ELSEVIER
Theoretical Computer Science 294 (2003) 103–149
Theoretical Computer Science
www.elsevier.com/locate/tcs

Combining word problems through rewriting in categories with products

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Abstract

We give an algorithm solving combined word problems (over non-necessarily disjoint signatures) based on rewriting of equivalence classes of terms. The canonical rewriting system we introduce consists of few transparent rules and is obtained by applying Knuth–Bendix completion procedure to presentations of pushouts among categories with products. It applies to pairs of theories which are both constructible over their common reduct (on which we do not make any special assumption). © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Combined word problems; Rewrite systems; Functorial semantics

1. Introduction

An essential problem in automated deduction consists in integrating theorem provers which are able to perform separated tasks. In the field of equational logic, this leads in particular to the following question: suppose that you are able to solve word problems for theories T_1, T_2; can you solve word problem for $T_1 \cup T_2$? Moreover, can one design an algorithm taking as input two arbitrary algorithms for word problems for T_1 and T_2 and realizing a decision procedure for word problem for $T_1 \cup T_2$?

In the case where T_1, T_2 have disjoint signatures the positive answer was known since long time [13], although it was only more recently discovered within automated deduction community (see e.g. [12]). In the general case, combining decidable word problems may lead to undecidability, even if we suppose that T_1, T_2 are both conservative over their common reduct T_0. To this aim, consider the following example. Let T_0 be the theory of join-semilattices with zero (i.e. of commutative idempotent monoids) and let T_1 be the theory of Boolean algebras. As T_2 we take the theory of semilattice-monoids, which are algebras having both a monoid and a join-semilattice with zero

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PII: S0304-3975(01)00248-1

structure and which satisfy the further equation:

$$\left(\bigvee_{i=1}^{n} x_i \right) \circ \left(\bigvee_{j=1}^{m} y_j \right) = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (x_i \circ y_j).$$

T_2 clearly has decidable word problem (elements in a free algebra are finite sets of lists of the generators), as well as T_1. The union theory (which we indicate better with $T_1 +_{T_0} T_2$) corresponds to the "distributive linear logic" of [9] and falls within the undecidability results of [1].

Clearly something must be assumed in order to have positive solution to combined word problems; in the literature it is usually assumed that T_1, T_2 share a set of constructors (we prefer the terminology "they are both constructible over T_0"). There are various definitions of constructors and depending on such definitions there are variable strength results. Main papers on the subject are [5] and [3]: the second has a weaker definition and consequently, a stronger result. Quite recently, Baader and Tinelli, working independent of us, were able to strengthen their previous work in [3] by extending the related methods to the case in which T_0 may not be collapse-free. The general result they obtained was presented at FROCOS2000 and will appear in extended full version in the detailed paper [4]. The statement of their result *coincides* with the result we have in the present paper [2] (which is a very remarkable fact, given that we worked independently and given that—as it will appear from the remarks below—we used quite different methods, both in the formulation of the combination algorithm and in its mathematical justification).

In [3, 4], the combined decision algorithm is obtained through a complex refutation technique manipulating equations according to certain non-deterministic rules. As such it has the advantage of being more flexible, although it does not provide normal forms. On the contrary, in [5] (and in the similar method of [12] for the disjoint case) one can directly manipulate terms by abstracting and collapsing alien subterms and the suggested algorithm follows a rigidly preassigned procedure. Our method is more similar to that of [5] (in the sense that it manipulates terms), but has the same flexibility advantages as the method of [3, 4]. The idea is simple: we build a canonical rewriting system which is able to normalize paths of mixed pure terms.

The realization of such a plan looks very hard at a first glance: terms from combined signatures are quite unreliable datatypes, basically because they can compose, decompose and even collapse in many uncontrolled and overlapping ways. However, we shall put such a complex combinatorics *under the control framework provided by the categorical approach to equational logic*: such an approach goes back to the classical pioneering paper of Lawvere [10] in functorial semantics. [3] Basically, equational theories are identified with categories with products, so that in our situation we need to manipulate *presentations of pushouts* among such categories. We get a first

[2] See Section 7.3 of [4] for comparison details.

[3] We recall that there is another quite interesting category-theoretic approach to universal algebra, namely the *monads* approach (which has also been significantly used in questions related to rewriting, see e.g. [11]).

general and simple presentation of these pushouts in Section 3 by means of two-sides rewrite rules. To this presentation we apply, in Section 5, *Knuth–Bendix completion procedure* and get the desired rewriting system, under some "constructors" hypothesis for our theories.

This constructors hypothesis is formulated within a categorical framework in Section 5 by means of (weak) factorization systems and translated in symbolic terms in Section 10: roughly speaking, T_i is said to be constructible over T_0 iff there is a class E_i of terms (including variables and closed under renamings) in the signature Ω^i of T_i so that any Ω^i-term $t(x_1,\dots,x_n)$ decomposes uniquely (up to provable identity) as $u(v_1,\dots,v_k)$, where the $v_i(x_1,\dots,x_n)$ are (always up to provable identity) distinct terms from E_i and u is a k-minimized term in the signature Ω^0 of T_0 (a term $u(x_1,\dots,x_k)$ is said to be k-minimized iff it is not provably identical to any term in which only variables coming from a proper subset of $\{x_1,\dots,x_k\}$ occur). Thus, u is a kind of T_0-*head normal form of* t. Examples are provided in Section 10 (a typical example is the case of commutative rings with unit which are constructible over abelian groups).

We briefly describe here the rewriting system $\mathscr{R}$ that we obtained. $\mathscr{R}$ consists of only four rules (for technical reasons concerning "colours" of terms, two of such rules are "duplicated"). The first rule (called *composition rule*) simply allows to compose equally coloured consecutive (equivalence classes of) terms. The second rule (called ε-*extraction rule*) minimizes terms by "moving left" projections (i.e. n-tuples of distinct variables). The fourth rule (called *products rule*) is suggested by the completion procedure and has the following meaning: any projection (i.e. any tuple of distinct variable terms) appearing in an internal position of a path of pure terms represents a "hole" and the normalization process is supposed to *fill such a hole* by "moving right" genuine terms (i.e. terms which are not projections). In addition, the normalization algorithm *propagates to the right of the path the* T_0-*chunks of terms coming from extraction of* T_0-*head normal forms*: this is done by the third rule (called μ-*extraction rule*). The complete table of rules of $\mathscr{R}$ is given in Section 5.

Although $\mathscr{R}$ is a quite simply described system, the confluence proof requires lot of work, because all critical pairs must be examined. This leads to a *large amount of details*, all consisting of elementary computations (in fact, once the technical tools are appropriately settled, single cases are treated in the most natural way).

The paper is organized as follows: in Section 2, we recall the necessary background and fix notations; in Section 3, we get a first presentation of pushouts among Lawvere categories. In Sections 4 and 5, we apply completion procedure and get the appropriate rewriting system $\mathscr{R}$. In Section 6, we provide local confluence and termination for a simple subsystem $\mathscr{R}_0$ of $\mathscr{R}$. In Section 7, a third rewriting system, called $\mathscr{R}^+$ is introduced ($\mathscr{R}^+$ is equivalent to $\mathscr{R}$, it normalizes slower but it is easier to manage); in addition, useful technical facts are collected. In Section 8, $\mathscr{R}^+$ is proved to be locally confluent, whereas in Section 9 termination of both $\mathscr{R}$ and $\mathscr{R}^+$ is established. Finally, equivalence between $\mathscr{R}$ and $\mathscr{R}^+$ and canonicity of the former are obtained. Section 10 provides examples of constructible theories and of normalizations of paths of terms.

Sections 6–9 can be skipped in a first reading by people mostly interested in our results (and less interested in their proofs).

For space reasons, some routine work is omitted in this paper; the reader may find all the details in the technical report [6]. We assume a certain familiarity with rewriting (for some unexplained notions readers may consult [2]) and with the elementary formalism of categories with products.

2. Preliminaries

An (equational) theory $T = \langle \Omega, Ax \rangle$ is just an ordinary signature Ω endowed with a set of pairs of terms ("the axioms" of T). We use letters $t, u, v, \dots$ for terms and letters $x_1, x_2, \dots$ for variables; $t(x_1, \dots, x_n)$ means that the term t contains at most the variables $x_1, \dots, x_n$. Notation $t(u_1/x_1, \dots, u_n/x_n)$ (or simply $t(u_i/x_i)$ or again $t(u_1, \dots, u_n)$) is used for substitutions; when we write $t(u/x_i)$ we mean $t(x_1/x_1, \dots, u/x_i, \dots, x_n/x_n)$. Notations like $\vdash_T t_1 = t_2$ refer to some sound and complete deduction system (e.g. equational logic). Deciding $\vdash_T t_1 = t_2$ is just the (uniform) *word problem* for T. In order to avoid irrelevant cases, *we shall always assume that our theories T match the following two requirements*:

- Ω always contains a constant symbol c_0 (this is harmless, because adding a free constant—if needed—does not change the nature of word problems);
- T is non-degenerate, namely $\nvdash_T x_1 = x_2$.

A basic point in categorical logic consists in treating theories as (small) categories. In our case, we have the notion of *Lawvere category*. Basically, this is nothing but any one-sorted (finite products) category.[4] Formally, a Lawvere category is a category having objects $\{X^n\}_{n \geqslant 0}$, in which X^n (endowed with specified projections $\pi_i : X^n \to X$) is the product of $X = X^1$ with itself n-times. In our context, (see below) π_i will be the (equivalence class of) the variable x_i. We fix the following convention about a Lawvere category: arrows $X^n \to X^m$ of the kind $\langle \pi_{i_1}, \dots, \pi_{i_m} \rangle$ (where $i_1, \dots, i_m \leqslant n$) are called

- (pure) *projections* iff the $i_1, \dots, i_m$ are all distinct (in this case we must have $m \leqslant n$);
- *diagonals* iff $\{i_1, \dots, i_m\}$ includes $\{1, \dots, n\}$ (in this case we must have $m \geqslant n$);
- *renamings* iff $i_1, \dots, i_m$ are just a permutation of $1, \dots, n$ (in this case we must have $n = m$).

Lawvere categories are essentially in one-to-one correspondence with equational theories (we said "essentially" because two equational theories differing in only the choice of the language and of the axioms are collapsed into the same "invariant" Lawvere category). In this paper, we need only one side of this correspondence, which we are going to recall. Let $T = (\Omega, Ax)$ be a theory; we build a Lawvere category $\mathbf{T}$ in

[4] In this paper, by "category" we always mean a category with finite products and by "functor" we always mean a finite products preserving functor. We use $\alpha \circ \beta$ to denote the composition of $\xrightarrow{\alpha}\xrightarrow{\beta}$ (contrary to some more customary notation).

the following way. We take as arrows $X^n \to X^m$ the m-tuples of equivalence classes of terms containing at most the variables $x_1, \dots, x_n$ (equivalence is intended through provable identity in T); composition is substitution and identity of X^n is the n-tuple of equivalence classes of $x_1, \dots, x_n$. As a consequence of its definition, $\mathbf{T}$ has finite products and the equivalence classes of the variables x_i are the specified projections $\pi_i : X^n \to X$.

3. Basic equations

We now fix our main data for the paper: we have three theories

$$T_0 = (\Omega^0, Ax_0), \quad T_1 = (\Omega^1, Ax_1), \quad T_2 = (\Omega^2, Ax_2),$$

such that T_1 and T_2 are conservative extensions of T_0 and $\Omega^0 = \Omega^1 \cap \Omega^2$; taking (non-disjoint) union of signatures and axioms we get a further theory which we call $T_1 +_{T_0} T_2$. *We suppose that we are able to solve the word problem for* T_1, T_2; in general, as explained in the introduction, this is not enough for solving the word problem for $T_1 +_{T_0} T_2$ however, we may look for sufficient conditions yielding a positive solution.

The category $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ can be built as usual, by using terms; however, we can characterize it intrinsically in terms of $\mathbf{T_0}, \mathbf{T_1}, \mathbf{T_2}$ as the *pushout* of $\mathbf{T_1}, \mathbf{T_2}$ over $\mathbf{T_0}$. Hence, we can try to describe it directly through its universal property. For this description we do not use terms anymore, but a more algebraic notion, namely mixed paths of arrows from $\mathbf{T_1}, \mathbf{T_2}$. To make the notation simpler, we act as if the syntactic expansion functors $I_1 : \mathbf{T_0} \to \mathbf{T_1}$ and $I_2 : \mathbf{T_0} \to \mathbf{T_2}$ (which are faithful by conservativity hypothesis) were just inclusions. Formally, a *path* $K : X^n \to X^m$ is a non-empty list of arrows coming from either $\mathbf{T_1}$ or $\mathbf{T_2}$ (or both)

$$K = \alpha_1, \dots, \alpha_k$$

such that
(i) the domain of α_1 is X^n;
(ii) the codomain of α_k is X^m;
(iii) for every $i = 1, \dots, k-1$, the codomain of α_i is equal to the domain of α_{i+1}.
Paths are just words (with "typing" restrictions). Equivalence relations on paths (*stable* with right and left concatenation) can be introduced by two-side rewrite rules. The plan is quite simple: **identify such rules, orient and complete them into a canonical rewrite system**.

In the remaining part of the paper, we make the following *conventions*:

- we shall use letters $\alpha, \beta, \dots$ for arrows from $\mathbf{T_1} \cup \mathbf{T_2}$, letters $\alpha^1, \beta^1, \dots$ for arrows from $\mathbf{T_1}$, letters $\alpha^2, \beta^2, \dots$ for arrows from $\mathbf{T_2}$ and letters $\alpha^0, \beta^0, \dots$ for arrows from $\mathbf{T_0}$; notice that any arrow like α^1 may happen to come from $\mathbf{T_0}$;
- instead of indicating types (i.e. objects of Lawvere categories) with X^n, $X^m, \dots$ we may use letters $Y, Z, U, \dots$ if the knowledge of the exponent does not matter; letter X however can only indicate X^1;

- Roman letters can be used to indicate arrows having codomain X, that is a^1 for instance, stands for an arrow in $\mathbf{T_1}$ (which might belong to $\mathbf{T_0}$ too) having as domain some $Y = X^n$, but whose codomain can be only $X = X^1$.

Next, we give our main definitions for path rewriting. Let $\mathscr{S}$ be a set of pairs of paths; we write

(i) $K \Rightarrow_{\mathscr{S}} L$ (or simply $K \Rightarrow L$, leaving $\mathscr{S}$ as understood from the context) iff $K = K_1, L', K_2$ and $L = K_1, R', K_2$ for some pair $\langle L', R'\rangle \in \mathscr{S}$;
(ii) $K \Leftrightarrow_{\mathscr{S}} L$ (or simply $K \Leftrightarrow L$) iff $K = K_1, L', K_2$ and $L = K_1, R', K_2$ for some pair $\langle L', R'\rangle$ such that either $\langle L', R'\rangle \in \mathscr{S}$ or $\langle R', L'\rangle \in \mathscr{S}$;
(iii) $K \Rightarrow^*_{\mathscr{S}} L$ (or simply $K \Rightarrow^* L$) for the reflexive-transitive closure of $\Rightarrow_{\mathscr{S}}$;
(iv) $K \Leftrightarrow^*_{\mathscr{S}} L$ (or simply $K \Leftrightarrow^* L$) for the smallest equivalence relation containing $\Rightarrow_{\mathscr{S}}$.

Clearly $\Leftrightarrow^*$ is the least stable equivalence relation extending $\mathscr{S}$. Pairs $\langle L, R\rangle \in \mathscr{S}$ will be directly written as $L \Rightarrow R$ and called *rules of* $\mathscr{S}$; alternatively, they might be written as $L \Leftrightarrow R$ (and called *basic equations of* $\mathscr{S}$), but in such a case we tacitly assume that $\mathscr{S}$ is symmetric, i.e. that $\mathscr{S}$ contains $\langle R, L\rangle$ in case it contains $\langle L, R\rangle$ (in such a case e.g. relations $\Rightarrow$ and $\Leftrightarrow$ obviously coincide).

The next theorem accomplishes our first goal (“finding appropriate basic equations”):

Theorem 3.1. *Let $\mathscr{P}$ be the set of the following two kinds of pairs of paths*:

$$\alpha^i, \beta^i \quad \Leftrightarrow \quad \alpha^i \circ \beta^i \quad (i = 1, 2)$$
$$1 \times \alpha_2, \alpha_1 \times 1 \quad \Leftrightarrow \quad \alpha_1 \times 1, 1 \times \alpha_2$$

(*where in the last pair we have* $\alpha_1 : Y_1 \to Z_1$, $\alpha_2 : Y_2 \to Z_2$ *and so e.g.* $1 \times \alpha_2 : Y_1 \times Y_2 \to Y_1 \times Z_2$). *We have that* $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ *is isomorphic to the Lawvere category having as arrows the equivalence classes of paths under the relation* $\Leftrightarrow^*_{\mathscr{P}}$.

Proof. Let $\mathbf{P}$ be the category having $\{X^n\}_{n \geqslant 0}$ as objects and as arrows $X^n \to X^m$ the equivalence classes (wrt $\Leftrightarrow^*_{\mathscr{P}}$) of paths of domain X^n and codomain X^m. Composition of $\{K\}$ and $\{L\}$ is $\{K, L\}$. Identity of X^n turns out to be just $\{1_{X^n}\}$.

We first show that $\mathbf{P}$ has finite products. $X^0 = \mathbf{1}$ is obviously terminal. Given objects $Y_1 = X^{n_1}, Y_2 = X^{n_2}$, we take $Y_1 \times Y_2$ (i.e. $X^{n_1+n_2}$) as binary product and $\{\pi_{Y_1}\}, \{\pi_{Y_2}\}$ as projections (here, π_{Y_1}, π_{Y_2} are obviously the projections in $\mathbf{T_0}$). Let us now take two paths K_1, K_2 of domain Z and codomains Y_1, Y_2, respectively. Suppose for instance that

$$K_1 = \alpha_1, \ldots, \alpha_r, \qquad K_2 = \beta_1, \ldots, \beta_s.$$

Let $\langle K_1, K_2\rangle$ be the path:

$$Z \overset{\langle 1_Z, 1_Z\rangle}{\to} Z \times Z \overset{1_Z \times K_2}{\to} Z \times Y_2 \overset{K_1 \times 1_{Y_2}}{\to} Y_1 \times Y_2,$$

where $1_Z \times K_2$ is $(1_Z \times \beta_1), \ldots, (1_Z \times \beta_s)$ ($K_1 \times 1_{Y_2}$ is defined analogously). We leave the reader to show that $\{\langle K_1, K_2\rangle\}$ enjoys the universal property for pairs.

In order to check that $\mathbf{P}$ is isomorphic to $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$, it is sufficient to observe it has the universal property of pushouts. □

In the applications, we should keep in mind that the isomorphism of categories between $\mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T_2}$ and $\mathbf{P}$ is the unique expansion to the signature $\Omega^1 \cup \Omega^2$ of the internal models $F_1 : \mathbf{T_1} \to \mathbf{P}$, $F_2 : \mathbf{T_2} \to \mathbf{P}$ associating with α^i the equivalence class $\{\alpha^i\}$. This means the following: given an $\Omega^1 \cup \Omega^2$-term t, the universal model (isomorphism) $U : \mathbf{T_1} +_{\mathbf{T_0}} \mathbf{T}_2 \to \mathbf{P}$ *interprets it as the equivalence class of any path obtained by expressing t as an iterated composition of terms which are pure*, i.e. which are either Ω^1 or Ω^2-terms. Such a path (called a *splitting path* for t) can be effectively computed from t in many ways (possibly yielding not the same path, but yielding in any case $\Leftrightarrow^*_{\mathscr{P}}$-equivalent paths); one might for instance adopt the usual abstraction of alien subterms, or alternatively make use of the following simply described inductive procedure (which applies to any tuple $\langle t_1, \dots, t_n \rangle$ of terms having variables included in some fixed list $x_1, \dots, x_m$):

- if $t_1, \dots, t_n$ are all Ω^1 or Ω^2-terms, a splitting path is the singleton path
$$\langle \{t_1\}, \dots, \{t_n\} \rangle,$$
having domain X^m and codomain X^n;
- otherwise, we have e.g. that $t_i = f(u_1, \dots, u_k)$; a splitting path K of
$$\langle t_1, \dots, t_{i-1}, u_1, \dots, u_k, t_{i+1}, \dots, t_n \rangle,$$
is given (we apply multiset induction on term complexities) and it has codomain X^{n-1+k}, so we can take
$$K, \langle \{x_1\}, \dots, \{f(x_i, \dots, x_{i+k-1})\}, \dots, \{x_{n-1+k}\} \rangle,$$
as a splitting path for $\langle t_1, \dots, t_n \rangle$.

It is now clear how we can deal with word problems: to decide whether t and u are $T_1 +_{T_0} T_2$-equal, it is sufficient to split them into paths K and L according to one of the above-mentioned procedures and then check whether $K \Leftrightarrow^*_{\mathscr{P}} L$ holds or not. Of course, this will become convenient only after turning our basic equations into a canonical rewriting system.

4. Adjusting datatypes

Before beginning orientation and completion, we make some modifications to our "datatypes". First, we do not want to bother distinguishing paths that are mere alphabetic variants of each other. Consider e.g. the paths

$$\mathbf{1} \overset{\langle c,d \rangle}{\to} X^2 \overset{f(x_1,x_2)}{\to} X,$$
$$\mathbf{1} \overset{\langle d,c \rangle}{\to} X^2 \overset{f(x_2,x_1)}{\to} X,$$

where f is a binary function symbol from Ω^2 and c, d are constants from Ω^1.[5] Clearly the two paths are splitting paths of the same term $f(c,d)$ and the system $\mathscr{P}$ is indeed

[5] Recall that paths are formed by equivalence classes of terms; however, in the examples, our practice is that of directly indicating equivalence classes by terms representing them.

able to deduce their equivalence, but in order to do it, it needs to extract a renaming, for instance as follows:

$$\langle c,d\rangle, f(x_1,x_2) \;\Leftrightarrow_{\mathscr{P}}\; \langle c,d\rangle, \langle x_2,x_1\rangle, f(x_2,x_1) \;\Leftrightarrow_{\mathscr{P}}\; \langle d,c\rangle, f(x_2,x_1),$$

where first basic equation of Theorem 3.1 has been used twice. What is wrong with this is that this “extraction of a renaming”, no matter in which precise form it is allowed, immediately produces non-termination. As it seems that there is no way of deducing the equivalence of paths $\langle c,d\rangle, f(x_1,x_2)$ and $\langle d,c\rangle, f(x_2,x_1)$ without extracting a renaming, we shall just consider them to be “the same path”. To do this, we need some further definitions.

Let K be the path $Y_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} Y_{k+1}$ and let L be the “parallel” path $Y_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_k} Y_{k+1}$, with α_i, β_i equally coloured and having the same domain and codomain;

- K is said to be a *ρ-renaming* of L (where $\rho = \{\rho_i : Y_i \to Y_i\}_{1 \leqslant i \leqslant k+1}$ is a list of renamings) iff the following squares

$$\begin{array}{ccc} Y_i & \xrightarrow{\alpha_i} & Y_{i+1} \\ {\scriptstyle \rho_i}\downarrow & & \downarrow{\scriptstyle \rho_{i+1}} \\ Y_i & \xrightarrow[\beta_i]{} & Y_{i+1} \end{array}$$

  commute for $i = 1,\ldots,k$ (otherwise said, we have $\beta_i = \rho_i^{-1} \circ \alpha_i \circ \rho_{i+1}$ for all i); we write $L = \rho(K)$ in order to express that K is (the) ρ-renaming of L;
- K is said to be the *ρ-alphabetic variant* of L (where $\rho = \{\rho_i : Y_i \to Y_i\}_{1 \leqslant i \leqslant k+1}$ is a list of renamings) iff it is the ρ-renaming of L and moreover $\rho_1 = 1_{Y_1}$ and $\rho_{k+1} = 1_{Y_{k+1}}$ (the reason for this definition is that variables in internal equivalence classes of terms in a path are considered bounded).

Example. For every permutation σ on the n-elements set, we have that the path

$$K_1, \langle a_1,\ldots,a_n\rangle, \alpha, K_2$$

is an alphabetic variant of the path

$$K_1, \langle a_{\sigma(1)},\ldots,a_{\sigma(n)}\rangle, \langle \pi_{\sigma^{-1}(1)},\ldots,\pi_{\sigma^{-1}(n)}\rangle \circ \alpha, K_2$$

(here K_1, K_2 might be empty). Thus applying alphabetic variants allows permuting the components of an arrow in a path (provided such an arrow is not the last arrow of the path).

Example. Path

$$W \xrightarrow{K_1} Y \times Z \times U \xrightarrow{\langle \alpha, \pi_Z\rangle} V \times Z \xrightarrow{K_2} T,$$

is an alphabetic variant of the path

$$W \xrightarrow{K_1 \circ \langle \pi_Y, \pi_U, \pi_Z \rangle} Y \times U \times Z \xrightarrow{\langle \langle \pi_Y, \pi_Z, \pi_U \rangle \circ \alpha, \pi_Z \rangle} V \times Z \xrightarrow{K_2} T$$

(here only K_2 might be empty and $K_1 \circ \langle \pi_Y, \pi_U, \pi_Z \rangle$ is K_1 with last arrow composed with $\langle \pi_Y, \pi_U, \pi_Z \rangle$). Thus, applying alphabetic variants allows to assume that certain projections (located not in the first arrow of the path) project, say, on last components of their domains.

We shall apply rewriting on equivalence classes of paths modulo "being an alphabetic variant of": thus, from now on, a "path" will be an equivalence class of paths, *two paths being considered the same in case they are alphabetic variants of each other*. This needs an additional convention on our rules:

Convention. We stipulate that *the renaming of any rule is always supposed to be available as a rule*: by this, we mean that whenever we introduce a rule $K \Rightarrow K'$, we tacitly suppose that $\rho(K) \Rightarrow \rho'(K')$ is also a rule, for any list ρ, ρ' of renamings such that first and last components of ρ, ρ' are, respectively equal.[6]

The reader may check that the following property is a direct consequence of the above convention: *if K rewrites to L by means of a certain rule*, *then any alphabetic variant of K rewrites to some alphabetic variant of L by means of the same rule* (this means, in particular, that it does not matter which path, within a given equivalence class of paths, we use for reduction and normalization). As a concrete example of application of our convention, notice that the passage from

$$Y_1 \times Z \times Y_2 \xrightarrow{1 \times \alpha \times 1} Y_1 \times Z' \times Y_2 \xrightarrow{\beta_1 \times 1 \times \beta_2} Y_1' \times Z' \times Y_2'$$

to

$$Y_1 \times Z \times Y_2 \xrightarrow{\beta_1 \times 1 \times \beta_2} Y_1' \times Z \times Y_2' \xrightarrow{1 \times \alpha \times 1} Y_1' \times Z' \times Y_2',$$

is now legal in $\mathscr{P}$, on the basis of the basic equations (i.e. of the two-side rules) of Theorem 3.1 (notice that we do not need any extraction of a renaming to justify the equivalence of these two paths).

A side effect of the choice of rewriting modulo alphabetic variants is that the normal forms we eventually obtain, are unique only up to alphabetic variants. Checking whether two paths are alphabetic variants of each other, in case we know they are both in normal form, does not substantially affect efficiency, given the particular structure of normal forms (we shall turn to that in Section 10).

[6] We shall of course always deal with rules $K \Rightarrow K'$ such that K and K' agree on domains and codomains. Thus, our Convention says that $\rho(K) \Rightarrow \rho'(K')$ is a rule in case $K \Rightarrow K'$ is a rule, $\rho = \{\rho_1, \ldots, \rho_n\}$, $\rho' = \{\rho_1', \ldots, \rho_m'\}$ and $\rho_1 = \rho_1'$ and $\rho_n = \rho_m'$.

Before going on, we need another preliminary indispensable decision about our datatypes. Notice that terms like $f(t_1,t_2)$, where $f \in \Omega^0$ and where $t_i(x_1)$ is a pure Ω^i-term, have (at least) two different splitting paths, namely

$$X \xrightarrow{\langle t_1(x_1),x_1\rangle} X^2 \xrightarrow{f(x_1,t_2(x_2))} X \quad \text{and} \quad X \xrightarrow{\langle x_1,t_2(x_1)\rangle} X^2 \xrightarrow{f(t_1(x_1),x_2)} X.$$

Our final aim is that of having (uniqueness of) normal forms for paths, so we must decide once and for all which one has to be considered in normal form. This choice is clearly conventional, but has to be done one way or another: we choose the former path and declare the latter path to be illegal. This yields the following notion: we say that a path is *well-coloured* iff it has the form K, α^2 (where K is possibly empty). This means that the last arrow in a well-coloured path must come from $\mathbf{T_2}$ (which does not exclude that it might come from $\mathbf{T_0}$ as well).

We modify our basic equations so that we need to consider only well-coloured paths. For a path $K: Y \to Z$, let K^+ be the well-coloured path $K, 1_Z$.

Let us reformulate our basic equations as follows:

$$(\text{E1})^1 \quad \alpha^1, \beta^1, \gamma \;\Leftrightarrow\; \alpha^1 \circ \beta^1, \gamma$$

$$(\text{E1})^2 \quad \alpha^2, \beta^2 \;\Leftrightarrow\; \alpha^2 \circ \beta^2$$

$$(\text{E2}) \quad 1 \times \alpha_2,\, \alpha_1 \times 1,\, \beta \;\Leftrightarrow\; \alpha_1 \times 1,\, 1 \times \alpha_2,\, \beta.$$

These new equations do not allow to rewrite a well-coloured path into a non-well-coloured path; notice also that the “interchange basic equation” $1 \times \alpha_2, \alpha_1 \times 1 \Leftrightarrow \alpha_1 \times 1, 1 \times \alpha_2$ now does not apply anymore in the last position of a path.

As we said, we shall consider from now on *only well-coloured paths subject to the new basic equations* $(\text{E1})^i, (\text{E2})$.[7] There is no loss of generality in that because for well-coloured paths K, L, we have $K \Leftrightarrow^* L$ (according to the old basic equations) iff $K \Leftrightarrow^* L$ (according to the new basic equations). In fact, one side is trivial; for the other side, let us consider a $\Leftrightarrow$-chain like

$$K = K_0 \Leftrightarrow K_1 \Leftrightarrow \cdots \Leftrightarrow K_n = L,$$

obtained according to the old basic equations. We thus have

$$K^+ = K_0^+ \Leftrightarrow K_1^+ \Leftrightarrow \cdots \Leftrightarrow K_n^+ = L^+,$$

according to the new basic equations; now two applications of $(\text{E1})^2$ yield $K \Leftrightarrow K^+$ and $L \Leftrightarrow L^+$, because K, L are well-coloured. Thus, $K \Leftrightarrow^* L$ holds by using the new equations too.

[7] Of course, this means also that, when computing the splitting path of a term, identity should be added at the end in case the top symbol of the term has the wrong colour.

5. Completion

The above modified basic equations $(\mathrm{E1})^1, (\mathrm{E1})^2, (\mathrm{E2})$ can be turned into a canonical rewriting system $\mathscr{R}$ by applying Knuth–Bendix completion procedure. Details of such a completion process are fully given in [6]; to save considerable space, here, we content ourselves by giving the final result and some hints.

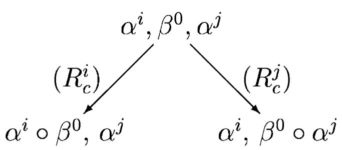
First of all, equations $(\mathrm{E1})^1, (\mathrm{E1})^2$ are obviously oriented as follows:

$(\mathrm{R}_\mathrm{c}^1)$ $\alpha^1, \beta^1, \gamma \Rightarrow \alpha^1 \circ \beta^1, \gamma$

$(\mathrm{R}_\mathrm{c}^2)$ $\alpha^2, \beta^2 \Rightarrow \alpha^2 \circ \beta^2$

and are called *composition rules*.

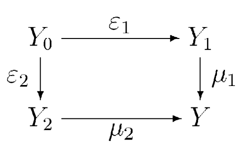
In order to deal with critical pairs like



we need to introduce factorization systems (because any naif orientation in one sense or in the other immediately produces infinite rewriting). There is a standard notion of factorization system in category theory (see [7]), however, such a notion is too strong in the present context, so that we weaken it.

Let **C** be any category; by a *weak factorization system* in **C**, we mean a pair of classes of arrows $(\mathscr{E}, \mathscr{M})$ from **C** such that:

(1) both $\mathscr{E}$ and $\mathscr{M}$ contain identities and are closed with respect to left and right composition with arrows in $\mathscr{E} \cap \mathscr{M}$;
(2) for every $\alpha \in \mathbf{C}$, there are $\varepsilon \in \mathscr{E}$, $\mu \in \mathscr{M}$ such that $\alpha = \varepsilon \circ \mu$;
(3) whenever we have a commutative square



with $\varepsilon_1, \varepsilon_2 \in \mathscr{E}$, $\mu_1, \mu_2 \in \mathscr{M}$, there is a unique $\rho \in \mathscr{E} \cap \mathscr{M}$ such that $\varepsilon_2 \circ \rho = \varepsilon_1$ and $\rho \circ \mu_1 = \mu_2$ (this condition denotes that the factorization given by (2) is essentially unique).

From the above axioms, it follows that arrows $\rho \in \mathscr{E} \cap \mathscr{M}$ are invertible (because they have two trivial factorizations, namely $\rho \circ 1$ and $1 \circ \rho$, hence…); such arrows will be just renamings in our cases. A weak factorization system becomes a usual factorization system when the classes $\mathscr{E}$ and $\mathscr{M}$ are required to be closed under composition and to contain all the isomorphisms. Observe that in this case, property (3) of weak factorization systems implies that every two morphisms $e \in \mathscr{E}$ and $m \in \mathscr{M}$ are "orthogonal" in the usual sense of factorization systems.[8]

Main Example. For any equational theory $T = (\Omega, Ax)$, the corresponding Lawvere category **T** always has a weak factorization system $(\mathscr{E}, \mathscr{M})$ (which we call the *standard weak factorization system* for **T**):

- arrows in $\mathscr{E}$ are just projections (i.e. tuples of distinct variables in symbolic presentations);
- arrows in $\mathscr{M}$ are those α such that in case it happens that $\alpha = \varepsilon \circ \alpha'$ (with $\varepsilon \in \mathscr{E}$), we must have that ε is just a renaming.

The factorizations $\alpha = \alpha_\varepsilon \circ \alpha_\mu$ (with $\alpha_\varepsilon \in \mathscr{E}, \alpha_\mu \in \mathscr{M}$) are obtained as follows. Let $\vec{t}(x_1, \ldots, x_n)$ be a tuple of terms containing at most the variables $x_1, \ldots, x_n$; consider that this tuple is *n-minimized* iff for no $i = 1, \ldots, n$ we have $\vdash_T \vec{t} = \vec{t}(c_0/x_i)$.[9] Now it is not difficult to see that the m-tuple of terms $\vec{t}$ *is n-minimized iff the arrow* $\alpha : X^n \to X^m$ *belongs to* $\mathscr{M}$, where α is the vector of the equivalence classes of terms represented by the m components of $\vec{t}$ (if, say, $\vec{t} = \langle t_1, \ldots, t_m \rangle$, then α is $\langle \{t_1\}, \ldots, \{t_m\} \rangle$).

Now, let α be arbitrary; in order to get the factorization $\alpha = \alpha_\varepsilon \circ \alpha_\mu$ (where $\alpha_\varepsilon \in \mathscr{E}$ and $\alpha_\mu \in \mathscr{M}$), it is sufficient to take any vector of terms in the equivalence classes of α containing a minimal set of variables: if such a vector is $\vec{t}(x_{i_1}, \ldots, x_{i_k})$, then the factorization is $\alpha = \langle \pi_{i_1}, \ldots, \pi_{i_k} \rangle \circ \beta$, where β is represented by the vector of terms $\vec{t}(x_1, \ldots, x_k)$. Notice that this process is effective, in case word problem for T is solvable: one takes any $\vec{t}$ representing α and then go on by replacing variables in it by c_0; the procedure stops when only terms not provably equal to $\vec{t}$ can be obtained.

Uniqueness of the above factorization (up to a renaming) is easily established.

Example. Let G be the group theory (we use $*$ for product, i for inverse, e for neutral element). Consider the standard weak factorization system of the associated Lawvere category **G**. Notice first that there are isomorphisms which are not renamings, e.g. $X \overset{i(x_1)}{\to} X$: this term is in $\mathscr{M}$, but not in $\mathscr{E}$. Moreover, terms

$$X \xrightarrow{\langle x_1, i(x_1) \rangle} X^2 \xrightarrow{x_1 * x_2} X,$$

are both minimized (i.e. are in $\mathscr{M}$), however their composition, i.e. the term $x_1 * i(x_1)$ is not in $\mathscr{M}$ anymore, it factorizes as

$$X \to \mathbf{1} \overset{e}{\to} X$$

(where first component is the unique arrow into the terminal, i.e. the empty list of terms).

[8] We thank the anonymous referee for this observation.

[9] Notations like $\vdash_T \vec{u} = \vec{v}$, for $\vec{u} = \langle u_1, \ldots, u_m \rangle$ and $\vec{v} = \langle v_1, \ldots, v_m \rangle$, mean that $\vdash_T \bigwedge_{j=1}^{m} u_j = v_j$. Recall that in Section 2, we assumed that there is at least one ground term c_0 in our signatures.

Let $\mathbf{C}$ be a subcategory of $\mathbf{C}'$ and let $(\mathscr{E},\mathscr{M})$ be a weak factorization system in $\mathbf{C}$. A weak factorization system $(\mathscr{E}',\mathscr{M})$ in $\mathbf{C}'$ (notice that $\mathscr{M}$ is the same!) is said to be a *left extension* of $(\mathscr{E},\mathscr{M})$ iff the following hold:
- $\mathscr{E}'\cap\mathbf{C}=\mathscr{E}$;
- if $\varepsilon_1,\varepsilon_2\in\mathscr{E}$ and if $e\in\mathscr{E}'$, then $\varepsilon_1\circ e\in\mathscr{E}'$ and $e\circ\varepsilon_2\in\mathscr{E}'$ (whenever compositions make sense).

Notice that this implies that $\mathscr{E}$—not necessarily $\mathscr{E}'$—is closed under composition. Let us say that T_i *is constructible over* T_0 iff in $\mathbf{T_i}$ there is a left extension $(\mathscr{E}_i,\mathscr{M}_0)$ of the standard weak factorization system $(\mathscr{E}_0,\mathscr{M}_0)$ of $\mathbf{T_0}$.

Assumption. We assume that T_1,T_2 are both constructible over T_0.

We postpone to Section 10 a symbolic translation of this assumption as well as the analysis of some examples (and counterexamples). For the moment, let us underline that, as an effect of the above assumption, we now have that any arrow α^i admits two factorizations, namely:
- it can be factored as $\alpha^i_\varepsilon\circ\alpha^i_m$ according to the standard weak factorization system $(\mathscr{E}_0,\mathscr{M}_i)$ of $\mathbf{T_i}$ (we recall that here $\mathscr{E}_0$ is formed by arrows which are projections, whereas $\mathscr{M}_i$ is formed by arrows represented by minimized—in the sense of the theory T_i—vectors of terms);
- it can be factored as $\alpha^i_e\circ\alpha^i_\mu$ according to the left extension $(\mathscr{E}_i,\mathscr{M}_0)$ of the standard weak factorization system of $\mathbf{T_0}$ (here, the class $\mathscr{E}_i$ is axiomatically given by the above assumption, whereas $\mathscr{M}_0$ is the class of arrows from $\mathbf{T_0}$ represented by minimized vectors of terms—in the sense of the theory T_0).[10]

The constructibility assumption over T_0 we made for T_1,T_2 essentially denotes that terms from T_i have a kind of "head normal form" relative to T_0; in terms of arrows β^i from $\mathbf{T_i}$, this head normal form is just β^i_μ. Notice the following fact (which will be repeatedly used within the paper, especially in the most technical parts): suppose that we want to factorize an arrow like $\beta^i\circ\alpha^0$ in the left extension. This is done as follows (see Fig. 1): we first take the factorization $\beta^i_e\circ\beta^i_\mu$ of β^i, then we decompose $\beta^i_\mu\circ\alpha^0$ as $\varepsilon\circ\mu$ in $\mathbf{T_0}$ (according to the standard weak factorization system of $\mathbf{T_0}$) and then we take $(\beta^i_e\circ\varepsilon)\circ\mu$; this decomposition is just $(\beta^i\circ\alpha^0)_e\circ(\beta^i\circ\alpha^0)_\mu$ (up to a renaming), by uniqueness and by the second condition of being a left extension. Easy counterexamples show that the decomposition $\beta^i_e\circ(\beta^i_\mu\circ\alpha^0)$ may not be okay, because $\beta^i_\mu\circ\alpha^0$ may not be minimized (i.e. may not belong to $\mathscr{M}_0$): minimization may delete some components of β^i_e, which is exactly what happens by taking the composition $\beta^i_e\circ\varepsilon$, which is the good e-component of the arrow $\beta^i\circ\alpha^0$.

The above-mentioned critical pairs are treated by the following pairs of rules:

$$(\mathrm{R}_\varepsilon)\quad \alpha,\beta\Rightarrow\alpha\circ\beta_\varepsilon,\ \beta_m,$$

$$(\mathrm{R}_\mu)\quad \alpha,\beta\Rightarrow\alpha_e,\ \alpha_\mu\circ\beta,$$

[10] These vectors of terms are also minimized in the sense of T_i, given that T_i is conservative over T_0.

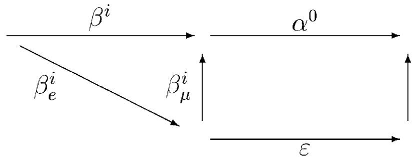


Fig. 1. Factorization of $\beta^i \circ \alpha^0$ according to $(\mathscr{E}_i, \mathscr{M}_0)$.

called *ε-extraction and μ-extraction rules*, respectively.[11] Let us call $\mathscr{R}_0$ the rewriting system formed by rules $(\mathrm{R}^i_{\mathrm{c}}), (\mathrm{R}_\varepsilon), (R_\mu)$. Notice that our notation says that only the "Greek parts" of a term β^i are exchanged during rewriting: precisely, β^i_ε is exchanged to the left, whereas β^i_μ is exchanged to the right. In fact, the reduction process propagates to the right the T_0-head normal forms of the kind β^i_μ. Such a propagation may have side effects, because β^i_μ composed to the right with the consecutive term α^j may cause the extraction of a certain $\varepsilon^0 \in \mathscr{E}_0$ from $\beta^i_\mu \circ \alpha^j$. Such an extraction may in its turn delete certain components of β^i_e, thus possibly collapsing β^i_e to a tuple of variables, a fact which might make consecutive terms now composable by $(\mathrm{R}^i_{\mathrm{c}})$, etc. Anyway, in Section 6 we shall prove that

Theorem 5.1. *$\mathscr{R}_0$ is canonical (i.e. confluent and terminating).*

In order to finish our completion process we need only to treat Eq. (E2). This is a more technical point; we just mention that, after suitable orientation, superposition, simplification and deletion steps, we get a couple of rules $(\mathrm{R}^1_{\mathrm{p}})$ and $(\mathrm{R}^2_{\mathrm{p}})$ (called *products rules*) which are so described. First member of rule $(\mathrm{R}^i_{\mathrm{p}})$ is

$$(\mathrm{I}) \quad Y \xrightarrow{\langle\gamma^i,\delta^i\rangle} Y_1 \times Z \xrightarrow{\langle\alpha,\pi_Z\rangle} Y_2 \times Z \xrightarrow{\beta^i} U \xrightarrow{(\theta)} V,$$

whereas second member is

$$(\mathrm{II}) \quad Y \xrightarrow{\langle\gamma^i,\delta^i,\delta^i_\varepsilon\rangle} Y_1 \times Z \times Y' \xrightarrow{\alpha\times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2}\times\delta^i_m)\circ\beta^i} U \xrightarrow{(\theta)} V$$

(the extra arrow θ is needed only if $i=1$). We add a provisio for these two rules: $\delta^i \notin \mathscr{E}_0$ (*that is, δ^i cannot be a projection*). The reason for this last proviso is that, in case δ^i is a projection, it may happen that the second member of $(\mathrm{R}^i_{\mathrm{p}})$ can be re-written to the first (thus causing termination problems).

In conclusion, we obtain the rewriting system $\mathscr{R}$ which is described by Table 1 (in the last two rules of the table, Z, Y' and Y_2 are the codomains of $\delta^i, \delta^i_\varepsilon$ and α, respectively, as in the fully displayed paths (I) and (II) above).

[11] It goes without saying that such rules do not apply in case of trivial factorizations (i.e. when α_ε—resp. α_μ—are, up to a renaming, just identities). Concerning this, recall from the previous Section that the exchange of a renaming simply produces an alphabetic variant.

Table 1
The system $\mathscr{R}$

$(\mathrm{R}^1_\mathrm{c})$	$\alpha^1, \beta^1, \gamma \Rightarrow \alpha^1 \circ \beta^1, \gamma$
$(\mathrm{R}^2_\mathrm{c})$	$\alpha^2, \beta^2 \Rightarrow \alpha^2 \circ \beta^2$
(R_ε)	$\alpha, \beta \Rightarrow \alpha \circ \beta_\varepsilon, \beta_m$
(R_μ)	$\alpha, \beta \Rightarrow \alpha_e, \alpha_\mu \circ \beta$
$(\mathrm{R}^1_\mathrm{p})$	$\langle \gamma^1, \delta^1 \rangle, \langle \alpha, \pi_Z \rangle, \beta^1, \theta \Rightarrow \langle \gamma^1, \delta^1, \delta^1_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times \delta^1_m) \circ \beta^1, \theta$ where $\delta^1 \notin \mathscr{E}_0$
$(\mathrm{R}^2_\mathrm{p})$	$\langle \gamma^2, \delta^2 \rangle, \langle \alpha, \pi_Z \rangle, \beta^2 \Rightarrow \langle \gamma^2, \delta^2, \delta^2_\varepsilon \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times \delta^2_m) \circ \beta^2$ where $\delta^2 \notin \mathscr{E}_0$

We also stipulate that *if $L \Rightarrow R$ is a rule, then $L' \Rightarrow R'$ is a rule too, where L' is any alphabetic variant of L and R' is any alphabetic variant of R.* [12]

The content of the present section is summarized in the following lemma (which comes from the fact that $\mathscr{R}$ can be obtained through Knuth–Bendix completion from $\mathscr{P}$, but which we prove directly):

Lemma 5.2. *For well-coloured paths K_1, K_2, we have $K_1 \Leftrightarrow^*_{\mathscr{R}} K_2$ iff $K_1 \Leftrightarrow^*_{\mathscr{P}} K_2$ (where $\mathscr{P}$ is the system introduced in the proof of Theorem* 3.1, *as modified with the new basic equations* $(E1)^1$, $(E1)^2$ *and* $(E2)$ *of Section* 4).

Proof. Let us show that the two members of $(\mathrm{R}^i_\mathrm{p})$ are $\Leftrightarrow^*_{\mathscr{P}}$-equivalent. This is obtained as follows. We let $Z \xrightarrow{\Delta_Z} Z \times Z$ to be $\langle 1_Z, 1_Z \rangle$, moreover, (for space reasons) we leave out of the pictures the fourth arrow θ which is needed in case $i = 1$:

$$Y \xrightarrow{\langle \gamma^i, \delta^i \rangle} Y_1 \times Z \xrightarrow{\langle \alpha, \pi_Z \rangle} Y_2 \times Z \xrightarrow{\beta^i} U$$
$$=$$
$$Y \xrightarrow{\langle \gamma^i, \delta^i \rangle} Y_1 \times Z \xrightarrow[\Leftrightarrow^*_{\mathscr{P}} \quad \text{two applications of } (\mathrm{E1})^i]{(1_{Y_1} \times \Delta_Z) \circ (\alpha \times 1_Z)} Y_2 \times Z \xrightarrow{\beta^i} U$$
$$Y \xrightarrow{\langle \gamma^i, \delta^i \rangle \circ (1_{Y_1} \times \Delta_Z)} Y_1 \times Z \times Z \xrightarrow{\alpha \times 1_Z} Y_2 \times Z \xrightarrow{\beta^i} U$$
$$=$$
$$Y \xrightarrow{\langle \gamma^i, \delta^i, \delta^i \rangle} Y_1 \times Z \times Z \xrightarrow[\Leftrightarrow_{\mathscr{P}} \quad \text{by } (\mathrm{E1})^i]{\alpha \times 1_Z} Y_2 \times Z \xrightarrow{\beta^i} U$$
$$Y \xrightarrow{\langle \gamma^i, \delta^i, \delta^i_\varepsilon \rangle} Y_1 \times Z \times Y' \xrightarrow[\Leftrightarrow_{\mathscr{P}} \quad \text{by } (\mathrm{E2})^i]{1_{Y_1} \times 1_Z \times \delta^i_m} Y_1 \times Z \times Z \xrightarrow{\alpha \times 1_Z} Y_2 \times Z \xrightarrow{\beta^i} U$$

[12] Given that the rules of $\mathscr{R}$ are all closed under the operation of composing first (or last) arrow in each member by the same single renaming, this stipulation is automatically sufficient to ensure the (slightly stronger) Convention we made in Section 4, namely that “the renaming of any rule is available as a rule”.

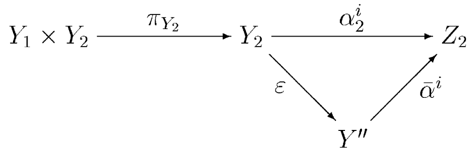


Fig. 2. Factorization of $\pi_{Y_2} \circ \alpha_2^i$.

$$Y \xrightarrow{\langle \gamma^i, \delta^i, \delta^i_\varepsilon \rangle} Y_1 \times Z \times Y' \underset{\Leftrightarrow_{\mathscr{P}} \quad \text{by (E1)}^i}{\xrightarrow{\alpha \times 1_{Y'}}} Y_2 \times Y' \xrightarrow{1_{Y_2} \times \delta^i_m} Y_2 \times Z \xrightarrow{\beta^i} U$$

$$Y \xrightarrow{\langle \gamma^i, \delta^i, \delta^i_\varepsilon \rangle} Y_1 \times Z \times Y' \xrightarrow{\alpha \times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times \delta^i_m) \circ \beta^i} U$$

(notice that the last application of $(\mathrm{E1})^i$ is correct because for $i = 1$ the further arrow θ belongs to the path).

Conversely, let us show that the two members

$$\text{(I)} \quad Y_1 \times Y_2 \xrightarrow{1_{Y_1} \times \alpha_2} Y_1 \times Z_2 \xrightarrow{\alpha_1 \times 1_{Z_2}} Z_1 \times Z_2 \xrightarrow{\beta} U,$$

$$\text{(II)} \quad Y_1 \times Y_2 \xrightarrow{\alpha_1 \times 1_{Y_2}} Z_1 \times Y_2 \xrightarrow{1_{Z_1} \times \alpha_2} Z_1 \times Z_2 \xrightarrow{\beta} U,$$

of (E2) are joinable in $\mathscr{R}$. This is clear when α_1, α_2 are equally coloured; otherwise, let e.g. α_2 and β have same colour i.[13] Notice that in case $\beta \in \mathbf{T}_1 \setminus \mathbf{T_0}$, there must be, in a well-coloured path, a further arrow θ: for space reasons, we do not indicate it in the displayed paths below, but it should be remarked that, just because of its presence, it is in any case possible to apply rule $(\mathrm{R}^i_\mathrm{p})$ to (I) and rule $(\mathrm{R}^i_\mathrm{c})$ to (II). We have for first member (I)

$$Y_1 \times Y_2 \xrightarrow{1_{Y_1} \times \alpha_2^i} Y_1 \times Z_2 \xrightarrow{\alpha_1^j \times 1_{Z_2}} Z_1 \times Z_2 \xrightarrow{\beta^i} U,$$

$$=$$

$$Y_1 \times Y_2 \xrightarrow{\langle \pi_{Y_1}, \pi_{Y_2} \circ \alpha_2^i \rangle} Y_1 \times Z_2 \xrightarrow{\langle \pi_{Y_1} \circ \alpha_1^j, \pi_{Z_2} \rangle} Z_1 \times Z_2 \xrightarrow{\beta^i} U. \tag{1}$$

Let us suppose that α_2^i factorizes in ε/m-components as $\varepsilon \circ \bar{\alpha}^i$ (see Fig. 2); by uniqueness, $\pi_{Y_2} \circ \alpha_2^i$ factorizes in ε/m-components as $(\pi_{Y_2} \circ \varepsilon) \circ \bar{\alpha}^i$.

An application of $(\mathrm{R}^i_\mathrm{p})$ to (1) yields

$$Y_1 \times Y_2 \xrightarrow{\langle \pi_{Y_1}, \pi_{Y_2} \circ \alpha_2^i, \pi_{Y_2} \circ \varepsilon \rangle} Y_1 \times Z_2 \times Y'' \xrightarrow{(\pi_{Y_1} \circ \alpha_1^j) \times 1_{Y''}} Z_1 \times Y'' \xrightarrow{(1_{Z_1} \times \bar{\alpha}^i) \circ \beta^i} U. \tag{2}$$

[13] In case it is α_1 which shares the same colour as β, the argument is the same (we need below an obvious alphabetic variant of $(\mathrm{R}^i_\mathrm{p})$). Notice that if $\beta \in \mathbf{T_0}$, we choose among α_1, α_2 the arrow having colour 2 in order to be able to apply $(\mathrm{R}^2_\mathrm{p})$.

Now,

$$Y_1 \times Z_2 \times Y'' \xrightarrow{(\pi_{Y_1} \circ \alpha_1^j) \times 1_{Y''}} Z_1 \times Y'',$$
$$=$$
$$Y_1 \times Z_2 \times Y'' \xrightarrow{\langle \pi_{Y_1}, \pi_{Y''} \rangle \circ (\alpha_1^j \times 1_{Y''})} Z_1 \times Y'',$$

hence,

$$Y_1 \times Z_2 \times Y'' \xrightarrow{\langle \pi_{Y_1}, \pi_{Y''} \rangle \circ (\alpha_1^j \times 1_{Y''})_\varepsilon} W \xrightarrow{(\alpha_1^j \times 1_{Y''})_m} Z_1 \times Y'',$$

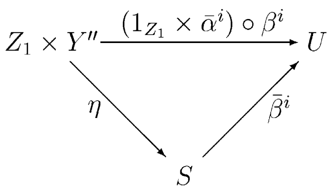
is, by uniqueness, the ε/m-factorization of $(\pi_{Y_1} \circ \alpha_1^j) \times 1_{Y''}$. Thus, by applying an (R_ε)-step to (2), we get

$$Y_1 \times Y_2 \xrightarrow{\langle \pi_{Y_1}, \pi_{Y_2} \circ \alpha_2^j, \pi_{Y_2} \circ \varepsilon \rangle \circ \langle \pi_{Y_1}, \pi_{Y''} \rangle \circ (\alpha_1^j \times 1_{Y''})_\varepsilon} W \xrightarrow{(\alpha_1^j \times 1_{Y''})_m} Z_1 \times Y'' \xrightarrow{(1_{Z_1} \times \bar{\alpha}^i) \circ \beta^i} U$$
$$=$$
$$Y_1 \times Y_2 \xrightarrow{\langle \pi_{Y_1}, \pi_{Y_2} \circ \varepsilon \rangle \circ (\alpha_1^j \times 1_{Y''})_\varepsilon} W \xrightarrow{(\alpha_1^j \times 1_{Y''})_m} Z_1 \times Y'' \xrightarrow{(1_{Z_1} \times \bar{\alpha}^i) \circ \beta^i} U.$$

Composing the first two arrows by $(\mathrm{R}_\mathrm{c}^j)$, we get

$$Y_1 \times Y_2 \xrightarrow{\langle \pi_{Y_1}, \pi_{Y_2} \circ \varepsilon \rangle \circ (\alpha_1^j \times 1_{Y''})} Z_1 \times Y'' \xrightarrow{(1_{Z_1} \times \bar{\alpha}^i) \circ \beta^i} U$$
$$=$$
$$Y_1 \times Y_2 \xrightarrow{\alpha_1^j \times \varepsilon} Z_1 \times Y'' \xrightarrow{(1_{Z_1} \times \bar{\alpha}^i) \circ \beta^i} U.$$

Let us assume that $(1_{Z_1} \times \bar{\alpha}^i) \circ \beta^i$ factorizes in ε/m-components as follows:



An (R_ε)-step produces

$$Y_1 \times Y_2 \xrightarrow{(\alpha_1^j \times \varepsilon) \circ \eta} S \xrightarrow{\bar{\beta}^i} U. \qquad (K)$$

Let us now operate on second member (II). We first apply an $(\mathrm{R}_\mathrm{c}^i)$-step thus getting

$$Y_1 \times Y_2 \xrightarrow{\alpha_1^j \times 1_{Y_2}} Z_1 \times Y_2 \xrightarrow{(1_{Z_1} \times \alpha_2^i) \circ \beta^i} U. \qquad (3)$$

Let us consider the commutative diagram in Fig. 3.

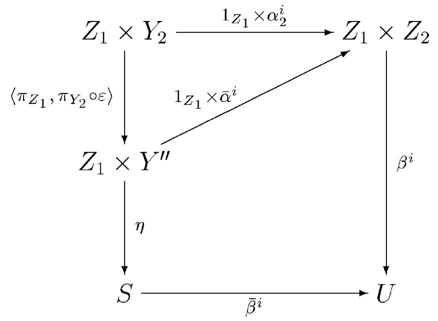


Fig. 3. Factorization of $1_{Z_1} \times \alpha_2^i$.

By uniqueness, $\langle \pi_{Z_1}, \pi_{Y_2} \circ \varepsilon \rangle \circ \eta$ and $\bar{\beta}^i$ are the ε/m-components of $(1_{Z_1} \times \alpha_2^i) \circ \beta^i$. An (R_ε)-step applied to (3) leads to

$$Y_1 \times Y_2 \xrightarrow{(\alpha_1^j \times 1_{Y_2}) \circ \langle \pi_{Z_1}, \pi_{Y_2} \circ \varepsilon \rangle \circ \eta} S \xrightarrow{\bar{\beta}^i} U,$$

which is precisely (K). □

In Section 9, we shall prove our main result, namely that

Theorem 5.3. *$\mathscr{R}$ is canonical.*

6. Local confluence, I

In this section, we will prove the canonicity of the system $\mathscr{R}_0$ which, we recall, is the system described by Table 2.

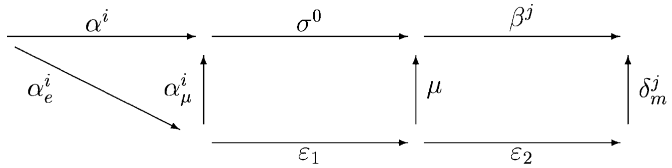
We begin by showing that $\mathscr{R}_0$ is locally confluent: we single out all critical pairs arising from superpositions between the rules of $\mathscr{R}_0$ and we prove that they are joinable. Most of the cases can be reduced to the critical pairs treated in the following lemma.

Lemma 6.1. *The paths $\alpha^i \circ \sigma^0, \beta^j$ and $\alpha^i, \sigma^0 \circ \beta^j$ are joinable in $\mathscr{R}_0$.*

Table 2
The system $\mathscr{R}_0$

(R_c^1)	$\alpha^1, \beta^1, \gamma \;\Rightarrow\; \alpha^1 \circ \beta^1, \gamma$
(R_c^2)	$\alpha^2, \beta^2 \;\Rightarrow\; \alpha^2 \circ \beta^2$
(R_ε)	$\alpha, \beta \;\Rightarrow\; \alpha \circ \beta_\varepsilon, \beta_m$
(R_μ)	$\alpha, \beta \;\Rightarrow\; \alpha_e, \alpha_\mu \circ \beta$

Proof. Let α_e^i and α_μ^i be the e/μ components of α^i and let us consider the following commutative diagram, where $\varepsilon_1 \circ \mu$ corresponds to the ε/μ factorization in $\mathbf{T_0}$ of $\alpha_\mu^i \circ \sigma^0$ and $\varepsilon_2 \circ \delta_m^j$ is the ε/m factorization of $\mu \circ \beta^j$ in $\mathbf{T}_j$.



Since $\alpha_e^i \circ \varepsilon_1$ belongs to $\mathscr{E}_i$ (recall the definition of left extensions of factorization systems) and δ_m^j belongs to $\mathscr{M}_j$, we have (up to a renaming):

$$(\alpha^i \circ \sigma^0)_e = \alpha_e^i \circ \varepsilon_1, \quad (\alpha_\mu^i \circ \sigma^0 \circ \beta^j)_\varepsilon = \varepsilon_1 \circ \varepsilon_2,$$

$$(\alpha^i \circ \sigma^0)_\mu = \mu, \qquad (\alpha_\mu^i \circ \sigma^0 \circ \beta^j)_m = \delta_m^j.$$

We can do the following rewriting steps:

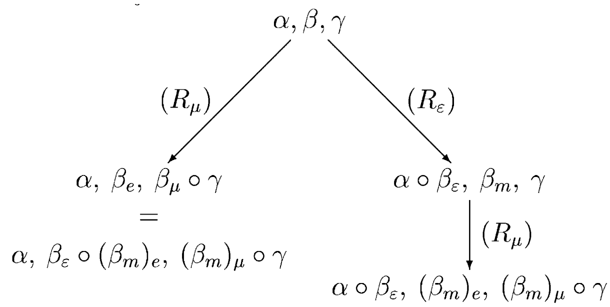
$$\alpha^i \circ \sigma^0, \beta^j \Rightarrow_{\mathrm{R}_\mu} \alpha_e^i \circ \varepsilon_1, \mu \circ \beta^j \Rightarrow_{\mathrm{R}_\varepsilon} \alpha_e^i \circ \varepsilon_1 \circ \varepsilon_2,\ \delta_m^j,$$

$$\alpha^i, \sigma^0 \circ \beta^j \Rightarrow_{\mathrm{R}_\mu} \alpha_e^i, \alpha_\mu^i \circ \sigma^0 \circ \beta^j \Rightarrow_{\mathrm{R}_\varepsilon} \alpha_e^i \circ \varepsilon_1 \circ \varepsilon_2,\ \delta_m^j$$

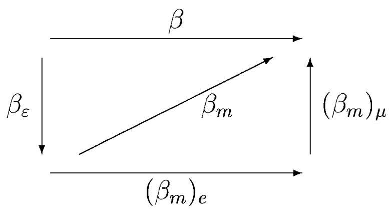
and this proves the lemma. □

Theorem 6.2. *$\mathscr{R}_0$ is locally confluent.*

Proof. We must show that all critical pairs arising from rules of $\mathscr{R}_0$ are joinable. For this, we need a systematic analyses of the cases: treating (R_c^1) and (R_c^2) together, there are 12 such cases, which are either easy or reducible to the above lemma. We just consider the most relevant one.



In the first member, we use the fact that the following diagram is commutative



Thus, reasoning as usual (by uniqueness of factorizations—up to a renaming), we can state that $\beta_e = \beta_\varepsilon \circ (\beta_m)_e$ and $\beta_\mu = (\beta_m)_\mu$. We can apply Lemma 6.1, with $\sigma^0 = \beta_\varepsilon$. □

It remains to show the termination of $\mathscr{R}_0$ (then Newman's Lemma applies, showing canonicity of $\mathscr{R}_0$). This result is a consequence of Theorem 9.7, however, here we give a direct proof, which uses less machinery. We need a complexity measure for paths which decreases with application of our rules. At this aim, we define:

$$\mu(\alpha^i) = \begin{cases} 0 & \text{if } \alpha^i \in \mathscr{E}_i, \\ 1 & \text{otherwise,} \end{cases} \qquad \varepsilon(\alpha^i) = \begin{cases} 0 & \text{if } \alpha^i \in \mathscr{M}_i, \\ 1 & \text{otherwise.} \end{cases}$$

Let K be the path $\alpha_1, \ldots, \alpha_n$. We define

$$\mu(K) = \langle \mu(\alpha_1), \ldots, \mu(\alpha_n) \rangle, \quad \varepsilon(K) = \langle \varepsilon(\alpha_n), \ldots, \varepsilon(\alpha_1) \rangle$$

(notice that $\mu(K) = \mu(K')$ and $\varepsilon(K) = \varepsilon(K')$ hold in case K and K' are alphabetic variants of each other).

Finally, we introduce the following order relation $\succ$ between paths K, L:

- $K \succ L$ if and only if either (i) or (ii) hold:
  (i) $|K| > |L|$ (where $|K|$ denotes the length of K);
  (ii) $|K| = |L|$ and $\langle \mu(K), \varepsilon(K) \rangle >_l \langle \mu(L), \varepsilon(L) \rangle$
  (where $>_l$ denotes the lexicographic order between n-tuple of integers).

Theorem 6.3. *$\mathscr{R}_0$ is terminating.*

Proof. It is a standard fact that $\succ$ is a terminating transitive relation. Moreover, it is easily shown that $\succ$ is stable, in the sense that $K \succ K'$ implies $L, K, R \succ L, K', R$ (for all L, R). It remains to prove that if $L \Rightarrow L'$ is a rule of $\mathscr{R}_0$, then $L \succ L'$. This is not difficult and is left to the reader. □

*This conclude*s *the proof of Theorem* 5.1. □

7. The system $\mathscr{R}^+$

Proving directly local confluence of $\mathscr{R}$ leads to unnecessary complications, this is why we prefer to introduce another system (which we call $\mathscr{R}^+$) and prove local confluence of the latter. In Section 9, we shall prove termination of both $\mathscr{R}$ and $\mathscr{R}^+$ and then we shall make a more precise comparison between $\mathscr{R}$ and $\mathscr{R}^+$: from this comparison, canonicity of $\mathscr{R}$ follows immediately. In order to introduce $\mathscr{R}^+$ we first consider slight modifications of rules (R_μ) and $(\mathrm{R}^i_\mathrm{p})$.

Rule (R_μ) is enlarged as follows:

$$(\mathrm{R}_\mu)^+ \quad \langle \alpha, \beta \rangle, \gamma \Rightarrow \langle \alpha_e, \beta \rangle, (\alpha_\mu \times 1) \circ \gamma$$

(notice that in case vector β is empty, we get ordinary (R_μ)-rule).

Rules $(\mathrm{R}^i_\mathrm{p})$ are on the other hand, restricted so that only 1-component arrows are “moved to the right” (let us call $(\mathrm{R}^i_\mathrm{p})^+$ the rules obtained by this restriction). In conclusion, we let $\mathscr{R}^+$ be the rewriting system of Table 3.

It should be noticed that (as for $\mathscr{R}$) even $\mathscr{R}^+$ alphabetic variants of the above rules are available as rules. For instance, rule $(\mathrm{R}^i_\mathrm{p})^+$ has the following alphabetic variant:

$$Y \xrightarrow{\langle \gamma^i_1, d^i, \gamma^i_2 \rangle} Y_1 \times X \times Y_2 \xrightarrow{\langle \alpha_1, \pi_X, \alpha_2 \rangle} Z_1 \times X \times Z_2 \xrightarrow{\beta^i} U$$
$$\Downarrow$$
$$Y \xrightarrow{\langle \gamma^i_1, d^i, d^i_\varepsilon, \gamma^i_2 \rangle} Y_1 \times X \times Y' \times Y_2 \xrightarrow{\langle \pi \circ \alpha_1, \pi_{Y'}, \pi \circ \alpha_2 \rangle} Z_1 \times Y' \times Z_2 \xrightarrow{(1 \times d^i_m \times 1) \circ \beta^i} U$$

(where a further arrow must be inserted to the right in case $i = 1$, where Y' is the codomain of d^i_ε and where π is the projection $Y_1 \times X \times Y' \times Y_2 \to Y_1 \times X \times Y_2$). Other alphabetic variants are possible, e.g. by permuting the components of $\langle \gamma^i_1, d^i, d^i_\varepsilon, \gamma^i_2 \rangle$.

In the remaining part of this section we collect useful technical facts. We first analyse the relationship between old and new μ-extraction rules.

Table 3
The system $\mathscr{R}^+$

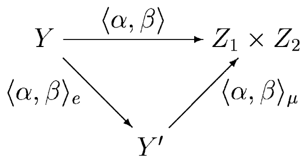
$(\mathrm{R}^1_\mathrm{c})$	$\alpha^1, \beta^1, \gamma \Rightarrow \alpha^1 \circ \beta^1, \gamma$
$(\mathrm{R}^2_\mathrm{c})$	$\alpha^2, \beta^2 \Rightarrow \alpha^2 \circ \beta^2$
(R_ε)	$\alpha, \beta \Rightarrow \alpha \circ \beta_\varepsilon, \beta_m$
$(\mathrm{R}_\mu)^+$	$\langle \alpha, \beta \rangle, \gamma \Rightarrow \langle \alpha_e, \beta \rangle, (\alpha_\mu \times 1) \circ \gamma$
$(\mathrm{R}^1_\mathrm{p})^+$	$\langle \gamma^1, d^1 \rangle, \langle \alpha, \pi_X \rangle, \beta^1, \theta \Rightarrow \langle \gamma^1, d^1, d^1_\varepsilon \rangle, \alpha \times 1, (1 \times d^1_m) \circ \beta^1, \theta$ where $d^1 \notin \mathscr{E}_0$
$(\mathrm{R}^2_\mathrm{p})^+$	$\langle \gamma^2, d^2 \rangle, \langle \alpha, \pi_X \rangle, \beta^2 \Rightarrow \langle \gamma^2, d^2, d^2_\varepsilon \rangle, \alpha \times 1, (1 \times d^2_m) \circ \beta^2$ where $d^2 \notin \mathscr{E}_0$

Lemma 7.1. *If $K \Rightarrow K'$ by a single $(\mathrm{R}_\mu)^+$-step, then there is K'' such that K' rewrites to K'' by (at most) 2 $(\mathrm{R}_\mu)^+$-rewrite steps and K rewrites to K'' by a single (R_μ)-rewrite step.*

Proof. We have the following three $(\mathrm{R}_\mu)^+$-rewrite steps:

$$\langle\alpha,\beta\rangle,\gamma \Rightarrow \langle\alpha_e,\beta\rangle,\ (\alpha_\mu\times 1)\circ\gamma \Rightarrow \langle\alpha_e,\beta_e\rangle,(\alpha_\mu\times\beta_\mu)\circ\gamma$$
$$\Rightarrow \langle\alpha_e,\beta_e\rangle_e, \langle\alpha_e,\beta_e\rangle_\mu\circ(\alpha_\mu\times\beta_\mu)\circ\gamma.$$

We need only to show that $\langle\alpha_e,\beta_e\rangle_e=\langle\alpha,\beta\rangle_e$ and $\langle\alpha_e,\beta_e\rangle_\mu\circ(\alpha_\mu\times\beta_\mu)=\langle\alpha,\beta\rangle_\mu$. Let us consider the factorization



and let us put $\langle\alpha,\beta\rangle_\mu=\langle\sigma,\tau\rangle$. We have

$$\alpha_e\circ\alpha_\mu=\alpha=\langle\alpha,\beta\rangle\circ\pi_{Z_1}=\langle\alpha,\beta\rangle_e\circ\langle\sigma,\tau\rangle\circ\pi_{Z_1}=(\langle\alpha,\beta\rangle_e\circ\sigma_\varepsilon)\circ\sigma_\mu,$$

hence, (by uniqueness of factorization)

$$\langle\alpha,\beta\rangle_e\circ\sigma_\varepsilon=\alpha_e \quad\text{and}\quad \alpha_\mu=\sigma_\mu$$

and similarly

$$\langle\alpha,\beta\rangle_e\circ\tau_\varepsilon=\beta_e \quad\text{and}\quad \beta_\mu=\tau_\mu.$$

Thus,

$$(*)\quad \langle\alpha,\beta\rangle=\langle\alpha,\beta\rangle_e\circ(\langle\sigma_\varepsilon,\tau_\varepsilon\rangle\circ(\alpha_\mu\times\beta_\mu)).$$

The arrow $\langle\sigma_\varepsilon,\tau_\varepsilon\rangle\circ(\alpha_\mu\times\beta_\mu)$ belongs to $\mathscr{M}_0$ as it is equal to $\langle\sigma,\tau\rangle=\langle\alpha,\beta\rangle_\mu$; so if we factorize $\langle\sigma_\varepsilon,\tau_\varepsilon\rangle$ as $\varepsilon\circ\mu$ and then $\mu\circ(\alpha_\mu\times\beta_\mu)$ as $\varepsilon'\circ\mu'$, we get that $\varepsilon\circ\varepsilon'$ is the identity (being equal to the first component of the ε/μ-factorization of an arrow in $\mathscr{M}_0$, namely $\langle\sigma_\varepsilon,\tau_\varepsilon\rangle\circ(\alpha_\mu\times\beta_\mu)$). This can happen only if ε itself (which is a projection) is in fact identity (up to a renaming); we thus established that $\langle\sigma_\varepsilon,\tau_\varepsilon\rangle$ belongs to $\mathscr{M}_0$—which means that

$(*)'$ $\langle\sigma_\varepsilon,\tau_\varepsilon\rangle$ is *a diagonal*

(this is clear as $\sigma_\varepsilon,\tau_\varepsilon$ are both projections). From $\langle\alpha,\beta\rangle_e\circ\sigma_\varepsilon=\alpha_e$ and $\langle\alpha,\beta\rangle_e\circ\tau_\varepsilon=\beta_e$, we get

$$\langle\alpha_e,\beta_e\rangle=\langle\alpha,\beta\rangle_e\circ\langle\sigma_\varepsilon,\tau_\varepsilon\rangle$$

As first component is in $\mathscr{E}_i$ and second component is in $\mathscr{M}_0$, we get by uniqueness of factorization,

$$\langle \alpha_e, \beta_e \rangle_e = \langle \alpha, \beta \rangle_e$$

and

$$(*)'' \quad \langle \alpha_e, \beta_e \rangle_\mu = \langle \sigma_\varepsilon, \tau_\varepsilon \rangle$$

which gives the claim (combined with $\langle \alpha, \beta \rangle_\mu = \langle \sigma_\varepsilon, \tau_\varepsilon \rangle \circ (\alpha_\mu \times \beta_\mu)$ coming from $(*)$). □

The above lemma guarantees that there is no need in the local confluence proof to compute superpositions between rule $(\mathrm{R}_\mu)^+$ and the other rules ($(\mathrm{R}_\mu)^+$ itself included): it is sufficient to compute superpositions between (R_μ) and the other rules.[14] Using $(\mathrm{R}_\mu)^+$ instead of (R_μ) allows us to apply a less restrictive rule during confluence proofs; this makes some passages shorter (the only little price we pay for that is that we shall need to prove termination of $(\mathrm{R}_\mu)^+$ too). The next corollary will be used in Section 9 and is a slightly more accurate reformulation of what comes from the proof of Lemma 7.1: recall that, according to $(*)'$ and $(*)''$, the third step was in fact an (R_μ)-step moving to the right a diagonal (we call such (R_μ)-steps "diagonalization" steps):

Lemma 7.2. *Let* $(\mathrm{R}_\mu)^{+1}$ *be the following special case of rule* $(\mathrm{R}_\mu)^+$:

$$(\mathrm{R}_\mu)^{+1} \quad \langle a, \alpha \rangle, \beta \Rightarrow \langle a_e, \alpha \rangle, (a_\mu \times 1) \circ \beta.$$

If $K \Rightarrow K'$ *by a single* (R_μ) *or* $(\mathrm{R}_\mu)^+$*-rewrite step, then* K *rewrites to* K' *by using a finite number of* $(\mathrm{R}_\mu)^{+1}$*-rewrite steps followed by a single diagonalization step.*

In words, the e/μ factorization of $\langle a_1, \ldots, a_n \rangle$ is obtained by taking the componentwise e/μ factorizations and then by applying a diagonalization step. The following fact is useful:

Lemma 7.3. *If* $\langle e_1^i, \ldots, e_n^i \rangle \in \mathscr{E}_i$, *then the* e_j^i *are pairwise distinct.*

Proof. As $\mathscr{E}_i$ is closed under composition with projections, all e_j^i are in $\mathscr{E}_i$. Let $\langle e_{j_1}^i, \ldots, e_{j_m}^i \rangle$ be a list of distinct arrows containing exactly all the arrows among $e_1^i, \ldots, e_n^i$. By the previous Lemma, $\langle e_{j_1}^i, \ldots, e_{j_m}^i \rangle \in \mathscr{E}_i$. According to the definition of $\langle e_{j_1}^i, \ldots, e_{j_m}^i \rangle$, there is a diagonal δ such that

$$\langle e_{j_1}^i, \ldots, e_{j_m}^i \rangle \circ \delta = \langle e_1^i, \ldots, e_n^i \rangle.$$

[14] If $K \Rightarrow K'$ and $K \Rightarrow K''$ give rise to the critical pair (K', K'') and, say, $K \Rightarrow K'$ is a $(\mathrm{R}_\mu)^+$-step, we can find K_0 such that $K' \Rightarrow_{\mathscr{R}^+}^+ K_0$ and the pair (K_0, K'') is a critical pair generated by rule (R_μ) (instead of rule $(\mathrm{R}_\mu)^+$).

As $\delta \in \mathscr{M}_0$, by uniqueness of e/μ-factorizations, δ is a renaming (thus showing the claim). □

Corollary 7.4. *$\alpha^i \in \mathscr{E}_i$ iff the components of α^i are pairwise distinct and all belong to $\mathscr{E}_i$.*

A consequence of the above results is that e/μ-factorizations are stable under certain pullbacks, in the sense of the following:

Lemma 7.5. *If $\alpha : Y_1 \to Y_2$ has factorization $\alpha_e \circ \alpha_\mu$, then for every Z, $\alpha \times 1_Z$ has factorization $(\alpha_e \times 1_Z) \circ (\alpha_\mu \times 1_Z)$.*

Proof. It is sufficient to show that the components of $\pi_{Y_1} \circ \alpha_e$ cannot be equal to the components of π_Z. This is clear, otherwise we would have in our theories provable equations of the kind $t = x_i$, where t is a term not containing the variable x_i: this cannot be, otherwise (after making the term t a ground term by a substitution, if you like) we would obtain degeneration, i.e. that all terms are provably equal. □

We now show that rule $(\mathrm{R}^i_{\mathrm{p}})$ also can be roughly achieved by finitely many $(\mathrm{R}^i_{\mathrm{p}})^+$-rewrite steps. Let us use *the notation $K \searrow L$ in order to express that there is K' such that $K \Rightarrow^*_{\mathscr{R}^+} K'$ and $K' \Leftrightarrow^*_{\mathscr{R}_0} L$.*

Lemma 7.6. *Let L be the path*

$$L = Y \xrightarrow{\langle \gamma^i, \delta^i \rangle} Y_1 \times Z \xrightarrow{\langle \alpha, \pi_Z \rangle} Y_2 \times Z \xrightarrow{\beta^i} U \xrightarrow{(\lambda)} V$$

(*where the arrow λ is missed if $i = 2$*) *and let R, R' be the following paths*:

$$R = Y \xrightarrow{\langle \gamma^i, \delta^i, \delta^i_\varepsilon \rangle} Y_1 \times Z \times Y' \xrightarrow{\alpha \times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times \delta^i_m) \circ \beta^i} U \xrightarrow{(\lambda)} V,$$

$$R' = Y \xrightarrow{\langle \gamma^i, \delta^i, 1_Y \rangle} Y_1 \times Z \times Y \xrightarrow{\alpha \times 1_Y} Y_2 \times Y \xrightarrow{(1_{Y_2} \times \delta^i) \circ \beta^i} U \xrightarrow{(\lambda)} V$$

(*where we supposed that Y' is the codomain of δ_ε*). *We have*:

(i) $R \Leftrightarrow^*_{\mathscr{R}_0} R'$;
(ii) *If $\delta^i \in \mathbf{T_0}$, then $L \Leftrightarrow^*_{\mathscr{R}_0} R$;*
(iii) *In the general case, $L \searrow R$ (and consequently $L \searrow R'$).*

Proof. (i) and (ii) are easy. (iii) is proved by induction on the number of components of δ. If such a number is 1, there is nothing to prove (because either $(\mathrm{R}^i_{\mathrm{p}})^+$ or (ii) applies). So suppose it is bigger than 1. If $\delta \in \mathbf{T_0}$, we just proved a stronger claim; otherwise L and R (up to an alphabetic variant) are

$$(1) \quad Y \xrightarrow{\langle \gamma, \delta, d \rangle} Y_1 \times Z \times X \xrightarrow{\langle \alpha, \pi_Z, \pi_X \rangle} Y_2 \times Z \times X \xrightarrow{\beta} U \xrightarrow{(\lambda)} V$$

and

$$(2) \quad Y \xrightarrow{\langle\gamma,\delta,d,\langle\delta,d\rangle_\varepsilon\rangle} Y_1 \times Z \times X \times Y' \xrightarrow{\alpha\times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2}\times\langle\delta,d\rangle_m)\circ\beta} U \xrightarrow{(\lambda)} V,$$

respectively (with $d \notin \mathbf{T_0}$). To the former, we can apply a $(\mathrm{R}^i_\mathrm{p})^+$-rewrite step thus getting (we leave (λ) as understood in displayed paths from now on)

$$(3) \quad Y \xrightarrow{\langle\gamma,\delta,d,d_\varepsilon\rangle} Y_1 \times Z \times X \times Y_0'' \xrightarrow{\langle\alpha,\pi_Z\rangle\times 1_{Y_0''}} Y_2 \times Z \times Y_0'' \xrightarrow{(1_{Y_2}\times 1_Z\times d_m)\circ\beta} U$$

(where we called Y_0'' the codomain of d_ε). By induction hypothesis, there is path K'' such that (3) $\Rightarrow^*_{\mathscr{R}^+} K''$ and $K'' \Leftrightarrow^*_{\mathscr{R}_0}$ (4), where (4) is the path (let Y_0' be the codomain of δ_ε):

$$Y \xrightarrow{\langle\gamma,\delta,d,\delta_\varepsilon,d_\varepsilon\rangle} Y_1 \times Z \times X \times Y_0' \times Y_0'' \xrightarrow{\alpha\times 1_{Y_0'}\times 1_{Y_0''}} Y_2 \times Y_0' \times Y_0'' \xrightarrow{(1_{Y_2}\times\delta_m\times d_m)\circ\beta} U.$$

As $\langle\gamma,\delta,d,\delta_\varepsilon,d_\varepsilon\rangle$ is equal to $\langle\gamma,\delta,d,1_Y\rangle \circ (1_{Y_1}\times 1_Z \times 1_X \times \langle\delta_\varepsilon,d_\varepsilon\rangle)$, we can move right $1_{Y_1}\times 1_Z\times 1_X\times\langle\delta_\varepsilon,d_\varepsilon\rangle$ by $\Leftrightarrow^*_{\mathscr{R}_0}$-equivalence, thus getting from (4) the path

$$(5) \quad Y \xrightarrow{\langle\gamma,\delta,d,1_Y\rangle} Y_1 \times Z \times X \times Y \xrightarrow{\alpha\times 1_Y} Y_2 \times Y \xrightarrow{(1_{Y_2}\times\langle\delta,d\rangle)\circ\beta} U,$$

which we know from (i) that it is $\Leftrightarrow^*_{\mathscr{R}_0}$-equivalent to (2). In conclusion, we have

$$(1) \Rightarrow_{\mathscr{R}^+} (3) \Rightarrow^*_{\mathscr{R}^+} K'' \Leftrightarrow^*_{\mathscr{R}_0} (4) \Leftrightarrow^*_{\mathscr{R}_0} (5) \Leftrightarrow^*_{\mathscr{R}_0} (2)$$

thus showing the claim. □

We need a final lemma for the next Section (the proof is left to the reader):

Lemma 7.7. *We have $R_1 \searrow R_2$, where R_1, R_2 are the paths*

$$R_1 = Y \times Z_1 \xrightarrow{\langle\alpha^i_1,\alpha^i_2\rangle\times 1} Y_1 \times Y_2 \times Z_1 \xrightarrow{\pi_{Y_1}\times\beta^j} Y_1 \times Z_2 \xrightarrow{\gamma^i} W \xrightarrow{(\lambda)} V,$$

$$R_2 = Y \times Z_1 \xrightarrow{1\times\beta^j} Y \times Z_2 \xrightarrow{(\alpha^i_1\times 1)\circ\gamma^i} W \xrightarrow{(\lambda)} V$$

(λ *is missed in case* $i=2$).

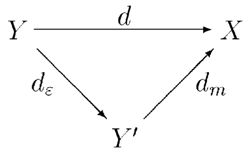
8. Local confluence, II

In this section, we prove that $\mathscr{R}^+$ is locally confluent. In order to show confluence of a pair of paths (R_1, R_2), *we shall use the following schema*: *we find L_1, L_2 such that $R_1 \searrow L_1$ and $R_2 \searrow L_2$ and $L_1 \Leftrightarrow^*_{\mathscr{R}_0} L_2$*. Canonicity of $\mathscr{R}_0$ (which was proved in Section 6) guarantees that in such a condition R_1, R_2 are joinable.

Throughout this section we shall mention arrows $\gamma, d, \alpha, \beta, \theta, \lambda$ whose domains and codomains are fixed as follows:

$$Y \stackrel{\langle\gamma,d\rangle}{\to} Y_1 \times X \stackrel{\langle\alpha,\pi_X\rangle}{\to} Y_2 \times X \stackrel{\beta}{\to} U \stackrel{\theta}{\to} V \stackrel{\lambda}{\to} T.$$

We also assume that d factorizes in ε/m-components as follows:



We first analyse some situations which are very frequent during local confluence proofs.

Lemma 8.1. *Let* K_i $(i \in \{1,2\})$ *be the following path*:

$$K_i = Y \xrightarrow{\langle\gamma^i,d^i,d^i_\varepsilon\rangle} Y_1 \times X \times Y' \xrightarrow{\alpha^j \times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times d^i_m)\circ\beta^0\circ\theta^i} V \stackrel{(\lambda)}{\to} T$$

(*where* λ *lacks in case* $i=2$). *Then*:

(i) *The path*

$$K_i' = Y \stackrel{\langle\gamma^i,d^i\rangle}{\longrightarrow} Y_1 \times X \xrightarrow{(\langle\alpha^j,\pi_X\rangle\circ\beta^0)_e} W \xrightarrow{(\langle\alpha^j,\pi_X\rangle\circ\beta^0)_\mu\circ\theta^i} V \stackrel{(\lambda)}{\to} T,$$

is joinable with K_i *in* $\mathscr{R}^+$.

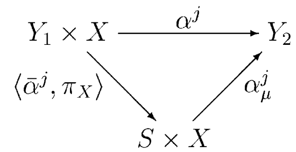
(ii) *The path*

$$K_i'' = Y \stackrel{\langle\gamma^i,d^i\rangle}{\longrightarrow} Y_1 \times X \xrightarrow{\langle\alpha^j,\pi_X\rangle\circ\beta^0} U \stackrel{\theta^i}{\to} V \stackrel{(\lambda)}{\to} T,$$

is joinable with K_i *in* $\mathscr{R}^+$.

Proof. (ii) is trivially reduced to (i) (just apply (R_μ) in K_i'' to decompose $\langle\alpha^j,\pi_X\rangle \circ \beta^0$).

To prove (i), we have to factorize the arrow $\langle\alpha^j,\pi_X\rangle \circ \beta^0$ in components e/μ. We first factorize $\langle\alpha^j,\pi_X\rangle$: by Lemmas 7.2 and 7.3, such a factorization is obtained by first factorizing α_j in e/μ components and then diagonalizing with π_X in case π_X appears among the components of α^j_e. We leave to the reader the easier case in which π_X is not among the components of α^j_e; so let α^j have the following factorization in e/μ-components:



It follows that the diagram in Fig. 4 is commutative, thus

$$(1) \quad \langle \alpha^j, \pi_X \rangle = \langle \bar{\alpha}^j, \pi_X \rangle \circ (1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X).$$

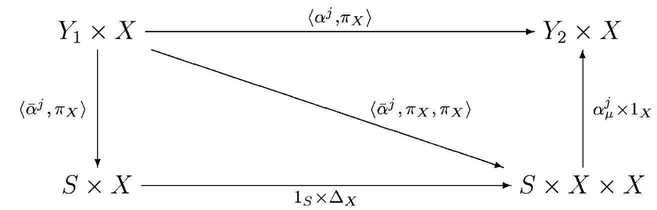


Fig. 4. Commutative diagram.

We have two cases, depending on whether π_X appears in the ε-component of $(1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0$ or not (again the easier negative case is left to the reader); let $(1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0$ factorize in $\mathbf{T_0}$ as in Fig. 5.

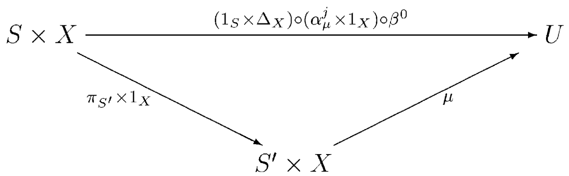


Fig. 5. Factorization of $(1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0$.

By (1), it follows that

$$\langle \alpha^j, \pi_X \rangle \circ \beta^0 = \langle \bar{\alpha}^j, \pi_X \rangle \circ (\pi_{S'} \times 1_X) \circ \mu.$$

By the fact that $\langle \bar{\alpha}^j, \pi_X \rangle \circ (\pi_{S'} \times 1_X)$ belongs to $\mathscr{E}_j$ and by the uniqueness of decomposition we have:

$$(\langle \alpha^j, \pi_X \rangle \circ \beta^0)_e = \langle \bar{\alpha}^j, \pi_X \rangle \circ (\pi_{S'} \times 1_X) = \langle \bar{\alpha}^j \circ \pi_{S'}, \pi_X \rangle,$$
$$(\langle \alpha^j, \pi_X \rangle \circ \beta^0)_\mu = \mu.$$

It follows that K'_i coincides with the path (we leave arrow (λ) as understood in displayed paths)

$$Y \xrightarrow{\langle \gamma^i, d^i \rangle} Y_1 \times X \xrightarrow{\langle \bar{\alpha}^j \circ \pi_{S'}, \pi_X \rangle} S' \times X \xrightarrow{\mu \circ \theta^i} V.$$

We can apply Lemma 7.6(iii) (in fact, if $i = 1$, the arrow λ belongs to the path) and we obtain that $K'_i \searrow L_1$, where L_1 is the path

$$(L_1) \quad Y \xrightarrow{\langle \gamma^i, d^i, 1_Y \rangle} Y_1 \times X \times Y \xrightarrow{(\bar{\alpha}^j \circ \pi_{S'}) \times 1_Y} S' \times Y \xrightarrow{(1_{S'} \times d^i) \circ \mu \circ \theta^i} V.$$

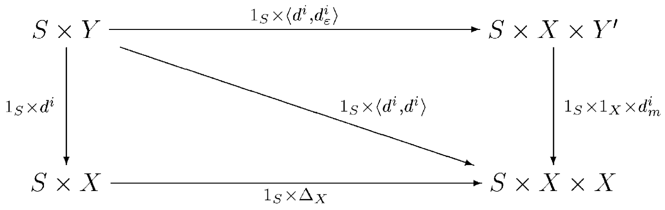


Fig. 6. Commutative diagram.

Let us consider K_i. We first observe that $\alpha^j \times 1_{Y'}$ can be decomposed in e/μ components as $(\alpha^j_e \times 1_{Y'}) \circ (\alpha^j_\mu \times 1_{Y'})$ by Lemma 7.5; therefore, an application of (R_μ) (which, we recall, is a special case of $(\mathrm{R}_\mu)^+$) yields

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \rangle} Y_1 \times X \times Y' \xrightarrow{\alpha^j_e \times 1_{Y'}} S \times X \times Y' \xrightarrow{(\alpha^j_\mu \times 1_{Y'}) \circ (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i} V$$
$$=$$
$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \rangle} Y_1 \times X \times Y' \xrightarrow{\langle \pi \circ \bar{\alpha}^j, \pi_X, \pi_{Y'} \rangle} S \times X \times Y' \xrightarrow{(\alpha^j_\mu \times 1_{Y'}) \circ (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i} V,$$

where $Y_1 \times X \times Y' \xrightarrow{\pi} Y_1 \times X$.

We can apply Lemma 7.6(iii) on $\langle d^i, d^i_\varepsilon \rangle$ and we get (recall that if $i = 1$, arrow λ belongs to the path):

$$K_i \searrow$$
$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle} Y_1 \times X \times Y' \times Y \xrightarrow{(\pi \circ \bar{\alpha}^j) \times 1_Y} S \times Y \xrightarrow{(1_S \times \langle d^i, d^i_\varepsilon \rangle) \circ (\alpha^j_\mu \times 1_{Y'}) \circ (1_{Y_2} \times d^i_m) \circ \beta^0 \circ \theta^i} V$$
$$=$$
$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle} Y_1 \times X \times Y' \times Y \xrightarrow[= \text{(see Fig. 6).}]{(\pi \circ \bar{\alpha}^j) \times 1_Y} S \times Y \xrightarrow{(1_S \times \langle d^i, d^i_\varepsilon \rangle) \circ (1_S \times 1_X \times d^i_m) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0 \circ \theta^i} V$$
$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle} Y_1 \times X \times Y' \times Y \xrightarrow[= \text{(see } \textit{Fig.}\ 5)]{(\pi \circ \bar{\alpha}^j) \times 1_Y} S \times Y \xrightarrow{(1_S \times d^i) \circ (1_S \times \Delta_X) \circ (\alpha^j_\mu \times 1_X) \circ \beta^0 \circ \theta^i} V$$
$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle} Y_1 \times X \times Y' \times Y \xrightarrow{(\pi \circ \bar{\alpha}^j) \times 1_Y} S \times Y \xrightarrow{(1_S \times d^i) \circ (\pi_{S'} \times 1_X) \circ \mu \circ \theta^i} V$$
$$=$$
$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle} Y_1 \times X \times Y' \times Y \xrightarrow[\Leftrightarrow^*_{\mathscr{R}_0}]{(\pi \times 1_Y) \circ (\bar{\alpha}^j \times 1_Y)} S \times Y \xrightarrow{(\pi_{S'} \times 1_Y) \circ (1_{S'} \times d^i) \circ \mu \circ \theta^i} V$$
$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon, 1_Y \rangle \circ (\pi \times 1_Y)} Y_1 \times X \times Y \xrightarrow{(\bar{\alpha}^j \times 1_Y) \circ (\pi_{S'} \times 1_Y)} S' \times Y \xrightarrow{(1_{S'} \times d^i) \circ \mu \circ \theta^i} V$$

which coincides with (L_1), and this proves (i). □

Lemma 8.2. *Let K_j $(j \in \{1, 2\})$ be the following path*:

$$K_j = Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \rangle} Y_1 \times X \times Y' \xrightarrow{\alpha^j \times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times d^i_m) \circ \beta^0} U \xrightarrow{\theta^j} V \xrightarrow{(\lambda)} T$$

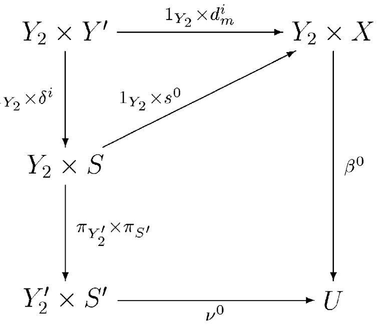


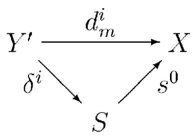
Fig. 7. Factorization of $1_{Y_2} \times d^i_m$.

(*where* λ *lacks in case* $j=2$). *Then the path*

$$K_i''' = Y \xrightarrow{\langle \gamma^i, d^i \rangle} Y_1 \times X \xrightarrow{\langle \alpha^j, \pi_X \rangle \circ \beta^0 \circ \theta^j} V \xrightarrow{(\lambda)} T,$$

is joinable with K_j *in* $\mathscr{R}^+$.

Proof. Here, we cannot apply the products rule on K_i''', therefore, we have to act on K_j; thus we have to decompose $(1_{Y_2} \times d^i_m) \circ \beta^0$ in e/μ components. Suppose that the e/μ-components of d^i_m are



Then by Lemma 7.5:

$$(1_{Y_2} \times d^i_m)_e = 1_{Y_2} \times \delta^i,$$
$$(1_{Y_2} \times d^i_m)_\mu = 1_{Y_2} \times s^0.$$

We decompose $(1_{Y_2} \times d^i_m) \circ \beta^0$ as in Fig. 7.

Since $1_{Y_2} \times \delta^i$ belongs to $\mathscr{E}_i$, we can state that

$$((1_{Y_2} \times d^i_m) \circ \beta^0)_e = (1_{Y_2} \times \delta^i) \circ (\pi_{Y_2'} \times \pi_{S'}) = \pi_{Y_2'} \times (\delta^i \circ \pi_{S'}),$$
$$((1_{Y_2} \times d^i_m) \circ \beta^0)_\mu = \nu^0.$$

By (R_μ), K_j rewrites to the following path (hereafter, we will leave out the last arrow (λ)).

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \rangle} Y_1 \times X \times Y' \xrightarrow{\alpha^j \times 1_{Y'}} Y_2 \times Y' \xrightarrow{\pi_{Y_2'} \times (\delta^i \circ \pi_{S'})} Y_2' \times S' \xrightarrow{\nu^0 \circ \theta^j} V.$$

Lemma 7.7 yields (by a $\searrow$-step): [15]

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \rangle} Y_1 \times X \times Y' \xrightarrow{1_{Y_1} \times 1_X \times (\delta^i \circ \pi_{S'})} Y_1 \times X \times S' \xrightarrow{((\alpha^j \circ \pi_{Y_2'}) \times 1_{S'}) \circ v^0 \circ \theta^j} V$$

$$\Leftrightarrow^*_{\mathscr{R}_0}$$

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \rangle} Y_1 \times X \times Y' \xrightarrow{1_{Y_1} \times 1_X \times \delta^i} Y_1 \times X \times S \xrightarrow{(1_{Y_1} \times 1_X \times \pi_{S'}) \circ ((\alpha^j \circ \pi_{Y_2'}) \times 1_{S'}) \circ v^0 \circ \theta^j} V$$

$$\Leftrightarrow^*_{\mathscr{R}_0}$$

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \circ \delta^i \rangle} Y_1 \times X \times S \xrightarrow{(\alpha^j \times 1_S) \circ (\pi_{Y_2'} \times \pi_{S'}) \circ v^0 \circ \theta^j} V$$

$$= (\text{see } \textit{Fig. } 7)$$

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \circ \delta^i \rangle} Y_1 \times X \times S \xrightarrow{(\alpha^j \times 1_S) \circ (1_{Y_2} \times s^0) \circ \beta^0 \circ \theta^j} V$$

$$=$$

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \circ \delta^i \rangle} Y_1 \times X \times S \xrightarrow{(1_{Y_1} \times 1_X \times s^0) \circ (\alpha^j \times 1_X) \circ \beta^0 \circ \theta^j} V$$

$$\Leftrightarrow^*_{\mathscr{R}_0}$$

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i_\varepsilon \circ \delta^i \rangle \circ (1_{Y_1} \times 1_X \times s^0)} Y_1 \times X \times X \xrightarrow{(\alpha^j \times 1_X) \circ \beta^0 \circ \theta^j} V$$

$$=$$

$$Y \xrightarrow{\langle \gamma^i, d^i, d^i \rangle} Y_1 \times X \times X \xrightarrow{(\alpha^j \times 1_X) \circ \beta^0 \circ \theta^j} V$$

$$=$$

$$Y \xrightarrow{\langle \gamma^i, d^i \rangle \circ (1_{Y_1} \times \Delta_X)} Y_1 \times X \times X \xrightarrow{(\alpha^j \times 1_X) \circ \beta^0 \circ \theta^j} V$$

$$\Leftrightarrow^*_{\mathscr{R}_0}$$

$$Y \xrightarrow{\langle \gamma^i, d^i \rangle} Y_1 \times X \xrightarrow{(1_{Y_1} \times \Delta_X) \circ (\alpha^j \times 1_X) \circ \beta^0 \circ \theta^j} V$$

which, with the addition of (λ), coincides with K_i'''. □

Theorem 8.3. *$\mathscr{R}^+$ is locally confluent.*

Proof. To prove local confluence of $\mathscr{R}^+$, by Section 6 results, it suffices to study the superpositions between the rule $(\mathrm{R}^i_\mathrm{p})^+$ and the other rules, itself included (see also the observation following the proof of Lemma 7.1). There are 4 superpositions among $(\mathrm{R}^i_\mathrm{p})^+$ and each of the rules $(\mathrm{R}^j_\mathrm{c})$, (R_ε) and (R_μ); in addition, there are 3 superpositions among $(\mathrm{R}^i_\mathrm{p})^+$ and itself. For space reasons, we only consider some cases. The relevant tools for confluence are provided by Lemmas 7.6, 8.1 and 8.2.

Example of superposition between $(\mathrm{R}^i_\mathrm{p})^+$ and $(\mathrm{R}^j_\mathrm{c})$. We have a path of three arrows $\theta_1, \theta_2, \theta_3$ and we apply $(\mathrm{R}^j_\mathrm{c})$ on θ_2, θ_3 and $(\mathrm{R}^i_\mathrm{p})^+$ on the whole path. Everything composes if $i = j$; otherwise θ_3 must belong to $\mathbf{T_0}$. As $i \neq j$, either $i = 1$ or $j = 1$, hence the table of rules of $\mathscr{R}^+$ requires in any case a fourth arrow θ_4 (θ_4, in its turn, must be followed *in a well-coloured path* by a further arrow λ in case θ_4 belongs to $\mathbf{T_1} \backslash \mathbf{T_0}$).

[15] We have a projection $\pi_{Y_2'} : Y_2 \to Y_2'$, hence α^j must be a pair (of vectors), whose component having codomain Y_2' is obviously $\alpha^j \circ \pi_{Y_2'}$.

We have

$$\langle \gamma^i, d^i \rangle, \langle \alpha^j, \pi_X \rangle, \beta^0, \theta, (\lambda)$$

$$\swarrow (R_c^j) \qquad\qquad (R_p^i)^+ \searrow$$

$$\langle \gamma^i, d^i \rangle, \langle \alpha^j, \pi_X \rangle \circ \beta^0, \theta, (\lambda) \qquad\qquad \langle \gamma^i, d^i, d_\varepsilon^i \rangle, \alpha^j \times 1_{Y'}, (1_{Y_2} \times d_m^i) \circ \beta^0, \theta, (\lambda)$$

If $\theta \in \mathbf{T_j} \backslash \mathbf{T_0}$, we compose $\langle \alpha^j, \pi_X \rangle \circ \beta^0$ with θ and then apply Lemma 8.2. If $\theta \in \mathbf{T_i} \backslash \mathbf{T_0}$, we compose $(1_{Y_2} \times d_m^i) \circ \beta^0$ with θ and the confluence immediately follows by Lemma 8.1(ii). If $\theta \in \mathbf{T_0}$, we can in any case apply one of the two previous solutions (because either i or j must be 2, hence lack of λ does not matter).

Example of superposition between $(\mathrm{R}_\mathrm{p}^i)^+$ and $(\mathrm{R}_\mathrm{c}^j)$. We have a path of four arrows $\theta_1, \theta_2, \theta_3, \theta_4$ and we apply $(\mathrm{R}_\mathrm{c}^j)$ on θ_3, θ_4 and $(\mathrm{R}_\mathrm{p}^i)^+$ on $\theta_1, \theta_2, \theta_3$. Suppose $j = i$; that is:

$$\langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^i, \theta^i, (\lambda)$$

$$\swarrow (R_c^i) \qquad\qquad (R_p^i)^+ \searrow$$

$$\langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^i \circ \theta^i, (\lambda) \qquad\qquad \langle \gamma^i, d^i, d_\varepsilon^i \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d_m^i) \circ \beta^i, \theta^i, (\lambda)$$

Then we can reduce both first member (by $(\mathrm{R}_\mathrm{p}^i)^+$) and second member (by $(\mathrm{R}_\mathrm{c}^i)^+$) to the path

$$\langle \gamma^i, d^i, d_\varepsilon^i \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d_m^i) \circ \beta^i \circ \theta^i, (\lambda).$$

Suppose that $i \neq j$; in this case $\theta_3 \in \mathbf{T_0}$ and we have

$$\langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^0, \theta^j, (\lambda)$$

$$\swarrow (R_c^j) \qquad\qquad (R_p^i)^+ \searrow$$

$$\langle \gamma^i, d^i \rangle, \langle \alpha, \pi_X \rangle, \beta^0 \circ \theta^j, (\lambda) \qquad\qquad \langle \gamma^i, d^i, d_\varepsilon^i \rangle, \alpha \times 1_{Y'}, (1_{Y_2} \times d_m^i) \circ \beta^0, \theta^j, (\lambda)$$

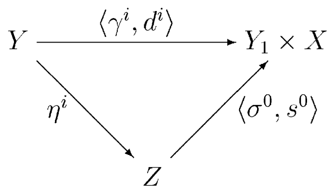
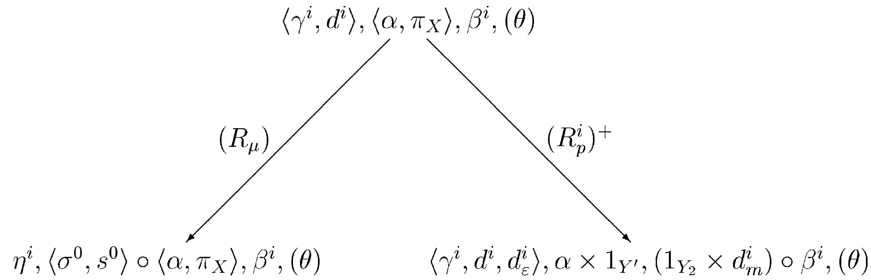


Fig. 8. Factorization of $\langle \gamma^i, d^i \rangle$.

If $\alpha \in \mathbf{T_i}$, the two members are $\Leftrightarrow^*_{\mathscr{R}_0}$-equivalent. The relevant case is when $\alpha \in \mathbf{T_j}$: here we can rewrite first member by $(\mathrm{R}^j_\mathrm{c})$ to $\langle \gamma^i, d^i \rangle, \langle \alpha^j, \pi_X \rangle \circ \beta^0 \circ \theta^j, (\lambda)$ and then we apply Lemma 8.2.

Example of superposition between $(\mathrm{R}^i_\mathrm{p})^+$ and (R_μ). We have three arrows, (R_μ) is applied to the first two and $(\mathrm{R}^i_\mathrm{p})^+$ to the whole path:

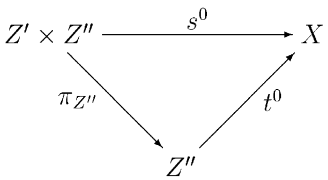


where we suppose $\langle \gamma^i, d^i \rangle$ to factorize in components e/μ as in Fig. 8.

We apply $(\mathrm{R}_\mu)^+$ on the second member to the component $\langle \gamma^i, d^i \rangle$ of $\langle \gamma^i, d^i, d^i_\varepsilon \rangle$ and we obtain (we leave arrow θ out of displayed paths)

$$Y \xrightarrow{\langle \eta^i, d^i_\varepsilon \rangle} Z \times Y' \xrightarrow{(\langle \sigma^0, s^0 \rangle \times 1_{Y'}) \circ (\alpha \times 1_{Y'})} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times d^i_m) \circ \beta^i} U$$
$$=$$
$$Y \xrightarrow{\langle \eta^i, d^i_\varepsilon \rangle} Z \times Y' \xrightarrow[\Leftrightarrow^*_{\mathscr{R}_0} \quad \text{(by Lemma 7.6(i) and (ii))}]{(\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times d^i_m) \circ \beta^i} U$$
$$Y \xrightarrow{\langle \eta^i, 1_Y \rangle} Z \times Y \xrightarrow{(\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_Y} Y_2 \times Y \xrightarrow{(1_{Y_2} \times d^i_\varepsilon) \circ (1_{Y_2} \times d^i_m) \circ \beta^i} U$$
$$=$$
$$Y \xrightarrow{\langle \eta^i, 1_Y \rangle} Z \times Y \xrightarrow{(\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_Y} Y_2 \times Y \xrightarrow{(1_{Y_2} \times d^i) \circ \beta^i} U. \qquad (L_1)$$

We need to factorize s^0 in components ε/μ in $\mathbf{T_0}$.



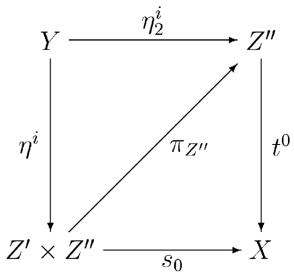


Fig. 9. Commutative diagram.

where $Z' \times Z'' = Z$. This implies that η^i has the form $\langle \eta^i_1, \eta^i_2 \rangle$, where $Y \xrightarrow{\eta^i_1} Z'$ and $Y \xrightarrow{\eta^i_2} Z''$. By applying $(\mathrm{R}_\mu)^+$ on the first member to the arrow $\langle \sigma^0, s^0 \rangle \circ \langle \alpha, \pi_X \rangle = \langle \langle \sigma^0, s^0 \rangle \circ \alpha, s^0 \rangle$, in order to decompose s^0, we obtain

$$Y \xrightarrow{\langle \eta^i_1, \eta^i_2 \rangle} Z' \times Z'' \xrightarrow{\langle \langle \sigma^0, s^0 \rangle \circ \alpha, \pi_{Z''} \rangle} Y_2 \times Z'' \xrightarrow{(1_{Y_2} \times t^0) \circ \beta^i} U,$$

which, by Lemma 7.6(iii), becomes (through a $\searrow$-step)

$$\begin{gathered} Y \xrightarrow{\langle \eta^i_1, \eta^i_2, 1_Y \rangle} Z' \times Z'' \times Y \xrightarrow{(\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_Y} Y_2 \times Y \xrightarrow{(1_{Y_2} \times \eta^i_2) \circ (1_{Y_2} \times t^0) \circ \beta^i} U \\ = \\ Y \xrightarrow{\langle \eta^i, 1_Y \rangle} Z' \times Z'' \times Y \xrightarrow{(\langle \sigma^0, s^0 \rangle \circ \alpha) \times 1_Y} Y_2 \times Y \xrightarrow{(1_{Y_2} \times \eta^i_2 \circ t^0) \circ \beta^i} U. \end{gathered} \tag{L_2}$$

Since the diagrams in Figs. 9 and 8 are commutative, we have

$$\eta^i_2 \circ t^0 = \eta^i \circ s^0 = d^i$$

and this implies that (L_2) coincides with (L_1). □

9. Termination

In order to show termination of $\mathscr{R}$ and of $\mathscr{R}^+$, we shall associate with our paths certain commutative labelled trees. Such trees are represented as terms built up from the countable set of variables $\{x_i\}_{i \geqslant 1}$ by using four[16] constructors f_i $(i \in \{0,1\}^2)$ of type $\mathit{TermMultiset} \rightarrow \mathit{Term}$.

$\mathscr{R}$-trees (or, briefly, trees) are inductively defined as follows:

- x_i is an $\mathscr{R}$-tree for every $i \geqslant 1$;
- if $\{T_1, \ldots, T_n\}$ is a multiset of $\mathscr{R}$-trees and $i \in \{0,1\}^2$, then $f_i(T_1, \ldots, T_n)$ is an $\mathscr{R}$-tree.

As a next step, we introduce a relation $>$ among our trees; we have $T_1 > T_2$ iff one of the following two conditions is satisfied:

- T_1 is $f_i(T'_1, \ldots, T'_n)$, T_2 is $f_j(T''_1, \ldots, T''_k)$ and $\{T'_1, \ldots, T'_n\} >_m \{T''_1, \ldots, T''_k\}$;

[16] Actually, only three such constructors will be really used ($f_{\langle 0,1 \rangle}$ is useless).

- T_1' is $f_i(T_1',\dots,T_n')$ and T_2 is $f_j(T_1',\dots,T_n')$ and $i>j$ (in the lexicographic sense).

Some comments are in order. First $>_m$ is the multiset extension of $>$; secondly, the definition is by induction on the height $h(T_1)$ of the tree T_1. It is easily seen that $T_1>T_2$ implies $h(T_1)\geqslant h(T_2)$. In the following, we use $\geqslant$ for the reflexive closure of $>$.

We have the following

Lemma 9.1. *$>$ is a transitive and terminating relation.*

As our trees are represented as terms, it makes sense to speak about substitutions. Substitutions are compatible with $>$ in the following sense:

Lemma 9.2. *Let a succession $\{T_i\}_{i\geqslant 1}$ of trees be given and let T',T'' be such that $T'>T''$; we then have $T'(T_i/x_i)>T''(T_i/x_i)$.*

Let us now turn to our paths. First, we need a definition. For an arrow α^i, let us put

$$e(\alpha^i)=\begin{cases}0 & \text{if } \alpha^i\in\mathscr{E}_0,\\ 1 & \text{otherwise,}\end{cases}\qquad m(\alpha^i)=\begin{cases}0 & \text{if } \alpha^i\in\mathscr{E}_i,\\ 1 & \text{otherwise,}\end{cases}$$

$$\chi(\alpha^i)=\langle m(\alpha^i),e(a^i)\rangle$$

Lemma 9.3. *For every arrow α and for every $\varepsilon\in\mathscr{E}_0$, we have $\chi(\varepsilon\circ\alpha)=\chi(\alpha)$ (whenever composition makes sense).*

Proof. If $e(\alpha)=0$ then clearly $e(\varepsilon\circ\alpha)=0$ too; vice versa, if $e(\varepsilon\circ\alpha)=0$, then the two ε/m factorizations $(\varepsilon\circ\alpha)\circ 1=(\varepsilon\circ\alpha_\varepsilon)\circ\alpha_m$ of $\varepsilon\circ\alpha$ must be equal so that α_m is the identity; hence $\alpha=\alpha_\varepsilon$, that is $\alpha\in\mathscr{E}_0$. The proof of $m(\alpha)=0$ iff $m(\varepsilon\circ\alpha)=0$ is similar. □

For a path $K:Y\to Z$ and for $\beta^0:Z\to V$, let $K\circ\beta^0$ be the path obtained by composing the last arrow of K with β^0 (that is, if $K=K',\alpha$, then $K\circ\beta$ is $K',\alpha\circ\beta^0$).

With a path $K:X^n\to X$ (resp. $L:X^n\to X^m$), we now associate an $\mathscr{R}$-tree $T(K)$ (resp. a multiset of $\mathscr{R}$-trees $T(L)$) as follows (definition is by induction on the lengths $|K|$, $|L|$ of K and L):

$$T(a)=f_{\chi(a)}(x_{i_1},\dots,x_{i_k}),\quad \text{if } a_\varepsilon=\langle\pi_{i_1},\dots,\pi_{i_k}\rangle;$$

$$T(\langle a_1,\dots,a_m\rangle)=\{T(a_1),\dots,T(a_m)\};$$

$$T(K',a)=f_{\chi(a)}(T(K'\circ a_\varepsilon));$$

$$T(L',\langle a_1,\dots,a_m\rangle)=\{T(L',a_1),\dots,T(L',a_m)\}.$$

Lemma 9.4. *Let* $L : Y \to X^n$ *and* $K : X^n \to X^m$. *We have that*

$$T(L,K) = T(K)(T(L_1)/x_1, \dots, T(L_n)/x_n),$$

where $L_1 = L \circ \pi_1, \dots, L_n = L \circ \pi_n$.

From Lemma 9.4 it is possible to establish that $T(K) = T(K')$, in case K' is an alphabetic variant of K.

Lemma 9.5. *Let* $\delta = \langle d_1, \dots, d_n \rangle : X^m \to X^n$ *be an arrow which is not in* $\mathscr{E}_0$ (*i.e. it is not a projection*); *suppose that* $\delta_\varepsilon = \langle \pi_{i_1}, \dots, \pi_{i_k} \rangle : X^m \to X^k$. *We have that* $T(\delta, 1_{X^n}) > T(\delta_\varepsilon, 1_{X^k})$.

Proof. We have

$$T(\delta_\varepsilon, 1_{X^k}) = \{ f_{\langle 0,0 \rangle}(f_{\langle 0,0 \rangle}(x_s)) \}_{s=i_1,\dots,i_k}$$

and

$$T(\delta, 1_{X^n}) = \{ f_{\langle 0,0 \rangle}(f_{\chi(d_j)}(x_{i_{j(1)}}, \dots, x_{i_{j(l_j)}})) \}_{j=1,\dots,n},$$

where we supposed that $(d_j)_\varepsilon = \langle \pi_{i_{j(1)}}, \dots, \pi_{i_{j(l_j)}} \rangle$. Now elements of the former multiset are all distinct and for every $s = i_1, \dots, i_k$, there is j such that s is among $j(1), \dots, j(l_j)$ (otherwise π_s would be missed in δ_ε). This means in particular that for such s, j we have $f_{\langle 0,0 \rangle}(x_s) \leqslant f_{\chi(d_j)}(x_{i_{j(1)}}, \dots, x_{i_{j(l_j)}})$ (where this inequality is strict in case the same j corresponds to different s). Consequently, the former multiset is less than or equal to the latter. It is strictly less indeed; in fact, δ cannot be in $\mathscr{E}_0$ for two independent reasons: some of the $\chi(d_j)$ is not $\langle 0, 0 \rangle$ or some projection among $\langle \pi_{i_1}, \dots, \pi_{i_k} \rangle$ appears at least twice in δ. In both cases, this is a sufficient reason for the latter multiset to be bigger. □

For a path $K = \alpha_1, \dots, \alpha_k$, we define $c(K)$ to be the vector

$$\langle T(\alpha_1, \dots, \alpha_k), T(\alpha_1, \dots, \alpha_{k-1}), \dots, T(\alpha_1) \rangle$$

and for paths K, L, we put

$$K > L \quad \text{iff} \quad c(K) > c(L)$$

where second member refers to the lexicographic extension of $>_m$. The next lemma says that c is "almost stable by concatenation" as a complexity measure:

Lemma 9.6. *Let* $K : X^m \to X^n$ *and* $K' : X^m \to X^n$ *be two paths such that* $K > K'$ (*notice that they agree on domains and codomains*); *then*

(i) *for every path* L *having codomain* X^m, *we have* $L, K > L, K'$;

(ii) *suppose that* $K = K_0, \langle a_1, \dots, a_n \rangle$, $K' = K'_0, \langle a'_1, \dots, a'_n \rangle$ *and that* $T(K_0, a_i) \geqslant T(K'_0, a'_i)$ *holds for all* $i = 1, \dots, n$; *then for every path* R *having domain* X^n, *we have* $K, R > K', R$.

Proof. By Lemmas 9.4, 9.2 and by induction on $|R|$. □

Theorem 9.7. *$\mathscr{R}$ and $\mathscr{R}^+$ are terminating.*

Proof. If we have $K \Rightarrow K'$ by rules (R_c^i), then $K > K'$ always holds, because such rules are length-reducing (recall that in lexicographic orders for variable length vectors, length is the principal parameter).

According to the above lemma, it is sufficient to show that for every other rule $L \Rightarrow R$ of $\mathscr{R}^+ \cup \mathscr{R}$, we have both

$$(1) \quad T(L \circ \pi_i) \geqslant T(R \circ \pi_i)$$

for every $i = 1, \ldots, n$ (here, X^n is the common codomain of L, R) and

$$(2) \quad c(L) > c(R).$$

Notice that any (R_ε)-rewrite step is a special case of an (Rpr)*-rewrite step, where (Rpr)* is the rewrite rule

$$(\text{Rpr})^* \quad \alpha, \varepsilon \circ \beta \Rightarrow \alpha \circ \varepsilon, \beta$$

(here ε is any projection which is not a renaming). Moreover, we know from Lemma 7.2 that any (R_μ) or $(R_\mu)^+$-rewrite step is a composition of a finite number of $(R_\mu)^{+1}$ and of $(\text{Rdi}^{+1})^*$-rewrite steps, where $(R_\mu)^{+1}$ is (any alphabetic variant of)

$$(R_\mu)^{+1} \quad \langle \alpha, a \rangle, \beta \Rightarrow \langle \alpha, a_e \rangle, (1 \times a_\mu) \circ \beta$$

and $(\text{Rdi}^{+1})^*$ is (any alphabetic variant of)

$$(\text{Rdi}^{+1})^* \qquad \langle \alpha, a, a \rangle, \beta \Rightarrow \langle \alpha, a \rangle, (1 \times \Delta_X) \circ \beta.$$

Consequently, it is sufficient to prove (1) and (2) for rules (Rpr)*, $(R_\mu)^{+1}$, $(\text{Rdi}^{+1})^*$ and (R_p^i). We show the argument for (Rpr)* and (R_p^i) and leave the remaining cases to the reader.[17]

Proof of (1) *for rule* (Rpr)*:

$$\alpha, \varepsilon \circ \beta \Rightarrow \alpha \circ \varepsilon, \beta.$$

Let b be any component of β; as $(\varepsilon \circ b_\varepsilon) \circ b_m$ is the factorization of $\varepsilon \circ b$, we have (taking into account Lemma 9.3):

$$T(\alpha, \varepsilon \circ b) = f_{\chi(b)}(T(\alpha \circ \varepsilon \circ b_\varepsilon)) = T(\alpha \circ \varepsilon, b),$$

as required.

Proof of (2) *for rule* (*Rpr*)*: By the previous point, we have $T(\alpha, \varepsilon \circ \beta) = T(\alpha \circ \varepsilon, \beta)$; however, $T(\alpha) > T(\alpha \circ \varepsilon)$ because the projection is strict.

[17] We only observe that the second clause in the definition of order for trees is used to deal with $(R_\mu)^{+1}$.

Notice that the above established fact that $T(\alpha, \varepsilon \circ \beta)$ and $T(\alpha \circ \varepsilon, \beta)$ are componentwise equal (together with Lemma 9.4), yields the following important information to be used in the sequel: let us write $K \Rightarrow_{\varepsilon}^{*} K'$ in order to express that K' is obtained from K by a sequence of (Rpr)*-rewrite steps; we have that

$$(*) \quad K \Rightarrow_{\varepsilon}^{*} K' \quad \text{implies} \quad T(K) = T(K').$$

Proof of (1) *for rule* (R_p^i): We recall that the first member of (R_p^i) is

$$Y \xrightarrow{\langle \gamma, \delta \rangle} Y_1 \times Z \xrightarrow{\langle \alpha, \pi_Z \rangle} Y_2 \times Z \xrightarrow{\beta} U,$$

whereas the second member is

$$Y \xrightarrow{\langle \gamma, \delta, \delta_{\varepsilon} \rangle} Y_1 \times Z \times Y' \xrightarrow{\alpha \times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times \delta_m) \circ \beta} U$$

(with an extra arrow to the right in case $i = 1$). This rule is subject to the proviso that δ cannot be a projection. Let b be any component of β; we first assume that b_{ε} is the identity (and then reduce to this case). We have that

$$\begin{aligned} T(\langle \gamma, \delta \rangle, \langle \alpha, \pi_Z \rangle, b) &= f_{\chi(b)}(T(\langle \gamma, \delta \rangle, \langle \alpha, \pi_Z \rangle)) \\ &= f_{\chi(b)}(T(\langle \gamma, \delta \rangle, \alpha) \cup T(\delta, 1_Z)), \end{aligned}$$

where $\cup$ refers to multiset union (notice that we used $(*)$ above in the missed intermediate passages). We do not know what is $((1 \times \delta_m) \circ b)_{\varepsilon}$: let us then consider the worst case (it is identity) and proceed as follows by using $(*)$ again:

$$\begin{aligned} T(\langle \gamma, \delta, \delta_{\varepsilon} \rangle, \alpha \times 1, (1 \times \delta_m) \circ b) &\leqslant f_{\chi((1 \times \delta_m) \circ b)}(T(\langle \gamma, \delta, \delta_{\varepsilon} \rangle, \alpha \times 1)) \\ &= f_{\chi((1 \times \delta_m) \circ b)}(T(\langle \gamma, \delta \rangle, \alpha) \cup T(\delta_{\varepsilon}, 1_{Y'})). \end{aligned}$$

This tree is indeed smaller than $f_{\chi(b)}(T(\langle \gamma, \delta \rangle, \alpha) \cup T(\delta, 1_Z))$ (by the first clause of the definition of trees order): in fact, by Lemma 9.5 we have $T(\delta, 1_Z) > T(\delta_{\varepsilon}, 1_{Y'})$.

Let us now turn to the general case (b_{ε} may not be identity). In such a case, let us transform both

$$Y \xrightarrow{\langle \gamma, \delta \rangle} Y_1 \times Z \xrightarrow{\langle \alpha, \pi_Z \rangle} Y_2 \times Z \xrightarrow{b} X$$

and

$$Y \xrightarrow{\langle \gamma, \delta, \delta_{\varepsilon} \rangle} Y_1 \times Z \times Y' \xrightarrow{\alpha \times 1_{Y'}} Y_2 \times Y' \xrightarrow{(1_{Y_2} \times \delta_m) \circ b} X,$$

by $\Rightarrow_{\varepsilon}^{*}$-rewriting and then apply $(*)$. Suppose that we have $Y_2 = Y_2' \times Y_2''$ and $Z = Z' \times Z''$ (consequently, δ and α are also splitted as δ', δ'' and α', α'', respectively); let b factor as follows:

$$Y_2' \times Y_2'' \times Z' \times Z'' \xrightarrow{b_{\varepsilon}} Y_2'' \times Z'' \xrightarrow{b_m} X,$$

where b_{ε} is the obvious projection. We then have for the first member

$$\langle \gamma, \delta \rangle, \langle \alpha, \pi_Z \rangle, b \Rightarrow_{\varepsilon}^{*} \langle \gamma, \delta', \delta'' \rangle, \langle \alpha'', \pi_{Z''} \rangle, b_m.$$

Let us also split $\delta_m : Y' \to Z' \times Z''$ as θ', θ'' (as a consequence, from $\langle \delta', \delta'' \rangle = \delta = \delta_\varepsilon \circ \delta_m$, we have in particular $\delta_\varepsilon \circ \theta'' = \delta''$); an analogous transformation on the second member gives

$$\langle \gamma, \delta, \delta_\varepsilon \rangle, \alpha \times 1, (1 \times \delta_m) \circ b \Rightarrow_\varepsilon^* \langle \gamma, \delta', \delta'', \delta_\varepsilon \rangle, \alpha'' \times 1, (1 \times \theta'') \circ b_m.$$

Let us now factorize $\theta'' = \theta''_\varepsilon \circ \theta''_m$; from $\delta_\varepsilon \circ \theta'' = \delta''$, by uniqueness of factorizations, we get $\delta''_\varepsilon = \delta_\varepsilon \circ \theta''_\varepsilon$ and $\delta''_m = \theta''_m$; thus, by further $\Rightarrow_\varepsilon^*$-rewrite steps, we get

$$\langle \gamma, \delta', \delta'', \delta_\varepsilon \rangle, \alpha'' \times 1, (1 \times \theta'') \circ b_m \Rightarrow_\varepsilon^* \langle \gamma, \delta', \delta'', \delta''_\varepsilon \rangle, \alpha'' \times 1, (1 \times \delta''_m) \circ b_m.$$

Now

$$\langle \gamma, \delta', \delta'' \rangle, \langle \alpha'', \pi_{Z''} \rangle, b_m$$

and

$$\langle \gamma, \delta', \delta'', \delta''_\varepsilon \rangle, \alpha'' \times 1, (1 \times \delta''_m) \circ b_m,$$

are the first and second member of an $(\mathrm{R}^i_\mathrm{p})$-rewrite rule and $(b_m)_\varepsilon$ is the identity. We can thus reduce the above particular case, except that now there is no guarantee that δ'' is not a projection: this further case has to be considered separately. However, in such a case, $1 \times \delta''_m$ is the identity, $\delta''_\varepsilon = \delta''$ and all that we need is to prove that trees corresponding to the paths

$$Y \xrightarrow{\langle \gamma, \delta', \delta'' \rangle} Y_1 \times Z' \times Z'' \xrightarrow{\langle \alpha'', \pi_{Z''} \rangle} Y''_2 \times Z'',$$

$$Y \xrightarrow{\langle \gamma, \delta', \delta'', \delta'' \rangle} Y_1 \times Z' \times Z'' \times Z'' \xrightarrow{\alpha'' \times 1} Y''_2 \times Z'',$$

are the same. Indeed, they are both equal to $T(\langle \gamma, \delta', \delta'' \rangle, \alpha'') \cup T(\delta'', 1_{Z''})$ (again by $(*)$).

Proof of (2) *for rule* $(\mathrm{R}^i_\mathrm{p})$: By the previous point, we have that the multiset of trees corresponding to the first member of the rule is greater or equal to the multiset of trees corresponding to the second member. This does not prevent them from being equal, in some cases; in such cases it is sufficient to observe that

$$T(\langle \gamma, \delta \rangle, \langle \alpha, \pi_Z \rangle) > T(\langle \gamma, \delta, \delta_\varepsilon \rangle, \alpha \times 1_{Y'})$$

by Lemma 9.5. □

From the previous section results, we immediately get:

Corollary 9.8. *$\mathscr{R}^+$ is canonical.*

We now compare rewrite systems $\mathscr{R}^+$ and $\mathscr{R}$: it will turn out that they are essentially the same, hence in particular, canonicity of $\mathscr{R}$ will follow.

Lemma 9.9. *If $K \Rightarrow^*_{\mathscr{R}^+} K'$, then there exists K'' such that $K' \Rightarrow^*_{\mathscr{R}^+} K''$ and $K \Rightarrow^*_{\mathscr{R}} K''$.*

Proof. The statement is proved by noetherian induction on K (with respect to the order $>$ among paths which has been used in the termination proof), by using Lemma 7.1 and confluence of $\mathscr{R}^+$. $\square$

Lemma 9.10. *If* $K \Rightarrow^*_{\mathscr{R}} K'$, *then* $K \Leftrightarrow^*_{\mathscr{R}^+} K'$.

Proof. The statement is again proved by noetherian induction on K. The only relevant case is when we have $K \Rightarrow_{\mathscr{R}} K'$ by a single $(\mathrm{R}^i_{\mathrm{p}})$-rewrite step, which is covered by Lemma 7.6(iii). $\square$

We can finally complete the

Proof of Theorem 5.3. As we know from Proposition 9.7 that $\mathscr{R}$ is terminating, we only have to prove its confluence. If we have that $K \Rightarrow^*_{\mathscr{R}} K'$ and $K \Rightarrow^*_{\mathscr{R}} K''$, then $K' \Leftrightarrow^*_{\mathscr{R}^+} K''$ by Lemma 9.10; as $\mathscr{R}^+$ is canonical, K' and K'' both $\Rightarrow^*_{\mathscr{R}^+}$-rewrite to their common normal form N. Now it is sufficient to apply Lemma 9.9. $\square$

10. Examples

In this Section, we illustrate our results in concrete cases. First, we gave in Section 5 a definition of constructibility for theories referring to their associated Lawvere categories. Now we give a useful equivalent purely symbolic definition:

Proposition 10.1. *A theory* $T' = \langle \Omega', Ax' \rangle$ *is constructible over a theory* $T = \langle \Omega, Ax \rangle$ *iff* T' *is a conservative extension of* T *and there exists a class* E' *of* Ω'*-terms such that*:

(i) E' *contains the variables and is closed under renamings of terms*;

(ii) *for every* Ω'*-term* $t(x_1, \dots, x_n)$ *there are a* k*-minimized* Ω*-term* $u(x_1, \dots, x_k)$ *and pairwise distinct* (*with respect to provable identity in* T') Ω'*-terms*

$$v_1(x_1, \dots, x_n), \dots, v_k(x_1, \dots, x_n)$$

belonging to E' *such that*

$$\vdash_{T'} t = u(v_1, \dots, v_k);$$

(iii) *whenever* u, u' *are* k (*resp.* k')*-minimized* Ω*-terms and we have*

$$\vdash_{T'} u(v_1, \dots, v_k) = u'(v'_1, \dots, v'_{k'})$$

for pairwise distinct (*wrt* T'*-provability*) *terms* $v_1, \dots, v_k \in E'$ *and pairwise distinct* (*wrt* T'*-provability*) *terms* $v'_1, \dots, v'_{k'} \in E'$, *then* $k = k'$ *and there is a permutation* σ *acting on the* k*-elements set, such that*

$$\vdash_{T'} v'_{\sigma(i)} = v_i \ (i = 1, \dots, k) \quad \textit{and} \quad \vdash_T u' = u(x_{\sigma(i)}/x_i).$$

Proof. We give the relevant hints and leave the details to the reader. If T' is constructible over T, in $\mathbf{T}'$ there is a left extension $(\mathscr{E}', \mathscr{M})$ of the standard weak factorization system $(\mathscr{E}, \mathscr{M})$ of $\mathbf{T}$. In order to find E' fulfilling the above requirements it is sufficient to take the set of terms $t(x_1, \ldots, x_n)$ such that the equivalence class of t (seen as an arrow $X^n \to X$ in $\mathbf{T}'$) belongs to $\mathscr{E}'$.

Vice versa, suppose that a class E' of Ω'-terms fulfilling the above requirements is given. We define a left extension $(\mathscr{E}', \mathscr{M})$ of the standard weak factorization system $(\mathscr{E}, \mathscr{M})$ of $\mathbf{T}$ by taking as $\mathscr{E}'$ the set of arrows $\langle e_1, \ldots, e_m \rangle : X^n \to X^m$ such that the e_i are represented by distinct (up to provable identity in T') terms in E'. □

We say that T' is *effectively constructible* over T iff it is constructible over T and moreover for every term t, terms $u, v_1, \ldots, v_k$ satisfying (ii) above are provided by a total recursive function. As an immediate corollary to our main Theorem 5.3, we have:

Theorem 10.2. *Suppose that T_1, T_2 are both effectively constructible over T_0 and that word problems for T_1, T_2 are solvable; then word problem for $T_1 +_{T_0} T_2$ is solvable too.*

Proof. By Theorem 3.1, Lemma 5.2 and Theorem 5.3, it is sufficient to observe that applicability of rules of $\mathscr{R}$ is effective whenever a path is given as a list of terms, representing their respective equivalence classes (in order to be able to compare normal forms, we also need the obvious fact that it is effectively recognizable whether two paths are alphabetic variants of each other).

For rules $(\mathrm{R}_\mathrm{c}^i)$ we need to be able to recognize whether a certain arrow i comes from $\mathbf{T_0}$: this happens iff $\alpha_e \in \mathscr{E}_0$ (by uniqueness of e/μ factorization and by the fact that $\mathscr{E}_0 \subseteq \mathscr{E}_i$), a fact which is effective by appealing to the solvability of word problem for T_i.[18] For rule (R_ε) we already observed in Section 5 that ε-extraction is effective in case word problem is decidable. For rule (R_μ), one just uses effective constructibility, together with the fact that the e/μ factorization of $\langle a_1, \ldots, a_n \rangle$ can be reduced to the e/μ factorization of components, see Lemma 7.2. Finally, in order to apply rules $(\mathrm{R}_\mathrm{p}^i)$ (and checking the relative proviso) it is sufficient to be able to recognize projections, a fact which is reduced once again to solvability of the input word problems.

Finally, we show that it is effectively recognizable whether two paths are alphabetic variants of each other. In case they are both in normal form (which is the relevant case), there is a quick procedure for that. First, for $\alpha_1, \ldots, \alpha_k$ to be an alphabetic variant of $\beta_1, \ldots, \beta_{k'}$ we need $k = k'$; secondly, as the components of α_1 and β_1 are distinct (because paths are in normal form and (R_μ) does not apply), it is easily computed—provided it exists—the renaming ρ_1 such that $\alpha_1 \circ \rho_1 = \beta_1$; at this point, we recursively need to check whether $\rho_1^{-1} \circ \alpha_2, \ldots, \alpha_k$ is an alphabetic variant of $\beta_2, \ldots, \beta_k$ and so on. □

[18] Clearly if the term t represents $a : X^n \to X$, then a is a projection iff t *collapses to* (i.e. it is provably equal to) a variable x_i (for $i = 1, \ldots, n$); a similar observation applies to a vector of terms.

Example 1. Commutative rings with unit are constructible over abelian groups. In fact, terms $t(x_1,\dots,x_n)$ in the theory of abelian groups can be represented as homogeneous linear polynomials in the indeterminates $x_1,\dots,x_n$ with integer coefficients (they are minimized iff no coefficient is zero); terms in the theory of commutative rings with unit can be represented as arbitrary polynomials with integer coefficients. Class E' needed for constructibility is formed by monic monomials (1 included): in fact, every integer polynomial can be uniquely expressed as a linear combination (with integer non-zero coefficients) of distinct monic monomials.

Example 2. Let T be the theory of join-semilattices with zero and let T' be the theory of semilattice-monoids we had seen in the Introduction. T' is constructible over T: class E' is given by terms of the form $x_{i_1} \circ \cdots \circ x_{i_k}$ (for $k \geqslant 0$).

Example 3. The theory of abelian groups endowed with an endomorphism f is constructible over the theory of abelian groups: class E' is given by terms of the form $f^n(x_i)$ (for $n \geqslant 0$).

Example 4. Differential rings (i.e. rings endowed with a differentiation operator ∂ satisfying usual laws for derivatives of sums and products) are constructible over commutative rings with unit: class E' is given by terms of the form $\{\partial^k x_i\}$ (for $k \geqslant 0$).

Notice that in the above examples the smaller theory *is not collapse-free*. Additional examples of different nature can be found in [3, 4]. In order to build counterexamples, a useful tool is given in the following proposition (clearly inspired from [3]):

Proposition 10.3. *If T' is constructible over T, then the T-reduct of any free T'-algebra is a free T-algebra (on a bigger set of generators).*

Proof. Let $F_{T'}(G)$ be the free T'-algebra on the set G of generators; we show that its T-reduct is free over the set of elements of the form $u(g_1,\dots,g_n)$, where $u(x_1,\dots,x_n) \in E'$ and $g_1,\dots,g_n$ are distinct elements from G. Clearly, the claim follows from the case in which G is finite. To have a quick proof we translate everything in the terminology of functorial semantics.

Let $(\mathscr{E},\mathscr{M})$ be the standard weak factorization system of $\mathbf{T}$ and let $(\mathscr{E}',\mathscr{M})$ be its left extension to $\mathbf{T}'$. For any functor F having domain $\mathbf{T}'$ let us call $|F|$ its restriction to $\mathbf{T}$; for any type Y let $\mathscr{E}'(Y,X)$ be $\mathbf{T}'(Y,X) \cap \mathscr{E}'$. Fix a type Y and a T-algebra $A : \mathbf{T} \to \mathbf{Set}$; we need to find a bijective natural correspondence between set-theoretic functions

$$\bar{N} : \mathscr{E}'(Y,X) \to A(X)$$

and natural transformations

$$N : |\mathbf{T}'(Y,-)| \to A.$$

Given N, let $\bar{N}$ be the restriction of N_X to $\mathscr{E}'(Y,X)$ in the domain. Conversely, if $\bar{N}$ is given, we define for every Z and $\alpha : Y \to Z$

$$N_Z(\alpha) = A(\alpha_\mu)(\bar{N}(e_1),\ldots,\bar{N}(e_k)),$$

where $\alpha_e = \langle e_1,\ldots,e_k\rangle$. □

Counterexample 5. Boolean algebras are not constructible over join-semilattices with zero. In fact, the free join-semilattice with zero over an infinite set G of generators is just the set of finite subsets of G; in this algebra, clearly the strict part of the partial order relation associated with the join is terminating. It is not so however in the countably generated free Boolean algebra, which is atomless.

Counterexample 6. Modal algebras (also $K4$-modal algebras, interior algebras, diagonalizable algebras, etc.) are not constructible over Boolean algebras: in fact, in such varieties, finitely generated free algebras are atomic and infinite,[19] whereas free Boolean algebras are either finite or atomless.

Proposition 10.3 can be inverted, thus giving another characterization of constructibility:[20]

Proposition 10.4. *Let T' be a conservative extension of T. We have that T' is constructible over T iff the T-reduct of any T'-free algebra $F_{T'}(G')$ is a free T-algebra over a set of generators G such that*
(i) $G' \subseteq G$;
(ii) *G is invariant under the T'-isomorphisms of $F_{T'}(G')$ which are the extension of a bijection on the set of free generators G'.*

Proof. The "only if" side is covered by Proposition 10.3 and its proof. For the "if" side, take as G' a countable set like $\{g_1, g_2, \ldots\}$. Let E' be the set of terms $e(x_1,\ldots,x_n)$ such that $e(g_1/x_1,\ldots,g_n/x_n) \in G$ (here we made a slight abuse of notation, clearly $e(g_1/x_1,\ldots,g_n/x_n)$ means the result of the function interpreting the term e in $F_{T'}(G')$ applied to $g_1,\ldots,g_n$). Notice that for all $a \in G$ there is $e(x_1,\ldots,x_n) \in E'$ such that $a = e(g_1,\ldots,g_n)$. We show that E' matches all requirements from Proposition 10.1.

Clearly E' is closed under renamings and contains variables by (i) and (ii). Let us first show uniqueness of factorizations. Suppose that we have k (resp. k')-minimized terms (in the signature of T) u, u' and that we have

$$(1) \qquad \vdash_{T'} u(v_1,\ldots,v_k) = u'(v'_1,\ldots,v'_{k'})$$

for pairwise distinct (wrt T'-provability) terms $v_1,\ldots,v_k \in E'$ and pairwise distinct (wrt T'-provability) terms $v'_1,\ldots,v'_{k'} \in E'$. Notice that if two terms in E' are distinct (wrt T'-provability) and if we "replace" in them the variables x_j by the corresponding free

[19] These are well-known results. For a proof making use of normal forms, see [8].
[20] It is an interesting question whether there exists a more conceptual characterization of constructibility (e.g. in terms of monads).

generators g_j, then we get distinct elements of $F_{T'}(G')$. Let $w_1,\dots,w_s$ be the terms which are common to the lists $v_1,\dots,v_k$ and $v'_1,\dots,v'_{k'}$. For simplicity, let us also rearrange such lists as

$$v_1,\dots,v_k = w_1,\dots,w_s,r_1,\dots,r_l,$$

$$v'_1,\dots,v'_{k'} = w_1,\dots,w_s,r'_1,\dots,r'_{l'}.$$

Let us call $a_1,\dots,a_s,b_1,\dots,b_l,b'_1,\dots,b'_{l'}$ the elements of $F_{T'}(G')$ which we get by, respectively, "replacing" in $w_1,\dots,w_s,r_1,\dots,r_l,r'_1,\dots,r'_{l'}$ the variables by the corresponding free generators. From (1), we get

$$u(a_1,\dots,a_s,b_1,\dots,b_l) = u'(a_1,\dots,a_s,b'_1,\dots,b'_{l'});$$

as the $a_i,b_j,b'_{j'}$'s are all distinct elements of G which freely generates the T-reduct, we can abstract them by distinct variables thus getting

$$\vdash_T u(x_1,\dots,x_s,y_1,\dots,y_l) = u'(x_1,\dots,x_s,z_1,\dots,z_{l'}),$$

which cannot be (unless $l = l' = 0$, yielding what we need) because u and u' are minimized.

Let us now show the existence of factorizations. Take any T'-term $t(x_1,\dots,x_n)$; as the T-reduct of $F_{T'}(G')$ is free over G, there is T-term $s(x_1,\dots,x_k)$ and $a_1,\dots,a_k \in G$ such that $t(g_1,\dots,g_n) = s(a_1,\dots,a_k)$. Without loss of generality, we can furthermore assume that $a_1,\dots,a_k$ are distinct and that s is k-minimized. As $a_1,\dots,a_k$ are distinct and in G, there are pairwise distinct (up to T'-provability) terms $r_1,\dots,r_k \in E'$ such that (we suppose that $r_1,\dots,r_k$ contain at most the variables $x_1,\dots,x_n,\dots,x_{n+m}$)

$$a_j = r_j(g_1,\dots,g_n,\dots,g_{n+m})$$

for all $j = 1,\dots,k$. Being the g_i's free generators, we get

$$\vdash_{T'} t(\vec{x}) = s(r_1(\vec{x},\vec{y}),\dots,r_k(\vec{x},\vec{y})),$$

where we used the abbreviations $\vec{x}$ for $x_1,\dots,x_n$ and $\vec{y}$ for $x_{n+1},\dots,x_{n+m}$. Although s is minimized and $r_1,\dots,r_k$ are distinct terms from E', this is not yet good, because we must eliminate the extra variables $\vec{y}$ (they are not in principle allowed by Proposition 10.1(ii)). Let $\vec{z}$ be a renaming of $\vec{y}$ (away from $\vec{y}$); we get

$$\vdash_{T'} t(\vec{x}) = s(r_1(\vec{x},\vec{z}),\dots,r_k(\vec{x},\vec{z}))$$

hence,

$$\vdash_{T'} s(r_1(\vec{x},\vec{y}),\dots,r_k(\vec{x},\vec{y})) = s(r_1(\vec{x},\vec{z}),\dots,r_k(\vec{x},\vec{z})).$$

As s is minimized and $r_1(\vec{x},\vec{y}),\dots,r_k(\vec{x},\vec{y})$ (consequently, even $r_1(\vec{x},\vec{z}),\dots,r_k(\vec{x},\vec{z})$) are pairwise distinct up to provable identity in T', uniqueness of factorization just proved denotes that we must have

$$\vdash_{T'} r_i(\vec{x}, \vec{y}) = r_i(\vec{x}, \vec{z})$$

for all $i = 1, \dots, k$ (possibly up to a permutation). Replacing all the $\vec{y}$ by ground terms $\vec{c}$ (we may use the same ground term for all of them), we get

$$\vdash_{T'} r_i(\vec{x}, \vec{c}) = r_i(\vec{x}, \vec{z}) = r_i(\vec{x}, \vec{y}).$$

Now $r_i(\vec{x}, \vec{c})$ is provably equal to $r_i(\vec{x}, \vec{y})$, hence as the latter is in E' so the former is (E' is closed under provably identical terms according to its definition). For the same reason, all the $r_i(\vec{x}, \vec{c})$ are pairwise distinct (with respect to provable identity in T') because the $r_i(\vec{x}, \vec{y})$ are. We finally get

$$\vdash_{T'} t(\vec{x}) = s(r_1(\vec{x}, \vec{c}), \dots, r_k(\vec{x}, \vec{c}))$$

which is a factorization matching all the requirements from Proposition 10.1(ii). □

Let us now give examples of normalization through our rewriting system $\mathscr{R}$. In order to apply normalization to paths of equivalence classes of terms, algebraic notation for rules must be converted into ordinary symbolic notation. This is not difficult (all needed information is contained in Section 2 above), however, some care is needed. Suppose that e.g. you want to apply products rule to the path

$$X^3 \xrightarrow{\langle t,u \rangle} X^2 \xrightarrow{\langle v,x_2 \rangle} X^2 \xrightarrow{w} X.$$

First $u(x_1, x_2, x_3)$ has to be minimized (this is the factorization $\delta = \delta_\varepsilon \circ \delta_m$ of Table 1 of Section 5). Suppose that it minimizes as $u'(x_1, x_3)$; the pair of projections $\langle x_1, x_3 \rangle$ stays in the first position, whereas $u'(x_1, x_2)$ is moved to the third position. However, the term moved to the last position for composition with $w(x_1, x_2)$ (the arrow $1 \times \delta_m$ of Table 1), requires a renaming away from x_1 and consequently it is the pair $\langle x_1, u'(x_2, x_3) \rangle$. Thus, the products rule rewrite step produces

$$X^3 \xrightarrow{\langle t,u,x_1,x_3 \rangle} X^4 \xrightarrow{\langle v,x_3,x_4 \rangle} X^3 \xrightarrow{w(x_1,u'(x_2,x_3))} X.$$

In the examples below, we consider the following theories, leaving the reader to check that T_1, T_2 are both constructible over T_0 (for the choice of appropriate E_1 and E_2 just imitate Examples 1 and 3 above):

$T_0 =$ *Abelian groups with period 2.*

$T_1 =$ *BBoolean rings.*

$T_2 = T_0 +$ *aan idempotent endomorphism* f (i.e. $f(f(x_1)) = f(x_1)$).

Example. Let us consider the following instance of word problem in the theory $T_1 +_{T_0} T_2$:

$$f(x_1 \cdot x_2 + x_2 + f(x_2)) \stackrel{?}{=} f(x_1 \cdot x_2).$$

Let us rewrite a splitting path of first member in $\mathscr{R}$.

$$X^2 \xrightarrow{\langle x_1,x_2,f(x_2)\rangle} X^3 \xrightarrow[\Downarrow_{R_\mu}\quad \text{(see (F1) of Table 4)}]{x_1\cdot x_2+x_2+x_3} X \xrightarrow{f(x_1)} X$$

$$X^2 \xrightarrow{\langle x_1,x_2,f(x_2)\rangle} X^3 \xrightarrow[\Downarrow_{(\mathrm{R}_\mathrm{p}^2)}]{\langle x_1\cdot x_2,x_2,x_3\rangle} X^3 \xrightarrow{f(x_1+x_2+x_3)} X$$

$$X^2 \xrightarrow{\langle x_1,x_2,f(x_2),x_2\rangle} X^4 \xrightarrow[\Downarrow_{\mathrm{R}_\varepsilon}\quad \text{(see (F2) of Table 4)}]{\langle x_1\cdot x_2,x_2,x_4\rangle} X^3 \xrightarrow{f(x_1+x_2+f(x_3))} X$$

$$X^2 \xrightarrow{\langle x_1,x_2,x_2\rangle} X^3 \xrightarrow[\Downarrow_{(\mathrm{R}_\mathrm{c}^1)}]{\langle x_1\cdot x_2,x_2,x_3\rangle} X^3 \xrightarrow{f(x_1+x_2+f(x_3))} X$$

$$X^2 \xrightarrow{\langle x_1\cdot x_2,x_2,x_2\rangle} X^3 \xrightarrow[\Downarrow_{\mathrm{R}_\mu}\quad \text{(see (F3) of Table 4)}]{f(x_1+x_2+f(x_3))} X$$

$$X^2 \xrightarrow{\langle x_1\cdot x_2,x_2\rangle} X^2 \xrightarrow[\Downarrow_{\mathrm{R}_\varepsilon}\quad \text{(see (F4) of Table 4)}]{f(x_1+x_2+f(x_2))} X$$

$$X^2 \xrightarrow{x_1\cdot x_2} X \xrightarrow{f(x_1)} X,$$

where the last path corresponds to the splitting path of the term $f(x_1\cdot x_2)$.

Example. Let us consider the following instance of word problem for $T_1 +_{T_0} T_2$:

$$f(x_1)\cdot f(x_2) + f(x_1)\cdot(f(x_1)+f(x_2)) \stackrel{?}{=} f(x_1).$$

We rewrite the first member as follows.

$$X^2 \xrightarrow[\Downarrow_{R_\mu}\quad \text{(see (F1) of Table 4)}]{\langle f(x_1),f(x_2),f(x_1)+f(x_2)\rangle} X^3 \xrightarrow{\langle x_1\cdot x_2,x_1\cdot x_3\rangle} X^2 \xrightarrow{x_1+x_2} X$$

$$X^2 \xrightarrow{\langle f(x_1),f(x_2)\rangle} X^2 \xrightarrow[=]{\langle x_1,x_2,x_1+x_3\rangle\circ\langle x_1\cdot x_2,x_1\cdot x_3\rangle} X^2 \xrightarrow{x_1+x_2} X$$

$$X^2 \xrightarrow{\langle f(x_1),f(x_2)\rangle} X^2 \xrightarrow[\Downarrow_{R_\mu}\quad \text{(see (F6) of Table 4)}]{\langle x_1\cdot x_2,x_1\cdot(x_1+x_2)\rangle} X^2 \xrightarrow{x_1+x_2} X$$

$$X^2 \xrightarrow{\langle f(x_1),f(x_2)\rangle} X^2 \xrightarrow[=]{\langle x_1\cdot x_2,x_1\rangle} X^2 \xrightarrow{\langle x_1,x_1+x_2\rangle\circ(x_1+x_2)} X$$

$$X^2 \xrightarrow{\langle f(x_1),f(x_2)\rangle} X^2 \xrightarrow[\Downarrow_{R_\varepsilon}\quad \text{(see (F7) of Table 4)}]{\langle x_1\cdot x_2,x_1\rangle} X^2 \xrightarrow{x_1+x_1+x_2} X$$

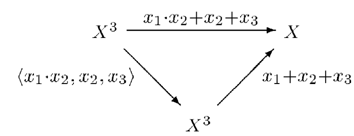
$$X^2 \xrightarrow{\langle f(x_1),f(x_2)\rangle} X^2 \xrightarrow[=]{\langle x_1\cdot x_2,x_1\rangle\circ x_2} X \xrightarrow{x_1} X$$

$$X^2 \xrightarrow{\langle f(x_1),f(x_2)\rangle} X^2 \xrightarrow[\Downarrow_{(R_c^2)}]{x_1} X \xrightarrow{x_1} X$$

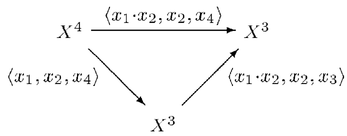
$$X^2 \xrightarrow{f(x_1)} X,$$

where the last path coincides with the second term of the problem.

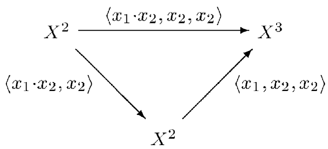
Table 4
Examples of factorizations



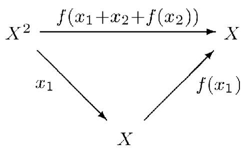
(F1) Factorization of $x_1 \cdot x_2 + x_2 + x_3$ according to $(\mathcal{E}_1, \mathcal{M}_0)$.



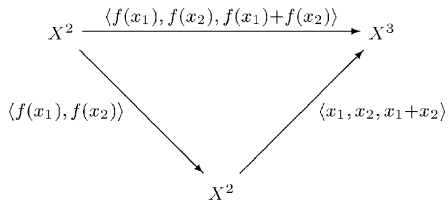
(F2) Factorization of $\langle x_1 \cdot x_2, x_2, x_4\rangle$ according to $(\mathcal{E}_0, \mathcal{M}_1)$.



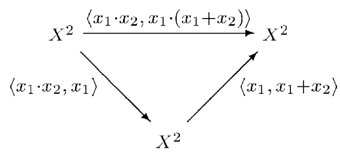
(F3) Factorization of $\langle x_1 \cdot x_2, x_2, x_2\rangle$ according to $(\mathcal{E}_1, \mathcal{M}_0)$.



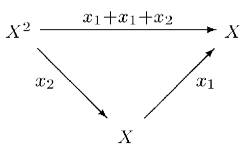
(F4) Factorization of $f(x_1 + x_2 + f(x_2))$ according to $(\mathcal{E}_0, \mathcal{M}_2)$.



(F5) Factorization of $\langle f(x_1), f(x_2), f(x_1) + f(x_2)\rangle$ according to $(\mathcal{E}_2, \mathcal{M}_0)$.



(F6) Factorization of $\langle x_1 \cdot x_2, x_1 \cdot (x_1 + x_2)\rangle$ according to $(\mathcal{E}_1, \mathcal{M}_0)$.



(F7) Factorization of $x_1 + x_1 + x_2$ according to $(\mathcal{E}_0, \mathcal{M}_1)$.

To conclude, let us mention some possible *directions for future research*. Of course, there is the problem of extending our results to combined unification. Secondly, one may try to generalize combined word problems to the case in which the definition of constructibility is related to a weak factorization system of the smaller theory which may not be the standard one (that is, class $\mathscr{E}_0$ is supposed to be larger than the class of projections). Results from Section 6 are still valid, however it is not clear what happens with critical pairs arising from superpositions with products rule. Such enlargements of the definition of constructibility are important because they could cover additional mathematically relevant examples. Finally, although quite difficult, it would be essential to be able to deal with theories extending $T_1 +_{T_0} T_2$ to further axioms. In principle, as our combination algorithm is obtained through rewriting, one may try to apply some form of Knuth–Bendix completion to get decision procedures in such situations too.

Acknowledgements

We wish to thank F. Baader and C. Tinelli for the fruitful e-mail correspondence we had with both of them after the submission of the present paper. We also thank the anonymous referee for his careful comments.

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