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On Maximal Intermediate Predicate Constructive Logics

Abstract. We extend to the predicate frame a previous characterization of the maximal intermediate propositional constructive logics. This provides a technique to get maximal intermediate predicate constructive logics starting from suitable sets of classically valid predicate formulae we call maximal nonstandard predicate constructive logics. As an example of this technique, we exhibit two maximal intermediate predicate constructive logics, yet leaving open the problem of stating whether the two logics are distinct. Further properties of these logics will be also investigated.

Key words: intermediate predicate logic, nonstandard intermediate predicate logic, predicate constructive logic, nonstandard predicate constructive logic, maximal predicate constructive logic, maximal nonstandard predicate constructive logic, smooth predicate constructive logic.

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1. Introduction

The maximal intermediate predicate logics with the disjunction property and the explicit definability property (we call predicate constructive logics) have been scarcely investigated in literature. In this frame, we know that the cardinality of their set is $2^{\aleph_0}$ [6]. However, the proof of this result cannot be used to provide a complete characterization of any maximal predicate constructive logic, since it is based on an indirect argument requiring the Axiom of Choice. The same paper [6] (and, independently, [1, 12]) proves that the cardinality of the set of maximal intermediate propositional logics with the disjunction property (we call propositional constructive logics) is $2^{\aleph_0}$ too. Also in this case (as in [1]), the proof is indirect and uses the Axiom of Choice.

In literature the study of the maximal propositional constructive logics is in a more advanced state than the study of the maximal predicate constructive logics. For instance, starting from Kirk's discovery that the greatest propositional constructive logic does not exist [14], the problem of counting the number of maximal propositional constructive logics has been considered in several papers [1, 6, 12, 18, 19, 23] (providing also some picture of

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the class of these logics, as in [23]), while the importance of such logics in connection with Minari's conjecture [21] has been pointed out in [2]. Also, Medvedev's maximal propositional constructive logic of the finite problems is rather well known [4, 5, 10, 19, 20, 22, 23, 26] (even if the problem of a recursive axiomatization of this logic is still open). Finally, in [26] a first explanation of a method to single out maximal propositional constructive logics is given, which in [4, 5] is systematically used to get Kripke frames semantics for infinitely many maximal propositional constructive logics (in [4, 5] also some points of possible interest for a classification of the set of the maximal propositional constructive logics are analyzed). On the contrary, the problem of counting the maximal predicate constructive logics has been proposed (and solved) only in [6]. Moreover, no example is known in literature of a maximal predicate constructive logic.

The purpose of this paper is both to extend the techniques for the investigation of the maximal propositional constructive logics, given in [26], to the predicate case and to present the first examples of maximal predicate constructive logics. In this frame, we will introduce the nonstandard predicate logics, which differ from the intermediate predicate logics for being closed only under special substitutions (such as the substitutions of atomic formulae with negated formulae). Then we will define four operators working on the standard and nonstandard predicate logics and, generalizing the results of [26], we will prove the existence of a one-to-one correspondence between standard and nonstandard maximal predicate constructive logics.

The paper is organized as follows. In the next section we will introduce the basic conventions and definitions, recalling notions such as intermediate predicate logic, disjunction property and explicit definability property. We will also introduce the notion of nonstandard predicate logic, to be used in the subsequent sections.

In Section 3 we will deal about the relations between standard and nonstandard predicate constructive logics, in particular between maximal standard predicate constructive logics and maximal nonstandard predicate constructive logics. In this frame, we will introduce the operators of extension (transforming any nonstandard predicate logic into a nonstandard predicate logic), standardization (transforming any nonstandard predicate logic into a standard predicate logic), reduction (transforming any nonstandard predicate logic including *Kuroda logic* [11, 24, 25, 32] into a standard predicate logic), and weak reduction (transforming any nonstandard predicate logic into a standard predicate logic).

Finally, in Section 4, we will introduce two nonstandard predicate logics $\mathcal{M}_1$ and $\mathcal{M}_2$ by means of variants of a "special kind" of non–kripkean semantics based on the notion of classical provability; the two logics will be provided by different definitions, but we have not yet proved that they are distinct. We will show that they are maximal nonstandard predicate constructive logics and will put into evidence some principles holding in them and in the (standard) maximal predicate logics related to them. We will also provide an interesting characterization of $\mathcal{M}_2$ and its standardization in terms of an appropriate generalization at the predicate level of the notion of "smoothness" considered in [26] and in [30] (in [30] called "structural completeness" and connected with a problem raised by H. Friedman on the propositional constructive logics [7]).

2. Preliminary notions

First of all, the set of the *predicate well formed formulae* (predicate wff's for short) is defined, as usual, starting from the propositional connectives $\neg, \vee, \wedge, \rightarrow$, the quantifiers $\forall$ and $\exists$, a denumerable set of individual variables $x_0, x_1, ..., x_n, ...$ (also denoted $x, y, z, v, w, ...$) and, for every $n \geq 0$, a denumerable set of *n-ary* predicate variables $P_0^n, P_1^n, ..., P_h^n, ...$ (also denoted $P^n, Q^n, R^n, ...$ where, for $n = 0$, one recovers the propositional variables). The notation $A \leftrightarrow B$ will be taken as an abbreviation of $(A \rightarrow B) \wedge (B \rightarrow A)$. The notions of *free individual variable* and *bounded individual variable* are defined as usual. We will say that a predicate wff is *open* if some free individual variable occurs in it; otherwise, the predicate wff will be said to be *closed.* The universal closure of an open formula (disregarding the order of the involved universal quantifiers) is defined as usual, while the universal closure of a closed formula will be taken as the formula itself; the notation A^* will indicate the universal closure of the predicate wff A. When only propositional variables are considered, one obtains the *propositional well formed formulae* (propositional wff's). Symbols such as $A, A', B, B', C, C', ...$ (with possible indexes) will denote predicate or propositional wff's. Notations such as $A(x)$ will be used to indicate that x is a possible free variable of A. More generally, $A(x_1, ..., x_n)$ (with $n \geq 1$) will indicate that $x_1, ..., x_n$ may be free in A. Also, notations such as $\underline{x}$ will indicate (possibly empty) sequences of distinct individual variables and $A(\underline{x})$ will indicate that the variables of $\underline{x}$ may occur free in A; in this line, if P is a predicate variable whose arity is not put into evidence and $\underline{x}$ is a sequence containing a number of variables coinciding with the arity of P, the notation $P(\underline{x})$ will indicate the atomic formula built up starting from P and the variables of $\underline{x}$, placed in $P(\underline{x})$ according to their order in $\underline{x}$. A *negated* wff will be any predicate or propositional wff A such that $A = \neg B$ for some B. A *predicate negatively saturated* wff will be any predicate wff A such that every occurrence in A of a predicate variable is in the scope of a negation. Correspondingly, a *propositional negatively saturated* wff will be any propositional wff A such that every occurrence in A of a propositional variable is in the scope of a negation [4, 5, 26]. A *predicate Harrop* wff [13, 28, 32] will be any predicate wff A such that A is either atomic or negated, or $A = B \wedge C$ with B and C predicate Harrop wff's, or $A = \forall x B$ with B a predicate Harrop wff, or $A = B \to C$ with C a predicate Harrop wff. In particular, a *propositional Harrop* wff will be either an atomic or negated propositional wff, or the conjunction of two propositional Harrop wff's, or an implication whose consequent is a propositional Harrop wff.

An *individual substitution* is any function η from the set of all the individual variables to the same set. Given any individual substitution η and any wff A, the result of the application of η to A (denoted ηA) is defined as the predicate wff obtained from A by correctly substituting each free individual variable x in A with $\eta(x)$ (see any textbook of mathematical logic for the definition of substitution of free individual variable in a predicate wff; if the substitution is not correct, i.e., for some x the variable $\eta(x)$ is not free in A, then we say that it is empty for A and we set $\eta A = A$).

A *predicate substitution* is any function σ associating, with every predicate variable P^n, a predicate wff $\sigma(P^n)$ such that at least n distinct free individual variables (in a one-to-one correspondence with the places involved in P^n) occur in $\sigma(P^n)$. Given any predicate substitution σ and any predicate wff A, the result of the application of σ to A (denoted σA) is the formula obtained from A by correctly replacing each atomic formula of the form $P^n(z_1, ..., z_n)$ with the predicate wff $S^{x_1,...,x_n}_{z_1,...,z_n} U(x_1, ..., x_n)$ (where $U(x_1, ..., x_n) = \sigma(P^n)$ may contain also free individual variables different from $x_1, ..., x_n$ and $S^{x_1,...,x_n}_{z_1,...,z_n} U(x_1, ..., x_n)$ is obtained from $U(x_1, ..., x_n)$ by replacing the free occurrences of $x_1, ..., x_n$ with $z_1, ..., z_n$; for a precise definition of correct substitution, see, e.g., [3, 27]). If the substitution σ cannot be correctly performed for A, we say that σ is empty for A and we set $\sigma A = A$.

A *predicate negatively satured substitution* (*NegSat–substitution* for short) will be any predicate substitution σ^{NS} replacing every predicate variable with a predicate negatively saturated wff. An *Harrop predicate substitution* (simply *H–substitution* when no confusion arises) will be any predicate substitution σ^H such that, for every predicate variable P^n, $\sigma^H(P^n)$ is a predicate Harrop wff. Finally, a *restricted predicate substitution* (*r–substitution* for short) will be any predicate substitution σ^r such that, for every predicate variable P^n, $\sigma^r(P^n)$ is a negated predicate wff. In a similar way, mutatis mutandis, one defines the notions of *propositional substitution* (denoted σ_{prop}), *propositional negatively satured substitution* (denoted σ^{NS}_{prop}), *Harrop propo-*

sitional substitution (denoted σ^{H}_{prop}) and *restricted propositional substitution* (denoted σ^{r}_{prop}) [4, 5, 26].

A *predicate transformation* (*transformation* for short) τ is either a restricted predicate substitution σ^r (in which case τ is empty for a wff A if σ^r is), or an individual substitution η (in which case τ is empty for a wff A if η is), or a composition of two transformations τ_1 and τ_2 (in which case τ is empty for a wff A if τ_1 is empty for $\tau_2 A$ or τ_2 is empty for A). If $\Gamma = \{B_1, ..., B_n\}$, $B_1, ..., B_n$ are predicate wff's and τ is a transformation, by $\tau\Gamma$ we will indicate the set of predicate wff's $\{\tau B_1, ..., \tau B_n\}$.

By INT (INT_{prop}) and CL (CL_{prop}) we will indicate the set of intuitionistically valid predicate (respectively, propositional) wff's and the set of classically valid predicate (respectively, propositional) wff's.

An *intermediate predicate logic* will be any set L of predicate wff's such that $INT \subseteq L \subseteq CL$, and L is closed under detachment (i.e., $A \in L$ and $A \to B \in L$ implies $B \in L$), generalization (i.e., $A \in L$ implies $\forall x A \in L$ for every individual variable x) and predicate substitution (i.e., $A \in L$ implies $\sigma A \in L$ for every predicate substitution σ). Likewise, an *intermediate propositional logic* will be any set L such that $INT_{prop} \subseteq L \subseteq CL_{prop}$, and L is closed under detachment and propositional substitution. We recall that, for every intermediate predicate logic L, the set $PROP(L) = \{A \mid A \in L$ and A is a propositional wff$\}$ is an intermediate propositional logic [6], called *the greatest intermediate propositional logic included in L.*

Near the notions of intermediate predicate logic and intermediate propositional logic, we will consider the notions of nonstandard propositional logic [4, 5, 26] and the new framework of nonstandard predicate logic. We say that a set L of predicate wff's is a *nonstandard predicate logic* iff $INT \subseteq L \subseteq CL$ and L is closed under detachment, generalization and r–substitution (i.e., $A \in L$ implies $\sigma^r A \in L$ for every r–substitution σ^r). Likewise, a *nonstandard propositional logic* will be any set L of propositional wff's such that $INT_{prop} \subseteq L \subseteq CL_{prop}$, and L is closed under detachment and restricted propositional substitution. As in the standard case, one can easily prove that, for every nonstandard predicate logic L, the set $PROP(L) = \{A \mid A \in L$ and A is a propositional wff$\}$ is a nonstandard propositional logic, called *the greatest nonstandard propositional logic included in L.*

We will sometimes identify a logic (predicate or propositional, standard or nonstandard) with the set of theorems of some formal system. In this sense, if Σ is a set of predicate axiom schemes and L is an intermediate predicate logic, the notation $L + \Sigma$ will indicate the intermediate predicate logic obtained by adding the instances of the axiom schemes of Σ to L, i.e., the smallest set of predicate wff's closed under detachment and generalization which includes the set $L \cup \Sigma^*$, where Σ^* is the set of all the instances of the elements of Σ. Note that the set Σ^* is a set of wff's closed under predicate substitution, and the same holds for L. Likewise, we will represent intermediate propositional logics by notations such as $L_{prop} + \Sigma_{prop}$, L_{prop} being an intermediate propositional logic and Σ_{prop} being a set of propositional axiom schemes. On the other hand, if L is a nonstandard predicate logic and Σ' is a set of classically valid predicate wff's closed under r–substitution or such that $\sigma^r A \in L \cup \Sigma'$ for every r–substitution σ^r and every $A \in \Sigma'$, then the notation $L \oplus \Sigma'$ will denote both the formal system (closed under detachment and generalization) obtained by adding (as axioms) the elements of Σ' to L and the related set of theorems, which is a nonstandard predicate logic. In a similar way, if L_{prop} is a nonstandard propositional logic and Σ'_{prop} is a set of classically valid propositional wff's closed under restricted propositional substitution or such that $\sigma^r_{prop} A \in L_{prop} \cup \Sigma'_{prop}$ for every restricted propositional substitution σ^r_{prop} and every $A \in \Sigma'_{prop}$, then $L_{prop} \oplus \Sigma'_{prop}$ will indicate both a formal system and a nonstandard propositional logic.

The notation $\vdash_{\overline{L}}$ will indicate provability in a (standard or nonstandard, predicate or propositional) logic L. Thus, $A \in L$ and $\vdash_{\overline{L}} A$ will have the same meaning. Further, if Γ is either a set of predicate wff's or a set of propositional wff's, then $\Gamma \vdash_{\overline{L}} A$ will indicate the fact that there are $B_1, ..., B_m$ (with $m \geq 0$) such that $\{B_1, ..., B_m\} \subseteq \Gamma$ and $B_1 \wedge ... \wedge B_m \to A \in L$ (if $m = 0$, $B_1 \wedge ... \wedge B_m \to A$ stands for A).

Given a set Ψ of predicate or propositional wff's, we will say that Ψ has the *disjunction property* iff $A \in \Psi$ or $B \in \Psi$ whenever $A \vee B \in \Psi$. Also, if Ψ is a set of predicate wff's, then Ψ will be said to have the *explicit definability property* iff $\exists x A(x) \in \Psi$ implies that $A(y) \in \Psi$ for some variable y (where $A(y)$ indicates the result of a correct substitution of the individual variable x with the individual variable y in A).

We will be interested in the disjunction and explicit definability properties when the set Ψ is a (standard or nonstandard, predicate or propositional) logic L. In this frame, we will say that L is an *intermediate predicate constructive* (*nonstandard predicate constructive*) logic iff L is an intermediate predicate (nonstandard predicate) logic and L has simultaneously the disjunction property and the explicit definability property. Also, L is an *intermediate propositional constructive* (*nonstandard propositional constructive*) logic iff L is an intermediate propositional (nonstandard propositional) logic and L has the disjunction property.

For the sake of readability, in the following we will omit the specification "intermediate"; for instance, "intermediate predicate logic" will be synonymous with "predicate logic" and "intermediate propositional logic" with "propositional logic". Also, "logic" will indicate a predicate or a propositional logic, and " nonstandard logic" will stand for a nonstandard predicate or propositional logic. In this line, we will have predicate constructive logics, nonstandard predicate constructive logics, propositional constructive logics, nonstandard propositional constructive logics, constructive logics and nonstandard constructive logics.

A *maximal predicate* (respectively, *nonstandard predicate*) *constructive logic* will be any predicate (respectively, nonstandard predicate) constructive logic L such that, for every predicate (respectively, nonstandard predicate) constructive logic L' , $L \subseteq L'$ implies $L = L'$. Also, L is a *maximal propositional* (respectively, *nonstandard propositional*) *constructive logic* iff L is a propositional (respectively, nonstandard propositional) constructive logic and $L \subseteq L'$ implies $L = L'$ for every propositional (respectively, nonstandard propositional) constructive logic L'.

To our further purposes, the following lemma will be useful (the proof is easy, see, e.g., [6]):

LEMMA 2.1. *Let L be a predicate logic or a nonstandard predicate logic. Then L has the explicit definability property iff, for every open predicate wff $A = \exists x B(x, y_1, ..., y_m)$ (with $m \geq 1$) such that $y_1, ..., y_m$ are exactly the free individual variables of A, $A \in L$ iff one of $B(y_1, y_1, ..., y_m)$,..., $B(y_m, y_1, ..., y_m)$ belongs to L.*

From now on, **K** (*Kuroda logic*) will stand for the predicate logic $INT + \{(KUR)\}$ where (KUR) is the following axiom schema:

$$\forall x \neg\neg A(x) \to \neg\neg\forall x A(x) \text{ (with } A \text{ any formula).}$$

Equivalent axiom schemas are: $\neg\neg\forall x(A(x) \vee \neg A(x))$ and $\neg(\forall x \neg\neg A(x) \wedge \neg\forall x A(x))$ (with A any formula).

It is well known that, for any predicate wff $\neg\neg A$ and for any set Γ of predicate wff's, $\Gamma \vdash_{\mathbf{K}} \neg\neg A$ iff $\Gamma \vdash_{\mathrm{CL}} A$ (see, e.g., [11, 24, 25, 32]). Note that this result holds for intuitionistic propositional logic as well (i.e., for any propositional wff $\neg\neg A$ and for any set Γ of propositional wff's, $\Gamma \vdash_{\mathrm{INTprop}} \neg\neg A$ iff $\Gamma \vdash_{\mathrm{CLprop}} A$, see, e.g., [15, 16]), but it generally does not hold in the predicate frame replacing **K** with INT. As a matter of fact, the above principle $\neg\neg\forall x(A(x) \vee \neg A(x))$ clearly does not belong to intuitionistic predicate logic.

We call *kurodian* any nonstandard predicate logic L such that $\mathbf{K} \subseteq L$. As we will see in the last part of Section 4, kurodian logics play an important role in the study of predicate smoothness.

3. Relations between standard and nonstandard predicate logics

In the previous section we have introduced the notion of non standard predicate logic. Now, in a strict parallelism with the results of [26] for the propositional case, we investigate the relations between standard and nonstandard predicate logics.

The *predicate extension operator* is the operator E associating, with every nonstandard predicate logic L, the set of predicate wff's $E(L) = L \oplus \{\neg\neg A \rightarrow A \mid A \text{ is an atomic predicate wff}\}$.

If $A(P_1, ..., P_n)$, for some $n \geq 1$, is a predicate wff containing exactly the predicate variables $P_1, ...P_n$ (whose arities are not put into evidence) and L is a nonstandard predicate logic, then, from the definition of $E(L)$, one easily gets that $A(P_1, ..., P_n) \in E(L)$ iff $(\neg\neg P_1(\underline{x}_1) \rightarrow P_1(\underline{x}_1))^* \wedge ... \wedge (\neg\neg P_n(\underline{x}_n) \rightarrow P_n(\underline{x}_n))^* \rightarrow A(P_1, ..., P_n) \in L$. Hence, a straightforward adaptation of the proofs of the corresponding propositional cases, given in [26], provides the two following facts:

PROPOSITION 3.1. *If L is a nonstandard predicate logic then so is E(L).*

PROPOSITION 3.2. *If L is a nonstandard predicate constructive logic then so is E(L).*

Generally, if L is a nonstandard predicate logic, then $L \neq E(L)$. From now on, we will call *regular* any nonstandard predicate logic L such that $L = E(L)$.

PROPOSITION 3.3. *Let L be any regular nonstandard predicate logic. Then L is closed under H–substitution.*

PROOF. Since $\neg\neg A \rightarrow A \in L$ for every A atomic, we have that $\neg\neg A \leftrightarrow A \in L$ for every A atomic; thus, since $\neg\neg A \leftrightarrow A \in INT$ for every negated wff $A = \neg B$, we have that $\neg\neg A \leftrightarrow A \in L$ for every A atomic or negated. This is the basis of an easy induction stating that, for every predicate Harrop wff Z, $\neg\neg Z \leftrightarrow Z \in L$. Now, let $A \in L$ and let σ^H be any H–substitution; also, let σ^r be the r–substitution associating, with every predicate variable P^n, the wff $\neg\neg\sigma^H(P^n)$; then we have that $\sigma^r A \in L$. On the other hand, by the above discussion, $(\sigma^r(P^n) \leftrightarrow \sigma^H(P^n))^* \in L$; thus, since the replacement theorem holds in L, we get $\sigma^H A \in L$. ■

The *predicate standardization operator* is the operator S associating, with every nonstandard predicate logic L, the set of predicate wff's S(L) such that, for every wff A, $A \in S(L)$ iff, for every predicate substitution σ, $\sigma A \in L$. The following Proposition 3.4 and Theorem 3.6 generalize the corresponding propositional cases explained in [26].

PROPOSITION 3.4. *If L is a nonstandard predicate logic, then $S(L)$ is a predicate logic.*

LEMMA 3.5. *Let A be any predicate wff without occurrences of $\neg$. Then there exists a H–substitution σ^H such that $\sigma^H A \in INT$.*

PROOF. Let σ^H be the H–substitution replacing every atomic formula $P(x_1, ..., x_n)$ with the predicate Harrop wff $P(x_1, ..., x_n) \to P(x_1, ..., x_n)$. Then, by an easy induction on the complexity of any predicate wff A without negation, one can prove that $\sigma^H A \in INT$. ■

THEOREM 3.6. *If L is a regular nonstandard predicate constructive logic, then $S(L)$ is a predicate constructive logic.*

PROOF. Suppose that $S(L)$ does not satisfy the explicit definability property. Then, by Lemma 2.1, there is an open predicate wff $\exists x B(x, y_1, ..., y_m)$ ($m \geq 1$) such that $\exists x B(x, y_1, y_2, ..., y_m) \in S(L)$, but $B(y_1, y_1, ..., y_m) \notin S(L)$ and...and $B(y_m, y_1, ..., y_m) \notin S(L)$.

Let us represent $B(x, y_1, ..., y_m)$ by $B(P_1, ..., P_k, x, y_1, ..., y_m)$, to put into evidence that $P_1, ..., P_k$ are the predicate variables occurring in B (we will not be interested, on the other hand, in indicating the arities of $P_1, ..., P_k$). Let $1 \leq i \leq m$; since $B(P_1, ..., P_k, y_i, y_1, ..., y_m) \notin S(L)$, by definition of $S(L)$ there is a predicate substitution σ_i such that $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m) \notin L$. Without loss of generality, we can assume that $y_1, ..., y_m$ are the only free individual variables included in $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m)$. For, if $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m)$ contains $v_1, ..., v_h$ as additional free individual variables, we can replace in $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m)$ every atomic formula involving $v_1, ..., v_h$ with an atomic formula not involving $v_1, ..., v_h$, so as to get a formula θ with the following properties: the only free individual variables of θ are $y_1, ..., y_m$; θ has the form $\sigma' B(P_1, ..., P_k, y_i, y_1, ..., y_m)$ for some σ'; there is an H–substitution σ^H such that $\sigma^H \theta$ coincides with $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m)$. Since L is closed under H–substitution, $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m) \notin L$ implies that $\theta \notin L$; thus, if $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m)$ does not satisfy the required properties, we can take θ in place of $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m)$. Now, by $B(C_1^i, ..., C_k^i, y_i, y_1, ..., y_m)$ we will indicate the predicate wff $\sigma_i B(P_1, ..., P_k, y_i, y_1, ..., y_m)$, where we put into evidence the only free individual variables $y_1, ..., y_m$, and where, for $1 \leq j \leq k$, C_j^i is the formula $\sigma_i(P_j)$. Since L is closed under H–substitution, we can also assume that, for i_1, i_2, j_1, j_2 such that $1 \leq i_1$, $i_2 \leq m$, $1 \leq j_1$, $j_2 \leq k$, and $i_1 \neq i_2$ or $j_1 \neq j_2$, no predicate variable occurring in the formula $C^{i_1}_{j_1} = \sigma_{i_1}(P_{j_1})$ occurs in the formula $C^{i_2}_{j_2} = \sigma_{i_2}(P_{j_2})$ (for every i with $1 \leq i \leq m$, there is a formula $B(Z^i_1, ..., Z^i_k, y_i, y_1, ..., y_m)$ such that: no predicate variable occurring in $Z^i_{j_1}$ occurs in $Z^i_{j_2}$ if $j_1 \neq j_2$; $B(C^i_1, ..., C^i_k, y_i, y_1, ..., y_m)$ can be obtained by applying to $B(Z^i_1, ..., Z^i_k, y_i, y_1, ..., y_m)$ a predicate substitution replacing predicate variables with predicate variables of the same arity; of course, if $B(C^i_1, ..., C^i_k, y_i, y_1, ..., y_m) \notin L$ then, *a fortiori*, $B(Z^i_1, ..., Z^i_k, y_i, y_1, ..., y_m) \notin L$). Finally, the fact that L is closed under H–substitution allows also to assume, without loss of generality, that, for every i with $1 \leq i \leq m$ and every j with $1 \leq j \leq k$, the formula C^i_j does not contain occurrences of negation. As a matter of fact, let D be any predicate wff, let P^0 be a 0–ary predicate variable not occurring in D, and let $\phi_{P_0}(D)$ be the predicate wff obtained from D by replacing in D every occurrence of a subformula such as $\neg K$ with an occurrence of the formula $K \rightarrow P^0$; let σ_{P^0} be the H–substitution leaving unchanged all the predicate variables different from P^0 and such that $\sigma_{P^0}(P^0) = P^0 \wedge \neg P^0$; then, considering the formula $\sigma_{P^0}(\phi_{P^0}(D))$, we have that $\vdash_{\overline{\text{INT}}} D \leftrightarrow \sigma_{P^0}(\phi_{P^0}(D))$. Thus, if some formula C^i_j does not satisfy our requirements, we can choose some appropriate P^0 and take $\phi_{P^0}(C^i_j)$ in place of C^i_j: from the above discussion it follows that the assumption that $B(C^i_1, ..., C^i_j, ..., C^i_k, y_i, y_1, ..., y_m) \notin L$ implies that $B(C^i_1, ..., \phi_{P^0}(C^i_j), ..., C^i_k, y_i, y_1, ..., y_m) \notin L$.

Now, consider the predicate substitution σ such that, for every j with $1 \leq j \leq k$, $\sigma(P_j) = C^1_j \wedge ... \wedge C^m_j$; let, in line with the above notations, $\exists x B(C^1_1 \wedge ... \wedge C^m_1, ..., C^1_k \wedge ... \wedge C^m_k, x, y_1, ..., y_m)$ indicate the formula $\sigma \exists x B(x, y_1, ..., y_m)$, where $y_1, ..., y_m$ are the only free individual variables. Since $\exists x B(x, y_1, ..., y_m) \in S(L)$, by definition of $S(L)$ we have that $\exists x B(C^1_1 \wedge ... \wedge C^m_1, ..., C^1_k \wedge ... \wedge C^m_k, x, y_1, ..., y_m) \in L$. Since L is constructive, by Lemma 2.1 one of $B(C^1_1 \wedge ... \wedge C^m_1, ..., C^1_k \wedge ... \wedge C^m_k, y_1, y_1, ..., y_m), ..., B(C^1_1 \wedge ... \wedge C^m_1, ..., C^1_k \wedge ... \wedge C^m_k, y_m, y_1, ..., y_m)$ belongs to L; let, for the sake of definiteness, $B(C^1_1 \wedge ... \wedge C^m_1, ..., C^1_k \wedge ... \wedge C^m_k, y_1, y_1, ..., y_m) \in L$. By Lemma 3.5 and the properties of the formulas $C^1_1, ..., C^m_1, ..., C^1_k, ..., C^m_k$, for every i and j with $1 \leq i \leq m$ and $1 \leq j \leq k$, we can choose an H–substitution $\sigma_{i,j}$ such that $\sigma_{i,j} C^i_j \in L$. Since no predicate variable occurring in $C^{i_1}_{j_1}$ occurs in $C^{i_2}_{j_2}$ for $1 \leq i_1, i_2 \leq m$, $1 \leq j_1, j_2 \leq k$, and $i_1 \neq i_2$ or $j_1 \neq j_2$, we can define an H–substitution σ^* with the following properties: for every i and j such that $2 \leq i \leq m$ and $1 \leq j \leq k$, if P is a predicate variable (of any arity) occurring in C^i_j then $\sigma^*(P) = \sigma_{i,j}(P)$; if P is not a predicate variable occurring in some C^i_j with $2 \leq i \leq m$ and $1 \leq j \leq k$, then $\sigma^*(P) = P$. It turns out that $\sigma^* C^1_j = C^1_j$ for every j such that

$1 \leq j \leq k$, while $\sigma^* C_j^i \in L$ for $2 \leq i \leq m$ and $1 \leq j \leq k$; hence, one has that $C_1^1 \leftrightarrow \sigma^*(C_1^1 \wedge ... \wedge C_1^m) \in L$, ...,$C_k^1 \leftrightarrow \sigma^*(C_k^1 \wedge ... \wedge C_k^m) \in L$. Being $\sigma^* B(C_1^1 \wedge ... \wedge C_1^m, ..., C_k^1 \wedge ... \wedge C_k^m, y_1, y_1, ..., y_m) \in L$, by the replacement theorem it follows that $B(C_1^1, ..., C_k^1, y_1, y_1, ..., y_m) \in L$. This contradicts what has been previously stated about $B(C_1^1, ..., C_k^1, y_1, y_1, ..., y_m)$.

In a quite similar way one can prove that $S(L)$ satisfies the disjunction property. ■

By Theorem 3.6 and Proposition 3.1 one immediately gets:

COROLLARY 3.7. *If L is any predicate constructive logic, then $S(E(L))$ is a predicate constructive logic and $L \subseteq S(E(L))$.*

As in the propositional case, if L is a maximal predicate constructive logic, then $L = S(E(L))$, i.e., L is a fixed point of the operator $S \circ E$ transforming predicate constructive logics into predicate constructive logics. Thus, calling *SE–stable* any (possibly non constructive) predicate logic L such that $L = S(E(L))$, we have that every maximal predicate constructive logic is SE–stable just as it happens in the propositional case (see [26]). On the other hand, in the propositional context SE–stability is equivalent to NegSat–determinatedness (see [26]). In the predicate frame the same can be stated, in general, only for the kurodian logics.

To be more precise, we say that a predicate logic L is *predicate negatively saturatedly determinated* (*NegSat–determinated* for short) iff the following condition holds for every predicate wff A:

if $\sigma^{NS} A \in L$ for every NegSat–substitution σ^{NS}, then $A \in L$.

Now, using the properties of the logic $\mathbf{K}$, we can prove:

LEMMA 3.8. *Let $A(P_1, ..., P_n)$ be any predicate negatively saturated wff, where $P_1, ..., P_n$ (with $n \geq 1$) are the predicate variables occurring in it (whose arities are not put into evidence). Then $A(P_1, ..., P_n) \leftrightarrow A(\neg\neg P_1, ..., \neg\neg P_n) \in \mathbf{K}$.*

PROOF. An easy induction on the complexity of the formula $A(P_1, ..., P_n)$, taking the negated subformulae of $A(P_1, ..., P_n)$ in the basis of the induction and using the replacement theorem. The starting point is that $\neg B(P_1, ..., P_n) \leftrightarrow \neg B(\neg\neg P_1, ..., \neg\neg P_n) \in \mathbf{K}$. To prove the latter fact one has that $\neg B(P_1, ..., P_n) \leftrightarrow \neg B(\neg\neg P_1, ..., \neg\neg P_n) \in CL$, which implies, by the properties of $\mathbf{K}$, $\neg\neg(\neg B(P_1, ..., P_n) \leftrightarrow \neg B(\neg\neg P_1, ..., \neg\neg P_n)) \in \mathbf{K}$; since $\neg\neg(\neg B(P_1, ..., P_n) \leftrightarrow \neg B(\neg\neg P_1, ..., \neg\neg P_n)) \leftrightarrow (\neg B(P_1, ..., P_n) \leftrightarrow \neg B(\neg\neg P_1, ..., \neg\neg P_n)) \in INT$, it follows that $\neg B(P_1, ..., P_n) \leftrightarrow \neg B(\neg\neg P_1, ..., \neg\neg P_n) \in \mathbf{K}$. ■

Now, we can prove:

THEOREM 3.9. *Let L be any kurodian predicate logic. Then L is SE–stable iff L is NegSat–determinated.*

PROOF. Let L be SE-stable and let $\sigma^{NS}A \in L$ for every NegSat–substitution σ^{NS}. Since $(P(\underline{x}) \leftrightarrow \neg\neg P(\underline{x}))^* \in E(L)$ for every predicate variable P, by the replacement theorem it follows that $\sigma A \in E(L)$ for every substitution σ; hence, $A \in S(E(L))$ and, since $L = S(E(L))$, $A \in L$. Thus, L is NegSat–determinated.

Conversely, let L be NegSat–determinated, $A \in S(E(L))$ and σ^{NS} be any NegSat–substitution. Then $\sigma^{NS}A \in E(L)$; hence (for some n) $(\neg\neg P_1(\underline{x}_1) \rightarrow P_1(\underline{x}_1))^* \wedge ... \wedge (\neg\neg P_n(\underline{x}_n) \rightarrow P_n(\underline{x}_n))^* \rightarrow (\sigma^{NS}A)(P_1, ..., P_n) \in L$, where the notation $(\sigma^{NS}A)(P_1, ..., P_n)$ indicates that the formula $\sigma^{NS}A$ contains the predicate variables $P_1, ..., P_n$. Since $(\neg\neg\neg\neg P_1(\underline{x}_1) \rightarrow \neg\neg P_1(\underline{x}_1))^* \wedge ... \wedge (\neg\neg\neg\neg P_n(\underline{x}_n) \rightarrow \neg\neg P_n(\underline{x}_n))^* \in INT$ and L is closed under r–substitution, it follows that $(\sigma^{NS}A)(\neg\neg P_1, ..., \neg\neg P_n) \in L$. Since $\mathbf{K} \subseteq L$, by Lemma 3.8 we get $\sigma^{NS}A \in L$. Hence, being σ^{NS} any NegSat–substitution, we have that $A \in L$. Since A is any element of $S(E(L))$, it follows that $L = S(E(L))$. ■

REMARKS:

- ▹ The proof of the above theorem shows that any SE–stable predicate logic L is always NegSat–determinated, whether or not L is kurodian. On the other hand, the hypothesis $\mathbf{K} \subseteq L$ seems to be necessary in order to prove that a NegSat–determinated predicate logic L is SE–stable, since, otherwise, Lemma 3.8 fails; for instance, $\neg\neg\forall x P(x) \leftrightarrow \neg\neg\forall x \neg\neg P(x) \notin INT$.
- ▹ Not only the maximal predicate constructive logics are NegSat–determinated. For, one can prove thsat also the non–maximal predicate constructive logic INT is SE–stable, and hence NegSat–determinated. Of course, the predicate non–constructive logic CL is SE–stable, and hence NegSat–determinated.
- ▹ We do not know examples of non–kurodian NegSat–determinated predicate logics which are not SE–stable.
- ▹ $\mathbf{K}$ can be seen to be an example of kurodian NegSat–determinated (and hence SE–stable) non–maximal predicate constructive logic.

Now, let L be any nonstandard predicate logic. Then $NS(L) = \{A \mid A$ is predicate negatively saturated and $A \in L\}$, while $WNS(L) = \{\sigma^r A \mid A \in L$ and σ^r is an r–substitution which is not empty for $A\}$. Moreover, we call $R(L) = INT \oplus NS(L)$ the *reduction* of L and $WR(L) = INT \oplus WNS(L)$ the *weak reduction* of L (recall that $INT \oplus NS(L)$ is the closure of $INT \cup NS(L)$ under generalization and modus ponens and the like for $INT \oplus WNS(L)$).

THEOREM 3.10. *Let L be a nonstandard predicate logic. Then:*

1. $WR(L) \subseteq R(L)$;
2. $WR(L)$ *is a predicate logic;*
3. *if L is kurodian then $R(L)$ is a predicate logic and* $R(L) = \mathbf{K} \oplus WNS(L) = \mathbf{K} \oplus WR(L)$.

PROOF.

1. Immediate, since if $A \in L$ and σ^r is not empty for A then $\sigma^r A \in L$ and A is predicate negatively saturated.

2. Let $\sigma^r A \in WNS(L)$ for some $A \in L$ and some r–substitution σ^r which is not empty for A. Let σ_1 be any substitution. Then (even if σ_1 is empty for $\sigma^r A$) there is a restricted substitution σ_2^r such that $\sigma_1 \sigma^r A = \sigma_2^r A$. It follows that $WNS(L)$ is closed under arbitrary substitutions, hence $WR(L) = INT \oplus WNS(L)$ is.

3. Let L be kurodian. Then, since all the instances of the axiom–schema (KUR) are predicate negatively saturated wff's we have $\mathbf{K} \subseteq R(L)$, hence $\mathbf{K} \oplus WNS(L) = \mathbf{K} \oplus WR(L) \subseteq R(L)$.
   On the other hand, $A(P_1, ..., P_n) \in NS(L)$ implies $A(P_1, ..., P_n) \in L$, which implies $A(\neg\neg P_1, ..., \neg\neg P_n) \in WNS(L)$, which implies, by Lemma 3.8, $A(P_1, ..., P_n) \in \mathbf{K} \oplus WNS(L)$; hence $R(L) = INT \oplus NS(L) \subseteq \mathbf{K} \oplus WNS(L) = \mathbf{K} \oplus WR(L)$. Thus, if L is kurodian then $R(L) = \mathbf{K} \oplus WNS(L) = \mathbf{K} \oplus WR(L)$; hence, since $\mathbf{K}$ and $WNS(L)$ are closed under arbitrary substitutions, $R(L)$ is a predicate logic.

REMARKS:

- ▷ Even for a kurodian nonstandard predicate logic L we may have $WR(L) \neq R(L)$. For instance, $WR(\mathbf{K}) = INT \oplus WNS(\mathbf{K}) \neq \mathbf{K} = R(\mathbf{K})$.
- ▷ We do not know whether there is a non–kurodian nonstandard predicate logic L such that $R(L)$ is not a predicate logic.
- ▷ Let the propositional counterparts of the operators R and WR be R_{prop} and WR_{prop} respectively. Then R_{prop} and WR_{prop} coincide for every nonstandard propositional logic.

Now, as in the corresponding propositional case [26], whether or not L is kurodian (and whether or not $R(L)$ is standard), we can prove:

THEOREM 3.11. *If L is a nonstandard predicate constructive logic then $R(L)$ is a nonstandard predicate constructive logic and $WR(L)$ is a predicate constructive logic.*

PROOF. We take into account only $R(L)$, since the proof for $WR(L)$ is quite similar. Also, we may consider $R(L)$ as the set of predicate wff's provable in the calculus $\mathcal{C}_{R(L)}$ consisting of the natural calculus for INT (see [29]), modified so as to treat $\neg$ as a primitive symbol, and of a set of zero–premisses rules (the intuitionistically unprovable formulae of $NS(L)$), which we call "axioms" of $R(L)$. So, a wff belongs to $R(L)$ iff there is a proof of it in $\mathcal{C}_{R(L)}$ without undischarged assumptions.

Now, we say that a predicate wff A is *well contained* (wc for short) in $R(L)$ iff $A \in R(L)$ and one of the following conditions is satisfied:

1. A is negated;
2. $A = B \wedge C$, and B is wc in $R(L)$ and C is wc in $R(L)$;
3. $A = B \vee C$, and B is wc in $R(L)$ or C is wc in $R(L)$;
4. $A = B \to C$, and if B is wc in $R(L)$ then C is wc in $R(L)$;
5. $A = \forall x B(x)$, and, for every individual variable y, $B(y)$ is wc in $R(L)$;
6. $A = \exists x B(x)$, and there is an individual variable y such that $B(y)$ is wc in $R(L)$.

Then, since L is constructive, first of all we have the following fact, whose proof is an easy induction on the complexity of A:

(P1) *if $A \in NS(L)$ then A is wc in $R(L)$.*

Using (P1) we can prove:

(P2) *let*

$$\begin{array}{c} A_1, ..., A_n \\ \Pi \\ B \end{array}$$

be a proof of the calculus $\mathcal{C}_{R(L)}$, where $A_1, ..., A_n$ are the undischarged assumptions and B is the consequence; let $A_1, ..., A_n$ be wc in $R(L)$; then B is wc in $R(L)$.

The proof of (P2) is by induction on the complexity of the proof $\prod$ and is similar to the proof of Point $\mathbf{p_2}$ of the proof of Theorem 5 of [26]. Thus, as an example of a typically predicative rule, we will illustrate only the case of the rule ($\forall I$).

Let

$$\begin{array}{c}\cdots\\ \prod\\ B\end{array} = \dfrac{\begin{array}{c}\cdots\\ \prod_1\\ C(y)\end{array}}{\forall x C(x)},$$

let z be any individual variable, and let $\begin{array}{c}\cdots\\ \prod_2\\ C(z)\end{array}$ be the tree obtained from $\begin{array}{c}\cdots\\ \prod_1\\ C(y)\end{array}$ by replacing in it every predicate wff such as $D(y)$ with the predicate wff $S_z^y D(y)$ (these individual substitutions can be correctly performed, modulo renaming of bounded variables). Since y does not occur free in the undischarged assumptions of $\prod_1$ (otherwise the $\forall I$–rule has not been correctly applied), it turns out that $\begin{array}{c}\cdots\\ \prod_2\\ C(z)\end{array}$ is a proof of the calculus $\mathcal{C}_{R(L)}$ whose undischarged assumptions coincide with the ones of $\begin{array}{c}\cdots\\ \prod_1\\ C(y)\end{array}$; then , by induction hypothesis, we have that $C(z)$ is wc in $R(L)$. Since z is any individual variable and since $\forall x C(x) \in R(L)$ (as a consequence of the fact that the undischarged assumptions of $\prod$, being wc in $R(L)$, belong to $R(L)$), $B = \forall x C(x)$ turns out to be wc in $R(L)$ by Point 5 of the definition of predicate wff wc in $R(L)$.

Now, using (P2), we can prove:

(P3) *All the formulae of* $R(L)$ *are wc in* $R(L)$.

To prove (P3), let $A \in R(L)$. Then there is a proof $\begin{array}{c}\prod\\ A\end{array}$ in $\mathcal{C}_{R(L)}$ without undischarged assumptions. Hence, by (P2), A is wc in $R(L)$.

Having (P3), we can conclude our proof as follows. If $A \vee B \in R(L)$, then A is wc in $R(L)$ or B is wc in $R(L)$; *a fortiori*, $A \in R(L)$ or $B \in R(L)$. Likewise, we can prove that if $\exists x A(x) \in R(L)$ then there is y such that $A(y) \in R(L)$. ■

PROPOSITION 3.12. *Let L be any nonstandard predicate logic. Then:*

1. $L \subseteq E(WR(L))$;
2. $E(WR(L)) = E(R(L))$;
3. *if L is regular then* $L = E(R(L)) = E(WR(L))$;
4. *if L is kurodian then* $R(L) = R(E(L)) \subseteq L$;
5. *if L can be expressed as $INT \oplus AX$, where AX is a set of predicate negatively satured wff's, then $R(L) = L$; moreover, if L is kurodian then $R(E(L)) = L$.*

PROOF.

1. If $A(P_1, ..., P_n) \in L$ then $A(\neg\neg P_1, ..., \neg\neg P_n) \in WR(L)$. Since, for every predicate variable P, $(\neg\neg P(\underline{x}) \to P(\underline{x}))^* \in E(WR(L))$ (where $\underline{x}$ is an appropriate sequence of individual variables), then $A(P_1, ..., P_n) \in E(WR(L))$.
2. From the previous case one gets $E(L) \subseteq E(E(WR(L))) = E(WR(L)) \subseteq E(R(L)) \subseteq E(L)$; hence the assertion.
3. Since $E(R(L)) \subseteq E(L)$, the assertion immediately comes from the previous cases.
4. The proof easily follows from Lemma 3.8.
5. Since $L = INT \oplus AX$ and the predicate wff's of AX are negatively saturated, $L \subseteq R(L)$; on the other hand, of course, $R(L) \subseteq L$, hence $L = R(L)$. If, in addition, L is kurodian, by the above Point 4 we have also $R(L) = R(E(L))$.

PROPOSITION 3.13. *If L is a nonstandard predicate logic, then $WR(L) \subseteq S(L)$.*

PROOF. By Theorem 3.10 we have that $WR(L)$ is a predicate logic; also, $WR(L) \subseteq L$. Since, by definition, $S(L)$ is the greatest predicate logic included in L, the assertion immediately follows. ■

THEOREM 3.14. *If L is a maximal nonstandard predicate constructive logic and L' is a predicate constructive logic such that $WR(L) \subseteq L'$, then $L' \subseteq S(L)$.*

PROOF. Let L' be any predicate constructive logic such that $WR(L) \subseteq L'$. Since L is a maximal nonstandard predicate constructive logic, $L = E(L)$ and, by Point 3 of Proposition 3.12, $L = E(WR(L))$. By hypothesis, $WR(L) \subseteq L'$ and hence $E(WR(L)) \subseteq E(L')$. Therefore, by the maximality of L and the fact that, by Proposition 3.1, $E(L')$ is a nonstandard predicate constructive logic, $L = E(L')$; hence $L' \subseteq L$. Since L' is a predicate logic included in L and $S(L)$ is the greatest predicate logic included in L, $L' \subseteq S(L)$. ■

COROLLARY 3.15. *If L is a maximal nonstandard predicate constructive logic, then $S(L)$ is a maximal predicate constructive logic.*

PROOF. By Proposition 3.13 we have $WR(L) \subseteq S(L)$. Hence, if $S(L) \subseteq L'$, we have $WR(L) \subseteq L'$. The assertion then follows from Theorem 3.14. ■

THEOREM 3.16. *If L is a maximal predicate constructive logic, then $E(L)$ is a maximal nonstandard predicate constructive logic.*

PROOF. By Proposition 3.1, $E(L)$ is a nonstandard predicate constructive logic. Now, let L' be any nonstandard predicate constructive logic such that $E(L) \subseteq L'$; then $S(E(L)) \subseteq S(L')$, from which, since by Corollary 3.7 we have $L \subseteq S(E(L))$, we get $L \subseteq S(L')$. Also, since $E(L) \subseteq L'$, $L' = E(L')$; hence L' is regular and, by Theorem 3.6, $S(L')$ is a predicate constructive logic. Since L is a maximal predicate constructive logic, $L = S(L')$. On the other hand, by Proposition 3.13, $WR(L') \subseteq S(L')$. Thus, $WR(L') \subseteq S(L') = L$, which implies $E(WR(L')) \subseteq E(L)$. But L' is regular, hence, by Point 3 of Proposition 3.12, $L' = E(WR(L'))$. It follows that $L' \subseteq E(L)$ and $E(L)$ turns out to be maximal. ■

REMARKS:

- ▷ Theorem 3.14 says more than Corollary 3.15. As a matter of fact, Theorem 3.14 allows to say that not only $S(L)$ is a maximal predicate constructive logic, but also that it is *the greatest* predicate constructive logic among the ones including $WR(L)$.
- ▷ By Corollary 3.15 and Theorem 3.16, there is a one–to–one correspondence between maximal predicate constructive logics and maximal nonstandard predicate constructive logics, the correspondence associating every standard object with the unique nonstandard object containing it.
- ▷ Of course, by Zorn's Lemma, i.e., by the Axiom of Choice, there is a maximal predicate constructive logic. Indeed, in the following section

we will exhibit examples of such logics (without using the Axiom of Choice). These logics will turn out to be kurodian. A question we leave open in this paper is whether there are maximal predicate constructive logics which are not kurodian.

4. Two examples of maximal predicate constructive logics

Let Neg_{fin} be the set of all finite sets of negated predicate wff's. Then the notion of Γ–sound predicate wff is inductively defined as follows.

Let A be a predicate wff and let $\Gamma \in Neg_{fin}$. Then A is Γ–sound iff $\Gamma \vdash_{\mathrm{CL}} A$ and one of the following conditions is satisfied:

1. A is atomic or negated;
2. $A = B \wedge C$, and B is Γ–sound and C is Γ–sound;
3. $A = B \vee C$, and B is Γ–sound or C is Γ–sound;
4. $A = B \to C$, and, for every $\Gamma' \in Neg_{fin}$ and every transformation τ such that, for every $Z \in \Gamma \cup \{B \to C\}$, τ is not empty for Z and $\tau\Gamma \subseteq \Gamma'$, if τB is Γ'–sound then τC is Γ'–sound;
5. $A = \exists x B(x)$, and there is an individual variable y such that $B(y)$ is Γ–sound;
6. $A = \forall x B(x)$, and, for every individual variable y, $B(y)$ is Γ–sound.

We point out that the empty set $\emptyset$ belongs to Neg_{fin}. Thus, we define the set $\mathcal{M}_1$ of predicate wff's as:

$$\mathcal{M}_1 = \{A \mid A \text{ is } \emptyset\text{–sound}\}.$$

REMARKS:

- ▷ Without affecting the definition of $\mathcal{M}_1$, we could restrict ourselves to all the consistent elements of Neg_{fin}, as it can be easily shown.
- ▷ Furthermore, since in Γ–soundness the involved provability is the classical one, we could as well define Neg_{fin} as the set of all finite sets of predicate wff's (negated or not). However, the definition of Neg_{fin} we have given above turns out to be more convenient in proving the maximality of $\mathcal{M}_1$.
- ▷ Note that the definition of "A is Γ–sound" involves, in Point 4, the predicate wff's $\tau\Gamma \cup \{\tau A\}$, in particular the wff τA. However, since any transformation τ is a composition of *individual substitutions* and

r–substitutions, the *relevant complexity* of τA does not exceed the one of A. For, in the definition of Γ–soundness, the basic Point 1 takes as elementary formulae both the atomic and the negated ones, while individual substitutions do not affect the complexity of formulae. In this sense, the inductive definition of Γ–soundness is quite correct and gives rise to the set of wff's $\mathcal{M}_1$, which, as we will see, turns out to be a kurodian maximal nonstandard predicate constructive logic, *but not* a maximal (standard) predicate constructive logic; to get a logic of the second kind (closed with respect to substitutions affecting in an essential way the complexity of formulae) one has to consider, according to Corollary 3.15, $S(\mathcal{M}_1)$.

Now let us show that $\mathcal{M}_1$ is a nonstandard predicate constructive logic. To this purpose, we first need some technical lemmas.

LEMMA 4.1. *For every predicate wff A, for every $\Gamma, \Gamma' \in Neg_{fin}$ and for every transformation τ such that, for every $Z \in \Gamma \cup \{A\}$, τ is not empty for Z and $\tau\Gamma \subseteq \Gamma'$, if A is Γ–sound then τA is Γ'–sound.*

PROOF. By induction on the complexity of A. First of all, if $\Gamma \vdash_{\mathrm{CL}} A$ then, obviously, $\Gamma' \vdash_{\mathrm{CL}} \tau A$. Thus, if A is an atomic or negated formula then the assertion follows trivially. Moreover, when $A = B \vee C$, or $A = B \wedge C$, or $A = \exists x B(x)$, or $A = \forall x B(x)$, the result easily follows from the induction hypothesis. Let $A = B \rightarrow C$, let A be Γ–sound and let us suppose that τ is such that, for every $Z \in \Gamma \cup \{B \rightarrow C\}$, τ is not empty for Z, but $\tau(B \rightarrow C)$ is not Γ'–sound. Then, since $\Gamma' \vdash_{\mathrm{CL}} \tau(B \rightarrow C)$ holds, by Point 4 of the definition of Γ'–sound wff there must be a Γ'' and a τ' such that, for every $Z \in \Gamma' \cup \{\tau(B \rightarrow C)\}$ τ' is not empty for Z, $\tau'\Gamma' \subseteq \Gamma''$, $\tau'\tau B$ is Γ''–sound, but $\tau'\tau C$ is not Γ''–sound. Since the composition $\tau'\tau$ is still a transformation, let us call it τ''. We have that there is Γ'' together with τ'' such that $\tau''\Gamma \subseteq \Gamma''$, for every $\theta \in \Gamma \cup \{B \rightarrow C\}$ τ'' is not empty for θ, $\tau'' B$ is Γ''–sound, but $\tau'' C$ is not Γ''–sound; hence $B \rightarrow C$ is not Γ–sound, a contradiction. ■

The next corollaries immediately follow from the previous lemma.

COROLLARY 4.2. *$\mathcal{M}_1$ is closed under r–substitution.*

PROOF. The proof follows trivially from Lemma 4.1, taking $\emptyset$ for Γ and Γ', and any σ^r for τ. ■

COROLLARY 4.3. *$\mathcal{M}_1$ is closed under generalization.*

PROOF. The proof follows trivially from Lemma 4.1, taking $\emptyset$ for Γ and Γ', and any individual substitution η for τ. ■

LEMMA 4.4. $INT \subseteq \mathcal{M}_1$.

PROOF. It will suffice to prove the following fact. Let

$$\begin{array}{c} A_1, ..., A_n \\ \prod \\ B \end{array}$$

be any proof in the natural calculus for INT (see [29]) modified so as to treat $\neg$ as a primitive symbol, where $A_1, ..., A_n$ $(n \geq 0)$ are the undischarged assumptions and B is the proved formula. Then, for every $\Gamma \in Neg_{fin}$, if $A_1, ..., A_n$ are Γ–sound then B is Γ–sound.

To show the assertion one goes on by induction on the complexity of the proof. The basis (introduction of assumption) is immediate and the various cases of the induction step are easily handled. Here we will consider only the most interesting case, namely the one corresponding to the inference rule $\rightarrow I$.

Let our proof $\prod$ have the form:

$$\frac{\begin{array}{c} A_1, ..., A_n, [C] \\ \prod_1 \\ D \end{array}}{C \rightarrow D}.$$

Since $A_1, ..., A_n$ are Γ–sound, we have $\Gamma \vdash_{\overline{CL}} A_1, ..., \Gamma \vdash_{\overline{CL}} A_n$.

On the other hand, since $\{A_1, ..., A_n\} \vdash_{\overline{INT}} B$ (by the existence of our proof) and since $INT \subseteq CL$, we have $\{A_1, ..., A_n\} \vdash_{\overline{CL}} B$. It follows that $\Gamma \vdash_{\overline{CL}} B$.

Now, let $\Gamma' \in Neg_{fin}$ and let τ be any transformation such that, for every $Z \in \Gamma \cup \{C \rightarrow D\}$ τ is not empty for Z, $\tau\Gamma \subseteq \Gamma'$, and τC is Γ'–sound. Then, first of all we can built up the proof

$$\begin{array}{c} \tau A_1, ..., \tau A_n, \tau C \\ \prod_2 \\ \tau D \end{array}$$

having the same tree–structure as $\prod_1$ (hence, having the same complexity as $\prod_1$), where we assume, without loss of generality, that, for every $\theta \in \{A_1, ..., A_n\}$, τ is not empty for θ (if τ is empty for some $\theta \in \{A_1, ..., A_n\}$, we can go on similarly, choosing some τ' and Γ'' such that, for every $\theta' \in$

$\Gamma \cup \{A_1, ..., A_n, C, D\}$, τ' is not empty for θ', $\tau\Gamma \subseteq \Gamma''$, and $\tau'C$, $\tau'D$ and Γ'' differ from τC, τD and Γ' only for a renaming of variables; here one proves that if $K \in \{C, D\}$ then τK is Γ'–sound if $\tau' K$ is Γ''–sound). Since $A_1, ..., A_n$ are Γ–sound, from Lemma 4.1 we get that $\tau A_1, ..., \tau A_n$ are Γ'–sound; hence, since τC is Γ'–sound, all the undischarged assumptions of $\prod_2$ are Γ'–sound, which implies, by induction hypothesis, that τD is Γ'–sound. Thus, since Γ' and τ are any, $B \to C$ turns out to be Γ–sound. ■

The three following facts come immediately from the definition of $\emptyset$–soundness.

LEMMA 4.5. $\mathcal{M}_1 \subseteq CL$.

LEMMA 4.6. $\mathcal{M}_1$ *is closed under detachment.*

LEMMA 4.7. $\mathcal{M}_1$ *has the disjunction property and the explicit definability property.*

From the previous lemmas the following corollary immediately follows:

COROLLARY 4.8. $\mathcal{M}_1$ *is a nonstandard predicate constructive logic.*

Now, to better understand what is involved in $\mathcal{M}_1$, we put into evidence some formulae and principles (axiom–schemes) belonging to it. The knowledge that these formulae and principles are in $\mathcal{M}_1$ will be also used to prove that $\mathcal{M}_1$ is a kurodian maximal nonstandard predicate constructive logic.

We start from the following proposition:

PROPOSITION 4.9. $\mathcal{M}_1 = E(\mathcal{M}_1)$.

PROOF. Let A be atomic. Then, for every $\Gamma \in Neg_{fin}$ and every transformation τ which is not empty for $\neg\neg A \to A$, $\tau\neg\neg A$ is Γ–sound iff τA is Γ-sound, by definition of Γ–soundness of atomic and negated formulae. Since $\emptyset \vdash_{CL} \neg\neg A \to A$ and $\tau\emptyset \subseteq \Gamma$ for every Γ, we therefore have that $\neg\neg A \to A$ is $\emptyset$–sound, i.e., $\neg\neg A \to A \in \mathcal{M}_1$. ■

By Proposition 4.9, $\mathcal{M}_1$ cannot be a predicate logic. As a matter of fact, if $\mathcal{M}_1$ were closed under arbitrary substitutions, then $\neg\neg A \to A$ would belong to $\mathcal{M}_1$ for every predicate wff A, i.e., $\mathcal{M}_1$ would coincide with CL. But $\mathcal{M}_1 \neq CL$, since $\mathcal{M}_1$ is constructive and CL is not.

Putting together Proposition 3.3 and Proposition 4.9, we have the following corollary:

Corollary 4.10. *$\mathcal{M}_1$ is closed under H–substitution.*

Coming to the axiom–schemes, we have:

Proposition 4.11. *Every instance of (KUR) belongs to $\mathcal{M}_1$.*

Proof. Immediate, since $\mathbf{K} = INT + \{(KUR')\}$, where $\{(KUR')\}$ is the set of all the instances of the form $\neg(\forall x \neg\neg A(x) \wedge \neg \forall x A(x))$. The latter formulae, of course, belong to $\mathcal{M}_1$, since they are negated formulae belonging to CL. ∎

Now we introduce Kreisel and Putnam disjunction principle, we will call $(KP_\vee)$, and Kreisel and Putnam existential principle, we will call $(KP_\exists)$. The former is well known to every people working in intermediate propositional logics [10, 11, 17], while the latter, which is also known in the area of constructivism as (IP) (see, e.g., [32]), naturally completes the meaning of the former at the predicate level. The two principles are so defined: $(KP_\vee)$ is the axiom–schema whose instances are all the formulae $(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$, where A, B and C are arbitrary predicate wff's; $(KP_\exists)$ is the axiom–schema whose instances are all the formulae $(\neg A \to \exists x B(x)) \to \exists x(\neg A \to B(x))$, where A and $B(x)$ are predicate wff's such that x is not free in A. We have:

Proposition 4.12. *All the instances of $(KP_\vee)$ and all the instances of $(KP_\exists)$ belong to $\mathcal{M}_1$.*

Proof. Consider, e.g., $(KP_\exists)$. Then, first of all, for every A and $B(x)$ with x not free in A, we have $\emptyset \vdash_{CL} (\neg A \to \exists x B(x)) \to \exists x(\neg A \to B(x))$. Now, let $\Gamma \in Neg_{fin}$, and let τ be any transformation such that τ is not empty for $(\neg A \to \exists x B(x)) \to \exists x(\neg A \to B(x))$ and $\tau(\neg A \to \exists x B(x))$ is Γ–sound. Then, by Lemma 4.1, $\tau(\neg A \to \exists x B(x))$ is $(\Gamma \cup \{\tau\neg A\})$–sound (where $\tau\neg A = \neg\tau A$ is a negated wff). Since $\tau\neg A$ is $(\Gamma \cup \{\tau\neg A\})$–sound, it follows that $\tau\exists x B(x) = \exists x \tau B(x)$ is $(\Gamma \cup \{\tau\neg A\})$–sound, i.e., there is y such that $\tau B(y)$ is $(\Gamma \cup \{\tau\neg A\})$–sound. The latter fact implies, in particular, that $\Gamma \vdash_{CL} \tau(\neg A \to B(y))$. On the other hand, let $\Gamma' \in Neg_{fin}$ and let τ' be any transformation such that, for every $Z \in \Gamma \cup \{\tau(\neg A \to B(y))\}$, τ' is not empty for Z, $\tau'\Gamma \subseteq \Gamma'$ and $\tau'\tau\neg A = \neg\tau'\tau A$ is Γ'–sound. Then, since $\Gamma' \vdash_{CL} \tau'\tau\neg A$, we have that $\{K \mid \Gamma' \vdash_{CL} K\} = \{K \mid \Gamma' \cup \{\tau'\tau\neg A\} \vdash_{CL} K\}$, which implies that a predicate wff θ is Γ'–sound iff θ is $(\Gamma' \cup \{\tau'\tau\neg A\})$–sound, as the reader can easily prove by induction on the complexity of θ. Since $\tau'\tau B(y)$ is $(\Gamma' \cup \{\tau'\tau\neg A\})$–sound, as a consequence of Lemma 4.1 and of the fact that $\tau'(\Gamma \cup \{\tau\neg A\}) \subseteq \Gamma' \cup \{\tau'\tau\neg A\}$ and $\tau B(y)$ is $(\Gamma \cup \{\tau\neg A\})$–sound, we conclude that $\tau'\tau B(y)$ is Γ'–sound. Since Γ' and τ' have been chosen in an arbitrary way, from $\Gamma\vdash_{\overline{\text{CL}}}\tau(\neg A \to B(y))$ and from the definition of Γ–soundness of a wff of the form $Z_1 \to Z_2$, we have that $\tau(\neg A \to B(y))$ is Γ–sound; thus, since $\Gamma\vdash_{\overline{\text{CL}}}\tau\exists x(\neg A \to B(x))$ (which immediately follows from $\Gamma\vdash_{\overline{\text{CL}}}\tau(\neg A \to B(y))$ previously stated, as well as from $\emptyset\vdash_{\overline{\text{CL}}}(\neg A \to \exists xB(x)) \to \exists x(\neg A \to B(x))$ and $\Gamma\vdash_{\overline{\text{CL}}}\tau(\neg A \to \exists xB(x)))$, we have that $\tau\exists x(\neg A \to B(x))$ is Γ–sound. Since τ and Γ are arbitrary elements such that $\Gamma \in Neg_{fin}$, τ is a transformation which is not empty for $(\neg A \to \exists xB(x)) \to \exists x(\neg A \to B(x))$, and $\tau(\neg A \to \exists xB(x))$ is Γ–sound, by definition of $\emptyset$–soundness we conclude that $(\neg A \to \exists xB(x)) \to \exists x(\neg A \to B(x))$ is $\emptyset$–sound; hence $(\neg A \to \exists xB(x)) \to \exists x(\neg A \to B(x)) \in \mathcal{M}_1$.

A quite similar argument shows that $(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C) \in \mathcal{M}_1$ for any A, B and C. ■

Now we want to put into evidence another interesting principle of $\mathcal{M}_1$ which, differently from $(KP_\vee)$ and $(KP_\exists)$, will not be used in the proof that $\mathcal{M}_1$ is a maximal nonstandard predicate constructive logic. Such a principle is the well known Grzegorczyk principle (GRZ), whose addition to INT gives rise to a (standard) predicate logic semantically characterized by the class of all the Kripke models with constant domains, see [8, 9, 31]. The principle is so defined: (GRZ) is the axiom–schema whose instance are all the formulae $\forall x(A \vee B(x)) \to A \vee \forall xB(x)$, where A and $B(x)$ are any predicate wff's and where x is not free in A.

We have:

PROPOSITION 4.13. *Every instance of* (GRZ) *belongs to* $\mathcal{M}_1$.

PROOF. First of all, for every A and $B(x)$ with x not free in A, we have $\emptyset\vdash_{\overline{\text{CL}}}\forall x(A \vee B(x)) \to A \vee \forall xB(x)$. Now, let $\Gamma \in Neg_{fin}$ and let τ be any transformation such that τ is not empty for $\forall x(A\vee B(x)) \to A\vee\forall xB(x)$ and $\tau\forall x(A \vee B(x))$ is Γ–sound. Then, we immediately get $\Gamma\vdash_{\overline{\text{CL}}}\tau(A \vee \forall xB(x))$. On the other hand, we have that, for every y, $\tau(A \vee B(y))$ is Γ–sound. If τA is Γ–sound, then from $\Gamma\vdash_{\overline{\text{CL}}}\tau(A \vee \forall xB(x))$ we immediately get that $\tau(A \vee \forall xB(x))$ is Γ–sound, and we are done. Otherwise, τA is not Γ–sound; thus, since $\tau(A \vee B(y))$ is Γ–sound for every y, we must have that $\tau B(y)$ is Γ–sound for every y. In the latter case, since Γ is finite, we can choose an individual variable z such that z does not occur free in any formula of Γ, z is different from the free variables of $\tau B(x)$ and $\tau B(z)$ is Γ–sound. Since $\Gamma\vdash_{\overline{\text{CL}}}\tau B(z)$ is a consequence of the fact that $\tau B(z)$ is Γ–sound, by the properties of the variable z we get $\Gamma\vdash_{\overline{\text{CL}}}\forall x\tau B(x)$; hence $\forall x\tau B(x)$ is Γ–sound, as a consequence of the definition of Γ–soundness and the fact that

$\tau B(y)$ is Γ–sound for every y. Being $\Gamma \vdash_{\mathrm{CL}} \tau(A \vee \forall x B(x))$, it follows that $\tau(A \vee \forall x B(x))$ is Γ–sound. Since Γ and τ have been chosen in an arbitrary way, we therefore have that $\forall x(A \vee B(x)) \to A \vee \forall x B(x)$ is $\emptyset$–sound, hence $\forall x(A \vee B(x)) \to A \vee \forall x B(x) \in \mathcal{M}_1$. ■

Now we are going to show that $\mathcal{M}_1$ is a maximal nonstandard predicate constructive logic. To do so, further lemmas are needed. We start from the following one:

LEMMA 4.14. *Let* $\Gamma = \{\neg A_1, ..., \neg A_n\} \in Neg_{fin}$ *and let* B *be any predicate wff. Then* B *is* Γ*–sound iff* $\neg A_1 \wedge ... \wedge \neg A_n \to B \in \mathcal{M}_1$ (*if* $n = 0$ *then* $\neg A_1 \wedge ... \wedge \neg A_n \to B$ *is equal to* B).

PROOF. Let $\neg A_1 \wedge ... \wedge \neg A_n \to B \in \mathcal{M}_1$. Then, since $\neg A_1 \wedge ... \wedge \neg A_n$ is immediately seen to be Γ–sound, by Lemma 4.1 we have that B is Γ–sound.

Conversely, let B be Γ–sound. Then $\Gamma \vdash_{\mathrm{CL}} B$, i.e., $\{\neg A_1, ..., \neg A_n\} \vdash_{\mathrm{CL}} B$, and hence $\emptyset \vdash_{\mathrm{CL}} \neg A_1 \wedge ... \wedge \neg A_n \to B$. On the other hand, let $\Gamma' \in Neg_{fin}$ and let τ be such that τ is not empty for $\neg A_1 \wedge ... \wedge \neg A_n \to B$ and $\tau(\neg A_1 \wedge ... \wedge \neg A_n)$ is Γ'–sound. Then $\Gamma' \vdash_{\mathrm{CL}} \tau \neg A_1, ..., \Gamma' \vdash_{\mathrm{CL}} \tau \neg A_n$, which implies that τB is Γ'–sound iff τB is $(\tau\Gamma \cup \Gamma')$–sound (as one easily proves by induction on the complexity of B). Since B is Γ–sound, by Lemma 4.1 τB is $(\tau\Gamma \cup \Gamma')$–sound. Hence τB is Γ'–sound. ■

Using Lemma 4.14 we can prove:

LEMMA 4.15. *If* $\Gamma \in Neg_{fin}$ *then* $\{B \mid \Gamma \cup \mathcal{M}_1 \vdash_{\mathrm{INT}} B\} = \{B \mid B$ *is* Γ*–sound*$\}$.

PROOF. Suppose that $\Gamma \cup \mathcal{M}_1 \vdash_{\mathrm{INT}} B$. Then there is a proof in the natural calculus of intuitionistic predicate logic whose undischarged assumptions, say $A_1, ..., A_n$, belong to $\Gamma \cup \mathcal{M}_1$. Since the elements of Γ are Γ–sound and the elements of $\mathcal{M}_1$, being $\emptyset$–sound, are Γ–sound, we have that $A_1, ..., A_n$ are Γ–sound. Hence, arguing as in Lemma 4.4, B turns out to be Γ–sound.

Conversely, let B be Γ–sound, with $\Gamma = \{\neg A_1, ..., \neg A_n\}$. Then, by Lemma 4.14, $\neg A_1 \wedge ... \wedge \neg A_n \to B \in \mathcal{M}_1$. Since $\Gamma \vdash_{\mathrm{INT}} \neg A_1 \wedge ... \wedge \neg A_n$, it follows that $\Gamma \cup \mathcal{M}_1 \vdash_{\mathrm{INT}} B$. ■

COROLLARY 4.16. *If* $\Gamma \in Neg_{fin}$ *then* $\{B \mid \Gamma \vdash_{\mathcal{M}_1} B\} = \{B \mid B$ *is* Γ*–sound*$\}$.

PROOF. Since, for every set Δ of predicate wff's, every predicate wff B and every nonstandard predicate logic L, one has $\Delta \cup L \vdash_{\mathrm{INT}} B$ iff $\Delta \vdash_{\mathrm{L}} B$, the assertion follows immediately from the previous lemma. ■

Finally, with the help of Corollary 4.16, we can prove:

LEMMA 4.17. *Let L be any nonstandard predicate constructive logic such that $\mathcal{M}_1 \subseteq L$. Then, for every $\Gamma \in Neg_{fin}$, $\{B \mid \Gamma \vdash_{\mathcal{M}_1} B\} = \{B \mid \Gamma \vdash_{\mathrm{L}} B\}$.*

PROOF. Of course, $\{B \mid \Gamma \vdash_{\mathcal{M}_1} B\} \subseteq \{B \mid \Gamma \vdash_{\mathrm{L}} B\}$. Thus, we will show, by induction on the complexity of B, that $\Gamma \vdash_{\mathrm{L}} B$ implies $\Gamma \vdash_{\mathcal{M}_1} B$. In treating the basis of this induction we will consider only the case $B = \neg C$. As a matter of fact, Proposition 4.9 and the hypothesis that $\mathcal{M}_1 \subseteq L$ imply that $L = E(L)$. Thus, for A atomic we have that $\Gamma \vdash_{\mathrm{L}} A$ iff $\Gamma \vdash_{\mathrm{L}} \neg\neg A$ and that $\Gamma \vdash_{\mathcal{M}_1} A$ iff $\Gamma \vdash_{\mathcal{M}_1} \neg\neg A$.

Now, let $B = \neg C$ and $\Gamma \vdash_{\mathrm{L}} B$; then, *a fortiori*, $\Gamma \vdash_{\mathrm{CL}} B$, i.e., $\Gamma \vdash_{\mathrm{CL}} \neg C$, which implies, being $\neg C$ negated, that $\neg C$ if Γ–sound. Hence, by Corollary 4.16, $\Gamma \vdash_{\mathcal{M}_1} \neg C$, i.e., $\Gamma \vdash_{\mathcal{M}_1} B$.

The case $B = C \wedge D$ is an immediate consequence of the induction hypothesis.

Let $B = C \vee D$ and let $\Gamma \vdash_{\mathrm{L}} B$, with $\{\neg A_1, ..., \neg A_n\} = \Gamma$ (with $n \geq 0$). Then $\neg A_1 \wedge ... \wedge \neg A_n \to B \in L$ (if $n = 0$, then $B \in L$), which implies (being $\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n)$ intuitionistically equivalent to $\neg A_1 \wedge ... \wedge \neg A_n$) that $\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to C \vee D \in L$. Now, by Proposition 4.12, $(\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to C \vee D) \to (\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to C) \vee (\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to D) \in \mathcal{M}_1$, which implies that $(\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to C \vee D) \to (\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to C) \vee (\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to D) \in L$. Thus, being L constructive, we get $\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to C \in L$ or $\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to D \in L$. Let, for the sake of definiteness, $\neg\neg(\neg A_1 \wedge ... \wedge \neg A_n) \to C \in L$; then, $\neg A_1 \wedge ... \wedge \neg A_n \to C \in L$, which implies $\Gamma \vdash_{\mathrm{L}} C$. By induction hypothesis we therefore have $\Gamma \vdash_{\mathcal{M}_1} C$, which yields $\Gamma \vdash_{\mathcal{M}_1} C \vee D$, i.e., $\Gamma \vdash_{\mathcal{M}_1} B$.

The case $B = \forall x C(x)$ immediately comes from the induction hypothesis and from the fact that $\Gamma \vdash_{\mathcal{M}_1} C(y)$, with y an individual variable different from the free individual variables of $\Gamma \cup \forall x C(x)$, implies $\Gamma \vdash_{\mathcal{M}_1} \forall x C(x)$, the latter fact, in turn, coming from the closure of $\mathcal{M}_1$ under generalization.

The case $B = \exists x C(x)$ is quite similar to the case $B = C \vee D$, since, by Proposition 4.12, all the instances of $(KP_\exists)$ belong to $\mathcal{M}_1$.

Finally, let $B = C \to D$, let $\Gamma \vdash_{\mathrm{L}} C \to D$, but suppose that $\Gamma \vdash_{\mathcal{M}_1} C \to D$ does not hold. Then, by Corollary 4.16, $C \to D$ is not Γ–sound, which implies (since $\Gamma \vdash_{\mathrm{CL}} C \to D$ follows from $\Gamma \vdash_{\mathrm{L}} C \to D$) that there is $\Gamma' \in Neg_{fin}$ and a transformation τ such that, for every $H \in \Gamma \cup \{C \to D\}$ τ is not empty for H, $\tau\Gamma \subseteq \Gamma'$, τC is Γ'–sound and τD is not Γ'–sound. Hence, by Corollary 4.16, we have that $\Gamma' \vdash_{\mathcal{M}_1} \tau C$, while $\Gamma' \vdash_{\mathcal{M}_1} \tau D$ does not hold. On the other hand, from $\Gamma \vdash_{\mathrm{L}} C \to D$, $\tau\Gamma \subseteq \Gamma'$ and the fact that L is closed under predicate transformation, we get $\Gamma' \vdash_{\mathrm{L}} \tau(C \to D)$. Moreover, from $\Gamma' \vdash_{\mathcal{M}_1} \tau C$ we get $\Gamma' \vdash_{\mathrm{L}} \tau C$. Thus, by detachment, $\Gamma' \vdash_{\mathrm{L}} \tau D$. It follows, by induction hypothesis, that $\Gamma' \vdash_{\mathcal{M}_1} \tau D$, a contradiction. ∎

Taking $\Gamma = \emptyset$, from the previous lemma we get:

THEOREM 4.18. *$\mathcal{M}_1$ is a maximal nonstandard predicate constructive logic.*

Finally, using Theorem 4.18 and Corollary 3.15, we get:

THEOREM 4.19. *$S(\mathcal{M}_1)$ is a maximal predicate constructive logic.*

REMARKS:

▷ By Proposition 4.12, $\mathcal{M}_1$ contains all the propositional instances of the axiom–schema $(KP_\vee)$; moreover, by Proposition 4.9, for every propositional variable p, the propositional wff $\neg\neg p \rightarrow p$ belongs to $\mathcal{M}_1$. Thus, according to the treatment of [26], $\mathcal{M}_1$ includes the maximal nonstandard propositional logic $E_{prop}(MV)$, where MV is Medvedev's logic [4, 5, 10, 19, 20, 22, 23, 26] and E_{prop} is the propositional extension operator described in [26] (there denoted E), which is the propositional counterpart of the operator E of the present paper. It follows that $E_{prop}(MV)$ is included in the greatest propositional logic included in $\mathcal{M}_1$, i.e., $E_{prop}(MV) \subseteq PROP(\mathcal{M}_1)$. On the other hand, since $\mathcal{M}_1$ is a nonstandard predicate constructive logic, $PROP(\mathcal{M}_1)$ is immediately seen to be a nonstandard propositional constructive logic. Hence, since $E_{prop}(MV)$ is a maximal nonstandard propositional constructive logic, $E_{prop}(MV) = PROP(\mathcal{M}_1)$; from this and from the treatment of [26], we immediately get $MV = S_{prop}(PROP(\mathcal{M}_1))$, where S_{prop} is the propositional standardization operator analyzed in [26] (there denoted S), which is the propositional counterpart of the operator S of the present paper.

However, from $MV = S_{prop}(PROP(\mathcal{M}_1))$ we get that MV includes $PROP(S(\mathcal{M}_1))$, but we do not know whether or not MV coincides with $PROP(S(\mathcal{M}_1))$.

Indeed, let $A \in S_{prop}(PROP(\mathcal{M}_1))$. Then, for every *propositional* substitution σ_{prop}, we have that $\sigma_{prop}A \in \mathcal{M}_1$; but this does not seem to necessarily imply that, for every *predicate* substitution σ, $\sigma A \in \mathcal{M}_1$. Thus, even if $PROP(S(\mathcal{M}_1))$ is a (standard) propositional constructive logic, *we do not know whether $PROP(S(\mathcal{M}_1))$ is a maximal propositional constructive logic.* Since $PROP(S(\mathcal{M}_1))$ includes the propositional logic WKP studied in [26], according to the results of [26] we only know that *MV is the only maximal propositional constructive logic including $PROP(S(\mathcal{M}_1))$.*

- ▷ The previous remark gives rise the problem of the existence of a maximal predicate constructive logic L such that $PROP(L)$ is a maximal propositional constructive logic. Well, such a problem has a positive answer, using the Axiom of Choice. As a matter of fact, consider the class $\mathcal{F}$ of all predicate Kripke frames $\underline{F} = \langle P, \leq, D \rangle$ so defined:

  1. the underlying poset $\langle P, \leq \rangle$ of $\underline{F}$ is a $\mathcal{MV}$ poset in the sense of [23];
  2. the function D associating, with every $\alpha \in P$, the domain $D(\alpha)$ of α is any (provided that the condition $D(\beta) \subseteq D(\gamma)$ for $\beta, \gamma \in P$ and $\beta \leq \gamma$ is fulfilled).

  Then, it is easily shown that the predicate logic L generated by the class of frames $\mathcal{F}$ (see, e.g., [6, 27, 28] for a definition) is constructive and that $MV = PROP(L)$. Thus, by Zorn's Lemma (i.e., by the Axiom of Choice) there is a maximal predicate constructive logic L' such that $L \subseteq L'$, hence $MV \subseteq PROP(L')$. Since MV is a maximal propositional constructive logic and $PROP(L')$ is constructive, we get $MV = PROP(L')$.
  More generally, for each of the infinitely many maximal propositional constructive logics presented in [4, 5], using the Axiom of Choice we can claim that there is a maximal predicate constructive logic including it.

- ▷ The method presented in this paper to get maximal predicate constructive logics from maximal nonstandard predicate constructive logics has been previously applied in [4, 5, 26] to get maximal propositional constructive logics from maximal nonstandard propositional constructive logics. Also, these propositional applications have been enriched by a method to get Kripke frames semantics for standard logics starting from special Kripke frames semantics for nonstandard logics. Combining these tools, infinitely many maximal propositional constructive logics have been presented as the logics generated by suitable classes of frames (posets), and the corresponding maximal nonstandard propositional constructive logics as the regular logics characterized by the same frames, considering only special Kripke models (called regular) built on them [4, 5]; moreover (differently from the maximal standard) such maximal nonstandard constructive logics have been recursively axiomatized (and have turned out to be decidable).
  Now, in the context of the present paper we are not able to get results comparable with the propositional ones. In this sense, we leave open the problem of *stating whether* $S(\mathcal{M}_1)$ *has a predicate Kripke frames*

semantics (in the sense, e.g., of [27, 28]). Also, *we do not know whether there is a recursive axiomatization of* $\mathcal{M}_1$ (the unique extraintuitionistic axioms we know are the formulae $\neg\neg A \to A$ for A atomic and the instances of (KUR), $(KP_\vee)$, $(KP_\exists)$ and (GRZ)). If such an axiomatization would exists, $\mathcal{M}_1$ would be recursively enumerable, hence a $\sum_1$–set in the arithmetical hierarchy, while $S(\mathcal{M}_1)$ would be at most a $\prod_2$–set (which would not exclude the possibility of proving that $S(\mathcal{M}_1)$ is a $\sum_1$–set too, even if, according to our previous experience in the propositional framework, the problem of providing a recursive axiomatization for $S(\mathcal{M}_1)$ should be much more difficult than for $\mathcal{M}_1$).
Note that, being axiom–schemes (their instances are closed under arbitrary substitutions) of $\mathcal{M}_1$, (KUR), $(KP_\vee)$, $(KP_\exists)$ and (GRZ) are axiom–schemes of $S(\mathcal{M}_1)$ too.

Now, we will consider a new set of predicate wff's and, as for $\mathcal{M}_1$, we will prove that it is a kurodian maximal nonstandard predicate constructive logic.

Let A be a predicate wff. We say that A is *constructively sound* (*Csound* for short) iff $\emptyset \vdash_{\mathrm{CL}} A$ and one of the following conditions hold:

1. A is negated;
2. $A = B \wedge C$, and B is Csound and C is Csound;
3. $A = B \vee C$, and B is Csound or C is Csound;
4. $A = B \to C$, and, for every transformation τ, if τ is not empty for $B \to C$ and τB is Csound then τC is Csound;
5. $A = \exists x B(x)$, and there is an individual variable y such that $B(y)$ is Csound;
6. $A = \forall x B(x)$, and, for every individual variable y, $B(y)$ is Csound.

As made for $\mathcal{M}_1$, we define the set $\mathcal{M}_2$ of predicate wff's as:

$$\mathcal{M}_2 = \{A \mid A \text{ is Csound}\}.$$

Following the ideas involved in the previous results related to $\mathcal{M}_1$, it is easy to show that $\mathcal{M}_2$ is a nonstandard predicate constructive logic, i.e., $\mathcal{M}_2$ is closed under generalization, restricted substitution (a trivial modification of Lemma 4.1) and detachment; it includes INT (a trivial modification of Lemma 4.4) and is included in CL.

As seen for $\mathcal{M}_1$, $\mathcal{M}_2 = E(\mathcal{M}_2)$ and hence $\mathcal{M}_2$ is closed under H–substitutions. Moreover, it is easy to prove that $\mathbf{K} \subseteq \mathcal{M}_2$ and that (GRZ) is a principle of $\mathcal{M}_2$.

Now, we prove that $\mathcal{M}_2$ is a maximal nonstandard predicate constructive logic.

THEOREM 4.20. *Let L be any nonstandard predicate constructive logic such that $\mathcal{M}_2$ is included in L. Then, for any predicate wff A, if $A \in L$ then $A \in \mathcal{M}_2$.*

PROOF. By induction on the complexity of A. Obviously A cannot be an atomic formula since $L \subseteq CL$. If $A = \neg B$, then, since $L \subseteq CL$, the assertion follows trivially from the definition of Csoundness of the negated formulae. If $A = B \wedge C$, or $A = \forall x B(x)$, or $A = B \vee C$, or $A = \exists x B(x)$ the assertion follows from the induction hypothesis and (for the last two cases) from the constructiveness of L. Now, suppose that $A = B \to C$ and $B \to C$ does not belong to $\mathcal{M}_2$. Since $\emptyset \Vdash_{\overline{\mathrm{CL}}} B \to C$, there exists a transformation τ such that τ is not empty for $B \to C$ and τB is Csound, hence $\tau B \in \mathcal{M}_2$, but τC is not Csound, hence $\tau C \notin \mathcal{M}_2$. Therefore, $\tau B \in L$ and $\tau C \notin L$ by induction hypothesis. But, since $B \to C \in L$ and thus $\tau(B \to C) \in L$, we have a contradiction. ■

Finally, using Theorem 4.20 and Corollary 3.15, we get:

THEOREM 4.21. *$S(\mathcal{M}_2)$ is a maximal predicate constructive logic.*

Of course, since $S(\mathcal{M}_1)$ and $S(\mathcal{M}_2)$ are maximal predicate constructive logics, $S(\mathcal{M}_2)$ cannot be properly included in $S(\mathcal{M}_1)$. Hence since (KUR) and (GRZ) are axiom–schemes of both logics, $S(\mathcal{M}_2)$ must contain some extraintuitionistic axiom–schema independent of (KUR) and (GRZ). From this point of view, we do not know whether the axiom–schemes $(KP_{\vee})$ and $(KP_{\exists})$ hold in $S(\mathcal{M}_2)$; more generally, we *do not even know whether or not $S(\mathcal{M}_2)$ and $S(\mathcal{M}_1)$ are different predicate logics* (i.e., whether or not $\mathcal{M}_1$ and $\mathcal{M}_2$ are different nonstandard predicate logics).

Now we end this section by giving an interesting characterization of $\mathcal{M}_2$ and $S(\mathcal{M}_2)$. To do so, we generalize at the predicate level two notions analyzed in [26].

First of all, a *generalized transformation* τ_g will be either a (possibly non restricted) predicate substitution σ, or an individual substitution η, or a composition $\tau_g \tau'_g$ of two generalized transformations τ_g and τ'_g. On the other hand, a transformation τ will be, as before, either a r–substitution σ^r, or an individual substitution η, or a composition $\tau\tau'$ of two transformations τ and τ'. The definition of "generalized transformation τ_g which is not empty for a predicate wff A" is given in a way quite similar to the one of "transformation τ which is not empty for a predicate wff A".

Now, let L be a nonstandard predicate logic, let Γ be a set of predicate wff's and let A be a predicate wff; then we define the relations $\Vdash\!\xrightarrow[\mathrm{pred}]{\mathrm{r}}_{\mathrm{L}}$ and $\Vdash\!\xrightarrow[\mathrm{pred}]{}_{\mathrm{L}}$ between Γ and A as follows:

$\Gamma \|\!\overset{r}{\underset{\text{pred}}{\longmapsto}}_{\text{L}} A$ (respectively, $\Gamma \|\!\underset{\text{pred}}{\longmapsto}_{\text{L}} A$) iff $\Gamma \vdash_{\text{CL}} A$ and, for every transformation τ (respectively, τ_g) such that, for every $B \in \Gamma \cup \{A\}$, τ is not empty for B (respectively, τ_g is not empty for B), if $\{\tau C \mid C \in \Gamma\} \subseteq L$ (respectively, $\{\tau_g C \mid C \in \Gamma\} \subseteq L$) then $\tau A \in L$ (respectively, $\tau_g A \in L$).

Of course, if $\Gamma \|\!\underset{\text{pred}}{\longmapsto}_{\text{L}} A$ then $\Gamma \|\!\overset{r}{\underset{\text{pred}}{\longmapsto}}_{\text{L}} A$, but in general the converse does not hold. We say that a nonstandard predicate logic L is *predicatively smooth* (respectively, *predicatively finitely smooth*) iff, for every set Γ of predicate wff's (respectively, for every *finite* set Γ of predicate wff's) and every predicate wff A, $\Gamma \|\!\underset{\text{pred}}{\longmapsto}_{\text{L}} A$ implies $\Gamma \vdash_{\text{L}} A$. We also say that a nonstandard predicate logic L is *predicatively strongly smooth* (respectively, *predicatively finitely strongly smooth*) iff, for every set Γ of predicate wff's (respectively, for every *finite* set Γ of predicate wff's) and every predicate wff A, $\Gamma \|\!\overset{r}{\underset{\text{pred}}{\longmapsto}}_{\text{L}} A$ implies $\Gamma \vdash_{\text{L}} A$.

Of course, since $\Gamma \|\!\underset{\text{pred}}{\longmapsto}_{\text{L}} A$ implies $\Gamma \|\!\overset{r}{\underset{\text{pred}}{\longmapsto}}_{\text{L}} A$, we have that if L is predicatively (finitely) strongly smooth then L is predicatively (finitely) smooth (but the converse in general does not hold). For a predicatively (finitely) smooth nonstandard predicate logic L, while $\Gamma \|\!\underset{\text{pred}}{\longmapsto}_{\text{L}} A$ implies $\Gamma \vdash_{\text{L}} A$ for every (finite) Γ and every A, $\Gamma \vdash_{\text{L}} A$ may fail to imply $\Gamma \|\!\underset{\text{pred}}{\longmapsto}_{\text{L}} A$, since a generalized transformation τ_g may be an arbitrary predicate substitution σ and L is not necessarily closed under arbitrary predicate substitutions. Thus, smoothness (finite smoothness) is appropriately considered only for predicate logics L; in this case, for every predicatively (finitely) smooth standard predicate logic L, every (finite) Γ and every A, we immediately get $\Gamma \|\!\underset{\text{pred}}{\longmapsto}_{\text{L}} A$ iff $\Gamma \vdash_{\text{L}} A$. On the other hand, note that, for every atomic predicate wff A and every nonstandard logic L, $\{\neg\neg A\} \|\!\overset{r}{\underset{\text{pred}}{\longmapsto}}_{\text{L}} A$; it follows that only a regular nonstandard predicate logic can be (finitely) strongly smooth. Note also that, since a nonstandard predicate logic is closed under transformations, for any predicatively (finitely) strongly smooth L, any (finite) Γ and any A, we have that $\Gamma \|\!\overset{r}{\underset{\text{pred}}{\longmapsto}}_{\text{L}} A$ iff $\Gamma \vdash_{\text{L}} A$.

The propositional counterpart of the previous definitions are respectively *propositional smoothness*, *propositional finite smoothness*, *propositional strong smoothness* and *propositional finite strong smoothness*. These propositional notions have been defined in [26] and are based on the relations $\Gamma \|\!\underset{\text{prop}}{\longmapsto}_{\text{L}} A$ and $\Gamma \|\!\overset{r}{\underset{\text{prop}}{\longmapsto}}_{\text{L}} A$, defined as obvious reductions to the propositional context of $\Gamma \|\!\underset{\text{pred}}{\longmapsto}_{\text{L}} A$ and $\Gamma \|\!\overset{r}{\underset{\text{pred}}{\longmapsto}}_{\text{L}} A$, recalling that in the propositional frame the generalized transformations are arbitrary propositional substitutions σ_{prop} and the transformations are propositional restricted substitutions σ^r_{prop} (in [26] the relations $\Gamma \|\!\underset{\text{prop}}{\longmapsto}_{\text{L}} A$ and $\Gamma \|\!\overset{r}{\underset{\text{prop}}{\longmapsto}}_{\text{L}} A$ are denoted, respectively, $\Gamma \|\!\longmapsto_{\text{L}} A$ and $\Gamma \|\!\overset{r}{\longmapsto}_{\text{L}} A$, while the qualification "propositional" is omitted in the various notions of smoothness). Note that in the propositional framework the condition "$\Gamma\vdash_{\mathrm{CLprop}}A$ and, for every propositional substitution σ_{prop} (respectively, for every propositional restricted substitution σ^r_{prop}), if $\{\sigma_{prop}C \mid C \in \Gamma\} \subseteq L$ (respectively, $\{\sigma^r_{prop}C \mid C \in \Gamma\} \subseteq L$) then $\sigma_{prop}A \in L$ (respectively, $\sigma^r_{prop}A \in L$)" is quite equivalent to "for every propositional substitution σ_{prop} (respectively, for every propositional restricted substitution σ^r_{prop}) if $\{\sigma_{prop}C \mid C \in \Gamma\} \subseteq L$ (respectively, $\{\sigma^r_{prop}C \mid C \in \Gamma\} \subseteq L$) then $\sigma_{prop}A \in L$ (respectively, $\sigma^r_{prop}A \in L$)"; thus, the requirement "$\Gamma\vdash_{\mathrm{CLprop}}A$" is omitted in the definition given in [26] of the relations $\Gamma\|\!\underset{\mathrm{prop}}{\longrightarrow}_{\mathrm{L}}A$ and $\Gamma\|\!\underset{\mathrm{prop}}{\overset{\mathrm{r}}{\longrightarrow}}_{\mathrm{L}}A$. We also recall that propositional finite smoothness is called structural completeness by the researchers of the polish school, see, e.g., [30].

We will be concerned with the finite smoothness of maximal (standard and nonstandard, propositional and predicate) constructive logics. From this point of view, we recall Prucnal's result, giving rise to a solution of Problem 42 of [7], according to which *Medvedev's logic MV is the greatest finitely smooth propositional constructive logic* [30]. As discussed in [26], we also have that *any smooth propositional constructive logic* includes $R_{prop}(MV)$ (taking into account smoothness in place of finite smoothness, we do not know whether there is some smooth propositional constructive logic; we do not even know whether MV is the unique finitely smooth propositional constructive logic). In [26] also propositional strong smoothness is taken into account; it turns out that $E_{prop}(MV)$ *is propositionally strongly smooth and the unique propositionally finitely strongly smooth nonstandard propositional constructive logic.* Now, turning to the kurodian predicate constructive logics, we want to show that $S(\mathcal{M}_2)$ and $\mathcal{M}_2$ play the same roles, with respect to predicate finite smoothness and predicate finite strong smoothness, as MV and $E_{prop}(MV)$ respectively do in the propositional framework.

First of all, we prove:

PROPOSITION 4.22. *$\mathcal{M}_2$ is predicatively finitely strongly smooth.*

PROOF. Let $\Gamma = \{B_1, ..., B_n\}$ be any finite set of predicate wff's, let A be a predicate wff and let $\Gamma\|\!\underset{\mathrm{pred}}{\overset{\mathrm{r}}{\longrightarrow}}_{\mathcal{M}_2}A$. Then $B_1 \wedge ... \wedge B_n \to A \in CL$ and, for every transformation τ such that, for every $C \in \Gamma \cup \{A\}$, τ is not empty for C, we have that if $\tau(B_1 \wedge ... \wedge B_n)$ is Csound then τA is Csound. It follows that $B_1 \wedge ... \wedge B_n \to A \in \mathcal{M}_2$, hence $\Gamma\vdash_{\mathcal{M}_2}A$. ■

We can also prove:

LEMMA 4.23. *Let L' and L'' be two kurodian predicatively finitely strongly smooth nonstandard predicate constructive logics. Then $L' = L''$.*

PROOF. We show, by induction on the complexity of a predicate wff A, that $A \in L'$ iff $A \in L''$. Let $A = \neg B$ (basis). Then the assertion immediately follows from the fact that $L', L'' \subseteq CL$ and both L' and L'' are kurodian (note that the case where A is atomic is not to be taken into account, since no atomic formula belongs to CL).

The cases $A = B \wedge C$ and $A = \forall x B(x)$ immediately come from the induction hypothesis (for the case $A = \forall x B(x)$ closure under generalization is needed). The cases $A = B \vee C$ and $A = \exists x B(x)$ are immediate consequences of the induction hypothesis and of the fact that L' and L'' are constructive.

Finally, let $A = B \to C$, and suppose, e.g., that $A \in L'$ but $A \notin L''$. Since L'' is finitely strongly smooth, we cannot have $\{B\} \|\!\xrightarrow[\text{pred}]{r}_{L''} C$; hence, since from $A \in L'$ we get $B \to C \in CL$, there must be some transformation τ such that τ is not empty for $B \to C$, $\tau B \in L''$, but $\tau C \notin L''$. Note that the relevant complexity of τB does not exceed the one of B and the same holds for τC (since τ involves only negated formulae and atomic formulae). Hence, we can apply the induction hypothesis, thus getting $\tau B \in L'$ and $\tau C \notin L'$, which implies $\tau(B \to C) \notin L'$. Since L' is a nonstandard predicate logic, L' is closed under transformations. Hence, being $B \to C \in L'$, we have $\tau(B \to C) \in L'$, a contradiction. ■

From Proposition 4.22 and Lemma 4.23, we immediately get:

THEOREM 4.24. *$\mathcal{M}_2$ is the unique kurodian predicatively finitely strongly smooth nonstandard predicate constructive logic.*

Now, considering the standard predicate constructive logic $S(\mathcal{M}_2)$, we can prove:

PROPOSITION 4.25. *$S(\mathcal{M}_2)$ is a predicatively finitely smooth predicate constructive logic.*

PROOF. Let $\Gamma = \{B_1, ..., B_n\}$ be a finite set of predicate wff's, let A be a predicate wff, let $\Gamma \|\!\xrightarrow[\text{pred}]{}_{S(\mathcal{M}_2)} A$ but suppose that $\Gamma \not\vdash_{S(\mathcal{M}_2)} A$. Then, first of all, $B_1 \wedge ... \wedge B_n \to A \notin S(\mathcal{M}_2)$, which implies, by definition of the operator S, that there is a predicate substitution σ such that $\sigma(B_1 \wedge ... \wedge B_n \to A) \notin \mathcal{M}_2$; since, by the $\mathcal{M}_2$–version of Proposition 4.9, we have $\mathcal{M}_2 = E(\mathcal{M}_2)$, there is also a NegSat–substitution σ^{NS} such that $\sigma^{NS}(B_1 \wedge ... \wedge B_n \to A) \notin \mathcal{M}_2$. The latter fact implies (since $\sigma^{NS}(B_1 \wedge ... \wedge B_n \to A) \in CL$ follows from $\Gamma \|\!\xrightarrow[\text{pred}]{}_{S(\mathcal{M}_2)} A$) that there is a transformation τ such that $\tau\sigma^{NS}(B_1 \wedge ... \wedge B_n) \in \mathcal{M}_2$ and $\tau\sigma^{NS} A \notin \mathcal{M}_2$ (by definition of $\mathcal{M}_2$). Since σ^{NS} is a NegSat–substitution, $\tau\sigma^{NS}(B_1 \wedge ... \wedge B_n \to A)$ is a negatively saturated predicate wff; hence, since $\tau\sigma^{NS}(B_1 \wedge ... \wedge B_n) \in \mathcal{M}_2$, we have that $\tau\sigma^{NS}(B_1 \wedge ... \wedge B_n) \in R(\mathcal{M}_2)$, which implies, by Theorem 3.10 and Proposition 3.13 (being $\mathcal{M}_2$ kurodian), that $\tau\sigma^{NS}(B_1 \wedge ... \wedge B_n) \in S(\mathcal{M}_2)$. On the other hand, from $\tau\sigma^{NS}A \notin \mathcal{M}_2$ we immediately get $\tau\sigma^{NS}A \notin S(\mathcal{M}_2)$. Thus, setting $\tau_g = \tau\sigma^{NS}$, there is a generalized transformation τ_g such that, for every $C \in \Gamma \cup \{A\}$, τ_g is not empty for C, $\{\tau_g D \mid D \in \Gamma\} \subseteq S(\mathcal{M}_2)$ and $\tau_g A \notin S(\mathcal{M}_2)$. It follows that $\Gamma \Vdash\!\xrightarrow[\text{pred}]{}_{S(\mathcal{M}_2)} A$ does not hold, a contradiction. ■

Now, to show that $S(\mathcal{M}_2)$ is the greatest kurodian predicatively finitely smooth predicate constructive logic, we need the following lemma:

LEMMA 4.26. *If L is a kurodian predicatively finitely smooth predicate logic then $E(L)$ is a predicatively finitely strongly smooth nonstandard predicate logic.*

PROOF. Let $\Gamma = \{B_1, ..., B_n\}$ be a finite set of predicate wff's, let A be a predicate wff, let $\Gamma \Vdash\!\xrightarrow[\text{pred}]{\text{r}}_{\mathrm{E(L)}} A$ but suppose that $\Gamma \not\vdash_{\mathrm{E(L)}} A$. Then $B_1 \wedge ... \wedge B_n \to A \notin E(L)$, from which, arguing, e.g., as in the proof of Theorem 3.9, we get the existence of a r–substitution σ^r such that $\sigma^r(B_1 \wedge ... \wedge B_n \to A) \notin L$, i.e., $\{\sigma^r B_1, ..., \sigma^r B_n\} \not\vdash_{\mathrm{L}} \sigma^r A$. Since L is finitely smooth, the latter fact implies that $\{\sigma^r B_1, ..., \sigma^r B_n\} \Vdash\!\xrightarrow[\text{pred}]{}_{\mathrm{L}} \sigma^r A$ does not hold, from which we get the existence of a generalized transformation τ_g such that, for every $C \in \{\sigma^r B_1, ..., \sigma^r B_n, \sigma^r A\}$, τ_g is not empty for C and $\{\tau_g \sigma^r B_1, ..., \tau_g \sigma^r B_n\} \subseteq L$ but $\tau_g \sigma^r A \notin L$. Hence, since $\sigma^r B_1, ..., \sigma^r B_n$ are negatively saturated predicate wff's and L is kurodian, it is not difficult to show that there exists a (non generalized) transformation τ such that, for every $C' \in \{\sigma^r B_1, ..., \sigma^r B_n, \sigma^r A\}$, τ is not empty for C' and $\{\tau\sigma^r B_1, ..., \tau\sigma^r B_n\} \subseteq L$ but $\tau\sigma^r A \notin L$. From $\{\tau\sigma^r B_1, ..., \tau\sigma^r B_n\} \subseteq L$ we get $\{\tau\sigma^r B_1, ..., \tau\sigma^r B_n\} \subseteq E(L)$; on the other hand, since $\tau\sigma^r A$ is a negatively saturated predicate wff, from $\tau\sigma^r A \notin L$ we get $\tau\sigma^r A \notin R(L)$, from which (by Proposition 3.12, being L kurodian) $\tau\sigma^r A \notin R(E(L))$, from which (being $\tau\sigma^r A$ a negatively saturated predicate wff) $\tau\sigma^r A \notin E(L)$. Hence, since $\tau\sigma^r$ is a transformation, there is a transformation τ' such that, for every $C'' \in \Gamma \cup \{A\}$, τ' is not empty for C'', $\{\tau' D \mid D \in \Gamma\} \subseteq E(L)$ but $\tau' A \notin E(L)$. It follows that $\Gamma \Vdash\!\xrightarrow[\text{pred}]{\text{r}}_{\mathrm{E(L)}} A$ does not hold, a contradiction. ■

Now, from Proposition 4.25, Theorem 4.24 and Lemma 4.26 we get the following theorem, stating that $S(\mathcal{M}_2)$ is the greatest kurodian predicatively finitely smooth predicate constructive logic and that $R(\mathcal{M}_2)$ is a lower bound for any logic of this kind:

THEOREM 4.27. *$S(\mathcal{M}_2)$ is a kurodian predicatively finitely smooth predicate constructive logic and, for every kurodian predicatively finitely smooth predicate constructive logic L, $R(\mathcal{M}_2) \subseteq L \subseteq S(\mathcal{M}_2)$.*

PROOF. The fact that $S(\mathcal{M}_2)$ is a kurodian predicatively finitely smooth predicate constructive logic follows from Proposition 4.25 and the $\mathcal{M}_2$–version of Proposition 4.11.

On the other hand, let L be a kurodian predicatively finitely smooth predicate constructive logic. Then, by Lemma 4.26 and Proposition 3.1, $E(L)$ is a kurodian predicatively finitely strongly smooth nonstandard predicate constructive logic. Hence, by Theorem 4.24, $E(L) = \mathcal{M}_2$, which implies $L \subseteq \mathcal{M}_2$. Since L is a standard predicate logic and $S(\mathcal{M}_2)$ is the greatest standard predicate logic included in $\mathcal{M}_2$, we get $L \subseteq S(\mathcal{M}_2)$.

Finally, let L be a kurodian predicatively finitely smooth predicate constructive logic. Then, as seen in the previous discussion, $E(L) = \mathcal{M}_2$. Hence, since L is kurodian, from Proposition 3.12 we get $R(L) = R(E(L)) = R(\mathcal{M}_2)$. Since $R(L) \subseteq L$, we immediately get $R(\mathcal{M}_2) \subseteq L$. ■

REMARKS:

- ▷ We leave *open the problem of stating whether $S(\mathcal{M}_2)$ is the unique kurodian predicatively finitely smooth predicate constructive logic.*
- ▷ We leave *open the problem of stating whether there is some kurodian predicatively smooth predicate constructive logic.*

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