$b \, \to \, u \, + \, l \, + \, \nu. \tag{27}$$

According to kinematics, any final state parton can take at most half of the initial energy, implying that

$$E_X^{(0)} \, = \, E_u \, \le \, \frac{m_b}{2}. \tag{28}$$

In lowest order, the final hadronic state consists indeed of the *up* quark only: $X_u = u$. To order α, a real gluon is radiated and the final hadronic state is a two-particle system: $X_u = u + g$. The final hadronic energy is not restricted anymore to half the beauty mass but can go up to the whole beauty mass:

$$E_X^{(1)} \, = \, E_u \, + \, E_g \, \le \, m_b. \tag{29}$$

For example, just consider an energetic *up* quark recoiling against the gluon, with a soft electron and a soft neutrino. The relevant case for us is a final state with the *up* quark of energy $\approx m_b/2$ and a soft and/or a collinear gluon. Such a state has a total energy slightly above $m_b/2$ and the matrix element is logarithmically enhanced because of the well-known infrared singularities. Such logarithmic enhancement cannot be cancelled by the $O(\alpha)$ virtual corrections, because of their tree-level kinematical limitation (28). We are left therefore with large infrared logarithms, of the form

$$\alpha \log^2 \left(E_X - \frac{m_b}{2} \right), \qquad \alpha \log \left(E_X - \frac{m_b}{2} \right) \qquad\qquad \left(E_X \geq \frac{m_b}{2} \right), \tag{30}$$

which are final and produce an infrared divergence for $E_X \to +m_b/2$. On the other hand, for $E_X < m_b/2$ there are no large logarithms of the form $\alpha \log^k(m_b/2 - E_X)$ $(k = 1, 2)$, because in this case real-virtual cancellation may occur, and it actually does. Let us summarize: the $O(\alpha)$ spectrum has an infrared singularity right in the middle of the domain, for $\overline{E}_X = m_b/2$, because the lowest order spectrum has a discontinuity in this point, above which it vanishes identically because of kinematics.

This infrared singularity is integrable, as

$$\int_{m_b/2}^{m_b/2+\delta} dE_X \, \alpha \, \log^k \left(E_X - \frac{m_b}{2} \right) \, < \, \infty, \tag{31}$$

where $\delta > 0$ is some energy-resolution parameter. The infrared divergence is therefore eliminated with some smearing over the hadronic energy, which experimentally is always the case. Furthermore, hadronization corrections certainly produce some smearing on the partonic final states because of parton recombination. In other words, non-perturbative mechanisms wash out this infrared divergence, which therefore does not present any problem of principle. As we are going to show, however, perturbation theory "saves itself" and no mechanism outside perturbation theory is needed to have a consistent prediction: resummation of the infrared logarithms in (30) to all orders completely eliminates the singularity, as in the cases of the usual infrared divergencies [6]. Since large logarithms occur for $E_X \sim m_b/2$, we have that the hard scale is given for this spectrum by the beauty mass,

$$Q \, = \, m_b \qquad\qquad \text{(hadron energy spectrum)}, \tag{32}$$

just like in radiative decays. This equality is noticeable, as it comes from completely independent kinematics with respect to the one in (5). There is therefore a pure short-distance relation between the hadron mass distribution in (5) and the hadron energy distribution in (7). This property remains true when we consider non-perturbative Fermi motion effects, which are factorized by the well-known structure function of the heavy flavors, also called the shape function.

This paper is organized as follows:

In sec. (2) we presents the results for the resummed triple-differential distribution, which is the most general distribution and the starting point of our analysis;

In sec. (3) we review the theory of threshold resummation in heavy flavor decays, giving explicit formulas for the QCD form factor in next-to-next-to-leading order (NNLO). The transformation to Mellin space in order to solve the kinematical constraints for multiple soft emission is discussed, together with the inverse transform to the original momentum space;

In sec. (4) we derive the double distribution in the hadronic energy and in the ratio (hadronic mass)/(hadronic energy), which are the most convenient variables for threshold resummation (these are the variables w and u defined there). The distribution in any hadronic variable can be obtained from this distribution by integration;

In sec. (5) we present the results for the resummed hadron energy spectrum in next-to-leading order, whose main physical properties have already been anticipated here. We also compute the average hadronic energy to first order and compare with the radiative decay. The hadron energy spectrum with an upper cutoff on the hadron mass, which is the easiest thing to measure in experiments, is derived in leading order;

In sec. (6) we derive the double distribution in the hadron and electron energies, i.e. in the two independent energies. A peculiarity of this spectrum is that it is characterized by the presence of two different series of large logarithms, which are factorized by two different QCD form factors. Another peculiarity is that this double differential distribution contains partially-integrated QCD form factors instead of differential ones. That implies that the infrared singularities occurring in this distribution are integrable, as in the case of the Sudakov shoulder which we have discussed before;

Finally, in sec. (7) we present our conclusions together with a discussion about natural developments.

2 Triple differential distribution

The triple differential distribution in the decay (7) is the starting point of our analysis. It has a resummed expression of the form [3]:[8]

$$\frac{1}{\Gamma}\frac{d^3\Gamma}{dxdudw} \;=\; C\left[x,w;\alpha(w\,m_b)\right]\,\sigma\left[u;\alpha(w\,m_b)\right] \,+\, d\left[x,u,w;\alpha(w\,m_b)\right], \tag{33}$$

where we have defined the following kinematical variables:

$$w \;=\; \frac{2E_X}{m_b} \qquad (0\le w\le 2), \qquad\qquad x \;=\; \frac{2E_l}{m_b} \qquad (0\le x\le 1) \tag{34}$$

and [9]

$$u \;=\; \frac{E_X-\sqrt{E_X^2-m_X^2}}{E_X+\sqrt{E_X^2-m_X^2}} \qquad (0\le u\le 1). \tag{35}$$

It is convenient to express u as:

$$u \;=\; \frac{1\,-\,\sqrt{1-4y}}{1\,+\,\sqrt{1-4y}}, \tag{36}$$

with

$$y \;=\; \frac{m_X^2}{4E_X^2} \qquad (0\le y\le 1/4). \tag{37}$$

The inverse formula of (36) reads:

$$y \;=\; \frac{u}{(1+u)^2}. \tag{38}$$

The functions entering the r.h.s. of eq. (33) are:

- $C\left[x,w;\alpha(w\,m_b)\right]$, a short-distance, process-dependent coefficient function, whose explicit expression will be given later. It depends on two independent energies x and w and on the QCD coupling α;
- $\sigma\left[u;\alpha(w\,m_b)\right]$, a process-independent, long-distance dominated, QCD form factor. It factorizes the threshold logarithms appearing in the perturbative expansion. At order α:

$$\sigma(u;\alpha) \;=\; \delta(u)\,-\,\frac{C_F\,\alpha}{\pi}\left(\frac{\log u}{u}\right)_+ \,-\,\frac{7\,C_F\,\alpha}{4\,\pi}\left(\frac{1}{u}\right)_+ \,+\,O(\alpha^2), \tag{39}$$

[8]We have normalized the distribution to the radiatively-corrected total semileptonic width $\Gamma = \Gamma_0\left[1+\alpha C_F/\pi\left(25/8-\pi^2/2\right)+O(\alpha^2)\right]$ and not to the Born width $\Gamma^{(0)}$, as originally done in [3]. We consider it to be a better choice because Γ, unlike $\Gamma^{(0)}$, is a physical quantity, directly measurable in the experiments and we are not interested in the prediction of total rates, but only in how a given rate distributes among different hadronic channels.

[9]Note that a similar variable simplifies two-loop computations with heavy quarks [9].

where C_F is the Casimir of the fundamental representation of $SU(3)_c$, $C_F = (N_c^2-1)/(2N_c)$ with $N_c = 3$ (the number of colors) and the plus distributions are defined as usual as:

$$P(u)_+ \,=\, P(u) - \delta(u) \int_0^1 du' P(u'). \tag{40}$$

The action on a test function $f(u)$ is therefore:

$$\int_0^1 du\, P(u)_+ \, f(u) \,=\, \int_0^1 du\, P(u)\, [f(u) \,-\, f(0)]. \tag{41}$$

The plus-distributions are sometimes called star-distributions and can also be defined as limits of ordinary functions as:

$$\begin{aligned} P(u)_+ \quad &= \quad \lim_{\epsilon\to 0^+} \left[\theta(u-\epsilon)\, P(u) \,-\, \delta(u) \int_\epsilon^1 du'\, P(u')\right] \\ &= \quad \lim_{\epsilon\to 0^+} \left[\theta(u-\epsilon)\, P(u) \,-\, \delta(u-\epsilon) \int_\epsilon^1 du'\, P(u')\right] \\ &= \quad \lim_{\epsilon\to 0^+} -\frac{d}{du}\left[\theta(u-\epsilon) \int_u^1 du'\, P(u')\right]. \end{aligned} \tag{42}$$

An important property of the plus-distributions is that their integral on the unit interval vanishes:

$$\int_0^1 P(u)_+ \, du \,=\, 0. \tag{43}$$

We have assumed a minimal factorization scheme in eq. (39), in which only terms containing plus-distributions are included in the form factor. The resummation of the logarithmically enhanced terms in σ to all orders in perturbation theory will be discussed in the next section;

- $d\,(x, u, w;\, \alpha)$ is a short-distance, process-dependent, remainder function, not containing large logarithms. Formally, it can have at most an integrable singularity for $u \to +0$, i.e. we require that:

$$\lim_{u\to+0} \int_0^u du' d(x, w, u'; \alpha) \,=\, 0. \tag{44}$$

This term is added to $C \cdot \sigma$ in order to correctly describe the region $u \sim O(1)$ and to reproduce the total rate. It depends on all the kinematical variables x, w and u and the explicit expression will be given later.

Eq. (33) is a generalization of the threshold resummation formula for the radiative decay in (5) [3, 10]:

$$\frac{1}{\Gamma_R}\frac{d\Gamma_R}{dt_s} \,=\, C_R\left[\alpha(w\, m_b)\right]\, \sigma\left[t_s;\, \alpha(w\, m_b)\right] \,+\, d_R\left[t_s;\, \alpha(w\, m_b)\right], \tag{45}$$

where[10]

$$t_s \,=\, \frac{m_{X_s}^2}{m_b^2}. \tag{46}$$

In this simpler case, the coefficient function $C_R(\alpha)$ does not depend on any kinematical variable but only on the QCD coupling α and has an expansion of the form[11]:

$$C_R(\alpha) \,=\, 1 \,+\, \alpha\, C_R^{(1)} \,+\, \alpha^2\, C_R^{(2)} \,+\, O(\alpha^3), \tag{47}$$

[10] The relation with the photon energy $x_\gamma = 2E_\gamma/m_b$ is $t_s = 1 - x_\gamma$.

[11] We perform expansions in powers of α, while the traditional expansion is in powers of $\alpha/(2\pi)$.

where $C_R^{(i)}$ are numerical coefficients. Basically, going from the 2-body decay (5) to the 3-body decay (7), the coefficient function acquires a dependence on the additional kinematical variables, namely two energies. The remainder function in eq. (45) depends on the (unique) variable t_s and has an expansion of the form:

$$d_R(t_s;\,\alpha)\;=\;\alpha\,d_R^{(1)}(t_s)\;+\;\alpha^2\,d_R^{(2)}(t_s)\;+\;O(\alpha^3). \tag{48}$$

The main point is that the QCD form factor σ in the same in both distributions (33) and (45), explicitly showing universality of long-distance effects in the two different decays. By universality we mean that we have the same function, evaluated at the argument u in the semileptonic case and at t_s in the radiative decay. This property is not explicit in the original formulation [11], in which the form factors differ in subleading order (see next section).

Since, as shown in the introduction, $w \sim 1$ in the radiative decay, we can make everywhere in eq. (45) the replacement

$$\alpha(w\,m_b)\;\rightarrow\;\alpha(m_b) \qquad\qquad \text{(radiative case only)}, \tag{49}$$

to obtain:

$$\frac{1}{\Gamma_R}\frac{d\Gamma_R}{dt_s}\;=\;C_R\left[\alpha(m_b)\right]\,\sigma\left[t_s;\,\alpha(m_b)\right]\;+\;d_R\left[t_s;\,\alpha(m_b)\right]. \tag{50}$$

The distribution contains now a constant coupling, independent on the kinematics $\alpha(m_b) \simeq 0.22$. The replacement (49) cannot be done in the semileptonic case.

In [3] the triple differential distribution was originally given in terms of the variable y instead of u, with the latter $u = 1-\xi$ being introduced in [10]. The variables u and y coincide in the threshold region in leading twist, i.e. at leading order in u in the expansion for $u \to 0$, as $y = u + O(u^2)$. Going from the variable y to the variable u only modifies the remainder function. The advantages of u over y are both technical and physical:

- u has, unlike y, unitary range;
- when we impose the kinematical relation between hadronic energy E_{X_s} and hadronic mass m_{X_s} of the radiative decay (5), u exactly equals t_s:

  $$u|_{E_{X_s}=m_b/2(1+m_{X_s}^2/m_b^2)}\;=\;t_s. \tag{51}$$

  This property suggests that some higher-twist effects may cancel in taking proper ratios of radiative and semileptonic spectra.

Let us now give the explicit expression of the coefficient function in the semileptonic case:

$$C(\overline{x},w;\,\alpha)\;=\;C^{(0)}(\overline{x},w)\;+\;\alpha\,C^{(1)}(\overline{x},w)\;+\;\alpha^2\,C^{(2)}(\overline{x},w)\;+\;O(\alpha^3), \tag{52}$$

where

$$C^{(0)}(\overline{x},w)\;=\;12(w-\overline{x})(1+\overline{x}-w); \tag{53}$$

$$C^{(1)}(\overline{x},w)\;=\;12\frac{C_F}{\pi}(w-\overline{x})\left\{(1+\overline{x}-w)\left[\mathrm{Li}_2(w)+\log w\log(1-w)-\frac{3}{2}\log w-\frac{w\log w}{2(1-w)}-\frac{35}{8}\right]+\right.$$
$$\left.+\;\frac{\overline{x}\,\log w}{2(1-w)}\right\} \tag{54}$$

with $\overline{x} = 1-x$ [12]. Note that the coefficient function contains the overall factor $w-\overline{x} = \overline{x}_\nu$, which vanishes linearly at the endpoint of the neutrino spectrum. We have defined $\overline{x}_\nu = 1-x_\nu$ and $x_\nu = 2E_\nu/m_b$.

[12] To avoid spurious imaginary parts for $w > 1$ one can use the relation $\mathrm{Li}_2(w) = -\mathrm{Li}_2(1-w) - \log w\log(1-w) + \pi^2/6$.

Unlike the coefficient function, the remainder function $d\,(x,u,w;\alpha)$ has an expansion starting at $O(\alpha)$:

$$d\,(x,w,u;\,\alpha)\;=\;\alpha\,d^{(1)}\,(x,w,u)\;+\;\alpha^2\,d^{(2)}\,(x,w,u)\;+\;O(\alpha^3). \tag{55}$$

Omitting the overall factor C_F/π, we obtain: [13]

$$\begin{aligned}
d^{(1)}(\overline{x},w,u) \;\;=\;\; & \frac{-3\,w^4\,\left(24+3\,w-8\,\overline{x}\right)}{4\,(1+u)^5}+\frac{9\,w^4\,\left(24+3\,w-8\,\overline{x}\right)}{8\,(1+u)^4}+\\
- \;\; & \frac{9\,\left(-12+w\right)\,\left(-2+w\right)^2\left(w-2\,\overline{x}\right)^2}{16\,(1-u)^3}+\frac{9\,\left(-12+w\right)\,\left(-2+w\right)^2\left(w-2\,\overline{x}\right)^2}{32\,(1-u)^2}+\\
- \;\; & \frac{3\,w^2\,\left(32-47\,w-8\,w^2+16\,\overline{x}+20\,w\,\overline{x}+w^2\,\overline{x}+8\,\overline{x}^2-3\,w\,\overline{x}^2\right)}{8\,(1+u)^2}+\\
- \;\; & \frac{3\,w^2\,\left(-64+94\,w+40\,w^2+3\,w^3-32\,\overline{x}-40\,w\,\overline{x}-10\,w^2\,\overline{x}-16\,\overline{x}^2+6\,w\,\overline{x}^2\right)}{8\,(1+u)^3}+\\
+ \;\; & \frac{3}{64\,(1+u)}\Big(640\,w-368\,w^2-200\,w^3-16\,w^4+3\,w^5-384\,\overline{x}+320\,w\,\overline{x}+528\,w^2\,\overline{x}+\\
+ \;\; & 112\,w^3\,\overline{x}-16\,w^4\,\overline{x}-256\,\overline{x}^2-48\,w\,\overline{x}^2-224\,w^2\,\overline{x}^2+24\,w^3\,\overline{x}^2\Big)+\\
+ \;\; & \frac{3}{64\,(1-u)}\Big(-256\,w+528\,w^2-200\,w^3-16\,w^4+3\,w^5+512\,\overline{x}-1472\,w\,\overline{x}+\\
+ \;\; & 528\,w^2\,\overline{x}+112\,w^3\,\overline{x}-16\,w^4\,\overline{x}+640\,\overline{x}^2-48\,w\,\overline{x}^2-224\,w^2\,\overline{x}^2+24\,w^3\,\overline{x}^2\Big)+\\
- \;\; & \frac{9\,w^5\,\log\,u}{4\,(1+u)^6}+\frac{9\,w^5\,\log\,u}{2\,(1+u)^5}-\frac{9\,\left(-12+w\right)\,\left(-2+w\right)^2\left(w-2\,\overline{x}\right)^2\,\log\,u}{16\,(1-u)^4}+\\
+ \;\; & \frac{9\,\left(-12+w\right)\,\left(-2+w\right)^2\left(w-2\,\overline{x}\right)^2\,\log\,u}{16\,(1-u)^3}+\\
+ \;\; & \frac{3\,w^3\,\left(-10+16\,w+w^2+8\,\overline{x}-2\,w\,\overline{x}-2\,\overline{x}^2\right)\,\log\,u}{8\,(1+u)^3}+\\
- \;\; & \frac{3\,w^3\,\left(-10+16\,w+7\,w^2+8\,\overline{x}-2\,w\,\overline{x}-2\,\overline{x}^2\right)\,\log\,u}{8\,(1+u)^4}+\\
- \;\; & \frac{3\log\,u}{64\,(1+u)^2}\,w\,\Big(-144\,w+208\,w^2+16\,w^3+w^4-64\,\overline{x}-80\,w\,\overline{x}-16\,w^2\,\overline{x}+\\
- \;\; & 8\,w^3\,\overline{x}+48\,\overline{x}^2-96\,w\,\overline{x}^2+16\,w^2\,\overline{x}^2\Big)+\\
+ \;\; & \frac{3\log\,u}{64\,(1-u)^2}\Big(-256\,w+624\,w^2-304\,w^3+16\,w^4+w^5+512\,\overline{x}-1856\,w\,\overline{x}+\\
+ \;\; & 944\,w^2\,\overline{x}-16\,w^3\,\overline{x}-8\,w^4\,\overline{x}+1024\,\overline{x}^2-464\,w\,\overline{x}^2-96\,w^2\,\overline{x}^2+16\,w^3\,\overline{x}^2\Big).
\end{aligned} \tag{57}$$

The remainder function is a combination of rational functions of u multiplied in some cases by $\log u$, with coefficients given by polynomials in w and $\overline{x}$.

[13] The $O(\alpha)$ function is obtained from that one given in [3] $d^{(1)}_{old}\,(x,w,y)$ in terms of the variables $z=1-y$ and $\zeta=1-4y$, by using a relation extending eq. (23) of [10]:

$$d^{(1)}\,(x,w,u)\;=\;d^{(1)}_{old}\,(x,w,y(u))\;\frac{dy}{du}(u)\;+\;C^{(0)}(x,w)\frac{C_F}{\pi}\left[\frac{\log u+7/4}{u}\;-\;\frac{\log y(u)+7/4}{y(u)}\,\frac{dy}{du}(u)\right]. \tag{56}$$

This function can also be obtained with a direct matching with the $O(\alpha)$ triple differential distribution computed in [8] after a change of variable (see the end of this section for a discussion about matching).

The main point about the semileptonic decay (7) is that it has — unlike the radiative decay (5) — $q^2 \neq 0$ and consequently the form factor depends not only on u but also on the hadronic energy w through the running coupling:

$$\sigma \,=\, \sigma[u;\, \alpha(w\, m_b)]. \tag{58}$$

The form factor is therefore a function of two variables.

We work in next-to-leading order (NLO), in which only the $O(\alpha)$ corrections to the coefficient function and remainder function are retained (see next section). Since the difference between $\alpha(w\, m_b)$ and $\alpha(m_b)$ is $O(\alpha^2)$, we can set $w = 1$ in the argument of the coupling entering the coefficient function and the remainder function. We then obtain the simpler expression:

$$\frac{1}{\Gamma}\frac{d^3\Gamma}{dxdudw} \,=\, C\left[x, w; \alpha(m_b)\right]\, \sigma\left[u; \alpha(w\, m_b)\right] \,+\, d\left[x, u, w; \alpha(m_b)\right] \qquad \text{(NLO)}. \tag{59}$$

Note that we cannot set $w = 1$ in the coupling entering the form factor, because in the latter case α is multiplied by large logarithms, which "amplify" $O(\alpha^2)$ differences in the couplings (see next section).

Let us make a few remarks about the final result of this section, eq. (59):

- it describes semi-inclusive decays, in which the internal structure of the hadronic final states is not observed, but only the total mass and energy are measured. Less inclusive quantities, such as for instance the energy distribution of the final *up* quark (i.e. the fragmentation function of the *up* quark), cannot be computed in this framework;
- it constitutes an improvement of the fixed-order $O(\alpha)$ result in all the cases in which there are large threshold logarithms. In all the other cases, where there are no threshold logarithms, such as for example the dilepton mass distribution [12], there is not any advantage of the resummed formula over the fixed-order one.

In the next sections we integrate the resummed triple-differential distribution to obtain double and single (resummed) spectra. There are two methods to accomplish this task which are completely equivalent:

1. The first method involves the direct integration of the complete triple-differential distribution. Schematically:

$$\text{(spectrum)} \,=\, \int C \cdot \sigma \,+\, \int d. \tag{60}$$

   Large logarithms come only from the first term on the r.h.s. of (60), while non-logarithmic, "small" terms come both from the first and the second term. To obtain a factorized form for the spectrum analogous to the one for the triple-distribution, in which the remainder function collects *all* the small terms, one rearranges the r.h.s. of (60): the small terms coming from the integration of $C \cdot \sigma$ are put in the remainder function;
2. In the second method, one integrates the block $C \cdot \sigma$ only and drops the small terms coming from the integration. The remainder function is obtained by expanding the resummed expression in powers of α and comparing with the fixed-order spectrum.

3 Threshold Resummation

It is convenient to define the partially integrated or cumulative form factor $\Sigma(u, \alpha)$:

$$\Sigma(u;\, \alpha) \,=\, \int_0^u du'\, \sigma(u';\, \alpha). \tag{61}$$

Performing the integrations, one obtains for the $O(\alpha)$ form factor:

$$\Sigma\,(u;\,\alpha)\;=\;1\,-\,\frac{C_F\,\alpha}{2\,\pi}\,L^2\,+\,\frac{7\,C_F\,\alpha}{4\,\pi}\,L\,+\,O(\alpha^2), \tag{62}$$

where

$$L\;=\;\log\frac{1}{u}. \tag{63}$$

Σ contains a double logarithm coming from the overlap of the soft and the collinear region and a single logarithm of soft or collinear origin. The normalization condition reads:

$$\Sigma(1;\,\alpha)\;=\;\int_0^1 du\,\sigma(u;\alpha)\;=\;1. \tag{64}$$

As already noted, we have assumed a minimal factorization scheme, in which only logarithms and not constants or other functions are contained in the form factor. The expression of the partially integrated form factor Σ is technically simpler than the one for the differential form factor σ, as it involves ordinary functions instead of generalized ones. Furthermore, in experiments one always measures some integral of σ around a central u value because of the binning.

In the limit $u \to 0^+$, no final states are included in the distribution and therefore one expects, on physical grounds, that

$$\lim_{u\to 0^+}\,\Sigma(u;\,\alpha)\;=\;0. \tag{65}$$

The $O(\alpha)$ expression (62) does not have this limit and it is actually divergent to $-\infty$ — a completely un-physical result. In general, a truncated expansion in powers of α is divergent for $u \to 0^+$, because the coefficients diverge in this limit. Therefore, one has to resum the infrared logarithms, i.e. the terms of the form $\alpha^n\,L^k$, to all orders in perturbation theory. In higher orders, Σ contains at most two logarithms for each power of α, one of soft origin and another one of collinear origin. Its general expression is then:

$$\Sigma\,(L,\alpha)\;=\;1\,+\,\sum_{n=1}^{\infty}\sum_{k=1}^{2n}\Sigma_{nk}\,\alpha^n\,L^k, \tag{66}$$

where Σ_{nk} are numerical coefficients. At present, a complete resummation of all the logarithmically-enhanced terms on the r.h.s. of eq. (66) is not feasible in QCD: one has to resort to approximate schemes. The most crude approximation consists of picking up the most singular term for $u \to 0^+$ for each power of α, i.e. all the terms of the form:

$$\alpha^n\,L^{2n}\qquad\qquad\text{(double logarithmic approximation)}. \tag{67}$$

In this approximation, we can neglect running coupling effects and effects related to the kinematical constraints: higher orders simply exponentiate the $O(\alpha)$ double logarithm and one obtains

$$\Sigma\,(u;\,\alpha)\;=\;e^{-\,C_F\,\alpha/(2\pi)\,L^2}\qquad\qquad\text{(double logarithmic approximation)}. \tag{68}$$

Let us note that the resummed expression (68), unlike the fixed-order one (62), does satisfy the condition (65). The exponent in the resummed form factor involves a single term, $-\,C_F\,\alpha/(2\pi)\,L^2$, and has therefore a simpler form than the form factor itself. This remains true when more accurate resummation schemes are constructed, so it is convenient to define G as:

$$\Sigma\;=\;e^G. \tag{69}$$

It can be shown that the expansion for the function G is of the form [13]:

$$G(L;\alpha)\;=\;\sum_{n=1}^{\infty}\sum_{k=1}^{n+1}G_{nk}\,\alpha^n\,L^k, \tag{70}$$

where G_{nk} are numerical coefficients. Let us note that the sum over k extends up to $n+1$ in (70), while it extends up to $2n$ in the form factor in eq. (66). This property is a generalization of the simple exponentiation of the $O(\alpha)$ logarithms which holds in QED and is called generalized exponentiation. In general, this property holds for quantities analogous to the semi-inclusive form factors, in which the gluon radiation is not directly observed. One sums therefore over all possible final states coming from the evolution of the emitted gluons (inclusive gluon decay quantities). The property expressed by eq. (70) does not hold for quantities in which gluon radiation is observed directly, as for example in parton multiplicities, where different evolutions of gluon jets give rise to different multiplicities.

3.1 N-space

A systematic resummation is consistently done in N-moment space or Mellin space, in which kinematical constraints are factorized in the soft limit and are easily integrated over [14]. One considers the Mellin transform of the form factor $\sigma(u;\,\alpha)$:

$$\sigma_N(\alpha)\,\equiv\,\int_0^1 du\,(1-u)^{N-1}\,\sigma(u;\,\alpha). \tag{71}$$

The threshold region is studied in moment space by taking the limit $N\to\infty$, because for large N the integral above takes contributions mainly from the region $u\ll 1$. For example, the Mellin transform of the spectrum in eq. (50) is of the form

$$\int_0^1 (1-t_s)^{N-1}\frac{1}{\Gamma_R}\frac{d\Gamma_R}{dt_s}\,dt_s\,=\,C_R(\alpha)\,\sigma_N(\alpha)\,+\,d_{R,N}(\alpha), \tag{72}$$

where

$$d_{R,N}(\alpha)\,\to\,0 \qquad\qquad \text{for } N\,\to\,\infty. \tag{73}$$

The total rate in Mellin space is obtained by taking $N=1$.

It can be shown [15, 1, 16] that the form factor in N-space has the following exponential structure:

$$\sigma_N(\alpha)\,=\,e^{G_N(\alpha)}, \tag{74}$$

where

$$G_N(\alpha)\,=\,\int_0^1 dz\frac{z^{N-1}-1}{1-z}\left\{\int_{Q^2(1-z)^2}^{Q^2(1-z)}\frac{dk_t^2}{k_t^2}A\left[\alpha(k_t^2)\right]\,+\,B\left[\alpha(Q^2(1-z))\right]\,+\,D\left[\alpha(Q^2(1-z)^2)\right]\right\}. \tag{75}$$

Let us note that a prescription has to be assigned to this formula since it involves integrations over the Landau pole [17]. The functions entering the resummation formula have a standard fixed-order expansion, with numerical coefficients:

$$A(\alpha)\;=\;\sum_{n=1}^{\infty}A_n\,\alpha^n\,=\,A_1\alpha+A_2\alpha^2+A_3\alpha^3+A_4\alpha^4+\cdots \tag{76}$$

$$B(\alpha)\;=\;\sum_{n=1}^{\infty}B_n\,\alpha^n\,=\,B_1\alpha+B_2\alpha^2+B_3\alpha^3+\cdots \tag{77}$$

$$D(\alpha)\;=\;\sum_{n=1}^{\infty}D_n\,\alpha^n\,=\,D_1\alpha+D_2\alpha^2+D_3\alpha^3+\cdots \tag{78}$$

The known values for the resummation constants read:

$$A_1 = \frac{C_F}{\pi}; \tag{79}$$
$$A_2 = \frac{C_F}{\pi^2}\left[C_A\left(\frac{67}{36}-\frac{z(2)}{2}\right)-\frac{5}{18}n_f\right]; \tag{80}$$
$$\begin{aligned} A_3 = \frac{C_F}{\pi^3}\Bigg[& C_A^2\Big(\frac{245}{96}+\frac{11}{24}z(3)-\frac{67}{36}z(2)+\frac{11}{8}z(4)\Big) - C_A\, n_f\Big(\frac{209}{432}+\frac{7}{12}z(3)-\frac{5}{18}z(2)\Big)+ \\ & - C_F\, n_f\Big(\frac{55}{96}-\frac{z(3)}{2}\Big)-\frac{n_f^2}{108}\Bigg]; \end{aligned} \tag{81}$$
$$B_1 = -\frac{3}{4}\frac{C_F}{\pi}; \tag{82}$$
$$B_2 = \frac{C_F}{\pi^2}\left[C_A\left(-\frac{3155}{864}+\frac{11}{12}z(2)+\frac{5}{2}z(3)\right)-C_F\left(\frac{3}{32}+\frac{3}{2}z(3)-\frac{3}{4}z(2)\right)+n_f\left(\frac{247}{432}-\frac{z(2)}{6}\right)\right]; \tag{83}$$
$$D_1 = -\frac{C_F}{\pi}; \tag{84}$$
$$D_2 = \frac{C_F}{\pi^2}\left[C_A\left(\frac{55}{108}-\frac{9}{4}z(3)+\frac{z(2)}{2}\right)+\frac{n_f}{54}\right], \tag{85}$$

where $C_A = N_c = 3$ is the Casimir of the adjoint representation. The coefficients A_1, B_1 and D_1 are renormalization-scheme independent, as they can be obtained from tree-level amplitudes with one-gluon emission (see later). The higher-order coefficients are instead renormalization-scheme dependent and are given in the $\overline{MS}$ scheme for the coupling constant [14].

To this approximation, the first three orders of the β-function are also needed [19, 20]:

$$\beta_0 = \frac{1}{4\pi}\Big[\frac{11}{3}C_A-\frac{2}{3}n_f\Big]; \tag{86}$$
$$\beta_1 = \frac{1}{24\pi^2}\Big[17\,C_A^2-\Big(5\,C_A+3\,C_F\Big)n_f\Big]; \tag{87}$$
$$\beta_2 = \frac{1}{64\pi^3}\Big[\frac{2857}{54}C_A^3-\left(\frac{1415}{54}C_A^2+\frac{205}{18}C_AC_F-C_F^2\right)n_f+\left(\frac{79}{54}C_A+\frac{11}{9}C_F\right)n_f^2\Big]. \tag{88}$$

As is well known, β_0 and β_1 are renormalization-scheme independent, while β_2 is not and has been given in the $\overline{MS}$ scheme. We define the β-function with an overall minus sign:

$$\frac{d\alpha}{d\log\mu^2} = -\,\beta(\alpha) = -\,\beta_0\,\alpha^2 - \beta_1\,\alpha^3 - \beta_2\,\alpha^4 - \cdots . \tag{89}$$

The running coupling reads:

$$\alpha(\mu) = \frac{1}{\beta_0\,\log\mu^2/\Lambda^2} - \frac{\beta_1}{\beta_0^3}\frac{\log\left(\log\mu^2/\Lambda^2\right)}{\log^2\mu^2/\Lambda^2} + \frac{\beta_1^2}{\beta_0^5}\frac{\log^2\left(\log\mu^2/\Lambda^2\right) - \log\left(\log\mu^2/\Lambda^2\right) - 1}{\log^3\mu^2/\Lambda^2} + \frac{\beta_2}{\beta_0^4}\frac{1}{\log^3\mu^2/\Lambda^2}. \tag{90}$$

The functions $A(\alpha)$, $B(\alpha)$ and $D(\alpha)$ have the following physical interpretation (see for example [21, 22]):

- The function $A(\alpha)$ involves a double integration over the transverse momentum k_t and the energy ω of the emitted gluon and represents emissions at small angle and at small energy from the light quark. The

[14] A discussion about the scheme dependence of the higher order coefficients A_2, B_2, etc. on the coupling constant can be found in [18].

leading term A_1 is the coefficient of that piece of the matrix element squared for one real gluon emission, which is singular in the small angle and small energy limit:

$$A_1 \, \alpha \, \frac{d\omega}{\omega} \, \frac{d\theta^2}{\theta^2} \cong A_1 \, \alpha \, \frac{d\omega}{\omega} \, \frac{dk_t^2}{k_t^2}, \tag{91}$$

where $k_t \simeq \omega \, \theta$ is the transverse momentum of the gluon. In (91) we have given the representation of the integral both in the angle θ and in the transverse momentum k_t. The subleading coefficients A_2, A_3, etc. represent corrections to the basic double-logarithmic emission. The function $A(\alpha)$ "counts" the number of light quark jets in different processes, i.e. we can write

$$A^{(P)}(\alpha) = n_q \, A(\alpha), \tag{92}$$

where n_q is the number of primary light quarks in the process P. For example, in e^+e^- annihilation into hadrons $n_q = 2$, while in the heavy flavor decays (2) $n_q = 1$. Since soft gluons only couple to the four-momentum of their emitters and not to their spin, the function $A_g(\alpha)$ for gluon jets is obtained from the quark one $A(\alpha)$ simply taking into account the change in the color charge, i.e. multiplying by C_A/C_F [23];

- the function $B(\alpha)$ represents emissions at small angle with a large energy from the light quark. B_1 is the coefficient of that piece of the matrix element squared which is singular in the small angle limit:

$$B_1 \, \alpha \, d\omega \, \frac{d\theta^2}{\theta^2} \cong B_1 \, \alpha \, d\omega \, \frac{dk_t^2}{k_t^2}. \tag{93}$$

The non logarithmic integration over the gluon energy ω has been done and does not appear explicitly in eq. (75); the integration over the angle θ or the transverse momentum k_t is rewritten as an integral over z. The function $B(\alpha)$ counts the number of final-quark jets, i.e.

$$B^{(P)}(\alpha) = n_l \, B(\alpha), \tag{94}$$

where n_l is the number of primary final quarks in the process P. For example in e^+e^- annihilation into hadrons $n_l = 2$, while in DIS or in the heavy flavor decays (2) $n_l = 1$. Since hard collinear emissions are sensitive to the spin of the emitting particles, the gluon function $B_g(\alpha)$ is not simply related to the quark one $B(\alpha)$ [23];

- the function $D(\alpha)$ represents emissions at large angle and small energy from the heavy quark. D_1 is the coefficient of that piece of the matrix element squared which is singular in the small energy limit:

$$D_1 \, \alpha \, \frac{d\omega}{\omega} \, d\theta^2. \tag{95}$$

The non logarithmic integration over the angle θ or the transverse momentum k_t has been done and does not appear explicitly in eq. (75); the integration over the energy ω is rewritten as an integral over z. $D_1 = 0$ in all the processes involving light partons only, as for instance DIS, Drell-Yan (DY) or e^+e^- annihilation into hadrons, while it is not zero in all the processes containing at least one heavy quark, such as for example the heavy flavor decays (2). Note that the effective coupling appearing in the D terms is $\alpha\left[Q^2(1-z)^2\right]$ and is therefore substantially larger for $1 - z \ll 1$ than the coupling entering the hard collinear terms, namely $\alpha\left[Q^2(1-z)\right]$.

Eq. (75) is therefore a generalization of the $O(\alpha)$ result, possessing a double logarithm coming from the overlap of the soft and the collinear region and a single logarithm of soft or collinear origin (see eqs. (91), (93) and (95)) [15]. The functions $A(\alpha)$ and $B(\alpha)$ are believed to by universal, i.e. process independent to any order in perturbation theory, as they represent the development of a parton into a jet, i.e. one-particle properties. The function $D(\alpha)$ on the contrary is process-dependent, as it describes soft emission at large angle, with interference

[15] Let us remember however that only two of the three functions appearing in eq. (75) are independent [16].

contributions from all the hard partons in the process, i.e. it describes global properties of the hadronic final states. Let us observe that A_2 and D_2, unlike B_2, do not have a C_F^2 contribution. That is a consequence of the eikonal identity, which holds in the soft limit [2]. According to this identity, the abelian contributions simply exponentiate the lowest order $O(\alpha\, C_F)$ term, just like in QED. That means that there are no higher order terms in the exponent G_N. Because of similar reasonings, A_3 does not have a C_F^3 contribution.

Despite its supposed asymptotic nature, the numerical values of the coefficients show a rather good convergence of the perturbative series. Note that all the double-logarithmic coefficients A_i are positive, implying an increasing suppression with the order of the expansion (up to the third one) of the rate in the threshold region. On the contrary, the single-logarithmic coefficients B_i and D_i – with the exception of B_2 — are all negative and therefore tend to enhance the rate in the threshold region [24]. We have:

$$\begin{aligned}
A_1 &= +\,0.424413; && (96)\\
A_2 &= +\,0.420947\,-\,0.0375264\,n_f\,=\,0.308367; && (97)\\
A_3 &= +\,0.592067\,-\,0.0923137 n_f\,-\,0.000398167\,n_f^2\,=\,0.311542; && (98)\\
B_1 &= -\,0.318310; && (99)\\
B_2 &= +\,0.229655\,+\,0.04020\,n_f\,=\,0.350269; && (100)\\
D_1 &= -\,0.424413; && (101)\\
D_2 &= -\,0.556416\,+\,0.002502\,n_f\,=\,-\,0.548911. && (102)
\end{aligned}$$

With our definition, the β-function coefficients are, as well known, all positive.

$$\begin{aligned}
\beta_0 &= +\,0.87535\,-\,0.05305\,n_f\,=\,+\,0.71620; && (103)\\
\beta_1 &= +\,0.64592\,-\,0.08021\,n_f\,=\,+\,0.40529; && (104)\\
\beta_2 &= +\,0.71986\,-\,0.140904\,n_f\,+\,0.003032\,n_f^2\,=\,+\,0.324436. && (105)
\end{aligned}$$

In the last member we have assumed 3 active flavors ($n_f = 3$).

Let us now discuss the computation of the coefficients entering the resummation formula. The occurrence of a Sudakov form factor in semileptonic B decays was acknowledged originally in [25], where a simple exponentiation involving A_1 and $B_1 + D_1$ was performed. The coefficient A_2 was computed for the first time, as far as we know, in [26]. It was denoted $A_1\,K$ since it was considered a kind of renormalization of the lowest-order contribution:

$$A_1\,\alpha\,\rightarrow\,A_1\,\alpha(1\,+\,K\,\alpha). \tag{106}$$

The coefficient A_2 was obtained from the soft-singular part of the $q \rightarrow q$ two-loop splitting function [27], that is as the coefficient of the $1/(1-z)$ term[16]. A_2 was subsequently recomputed in [29] in the framework of Wilson line theory, where the function $A(\alpha)$ has a geometrical meaning: it is the anomalous dimension of a cusp operator, representing the radiation emitted because of a sudden change of velocity of a heavy quark,

$$\Gamma_{cusp}(\alpha)\,=\,\sum_{n=1}^{\infty}\Gamma_{cusp}^{(n)}\,\alpha^n\,=\,\Gamma_{cusp}^{(1)}\,\alpha\,+\,\Gamma_{cusp}^{(2)}\,\alpha^2\,+\,\cdots \tag{107}$$

Indeed, it has been explicitly checked up to second order that these two functions coincide:

$$A(\alpha)\,=\,\Gamma_{cusp}(\alpha). \tag{108}$$

Let us note that:

- the theory of Wilson lines and Wilson loops;

[16] This is exactly the same procedure which has been followed to derive the third-order coefficient A_3 from the three-loop splitting function [28].

- the eikonal or soft approximation in perturbative QCD;
- the heavy quark effective theory (HQET) and the large energy effective theory (LEET),

all involve basically the same structure, i.e. the same propagators and vertices and the same amplitudes. Since the same structure has been studied in different frameworks, there is multiple notation and terminology for the same objects. Let us stress however that in ordinary QCD the function $A(\alpha)$ is not an anomalous dimension, since it is not obtained from ultraviolet $1/\epsilon$ poles in renormalization constants but from infrared poles or from finite parts of scattering amplitudes. $A(\alpha)$ becomes an anomalous dimension *in* the effective theory because the latter has additional ultraviolet divergencies with respect to QCD. While in QCD one has to subtract only ultraviolet divergencies related to coupling constant renormalization, in the effective theory one has also to subtract *additional* ultraviolet divergencies related to the cusp operators. A scheme dependence is therefore introduced in the effective theory, which is not present in full QCD. It seems to us therefore that the equality (108) is not guaranteed *a priori* in higher orders and may require a specific scheme for the subtractions in the effective theory. At present, A_3 has only been derived in full QCD and not in the effective theory.

The coefficient B_2 has been computed by means of the second order correction to the inclusive DIS cross section, which contains the combination $B_2 + D_2^{DIS}$ (the DIS analogue of eq. (136), see later) and by means of the third order correction, which contains the different combination $B_2 + 2D_2^{DIS}$ (the DIS analogue of eq. (139), see later). The knowledge of the fermionic contribution to the $O(\alpha^3)$ DIS cross section was sufficient for a complete determination of B_2 [30], with later checks offered by the complete computation [28, 31].

An incorrect value for the coefficient D_2 for heavy favor decays has been obtained in the original computation in [32], where the technique to compute real and virtual diagrams in the effective-theory in configuration space has also been developed. In [33] the coefficient of the single logarithm in the radiative decay (5) to order α^2 has been presented, from which the correct value of D_2 can be extracted (let us note however that numerically the two values are not very different). A second order computation of heavy flavor fragmentation in ordinary QCD was presented in [34], which allows the determination of the sum $B_2 + D_2^{frag}$ (the analogue of eq. (136), see later). Using an identity relating the coefficient for heavy flavor fragmentation with that one for heavy flavor decays, and subtracting the known value for the universal coefficient B_2, the correct value for D_2 was explicitly derived in [35] (see also [36]). Still in [35], by repeating the Wilson line computation of [32], errors were found and the same value of D_2 extracted from heavy flavor fragmentation was re-obtained. Recently, the second order contribution of the chromomagnetic operator O_7 to the photon spectrum in the radiative decay (5) has been calculated [37], confirming these results (see also [38]).

According to the previous remarks concerning the relation between $A(\alpha)$ and $\Gamma_{cusp}(\alpha)$, we believe it is a non-trivial fact that the same value of D_2 is obtained with two completely different methods:

- a direct computation in the effective theory, which describes the soft region only;
- an extraction from an ordinary QCD computation, which gives the sum of the soft and the collinear contributions $B_2 + D_2$, by subtracting the collinear contribution B_2 obtained from second order and third order DIS computations.

The following expansion holds true for the exponent:

$$G_N(\alpha) = \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} G_{nk} \alpha^n l^k + O\left(\frac{1}{N}\right), \tag{109}$$

where

$$l = \log N. \tag{110}$$

The expansion of the logarithm of the form factor has a similar structure in physical space and in N-space; roughly speaking, going to N-space, $\log 1/u \rightarrow \log N$.

As already discussed, we are interested in the large-N limit; the $O(1/N)$ terms can be neglected in our leading-twist analysis. A resummation of all the logarithmically-enhanced terms in (109) is at present unfeasible in QCD even in N-space, so one has to rely on approximate schemes. Let us discuss the fixed-logarithmic accuracy scheme:

- Leading order (LO). One keeps in the exponent $G_N(\alpha)$ only the leading power of the logarithm for each power of α, i.e. $k = n + 1$:

$$G_N^{LO} = \sum_{n=1}^{\infty} G_{n\,n+1} \alpha^n l^{n+1} = G_{12}\, \alpha\, l^2 + G_{23}\, \alpha^2\, l^3 + O(\alpha^3). \tag{111}$$

  The coefficient function is kept in lowest order, i.e. $C^{LO} = 1$ and the remainder function is completely neglected, i.e. $d^{LO} = 0$;

- Next-to-leading order (NLO). One keeps in $G_N(\alpha)$ also the terms with $n = k$, i.e.:

$$\begin{aligned} G_N^{NLO} &= \sum_{n=1}^{\infty} \left[G_{n\,n+1}\, \alpha^n\, l^{n+1} + G_{n\,n}\, \alpha^n l^n \right] \\ &= G_{12}\alpha l^2 + G_{11}\alpha l + G_{23}\alpha^2 l^3 + G_{22}\alpha^2 l^2 + O(\alpha^3). \end{aligned} \tag{112}$$

  To $O(\alpha)$ one retains both the double and the single logarithm. In general for each order in α one keeps the principal two logarithms. One also keeps the $O(\alpha)$ terms both in the coefficient function and in the remainder function:

$$C^{NLO} = 1 + \alpha\, C^{(1)}; \qquad d^{NLO} = \alpha\, d^{(1)}. \tag{113}$$

  The one-loop coefficient function is needed because of the factorized form of the QCD form factor. One has indeed a resummed expression of the form:

$$\left[1 + \alpha\, C^{(1)} \right] e^{G_{12}\, \alpha l^2 + \cdots} \tag{114}$$

  By expanding the exponent in powers of α, a term coupling the coefficient function and the double logarithm is obtained:

$$\alpha^2\, C^{(1)}\, G_{12}\, l^2, \tag{115}$$

  which must be included in the NLO approximation;

- Next-to-next-to-leading order (NNLO). One keeps in G_N also the terms with $n = k - 1$, i.e.:

$$\begin{aligned} G_N^{NNLO} &= \sum_{n=1}^{\infty} \left[G_{n\,n+1}\alpha^n l^{n+1} + G_{n\,n}\alpha^n l^n + G_{n\,n-1}\alpha^n l^{n-1} \right] \\ &= G_{12}\alpha l^2 + G_{11}\alpha l + G_{23}\alpha^2 l^3 + G_{22}\alpha^2 l^2 + G_{21}\alpha^2 l + G_{34}\alpha^3 l^4 + G_{33}\alpha^3 l^3 + G_{32}\alpha^3 l^2 + \\ &+ \; O(\alpha^4). \end{aligned} \tag{116}$$

  To $O(\alpha^2)$, all the infrared logarithms are included. In general, for each order in α, one keeps the principal three logarithms. The first omitted term is the single logarithm to order α^3. One has also to keep the $O(\alpha^2)$ terms both in the coefficient function and in the remainder function:

$$C^{NNLO} = 1 + \alpha\, C^{(1)} + \alpha^2\, C^{(2)}; \qquad d^{NNLO} = \alpha\, d^{(1)} + \alpha^2\, d^{(2)}. \tag{117}$$

The classes of logarithms discussed above can be explicitly resummed by means of a function series expansion of $G_N(\alpha)$ [1]:

$$G_N(\alpha) = l\, g_1(\lambda) + \sum_{n=0}^{\infty} \alpha^n\, g_{2+n}(\lambda) = l\, g_1(\lambda) + g_2(\lambda) + \alpha\, g_3(\lambda) + \alpha^2\, g_4(\lambda) + \cdots, \tag{118}$$

where

$$\lambda = \beta_0\,\alpha\,l. \tag{119}$$

The $g_i(\lambda)$ are homogeneous functions of λ and have a series expansion around $\lambda = 0$:

$$g_i(\lambda) = \sum_{n=1}^{\infty} g_{i\,n}\,\lambda^n. \tag{120}$$

In LO one needs only the function g_1, in NLO one need also g_2, in NNLO also g_3 is needed and so on. The explicit expressions read:

$$\begin{aligned}
g_1(\lambda) &= -\frac{A_1}{2\,\beta_0\,\lambda}\left[(1-2\,\lambda)\,\log(1-2\,\lambda) - 2\,(1-\lambda)\,\log(1-\lambda)\right]; && (121)\\
g_2(\lambda) &= \frac{D_1}{2\,\beta_0}\log(1-2\,\lambda) + \frac{B_1}{\beta_0}\log(1-\lambda) + \frac{A_2}{2\,{\beta_0}^2}\left[\log(1-2\,\lambda) - 2\,\log(1-\lambda)\right] + && (122)\\
&- \frac{A_1\,\beta_1}{4\,{\beta_0}^3}\left[2\,\log(1-2\,\lambda) + \log^2(1-2\,\lambda) - 4\,\log(1-\lambda) - 2\log^2(1-\lambda)\right] + \\
&+ \frac{A_1\,\gamma_E}{\beta_0}\left[\log(1-2\,\lambda) - \log(1-\lambda)\right] + \frac{A_1}{2\,\beta_0}\left[\log(1-2\,\lambda) - 2\,\log(1-\lambda)\right]\log\frac{\mu^2}{Q^2}.
\end{aligned}$$

The function $g_1(\lambda)$ in [3] is in agreement with that one obtained originally in [11]. $g_2(\lambda)$ in [3] differs instead from the corresponding $g_2^{sl}(\lambda)$ obtained in [11] and it is equal to the corresponding function entering the $B \to X_s\gamma$ spectrum; the formalism we use makes explicit the universality of soft gluon dynamics in semileptonic and radiative decays. The NNLO function g_3 has the rather lengthy expression:

$$\begin{aligned}
g_3(\lambda) &= -\frac{D_2\,\lambda}{\beta_0\,(1-2\,\lambda)} - \frac{2\,D_1\,\gamma_E\,\lambda}{1-2\,\lambda} + \frac{D_1\,\beta_1}{2\,{\beta_0}^2}\left(\frac{2\,\lambda}{1-2\,\lambda} + \frac{\log(1-2\,\lambda)}{1-2\,\lambda}\right) - \frac{B_2\,\lambda}{\beta_0\,(1-\lambda)} - \frac{B_1\,\gamma_E\,\lambda}{1-\lambda} + \\
&+ \frac{B_1}{{\beta_0}^2}\,\beta_1\left(\frac{\lambda}{1-\lambda} + \frac{\log(1-\lambda)}{1-\lambda}\right) - \frac{A_3}{2\,{\beta_0}^2}\left(\frac{\lambda}{1-2\,\lambda} - \frac{\lambda}{1-\lambda}\right) - \frac{A_2\,\gamma_E}{\beta_0}\left(\frac{1}{1-2\,\lambda} - \frac{1}{1-\lambda}\right) + \\
&+ \frac{A_2\,\beta_1}{2\,{\beta_0}^3}\left(\frac{3\,\lambda}{1-2\,\lambda} - \frac{3\,\lambda}{1-\lambda} + \frac{\log(1-2\,\lambda)}{1-2\,\lambda} - \frac{2\,\log(1-\lambda)}{1-\lambda}\right) + \\
&- \frac{A_1\,{\gamma_E}^2}{2}\left(\frac{4\,\lambda}{1-2\,\lambda} - \frac{\lambda}{1-\lambda}\right) - \frac{A_1\,\pi^2}{12}\left(\frac{4\,\lambda}{1-2\,\lambda} - \frac{\lambda}{1-\lambda}\right) + \\
&- \frac{A_1\,\beta_2}{4\,{\beta_0}^3}\left(\frac{2\,\lambda}{1-2\,\lambda} - \frac{2\,\lambda}{1-\lambda} + 2\,\log(1-2\,\lambda) - 4\,\log(1-\lambda)\right) + \\
&+ \frac{A_1\,\beta_1\,\gamma_E}{{\beta_0}^2}\left(\frac{1}{1-2\,\lambda} - \frac{1}{1-\lambda} + \frac{\log(1-2\,\lambda)}{1-2\,\lambda} - \frac{\log(1-\lambda)}{1-\lambda}\right) + \\
&- \frac{A_1\,{\beta_1}^2}{2\,{\beta_0}^4}\left(\frac{\lambda}{1-2\,\lambda} - \frac{\lambda}{1-\lambda} - \log(1-2\,\lambda) + \frac{\log(1-2\,\lambda)}{1-2\,\lambda}\right. \\
&+ \left.\frac{\log(1-2\,\lambda)^2}{2\,(1-2\,\lambda)} + 2\,\log(1-\lambda) - \frac{2\,\log(1-\lambda)}{1-\lambda} - \frac{\log(1-\lambda)^2}{1-\lambda}\right) + \\
&- \frac{D_1\,\lambda}{\beta_0\,(1-2\,\lambda)}\log\frac{\mu^2}{Q^2} - \frac{B_1\,\lambda}{\beta_0\,(1-\lambda)}\log\frac{\mu^2}{Q^2} - \frac{A_2}{{\beta_0}^2}\left(\frac{\lambda}{1-2\,\lambda} - \frac{\lambda}{1-\lambda}\right)\log\frac{\mu^2}{Q^2} + \\
&- \frac{A_1\,\gamma_E}{\beta_0}\left(\frac{2\,\lambda}{1-2\,\lambda} - \frac{\lambda}{1-\lambda}\right)\log\frac{\mu^2}{Q^2} + \\
&+ \frac{A_1\,\beta_1}{{\beta_0}^3}\left(\frac{\lambda}{1-2\,\lambda} - \frac{\lambda}{1-\lambda} + \frac{\log(1-2\,\lambda)}{2} + \frac{\lambda\,\log(1-2\,\lambda)}{1-2\,\lambda} - \log(1-\lambda) - \frac{\lambda\,\log(1-\lambda)}{1-\lambda}\right)\log\frac{\mu^2}{Q^2} + \\
&- \frac{A_1}{2\,\beta_0}\left(\frac{2\,\lambda^2}{1-2\,\lambda} - \frac{\lambda^2}{1-\lambda}\right)\log^2\frac{\mu^2}{Q^2}. && (123)
\end{aligned}$$

The function $g_3(\lambda)$ was originally computed in [39], where the first NNLO resummation in heavy flavor decays was presented. At the time of that work, not all the fixed-order computations were available from which to extract the coefficients entering the resummation formula, namely A_3, B_2 and D_2. A numerical estimate of the three-loop coefficient A_3 was used, which was obtained in [40] by fitting the known moments of the 3-loop splitting kernels and which has been later confirmed by the exact analytic evaluation [28]. As far as B_2 is concerned, an approximation based on the $q \to q$ splitting function at two loops has been assumed, which was shown to be rather poor by the subsequent exact computation in [30]. The coefficient D_2 was taken from its original computation in [32]. There is a misprint in $g_3(\lambda)$ in [39] in two terms proportional to $A_1\beta_2$: $\log[1-\lambda] - 1/2\log[1-2\lambda]$ has to be multiplied by a factor 2, as found indeed in the recent recomputation of the μ-independent terms [38]. With the misprint, the terms proportional to $A_1\beta_2$ would indeed appear at α^3, while they have to appear only at order α^4, as shown correctly in the α expansion of the g_3 in eq. (42) of [39].

Let us note that the soft terms, i.e. the terms proportional to the coefficients A_i and D_i, have the singularity closest to the origin in $\lambda = 1/2$ while the collinear terms, proportional to B_i, have only a singularity in $\lambda = 1$.

3.2 Inverse transform to physical space

The original form factor in u space is recovered by an inverse Mellin transform:

$$\sigma(u;\,\alpha) \;=\; \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i}(1-u)^{-N}\,\sigma_N(\alpha), \tag{124}$$

where c is a real constant chosen in such a way that all the singularities of σ_N lie to the left of the integration contour. The inverse transform can be done to any given logarithmic accuracy in closed analytic form, where now the logarithmic accuracy is defined as before but in terms of powers of α and $L = \log 1/u$ instead of $l = \log N$. To NNLO accuracy, one can write [39] [17]:

$$\Sigma\,[u;\alpha] \;=\; \frac{e^{L\,g_1(\tau)\,+\,g_2(\tau)}}{\Gamma\,[1-h_1(\tau)]}\delta\Sigma, \tag{125}$$

where

$$\tau \;=\; \beta_0\,\alpha\,L \tag{126}$$

and we have defined

$$h_1(\tau) \;=\; \frac{d}{d\tau}\,[\tau g_1(\tau)] \;=\; g_1(\tau) + \tau\,g_1'(\tau). \tag{127}$$

$\delta\Sigma$ is a NNLO correction factor which can be set equal to one in NLO:

$$\delta\Sigma_{NLO} \;=\; 1. \tag{128}$$

Its NNLO expression reads:

$$\delta\Sigma \;=\; S/S|_{L\to 0} \tag{129}$$

with

$$S \;=\; e^{\alpha\,g_3(\tau)}\bigg\{1 \,+\, \beta_0\,\alpha\,g_2'(\tau)\,\psi\,[1-h_1(\tau)] \,+\, \frac{1}{2}\beta_0\,\alpha\,h_1'(\tau)\Big\{\psi^2\,[1-h_1(\tau)] - \psi'\,[1-h_1(\tau)]\Big\}\bigg\}. \tag{130}$$

In [39] inhomogeneous terms were included in $\delta\Sigma$, which have been subtracted here. $\Gamma(x)$ is the Euler Gamma function and

$$\psi(x) \;=\; \frac{d}{dx}\log\Gamma(x) \tag{131}$$

is the digamma function.

[17] A factor $1-u \approx 1$ has been neglected in our leading twist accuracy.

Expanding the r.h.s. of eq. (125) up to third order, one obtains the following relations:

$$
\begin{aligned}
G_{12} &= -\frac{1}{2}A_1; && (132)\\
G_{11} &= -(B_1+D_1); && (133)\\
G_{23} &= -\frac{1}{2}A_1\beta_0; && (134)\\
G_{22} &= -\frac{1}{2}A_2-\frac{1}{2}\beta_0(B_1+2\,D_1)-\frac{1}{2}A_1^2 z(2); && (135)\\
G_{21} &= -(B_2+D_2)-A_1\,(B_1+D_1)\,z(2)-A_1^2 z(3); && (136)\\
G_{34} &= -\frac{7}{12}A_1\beta_0^2; && (137)\\
G_{33} &= -A_2\beta_0-\frac{1}{2}A_1\beta_1-\frac{1}{3}\beta_0^2\,(B_1+4\,D_1)-\frac{3}{2}A_1^2\beta_0 z(2)+\frac{1}{3}A_1^3 z(3); && (138)\\
G_{32} &= -\frac{1}{2}A_3-\beta_0\,(B_2+2\,D_2)-\frac{\beta_1}{2}\,(B_1+2\,D_1)-A_1\,A_2\,z(2)-\frac{A_1\,\beta_0}{2}\,(5\,B_1\,+7\,D_1)\,z(2)+\\
&+ \frac{{A_1}^3}{4}\,z(4)-\frac{9\,{A_1}^2\,\beta_0\,z(3)}{2}+{A_1}^2\,(B_1\,+D_1)z(3), && (139)
\end{aligned}
$$

where $z(a)=\sum_{n=1}^{\infty}1/n^a$ is Riemann Zeta function with $z(2)=\pi^2/6=1.64493\cdots$, $z(3)=1.20206\cdots$ and $z(4)=\pi^4/90=1.08232\cdots$. Note that the leading coefficients G_{23} and G_{34} involve products of the one-loop coefficients A_1 and β_0 only. The explicit expressions of the G_{ij} read:

$$
\begin{aligned}
G_{12} &= -\frac{C_F}{2\pi}; && (140)\\
G_{11} &= \frac{7C_F}{4\pi}; && (141)\\
G_{23} &= -\frac{C_F}{8\,\pi^2}\left(\frac{11\,C_A}{3}-\frac{2\,n_f}{3}\right); && (142)\\
G_{22} &= \frac{C_F}{4\,\pi^2}\left[C_A\left(\frac{95}{72}+z(2)\right)-\frac{13\,n_f}{36}-2\,C_F\,z(2)\right]; && (143)\\
G_{21} &= \frac{C_F}{6\,\pi^2}\left[n_f\left(-\frac{85}{24}+z(2)\right)+C_A\left(\frac{905}{48}-\frac{17}{2}z(2)-\frac{3\,z(3)}{2}\right)+C_F\left(\frac{9}{16}+6\,z(2)+3\,z(3)\right)\right]; && (144)\\
G_{34} &= \frac{C_F}{48\,\pi^3}\left(-\frac{847\,{C_A}^2}{36}+\frac{77\,C_A\,n_f}{9}-\frac{7\,{n_f}^2}{9}\right); && (145)\\
G_{33} &= \frac{C_F}{4\,\pi^3}\left[-\frac{{n_f}^2}{108}+C_A\,n_f\left(\frac{20}{27}-\frac{z(2)}{3}\right)+{C_A}^2\left(-\frac{1261}{432}+\frac{11\,z(2)}{6}\right)+\right.\\
&- \left.\frac{11\,z(2)}{2}C_A\,C_F\,+C_F\,n_f\left(\frac{1}{4}+z(2)\right)+\frac{4\,{C_F}^2\,z(3)}{3}\right]; && (146)\\
G_{32} &= \frac{C_F}{4\,\pi^3}\left[{n_f}^2\left(\frac{275}{648}-\frac{z(2)}{9}\right)+C_A\,n_f\left(-\frac{5399}{1296}+\frac{4\,z(2)}{3}-\frac{z(3)}{6}\right)+\right.\\
&+ {C_A}^2\left(\frac{21893}{2592}-\frac{119\,z(2)}{36}+\frac{77\,z(3)}{12}-\frac{11\,z(4)}{4}\right)+{C_F}^2\Big(-7\,z(3)+z(4)\Big)+\\
&+ \left.C_F\,n_f\left(\frac{19}{48}-\frac{71\,z(2)}{36}+z(3)\right)+C_A\,C_F\left(\frac{11}{32}+\frac{685\,z(2)}{72}-11\,z(3)+5\,z(4)\right)\right]. && (147)
\end{aligned}
$$

The numerical values of the coefficients show a good convergence of the perturbative series also in configuration space:

$$
\begin{aligned}
G_{12} &= -0.212207; && (148)\\
G_{11} &= +0.742723; && (149)\\
G_{23} &= -0.185756 + 0.011258\, n_f = -0.151982; && (150)\\
G_{22} &= +0.152206 - 0.012196\, n_f = +0.115618; && (151)\\
G_{21} &= +0.628757 - 0.0427065\, n_f = +0.500638; && (152)\\
G_{34} &= -0.189702 + 0.022994\, n_f - 0.0006968\, n_f^2 = -0.126990; && (153)\\
G_{33} &= -0.349055 + 0.033368\, n_f - 0.0000995\, n_f^2 = -0.249846; && (154)\\
G_{32} &= +0.96117 - 0.09368\, n_f + 0.0025974\, n_f^2 = +0.703506, && (155)
\end{aligned}
$$

where on the last member of the r.h.s. we have set $n_f = 3$.

4 Distribution in the hadronic variables

The distribution in the hadronic variables u and w is obtained integrating the triple differential distribution (59) over the electron energy $\overline{x} = 1 - x$. The integration range is

$$\overline{x}_1(w,u) \le \overline{x} \le \overline{x}_2(w,u), \tag{156}$$

where:

$$\overline{x}_1(w,u) = \frac{w\,u}{1+u} \qquad \text{and} \qquad \overline{x}_2(w,u) = \frac{w}{1+u}. \tag{157}$$

Let us use the second method of integration of the triple-differential distribution discussed at the end of sec. (2), i.e. let us neglect at first the remainder function. Since the QCD form factor $\sigma\,[u;\,\alpha(w\,m_b)]$ does not depend on the electron energy $\overline{x}$, the integration only involves the coefficient function:

$$\int_{\overline{x}_1}^{\overline{x}_2} d\overline{x}\, C(\overline{x}, w;\, \alpha). \tag{158}$$

We eliminate small terms $O(u)$ from the integral above by integrating over the range which is the limit $u \to 0$ of (156): [18]

$$\overline{x}_1(w,0) \le \overline{x} \le \overline{x}_2(w,0). \tag{159}$$

In fact, these terms $O(u)$, when multiplied with the plus distributions of u contained in the QCD form factor $\sigma(u;\,\alpha)$, give at worse terms of the form $\log u$, which miss the $1/u$ enhancement and therefore are to be considered as "small". Let us define therefore the coefficient function of the double hadronic distribution as:

$$C_H\,(w;\,\alpha) = \int_0^w d\overline{x}\, C\,(w, \overline{x}; \alpha)\,, \tag{160}$$

having the usual α expansion

$$C_H\,(w;\,\alpha) = C_H^{(0)}(w) + \alpha\, C_H^{(1)}(w) + \alpha^2\, C_H^{(2)}(w) + O(\alpha^3). \tag{161}$$

One easily obtains:

$$
\begin{aligned}
C_H^{(0)}(w) &= 2w^2(3-2w); && (162)\\
C_H^{(1)}(w) &= \frac{C_F}{\pi} w^2 \left\{ -(9-4w)\log w + 2(3-2w)\left[\mathrm{Li}_2(w) + \log w \log(1-w) - \frac{35}{8} \right] \right\}.
\end{aligned}
$$

[18] The latter are actually the integration regions for $e^+e^- \to q + \overline{q} + g$ with massless quarks.

The first two orders of the coefficient function vanish as w^2 for $w \to 0$, implying a suppression of the states with a small hadronic energy (i.e. with a small hard scale), as anticipated in the introduction.

The resummed distribution in the hadronic variables u and w then reads:

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{du dw} \,=\, C_H\left[w;\,\alpha(m_b)\right]\,\sigma\left[u;\,\alpha(w\,m_b)\right] \,+\, d_H\left[u,w;\,\alpha(m_b)\right], \tag{163}$$

where the remainder function has an expansion analogous to the one in the triple differential distribution:

$$d_H\left(u,w;\,\alpha\right) \,=\, \alpha\, d_H^{(1)}(u,w) \,+\, \alpha^2\, d_H^{(2)}(u,w) \,+\, O(\alpha^3). \tag{164}$$

Expanding to first order the above distribution and comparing (matching) with the known $O(\alpha)$ distribution, the following remainder function is obtained — an over-all factor C_F/π is omitted:

$$\begin{aligned} d_H^{(1)}(w,u) = \quad & -\quad \frac{4\,w^6\,\log u}{(1+u)^7} + \frac{4\,w^2\,(3-2\,w)\,\log u}{1+u} - \frac{32\,w^5 - 10\,w^6\,\log u}{(1+u)^6} + \\ & +\quad \frac{3\,w^2\,\left(14 - 6\,w - 5\,w^2\right) - 2\,w^3\,(3-4\,w)\,\log u}{(1+u)^2} + \\ & +\quad \frac{20\,w^3\,(2+w)\,(1-2\,w) - w^4\,\left(9 - 18\,w - 2\,w^2\right)\,\log u}{(1+u)^4} + \\ & +\quad \frac{64\,w^5 + 2\,w^4\,\left(3 - 6\,w - 4\,w^2\right)\,\log u}{(1+u)^5} + \\ & -\quad \frac{4\,w^3\,\left(10 - 15\,w - 2\,w^2\right) - w^3\,\left(12 - 13\,w - 6\,w^2\right)\,\log u}{(1+u)^3}. \end{aligned} \tag{165}$$

Eq. (163) provides a complete NLO resummation of the distribution in the two hadronic variables u and w, from which the distribution in any other pair of hadronic variables can be obtained by a change of variables. One can insert in eq. (163) the NNLO form factor σ, whose properties have been discussed in sec. (3), allowing an approximate NNLO resummation. In fact, for a complete NNLO resummation, one also needs the second order corrections to the coefficient function $C_H^{(2)}(w)$ and the remainder function $d_H^{(2)}(u,w)$, which are unknown at present.

5 Hadron energy spectrum

The distribution in the total hadron energy w is obtained by integrating the distribution in the hadronic variables (163). The integration range in u is:

$$\max(0, w-1) \,\le\, u \,\le\, 1. \tag{166}$$

Since the coefficient function $C_H(w;\,\alpha)$ does not depend on u, the integration only involves the QCD form factor and the remainder function:

$$\frac{1}{\Gamma}\frac{d\Gamma}{dw} \,=\, C_H\left(w;\,\alpha\right)\Big\{1 \,-\, \theta(w-1)\Sigma\left[w-1;\,\alpha(w\,m_b)\right]\Big\} \,+\, \int_{\max(0,w-1)}^{1} dw\, d_H(u,w;\,\alpha), \tag{167}$$

where $\Sigma(u;\,\alpha)$ is the partially-integrated form factor defined in section (3).

Because of the $\theta(w-1)$ multiplying $\Sigma(w-1;\,\alpha)$, there are large logarithms only for $w > 1$, as anticipated in the qualitative discussion in the introduction. We may therefore consider the parts of the spectrum for $w < 1$ and $w > 1$ as two different spectra, merging in the point $w = 1$. Let us consider the simpler case $w < 1$ first. Since, as already noted, there are no large logarithms, no resummation is required and the $O(\alpha)$ fixed-order result coincides with the NLO one. There is no QCD form factor and therefore there is no way to distinguish between the coefficient function and the remainder function. The spectrum for $w < 1$ can then be written as an ordinary α expansion:

$$\frac{1}{2\Gamma}\frac{d\Gamma}{dw} \;=\; L(w;\,\alpha) \qquad\qquad (w<1), \tag{168}$$

where:

$$L(w;\,\alpha) \;=\; L^{(0)}(w) \,+\, \alpha\, L^{(1)}(w) \,+\, \alpha^2\, L^{(2)}(w) \,+\, O(\alpha^3). \tag{169}$$

The first two orders read [7, 8]:

$$L^{(0)}(w) \;=\; w^2(3-2w); \tag{170}$$

$$\begin{aligned} L^{(1)}(w) \;=\; \frac{C_F}{\pi}\Bigg\{ & - w^2(3-2w)\left[\frac{25}{8}+\mathrm{Li}_2(1-w)\right] + \\ & + \frac{1}{720}w^2\left(4w^4-42w^3+585w^2-3720w+4860+1440w\log w-3240\log w\right)\Bigg\}. \end{aligned} \tag{171}$$

Let us now consider the more interesting case $w > 1$, where resummation is effective and one has to keep the resummed form of the distribution in (167). In a minimal scheme we have to subtract small terms from the first term on the r.h.s. of eq. (167), since the form factor must contain large logarithms only. This is done setting $w = 1$ in the argument of the coupling entering the form factor Σ as well as in the coefficient function C_H, obtaining the simpler expression:

$$\frac{1}{\Gamma}\frac{d\Gamma}{dw} \;=\; C_H\,(1;\,\alpha)\Big\{1 \,-\, \Sigma\,[w-1;\,\alpha(m_b)]\Big\} \,+\, \cdots\,, \tag{172}$$

where the dots denote terms not containing large logs of $w-1$. Let us prove the legitimacy of the transformation from (167) to (172). As far as the argument of the coupling is concerned, we expand the QCD form factor Σ in powers of $\alpha(w\,m_b)$. One obtains terms of the form

$$\alpha(w\,m_b)\,\log^2(w-1) \;=\; \alpha(m_b)\log^2(w-1) - 2\beta_0\alpha(m_b)^2\log w\log^2(w-1) + 4\beta_0^2\alpha(m_b)^3\log^2 w\log^2(w-1) + \cdots\,, \tag{173}$$

where on the r.h.s. an expansion of $\alpha(w\,m_b)$ around the point $w = 1$ has been performed. All the terms on the r.h.s except the first one vanish for $w \to 1^+$, therefore they are not large logarithms and can be dropped. The only large logarithm is the first term on the r.h.s., which is obtained by setting $w = 1$ in the coupling in the original expression on the l.h.s. All this implies that the coupling can be evaluated in the infrared-singular point $w = 1$. As far as the coefficient function is concerned, one just notices that the neglected terms,

$$\Big[C_H\,(w;\,\alpha) \,-\, C_H\,(1;\,\alpha)\Big]\Big\{1 \,-\, \Sigma\,[w-1;\,\alpha(m_b)]\Big\}, \tag{174}$$

are again vanishing for $w \to 1^+$, because $C_H\,(w;\,\alpha) - C_H\,(1;\,\alpha) = O(w-1)$ and therefore can be neglected in this limit.

In NLO one has also to add a remainder function to be determined via a matching procedure. That, as already discussed in other cases, is in order to take into account also the region $w - 1 \sim O(1)$. One then has the resummed expression:

$$\frac{1}{2\Gamma}\frac{d\Gamma}{dw} \;=\; C_W\,(\alpha)\Big\{1 \,-\, \Sigma\,[w-1;\,\alpha(m_b)]\Big\} \,+\, H(w;\,\alpha) \qquad\qquad (w>1), \tag{175}$$

where we have defined:

$$C_W(\alpha) \;\equiv\; \frac{1}{2}\,C_H(1;\,\alpha_s). \tag{176}$$

The coefficient function and the remainder function have a standard α expansion:

$$
\begin{aligned}
C_W(\alpha) &= 1 + \alpha\, C_W^{(1)} + + \alpha^2\, C_W^{(2)} + O(\alpha^3), && (177)\\
H(w;\,\alpha) &= \alpha\, H^{(1)}(w) + \alpha^2\, H^{(2)}(w) + O(\alpha^3). && (178)
\end{aligned}
$$

The first order correction to the coefficient function reads:

$$
C_W^{(1)} = \frac{C_F}{\pi}\left(\frac{\pi^2}{6} - \frac{35}{8}\right) = -\,1.15868. \tag{179}
$$

Note that $C_W^{(1)}$ is negative and has a rather large size; for $\alpha(m_b) = 0.22$ it gives a negative correction of $\approx -\,25\%$. By using the matching procedure described at the end of sec. (3), we obtain:

$$
\begin{aligned}
H^{(1)}(w) = \frac{C_F}{\pi}\Big\{ &-\frac{1}{2}\,(2w+1)\,(w-1)^2\log^2(w-1) - \frac{1}{3}(w-1)(2w^2-w-4)\log(w-1) + \\
&-\, w^2(3-2w)\left[2\mathrm{Li}_2\,(1-w) + 2\log(w-1)\log w + \frac{\pi^2}{6}\right] + \\
&+\frac{1}{720}(2-w)(4w^5 - 34w^4 + 517w^3 - 2946w^2 + 3798w + 1248)\Big\}.
\end{aligned} \tag{180}
$$

The above function is positive in all the kinematical range $1 < w < 2$ and goes to zero for $w \to 2$, as expected on the basis of the vanishing of the phase space in this point.

Let us make a few remarks about eq. (175). If we expand the r.h.s. of eq. (175) in powers of α, we find that $C_W^{(1)}$ only appears in order α^2 — this occurs because the form factor multiplying the coefficient function is in this case $1 - \Sigma = O(\alpha)$ and not $\Sigma = O(1)$. At present, only a full $O(\alpha)$ computation is available, implying that $C_W^{(1)}$ cannot be determined by the matching: only the remainder function can be fixed by this procedure. The value of $C_W^{(1)}$ came out "automatically" as a consequence of our resummation formula (see. eq. (176)). There is however another method to fix $C_W^{(1)}$: we require that the resummed spectrum is continuous in $w = 1$. Since $\Sigma(w-1) \to 0$ for $w \to 1^+$, we obtain the equation:

$$
1 + \alpha\, L^{(1)}(1) = 1 + \alpha\left[C_W^{(1)} + H^{(1)}(1)\right], \tag{181}
$$

to be solved in $C_W^{(1)}$:

$$
C_W^{(1)} = L^{(1)}(1) - H^{(1)}(1) \tag{182}
$$

and giving again the value (179). The condition of continuity of the resummed spectrum in $w = 1$ is very reasonable from the physical viewpoint and it is remarkable that the two methods give the same value for the coefficient function.

Even though we are considering a differential spectrum, its resummation involves, as we have explicitly seen, the partially integrated form factor. Σ usually enters event fractions in expressions of the form

$$
R(y;\,\alpha) = C(\alpha)\,\Sigma(y;\,\alpha) + D(y;\,\alpha), \tag{183}
$$

with a remainder function vanishing for $y \to 0$, where y is a general kinematical variable entering the large logarithms $\log 1/y$. In the case of the hadron energy spectrum, its resummation is different from (183) because it involves the combination $1 - \Sigma$ instead of Σ: there is an additive constant, namely one, which makes the spectrum non vanishing for $w \to 1^+$, as it should. It seem however reasonable to impose the vanishing of the remainder function $H(w;\alpha)$ for $w \to 1^+$ also in this case. The previous factorization scheme does not satisfy this condition, because:

$$
H^{(1)}(1) = \frac{C_F}{\pi}\left(\frac{2587}{720} - \frac{\pi^2}{6}\right). \tag{184}
$$

We can construct an improved scheme satisfying this condition by introducing two coefficient functions instead of one:

$$\frac{1}{2\Gamma}\frac{d\Gamma}{dw} = C_{W1}(\alpha)\left\{1 - C_{W2}(\alpha)\,\Sigma\left[w-1;\,\alpha(m_b)\right] + \tilde{H}(w;\,\alpha)\right\} \qquad \text{(improved scheme, } w>1), \tag{185}$$

where the new remainder function, vanishing in $w = 1$, reads:

$$\tilde{H}(w;\,\alpha) = H(w;\,\alpha) - H(1;\,\alpha). \tag{186}$$

The coefficient functions have the usual fixed-order expansions:

$$C_{W1}(\alpha) = 1 + \alpha\, C_{W1}^{(1)} + \alpha^2\, C_{W1}^{(2)} + O(\alpha^3); \tag{187}$$

$$C_{W2}(\alpha) = 1 + \alpha\, C_{W2}^{(1)} + \alpha^2\, C_{W2}^{(2)} + O(\alpha^3). \tag{188}$$

By imposing the continuity in $w = 1$ as in the previous scheme, we obtain for the first coefficient function at first order in α:

$$C_{W1}^{(1)} = L^{(1)}(1) = -\frac{C_F}{\pi}\frac{563}{720} = -0.331868. \tag{189}$$

The second coefficient function is obtained by imposing the usual matching with the first order computation:

$$C_{W2}^{(1)} = -H^{(1)}(1) = -\frac{C_F}{\pi}\left(\frac{2587}{720} - \frac{\pi^2}{6}\right) = -0.826808. \tag{190}$$

The improved resummed expression (185) is positive in all the kinematical range $1 < w < 2$ and vanishes for $w \to 2$.

We can compare the hadron energy spectrum for $w > 1$ given in eq. (175) or in eq. (185) with the hadron mass distribution in the radiative decay (5) given in eq. (50). The hadron energy distribution contains Σ, i.e. just the integral of the form factor σ entering the radiative decay spectrum. The hadron energy spectrum is therefore a very good quantity on the theoretical side — it is exceptional in this respect — being directly connected, via integration, to the radiative decay. By that we mean that the connection between the two spectra only involves short-distance coefficients. As show in [5], this is to be contrasted with the case of other single-differential spectra.

5.1 Average energy

As discussed in the introduction, the infrared singularity in $w = 1$ of the $O(\alpha)$ spectrum is integrable, so one can calculate directly the average hadronic energy as a truncated expansion in α:

$$\langle w\rangle = \frac{7}{10}\left[1 + \frac{\alpha\, C_F}{\pi}\frac{137}{840}\right] = 0.71. \tag{191}$$

The $O(\alpha)$ correction is very small, of the order of 1%, due to a large cancellation between the contribution for $w < 1$, which is negative, and the one for $w > 1$, which is positive. Setting for instance $m_b = m_B$ one obtains in leading order:

$$\langle E_X\rangle = \frac{7}{10}\frac{m_B}{2} = 1.843\,\mathrm{GeV} \tag{192}$$

with a tiny first-order correction of $+\,26$ MeV. This quantity can be directly compared with the experimental value. In the radiative decay (5) there is a larger final hadronic energy: in lowest order

$$\langle E_X\rangle_{B\to X_s\gamma} = \frac{m_B}{2} = 2.634\,\mathrm{GeV}. \tag{193}$$

The average hadronic energy is $\sim 30\%$ larger in the radiative decay than in the semileptonic decay, in line with the qualitative discussion about the differences of the two decays given in the introduction.

5.2 Upper cut on hadron masses

In experimental analysis an upper cut on invariant masses

$$m_X < \overline{m}_X \tag{194}$$

is imposed in order to kill the large background from semileptonic $b \to c$ transitions. Let us define:

$$k = 2\frac{\overline{m}_X}{m_b}. \tag{195}$$

In practice, $\overline{m}_X = 1.6 \div 1.8$ GeV, so we can assume $k < 1$. A leading order evaluation of the spectrum with the above cut gives:

$$\frac{1}{2\Gamma}\frac{d\Gamma}{dw} = \begin{cases} w^2(3-2w)\left\{\theta(k-w) + \theta(w-k)\Sigma\left[\frac{1-\sqrt{1-(k/w)^2}}{1+\sqrt{1-(k/w)^2}}; \alpha(wm_b)\right] - \theta(w-1)\Sigma\left[w-1; \alpha(m_b)\right]\right\} & w < w_M \\ 0 & w > w_M \end{cases} \tag{196}$$

where

$$w_M = 1 + \frac{k^2}{4} \tag{197}$$

is the maximal hadronic energy above which the spectrum vanishes; as expected on physical ground, cutting large hadron masses also acts as an upper cut on hadron energies. The spectrum is continuous in $w = w_M$ and it develops large logarithms for $k \to 0$. Let us observe that the argument of the first QCD form factor Σ has a similar form to the variable u defined in eq. (36). In fact,

$$\left(\frac{k}{w}\right)^2 = \left(\frac{\overline{m}_X}{E_X}\right)^2 \tag{198}$$

is the analogue of the variable $4y$ with y defined in eq. (37).

6 Distribution in hadron and electron energies

In this section we derive the distribution in the hadron and electron energies w and $\overline{x}$ by integrating the triple differential distribution (59) over u. In general, there are two independent energies in the semileptonic decay (7). That is because the hadronic final state X_u is basically a pseudoparticle, i.e. a single entity possessing an energy E_X and a (variable) mass m_X. We have therefore 3 particles/pseudoparticles in the final state and 3 energies, related by energy conservation:

$$x_e + x_\nu + w = 2, \tag{199}$$

where

$$x_\nu = \frac{2E_\nu}{m_b} \tag{200}$$

and we have written x_e instead of x for aesthetical reasons. Since the neutrino energy is not usually measured, let us take as independent energies the electron and the hadron energies. We have to integrate over u in the range

$$\max[0, w-1] \le u \le \min\left[\frac{w-\overline{x}}{\overline{x}}, \frac{\overline{x}}{w-\overline{x}}\right]. \tag{201}$$

As in the previous section, let us use the second method of integration, i.e. let us omit at first the remainder function. Since the coefficient function $C(x, w; \alpha)$ does not depend on u, the integration only involves the form factor σ — a complementary situation with respect to the one in the previous section — and we obtain:

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{dxdw} = C[\overline{x}, w;\, \alpha(m_b)]\Big\{\theta(2\overline{x}-w)\Sigma\left[\frac{w-\overline{x}}{\overline{x}};\, \alpha(w\, m_b)\right] + \theta(w-2\overline{x})\Sigma\left[\frac{\overline{x}}{w-\overline{x}};\, \alpha(w\, m_b)\right] + \\ -\ \theta(w-1)\Sigma\Big[w-1;\, \alpha(m_b)\Big]\Big\} + \cdots, \tag{202}$$

where the dots denote non logarithmic terms to be included later. The decay (7) involves an hadronic subprocess with a heavy quark decaying into a light quark evolving later into a jet. Hadron dynamics is therefore symmetric under the exchange of the electron and the neutrino momenta, since it is "blind" to W decay. That is clearly seen by expressing w through x_ν by means of eq. (199):

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{dx_e dx_\nu} = C[x_e, x_\nu;\, \alpha(m_b)]\Bigg\{\theta(x_\nu - x_e)\Sigma\left[\frac{1-x_\nu}{1-x_e};\, \alpha((2-x_e-x_\nu)m_b)\right] + \\ +\ \theta(x_e - x_\nu)\Sigma\left[\frac{1-x_e}{1-x_\nu};\, \alpha((2-x_e-x_\nu)m_b)\right] + \\ -\ \theta(1-x_e-x_\nu)\Sigma\Big[1-x_e-x_\nu;\, \alpha(m_b)\Big]\Bigg\} + \cdots. \tag{203}$$

Soft-gluon dynamics — i.e. the expression above in curly brackets — is symmetric under exchange of x_e with x_ν. The coefficient function $C[x_e, x_\nu;\, \alpha(m_b)]$ however is not symmetric under the exchange of the lepton energies because it does depend on the whole process, involving the decay of the W boson into the lepton pair, and not only on the hadronic subprocess.

To proceed with resummation, however, let us go back to the more familiar variable w, i.e. to eq. (202). Large logarithms can in principle be obtained by sending to zero the argument of any of the QCD form factors Σ's entering (202), i.e. in the following three cases:

$$1.\ w - \overline{x} \to 0; \qquad 2.\ \overline{x} \to 0; \qquad 3.\ w \to 1^+. \tag{204}$$

The coefficient function $C[\overline{x}, w;\, \alpha(m_b)]$ vanishes in the first limit as $O(w-\overline{x})$, implying that in this case there are actually no large logarithms. This limit corresponds to $E_\nu \to m_b/2$, a point where the tree-level spectrum vanishes suppressing soft-gluon effects. The only relevant limits are therefore the second and the third ones. It is therefore natural to write a factorization formula dropping the form factor not associated to large logarithms:

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{dxdw} = C[\overline{x}, w;\, \alpha(m_b)]\left\{\Sigma\left[\overline{x}/w;\, \alpha(w\, m_b)\right] - \theta(w-1)\,\Sigma\Big[w-1;\, \alpha(m_b)\Big]\right\} + \cdots. \tag{205}$$

We have taken the limit $\overline{x} \to 0$ in the theta functions containing $\overline{x}$ in the argument.

Let us consider separately the cases $w \le 1$ and $w > 1$. In the simpler case $w \le 1$ [19] there is a single form factor and one can write a factorized expression of the form:

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{dxdw} = C_L\left(\overline{x}, w;\, \alpha\right)\Sigma\left[\overline{x}/w;\, \alpha(w\, m_b)\right] + d_<(w, \overline{x};\, \alpha) \qquad (w < 1). \tag{206}$$

We require that the remainder function vanishes for $\overline{x} \to 0$:

$$\lim_{\overline{x}\to 0} d_<(w, \overline{x};\, \alpha) = 0. \tag{207}$$

[19] Note that this case is a "complication" of the analogous case for the single distribution in w, where the integration over $\overline{x}$ has been made and therefore there are no large logarithms of $\overline{x}$.

The coefficient function $C_L\left(\overline{x},w;\,\alpha\right)$ can be taken as:

$$
\begin{aligned}
C_L^{(0)}(w,\overline{x}) &= 12(w-\overline{x})(1+\overline{x}-w); && (208)\\
C_L^{(1)}(w,\overline{x}) &= \frac{C_F}{\pi}12(w-\overline{x})(1+\overline{x}-w)\left[\mathrm{Li}_2(w)+\log w\log(1-w)-\frac{3}{2}\log w-\frac{w\log w}{2(1-w)}-\frac{35}{8}\right]. && (209)
\end{aligned}
$$

In $C_L^{(0)}(w,\overline{x})$ we have put the factor $12(w-\overline{x})(1+\overline{x}-w)$, equal to the spectrum in lowest order, in order to have a vanishing remainder function in $O(\alpha^0)$: this is a non minimal choice, since the minimal choice would imply to set $\overline{x}=0$ in the coefficient function. We have inserted a similar factor also in $C_L^{(1)}(w,\overline{x})$, in order to have a simple multiplicative form of the correction [20]. As in previous cases, by matching with the full $O(\alpha)$ result [8], we determine the remainder function

$$
d_<(w,\overline{x};\,\alpha) \;=\; \alpha\, d_<^{(1)}(w,\overline{x}) + \;\alpha^2\, d_<^{(2)}(w,\overline{x}) \;+\; O(\alpha^3). \qquad (210)
$$

Omitting the over-all factor C_F/π, we obtain for the leading contribution:

$$
\begin{aligned}
d_<^{(1)}(w,\overline{x}) \;=\; & -\frac{1}{10}\,(w-\overline{x})\,\overline{x}\,\big(-210+280\,w-10\,w^2+2\,w^3-60\,\overline{x}-125\,w\,\overline{x}-7\,w^2\,\overline{x}+15\,\overline{x}^2\;+\\
& +\;32\,w\,\overline{x}^2-15\,\overline{x}^3\big)\;+\\
& +\;\frac{1}{5\;(-1+w)}\big(-45\,w+60\,w^2-20\,w^3+10\,w^4-6\,w^5+w^6-15\,\overline{x}+135\,w\,\overline{x}-255\,w^2\,\overline{x}\,+\\
& +\;85\,w^3\,\overline{x}+25\,w^4\,\overline{x}-5\,w^5\,\overline{x}-15\,\overline{x}^2+45\,w\,\overline{x}^2+75\,w^2\,\overline{x}^2-85\,w^3\,\overline{x}^2+10\,w^4\,\overline{x}^2\big)\,\log w\;+\\
& -\;6\;(-1+w-\overline{x})\;(w-\overline{x})\;\log^2 w\;+\;6\;(-1+w-\overline{x})\;(w-\overline{x})\;\log^2(w-\overline{x})\;+\\
& -\;\frac{1}{5}\,(w-\overline{x})\;\big(45-15\,w+5\,w^2-5\,w^3+w^4+15\,\overline{x}-10\,w\,\overline{x}+15\,w^2\,\overline{x}-4\,w^3\,\overline{x}+5\,\overline{x}^2\;+\\
& -\;15\,w\,\overline{x}^2+6\,w^2\,\overline{x}^2+5\,\overline{x}^3-4\,w\,\overline{x}^3+\overline{x}^4\big)\;\log(w-\overline{x})\;+\\
& -\;\frac{1}{5}\,\overline{x}\,\big(60-180\,w+120\,w^2+60\,\overline{x}-15\,w\,\overline{x}-45\,w^2\,\overline{x}+5\,\overline{x}^2-20\,w\,\overline{x}^2+10\,w^2\,\overline{x}^2\;+\\
& +\;5\,\overline{x}^3-5\,w\,\overline{x}^3+\overline{x}^4\big)\;\log\overline{x}\;+\\
& +\;12\;(-1+w-\overline{x})\;(w-\overline{x})\;\log w\;\log\overline{x}-12\;(-1+w-\overline{x})\;(w-\overline{x})\;\log(w-\overline{x})\;\log\overline{x}\;. \qquad (211)
\end{aligned}
$$

Let us now consider the case $w>1$:

$$
\frac{1}{\Gamma}\frac{d^2\Gamma}{dxdw} \;=\; C\left(\overline{x},w;\,\alpha\right)\Big\{\Sigma\left[\overline{x}/w;\,\alpha(w\,m_b)\right]\;-\;\Sigma\Big[\Delta w;\,\alpha(m_b)\Big]\Big\}\;+\;\cdots \qquad (w>1), \qquad (212)
$$

where we have defined

$$
\Delta w \;=\; w\;-\;1\;>\;0. \qquad (213)
$$

There are two form factors and large logarithms can be obtained in the following three kinematical configurations:

1. $\overline{x}\ll\Delta w\sim 1$: large logarithms of the form $\alpha^n\log^k\overline{x}$ have to be resummed;
2. $\Delta w\sim\overline{x}\ll 1$: large logarithms of the form $\log\Delta w\sim\log\overline{x}$ have to be resummed;
3. $\Delta w\ll\overline{x}\sim 1$: large logarithms of the form $\log\Delta w$ have to be resummed.

[20] We could have taken as coefficient function the original one $C(\overline{x},w;\,\alpha)$ as well, which however does not always contain the factor $12(w-\overline{x})(1+\overline{x}-w)$.

The first case is kinematically forbidden because

$$\Delta w \,\leq\, \overline{x}. \tag{214}$$

The second case does not give large logarithms because the coefficient function $C(\overline{x}, w;\, \alpha)$ vanishes linearly in this limit:

$$C(\lambda\, \overline{x}, 1 + \lambda\, \Delta w;\, \alpha) \,=\, O(\lambda) \qquad \text{for } \lambda \to 0. \tag{215}$$

The only relevant limit is therefore the third one, implying that one can drop the form factor $\Sigma(\overline{x}/w;\, \alpha)$. We propose then a resummed form for this distribution which is a generalization of that one for the hadron energy spectrum:

$$\frac{1}{\Gamma}\frac{d^2\Gamma}{dx dw} \,=\, C_{XW1}\,(\overline{x};\, \alpha)\left\{1 \,-\, C_{XW2}\,(\overline{x};\, \alpha)\,\Sigma\Big[\Delta w;\, \alpha(m_b)\Big]\right\} \,+\, d_{>}(\Delta w, \overline{x};\, \alpha) \qquad (w > 1). \tag{216}$$

We require that the remainder function vanishes for $\Delta w \to 0^+$:

$$\lim_{\Delta w \to 0^+} d_{>}(\Delta w, \overline{x};\, \alpha) \,=\, 0. \tag{217}$$

The first coefficient function is obtained by imposing the continuity of the spectrum for $w \to 1$ from both sides $w < 1$ and $w > 1$ and for any $\overline{x}$ [21]. We obtain:

$$C_{XW1}(\overline{x};\, \alpha) \,=\, C_L\,(\overline{x}, 1;\, \alpha)\,\Sigma\,(\overline{x};\, \alpha) \,+\, d_{<}(1, \overline{x};\, \alpha). \tag{218}$$

We can expand in the above equation $\Sigma\,(\overline{x};\, \alpha)$ in powers of α (up to first order) because the coefficient function for $w = 1$, $C_L\,(\overline{x}, 1;\, \alpha)$, vanishes linearly for $\overline{x} \to 0$, killing the large logarithms in the form factor. We then obtain:

$$C_{XW1}(\overline{x};\, \alpha) \,=\, C^{(0)}_{XW1}(\overline{x}) \,+\, \alpha\, C^{(1)}_{XW1}(\overline{x}) \,+\, \alpha^2\, C^{(2)}_{XW2}(\overline{x}) \,+\, O(\alpha^3) \tag{219}$$

where

$$\begin{aligned}
C^{(0)}_{XW1}(\overline{x}) \;&=\; 12\,(1 - \overline{x})\,\overline{x}; \qquad (220)\\
C^{(1)}_{XW1}(\overline{x}) \;&=\; \frac{C_F}{\pi}\left\{\frac{1}{10}\,(1 - \overline{x})\;\overline{x}\;\left(-587 + 192\,\overline{x} - 47\,\overline{x}^2 + 15\,\overline{x}^3\right) - \frac{1}{5}\,\overline{x}\;\left(105 - 105\,\overline{x} - 5\,\overline{x}^2 + \overline{x}^4\right)\,\log\overline{x} \;+\right.\\
&-\; \frac{1}{5}\,(1 - \overline{x})\;\left(31 + 16\,\overline{x} - 4\,\overline{x}^2 + \overline{x}^3 + \overline{x}^4\right)\;\log(1 - \overline{x}) - 6\;(1 - \overline{x})\;\overline{x}\;\log^2(1 - \overline{x}) \;+\\
&+\; \left. 12\;(1 - \overline{x})\;\overline{x}\;\log(1 - \overline{x})\;\log\overline{x} - 6\;(1 - \overline{x})\;\overline{x}\;\log^2\overline{x} + 12\;(1 - \overline{x})\;\overline{x}\,z(2)\right\}. \qquad (221)
\end{aligned}$$

The second coefficient function $C_{XW2}(\overline{x};\, \alpha)$ is obtained by matching with the fixed-order distribution in the limit $\Delta w \to 0^+$:

$$C_{XW2}(\overline{x};\, \alpha) \,=\, C^{(0)}_{XW2}(\overline{x}) \,+\, \alpha\, C^{(1)}_{XW2}(\overline{x}) \,+\, \alpha^2\, C^{(2)}_{XW2}(\overline{x}) \,+\, O(\alpha^3) \tag{222}$$

with

$$\begin{aligned}
C^{(0)}_{XW2}(\overline{x}) \;&=\; 1; \qquad (223)\\
C^{(1)}_{XW2}(\overline{x}) \;&=\; \frac{C_F}{\pi}\left\{\frac{1}{120}\left(62 - 192\,\overline{x} + 47\,\overline{x}^2 - 15\,\overline{x}^3\right) \,+\, \frac{1}{60\,\overline{x}}\left(31 + 16\,\overline{x} - 4\,\overline{x}^2 + \overline{x}^3 + \overline{x}^4\right)\,\log(1 - \overline{x}) \;+\right.\\
&+\; \frac{1}{2}\,\log^2(1 - \overline{x}) \,+\, \frac{1}{60\;(1 - \overline{x})}\,\left(105 - 105\,\overline{x} - 5\,\overline{x}^2 + \overline{x}^4\right)\,\log\overline{x} \,-\, \log(1 - \overline{x})\,\log\overline{x} \;+\\
&+\; \left.\frac{1}{2}\,\log^2\overline{x}\right\}. \qquad (224)
\end{aligned}$$

[21] This continuity condition, which involves a single point $w = 1$ for the hadron energy spectrum, involves in this more complicated case the line $(w = 1,\, \bar{x})$.

The remainder function $d_>(w,\overline{x};\,\alpha)$ is obtained by matching with the fixed-order distribution for $\Delta w \sim O(1)$:

$$d_>(w,\overline{x};\,\alpha) \,=\, \alpha\, d_>^{(1)}(w,\overline{x}) \,+\, \alpha^2\, d_>^{(2)}(w,\overline{x}) \,+\, O(\alpha^3), \tag{225}$$

where, omitting the overall factor C_F/π:

$$\begin{aligned}
d_>^{(1)}(w,\overline{x}) \;=\;& \frac{1}{10}\,(-1+w)\,(1-\overline{x})\,\big(-75+142\,w-7\,w^2+2\,w^3-212\,\overline{x}-105\,w\,\overline{x}-9\,w^2\,\overline{x}\,+\\
&+\;132\,\overline{x}^2+39\,w\,\overline{x}^2-47\,\overline{x}^3\big)\,+\,\frac{1}{5}\,(-1+w)\,\big(-4-99\,w+w^2-4\,w^3+w^4+140\,\overline{x}\,+\\
&+\;120\,w\,\overline{x}+15\,w^2\,\overline{x}-5\,w^3\,\overline{x}-65\,\overline{x}^2-65\,w\,\overline{x}^2+10\,w^2\,\overline{x}^2\big)\,\log(-1+w)\,+\\
&-\;6\,(-1+w)\,(w-2\,\overline{x})\,\log^2(-1+w)+\frac{1}{5}\,(1-\overline{x})\,\big(31+16\,\overline{x}-4\,\overline{x}^2+\overline{x}^3+\overline{x}^4\big)\,\log(1-\overline{x})\,+\\
&+\;6\,(1-\overline{x})\,\overline{x}\,\log^2(1-\overline{x})+6\,(-1+w-\overline{x})\,(w-\overline{x})\,\log^2(w-\overline{x})\,+\\
&-\;(-1+w)\,\big(-21\,w+30\,\overline{x}+24\,w\,\overline{x}-12\,\overline{x}^2-9\,w\,\overline{x}^2-2\,\overline{x}^3+2\,w\,\overline{x}^3-\overline{x}^4\big)\,\log\overline{x}\,+\\
&-\;12\,(1-\overline{x})\,\overline{x}\,\log(1-\overline{x})\,\log\overline{x}+6\,(-1+w)\,(w-2\,\overline{x})\,\log^2\overline{x}\,+\\
&-\;\frac{1}{5}\,(w-\overline{x})\,\big(45-15\,w+5\,w^2-5\,w^3+w^4+15\,\overline{x}-10\,w\,\overline{x}+15\,w^2\,\overline{x}-4\,w^3\,\overline{x}+5\,\overline{x}^2\,+\\
&-\;15\,w\,\overline{x}^2+6\,w^2\,\overline{x}^2+5\,\overline{x}^3-4\,w\,\overline{x}^3+\overline{x}^4\big)\,\log(w-\overline{x})\,+\\
&-\;12\,(-1+w-\overline{x})\,(w-\overline{x})\,\log\overline{x}\,\log(w-\overline{x})\,.
\end{aligned} \tag{226}$$

To summarize, we have presented a complete NLO resummation of the distribution in the hadron and electron energies w and x, which is a generalization of the resummation of the hadron energy spectrum of the previous section. Resummation takes a different form in the cases $w \le 1$ and $w > 1$. In the first case there is a series of threshold logarithms of the form

$$\alpha^n\,\log^k\frac{\overline{x}}{w} \qquad\qquad (w<1), \tag{227}$$

while in the second case the infrared logarithms are of the form

$$\alpha^n\,\log^k(w-1) \qquad\qquad (w>1). \tag{228}$$

Unlike the distribution in sec. 4, we have here a differential distribution involving the partially-integrated form factor Σ.

7 Conclusions

It is a rather old idea that semi-inclusive B decays can be related to each other because of some universal long-distance component [25]. We have presented in this paper a critical analysis of this idea, based on a resummation formula for the triple differential distribution in the semileptonic decay (7). Long-distance effects manifest themselves in perturbation theory in the form of series of large infrared logarithms, coming from the multiple emission of soft and/or collinear gluons. The universality of long-distance effects has therefore to show up in perturbation theory in the form of identical series of large logarithms in different distributions. Semi-inclusive B decays have been defined in all generality as decays of the form

$$B \,\rightarrow\, X_q \,+\, (\text{non QCD partons}), \tag{229}$$

in the kinematical region close to the threshold $m_X = 0$, i.e. for

$$m_X \,\ll\, E_X. \tag{230}$$

We have shown that semileptonic distributions are naturally divided into two classes.

The first class contains distributions which are not integrated over the hadronic energy E_X and consequently have a long-distance structure similar to the one in radiative decays (5). These are the (simpler) distributions to attack and have been treated in this paper. We have resummed to next-to-leading order:

1. the distribution in the hadronic energy E_X and in the variable u defined in sec. (3), which is basically the ratio $m_X^2/(4E_X^2)$, i.e. the hadron invariant mass squared in unit of the hard scale;
2. the hadron energy distribution, which is a case of the so-called Sudakov shoulder. This is the only single distribution which can be related to the radiative decay via short-distance factors only. The large logarithms which appear in this distribution are indeed equal to the ones which appear in the radiative decay (5). We have studied in detail the relation between the hadron energy spectrum and the photon spectrum in the radiative decay. It is remarkable that the large logarithms in the hadron energy spectrum occur at $E_X = m_b/2$, i.e. when the hard scale $Q = 2E_X$ equals m_b, as in the radiative decays;
3. the distribution in the hadron and in the charged lepton energies, which contains two different classes of large logarithms according to the cases $w \leq 1$ or $w > 1$. The resummation of this distribution is the most complicated and is a generalization of the resummation of the hadron energy spectrum.

The second class contains semileptonic distributions in which the hadronic energy is integrated over, such as for example the hadronic mass distribution or the charged lepton energy distribution. These distributions have a complicated logarithmic structure, which is not simply related to the one in the radiative decay and there is not a pure short-distance relation with the radiative decay spectrum. The resummation of these distributions to NLO is presented in [5].

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