UNIVERSITÀ
DEGLI STUDI
DI MILANO

Dipartimento di Matematica Federigo Enriques
Scuola di Dottorato in Scienze Matematiche
Dottorato di Ricerca in Matematica – Ciclo XXIV

Exponential-type inequalities in $\mathbb{R}^n$ and applications to elliptic and biharmonic equations

Ph.D. Thesis

Advisor
Prof. Bernhard Ruf

Candidate
Federica Sani

Academic Year 2010/2011

Introduction

Adams' inequality [2] in its original form is nothing but the Trudinger-Moser inequality for Sobolev spaces involving higher order derivatives. In this Thesis we present Adams-type inequalities for unbounded domains in $\mathbb{R}^n$ and some applications to existence and multiplicity results for elliptic and biharmonic problems involving nonlinearities with exponential growth.

The Thesis is divided into two parts. Part I is devoted to the study of *higher order exponential-type inequalities in* $\mathbb{R}^n$ and Chapter 1 is an introduction to this part, containing a brief historical overview of Trudinger-Moser and Adams inequalities. In Chapter 2, we introduce a sharp Adams-type inequality in $\mathbb{R}^n$ proved in the following paper

- B. Ruf, F. Sani, *Sharp Adams-type inequalities in* $\mathbb{R}^n$, Trans. Amer. Math. Soc. to appear

In view of applications to a class of biharmonic problems in $\mathbb{R}^4$, in Chapter 3, we focus our attention in the 4-dimensional case proving some consequences of the Adams-type inequality in $\mathbb{R}^4$.

Part II is devoted to *applications of Trudinger-Moser and Adams inequalities to elliptic and biharmonic equations*. Chapter 4 is a review of past developments in the study of elliptic and biharmonic problems involving nonlinearities in the critical growth range. In Chapter 5, we give a mountain pass characterization of groud state solutions of a nonlinear scalar field equation in $\mathbb{R}^2$ with critical exponential nonlinearity. This characterization is obtained in the following paper

- B. Ruf, F. Sani, *Ground states for elliptic equations in* $\mathbb{R}^2$ *with exponential critical growth*, submitted

Finally in Chapter 6 and 7, we study a class of biharmonic problems in the whole space $\mathbb{R}^4$ both in the case when the nonlinear term has subcritical and critical exponential growth. These results are contained in the following papers

- F. Sani, *A biharmonic equation in* $\mathbb{R}^4$ *involving nonlinearities with subcritical exponential growth*, Adv. Nonlinear Stud. 11 (2011), No. 4, 889–904

- F. Sani, *A biharmonic equation in* $\mathbb{R}^4$ *involving nonlinearities with critical exponential growth*, Commun. Pure Appl. Anal. to appear

Acknowledgements

Looking back at the last three years, I feel amazed when I realize how many interesting people have crossed my life. People that, in many ways, have influenced me and this work. It is time to thank everyone.

It is a pleasure for me to express deep gratitude to my advisor, Prof. Bernhard Ruf, for having proposed to me the topic of the present thesis. I found the problems that we studied extremely interesting, in these years he really made me love math. I wish to warmly thank him for his invaluable help during the whole time of my PhD and for having constantly supported me with his positive attitude. I feel fortunate to have had him as my mentor.

I am grateful to Massimo Villarini for the time we spent together collaborating and for sharing with me his passionate vision of mathematics.

I am also indebted with Daniele Cassani for his useful suggestions and with Cristina Tarsi for kindly answering my questions. A special mention also to Daniela Lupo, I thank her mostly for showing me how to work with a smile.

I would like to thank the *Dipartimento di Matematica di via Saldini* for having made possible my PhD studies and in particular the research group *Analisi non lineare ed equazioni alle derivate parziali non lineari* for providing me such a pleasant research environment. Many thanks to all the friends who shared with me these unforgettable three years at *Studio 1034* and a special thank goes to EmanuLele (my wonderful office mate), Patricio and Oscar.

Many thanks also to Prof. Nader Masmoudi for his kind hospitality at the *Courant Institute of Mathematical Sciences*. This was a great opportunity for me and I really enjoyed working with, and learning from, him. I met a lot of good people in New York, a special thank goes to Benoit, Jacob and Zaher.

Some people needs to be mentioned here with no explanation. Bea and Luisa, thank you especially for always believing in me and bearing me during these last months. But also, how to forget Sara, Gigio, Tommy, cugi-Wolly, Deanna, Leo, Caba, Elody, Caccia, Sonia, Miguel, Nigó, Baby, Daniele, Cesco, Pongo, Fede and Thomas . . . and don't get disappointed if I forgot someone!

Finally, words are not enough to show how grateful I am to my parents, to whom I dedicate this thesis.

Contents

Part I

Exponential-type inequalities in $\mathbb{R}^n$

CHAPTER 1

A brief history of the problem

First order Sobolev embedding theorem and the limiting case

Sobolev inequalities are among the most famous and useful functional inequalities in analysis. They express a strong integrability or regularity property for a function u in terms of some integrability properties for some derivatives of u. The most basic and important applications of Sobolev inequalities are to the study of partial differential equations. These inequalities provide some of the very basic tools in the study of existence, regularity and uniqueness of solutions of all sorts of partial differential equations, linear and nonlinear, elliptic, parabolic and hyperbolic.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. The Sobolev embedding theorem asserts that if $p < n$ then

$$W_0^{1,p}(\Omega) \subset L^q(\Omega) \qquad 1 \leq q \leq p^* := \frac{np}{n-p}\ .$$

Equivalently,

$$\sup_{u \in W_0^{1,\,p}(\Omega),\, \|\nabla u\|^p \leq 1} \int_\Omega |u|^q\, dx < +\infty \qquad \text{for } 1 \leq q \leq p^*\,,$$

where $\|\nabla u\|_p^p = \int_\Omega |\nabla u|^p\, dx$ denotes the Dirichlet norm of u, while

$$\sup_{u \in W_0^{1,\,p}(\Omega),\, \|\nabla u\|^p \leq 1} \int_\Omega |u|^q\, dx = +\infty \qquad \text{for any } q > p^*\,.$$

The maximal growth $|u|^{p^*}$ is the so called *critical Sobolev growth.* If we look at the limiting Sobolev case $p = n$ then

$$W_0^{1,n}(\Omega) \subset L^q(\Omega) \qquad \forall q \geq 1$$

and every polynomial growth is allowed. Since formally $p^* = \frac{np}{n-p} \sim +\infty$ as $p \to n$, one may expect that a function $u \in W_0^{1,\,n}(\Omega)$ is bounded, but it is well known that

$$W_0^{1,n}(\Omega) \nsubseteq L^\infty(\Omega)\ .$$

For instance, denoted by $|\cdot|$ the standard Euclidean norm in $\mathbb{R}^n$, we can define

$$u(x) := \begin{cases} \log|\log|x|\,| & \text{for any } x \in \mathbb{R}^n \text{ with } 0 < |x| < \frac{1}{e}\,, \\ 0 & \text{elsewhere}\,. \end{cases}$$

It is easy to see that

$$\|\nabla u\|_n^n = \omega_{n-1} \int_0^{\frac{1}{e}} \frac{dr}{r|\log r|^n} = \frac{\omega_{n-1}}{n-1}\,,$$

where ω_{n-1} is the surface measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and hence $u \in W_0^{1,n}(\Omega)$ for any domain $\Omega \subset \mathbb{R}^n$ containing the unit ball centered at the origin; but clearly $u \notin L^\infty(\Omega)$. However e^u is integrable:

$$\|e^u\|_1 = \omega_{n-1} \int_0^{\frac{1}{e}} r^{n-1}|\log r|\,dr = \frac{\omega_{n-1}}{n}\left(\frac{1}{e} + \frac{1}{n^2 e^{n-1}}\right)\,.$$

The *Trudinger-Moser inequality* concerns this borderline case $p = n$.

The Trudinger-Moser inequality

To fill in the gap of the Sobolev embedding theorem, it is natural to look for the maximal growth function $g : \mathbb{R} \to \mathbb{R}^+$ such that

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \int_\Omega g(u)\,dx < +\infty\,,$$

where $\Omega \subset \mathbb{R}^n$, $n \ge 2$ is a bounded domain. V. I. Yudovich [71], S. I. Pohozaev [56] and N. S. Trudinger [70] proved independently that the maximal growth is of exponential type and more precisely that there exist constants $\alpha_n > 0$ and $C_n > 0$ depending only on n such that

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}}\,dx \le C_n|\Omega|\,. \tag{1.1}$$

The proofs of Yudovich, Pohozaev and Trudinger relied on the same idea, namely developing the exponential function in power series, the problem reduces to show that a series of L^p-norms converges. These proofs, however, will not produce the optimal exponent α_n.

More precisely, the key tool in Trudinger's proof is the Sobolev estimate

$$\|u\|_q \le c_n|\Omega|^{\frac{1}{q}} q^{1-\frac{1}{n}} \|\nabla u\|_n \qquad \forall u \in W_0^{1,n}(\Omega),\ \forall q > 1\,, \tag{1.2}$$

where $c_n > 0$ is a constant depending only on n. Once proved inequality (1.2), then (1.1) follows easily using the power series expansion of the exponential function. In fact

$$\begin{aligned} \sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}}\,dx \quad &\le \quad \sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \sum_{k=0}^{+\infty} \frac{\alpha^k}{k!} \int_\Omega |u|^{k\frac{n}{n-1}}\,dx \le \\ &\le \quad |\Omega| \sum_{k=0}^{+\infty} \tilde{c}_n^k \frac{\alpha^k}{k!} k^k \end{aligned}$$

where the constant $\tilde{c}_n > 0$ depends on n only; applying Stirling's formula $k! \geq \left(\frac{k}{e}\right)^k$,

$$\sup_{u\in W_0^{1,n}(\Omega),\, \|\nabla u\|_n\leq 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}}\, dx \leq |\Omega| \sum_{k=0}^{+\infty} (\alpha \tilde{c}_n e)^k < +\infty$$

provided that $\alpha < \frac{1}{\tilde{c}_n e}$.

Later J. Moser in [50] replaced these proofs by a more refined one and, at the same time, he found the best exponent α_n proving the following sharp result

Theorem 1.1 ([50], Theorem 1)**.** *There exists a constant $C_n > 0$ such that*

$$\sup_{u\in W_0^{1,n}(\Omega),\, \|\nabla u\|_n\leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}}\, dx \leq C_n|\Omega| \qquad \forall \alpha \leq \alpha_n \tag{1.3}$$

where $\alpha_n := n\omega_{n-1}^{1/(n-1)}$ and ω_{n-1} is the surface measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Furthermore (1.3) is sharp, i.e. if $\alpha > \alpha_n$ then the supremum in (1.3) is infinite.

Therefore it turns out that there exists a positive number α_n such that (1.3) holds for $\alpha \leq \alpha_n$ and is false for $\alpha > \alpha_n$. The remarkable phenomenon is that the inequality still holds for the critical value α_n itself.

In the literature (1.3) is known under the name *Trudinger-Moser inequality*. In what follows we will refer to the sharpness of an inequality in the sense expressed in the second part of Theorem 1.1.

Moser's proof of inequality (1.3) relies strongly on Schwarz spherical symmetrization, which preserves integrals of functions and does not increase the Dirichlet norm in $W_0^{1,p}(\Omega)$ with $p \geq 1$. To every function $u \in W_0^{1,p}(\Omega)$ is associated a spherically symmetric function $u^\sharp$ such that the sublevel-sets of $u^\sharp$ are balls with the same measure as the corresponding sublevel-sets of $|u|$, that is

$$\Big|\{x \in \mathbb{R}^n \mid u^\sharp(x) < c\}\Big| = \Big|\{x \in \Omega \mid |u(x)| < c\}\Big| \quad \forall c \geq 0.$$

Then $u^\sharp$ is a nonnegative spherically nonincreasing function defined on a ball $B_R \subset \mathbb{R}^n$ centered at the origin with radius $R > 0$, satisfying $|B_R| = |\Omega|$, and $u^\sharp \in W_0^{1,p}(B_R)$. We recall also that

$$u^\sharp(x) = u^*\Big(\frac{\omega_{n-1}}{n}|x|^n\Big) \qquad x \in B_R$$

where u^* is the onedimensional decreasing rearrangement of u.

By construction, if $F : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that either $F \geq 0$ or $F(u) \in L^1(\Omega)$ then

$$\int_\Omega F(u)\, dx = \int_{B_R} F(u^\sharp)\, dx\,.$$

The monotonicity of Dirichlet norms under such a symmetrization is known as the *Pólya-Szëgo principle*

$$\|\nabla u\|_p \geq \|\nabla u^\sharp\|_p \qquad \forall u \in W_0^{1,p}(\Omega) \tag{1.4}$$

where $u^\sharp$ is the decreasing rearrangement of u.

Therefore, making use of symmetrizations,

$$\sup_{u\in W_0^{1,n}(\Omega),\,\|\nabla u\|_n\le 1}\int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}}\,dx \le \sup_{u^\sharp\in W_0^{1,n}(B_R),\,\|\nabla u^\sharp\|_n\le 1}\int_{B_R} e^{\alpha(u^\sharp)^{\frac{n}{n-1}}}\,dx$$

and hence to prove Theorem 1.1 it is sufficient to consider the radial case. Moser, after the change of variables

$$r=|x|=Re^{-\frac{t}{n}} \quad \text{and} \quad w(t):=n^{\frac{n-1}{n}}\omega_{n-1}^{\frac{1}{n}}u^\sharp(r),$$

$$\int_{B_R}|\nabla u^\sharp|^n\,dx=\int_0^{+\infty}(w')^n\,dt, \quad \int_{B_R}e^{\alpha(u^\sharp)^{\frac{n}{n-1}}}\,dx=|B_R|\int_0^{+\infty}e^{\frac{\alpha}{\alpha_n}|w|^{\frac{n}{n-1}}-t}\,dt$$

reduced the estimate to the following subtle one-dimensional calculus inequality.

Lemma 1.2 ([50], inequality (6)). *Let $\phi:[0,+\infty)\to\mathbb{R}$ be a nonnegative measurable function such that*

$$\int_0^{+\infty}\phi^n(t)\,dt\le 1\,.$$

Then

$$\int_0^{+\infty}e^{-F(t)}\,dt\le C_n$$

where $C_n>0$ is the same as in Theorem 1.1 and

$$F(t):=t-\left(\int_0^t\phi(s)\,ds\right)^{\frac{n}{n-1}}\,.$$

In 2005, A. Cianchi [25] complemented the classical result of Moser obtaining a sharp inequality for the space $W^{1,n}(\Omega)$, i.e. without boundary conditions. However, the reduction of the problem to a onedimensional inequality is more delicate, in fact, since functions which do not necessarily vanish on $\partial\Omega$ are allowed, Schwarz symmetrization is of no use in this case. A key tool in the proof of Cianchi is instead an asymptotically sharp relative isoperimetric inequality for domains in $\mathbb{R}^n$.

Trudinger-Moser inequalities for unbounded domains

An interesting extension of (1.3) is to construct Trudinger-Moser type inequalities for domains with infinite measure. In fact, we can notice that the supremum in (1.3) becomes infinite, even in the case $\alpha\le\alpha_n$, for domains $\Omega\subseteq\mathbb{R}^n$ with $|\Omega|=+\infty$ and consequently the original form of the Trudinger-Moser inequality is not available in these cases. A weaker inequality for unbounded domains has been proposed by D. M. Cao [19] for the case $n=2$ and by J. M. do Ó [30] for the general case $n\ge 2$. More precisely they proved that for any $u\in W^{1,n}(\mathbb{R}^n)$ with $\|\nabla u\|_n\le m<1$ and $\|u\|_n\le M<+\infty$

$$\int_{\mathbb{R}^n}\psi(\alpha_n|u|^{\frac{n}{n-1}})\,dx\le C(m,\,M) \tag{1.5}$$

where $C(m,\, M) > 0$ is a constant independent of u and

$$\psi(u) := e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}\,. \tag{1.6}$$

Later S. Adachi and K. Tanaka [1] studied the best possible exponent in this weaker type of inequalities, proving that for any $\alpha \in (0,\, \alpha_n)$ there exists a constant $C(\alpha) > 0$ such that

$$\int_{\mathbb{R}^n} \psi(\alpha |u|^{\frac{n}{n-1}})\, dx \leq C(\alpha) \|u\|_n^n \quad \forall u \in W^{1,\,n}(\mathbb{R}^n) \text{ with } \|\nabla u\|_n \leq 1\,, \tag{1.7}$$

and this inequality is false for $\alpha \geq \alpha_n$. The proof of Adachi and Tanaka is based on Moser's idea, that is, making use of symmetrization of functions and Moser's change of variables, the estimate reduces to a one-dimensional calculus inequality. The limit exponent α_n is excluded in (1.7), which is quite different from Moser's result (see Theorem 1.1). This reveals the *subcritical* aspect of such inequalities.

However, in the case $n = 2$ (i.e. for $W_0^{1,\,2}(\Omega)$ with $\Omega \subseteq \mathbb{R}^2$), B. Ruf [61] showed that if the Dirichlet norm is replaced by the standard Sobolev norm, namely

$$\|u\|^n_{W^{1,\,n}} := \|\nabla u\|_n^n + \|u\|_n^n\,,$$

then the result of Moser (Theorem 1.1) can be fully extended to unbounded domains and the supremum in (1.3) is uniformly bounded independently of the domain Ω:

Theorem 1.3 ([61], Theorem 1.1)**.** *There exists a constant $C > 0$ such that for any domain $\Omega \subseteq \mathbb{R}^2$*

$$\sup_{u \in W_0^{1,\,2}(\Omega),\, \|u\|_{W^{1,\,2}} \leq 1} \int_\Omega (e^{4\pi u^2} - 1)\, dx \leq C \tag{1.8}$$

and this inequality is sharp.

In [44], Y. Li and B. Ruf extended Theorem 1.3 to arbitrary dimensions $n \geq 2$, i.e. to $W_0^{1,n}(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ not necessarily bounded and $n \geq 2$.

The proof given in [61] for the 2-dimensional case is based on symmetrization techniques, more precisely it is sufficient to consider the case $\Omega = \mathbb{R}^2$ and, as in the proof of Moser, without loss of generality we may assume that u is spherically symmetric and nonincreasing. Then, the integral in (1.8) can be divided into two parts

$$\int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)\, dx = \int_{\mathbb{R}^2 \setminus B_{r_0}} (e^{4\pi u^2} - 1)\, dx + \int_{B_{r_0}} (e^{4\pi u^2} - 1)\, dx$$

with $r_0 > 0$ to be chosen. Concerning the integral on $\mathbb{R}^2 \setminus B_{r_0}$, using the power series expansion of the exponential function and, to estimate the single terms, a pointwise estimate for nonincreasing radial functions (see [15], Lemma A.IV), i.e.

$$|u(r)| \leq \frac{1}{r\sqrt{\pi}} \|u\|_2 \qquad \forall r > 0\,, \tag{1.9}$$

it is easy to obtain an upper bound which depends on r_0 only. The key point in the proof of Ruf is the estimate of the integral on the ball. Writing

$$u(r) = u(r) \pm u(r_0) =: v(r) + u(r_0)\,,$$

we obtain

$$u^2(r) \le v^2(r)\left(1 + \frac{1}{\pi r_0^2}\|u\|_2^2\right) + d(r_0) =: w^2(r) + d(r_0)$$

where $d(r_0) > 0$ is a constant depending only on $r_0 > 0$ and $w \in W_0^{1,2}(B_{r_0})$. Hence

$$\int_{B_{r_0}} (e^{4\pi u^2} - 1)\,dx \le e^{4\pi d(r_0)} \int_{B_{r_0}} e^{4\pi w^2}\,dx \le C$$

provided that $\|\nabla w\|_2 \le 1$. But this is indeed the case if we choose $r_0 > 0$ sufficiently large, in fact

$$\|\nabla w\|_2^2 = \left(1 + \frac{1}{\pi r_0^2}\|u\|_2^2\right)\|\nabla v\|_2^2 = \left(1 + \frac{1}{\pi r_0^2}\|u\|_2^2\right)\|\nabla u\|_2^2 \le \left(1 + \frac{1}{\pi r_0^2}\|u\|_2^2\right)(1 - \|u\|_2^2) \le 1$$

provided that $\pi r_0^2 \ge 1$.

We also recall that the proof given in [44] for the n-dimensional case is based on subtle tecniques of blow-up analysis, however the enlightening method of proof introduced in [61] can also be adapted to recover the general n-dimensional case (see Adimurthi and Y. Yang [7]) and will enable us to treat the case of higher order derivatives in unbounded domains. Moreover, arguing as in [61] it is easy to see that

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\, \|u\|_{W^{1,n},\tau} \le 1} \int_{\mathbb{R}^n} \psi(\alpha_n |u|^{\frac{n}{n-1}})\,dx < +\infty \tag{1.10}$$

where $\tau > 0$ and

$$\|u\|_{W^{1,n},\tau}^n := \|\nabla u\|_n^n + \tau\|u\|_n^n\,.$$

In fact, as shown also by Adimurthi and Y. Yang in [7], the value $\tau = 1$, appearing in $\|\cdot\|_{W^{1,n}} = \|\cdot\|_{W^{1,n},1}$ as a multiplicative constant for the L^n-norm, does not play any role and can be replaced by any $\tau > 0$.

We can notice that for any $u \in W^{1,n}(\mathbb{R}^n)$ with $\|\nabla u\|_n \le m < 1$ and $\|u\|_n \le M < +\infty$, choosing

$$0 < \tau \le \frac{1-m}{M}$$

we have that $\|u\|_{W^{1,n},\tau} \le 1$ and thus (1.10) implies (1.5). Nevertheless, inequality (1.5) implies only that fixed $\tau > 0$

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\, \|u\|_{W^{1,n},\tau \le 1}} \int_{\mathbb{R}^n} \psi(\alpha |u|^{\frac{n}{n-1}})\,dx < +\infty$$

provided that $\alpha \in (0, \alpha_n)$. Hence, (1.5) can be viewed as a *subcritical* inequality, while (1.10) as a *critical* one.

The Adams' inequality

In 1988 D. R. Adams [2] obtained another interesting extension of (1.3) for Sobolev spaces with higher order derivatives. For these spaces the Sobolev embedding theorem says that if $\Omega \subset \mathbb{R}^n$ then

$$W_0^{m,\,p}(\Omega) \subset L^{\frac{pn}{n-pm}}(\Omega)$$

and hence the limiting case is $p = \frac{n}{m}$. Let m be an integer and $\Omega \subset \mathbb{R}^n$ with $m < n$, Adams' result can be stated as follows: for $u \in W^{m,\,p}(\Omega)$ with $1 \le p < +\infty$, we will denote by $\nabla^j u$, $j \in \{1,\, 2\, \ldots,\, m\}$, the j-th order gradient of u, namely

$$\nabla^j u := \begin{cases} \Delta^{\frac{j}{2}} u & j \text{ even} \\ \nabla \Delta^{\frac{j-1}{2}} u & j \text{ odd} \end{cases}.$$

Theorem 1.4 ([2], Theorem 1)**.** *Let m be an integer and let $\Omega \subset \mathbb{R}^n$ with $m < n$. There exists a constant $C_{m,\,n} > 0$ such that*

$$\sup_{u \in W_0^{m,\,\frac{n}{m}}(\Omega),\, \|\nabla^m u\|_{\frac{n}{m}} \le 1} \int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}}\, dx \le C_{m,\,n} |\Omega| \tag{1.11}$$

where

$$\beta_0 = \beta_0(m,\, n) := \frac{n}{\omega_{n-1}} \begin{cases} \left[\dfrac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n}{n-m}} & \textit{if } m \textit{ is even,} \\ \\ \left[\dfrac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right]^{\frac{n}{n-m}} & \textit{if } m \textit{ is odd.} \end{cases}$$

Furthermore inequality (1.11) is sharp.

Adams' approach to the problem is to express u as the Riesz potential of its gradient of order m and then apply the following theorem

Theorem 1.5 ([2], Theorem 2)**.** *Let $1 < p < +\infty$. There exists a constant $c_0 = c_0(p,\, n)$ such that for all $f \in L^p(\mathbb{R}^n)$ with support contained in $\Omega \subset \mathbb{R}^n$*

$$\int_\Omega e^{\frac{n}{\omega_{n-1}} \left| \frac{I_\alpha * f(x)}{\|f\|_p} \right|^{p'}} \le c_0$$

where $\alpha = \frac{n}{p}$, $\frac{1}{p} + \frac{1}{p'} = 1$ and

$$I_\alpha * f(x) := \int_{\mathbb{R}^n} |x-y|^{\alpha - n} f(y)\, dy$$

is the Riesz potential of order α.

The reason why it is convenient to write u in terms of Riesz potential is that one cannot use directly the idea of decreasing rearrangement $u^\sharp$ to treat the higher order case, because no inequality of the type (1.4) is known to hold for higher order derivatives. To avoid this problem, Adams applied a result of R. O'Neil [53] on nonincreasing rearrangements for convolution integrals, if $h := g * f$ then

$$h^{**}(t) \le t g^{**}(t) f^{**}(t) + \int_t^{+\infty} g^*(s) f^*(s)\, ds$$

where $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\, ds$. Then, a change of variables reduces the estimate to a one-dimensional calculus inequality

Lemma 1.6 ([2], Lemma 1)**.** *Let* $a : \mathbb{R}\times[0,+\infty) \to \mathbb{R}$ *be a nonnegative measurable function such that a.e.*

$$a(s,\, t) \le 1 \quad \text{when } 0 < s < t\,, \qquad \text{and} \qquad \sup_{t>0} \left(\int_{-\infty}^{0} + \int_t^{+\infty} a^{p'}(s,\, t)\, ds \right)^{\frac{1}{p'}} =: b < +\infty$$

where $\frac{1}{p} + \frac{1}{p'} = 1$*. Then there exists a constant* $c_0 = c_0(p,\, b)$ *such that for any* $\phi : \mathbb{R} \to \mathbb{R}$ *satisfying* $\phi \ge 0$ *and*

$$\int_{-\infty}^{+\infty} \phi^p(s)\, ds \le 1\,,$$

the following inequality holds

$$\int_0^{+\infty} e^{-F(t)}\, dt \le c_0$$

where

$$F(t) := t - \left(\int_{-\infty}^{+\infty} a(s,\, t)\phi(s)\, ds \right)^{p'} .$$

We can notice that the one-dimensional technical inequality of Moser, see Lemma 1.2 above, corresponds to the case

$$a(s,\, t) = \begin{cases} 1 & \text{when } 0 < s < t, \\ 0 & \text{otherwise.} \end{cases}$$

Later L. Fontana [36] proved that the complete analogues of Adams' theorem, Theorem 1.4, is valid for every compact smooth Riemannian manifold M. In fact, the optimal exponent β_0 turn out to be the same for every such M as it is for domains in $\mathbb{R}^n$.

Adams' inequalities for unbounded domains

As in the case of first order derivatives, one notes that the bound in (1.11) becomes infinite for domains Ω with $|\Omega| = +\infty$. Let

$$\phi(t) := e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2} \frac{t^j}{j!} \tag{1.12}$$

where

$$j_{\frac{n}{m}} := \min\left\{ j \in \mathbb{N} \,\middle|\, j \ge \frac{n}{m} \right\} \ge \frac{n}{m}\,.$$

If m is an even integer, T. Ogawa and T. Ozawa [52] in the case $\frac{n}{m}=2$ and T. Ozawa [54] in the general case proved the existence of positive constants α and C such that

$$\int_{\mathbb{R}^n}\phi(\alpha|u|^{\frac{n}{n-m}})\,dx\le C\|u\|_{\frac{n}{m}}^{\frac{n}{m}}\quad\forall u\in W^{m\,\frac{n}{m}}(\mathbb{R}^n)\text{ with }\|\nabla^m u\|_{\frac{n}{m}}\le 1\,.$$

The proof of this result follows the original idea of Yudovich, Pohozaev and Trudinger; making use of the power series expansion of the exponential function, the problem reduces to majorizing each term of the expansion in terms of the Sobolev norms in order that the resulting power series should converge. More precisely the following sharp Sobolev inequalities are involved

$$\|u\|_q\le C(m,\,n)q^{1-\frac{m}{n}}\|\nabla^m u\|_{\frac{n}{m}}^{1-\frac{n}{qm}}\|u\|_{\frac{n}{m}}^{\frac{n}{qm}}\qquad\forall u\in W^{m,\,\frac{n}{m}}(\mathbb{R}^n),\ \forall q\in\left[\frac{n}{m},\,+\infty\right)\,.$$

However the problem concerning the best possible exponent for this type of inequalities is not solved with such a proof and it is still an open problem. We point out that indeed the results of [52] and [54] are more general and more precisely recover also the case of fractional derivatives.

In the case that m is even, namely $m=2k$, in Chapter 2 we will show that replacing the norm $\|\nabla^m u\|_{\frac{n}{m}}$ with the norm

$$\|u\|_{m,\,n}:=\|(-\Delta+I)^{\frac{m}{2}}u\|_{\frac{n}{m}}=\|(-\Delta+I)^k u\|_{\frac{n}{m}}$$

where I denotes the identity operator, the supremum in (1.11) is bounded by a constant independent of Ω.

The main result that we will prove in Chapter 2 is the following:

Theorem 1.7. *Let m be an even integer less than n. There exists a constant $C_{m,\,n}>0$ such that for any domain $\Omega\subseteq\mathbb{R}^n$*

$$\sup_{u\in W_0^{m,\,\frac{n}{m}}(\Omega),\,\|u\|_{m,\,n}\le 1}\int_\Omega\phi\left(\beta_0|u|^{\frac{n}{n-m}}\right)\,dx\le C_{m,\,n}$$

and this inequality is sharp.

In [43] (see Theorem 1.2), Kozono et al. explicitely exhibit a constant $\beta^*_{m,\,n}\le\beta_0$, with $\beta^*_{m,\,2m}=\beta_0(m,\,2m)$, such that if $\beta<\beta^*_{m,\,n}$ then

$$\sup_{u\in W^{m,\,\frac{n}{m}}(\mathbb{R}^n),\,\|u\|_{m,\,n}\le 1}\int_{\mathbb{R}^n}\phi\left(\beta|u|^{\frac{n}{n-m}}\right)\,dx\le C(\beta,\,m,\,n)\tag{1.13}$$

where $C(\beta,\,m,\,n)>0$ is a constant depending on β, m and n, while if $\beta>\beta_0$ the supremum is infinite. To do this they reduce the inequality to some equivalent form by means of Bessel potentials, then they apply techniques of symmetric decreasing rearrangements and, following a procedure similar to Adams', they make use of O'Neil's result [53] on the rearrangement of convolution functions. But with these arguments they did not answer the question whether or not the uniform boundedness in (1.13) holds also for the limiting case $\beta=\beta_0$.

CHAPTER 2

Sharp Adams-type inequalities in $\mathbb{R}^n$

Adams' inequality for bounded domains $\Omega \subset \mathbb{R}^4$ states that the supremum of

$$\int_\Omega e^{32\pi^2 u^2}\, dx$$

over all functions $u \in W_0^{2,\,2}(\Omega)$ with $\|\Delta u\|_2 \leq 1$ is bounded by a constant depending on Ω only. This bound becomes infinite for unbounded domains and in particular for $\mathbb{R}^4$. In this Chapter we prove that if $\|\Delta u\|_2$ is replaced by a suitable norm, namely $\|-\Delta u + u\|_2$, then the supremum of

$$\int_\Omega (e^{32\pi^2 u^2} - 1)\, dx$$

over all functions $u \in W_0^{2,\,2}(\Omega)$ with $\|-\Delta u + u\|_2 \leq 1$ is bounded by a constant independent of the domain Ω. Furthermore, we generalize this result to any $W_0^{m,\,\frac{n}{m}}(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ and m an even integer less than n. More precisely setting

$$\phi(t) := e^t - \sum_{j=0}^{j_{\frac{n}{m}}-2} \frac{t^j}{j!}\ ,$$

where

$$j_{\frac{n}{m}} := \min\left\{ j \in \mathbb{N} \;\middle|\; j \geq \frac{n}{m} \right\} \geq \frac{n}{m}\ ,$$

we will prove the following result:

Theorem 2.1. *Let m be an even integer less than n, $m = 2k < n$ with $k \geq 1$. There exists a constant $C_{m,\,n} > 0$ such that for any domain $\Omega \subseteq \mathbb{R}^n$*

$$\sup_{u \in W_0^{m,\,\frac{n}{m}}(\Omega),\, \|u\|_{m,\,n} \leq 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx \leq C_{m,\,n} \tag{2.1}$$

where

$$\|u\|_{m,\,n} := \|(-\Delta + I)^{\frac{m}{2}} u\|_{\frac{n}{m}} = \|(-\Delta + I)^k u\|_{\frac{n}{m}}$$

and

$$\beta_0 = \beta_0(m,\, n) := \frac{n}{\omega_{n-1}} \left[\frac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n}{n-m}} .$$

Furthermore inequality (2.1) is sharp.

To prove this theorem, the idea is to adapt the arguments in [61], but in order to do this one encounters difficulties in the use of symmetrization techniques to reduce the general problem to the radial case. Indeed, this cannot be done directly as in [61], since one would have to establish inequalities between $\|\nabla^m u\|_{\frac{n}{m}}$ and $\|\nabla^m u^\sharp\|_{\frac{n}{m}}$, where $u^\sharp$ denotes the symmetrized function of u, and such estimates are unknown in general for higher order derivatives. To get around this problem, the idea is to apply a suitable comparison principle. For example, in [22] and [21], the authors used the well known Talenti comparison principle (see [67]). Under suitable assumptions, this comparison principle leads to compare a function u, not necessarily radial, with a radial function v in such a way that $\|\nabla^m u\|_p = \|\nabla^m v\|_p$ and $\|u\|_p \le \|v\|_p$ for any $p \in [1, +\infty)$. Therefore, the Talenti comparison principle is a suitable tool if one works with the L^p-norm of the m-th order gradient. In our case, since we want to obtain an estimate independent of the domain, we need to replace the Dirichlet norm $\|\nabla^m u\|_{\frac{n}{m}}$ by a larger norm, and a natural choice is the norm

$$\|u\|_{m,\,n} := \|(-\Delta + I)^{\frac{m}{2}} u\|_{\frac{n}{m}} .$$

It is easy to check that the norm $\|u\|_{m,\,n}$ is equivalent to the Sobolev norm

$$\|u\|_{W^{m,\,\frac{n}{m}}} := \left(\|u\|_{\frac{n}{m}}^{\frac{n}{m}} + \sum_{j=1}^{m} \|\nabla^j u\|_{\frac{n}{m}}^{\frac{n}{m}} \right)^{\frac{m}{n}}$$

and in particular, if $u \in W_0^{m\,\frac{n}{m}}(\Omega)$ (or $u \in W^{m,\,\frac{n}{m}}(\mathbb{R}^n)$) then

$$\|u\|_{W^{m,\,\frac{n}{m}}} \le \|u\|_{m,\,n} \tag{2.2}$$

But the Talenti comparison principle cannot be applied to the norm $\|u\|_{m,n}$ since it increases the $\|\cdot\|_{m,\,n}$-norm; however, the norm $\|u\|_{m,n}$ is well-suited to apply (an iterated version of) a comparison principle due to G. Trombetti and J. L. Vázquez which appears in [69] (see also G. Chiti [24]). The iterated version of this comparison principle is stated in Proposition 2.8.

Having reduced the problem to the radial case, in order to prove Theorem 2.1, we will show that the supremum of

$$\int_{B_R} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx$$

over all radial functions with homogeneous Navier boundary conditions belonging to the unit ball of

$$\left(W^{m,\,\frac{n}{m}}_{N,\,\mathrm{rad}}(B_R) := W_N^{m,\,\frac{n}{m}}(B_R) \cap W_{\mathrm{rad}}^{m,\,\frac{n}{m}}(B_R),\; \|\cdot\|_{W^{m,\,\frac{n}{m}}} \right)$$

is bounded by a constant independent of $R>0$. Here and below, $B_R := \{x \in \mathbb{R}^n \mid |x| < R\}$ is the ball of radius $R>0$, and

$$\begin{aligned} W_N^{m,\frac{n}{m}}(B_R) &:= \left\{ u \in W^{n,\frac{n}{m}}(B_R) \,\middle|\, \Delta^j u|_{\partial B_R} = 0 \text{ in the sense of traces for } 0 \le j < \frac{m}{2} \right\} \\ W_{\mathrm{rad}}^{m,\frac{n}{m}}(B_R) &:= \left\{ u \in W^{m,\frac{n}{m}}(B_R) \mid u(x) = u(|x|) \ \text{ a.e. in } B_R \right\} \end{aligned}$$

are respectively the space of $W^{m,\frac{n}{m}}(B_R)$*-functions with homogeneous Navier boundary conditions* and the space of *radial* $W^{m,\frac{n}{m}}(B_R)$*-functions*. This result is expressed in the following:

Proposition 2.2. *Let m be an even integer less than n. There exists a constant $C_{m,n}>0$ such that*

$$\sup_{u \in W_{N,\,rad}^{m,\,\frac{n}{m}}(B_R),\, \|u\|_{W^{m,\,n/m}} \le 1} \int_{B_R} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le C_{m,n} \tag{2.3}$$

independently of $R>0$ and this inequality is sharp.

This ends an outline of the proof of Theorem 2.1.

We point out that, as $W_0^{m,\frac{n}{m}}(\Omega) \subset W_N^{m,\frac{n}{m}}(\Omega)$, we have

$$\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega),\, \|u\|_{m,n} \le 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le \sup_{u \in W_N^{m,\frac{n}{m}}(\Omega),\, \|u\|_{m,n} \le 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx$$

and actually we will also prove the following stronger version of the Adams-type inequality (2.1):

Proposition 2.3. *Let m be an even integer less than n. There exists a constant $C_{m,n}>0$ such that for any bounded domain $\Omega \subset \mathbb{R}^n$*

$$\sup_{u \in W_N^{m,\frac{n}{m}}(\Omega),\, \|u\|_{m,n} \le 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le C_{m,n} \tag{2.4}$$

and this inequality is sharp.

Comparing this last result with Theorem 2.1, in the case of bounded domains, it is remarkable that the sharp exponent β_0 does not depend on all the traces but only on the zero Navier boundary conditions. This is not obvious, as shown by A. Cianchi in [25] in the case of first order derivatives: with zero Neumann boundary conditions (i.e. in $W^{1,n}(\Omega)$ instead of $W_0^{1,n}(\Omega)$) the sharp exponent α_n in Theorem 1.1 strictly decreases.

The plan of the proofs of the results stated above is the following. In Section 2.1 we recall the comparison principle of G. Trombetti and J. L. Vázquez [69] (see also G. Chiti [24]) and we introduce an iterated version of it. In the following sections (Section 2.2 and 2.3), firstly we prove that the supremum of

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx$$

over all radial functions belonging to the unit ball of $(W^{m,\frac{n}{m}}(\mathbb{R}^n), \|\cdot\|_{W^{m,\frac{n}{m}}})$ is bounded:

Theorem 2.4. *Let m be an even integer less than n. There exists a constant $C_{m,n} > 0$ such that*

$$\sup_{u \in W^{m,\frac{n}{m}}_{rad}(\mathbb{R}^n),\, \|u\|_{W^{m,\,n/m}} \leq 1} \int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \leq C_{m,n} \tag{2.5}$$

where

$$W^{m,\frac{n}{m}}_{rad}(\mathbb{R}^n) := \left\{ u \in W^{m,\frac{n}{m}}(\mathbb{R}^n) \mid u(x) = u(|x|) \ \ a.e. \ in \ \mathbb{R}^n \right\}$$

Furthermore this inequality is sharp.

Secondly we will see that the proof of Theorem 2.4 can be easily adapted to prove Proposition 2.2. To make transparent the main ideas of the proof, in Section 2.2 we prove Theorem 2.4 and Proposition 2.2 in the simplest case $m = 2$, $n = 4$ and we give a general proof for $m \geq 2$ even and $n > m$ in Section 2.3. In Section 2.4 we prove the main theorem (Theorem 2.1), and we end the Section with the proof of Proposition 2.3. The proof of the sharpness of (2.1), (2.3), (2.4) and (2.5) is given in Section 2.5.

2.1. An iterated comparison principle

A crucial tool for the proof of Theorem 2.1 in the case $m = 2$ is the following comparison principle of G. Trombetti and J. L. Vázquez [69] (see also G. Chiti [24]) which we state only for balls $B_R \subset \mathbb{R}^n$, $n \geq 2$, in order to simplify the notations and as this is the case of our main interest. We will denote by $|B_R|$ the Lebesgue measure of B_R, namely $|B_R| := \sigma_n R^n$ where σ_n is the volume of the unit ball in $\mathbb{R}^n$.

Let $u : B_R \to \mathbb{R}$ be a measurable function. The *distribution funtion* of u is defined by

$$\mu_u(t) := |\{x \in B_R \mid |u(x)| > t\}| \qquad \forall t \geq 0 \,.$$

The *decreasing rearrangement* of u is defined by

$$u^*(s) := \inf\{t \geq 0 \mid \mu_u(t) < s\} \qquad \forall s \in [0, |B_R|] \,,$$

and the *spherically symmetric decreasing rearrangement* of u by

$$u^\sharp(x) := u^*(\sigma_n |x|^n) \qquad \forall x \in B_R \,.$$

The function $u^\sharp$ is the unique nonnegative integrable function which is radially symmetric, nonincreasing and has the same distribution function as $|u|$.

Let u be a weak solution of

$$\begin{cases} -\Delta u + u = f & \text{in } B_R \\ u \in W^{1,2}_0(B_R) \end{cases} \tag{2.6}$$

where $f \in L^{\frac{2n}{n+2}}(B_R)$.

Proposition 2.5 ([69], Inequality (2.20))**.** *If u is a nonnegative weak solution of (2.6) then*

$$-\frac{du^*}{ds}(s) \leq \frac{1}{n^2\sigma_n^{\frac{2}{n}}}\, s^{\frac{2}{n}-2} \int_0^s (f^* - u^*)\, d\tau \qquad \forall s \in (0,\, |B_R|)\ . \tag{2.7}$$

We now consider the problem

$$\begin{cases} -\Delta v + v = f^\sharp & \text{in } B_R \\ v \in W_0^{1,2}(B_R) \end{cases} . \tag{2.8}$$

Due to the radial symmetry of the equation the unique solution v of (2.8) is radially symmetric and it is easy to see that

$$-\frac{d\hat{v}}{ds}(s) = \frac{1}{n^2\sigma_n^{\frac{2}{n}}}\, s^{\frac{2}{n}-2} \int_0^s (f^* - \hat{v})\, d\tau \qquad \forall s \in (0,\, |B_R|) \tag{2.9}$$

where $\hat{v}(\sigma_n|x|^n) := v(x) \quad \forall x \in B_R$.

The maximum principle, together with inequalities (2.7) and (2.9), leads as proved in [69] to the following *comparison of integrals in balls*:

Proposition 2.6 ([69], Theorem 1)**.** *Let u, v be weak solutions of (2.6) and (2.8) respectively. For every $r \in (0,\, R)$ we have*

$$\int_{B_r} u^\sharp\, dx \leq \int_{B_r} v\, dx\ .$$

We are now interested to obtain a *comparison principle* for the polyharmonic operator, which will allow us to reduce the proof of Theorem 2.1 to the radial case. To this aim let $m = 2k$ with k a positive integer and let $u \in W^{m,2}(B_R)$ be a weak solution of

$$\begin{cases} (-\Delta + I)^k u = f & \text{in } B_R \\ u \in W_N^{m,2}(B_R) \end{cases} \tag{2.10}$$

where $f \in L^{\frac{2n}{n+2}}(B_R)$. If we consider the problem

$$\begin{cases} (-\Delta + I)^k v = f^\sharp & \text{in } B_R \\ v \in W_N^{m,2}(B_R) \end{cases} \tag{2.11}$$

then the following *comparison of integrals in balls* holds.

Proposition 2.7. *Let u, v be weak solutions of the polyharmonic problems (2.10) and (2.11) respectively. For every $r \in (0,\, R)$ we have*

$$\int_{B_r} u^\sharp\, dx \leq \int_{B_r} v\, dx\ .$$

Proof. Since equations in (2.10) and (2.11) are considered with homogeneous Navier boundary conditions, they may be rewritten as second order systems:

$$(P_1)\begin{cases}-\Delta u_1 + u_1 = f & \text{in } B_R\\ u_1 \in W_0^{1,2}(B_R)\end{cases} \qquad (P_i)\begin{cases}-\Delta u_i + u_i = u_{i-1} & \text{in } B_R\\ u_i \in W_0^{1,2}(B_R)\end{cases} \qquad i \in \{2, 3, \dots, k\}$$

$$(\overline{P}_1)\begin{cases}-\Delta v_1 + v_1 = f^\sharp & \text{in } B_R\\ v_1 \in W_0^{1,2}(B_R)\end{cases} \qquad (\overline{P}_i)\begin{cases}-\Delta v_i + v_i = v_{i-1} & \text{in } B_R\\ v_i \in W_0^{1,2}(B_R)\end{cases} \qquad i \in \{2, 3, \dots, k\}$$

where $u_k = u$ and $v_k = v$. Thus we have to prove that for every $r \in (0, R)$

$$\int_{B_r} u_k^\sharp \, dx \le \int_{B_r} v_k \, dx \ . \tag{2.12}$$

When $k = 1$, inequality (2.12) is the inequality in Proposition 2.6. When $k \ge 2$ we proceed by finite induction, proving that

$$\int_{B_r} u_i^\sharp \, dx \le \int_{B_r} v_i \, dx \tag{2.13}$$

holds for every $i \in \{1, 2, \dots, k\}$. By Proposition 2.6 it follows that if $i = 1$ then (2.13) holds. Now, assuming that inequality (2.13) has been proved for some $i \in \{1, 2, \dots, k-1\}$, we show that

$$\int_{B_r} u_{i+1}^\sharp \, dx \le \int_{B_r} v_{i+1} \, dx \ . \tag{2.14}$$

Without loss of generality we may assume that $u_{i+1} \ge 0$. Infact, let $\overline{u}_{i+1}$ be a weak solution of

$$\begin{cases}-\Delta \overline{u}_{i+1} + \overline{u}_{i+1} = |u_i| & \text{in } B_R\\ \overline{u}_{i+1} \in W_0^{1,2}(B_R)\end{cases}$$

then by the maximum principle $\overline{u}_{i+1} \ge 0$ and $\overline{u}_{i+1} \ge u_{i+1}$ in B_R.

Since u_{i+1} is a nonnegative weak solution of (P_{i+1}) then (2.7) holds and since v_{i+1} is a weak solution of $(\overline{P}_{i+1})$ also an analogue of (2.9) holds, namely

$$-\frac{du_{i+1}^*}{ds}(s) \le \frac{1}{n^2\sigma_n^{\frac{2}{n}}} \ s^{\frac{2}{n}-2} \int_0^s (u_i^* - u_{i+1}^*)\, d\tau \qquad \forall s \in (0, |B_R|) \ ,$$

$$-\frac{d\hat{v}_{i+1}}{ds}(s) = \frac{1}{n^2\sigma_n^{\frac{2}{n}}} \ s^{\frac{2}{n}-2} \int_0^s (\hat{v}_i - \hat{v}_{i+1})\, d\tau \qquad \forall s \in (0, |B_R|) \ .$$

Therefore for any $s \in (0, |B_R|)$

$$\frac{d\hat{v}_{i+1}}{ds}(s) - \frac{du_{i+1}^*}{ds}(s) - \frac{1}{n^2\sigma_n^{\frac{2}{n}}} \ s^{\frac{2}{n}-2} \int_0^s (\hat{v}_{i+1} - u_{i+1}^*)\, d\tau \le \frac{1}{n^2\sigma_n^{\frac{2}{n}}} \ s^{\frac{2}{n}-2} \int_0^s (u_i^* - \hat{v}_i)\, d\tau \ .$$

But as a consequence of the fact that inequality (2.13) holds for i we have that

$$\int_0^s (u_i^* - \hat{v}_i)\, d\tau \le 0 \quad \forall s \in (0, |B_R|)$$

and we get

$$\frac{d\hat{v}_{i+1}}{ds}(s) - \frac{du^*_{i+1}}{ds}(s) - \frac{1}{n^2\sigma_n^{\frac{2}{n}}}\, s^{\frac{2}{n}-2} \int_0^s (\hat{v}_{i+1} - u^*_{i+1})\, d\tau \le 0 \quad \forall s \in (0,\, |B_R|)\ .$$

We can now proceed as in [69], setting

$$y(s) := \int_0^s (\hat{v}_{i+1} - u^*_{i+1}) \quad \forall s \in (0,\, |B_R|)$$

so that

$$\begin{cases} y'' - \frac{1}{n^2\sigma_n^{\frac{2}{n}}}\, y \le 0 \quad \text{in } (0,\, |B_R|) \\ y(0) = y'(|B_R|) = 0 \end{cases}$$

and the maximum principle leads us to conclude that $y \ge 0$ which is equivalent to (2.14). □

Actually, in the proof of Theorem 2.1, we will not directly use the *comparison of integrals in balls*, Proposition 2.7, to reduce the problem to the radial case, but the following *comparison principle*:

Proposition 2.8 ([69], Corollary 1)**.** *Let u, v be weak solutions of (2.10) and (2.11) respectively. For every convex nondecreasing function $\phi :\, [0, +\infty) \to [0, +\infty)$ we have*

$$\int_{B_R} \phi(|u|)\, dx \le \int_{B_R} \phi(v)\, dx\ .$$

This result is a direct consequence of Proposition 2.7 and a well-known result of G. H. Hardy, J. E. Littlewood and G. Pólya [**?**].

Proposition 2.9 ([38])**.** *Let $h,\, g :\, [a,\, b] \to \mathbb{R}$ be non-negative measurable functions. The following conditions are equivalent:*

- *for any convex function ϕ we have $\int_a^b \phi(g(t))\, dt \le \int_a^b \phi(h(t))\, dt$;*
- *for any $x \in [a,\, b]$ we have $\int_a^x g^*(t)\, dt \le \int_a^x h^*(t)\, dt$ and equality holds when $x = b$.*

Proof of Proposition 2.8. From Proposition 2.7,

$$\int_0^x u^*(t)\, dt \le \int_0^x v(t)\, dt \qquad \forall x \in [0,\, |B_R|)\,.$$

Consequently, if

$$\int_0^{|B_R|} u^*(t)\, dt = \int_0^{|B_R|} v(t)\, dt \tag{2.15}$$

then, applying Proposition 2.9, we have

$$\int_{B_R} \phi(|u|)\, dx = \int_0^{|B_R|} \phi(u^*(t))\, dt \le \int_0^{|B_R|} \phi(v(t))\, dt = \int_{B_R} \phi(v)\, dx\,.$$

If (2.15) does not hold, it suffices to define

$$\overline{v}(t) := \begin{cases} v(t) & \forall t \in [0,\, M], \\ 0 & \forall t \in (M,\, |B_R|], \end{cases}$$

with $0 < M \leq |B_R|$ such that

$$\int_0^{B_R} u^*(t)\, dt = \int_0^M v(t)\, dt\,.$$

Then, applying again Proposition 2.9 and exploiting the monotonicity of ϕ, we obtain the desired estimate. □

Remark 2.10. *It is easy to adapt the previous arguments to obtain a result for general bounded domains. Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$ *be a bounded domain, we consider the problems:*

$$\begin{cases} (-\Delta + I)^k u = f & \text{in } \Omega \\ u \in W_N^{m,\,2}(\Omega) \end{cases} \qquad \begin{cases} (-\Delta + I)^k v = f^\sharp & \text{in } \Omega^\sharp \\ v \in W_N^{m,\,2}(\Omega^\sharp) \end{cases}$$

where $f \in L^{\frac{2n}{n+2}}(\Omega)$ *and* $\Omega^\sharp$ *is the ball in* $\mathbb{R}^n$ *centered at* $0 \in \mathbb{R}^n$ *with the same measure as* Ω*. Then for every convex nondecreasing function* $\phi\colon\, [0, +\infty) \to [0, +\infty)$ *we have*

$$\int_\Omega \phi(|u|)\, dx \leq \int_{\Omega^\sharp} \phi(v)\, dx\ .$$

Remark 2.11. *We can now explain how this last proposition may be used in the proof of Theorem 2.1. Let* $m = 2k < n$ *with* k *a positive integer. Let* $u \in \mathcal{C}_0^\infty(B_R)$ *with* $B_R \subset \mathbb{R}^n$ *and define* $f := (-\Delta + I)^k u$ *in* B_R*. By construction* u *is the unique solution of (2.10). Let* v *be the unique radial solution of (2.11), then by Proposition 2.8 it follows that*

$$\int_{B_R} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right)\, dx \leq \int_{B_R} \phi\left(\beta_0 |v|^{\frac{n}{n-m}}\right)\, dx\ .$$

Since $f \in L^{\frac{n}{m}}(B_R)$*, we have that* $f^\sharp \in L^{\frac{n}{m}}(B_R)$ *and thus* $v \in W^{m,\,\frac{n}{m}}_{N,\,rad}(B_R)$*. Furthermore*

$$\|v\|_{m,\,n} = \|(-\Delta + I)^k v\|_{\frac{n}{m}} = \|f^\sharp\|_{\frac{n}{m}} = \|f\|_{\frac{n}{m}} = \|(-\Delta + I)^k u\|_{\frac{n}{m}} = \|u\|_{m,\,n}\ .$$

This means that, starting with a function $u \in C_0^\infty(B_R)$*, we can always consider a radial function* $v \in W^{m,\,\frac{n}{m}}_{N,\,rad}(B_R)$ *which increases the integral we are interested in and which has the same* $\|\cdot\|_{m,\,n}$*-norm as* u*.*

2.2. An Adams-type inequality for radial functions in $W^{2,2}(\mathbb{R}^4)$

In this Section we will prove the first part of Theorem 2.4 in the case $m = 2$ and $n = 4$, namely we will prove the existence of a constant $C > 0$ such that

$$\sup_{u\in W^{2,2}_{\mathrm{rad}}(\mathbb{R}^4),\, \|u\|_{W^{2,2}}\le 1} \int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)\, dx \le C\,. \tag{2.16}$$

To do this we follow the techniques adopted in [61] for the proof of Theorem 1.3, and the key to adapt these arguments to the case of second order derivatives is the following stronger version of Adams' inequality:

Theorem 2.12 ([68])**.** *Let* $\Omega \subset \mathbb{R}^4$ *be a bounded domain, then there exists a constant* $C > 0$ *such that*

$$\sup_{u\in W^{2,2}(\Omega)\cap W^{1,2}_0(\Omega),\, \|\Delta u\|_2\le 1} \int_{\Omega} e^{32\pi^2 u^2}\, dx \le C|\Omega|$$

and this inequality is sharp.

Remark 2.13. *We point out that Adams' inequality, in its original form, deals with functions in* $W^{2,2}_0(\Omega)$ *(see Theorem 1.4) which is the closure of the space of smooth compactly supported functions. Note that* $W^{2,2}_0(\Omega)$ *is strictly contained in* $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ *and*

$$\sup_{u\in W^{2,2}_0(\Omega),\, \|\Delta u\|_2\le 1} \int_{\Omega} e^{32\pi^2 u^2}\, dx \le \sup_{u\in W^{2,2}(\Omega)\cap W^{1,2}_0(\Omega),\, \|\Delta u\|_2\le 1} \int_{\Omega} e^{32\pi^2 u^2}\, dx\,,$$

therefore Theorem 2.12 improves Adams' inequality showing that the sharp exponent $32\pi^2$ *does not depend on all the traces.*

In [68] C. Tarsi obtained more general embeddings in Zygmund spaces and Theorem 2.12 is a particular case of these results. For the convenience of the reader, we give here an alternative proof (see also C. S. Lin and J. Wei [46]). To do this we will follow an argument introduced by H. Brezis and F. Merle (see the proof of Theorem 1 in [16]) constructing an auxiliary function written in Riesz potential form and we will apply to this auxiliary function the following theorem due to D. R. Adams:

Theorem 2.14 ([2], Theorem 2)**.** *For* $1 < p < +\infty$*, there is a constant* $c_0 = c_0(p, n)$ *such that for all* $f \in L^p(\mathbb{R}^n)$ *with support contained in* Ω, $|\Omega| < +\infty$

$$\int_{\Omega} e^{\frac{n}{\omega_{n-1}}\left|\frac{I_\alpha * f(x)}{\|f\|_p}\right|^{p'}}\, dx \le c_0$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, ω_{n-1} *is the surface measure of the unit sphere* $S^{n-1} \subset \mathbb{R}^n$ *and*

$$I_\alpha * f(x) := \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, dy$$

is the Riesz potential of order $\alpha := \frac{n}{p}$.

Proof of Theorem 2.12. Let

$$C_D^\infty(\Omega) := \left\{ u \in C^\infty(\Omega) \cap C^0(\overline{\Omega}) \mid u|_{\partial\Omega} = 0 \right\} .$$

By density arguments, it suffices to prove that

$$\sup_{u \in C_D^\infty(\Omega),\, \|\Delta u\|_2 \le 1} \int_\Omega e^{32\pi^2 u^2}\, dx \le C|\Omega| .$$

Let $u \in C_D^\infty(\Omega)$ be such that $\|\Delta u\|_2 \le 1$ and set $f := \Delta u$ in Ω, so that u is a solution of the Dirichlet boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} .$$

We extend f to be zero outside Ω

$$\overline{f}(x) := \begin{cases} f(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^4 \setminus \Omega \end{cases}$$

and we define

$$\overline{u} := \left(\frac{4}{\omega_3 32\pi^2} \right)^{\frac{1}{2}} I_2 * |\overline{f}| \qquad \text{in } \mathbb{R}^4$$

so that $-\Delta \overline{u} = |\overline{f}|$ in $\mathbb{R}^4$. By construction $\overline{u} \ge 0$ in $\mathbb{R}^4$ and from the maximum principle it follows that $\overline{u} \ge |u|$ in Ω. Furthermore

$$32\pi^2 \overline{u}^2 \le \frac{4}{\omega_3} \left(\frac{I_2 * |\overline{f}|}{\|\overline{f}\|_2} \right)^2 \qquad \text{in } \mathbb{R}^4 .$$

Therefore

$$\int_\Omega e^{32\pi^2 u^2}\, dx \le \int_\Omega e^{32\pi^2 \overline{u}^2}\, dx \le \int_\Omega e^{\frac{4}{\omega_3} \left(\frac{I_2 * |\overline{f}|(x)}{\|\overline{f}\|_2} \right)^2}\, dx$$

and the last integral is bounded by a constant which depends on Ω only as a consequence of Theorem 2.14 with $n = 4$ and $p = 2$. $\square$

We can now begin the proof of (2.16). Let $u \in W^{2,2}_{\text{rad}}(\mathbb{R}^4)$ be such that $\|u\|_{W^{2,2}} \le 1$. Fixed $r_0 > 0$, set

$$I_1 := \int_{B_{r_0}} (e^{32\pi^2 u^2} - 1)\, dx , \qquad I_2 := \int_{\mathbb{R}^4 \setminus B_{r_0}} (e^{32\pi^2 u^2} - 1) dx$$

so that

$$\int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)\, dx = I_1 + I_2 .$$

During the proof we will show that it is possible to choose a suitable $r_0 > 0$ independent of u such that I_1 and I_2 are bounded by a constant which depends on r_0 only, and so we can conclude that (2.16) holds.

Firstly, we write I_2 using the power series expansion of the exponential function

$$I_2 = \sum_{k=1}^{+\infty} \frac{(32\pi^2)^k}{k!} I_{2,k} \,, \qquad I_{2,k} := \int_{\mathbb{R}^4 \setminus B_{r_0}} |u|^{2k}\, dx \,.$$

We estimate the single terms $I_{2,k}$ applying the following radial lemma.

Lemma 2.15 ([41], Lemma 1.1, Chapter 6)**.** *If* $u \in W^{1,2}_{rad}(\mathbb{R}^4)$ *then*

$$|u(x)| \le \frac{1}{\sqrt{\omega_3}} \frac{1}{|x|^{3/2}} \|u\|_{W^{1,2}}$$

for a.e. $x \in \mathbb{R}^4$*, where* $\omega_3 = 2\pi^2$ *is the surface measure of the unit sphere* $S^3 \subset \mathbb{R}^4$*.*

Hence for $k \ge 2$ we obtain

$$I_{2,k} \le \frac{\|u\|^{2k}_{W^{1,2}}}{(\omega_3)^k}\, \omega_3 \int_{r_0}^{+\infty} \frac{1}{\rho^{3k}}\, \rho^3 \, d\rho = \frac{\|u\|^{2k}_{W^{1,2}}}{(\omega_3)^{k-1}} \cdot \frac{{r_0}^{4-3k}}{3k-4} < \frac{\|u\|^{2k}_{W^{1,2}}}{(\omega_3)^{k-1}}\, {r_0}^{4-3k} \,.$$

This implies that

$$I_2 \le 32\pi^2 \|u\|_2^2 + \omega_3 r_0^4 \sum_{k=2}^{+\infty} \frac{1}{k!} \left(\frac{32\pi^2 \|u\|^2_{W^{1,2}}}{\omega_3 {r_0}^3} \right)^k \le c(r_0)$$

where the constant $c(r_0) > 0$ depends only on r_0 since by assumption $\|u\|_2 \le 1$ and $\|u\|_{W^{1,2}} \le 1$.

To estimate I_1, the idea is to use Theorem 2.12, and in order to do this we have to associate to $u \in W^{2,2}(B_{r_0})$ an auxiliary function $w \in W^{2,2}(B_{r_0}) \cap W^{1,2}_0(B_{r_0})$ such that $\|\Delta w\|_2 \le 1$. Recalling that $u \in W^{2,2}_{\mathrm{rad}}(\mathbb{R}^4)$, we define a radial function $v = v(|x|)$ as

$$v(|x|) =: u(|x|) - u(r_0) \qquad \text{for } 0 \le |x| \le r_0$$

and we can notice that $v \in W^{2,2}(B_{r_0}) \cap W^{1,2}_0(B_{r_0})$. Applying again the radial lemma, we get for $0 < |x| \le r_0$

$$\begin{aligned} u^2(|x|) &= v^2(|x|) + 2v(|x|)u(r_0) + u^2(r_0) \le v^2(|x|) + [v^2(|x|)u^2(r_0) + 1] + u^2(r_0) \le \\ &\le v^2(|x|) + v^2(|x|) \left[\frac{1}{2\pi^2} \frac{1}{{r_0}^3} \|u\|^2_{W^{1,2}} \right] + 1 + \frac{1}{2\pi^2} \frac{1}{{r_0}^3} \|u\|^2_{W^{1,2}} \le \\ &\le v^2(|x|) \left[1 + \frac{1}{2\pi^2} \frac{1}{{r_0}^3} \|u\|^2_{W^{1,2}} \right] + d(r_0) \,. \end{aligned}$$

Now we define

$$w(|x|) := v(|x|) \sqrt{1 + \frac{1}{2\pi^2} \frac{1}{{r_0}^3} \|u\|^2_{W^{1,2}}} \qquad \text{for all } \ 0 \le |x| \le r_0$$

so that $w \in W^{2,2}(B_{r_0}) \cap W^{1,2}_0(B_{r_0})$ and

$$u^2(|x|) \le w^2(|x|) + d(r_0) \qquad \text{for all } \ 0 < |x| \le r_0 \,. \tag{2.17}$$

By construction

$$\int_{B_{r_0}} (\Delta v)^2\, dx = \int_{B_{r_0}} (\Delta u)^2\, dx \le \|\Delta u\|_2^2 \le 1 - \|u\|_{W^{1,2}}^2$$

and hence

$$\begin{aligned}\int_{B_{r_0}} (\Delta w)^2\, dx &= \int_{B_{r_0}} \left[\Delta\left(v\sqrt{1+\frac{1}{2\pi^2}\frac{1}{{r_0}^3}\|u\|_{W^{1,2}}^2}\right)\right]^2 dx = \\ &= \left(1+\frac{1}{2\pi^2}\frac{1}{{r_0}^3}\|u\|_{W^{1,2}}^2\right)\int_{B_{r_0}} (\Delta v)^2\, dx \le \\ &\le \left(1+\frac{1}{2\pi^2}\frac{1}{{r_0}^3}\|u\|_{W^{1,2}}^2\right)(1-\|u\|_{W^{1,2}}^2) \le \\ &\le 1-\left(1-\frac{1}{2\pi^2}\frac{1}{{r_0}^3}\right)\|u\|_{W^{1,2}}^2 \le 1\end{aligned}$$

provided that ${r_0}^3 \ge \frac{1}{2\pi^2}$. From (2.17) it follows that

$$I_1 \le e^{32\pi^2 d(r_0)} \int_{B_{r_0}} e^{32\pi^2 w^2}\, dx$$

and if $r_0 \ge \sqrt[3]{\frac{1}{2\pi^2}}$, then the right hand side of this last inequality is bounded by a constant which depends on r_0 only, as a consequence of Theorem 2.12. This ends the proof of the first part of Theorem 2.4 in the case $m = 2$ and $n = 4$; for the sharpness see Section 2.5.

Remark 2.16. *In the estimate of I_1 we might expect to apply Adams' inequality (1.11). But to do this one would need to construct an auxiliary function w which is in $W_0^{2,2}(B_{r_0})$ and this is not an easy task. However in view of Theorem 2.12 it is sufficient that $w \in W^{2,2}(B_{r_0}) \cap W_0^{1,2}(B_{r_0})$, $\|\Delta w\|_2 \le 1$ to conclude that $\int_{B_{r_0}} (e^{32\pi^2 w^2} - 1)\, dx$ is bounded by a constant which depends on r_0 only.*

We can easily adapt the arguments above to obtain a proof of Proposition 2.2 in the case $m = 2$ and $n = 4$.

Proof of Proposition 2.2 in the case $m = 2$ and $n = 4$. Fix $R > 0$ and let $u \in W^{2,2}_{N,\mathrm{rad}}(B_R)$ be radial and such that $\|u\|_{W^{2,2}} \le 1$. First of all we recall that

$$W_N^{2,2}(B_R) = W^{2,2}(B_R) \cap W_0^{1,2}(B_R)$$

and so $u \in W^{2,2}_{\mathrm{rad}}(B_R) \cap W_0^{1,2}(B_R)$. To prove Proposition 2.2, we have to show that there exists a constant $C > 0$ independent of R and u such that

$$\int_{B_R} (e^{32\pi^2 u^2} - 1)\, dx \le C\ . \tag{2.18}$$

We have two alternatives:

(I) $R \le \sqrt[3]{\frac{1}{2\pi^2}}$. As in particular $\|\Delta u\|_2^2 \le 1$, we can apply Theorem 2.12 obtaining that

$$\int_{B_R} (e^{32\pi^2 u^2} - 1)\, dx \le C\,|B_R| \le C \left| B_{\sqrt[3]{\frac{1}{2\pi^2}}} \right| .$$

(II) $R > \sqrt[3]{\frac{1}{2\pi^2}}$. In this case we set

$$I_1 := \int_{B_{r_0}} (e^{32\pi^2 u^2} - 1)\, dx \,, \qquad I_2 := \int_{B_R \setminus B_{r_0}} (e^{32\pi^2 u^2} - 1)\, dx$$

where $\sqrt[3]{\frac{1}{2\pi^2}} \le r_0 < R$, so that

$$\int_{B_R} (e^{32\pi^2 u^2} - 1)\, dx = I_1 + I_2 \,.$$

To estimate I_1 and I_2 with a constant independent of R and u, we can use the same arguments as in the proof of Theorem 2.4. It suffices to notice that the radial lemma (Lemma 2.15) holds for any radial function in $W^{1,2}(\mathbb{R}^4)$ and, as $u \in W_0^{1,2}(B_R)$, we can extend u to be zero outside the ball B_R obtaining that $u \in W^{1,2}(\mathbb{R}^4)$, furthermore:

$$\|u\|_{W^{1,2}(\mathbb{R}^4)} = \|u\|_{W^{1,2}(B_R)}$$

□

2.3. An Adams-type inequality for radial functions in $W^{m,\frac{n}{m}}(\mathbb{R}^n)$

In this Section we will prove the first part of Theorem 2.4 in the case $m = 2k$ with k a positive integer and $m < n$. To this aim a crucial tool is the following extension of Adams' inequality to functions with homogeneous Navier boundary conditions.

Theorem 2.17 ([68])**.** *Let $m = 2k$ with k a positive integer and let $\Omega \subset \mathbb{R}^n$, with $m < n$, be a bounded domain. There exists a constant $C_{m,n} > 0$ such that*

$$\sup_{u \in W_N^{m,\frac{n}{m}}(\Omega),\, \|\nabla^m u\|_{\frac{n}{m}} \le 1} \int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}}\, dx \le C_{m,n} |\Omega|$$

and this inequality is sharp.

We give an alternative proof, following the idea of the proof of Theorem 2.12:

Proof of Theorem 2.17. By density arguments, it suffices to prove that

$$\sup_{u \in C_N^\infty(\Omega),\, \|\nabla^m u\|_{\frac{n}{m}} \le 1} \int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}}\, dx \le C_{m,n} |\Omega|$$

where

$$C_N^\infty(\Omega) := \{u \in C^\infty(\Omega) \cap C^{m-2}(\overline{\Omega}) \;\big|\; u|_{\partial\Omega} = \Delta^j u|_{\partial\Omega} = 0\;,\; 1 \le j < k\}\;.$$

Let $u \in C_N^\infty(\Omega)$ be such that $\|\nabla^m u\|_{\frac{n}{m}} = \|\Delta^k u\|_{\frac{n}{m}} \le 1$ and set $f := \Delta^k u$ in Ω, so that u is a solution of the Navier boundary value problem

$$\begin{cases} \Delta^k u = f & \text{in } \Omega \\ u = \Delta^j u = 0 & \text{on } \partial\Omega, \quad \forall j \in \{1, 2, \ldots, k-1\} \end{cases}.$$

We extend f by zero outside Ω

$$\overline{f}(x) := \begin{cases} f(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^{2m} \setminus \Omega \end{cases}$$

and we define

$$\overline{u} := \left(\frac{n}{\omega_{n-1}\beta_0}\right)^{\frac{n-m}{n}} I_m * |\overline{f}| \qquad \text{in } \mathbb{R}^{2m}\;,$$

so that $(-1)^k \Delta^k \overline{u} = |\overline{f}|$ in $\mathbb{R}^n$. By construction $\overline{u} \ge 0$ in $\mathbb{R}^n$ and

$$\beta_0 |\overline{u}|^{\frac{n}{n-m}} \le \frac{n}{\omega_{n-1}} \left(\frac{I_m * |\overline{f}|}{\|\overline{f}\|_{\frac{n}{m}}}\right)^{\frac{n}{n-m}} \qquad \text{in } \mathbb{R}^{2m}\;.$$

To end the proof it suffices to show that $\overline{u} \ge |u|$ in Ω. Indeed, if $\overline{u} \ge |u|$ in Ω, then

$$\int_\Omega e^{\beta_0 |u|^{\frac{n}{n-m}}}\, dx \le \int_\Omega e^{\beta_0 |\overline{u}|^{\frac{n}{n-m}}}\, dx \le \int_\Omega e^{\frac{n}{\omega_{n-1}}\left(\frac{I_m * |\overline{f}|}{\|\overline{f}\|_{\frac{n}{m}}}\right)^{\frac{n}{n-m}}}\, dx$$

and the last integral is bounded by a constant depending on Ω only, as a consequence of Theorem 2.14 with $p = \frac{n}{m} > 1$.

To see that $\overline{u} \ge |u|$, consider the following systems:

$$\begin{cases} \Delta u_1 = f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta u_i = u_{i-1} & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases} \quad i \in \{2, \ldots, k\}$$

$$\begin{cases} \Delta \overline{u}_1 = (-1)^k |\overline{f}| & \text{in } \Omega \\ \overline{u}_1 = \Delta^{k-1}\overline{u} & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta \overline{u}_i = \overline{u}_{i-1} & \text{in } \Omega \\ \overline{u}_i = \Delta^{k-i}\overline{u} & \text{on } \partial\Omega \end{cases} \quad i \in \{2, \ldots, k\}$$

where obviously $u_k = u$ and $\overline{u}_k = \overline{u}$ in Ω. Since for $i \in \{1, 2, \ldots, k-1\}$ we have

$$(-1)^k \Delta^{k-i} \overline{u} \begin{cases} \ge 0 & i \text{ even} \\ \le 0 & i \text{ odd} \end{cases} \qquad \text{in } \mathbb{R}^n,$$

by finite induction, and with the aid of the maximum principle we can conclude that $\overline{u} \ge |u|$ in Ω and this ends the proof. □

Now we begin the proof of the first part of Theorem 2.4. Let $u \in W^{m,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$ be such that $\|u\|_{W^{m,\frac{n}{m}}} \le 1$. Fixed $r_0 > 0$, set

$$I_1 := \int_{B_{r_0}} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx\ , \qquad I_2 := \int_{\mathbb{R}^n \setminus B_{r_0}} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx$$

so that

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx = I_1 + I_2\ .$$

We can notice that the starting point is the same as in the proof of the case $m = 2$, $n = 4$ and, as before, we will show that it is possible to choose a suitable $r_0 > 0$ independent of u such that I_1 and I_2 are bounded by a constant which depends on r_0 only.

In the estimate of I_2 there are no substantial differences to the case $m = 2$ and $n = 4$, we first need a suitable radial lemma, namely an adaptation of [41], Lemma1.1, Chapter 6:

Lemma 2.18. *If* $u \in W^{1,\frac{n}{m}}(\mathbb{R}^n)$ *then*

$$|u(x)| \le \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{n}m}} \|u\|_{W^{1,\frac{n}{m}}}$$

for a.e. $x \in \mathbb{R}^n$*, where* σ_n *is the volume of the unit ball in* $\mathbb{R}^n$*.*

Applying this radial lemma and using the power series expansion of the exponential function we get

$$\begin{aligned}
I_2 \quad &\le \quad \frac{\beta_0^{j\frac{n}{m}-1}}{\left(j\frac{n}{m}-1\right)!} \int_{\mathbb{R}^n \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j\frac{n}{m}-1\right)} dx + \\
&+ \quad \frac{n^2(m-1)}{n-m} \sigma_n r_0^n \sum_{j=j\frac{n}{m}}^{+\infty} \frac{1}{j!} \left(\frac{\beta_0 \|u\|_{W^{1,\frac{n}{m}}}^{\frac{n}{n-m}}}{(m\sigma_n)^{\frac{m}{n-m}} r_0^{\frac{n-1}{n-m}m}}\right)^j \le \\
&\le \quad \frac{\beta_0^{j\frac{n}{m}-1}}{\left(j\frac{n}{m}-1\right)!} \int_{\mathbb{R}^n \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j\frac{n}{m}-1\right)} dx + c(m,\, n,\, r_0)\ .
\end{aligned}$$

To estimate the first term on the right hand side of this last inequality, we need the continuity of the embedding of $W^{m,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$ in suitable L^q-spaces:

Lemma 2.19 ([48], Théorème II.1)**.** *The embedding* $W^{m\,\frac{n}{m}}_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ *is continuous for* $\frac{n}{m} \le q < +\infty$*.*

Now it suffices to notice that $\frac{n}{n-m}\left(j\frac{n}{m} - 1\right) \ge \frac{n}{m}$ to conclude that $I_2 \le \tilde{c}(m,\, n,\, r_0)$.

To estimate I_1 we apply, as in the case $m = 2$ and $n = 4$, Theorem 2.17 to an auxiliary radial function $w \in W^{m,\frac{n}{m}}_N(B_{r_0})$ with $\|\nabla^m w\|_{\frac{n}{m}} \le 1$ which increases the integral we are interested in. But the construction of this auxiliary function is rather difficult with respect to the case $m = 2$ and $n = 4$. In fact, in the case of second order derivatives, we only need to construct an auxiliary radial function which is zero on the boundary of B_{r_0}, while when dealing with m-th order derivatives, with $m > 2$, the auxiliary radial function has to be zero on the boundary of B_{r_0} together with its j-th order Laplacian for any $j \in \{1, 2, \dots, k-1\}$.

If $m = 2k > 2$ then for each $i \in \{1, 2, \dots, k-1\}$ we define

$$g_i(|x|) := |x|^{m-2i} \qquad \forall x \in B_{r_0}$$

so that $g_i \in W^{m, \frac{n}{m}}_{\text{rad}}(B_{r_0})$ and

$$\Delta^j g_i(|x|) = \begin{cases} c_i^j \, |x|^{m-2(i+j)} & \text{for } j \in \{1, \dots, k-i\} \\ 0 & \text{for } j \in \{k-i+1, \dots, k\} \end{cases} \qquad \forall x \in B_{r_0}$$

where

$$c_i^j := \prod_{h=1}^{j} [n + m - 2(h+i)]\,[m - 2(i+h-1)] \qquad \forall j \in \{1, 2, \dots, k-i\} .$$

These functions will be helpful in the construction of the auxiliary radial function w. A similar device was used in [37] to prove an embedding result for higher order Sobolev spaces, but with another aim, namely to show that a radial function defined in a ball may be extended to the whole space without increasing the Dirichlet norm while increasing the L^p-norm.

Let

$$v(|x|) := u(|x|) - \sum_{i=1}^{k-1} a_i g_i(|x|) - a_k \qquad \forall x \in B_{r_0}$$

where

$$\begin{aligned} a_i &:= \frac{\Delta^{k-i} u(r_0) - \sum_{j=1}^{i-1} a_j \Delta^{k-i} g_j(r_0)}{\Delta^{k-i} g_i(r_0)} \qquad \forall i \in \{1, 2, \dots, k-1\} , \\ a_k &:= u(r_0) - \sum_{i=1}^{k-1} a_i g_i(r_0) . \end{aligned}$$

We point out that if $m = 2k = 2$, namely when we deal with second order derivatives, then v reduces to

$$v(|x|) := u(|x|) - u(r_0) \qquad \forall x \in B_{r_0} .$$

By construction $v \in W^{m, \frac{n}{m}}_N(B_{r_0}) \cap W^{m, \frac{n}{m}}_{\text{rad}}(B_{r_0})$ and $\Delta^k v = \Delta^k u$ in B_{r_0} or equivalently $\nabla^m v = \nabla^m u$ in B_{r_0}. Furthermore

Lemma 2.20. *For* $0 < |x| \le r_0$ *we have*

$$\begin{aligned} \Big| u(|x|) \Big|^{\frac{n}{n-m}} \le{}& \Big| v(|x|) \Big|^{\frac{n}{n-m}} \left(1 + c_{m,n} \sum_{j=1}^{k-1} \frac{1}{r_0^{2j\frac{n}{m}-1}} \|\Delta^{k-j} u\|^{\frac{n}{m}}_{W^{1, \frac{n}{m}}} + \frac{c_{m,n}}{r_0^{n-1}} \|u\|^{\frac{n}{m}}_{W^{1, \frac{n}{m}}} \right)^{\frac{n}{n-m}} + \\ &+ d(m, n, r_0) \end{aligned}$$

where $c_{m,n} > 0$ *depends only on* m *and* n *and* $d(m, m, r_0) > 0$ *depends only on* m, n *and* r_0.

Proof. To simplify the notations let

$$g(|x|) := \sum_{i=1}^{k-1} a_i g_i(|x|) + a_k \qquad \forall x \in B_{r_0}$$

so that $v(|x|) = u(|x|) - g(|x|)$ for all $x \in B_{r_0}$. Fixed $0 < |x| \le r_0$, we set $r := |x|$ and so $0 < r \le r_0$.

Step 1. We want to dominate $|u(r)|^{\frac{n}{n-m}}$ with $|v(r)|^{\frac{n}{n-m}}$ up to multiplicative and additive constants depending only on m, n, r_0 and $g(r)$, and more precisely we will prove that

$$|u(r)|^{\frac{n}{n-m}} \le |v|^{\frac{n}{n-m}} \left(1 + \frac{m}{n-m} 2^{\frac{m}{n-m}} |g(r)|^{\frac{n}{m}}\right) + 2^{\frac{m}{n-m}} \left(1 + \frac{n}{n-m} |g(r)|^{\frac{n}{n-m}}\right) . \tag{2.19}$$

To this aim we recall that the binomial estimate

$$(a+b)^q \le a^q + q 2^{q-1} (a^{q-1} b + b^q)$$

is valid for $q \ge 1$ and a, $b \ge 0$. Using the definition of v and applying this binomial estimate we get

$$|u(r)|^{\frac{n}{n-m}} \le |v(r)|^{\frac{n}{n-m}} + \frac{n}{n-m} 2^{\frac{m}{n-m}} \left(|v(r)|^{\frac{m}{n-m}} |g(r)| + |g(r)|^{\frac{n}{n-m}}\right) . \tag{2.20}$$

As Young's inequality says that

$$ab \le \frac{m}{n} (ab)^{\frac{n}{m}} + \frac{n-m}{n}$$

provided that $ab \ge 0$, we can estimate

$$|v(r)|^{\frac{m}{n-m}} |g(r)| \le \frac{m}{n} |v(r)|^{\frac{n}{n-m}} |g(r)|^{\frac{n}{m}} + \frac{n-m}{n} \tag{2.21}$$

and this together with inequality (2.20) gives (2.19).

Step 2. We have to obtain a suitable estimate for $|g(r)|^\alpha$ and in particular we are interested in the cases $\alpha = \frac{n}{m}$ and $\alpha = \frac{n}{n-m}$, so we will assume that $\alpha > 1$. By convexity arguments

$$|g(r)|^\alpha \le 2^{k(\alpha-1)+1} \sum_{i}^{k-1} |a_i|^\alpha g_i^\alpha(r_0) + 2^{\alpha-1} |u(r_0)|^\alpha .$$

We will prove in *Step 3* below that

$$|a_i|^\alpha \le \bar{c}_i \sum_{j=1}^{i} r_0^{2\alpha(i-j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha \qquad \forall i \in \{1, 2, \ldots, k-1\} \tag{2.22}$$

where the constants $\bar{c}_i > 0$ depend on m and n only. As a consequence of (2.22) we get

$$\begin{aligned}
|g(r)|^\alpha &\le 2^{k(\alpha-1)+1} \sum_{i=1}^{k-1} \sum_{j=1}^{i} \bar{c}_i r_0^{\alpha(m-2j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha + 2^{\alpha-1} |u(r_0)|^\alpha = \\
&= 2^{k(\alpha-1)+1} \sum_{j=1}^{k-1} \left(r_0^{\alpha(m-2j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha \sum_{i=j}^{k-1} \bar{c}_i \right) + 2^{\alpha-1} |u(r_0)|^\alpha = \\
&= 2^{k(\alpha-1)+1} \sum_{j=1}^{k-1} \tilde{c}_j r_0^{\alpha(m-2j)} \left|\Delta^{k-j} u(r_0)\right|^\alpha + 2^{\alpha-1} |u(r_0)|^\alpha
\end{aligned}$$

with

$$\tilde{c}_j := \sum_{i=j}^{k-1} \overline{c}_i \qquad \forall j \in \{1,\, 2,\, \dots,\, k-1\}\ .$$

Now the radial lemma, Lemma 2.18, leads to

$$\begin{aligned} |g(r)|^\alpha \le 2^{k(\alpha-1)+1} \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}\alpha} \sum_{j=1}^{k-1} \tilde{c}_j r_0^{\alpha(m-2j-\frac{n-1}{n}m)} \|\Delta^{k-j} u\|^\alpha_{W^{1,\,\frac{n}{m}}} + \\ + 2^{\alpha-1} \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}\alpha} \frac{1}{r_0^{\frac{n-1}{n}m\alpha}} \|u\|^\alpha_{W^{1,\,\frac{n}{m}}}\ . \end{aligned} \tag{2.23}$$

Step 3. We have to show that (2.22) holds. We proceed by finite induction on i. When $i = 1$, by the definition of a_1 and g_1 we have

$$|a_1|^\alpha = \left|\frac{\Delta^{k-1}u(r_0)}{\Delta^{k-1}g_1(r_0)}\right|^\alpha = \frac{1}{(c_1^{k-1})^\alpha} \left|\Delta^{k-1}u(r_0)\right|^\alpha$$

which is nothing but (2.22) provided that $\overline{c}_1 := (c_1^{k-1})^{-\alpha}$. We now assume that (2.22) holds for any $j \in \{1,\, 2,\, \dots,\, i\}$ with $i \in \{1,\, 2,\, \dots,\, k-2\}$ and we show that

$$|a_{i+1}|^\alpha \le \overline{c}_{i+1} \sum_{j=1}^{i+1} r_0^{2\alpha(i+1-j)} \left|\Delta^{k-j}u(r_0)\right|^\alpha\ .$$

Using the definition of a_{i+1} and g_{i+1} we get

$$|a_{i+1}|^\alpha \le \frac{2^{\alpha-1}}{(c_{i+1}^{k-i-1})^\alpha} \left|\Delta^{k-i-1}u(r_0)\right|^\alpha + \frac{2^{i(\alpha-1)}}{(c_{i+1}^{k-i-1})^\alpha} \sum_{j=1}^{i} |a_j|^\alpha \left|\Delta^{k-i-1}g_j(r_0)\right|^\alpha\ .$$

By finite induction assumption and by definition of g_j with $j \in \{1,\, 2,\, \dots,\, i\}$ we can estimate

$$\begin{aligned} \sum_{j=1}^{i} a_j^\alpha \left(\Delta^{k-i-1}g_j(r_0)\right)^\alpha \quad &\le \quad \sum_{j=1}^{i}\sum_{h=1}^{j} \overline{c}_j (c_j^{k-i-1})^\alpha r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h}u(r_0)\right|^\alpha = \\ &= \quad \sum_{h=1}^{i} r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h}u(r_0)\right|^\alpha \left(\sum_{j=h}^{i} \overline{c}_j (c_j^{k-i-1})^\alpha\right) = \\ &= \quad \sum_{h=1}^{i} \hat{c}_h r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h}u(r_0)\right|^\alpha \end{aligned}$$

with

$$\hat{c}_h := \sum_{j=h}^{i} \overline{c}_j (c_j^{k-i-1})^\alpha\ .$$

In conclusion

$$
\begin{aligned}
|a_{i+1}|^{\alpha} &\leq \frac{2^{\alpha-1}}{(c_{i+1}^{k-i-1})^{\alpha}} \left|\Delta^{k-i-1} u(r_0)\right|^{\alpha} + \frac{2^{i(\alpha-1)}}{(c_{i+1}^{k-i-1})^{\alpha}} \sum_{h=1}^{i} \hat{c}_h r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h} u(r_0)\right|^{\alpha} \leq \\
&\leq \bar{c}_{i+1} \sum_{h=1}^{i+1} r_0^{2\alpha(i+1-h)} \left|\Delta^{k-h} u(r_0)\right|^{\alpha} .
\end{aligned}
$$

Step 4. Combining (2.19) and inequality (2.23) with $\alpha = \frac{n}{n-m}$, we obtain that

$$
|u(r)|^{\frac{n}{n-m}} \leq |v|^{\frac{n}{n-m}} \left(1 + \frac{m}{n-m} 2^{\frac{m}{n-m}} |g(r)|^{\frac{n}{m}}\right) + d(m,\, n,\, r_0) \ ,
$$

as $\|\Delta^{k-j} u\|_{W^{1,\,\frac{n}{m}}} \leq 1$ for $j \in \{1,\, \ldots,\, k-1\}$ and $\|u\|_{W^{1,\,\frac{n}{m}}} \leq 1$. Now, a further application of inequality (2.23) with $\alpha = \frac{n}{m}$ leads to

$$
|u(r)|^{\frac{n}{n-m}} \leq |v(r)|^{\frac{n}{n-m}} \left(1 + c_{m,\,n} \sum_{j=1}^{k-1} \frac{1}{r_0^{2j\frac{n}{m}-1}} \|\Delta^{k-j} u\|_{W^{1,\,\frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,\,n}}{r_0^{n-1}} \|u\|_{W^{1,\,\frac{n}{m}}}^{\frac{n}{m}}\right) + d(m,\, n,\, r_0)
$$

which easily implies the inequality expressed by the lemma. □

Now we define

$$
w(|x|) := v(|x|)\left(1 + c_{m,\,n} \sum_{j=1}^{k-1} \frac{1}{r_0^{2j\frac{n}{m}-1}} \|\Delta^{k-j} u\|_{W^{1,\,\frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,\,n}}{r_0^{n-1}} \|u\|_{W^{1,\,\frac{n}{m}}}^{\frac{n}{m}}\right) \qquad \forall x \in B_{r_0} \ .
$$

As $v \in W_N^{m,\,\frac{n}{m}}(B_{r_0}) \cap W_{\mathrm{rad}}^{m,\,\frac{n}{m}}(B_{r_0})$, we have that $w \in W_N^{m,\,\frac{n}{m}}(B_{r_0}) \cap W_{\mathrm{rad}}^{m,\,\frac{n}{m}}(B_{r_0})$ and from Lemma 2.20 it follows that

$$
\left|u(|x|)\right|^{\frac{n}{n-m}} \leq \left|w(|x|)\right|^{\frac{n}{n-m}} + d(m,\, n,\, r_0) \qquad \forall 0 < |x| \leq r_0 \ .
$$

Since

$$
\|\nabla^m v\|_{\frac{n}{m}} = \|\nabla^m u\|_{\frac{n}{m}} \leq \left(1 - \sum_{j=1}^{k-1} \|\Delta^{k-j} u\|_{W^{1,\,\frac{n}{m}}}^{\frac{n}{m}} - \|u\|_{W^{1,\,\frac{n}{m}}}^{\frac{n}{m}}\right)^{\frac{m}{n}} ,
$$

and the inequality

$$
(1-A)^q \leq 1 - qA
$$

holds for $0 \leq A \leq 1$ and for $0 < q \leq 1$, we have that

$$
\|\nabla^m v\|_{\frac{n}{m}} \leq \left(1 - \frac{m}{n} \sum_{j=1}^{k-1} \|\Delta^{k-j} u\|_{W^{1,\,\frac{n}{m}}}^{\frac{n}{m}} - \frac{m}{n} \|u\|_{W^{1,\,\frac{n}{m}}}^{\frac{n}{m}}\right) .
$$

Therefore

$$\begin{aligned}\|\nabla^m w\|_{\frac{n}{m}} &= \|\nabla^m v\|_{\frac{n}{m}}\left(1 + c_{m,n}\sum_{j=1}^{k-1}\frac{1}{r_0^{2j\frac{n}{m}-1}}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\frac{n}{m}}} + \frac{c_{m,n}}{r_0^{n-1}}\|u\|^{\frac{n}{m}}_{W^{1,\frac{n}{m}}}\right) \le \\ &\le \left(1 - \frac{m}{n}\sum_{j=1}^{k-1}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\frac{n}{m}}} - \frac{m}{n}\|u\|^{\frac{n}{m}}_{W^{1,\frac{n}{m}}}\right)\cdot \\ &\cdot \left(1 + c_{m,n}\sum_{j=1}^{k-1}\frac{1}{r_0^{2j\frac{n}{m}-1}}\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\frac{n}{m}}} + \frac{c_{m,n}}{r_0^{n-1}}\|u\|^{\frac{n}{m}}_{W^{1,\frac{n}{m}}}\right)\end{aligned}$$

and in conclusion

$$\|\nabla^m w\|_{\frac{n}{m}} \le 1 + \sum_{j=1}^{k-1}\left(\frac{c_{m,n}}{r_0^{2j\frac{n}{m}-1}} - \frac{m}{n}\right)\|\Delta^{k-j}u\|^{\frac{n}{m}}_{W^{1,\frac{n}{m}}} + \left(\frac{c_m}{r_0^{n-1}} - \frac{m}{n}\right)\|u\|^{\frac{n}{m}}_{W^{1,\frac{n}{m}}} \le 1 \quad (2.24)$$

provided that $r_0 > 0$ is sufficiently large: this is our choice of $r_0 > 0$. In conclusion

$$I_1 \le e^{\beta_0 d(m,n,r_0)}\int_{B_{r_0}} e^{\beta_0|w|^{\frac{n}{n-m}}}\,dx$$

and the right hand side of this inequality is bounded by a constant depending on r_0 only as a consequence of Theorem 2.17.

We end this Section with the

Proof of Proposition 2.2. We want to adapt the above arguments to obtain a proof of Proposition 2.2. The idea is to proceed exactly as in the case $m = 2$ and $n = 4$, but for this we have to specify:

- how the radial lemma (Lemma 2.18) can be used to obtain pointwise estimates for u and $\Delta^j u$ with $j \in \{1, 2, \dots, k-1\}$,
- how to modify the argument (Lemma 2.19) used in the estimate of I_2 to obtain an uppur bound for the integral

$$\int_{B_R\setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j\frac{n}{m}-1\right)}\,dx \quad (2.25)$$

  independent of u and R.

Let $u \in W^{m,\frac{n}{m}}_{N,\mathrm{rad}}(B_R)$ with $B_R \subset \mathbb{R}^n$. Since

$$W^{m,\frac{n}{m}}_N(B_R) \subset W^{1,\frac{n}{m}}_0(B_R)\,,$$

we may extend u by zero outside B_R, and obtain $u \in W^{1,\frac{n}{m}}_{\mathrm{rad}}(\mathbb{R}^n)$ with

$$\|u\|_{W^{1,\frac{n}{m}}(\mathbb{R}^n)} = \|u\|_{W^{1,\frac{n}{m}}(B_R)}$$

Thus we can apply Lemma 2.18 to u.

Similarly, for fixed $j \in \{1, 2, \ldots, k-1\}$, we have

$$W_N^{m-2j, \frac{n}{m}}(B_R) \subset W_0^{1, \frac{n}{m}}(B_R)$$

and since $\Delta^j u \in W_{N,\mathrm{rad}}^{m-2j, \frac{n}{m}}(B_R)$, we have in particular $\Delta^j u \in W_0^{1, \frac{n}{m}}(B_R)$. We extend $\Delta^j u$ to be zero outside B_R

$$f_j := \begin{cases} \Delta^j u & \text{in } B_R \\ 0 & \text{in } \mathbb{R}^n \setminus B_R \end{cases} .$$

As $\Delta^j u \in W_0^{1, \frac{n}{m}}(B_R)$ is radial, we have that $f_j \in W_{\mathrm{rad}}^{1, \frac{n}{m}}(\mathbb{R}^n)$ and f_j satisfies the assumption of Lemma 2.18. Therefore, for a.e. $x \in B_R$ we have

$$\begin{aligned} |\Delta^j u(x)| &= |f_j(x)| \leq \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{n}m}} \|f_j\|_{W^{1, \frac{n}{m}}(\mathbb{R}^n)} = \\ &= \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{n}m}} \|\Delta^j u\|_{W^{1, \frac{n}{m}}(B_R)} . \end{aligned}$$

It remains only to specify how to obtain an upper bound indepentent of u and R for the integral (2.25). Let $u \in W_{N,\mathrm{rad}}^{m, \frac{n}{m}}(B_R)$ be such that $\|u\|_{W^{m, \frac{n}{m}}} \leq 1$. As $u \in W_{\mathrm{rad}}^{1, \frac{n}{m}}(\mathbb{R}^n)$, from Lemma 2.18, it follows that there exists $r_1 = r_1(m, n) > 0$ independent of u and R such that

$$|u(x)| < 1 \qquad \text{for a.e. } x \in \mathbb{R}^n \setminus B_{r_1} .$$

Therefore for $R > r_1$ we can choose $0 < r_1 \leq r_0 < R$ so that

$$|u(x)| < 1 \qquad \text{for a.e. } x \in \mathbb{R}^n \setminus B_{r_0} ,$$

and since

$$\frac{n}{n-m}\left(j_{\frac{n}{m}} - 1\right) \geq \frac{n}{m}$$

we obtain that

$$\int_{B_R \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j_{\frac{n}{m}}-1\right)} \, dx \leq \int_{\mathbb{R}^n \setminus B_{r_0}} |u|^{\frac{n}{n-m}\left(j_{\frac{n}{m}}-1\right)} \, dx \leq \int_{\mathbb{R}^n \setminus B_{r_0}} |u|^{\frac{n}{m}} \, dx \leq 1 .$$

To conclude we can argue as in the proof of Proposition 2.2 in the case $m = 2$ and $n = 4$, but now the two alternatives that we have to distinguish are $R < \tilde{R}$ and $R \geq \tilde{R}$ with $\tilde{R} > r_1$ and such that (2.24) holds. □

2.4. Proof of the main theorem (Theorem 2.1)

Let $m = 2k$ with k a positive integer, $m < n$ and $\Omega \subseteq \mathbb{R}^n$ be a domain. Since any function $u \in W_0^{m, \frac{n}{m}}(\Omega)$ can be extended to be zero outside Ω obtaining a function in $(W^{m, \frac{n}{m}}(\mathbb{R}^n), \|\cdot\|_{m,n})$, we have that

$$\sup_{u \in W_0^{m, \frac{n}{m}}(\Omega),\, \|u\|_{m,n} \leq 1} \int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \leq \sup_{u \in W^{m, \frac{n}{m}}(\mathbb{R}^n),\, \|u\|_{m,n} \leq 1} \int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx$$

and the proof of the first part of Theorem 2.1 reduces to the following inequality

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le C_{m,n} \qquad \forall u \in W^{m,\frac{n}{m}}(\mathbb{R}^n),\ \|u\|_{m,n} = 1 \tag{2.26}$$

for some constant $C_{m,n} > 0$.

Let $u \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$ be such that $\|u\|_{m,n} = 1$, then there exists $\{u_j\}_{j\ge 1} \subset C_0^\infty(\mathbb{R}^n)$ such that $u_j \to u$ in $(W^{m,\frac{n}{m}}(\mathbb{R}^n), \|\cdot\|_{m,n})$ and $\|u_j\|_{m,n} = 1\ \forall j \ge 1$. Therefore $u_j \to u$ a.e. in $\mathbb{R}^n$, up to subsequences, and by Fatou's lemma

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le \liminf_{j\to+\infty} \int_{\mathbb{R}^n} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx\ .$$

But, for each fixed $j \ge 1$, there exists $R_j > 0$ such that supp $u_j \subset B_{R_j}$, so:

$$\int_{\mathbb{R}^n} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx = \int_{B_{R_j}} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx\ .$$

It is clear that if we can bound the integral on the right hand side of this last equality with a constant independent of j, then the proof of (2.26) is completed and hence Theorem 2.1 is thus proved. So it suffices to show that there exists a constant $C_{m,n} > 0$ independent of j such that

$$\int_{B_{R_j}} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx \le C_{m,n} \qquad \forall j \ge 1\ . \tag{2.27}$$

To this aim, for fixed $j \ge 1$, we define

$$f_j := (-\Delta + I)^k u_j$$

and consider the problem

$$\begin{cases} (-\Delta + I)^k v_j = f_j^\sharp & \text{in } B_{R_j} \\ v_j \in W_N^{m,2}(B_{R_j}) \end{cases}. \tag{2.28}$$

We now apply Proposition 2.8 which leads to a comparison between the integral in (2.27) and an analogous one involving v_j, as pointed out in Remark 2.11. In this way we obtain the following estimate

$$\int_{B_{R_j}} \phi\left(\beta_0 |u_j|^{\frac{n}{n-m}}\right) dx \le \int_{B_{R_j}} \phi\left(\beta_0 |v_j|^{\frac{n}{n-m}}\right) dx\ .$$

This estimate reduces the proof of (2.27) to the following inequality

$$\int_{B_{R_j}} \phi\left(\beta_0 |v_j|^{\frac{n}{n-m}}\right) dx \le C_{m,n} \tag{2.29}$$

for some constant $C_{m,n} > 0$ independent of j. But, as already noticed in Remark 2.11, $v_j \in W^{m,\frac{n}{m}}_{N,\mathrm{rad}}(B_R)$ and by (2.2)

$$\|v_j\|_{W^{m,\frac{n}{m}}} \le \|v_j\|_{m,n} = \|u_j\|_{m,n} = 1\ .$$

Thus (2.29) is a consequence of Proposition 2.2. □

We end the Section with the proof of Proposition 2.3.

Proof of Proposition 2.3. As in the proof of Theorem 2.17, by density arguments it suffices to prove that (2.4) holds for functions in

$$C^\infty_N(\Omega) := \left\{ u \in C^\infty(\Omega) \cap C^{m-2}(\overline{\Omega}) \;\middle|\; u|_{\partial\Omega} = \Delta^j u|_{\partial\Omega} = 0\ ,\ 1 \le j < k := \frac{m}{2} \right\} .$$

Let $u \in C^N_\infty(\Omega)$ be such that $\|u\|_{m,n} \le 1$. We define

$$f := (-\Delta + I)^k u$$

and we consider the problem

$$\begin{cases} (-\Delta + I)^k v = f^\sharp & \text{in } \Omega^\sharp \\ v \in W^{m,2}_N(\Omega^\sharp) \end{cases} \tag{2.30}$$

where $\Omega^\sharp$ is the ball in $\mathbb{R}^n$ centered at $0 \in \mathbb{R}^n$ with the same measure as Ω. Thus, as Ω is a bounded domain, we can apply the iterated version of the Trombetti-Vazquez comparison principle (see Remark 2.10) obtaining that

$$\int_\Omega \phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \le \int_{\Omega^\sharp} \phi\left(\beta_0 |v|^{\frac{n}{n-m}}\right) dx$$

and the last integral is bounded by a constant $C_{m,n} > 0$ independent of the domain Ω as a consequence of Proposition 2.2. □

2.5. Sharpness

We have already mentioned in Chapter 1 that Kozono et al. ([43], Corollary 1.3) proved that the supremum

$$\sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n),\, \|u\|_{m,n} \le 1} \int_{\mathbb{R}^n} \phi\left(\beta |u|^{\frac{n}{n-m}}\right) dx$$

is infinite for $\beta > \beta_0$. To do this they argue by contradiction using Bessel potentials and the sharpness of Adams' inequality (1.11), while here we will exhibit a sequence of test functions for which the integral in (2.1) can be made arbitrarily large, if the exponent β_0 is replaced by a number $\beta > \beta_0$.

In the case $m = 2$ and $n = 4$, we will consider a sequence of test functions that was used in [49] to prove a generalized version of Adams' inequality for bounded domains in $\mathbb{R}^4$. The following Proposition gives the sharpness of inequality (2.1) in the case $m = 2$ and $n = 4$.

Proposition 2.21. *Assume that* $\beta > 32\pi^2$. *Then, for any domain* $\Omega \subseteq \mathbb{R}^4$

$$\sup_{u \in W^{2,2}_0(\Omega),\, \|u\|_{2,4} \le 1} \int_\Omega (e^{\beta u^2} - 1)\, dx = +\infty .$$

Proof. Without loss of generality we assume that the unit ball $B_1 \subset \Omega$. For $\varepsilon > 0$ we define

$$u_\varepsilon(x) := \begin{cases} \sqrt{\frac{1}{32\pi^2}\log\frac{1}{\varepsilon}} - \frac{|x|^2}{\sqrt{8\pi^2\varepsilon\log\frac{1}{\varepsilon}}} + \frac{1}{\sqrt{8\pi^2\log\frac{1}{\varepsilon}}} & |x| \le \sqrt[4]{\varepsilon} \\ \frac{1}{\sqrt{2\pi^2\log\frac{1}{\varepsilon}}}\log\frac{1}{|x|} & \sqrt[4]{\varepsilon} < |x| \le 1 \\ \eta_\varepsilon & |x| > 1 \end{cases} \tag{2.31}$$

where $\eta_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ is such that $\eta_\varepsilon|_{\partial B_1} = \eta_\varepsilon|_{\partial\Omega} = 0\,,\ \frac{\partial\eta_\varepsilon}{\partial\nu}|_{\partial B_1} = \frac{1}{\sqrt{2\pi^2\log\frac{1}{\varepsilon}}}\,,\ \frac{\partial\eta_\varepsilon}{\partial\nu}|_{\partial\Omega} = 0$ and η_ε, $|\nabla\eta_\varepsilon|$, $\Delta\eta_\varepsilon$ are all $O\left(1/\sqrt{\log\frac{1}{\varepsilon}}\right)$. If $0 < \varepsilon < 1$ then we have that $u_\varepsilon \in W_0^{2,\,2}(\Omega)$, easy computations give

$$\|u_\varepsilon\|_2^2 = o\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big),\ \|\nabla u_\varepsilon\|_2^2 = o\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big),\ \|\Delta u_\varepsilon\|_2^2 = 1 + o\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)$$

and $\|u_\varepsilon\|_{2,\,4} = \left(\|\Delta u_\varepsilon\|_2^2 + 2\|\nabla u_\varepsilon\|_2^2 + \|u_\varepsilon\|_2^2\right)^{1/2} \to 1$ as $\varepsilon \to 0^+$. Now we normalize u_ε, setting

$$\tilde{u}_\varepsilon := \frac{u_\varepsilon}{\|u_\varepsilon\|_{2,\,4}} \in W_0^{2,\,2}(\Omega)$$

for $\varepsilon > 0$ sufficiently small. Since

$$\tilde{u}_\varepsilon \ge \frac{1}{\|u_\varepsilon\|_{2,\,4}}\sqrt{\frac{1}{32\pi^2}\log\frac{1}{\varepsilon}} \quad \text{ on } B_{\sqrt[4]{\varepsilon}}\,,$$

we have

$$\begin{aligned} \sup_{u\in W_0^{2,\,2}(\Omega),\,\|u\|_{2,\,4}\le 1}\int_\Omega (e^{\beta u^2} - 1)\,dx \quad &\ge \quad \lim_{\varepsilon\to 0^+}\int_{B_{\sqrt[4]{\varepsilon}}} (e^{\beta\tilde{u}_\varepsilon^2} - 1)\,dx \ge \\ &\ge \quad \lim_{\varepsilon\to 0^+} 2\pi^2\Big(e^{\frac{1}{\|u_\varepsilon\|^2}\frac{\beta}{32\pi^2}\log\frac{1}{\varepsilon}} - 1\Big)\left[\frac{r^4}{4}\right]_0^{\sqrt[4]{\varepsilon}} = +\infty\,. \end{aligned}$$

□

The test functions u_ε with $\varepsilon > 0$ defined in (2.31) of the above proof give also the sharpness of inequalities (2.3), (2.4) and (2.5) in the case $m = 2$ and $n = 4$.

We now consider the general case $m = 2k < n$ with k a positive integer. In this case the sequence of test functions which gives the sharpness of Adams' inequality in bounded domains in [2] gives also the sharpness of Adams' inequality in unbounded domains.

Proposition 2.22. *Assume that* $\beta > \beta_m$. *Then, for any domain* $\Omega \subseteq \mathbb{R}^n$

$$\sup_{u\in W_0^{m,\,\frac{n}{m}}(\Omega),\,\|u\|_{m,\,n}\le 1}\int_\Omega \phi\left(\beta|u|^{\frac{n}{n-m}}\right)\,dx = +\infty\,.$$

Proof. Without loss of generality we assume that the unit ball $B_1 \subset \Omega$. Let $\phi \in C^\infty([0,\,1])$ be such that

$$\begin{aligned}\phi(0) &= \phi'(0) = \cdots = \phi^{m-1}(0) = 0\ ,\\ \phi(1) &= \phi'(1) = 1, \quad \phi''(1) = \cdots = \phi^{(m-1)}(1) = 0\ .\end{aligned}$$

For $0 < \varepsilon < \frac{1}{2}$ we set

$$H(t) := \begin{cases} \varepsilon\phi\left(\frac{t}{\varepsilon}\right) & 0 < t \le \varepsilon \\ t & \varepsilon < t \le 1-\varepsilon \\ 1 - \varepsilon\phi\left(\frac{1-t}{\varepsilon}\right) & 1-\varepsilon < t \le 1 \\ 1 & 1 < t \end{cases}$$

and the choice of $0 < \varepsilon < \frac{1}{2}$ will be made during the proof. We introduce Adams' test functions

$$\psi_r(|x|) := H\left(\frac{\log\frac{1}{|x|}}{\log\frac{1}{r}}\right) \qquad \forall x \in \mathbb{R}^n \setminus \{0\}\ .$$

By construction, for $r > 0$ sufficiently small, $\psi_r \in W_0^{m,\,\frac{n}{m}}(\Omega)$, $\psi(|x|) = 1$ for $x \in B_r \setminus \{0\}$, and Adams in [2] proved that

$$\|\nabla^m \psi_r\|_{\frac{n}{m}}^{\frac{n}{m}} \le \omega_{n-1} a(m,\,n)^{\frac{n}{m}} \left(\log\frac{1}{r}\right)^{1-\frac{n}{m}} A_r$$

where

$$a(m,\,n) := \frac{\beta_0^{\frac{n-m}{n}}}{n\sigma_n^{\frac{m}{n}}}\ , \qquad A_r = A_r(m,\,n) := \left[1 + 2\varepsilon\Big(\|\psi'\|_\infty + O\big((\log 1/r)^{-1}\big)\Big)^{\frac{n}{m}}\right]\ .$$

Easy computations give also that for $r > 0$ sufficiently small

$$\|\psi_r\|_{\frac{n}{m}}^{\frac{n}{m}} = o\Big(\big(\log\frac{1}{r}\big)^{-\frac{n-m}{m}}\Big)\ , \quad \|\nabla^j \psi_r\|_{\frac{n}{m}}^{\frac{n}{m}} = o\Big(\big(\log\frac{1}{r}\big)^{-\frac{n-m}{m}}\Big) \qquad \forall j \in \{1,\,2,\,\ldots,\,m-1\}\ .$$

Now we define

$$u_r(|x|) := \left(\log\frac{1}{r}\right)^{\frac{n-m}{n}} \cdot \psi_r(|x|) \qquad \forall x \in \mathbb{R}^n \setminus \{0\}\ .$$

We can notice that for $r > 0$ sufficiently small $u_r \in W_0^{m,\,\frac{n}{m}}(\Omega)$, $u_r(|x|) = \left(\log\frac{1}{r}\right)^{\frac{n-m}{n}}$ for $x \in B_r \setminus \{0\}$ and

$$\|u_r\|_{m,\,n}^{\frac{n}{m}} \;\le\; \|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} + c_{m,\,n}\Big(\|u_r\|_{\frac{n}{m}}^{\frac{n}{m}} + \sum_{j=1}^{m-1} \|\nabla^j u_r\|_{\frac{n}{m}}^{\frac{n}{m}}\Big) \le \omega_{n-1} a^{\frac{n}{m}}(m,\,n)\big(A_r + o(1)\big)\ ,$$

so in particular

$$\|u_r\|_{m,\,n}^{\frac{n}{n-m}} \le \omega_{n-1}^{\frac{m}{n-m}} a^{\frac{n}{n-m}}(m,\,n)\big(A_r + o(1)\big)^{\frac{m}{n-m}} = \frac{\beta_0}{n}\big(A_r + o(1)\big)^{\frac{m}{n-m}}\ .$$

Therefore, for $r > 0$ sufficiently small, we have

$$\begin{aligned}
\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega),\, \|u\|_{m,n} \leq 1} \int_\Omega \phi\left(\beta |u|^{\frac{n}{n-m}}\right) dx &\geq \lim_{r \to 0^+} \int_{B_r} \phi\left(\beta\Big(\frac{|u_r|}{\|u_r\|_{m,n}}\Big)^{\frac{n}{n-m}}\right) dx \\
&\geq \lim_{r \to 0^+} \sigma_n \phi\left(\frac{\beta}{\|u_r\|_{m,n}^{\frac{n}{n-m}}} \log\frac{1}{r}\right) r^n \\
&\geq \lim_{r \to 0^+} \sigma_n e^{\log r\left(n - \frac{\beta}{\|u_r\|_{m,n}^{\frac{n}{n-m}}}\right)}.
\end{aligned}$$

If we choose $0 < \varepsilon < \frac{1}{2}$ so that

$$\beta_0 < \beta_0\big(1 + 2\varepsilon \|\phi'\|_\infty^{\frac{n}{m}}\big)^{\frac{m}{n-m}} < \beta$$

then

$$\lim_{r \to 0^+}\left(n - \frac{\beta}{\|u_r\|_{m,n}^{\frac{n}{n-m}}}\right) \leq n\left(1 - \frac{\beta}{\beta_0(1 + 2\varepsilon\|\phi'\|_\infty^{\frac{n}{m}})^{\frac{m}{n-m}}}\right) < 0$$

and

$$\lim_{r \to 0^+} \sigma_n e^{\log r\left(n - \frac{\beta}{\|u_r\|_{m,n}^{\frac{n}{n-m}}}\right)} = +\infty$$

$\square$

The same proof gives also the sharpness of inequalities (2.3), (2.4) and (2.5) in the general case $m = 2k < n$ with k a positive integer.

CHAPTER 3

Consequences of the Adams-type inequality in $\mathbb{R}^4$

In view of applications to biharmonic equations in $\mathbb{R}^4$, in order to simplify the notations, we will write $H^2(\mathbb{R}^4)$ instead of $W^{2,2}(\mathbb{R}^4)$ and we will denote by $\|\cdot\|_{H^2}$ the Sobolev norm

$$\|u\|_{H^2}^2 := \|(-\Delta + I)u\|_2^2 = \|\Delta u\|_2^2 + 2\|\nabla u\|_2^2 + \|u\|_2^2 \quad \forall u \in H^2(\mathbb{R}^4) \,.$$

In this Chapter we will prove some direct consequences of the Adams-type inequality in $\mathbb{R}^4$ (Theorem 2.1),

$$\sup_{u \in H^2(\mathbb{R}^4),\, \|u\|_{H^2} \leq 1} \int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)\, dx < +\infty \,, \tag{3.1}$$

which will allow us to study biharmonic equations in $\mathbb{R}^4$ involving nonlinearities with exponential growth.

We point out that (3.1) holds also if we replace the Sobolev norm $\|\cdot\|_{H^2}$ with the equivalent norm

$$\|u\|_{H^2,\,\tau}^2 := \|(-\Delta + \tau I)u\|_2^2 = \|\Delta u\|_2^2 + \tau(2\|\nabla u\|_2^2 + \tau\|u\|_2^2) \quad \forall u \in H^2(\mathbb{R}^4)$$

where $\tau > 0$. In fact, carefully reading the proof of (3.1), we can notice that the value $\tau = 1$, appearing in $\|\cdot\|_{H^2} = \|\cdot\|_{H^2,\,1}$ as a multiplicative constant for the L^2-norm of the gradient and for the L^2-norm of the function itself, does not play any role and can be replaced by any $\tau > 0$. Hence we have indeed that the following inequality holds

$$\sup_{u \in H^2(\mathbb{R}^4),\, \|u\|_{H^2,\,\tau} \leq 1} \int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)\, dx < +\infty \tag{3.2}$$

where $\tau > 0$ is arbitrarily fixed. Consequently, for fixed a, $b > 0$, we have also that

$$\sup_{u \in H^2(\mathbb{R}^4),\, \|u\|_{H^2,\,a,\,b} \leq 1} \int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)\, dx < +\infty \tag{3.3}$$

where

$$\|u\|_{H^2,\,a,\,b}^2 := \|\Delta u\|_2^2 + a\|\nabla u\|_2^2 + b\|u\|_2^2 \quad \forall u \in H^2(\mathbb{R}^4) \,.$$

In fact, setting $\tau := \min\left\{\frac{a}{2},\, \sqrt{b}\right\}$, it suffices to notice that

$$\|u\|_{H^2,\,\tau} \leq \|u\|_{H^2,\,a,\,b} \quad \forall u \in H^2(\mathbb{R}^4)\,.$$

In [49], G. Lu and Y. Yang proved the following Lions-type concentration-compactness result

Lemma 3.1 ([49], Proposition 3.1)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^4$*. Let* $\{u_n\}_n \subset H_0^2(\Omega)$ *be such that* $\|\Delta u_n\|_2 = 1$ *for any* $n \geq 1$ *and* $u_n \rightharpoonup u$ *in* $H_0^2(\Omega)$*. If* $\|\Delta u\|_2 < 1$ *then*

$$\sup_n \int_\Omega e^{pu_n^2}\,dx < +\infty \qquad \forall p \in \left(0,\, \frac{32\pi^2}{1-\|\Delta u\|_2^2}\right)\,.$$

Due to additional informations about the sequence, the exponent p appearing in this concentration-compactness estimate is above Adams' sharp exponent, see inequality (1.11), provided $\|\Delta u\|_2 \neq 0$. Now, we will establish a version of Lemma 3.1 for the whole space $\mathbb{R}^4$.

Lemma 3.2. *Let* $X \subseteq H^2(\mathbb{R}^4)$ *be a Hilbert space endowed with the norm* $|\cdot|$*. Let* $\{u_n\}_n \subset X$ *be such that* $|u_n| = 1$ *for any* $n \geq 1$ *and let* $u \in X$ *be the weak limit of* $\{u_n\}_n$ *in* X*. If* $|u| < 1$ *and*

$$\lim_{n\to+\infty} \|u_n - u\|_{H^2}^2 \leq \lim_{n\to+\infty} |u_n - u|^2 \tag{3.4}$$

then

$$\sup_n \int_{\mathbb{R}^4} (e^{pu_n^2}-1)\,dx < +\infty \qquad \forall p \in \left(0,\, \frac{32\pi^2}{1-|u|^2}\right)\,.$$

Proof of Lemma 3.2. We first consider the case $u = 0$. If $p\|u_n\|_{H^2} \leq 32\pi^2$, at least for any n sufficiently large, then nothing needs to be proved because of the Adams-type inequality (3.1).

From (3.4) it follows that

$$\lim_{n\to+\infty} \|u_n\|_{H^2}^2 \leq 1\,,$$

therefore for any $\varepsilon > 0$ there exists $n_\varepsilon \geq 1$ such that

$$\|u_n\|_{H^2}^2 \leq 1 + \varepsilon \quad \forall n \geq n_\varepsilon$$

and, since $p < 32\pi^2$, there exists $\varepsilon > 0$ such that

$$p\|u_n\|_{H^2}^2 \leq 32\pi^2 \quad \forall n \geq n_\varepsilon\,.$$

Now, we consider the case $u \neq 0$. Since, for arbitrarily fixed $\varepsilon > 0$, the inequality

$$a^2 \leq (1+\varepsilon^2)(a-b)^2 + \left(1+\frac{1}{\varepsilon^2}\right)b^2 \tag{3.5}$$

holds for any $a,\, b \in \mathbb{R}$, for any $\varepsilon > 0$ we have that

$$u_n^2 \leq (1+\varepsilon^2)(u_n-u)^2 + \left(1+\frac{1}{\varepsilon^2}\right)u^2 \qquad \forall n \geq 1$$

and hence

$$\int_{\mathbb{R}^4} (e^{pu_n^2} - 1)\,dx \le \int_{\mathbb{R}^4} \left(e^{p(1+\varepsilon^2)(u_n-u)^2} e^{p\left(1+\frac{1}{\varepsilon^2}\right)u^2} - 1 \right) dx \qquad \forall n \ge 1 \,.$$

Using Young's inequality,

$$ab - 1 \le \frac{1}{q}(a^q - 1) + \frac{1}{q'}(b^{q'} - 1) \qquad \forall a,\, b \ge 0,\ \forall q > 1 \tag{3.6}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, we get for $q > 1$ and for any $\varepsilon > 0$:

$$\int_{\mathbb{R}^4} (e^{pu_n^2} - 1)\,dx \le \frac{1}{q} \int_{\mathbb{R}^4} \left(e^{pq(1+\varepsilon^2)(u_n-u)^2} - 1 \right) dx + \frac{1}{q'} \int_{\mathbb{R}^4} \left(e^{pq'\left(1+\frac{1}{\varepsilon^2}\right)u^2} - 1 \right) dx \quad \forall n \ge 1 \,.$$

Therefore if we prove that for some $q > 1$ and for some $\varepsilon > 0$

$$\mathcal{I}_n^{q,\varepsilon} := \int_{\mathbb{R}^4} \left(e^{pq(1+\varepsilon^2)(u_n-u)^2} - 1 \right) dx \le C \qquad \forall n \ge 1 \,,$$

where $C > 0$ is a constant independent of n, then we obtain the desired inequality. For any $n \ge 1$ we can notice that

$$\mathcal{I}_n^{q,\varepsilon} = \int_{\mathbb{R}^4} \left(e^{pq(1+\varepsilon^2)\|u_n-u\|_{H^2}^2 \left(\frac{u_n-u}{\|u_n-u\|_{H^2}} \right)^2} - 1 \right) dx \le C$$

provided that

$$pq(1+\varepsilon^2)\|u_n - u\|_{H^2}^2 \le 32\pi^2 \tag{3.7}$$

at least for any $n \ge 1$ sufficiently large, and thus to conclude it remains only to prove the existence of $q > 1$ and $\varepsilon > 0$ such that (3.7) holds. Since $u_n \rightharpoonup u$ in X and $|u_n| = 1$ for any $n \ge 1$, we have that

$$\lim_{n\to+\infty} |u_n - u|^2 = 1 - |u|^2 \,.$$

Consequently

$$\lim_{n\to+\infty} \|u_n - u\|_{H^2}^2 \le 1 - |u|^2$$

and, using the fact that $p < \frac{32\pi^2}{1-|u|^2}$, we get the existence of $\sigma > 0$ such that

$$p\|u_n - u\|_{H^2}^2 < p(1 - |u|^2)(1+\sigma) < 32\pi^2 \qquad \forall n \ge \overline{n}$$

where $\overline{n} \ge 1$ is sufficiently large. Therefore choosing $q > 1$ sufficiently close to 1 and $\varepsilon > 0$ sufficiently close to 0 we have

$$pq(1+\varepsilon^2)\|u_n - u\|_{H^2}^2 \le 32\pi^2 \quad \forall n \ge \overline{n} \,.$$

$\square$

Remark 3.3. *We point out that, as a direct consequence of Lemma 3.2, we have indeed a Lions-type concentration-compactness result in* $(H^2(\mathbb{R}^4),\, \|\cdot\|_{H^2})$.

Now we introduce the subspace E of $H^2(\mathbb{R}^4)$ defined as

$$E := \left\{ u \in H^2(\mathbb{R}^4) \mid \int_{\mathbb{R}^4} V(x)u^2\,dx < +\infty \right\}$$

where $V : \mathbb{R}^4 \to \mathbb{R}$ is a continuous function bounded from below by a positive constant, namely V satisfies

(V_0) $V : \mathbb{R}^4 \to \mathbb{R}$ is continuous and $V(x) \geq V_0 > 0$ for any $x \in \mathbb{R}^4$.

From (V_0), it follows that E is a Hilbert space endowed with the scalar product

$$\langle u, v\rangle := \int_{\mathbb{R}^4} \Delta u \Delta v\,dx + \int_{\mathbb{R}^4} V(x)uv\,dx \qquad u, v \in E$$

to which corresponds the norm $\|u\| := \sqrt{\langle u, u\rangle}$. Applying an interpolation inequality, it is easy to see that the embedding $E \hookrightarrow H^2(\mathbb{R}^4)$ is continuous.

Since our aim is to study biharmonic equations in $\mathbb{R}^4$ which can be treated variationally in E, we will need the following consequence of Lemma 3.2 for the space $(E, \|\cdot\|)$

Lemma 3.4. *Assume* (V_0). *Let* $\{u_n\}_n \subset E$ *be such that* $\|u_n\| = 1$ *for any* $n \geq 1$ *and let* $u \in E$ *be the weak limit of* $\{u_n\}_n$ *in* E. *If* $\|u\| < 1$ *and* $u_n \to u$ *in* $L^2(\mathbb{R}^4)$ *then*

$$\sup_n \int_{\mathbb{R}^4} (e^{pu_n^2} - 1)\,dx < +\infty \quad \forall p \in \left(0, \frac{32\pi^2}{1 - \|u\|^2}\right) .$$

Proof. The strong convergence $u_n \to u$ in $L^2(\mathbb{R}^4)$ together with the following interpolation inequality

$$\|\nabla(u_n - u)\|_2^2 \leq C\|\Delta(u_n - u)\|_2\|u_n - u\|_2 \leq \overline{C}\|u_n - u\|_2 \quad \forall n \geq 1 ,$$

where $C, \overline{C} > 0$ are constants independent of n, leads us to conclude that

$$\lim_{n\to+\infty} \|u_n - u\|_{H^2}^2 = \lim_{n\to+\infty} \|\Delta(u_n - u)\|_2^2 \leq \lim_{n\to+\infty} \|u_n - u\|^2 .$$

The proof is complete in view of Lemma 3.2. □

Let

$$H^k_{\mathrm{rad}}(\mathbb{R}^4) := \{u \in H^k(\mathbb{R}^4) \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^4\}$$

with $k \geq 1$, we recall the *radial lemma*

Lemma 3.5. *For any* $u \in H^1_{rad}(\mathbb{R}^4)$

$$|u(x)| \leq \frac{1}{\sqrt{2\pi^2}} \frac{1}{|x|^{\frac{3}{2}}} \|u\|_{H^1} \qquad a.e. \text{ in } \mathbb{R}^4 \tag{3.8}$$

where $\|\cdot\|_{H^1}$ *is the standard Dirichlet norm, namely* $\|u\|_{H^1}^2 := \|\nabla u\|_2^2 + \|u\|_2^2$ *for any* $u \in H^1(\mathbb{R}^4)$.

Let

$$E_{\text{rad}} := E \cap H^2_{\text{rad}}(\mathbb{R}^4) = \left\{ u \in H^2_{\text{rad}}(\mathbb{R}^4) \mid \int_{\mathbb{R}^4} V(x) u^2 \, dx < +\infty \right\} ,$$

we have the following result

Lemma 3.6. *Assume* (V_0)*. Let* $\{u_n\}_n \subset E_{rad}$ *be such that* $\|u_n\| = 1$ *for any* $n \geq 1$ *and let* $u \in E_{rad}$ *be the weak limit of* $\{u_n\}_n$ *in* E*. If* $0 < \|u\|^2 < 1$ *then*

$$\sup_n \int_{\mathbb{R}^4} (e^{pu_n^2} - 1) \, dx < +\infty \qquad \forall p \in \left(0, \, \frac{32\pi^2}{1 - \|u\|^2}\right) .$$

Proof. The idea is to follow the proof of [63], Theorem 1.4 which is based on the techniques introduced in [61]. Let $R > 0$ be arbitrarily fixed, we split the integral into two parts

$$\int_{\mathbb{R}^4} (e^{pu_n^2} - 1) \, dx = \int_{\mathbb{R}^4 \setminus B_R} (e^{pu_n^2} - 1) \, dx + \int_{B_R} (e^{pu_n^2} - 1) \, dx \quad \forall n \geq 1 .$$

Applying the *radial lemma* (3.8), we can estimate

$$\int_{\mathbb{R}^4 \setminus B_R} (e^{pu_n^2} - 1) \, dx \leq \frac{p}{V_0} + 2\pi^2 R^4 e^{\frac{p}{2\pi^2 R^3} C} \quad \forall n \geq 1$$

where $C > 0$ is a constant independent of n. Therefore to end the proof it remains only to show that

$$\sup_n \int_{B_R} (e^{pu_n^2} - 1) \, dx < +\infty \qquad \forall p \in \left(0, \, \frac{32\pi^2}{1 - \|u\|^2}\right) . \tag{3.9}$$

Applying inequality (3.5) with $a = u_n$ and $b = u - u(R) + u_n(R)$, observing that using convexity arguments and the *radial lemma* (3.8) we have

$$b^2 \leq 2^2(|u|^2 + |u(R)|^2 + |u_n(R)|^2) \leq 2^2(|u|^2 + C(R)) ,$$

for any $\varepsilon > 0$ we obtain the following estimate

$$u_n^2 \leq (1 + \varepsilon^2) v_n^2 + 2^2 \left(1 + \frac{1}{\varepsilon^2}\right) (|u|^2 + C(R)) \quad \forall n \geq 1$$

where

$$v_n := ([u_n - u_n(R)] - [u - u(R)]) \in H^2_{\text{rad}}(B_R) \cap H^1_0(B_R) \quad \forall n \geq 1 .$$

Now, applying the Young's inequality (3.6), we get for any $\varepsilon > 0$

$$\int_{B_R} (e^{pu_n^2} - 1) \, dx \leq \frac{1}{q} \int_{B_R} (e^{pq(1+\varepsilon^2)v_n^2} - 1) \, dx + \frac{1}{q'} \int_{B_R} (e^{pq' 2^2 \left(1 + \frac{1}{\varepsilon^2}\right)(u^2 + C(R))} - 1) \, dx \quad \forall n \geq 1$$

where $1 < q, \, q' < +\infty$, $\frac{1}{q} + \frac{1}{q'} = 1$. As a consequence of the Adams' inequality with zero Navier boundary conditions (see Theorem 2.12), we have

$$\sup_n \int_{B_R} (e^{pq(1+\varepsilon^2)v_n^2} - 1) \, dx < +\infty$$

provided that

$$pq(1+\varepsilon^2)\|\Delta v_n\|_2^2 \le 32\pi^2 \quad \forall n \ge \overline{n} \tag{3.10}$$

with $\overline{n} \ge 1$ sufficiently large. Therefore if we prove that for some $q > 1$ and $\varepsilon > 0$ inequality (3.10) holds for any $n \ge 1$ sufficiently large then (3.9) follows and the proof is complete. But, since by construction

$$\|\Delta v_n\|_2^2 = \|\Delta(u_n - u)\|_2^2 \le \|u_n - u\|^2 \quad \forall n \ge 1 \ ,$$

passing to the limit as $n \to +\infty$ we obtain

$$\lim_{n\to+\infty} \|\Delta v_n\|_2^2 \le 1 - \|u\|^2$$

and choosing $q > 1$ sufficiently close to 1 and $\varepsilon > 0$ sufficiently close to 0 it is easy to see (as in the proof of Lemma 3.4) that (3.10) holds at least for any $n \ge 1$ sufficiently large. □

In the proof of the next result we will use the following

Lemma 3.7 ([31], Lemma 2.2)**.** *Let $\alpha > 0$ and $r > 1$. Then for any $\beta > r$ there exists a constant $C(\beta) > 0$ such that*

$$\left(e^{\alpha s^2} - 1\right)^r \le C(\beta)\left(e^{\alpha\beta s^2} - 1\right) \qquad \forall s \in \mathbb{R} \ .$$

For a proof of Lemma 3.7, the reader is referred to the proof of Lemma 2.2 in [31].

Remark 3.8. *As a consequence of Lemma 3.7 and Hölder's inequality, it is easy to see that if $\alpha > 0$ and $q \ge 1$ then the function $|u|^q(e^{\alpha u^2} - 1)$ belongs to $L^1(\mathbb{R}^4)$ for all $u \in H^2(\mathbb{R}^4)$.*

Lemma 3.9. *Let $\alpha > 0$ and $q \ge 2$. If $M > 0$ and $\alpha M^2 < 32\pi^2$ then there exists a constant $C(\alpha, q, M) > 0$ such that*

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)|u|^q \, dx \le C(\alpha, q, M)\|u\|_{H^2}^q \tag{3.11}$$

holds for any $u \in H^2(\mathbb{R}^4)$ with $\|u\|_{H^2} \le M$.

Proof. As $\alpha M^2 < 32\pi^2$, there exists $r \in \mathbb{R}$ such that $1 < r < \frac{32\pi^2}{\alpha M^2}$. Furthermore there exists $\beta \in \mathbb{R}$ such that

$$1 < r < \beta \le \frac{32\pi^2}{\alpha M^2}$$

and in particular $\alpha\beta M^2 \le 32\pi^2$. Let $u \in H^2(\mathbb{R}^4)$ be such that $\|u\|_{H^2} \le M$, then by Lemma 3.7 it follows that

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)^r \, dx \le C(\alpha, M) \int_{\mathbb{R}^4} (e^{\alpha\beta u^2} - 1) \, dx \le C(\alpha, M) \int_{\mathbb{R}^4} (e^{\alpha\beta M^2 \tilde{u}^2} - 1) \, dx$$

where $\tilde{u} := \frac{u}{\|u\|_{H^2}}$. Therefore

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)^r \, dx \le C(\alpha, M) \tag{3.12}$$

as a consequence of the Adams' type inequality (3.1).

Now, applying Hölder's inequality with $\frac{1}{r} + \frac{1}{r'} = 1$ and (3.12), we get

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)|u|^q \, dx \le \|u\|^q_{qr'} \left(\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)^r \, dx \right) \le C(\alpha, M) \|u\|^q_{qr'}$$

and (3.11) follows easily as $qr' \ge 2$ and the Sobolev embedding theorem states that $H^2(\mathbb{R}^4)$ is continuously embedded in $L^p(\mathbb{R}^4)$ for any $p \in [2, +\infty)$. □

Lemma 3.9 is a generalization of Lemma 2.4 in [31] for second order derivatives. We will also use the following version of Lemma 3.9.

Lemma 3.10. *Let* $\alpha > 0$, $r > 1$ *and* $q \ge 2$. *If* $M > 0$ *and* $\alpha r M^2 < 32\pi^2$ *then there exists a constant* $C(\alpha, r, q, M) > 0$ *such that*

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)^{\frac{r}{2}} |u|^{\frac{q}{2}} \, dx \le C(\alpha, r, q, M) \, \|u\|^{\frac{q}{2}}_{H^2} \tag{3.13}$$

holds for any $u \in H^2(\mathbb{R}^4)$ *with* $\|u\|_{H^2} \le M$.

Proof. As $\alpha r M^2 < 32\pi^2$ there exists $\beta > r$ such that $\alpha \beta M^2 \le 32\pi^2$. Let $u \in H^2(\mathbb{R}^4)$ be such that $\|u\|_{H^2} \le M$. As in the proof of Lemma 3.9, applying Lemma 3.7 and the Adams' type inequality (3.1), we get

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)^r \, dx \le C(\alpha, r, M) \, .$$

Now

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)^{\frac{r}{2}} |u|^{\frac{q}{2}} \, dx \le \left(\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)^r \, dx \right)^{\frac{1}{2}} \|u\|^{\frac{q}{2}}_q \le \sqrt{C(\alpha, r, M)} \, \|u\|^{\frac{q}{2}}_q$$

and this ends the proof, in fact (3.13) follows by the Sobolev embedding theorem. □

Part II

Applications to elliptic and biharmonic equations

CHAPTER 4

Elliptic and biharmonic equations with exponential nonlinearities

Elliptic and polyharmonic equations with critical growth nonlinearities have been widely investigated in the last decades. In dimension $n > 2k$ with $k \in \mathbb{N}$, the critical growth for problems of the form

$$(-\Delta)^k u = f(x,\, u)\,, \quad \text{in } \Omega \subseteq \mathbb{R}^n \tag{4.1}$$

is given by the Sobolev embeddings. While equations with subcritical growth are solved by standard variational methods, equations with critical growth need more specific methods due to the loss of compactness. While the situation in dimension $n > 2k$ is by now well understood, the case $n = 2k$ is quite different, and there are less results available. In this case the natural space for a variational treatment of problems of the form (4.1) is the Sobolev space H^k and $n = 2k$ is the limiting case for the corresponding Sobolev embeddings. Therefore the notion of critical growth for such problems is governed by the Trudinger-Moser and Adams inequalities introduced in Part I. In what follows we will consider elliptic and biharmonic equations in the whole space $\mathbb{R}^2$ and $\mathbb{R}^4$, respectively, involving nonlinearities with exponential growth.

In this Chapter we firstly give a review of past developments in the study of elliptic problems in $\mathbb{R}^2$ with nonliearities having an exponential behaviour. In particular, we focus our attention on results concerned with a mountain pass characterization of ground state solutions, that we will treat in Chapter 5 for a particular nonlinear scalar field equation. Secondly, after having specified the notion of exponential critical growth for problems which can be treated variationally in the Sobolev space $H^2(\mathbb{R}^4)$, we will introduce the biharmonic problems that we will study in Chapter 6 and 7.

The problem of loss of compactness

The problem of the loss of compactness in the Sobolev spaces has been extensively studied in the last decades. In particular, it is well known that, given a bounded domain $\Omega \subset \mathbb{R}^n$, for $1 \le p < n$ the *subcritical* embeddings

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \text{ with } 1 < q < \frac{np}{n-p}$$

are compact and this implies that the supremum

$$S_{p,q}(\Omega) := \sup_{u \in W_0^{1,p}(\Omega),\, \|\nabla u\|_p \le 1} \int_\Omega |u|^q \, dx$$

is attained. Instead in the *critical case*, namely for $p^* := \frac{np}{n-p}$, the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

is no longer compact and this loss of compactness is at present well understood. Moreover the supremum

$$S_{p,p^*}(\Omega) := \sup_{u \in W_0^{1,p}(\Omega),\, \|\nabla u\|_p \le 1} \int_\Omega |u|^{p^*} \, dx$$

is independent of the domain $\Omega \subset \mathbb{R}^n$ and is *never* attained for any domain different from $\mathbb{R}^n$. More precisely, for any $\Omega \subset \mathbb{R}^n$

$$S_{p,p^*}(\Omega) = S_{p,p^*}(\mathbb{R}^n)$$

and $S_{p,p^*}(\Omega)$ is attained only for $\Omega = \mathbb{R}^n$.

On the other hand if we look at the Trudinger-Moser embeddings

$$C_{n,\alpha}(\Omega) := \sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} \, dx \begin{cases} < +\infty & \text{if } \alpha \le \alpha_n \,, \\ = +\infty & \text{if } \alpha > \alpha_n \,, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, in the *subcritical case*, namely when $\alpha \in (0, \alpha_n)$, it is easy to see that the supremum $C_{n,\alpha}(\Omega)$ is attained, and in the *critical case* we have the following surprising result due to L. Carleson and S. Y. A. Chang [20] which is in striking contrast with the Sobolev case

Theorem 4.1 ([20]). *Let $\Omega := B_1 \subset \mathbb{R}^n$ be the unit ball in $\mathbb{R}^n$ then the supremum $C_{n,\alpha_n}(B_1)$ is attained.*

After the celebrated paper [20], M. Struwe [66] proved that the supremum $C_{n,\alpha_n}(\Omega)$ is attained for domains $\Omega \subset \mathbb{R}^n$ which are close to a ball in measure and M. Flucher showed that this is indeed true for any bounded domain $\Omega \subset \mathbb{R}^2$. Finally K. C. Lin [47] extended these results for smooth domains in all dimensions.

Concerning the supremum

$$D_{n,\alpha}(\Omega) := \sup_{u \in W_0^{1,n}(\Omega),\, \|u\|_{W^{1,n}} \le 1} \int_\Omega (e^{\alpha |u|^{\frac{n}{n-1}}} - 1) \, dx \begin{cases} < +\infty & \text{if } \alpha \le \alpha_n \,, \\ = +\infty & \text{if } \alpha > \alpha_n \,, \end{cases}$$

B. Ruf [61] proved that $D_{2,4\pi}(\Omega)$ is attained on balls and on $\mathbb{R}^2$, namely in the cases $\Omega = B_R \subset \mathbb{R}^2$ and $\Omega = \mathbb{R}^2$, and subsequently Y. Li and B. Ruf [44] showed that $D_{n,\alpha_n}(\mathbb{R}^n)$

is also attained in any dimension $n > 2$. The question, whether or not $D_{n,\alpha_n}(\Omega)$ is attained for general domains $\Omega \subseteq \mathbb{R}^n$, is still an open problem in any dimension $n \geq 2$.

In the case of higher order derivatives there is a long way to go yet. At our knowledge, the only result that has already been proved is due to G. Lu and Y. Yang [49]. More precisely in [49] the authors proved that the supremum

$$\sup_{u \in W_0^{2,2}(\Omega),\, \|\Delta u\|_2 \leq 1} \int_\Omega e^{32\pi^2 u^2}\, dx$$

is attained for any smooth bounded domain $\Omega \subset \mathbb{R}^4$.

Elliptic problems with critical growth

Problems involving critical growth in second-order elliptic equations in bounded domains of $\mathbb{R}^n$ with $n \geq 3$, i.e.

$$-\Delta u = f(u) \quad \text{ in } \Omega \subset \mathbb{R}^n\,, \tag{4.2}$$

have been extensively studied starting with the celebrated result due to H. Brezis and L. Nirenberg [17]. In dimension $n \geq 3$ and in the case when the nonlinearity f has critical polynomial growth, the functional associated to a variational approach of problem (4.2) reveals a loss of compactness, in fact at certain levels the Palais-Smale compactness condition fails. To overcome this difficulty, in [17] the authors uses special sequences of functions to show that the critical levels of the functional avoid these noncompactness levels. These sequences are obtained from the maximizing sequence for

$$S_{2,2^*}(\Omega) := \sup_{u \in H_0^1(\Omega),\, \|\nabla u\|_2 \leq 1} \int_\Omega |u|^{2^*}\, dx, \quad 2^* := \frac{2n}{n-2}\,,$$

and are explicit concentrating functions converging weakly to zero.

Let $\Omega \subseteq \mathbb{R}^2$, in dimension $n = 2$ the critical growth is given by the well known Trudinger-Moser inequality (see Theorem 1.1 and Theorem 1.3). The Trudinger-Moser embedding is critical and involves a lack of compactness similar to that of the Sobolev embeddings in dimension $n \geq 3$.

Exploring the approach introduced in [17], Adimurthi et al. in [3], [4], [6] and [5] obtained the solvability of second-order elliptic equations in bounded domains $\Omega \subset \mathbb{R}^2$ involving subcritical and critical nonlinearities. In the critical case, one again finds levels of noncompactness; however, due to the fact that the best constant

$$C_{2,4\pi}(\Omega) := \sup_{u \in H_0^1,\, \|\nabla u\|_2 \leq 1} \int_\Omega e^{4\pi u^2}\, dx$$

is attained, there is no natural concentrating sequence to be used to show that these levels are avoided. Thus, it is difficult to obtain optimal existence results. The sequence used in [3], [4], [6] and [5] is the so-called *Moser's sequence* which was proposed by Moser in [50] to prove that the inequality (1.3) is sharp with respect to the constant 4π in the exponent.

Later D. G. de Figueiredo, O. H. Miyagaki and B. Ruf [28] improved the existence conditions in [4] and extended the result to more general nonlinearities. Motivated by the

Trudinger-Moser inequality, the authors in [28] introduced the notion of critical exponential growth as follows. A nonlinearity $f : \mathbb{R} \to \mathbb{R}$ has *subcritical exponential growth* if

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0 \quad \forall \alpha > 0 \,,$$

while f has *critical exponential growth* if for some $\alpha_0 > 0$

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0 & \text{for } \alpha > \alpha_0 \,, \\ +\infty & \text{for } \alpha \leq \alpha_0 \,. \end{cases}$$

Among the subsequent works, concerning elliptic equations in bounded domains of $\mathbb{R}^2$, we mention in chronological order [27], [18], [51] and [29].

Nonlinear scalar field equations

In the study of nonlinear scalar field equations of the form

$$-\Delta u = g(u) \quad \text{ in } \mathbb{R}^n \,, \tag{4.3}$$

independently of the subcritical or critical behaviour of the nonlinearity, we have to tackle the problem of the loss of compactness due to the unboundedness of the domain. Problem (4.3) has been widely investigated starting from the fundamental papers due to H. Berestycki and P. L. Lions [15] and to H. Berestycki, T. Gallouët and O. Kavian [14]. We recall that these papers are both concerned with *subcritical* nonlinearities, in particular in [15] the authors treated nonlinearities with subcritical *polynomial* growth, while in [14] the authors treated nonlinearities with subcritical *exponential* growth. From now on, we will focus our attention in the case when the nonlinear term is of exponential type, since one of our aims is to study problem (4.3) with a nonlinearity exhibiting a critical exponential growth. To be more precise, in Chapter 5, we will study the following nonlinear scalar field equation

$$-\Delta u + u = f(u) \text{ in } \mathbb{R}^2 \tag{4.4}$$

in the case when the nonlinearity f has critical exponential growth.

The study of this kind of problems is motivated by applications in many areas of mathematical physics. In particular, these problems appear in the search for stationary states in nonlinear Klein-Gordon equations,

$$\frac{\partial \phi}{\partial t^2} - \Delta\phi + V(x)\phi = f(\phi)$$

where $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$, $\phi = \phi(t, x)$ and $V : \mathbb{R}^2 \to \mathbb{R}$, $V = V(x)$ is a given potential. Also, solutions of (4.4) provide stationary states for the nonlinear Schrödinger equation

$$i\frac{\partial \phi}{\partial t} - \Delta\phi + V(x)\phi = f(\phi)$$

where $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$, $\phi = \phi(t, x)$ and $V : \mathbb{R}^2 \to \mathbb{R}$, $V = V(x)$.

Searching for stationary states for both the nonlinear Klein-Gordon and Schrödinger equations is equivalent to solve

$$-\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^2 .$$

If the potential V is constant and $V(x) = 1$ for any $x \in \mathbb{R}^2$ then this last equation is nothing but (4.4) and, from a variational point of view, the energy functional associated to (4.4) presents a loss of compactness due to the unboundedness of the domain $\mathbb{R}^2$. But, if the potential V is not constant and satisfies suitable assumptions then this loss of compactness can be overcome. For existence results concerning potentials bounded away from zero and large at infinitiy, in the case when the nonlinear term f has a critical exponential behaviour, we refer the reader to the paper of J. M. do Ó, E. Medeiros and U. Severo [31], and the references therein.

A first result concerning the existence of solutions of problem (4.4), in the case when the nonlinearity f has critical exponential growth is due to D. M. Cao. [19]. Our aim is, indeed, not only an existence result, but to obtain a mountain pass characterization of ground state solutions of problem (4.4). Denoting $I : H^1(\mathbb{R}^2) \to \mathbb{R}$ the natural functional corresponding to a variational approach to problem (4.4)

$$\begin{aligned} I(u) \; &:= \; \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)\, dx - \int_{\mathbb{R}^2} F(u)\, dx = \\ &= \; \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \int_{\mathbb{R}^2} G(u)\, dx \end{aligned}$$

where

$$F(s) := \int_0^s f(t)\, dt \quad \text{and} \quad G(s) := \int_0^s g(t)\, dt ,$$

we recall that a solution u of problem (4.4) is a *ground state* if $I(u) = m$ with

$$m := \inf \left\{ I(u) \mid u \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ is a solution of (4.4)} \right\} .$$

In [40], L. Jeanjean and K. Tanaka enlighten a mountain pass characterization of ground state solutions of the more general nonlinear scalar field equation (4.3) in the case when the nonlinearity g (not necessarily of the form $f(s) - s$) has a *subcritical* exponential growth.

Theorem 4.2 ([40]). *Assume*

(g_0) $g : \mathbb{R} \to \mathbb{R}$ *is continuous and odd;*

(g_1) $\displaystyle\lim_{s\to 0} \frac{g(s)}{s} = -\nu < 0$ *;*

(g_2) *for any* $\alpha > 0$ *there exists* $C_\alpha > 0$ *such that* $|g(s)| \le C_\alpha e^{\alpha s^2}$ *for all* $s \ge 0$*;*

(g_3) *there exists* $s_0 > 0$ *such that* $G(s_0) > 0$.

Then the functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \int_{\mathbb{R}^2} G(u)\, dx$$

is in $\mathcal{C}^1(H^1(\mathbb{R}^2),\,\mathbb{R})$ *and has a mountain pass geometry. Moreover the mountain pass value*

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,\,1]} I(\gamma(t))\ ,$$

where

$$\Gamma := \left\{\gamma \in \mathcal{C}([0,\,1],\,H^1(\mathbb{R}^2)) \mid \gamma(0)=0,\,\gamma(1)=u_0\in H^1(\mathbb{R}^2)\right\} \qquad \textit{with}\ \ I(u_0)<0\ ,$$

is a critical value and

$$0 < c = m\ .$$

Later in [10], C. O. Alves, M. Montenegro and M. A. S. Souto improved the arguments in [40] obtaining, under suitable assumptions (see Chapter 5, Theorem 5.2), a mountain pass characterization of ground state solutions of problem (4.4) in the case when the nonlinearity f exhibits a *critical* exponential growth. In Chapter 5 (see Theorem 5.1), we will follow the ideas introduced in [10] to obtain a similar result in the case when

$$f(s) := \lambda s e^{4\pi s^2} \qquad \forall s\in\mathbb{R}$$

with $0<\lambda<1$. We will also prove (see Theorem 5.3) that the result of Alves, Montenegro and Souto still holds under the classical assumption

$$\lim_{|s|\to+\infty}\frac{sf(s)}{e^{4\pi s^2}} \ge \beta_0 > 0\ ,$$

introduced by D. G. de Figueiredo, O. H. Miyagaki and B. Ruf in [28] (see also [31]).

Biharmonic problems with critical growth

Recently, due to applications of higher order elliptic equations to conformal geometry, there has been considerable interest in the Paneitz operator which enjoys the property of conformal invariance. In $\mathbb{R}^4$, the Paneitz operator is the biharmonic operator Δ^2 where Δ is the Laplacian in $\mathbb{R}^4$. The study of superlinear problems involving powers of the Laplacian started with the works [34], [33] of D. E. Edmunds, D. Fortunato and E. Jannelli, and [57], [58] of P. Pucci and J. Serrin. We refer the reader to the paper [12] and the references therein for various results on the polyharmonic operator.

The Adams-type inequality in $\mathbb{R}^4$ expressed by Theorem 2.1 (see also (3.1) for easy reference) will be a fundamental tool in the study of biharmonic problems of the form

$$\Delta^2 u + V(x)u = f(u) \qquad \text{in}\ \ \mathbb{R}^4 \tag{4.5}$$

in the case when the nonlinear term f exhibits an exponential growth. We recall that equations of the form

$$\Delta^2 u + V(x)u = f(x,\,u) \quad \text{in}\ \ \mathbb{R}^n$$

with $n\ge 5$ and involving nonlinearities with polynomial growth, have been studied in [23], [9] and [8].

The natural space for a variational treatment of problem (4.5) is the Sobolev space H^2 and $\mathbb{R}^4$ is the limiting case for the corresponding Sobolev embeddings. Indeed, for $\mathbb{R}^4$ the notion of critical growth is given by the Adams-type inequality

$$\sup_{u\in H^2(\mathbb{R}^4),\,\|u\|_{H^2}\le 1}\int_{\mathbb{R}^4}(e^{\alpha u^2}-1)\,dx\ \begin{cases}<+\infty & \text{for } \alpha\le 32\pi^2\,,\\ =+\infty & \text{for } \alpha>32\pi^2\,,\end{cases}$$

where $\|u\|_{H^2}^2:=\|(-\Delta+I)u\|_2^2=\|\Delta u\|_2^2+2\|\nabla u\|_2^2+\|u\|_2^2$. In view of this inequality it is natural to say that a nonlinearity f has *subcritical exponential growth* if

$$\lim_{|s|\to+\infty}\frac{|f(s)|}{e^{\alpha s^2}}=0\qquad\forall\alpha>0\,,$$

while f has *critical exponential growth* if it behaves like $e^{\alpha_0 u^2}$ as $|s|\to+\infty$ for some $\alpha_0>0$, namely if there exists $\alpha_0>0$ such that

$$\lim_{|s|\to+\infty}\frac{|f(s)|}{e^{\alpha s^2}}=\begin{cases}0 & \text{if } \alpha>\alpha_0\,,\\ +\infty & \text{if } \alpha<\alpha_0\,.\end{cases}$$

We will always assume that V is a continuous positive potential bounded from below by a positive constant, more precisely

(V_0) $V:\mathbb{R}^4\to\mathbb{R}$ is continuous and $V(x)\ge V_0>0$ for any $x\in\mathbb{R}^4$,

and we will handle problem (4.5) by means of a variational approach.

Assuming (V_0) and some symmetry conditions on the potential V, in the case when the nonlinearity f has subcritical exponential growth we will obtain a multiplicity result for problem (4.5) (see Theorem 6.3, Theorem 6.4 and Theorem 6.5), while in the case when the nonlinearity f has critical exponential growth we will obtain the existence of a nontrivial radial solution of problem (4.5) (see Theorem 7.3). We point out that in both these results the potential V is allowed to be constant, hence in particular for the biharmonic problem

$$\Delta^2 u+V_0 u=f(u)\quad\text{ in } \mathbb{R}^4\,,$$

where $V_0>0$ is a positive constant, we will obtain a multiplicity result in the case when the nonlinear term f has subcritical exponential growth and the existence of a nontrivial radial solution in the case when the nonlinear term f has critical exponential growth.

Finally, assuming (V_0) and that the potential V is large at infinity, if the nonlinearity f has critical exponential growth we will obtain the existence of a nontrivial solution of problem (4.5), see Theorem 7.1.

In view of the Adams–type inequality (3.3) with the modified norm $\|\cdot\|_{H^2,\,a,\,b}$ for some a, $b>0$, the methods of proofs adopted to study equation (4.5), both in the case when f exhibits a subcritical and critical exponential growth, apply to

$$\Delta^2 u-\operatorname{div}\,(U(x)\nabla u)+V(x)u=f(u)\qquad\text{in } \mathbb{R}^4$$

where U, $V:\mathbb{R}^4\to\mathbb{R}$ are continuous and positive functions satisfying suitable assumptions, see Theorem 6.14 and Theorem 7.15.

CHAPTER 5

An elliptic equation in $\mathbb{R}^2$ with exponential critical growth: ground state solutions

This Chapter is concerned with the existence of solutions of a nonlinear scalar field equation of the form

$$\begin{cases} -\Delta u = g(u) \quad \text{in } \mathbb{R}^2 \\ u \in H^1(\mathbb{R}^2) \end{cases} \tag{5.1}$$

and in particular we will study the following problem

$$\begin{cases} -\Delta u + u = f(u) \quad \text{in } \mathbb{R}^2 \\ u \in H^1(\mathbb{R}^2) \end{cases} \tag{5.2}$$

which is nothing but problem (5.1) with $g(s) := f(s) - s$.

We recall that the natural functional corresponding to a variational approach to problem (5.2) is the functional $I : H^1(\mathbb{R}^2) \to \mathbb{R}$ defined as follows

$$\begin{aligned} I(u) \; &:= \; \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + u^2)\,dx - \int_{\mathbb{R}^2} F(u)\,dx = \\ &= \; \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \int_{\mathbb{R}^2} G(u)\,dx \end{aligned}$$

where

$$F(s) := \int_0^s f(t)\,dt \quad \text{and} \quad G(s) := \int_0^s g(t)\,dt\,.$$

We will say that I has a *mountain pass geometry*, if the following conditions hold:

(I_0) $I(0) = 0$;

(I_1) there exist ϱ, $a > 0$ such that $I(u) \geq a > 0$ for any $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{H^1} = \varrho$;

(I_2) there exists $u_0 \in H^1(\mathbb{R}^2)$ such that $\|u_0\|_{H^1} > \varrho$ and $I(u_0) < 0$.

Our main result is concerned with the particular case when $f(s) = \lambda s e^{4\pi s^2}$ where $0 < \lambda < 1$.

Theorem 5.1. *Let $0 < \lambda < 1$ and let*

$$f(s) := \lambda s e^{4\pi s^2} \qquad \forall s \in \mathbb{R} \; . \tag{5.3}$$

Then $I \in \mathcal{C}^1(H^1(\mathbb{R}^2),\, \mathbb{R})$ has a mountain pass geometry, the mountain pass value

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,\, 1]} I(\gamma(t)) \; ,$$

where

$$\Gamma := \left\{ \gamma \in \mathcal{C}([0,\, 1],\, H^1(\mathbb{R}^2)) \mid \gamma(0) = 0,\, \gamma(1) = u_0 \in H^1(\mathbb{R}^2) \right\} \qquad \textit{with} \;\; I(u_0) < 0 \; ,$$

is a critical value and gives the ground state level, namely $0 < c = m$ where

$$m := \inf \left\{ I(u) \mid u \in H^1(\mathbb{R}^2) \setminus \{0\} \textit{ is a solution of (5.2)} \right\} \; .$$

We also give a mountain pass characterization of ground state solutions of problem (5.2) in the case when the nonlinearity f satisfies the following assumptions:

(f_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous and has critical exponential growth with $\alpha_0 = 4\pi$, i.e.

$$\lim_{|s| \to +\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if } \alpha > 4\pi \, , \\ +\infty & \text{if } \alpha < 4\pi \, ; \end{cases}$$

(f_1) $\displaystyle\lim_{s \to 0} \frac{f(s)}{s} = 0$;

(f_2) there exists $\mu > 2$ such that $0 < \mu F(s) < f(s)s$ for any $s \in \mathbb{R} \setminus \{0\}$.

In [10] the authors obtained a mountain pass characterization of ground state solutions of problem (5.2) assuming the further assumption on f

(f_λ) there exist $\lambda > 0$ and $q \in (2,\, +\infty)$ such that $f(s) \geq \lambda s^{q-1}$ for all $s \geq 0$.

More precisely they prove

Theorem 5.2 ([10])**.** *Assume (f_0), (f_1) and (f_2). Assume also that (f_λ) holds with*

$$\lambda > \left(\frac{q-2}{q} \right)^{\frac{q-2}{q}} C_q^{\frac{q}{2}} \tag{5.4}$$

where $C_q > 0$ is the best constant of the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$, namely

$$C_q \|u\|_q^2 \leq \|u\|_{H_1}^2 \qquad \forall u \in H^1(\mathbb{R}^2) \; .$$

Then the mountain pass value c is a critical value and gives the ground state level, namely

$$0 < c = m \; .$$

Indeed in [10] the authors proved that Theorem 5.2 holds assuming, instead of the Ambrosetti-Rabinowitz condition (f_2), the weaker assumption:

(f_2') $f(s)s \geq 2F(s) \geq 0$ for all $s \in \mathbb{R}$.

In the present paper, we need to assume the Ambrosetti-Rabinowitz condition (f_2) only to prove that the functional I behaves like a mountain pass and more precisely to prove that I satisfies (I_2).

Replacing assumption (f_λ) with the following more natural assumption

(f_3) $\displaystyle\lim_{|s|\to+\infty} \frac{sf(s)}{e^{4\pi s^2}} \geq \beta_0 > 0$

we obtain the same result as in Theorem 5.2.

Theorem 5.3. *Assume (f_0), (f_1), (f_2) and (f_3). Then $I \in \mathcal{C}^1(H^1(\mathbb{R}^2), \mathbb{R})$ has a mountain pass geometry, the mountain pass value c is a critical value and*

$$0 < c = m\ .$$

We recall that assumption (f_3) for bounded domains was introduced in [28] to obtain an existence result for elliptic equations with nonlinearities in the critical exponential growth range in bounded domains of $\mathbb{R}^2$. In a subsequent paper, [31], (f_3) was taken into account to prove an existence result for analogous equations in the hole space $\mathbb{R}^2$.

To prove Theorem 5.1 and Theorem 5.3 we will follow the methods used in [10] which are based on the ideas introduced in [40] to obtain the following result

Theorem 5.4 ([40])**.** *Assume*

(g_0) $g : \mathbb{R} \to \mathbb{R}$ *is continuous and odd;*

(g_1) $\displaystyle\lim_{s\to 0} \frac{g(s)}{s} = -\nu < 0$ *;*

(g_2) *for any $\alpha > 0$ there exists $C_\alpha > 0$ such that $|g(s)| \leq C_\alpha e^{\alpha s^2}$ for all $s \geq 0$;*

(g_3) *there exists $s_0 > 0$ such that $G(s_0) > 0$.*

Then the functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \int_{\mathbb{R}^2} G(u)\,dx$$

is in $\mathcal{C}^1(H^1(\mathbb{R}^2), \mathbb{R})$ and has a mountain pass geometry. Moreover the mountain pass value c is a critical value and

$$0 < c = m\ .$$

In the proof of Theorem 5.4 a key argument is the existence of a solution of problem (5.1) given in [14]. In [14] it was shown that under the assumptions (g_0), (g_1), (g_2) and (g_3) the nonlinear scalar field equation (5.1) possesses a nontrivial ground state solution by means of the constrained minimization method

$$\inf\left\{\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \mid u \in H^1(\mathbb{R}^2)\setminus\{0\},\ \int_{\mathbb{R}^2} G(u)\,dx = 0\right\}\ .$$

The main difficulty highlighted in [10] for the proof of Theorem 5.2 is indeed to show that the infimum

$$A := \inf\left\{\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \mid u\in H^1(\mathbb{R}^2)\setminus\{0\},\ \int_{\mathbb{R}^2}G(u)\,dx=0\right\}$$

is achieved, provided that (f_0), (f_1), (f_2) and (f_λ) with $\lambda>0$ as in (5.4) hold. Therefore we point out that, following [10], as a by-product of the proofs of Theorem 5.1 and Theorem 5.3 we have

Proposition 5.5. *Assume either f is of the form (5.3) with $0<\lambda<1$ or assume (f_0), (f_1), (f_2) and (f_3). Then A is attained and the minimizer is, under a suitable change of scale, a solution of problem (5.2). In particular $m\le A$.*

This Chapter is organized as follows. In Section 5.1 we show that the functional I has a mountain pass geometry and in Section 5.2 we introduce some preliminary results. In Section 5.3 we obtain a precise estimate for the mountain pass level c that will enable us to prove, in Section 5.4, Proposition 5.5. Finally in Section 5.5 we prove the main theorems, Theorem 5.1 and Theorem 5.3, and the following

Proposition 5.6. *Assume either f is of the form (5.3) with $0<\lambda<1$ or assume (f_0), (f_1), (f_2) and (f_3). Then the minimizer $u\in H^1(\mathbb{R}^2)$ of A is a ground state solution of problem (5.2), that is $m=A$.*

5.1. Mountain pass geometry

If f is as in (5.3) with $0<\lambda<1$ then fixed $q>1$ we have the existence of two constants c_1, $c_2>0$ such that

$$f(s)\le c_1 s+c_2|s|^q(e^{4\pi s^2}-1)\qquad \forall s\in\mathbb{R}\ . \tag{5.5}$$

Since in this case

$$F(s)=\frac{\lambda}{8\pi}(e^{4\pi s^2}-1)\qquad \forall s\in\mathbb{R}\,,$$

fixed $q>2$ we have that for any $\varepsilon>0$ there exists a constant $C(q,\varepsilon)>0$ such that

$$F(s)\le\left(\frac{\lambda}{2}+\varepsilon\right)s^2+C(q,\varepsilon)|s|^q(e^{4\pi s^2}-1)\qquad \forall s\in\mathbb{R}\ . \tag{5.6}$$

We can notice that (5.6) implies that $F(u)\in L^1(\mathbb{R}^2)$ for any $u\in H^1(\mathbb{R}^2)$ and thus the functional $I:\,H^1(\mathbb{R}^2)\to\mathbb{R}$ is well defined. Furthermore, from (5.5) and using standard arguments (see in [15], Theorem A.VI), it follows that $I\in\mathcal{C}^1(H^1(\mathbb{R}^2),\mathbb{R})$.

Similarly, in the case when (f_0) and (f_1) holds, fixed $q>2$ we have for any $\varepsilon>0$ the existence of a constant $C(q,\varepsilon)>0$ such that

$$|f(s)|\le\varepsilon|s|+C(q,\varepsilon)|s|^{q-1}(e^{4\pi s^2}-1)\qquad \forall s\in\mathbb{R}\,,$$

and if in addition (f_2) holds then

$$F(s)\le\frac{\varepsilon}{2}s^2+C(q,\varepsilon)|s|^q(e^{4\pi s^2}-1)\qquad \forall s\in\mathbb{R}\ . \tag{5.7}$$

Therefore also in the case when (f_0), (f_1) and (f_2) holds we have that the functional I is well defined and of class $\mathcal{C}^1$ on $H^1(\mathbb{R}^2)$.

Obviously $I(0) = 0$, namely (I_0) holds. Now we prove that I satisfies also (I_1).

Lemma 5.7. *Assume either f is of the form (5.3) with $0 < \lambda < 1$ or assume (f_0), (f_1) and (f_2). Then there exist ϱ, $a > 0$ such that $I(u) \geq a > 0$ for any $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{H^1} = \varrho$.*

Proof. We begin considering the case when f is of the form (5.3) with $0 < \lambda < 1$. From (5.6) it follows that, fixed $q > 2$, for any $\varepsilon > 0$

$$\int_{\mathbb{R}^2} F(u)\,dx \leq \left(\frac{\lambda}{2} + \varepsilon\right) \|u\|^2_{H^1} + C(q,\,\varepsilon) \int_{\mathbb{R}^2} |u|^q (e^{4\pi u^2} - 1)\,dx \quad \forall u \in H^1(\mathbb{R}^2)\,.$$

In particular for any $u \in H^1(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} |u|^q (e^{4\pi u^2} - 1)\,dx \leq \|u\|^q_{2q} \left(\int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)^2\,dx\right)^{\frac{1}{2}} \leq \overline{C}_1 \|u\|^q_{H^1} \left(\int_{\mathbb{R}^2} (e^{8\pi u^2} - 1)\,dx\right)^{\frac{1}{2}}$$

where $\overline{C}_1 > 0$ is a constant independent of u and we used the fact that the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^{2q}(\mathbb{R}^2)$ is continuous for any $q > 2$. Moreover, recalling the *Trudinger-Moser inequality* in [61] (see also Theorem 1.3), we have the existence of a constant $\overline{C}_2 > 0$ such that

$$\int_{\mathbb{R}^2} (e^{8\pi u^2} - 1)\,dx = \int_{\mathbb{R}^2} \left(e^{8\pi \|u\|^2_{H^1} \left(\frac{u}{\|u\|_{H^1}}\right)^2} - 1\right) dx \leq \overline{C}_2$$

for any $u \in H^1(\mathbb{R}^2)$ with $8\pi\|u\|^2_{H^1} \leq 4\pi$. Therefore, fixed $q > 2$, for any $\varepsilon > 0$ we have that

$$\int_{\mathbb{R}^2} F(u)\,dx \leq \left(\frac{\lambda}{2} + \varepsilon\right) \|u\|^2_{H^1} + \overline{C}(q,\,\varepsilon) \|u\|^q_{H^1} \qquad \forall u \in H^1(\mathbb{R}^2),\ \|u\|_{H^1(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}}\,.$$

Let $0 < \varrho < \frac{1}{\sqrt{2}}$. Fixed $q > 2$, for any $\varepsilon > 0$

$$I(u) \geq \frac{1}{2}(1 - \lambda - 2\varepsilon)\varrho^2 - \overline{C}(q,\,\varepsilon)\varrho^q \qquad \forall u \in H^1(\mathbb{R}^2),\ \|u\|_{H^1(\mathbb{R}^2)} = \varrho\,,$$

and choosing $\varepsilon > 0$ so that $1 - \lambda - 2\varepsilon > 0$ and ϱ sufficiently small we have that

$$a := \frac{1}{2}(1 - \lambda - 2\varepsilon)\varrho^2 - \overline{C}(q,\,\varepsilon)\varrho^q > 0\,.$$

In the case when (f_0), (f_1) and (f_2) holds, using (5.7) and arguing as before we obtain, for fixed $q > 2$ and for any $\varepsilon > 0$, that

$$\int_{\mathbb{R}^2} F(u)\,dx \leq \frac{\varepsilon}{2} \|u\|^2_{H^1} + \overline{C}(q,\,\varepsilon) \|u\|^q_{H^1} \qquad \forall u \in H^1(\mathbb{R}^2),\ \|u\|_{H^1(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}}\,.$$

Therefore, for fixed $q > 2$ and for any $\varepsilon \in (0,\,1)$, we have

$$I(u) \geq \frac{1}{2}(1 - \varepsilon)\varrho^2 - \overline{C}(q,\,\varepsilon)\varrho^q \qquad \forall u \in H^1(\mathbb{R}^2),\ \|u\|_{H^1(\mathbb{R}^2)} = \varrho$$

where $0 < \varrho < \frac{1}{\sqrt{2}}$, and this leads to the desired conclusion choosing ϱ sufficiently small. $\square$

We end this section with the proof of (I_2).

Lemma 5.8. *Assume either f is of the form (5.3) with $0 < \lambda < 1$ or assume (f_0), (f_1) and (f_2). For any $u \in H^1(\mathbb{R}^2)$, we have that $I(tu) \to -\infty$ as $t \to +\infty$. In particular, there exists $u_0 \in H^1(\mathbb{R}^2)$ such that $\|u_0\|_{H^1} > \varrho$ and $I(u_0) < 0$.*

Proof. We begin with the case when f is of the form (5.3). We fix $u \in H^1(\mathbb{R}^2)$. Then for any $t \geq 0$, using the power series expansion of the exponential function, we get

$$\int_{\mathbb{R}^2} F(tu)\, dx \geq \frac{\lambda}{2} t^2 \|u\|_2^2 + \lambda\pi t^4 \|u\|_4^4$$

Thus

$$I(tu) \leq \frac{1}{2} t^2 \|u\|_{H^1} - \frac{\lambda}{2} t^2 \|u\|_2^2 - \lambda\pi t^4 \|u\|_4^4 \quad \forall t \geq 0$$

from which we deduce that $I(tu) \to -\infty$ as $t \to +\infty$. In the case when (f_0), (f_1) and (f_2) holds, in particular we have that

$$F(s) \geq c_1 |s|^\mu - c_2 |s|^2 \qquad \forall s \in \mathbb{R}$$

with c_1, $c_2 > 0$. Therefore, fixed $u \in H^1(\mathbb{R}^2)$, for any $t \geq 0$ we can estimate

$$I(tu) \leq \frac{1}{2} t^2 \|u\|_{H^1} + c_2 t^2 \|u\|_2^2 - c_1 t^\mu \|u\|_\mu^\mu$$

and recalling that $\mu > 2$ we can conclude that $I(tu) \to -\infty$ as $t \to +\infty$. □

5.2. Preliminary results

Let

$$H^1_{\mathrm{rad}}(\mathbb{R}^2) := \left\{ u \in H^1(\mathbb{R}^2) \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^2 \right\} ,$$

we recall that the following *radial lemma* holds

Lemma 5.9 ([41], Chapitre 6, Lemme 1.1))**.** *For any $u \in H^1_{rad}(\mathbb{R}^2)$*

$$|u(x)| \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{|x|}} \|u\|_{H^1} \qquad \text{a.e. in } \mathbb{R}^2 \ . \tag{5.8}$$

This *radial lemma* will be a useful tool to prove the following result

Lemma 5.10. *Assume that f is of the form (5.3) with $0 < \lambda < 1$. Let $\{u_n\}_n \subset H^1_{rad}(\mathbb{R}^2)$ be a sequence satisfying*

(i) $\sup_n \|\nabla u_n\|_2^2 = \varrho < 1$,

(ii) $\sup_n \|u_n\|_2^2 = M < +\infty$.

Then

$$\int_{\mathbb{R}^2} F(u_n)\, dx - \frac{\lambda}{2} \|u_n\|_2^2 \quad \overset{n \to +\infty}{\longrightarrow} \quad \int_{\mathbb{R}^2} F(u)\, dx - \frac{\lambda}{2} \|u\|_2^2$$

where $u \in H^1_{rad}(\mathbb{R}^2)$ is the weak limit of $\{u_n\}_n$ in $H^1(\mathbb{R}^2)$.

Proof. Recalling that

$$F(s) = \frac{\lambda}{8\pi}(e^{4\pi s^2} - 1) ,$$

it suffices to prove that

$$\int_{\mathbb{R}^2} (e^{4\pi u_n^2} - 1)\, dx - 4\pi \|u_n\|_2^2 \quad \overset{n\to+\infty}{\longrightarrow} \quad \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)\, dx - 4\pi \|u\|_2^2 .$$

The proof consists in three steps.

STEP 1 - There exists $\alpha > 4\pi$ such that

$$\sup_n \int_{\mathbb{R}^2} (e^{\alpha u_n^2} - 1)\, dx < +\infty . \tag{5.9}$$

In fact, since $\varrho < 1$, there exists $\sigma > 0$ such that $\varrho < 1 - \sigma < 1$. Choosing

$$0 < \tau < \frac{1 - (\sigma + \varrho)}{M}$$

we have that $\|u_n\|_{H^1,\tau}^2 < 1 - \sigma$ for any $n \geq 1$. Therefore applying inequality (1.10), we can conclude that inequality (5.9) holds for any

$$0 < \alpha \leq \frac{4\pi}{1 - \sigma} .$$

STEP 2 - We prove that for any $R > 0$

$$\int_{B_R} (e^{4\pi u_n^2} - 1)\, dx \quad \overset{n\to+\infty}{\longrightarrow} \quad \int_{B_R} (e^{4\pi u^2} - 1)\, dx . \tag{5.10}$$

The idea is to apply the *compactness lemma of Strauss* (see Theorem A.I in [15]). Let $\alpha > 4\pi$ be as Step 1, so that

$$\sup_n \int_{\mathbb{R}^2} (e^{\alpha u_n^2} - 1)\, dx < +\infty$$

and moreover, as $\alpha > 4\pi$, we have that

$$\lim_{|s|\to+\infty} \frac{e^{4\pi s^2} - 1}{e^{\alpha s^2} - 1} = 0 .$$

Since the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ is compact for any $p \in (2, +\infty)$, we have that $u_n \to u$ in $L^p(\mathbb{R}^2)$ for any $p \in (2, +\infty)$ up to a subsequence that we still denote with $\{u_n\}_n$. Therefore in particular $u_n \to u$ a.e. in $\mathbb{R}^2$ and thus

$$(e^{4\pi u_n^2} - 1) \quad \overset{n\to+\infty}{\longrightarrow} \quad (e^{4\pi u^2} - 1) \qquad \text{a.e. in } \mathbb{R}^2 .$$

Then, applying the *compactness lemma of Strauss*, we can conclude that for any bounded Borel set $B \subset \mathbb{R}^2$

$$\int_B |e^{4\pi u_n^2} - e^{4\pi u^2}|\, dx \quad \overset{n\to+\infty}{\longrightarrow} \quad 0 .$$

STEP 3 - Arbitrarily fixed $R > 1$, for any $n \geq 1$ we have

$$\int_{\mathbb{R}^2 \setminus B_R} (e^{4\pi u_n^2} - 1)\, dx - 4\pi \int_{\mathbb{R}^2 \setminus B_R} u_n^2\, dx \leq \frac{4\pi^2}{2} \int_{\mathbb{R}^2 \setminus B_R} u_n^4\, dx + \frac{2\pi}{R} e^{2(1+M)} \ . \tag{5.11}$$

Using the power series expansion of the exponential function we get

$$\int_{\mathbb{R}^2 \setminus B_R} (e^{4\pi u_n^2} - 1)\, dx - 4\pi \int_{\mathbb{R}^2 \setminus B_R} u_n^2\, dx \leq \frac{4\pi^2}{2} \int_{\mathbb{R}^2 \setminus B_R} u_n^4\, dx + \sum_{j=3}^{+\infty} \frac{(4\pi)^j}{j!} \int_{\mathbb{R}^2 \setminus B_R} u^{2j}\, dx \ .$$

For any $j \geq 3$, applying the *radial lemma* (5.8), we can estimate

$$\frac{(4\pi)^j}{j!} \int_{\mathbb{R}^2 \setminus B_R} u^{2j}\, dx \leq \frac{2^j}{j!} \|u_n\|_{H^1}^{2j} \int_{\mathbb{R}^2 \setminus B_R} \frac{1}{|x|^j}\, dx \leq \frac{2\pi}{R} \cdot \frac{1}{j!} \left(2\|u_n\|_{H^1}^2\right)^j$$

and thus

$$\int_{\mathbb{R}^2 \setminus B_R} (e^{4\pi u_n^2} - 1)\, dx - 4\pi \int_{\mathbb{R}^2 \setminus B_R} u_n^2\, dx \leq \frac{4\pi^2}{2} \int_{\mathbb{R}^2 \setminus B_R} u_n^4\, dx + \frac{2\pi}{R} e^{2\|u_n\|_{H^1}^2} \ .$$

STEP 4 - Let for any $n \geq 1$

$$\mathcal{I}_n := \left| \left[\int_{\mathbb{R}^2} (e^{4\pi u_n^2} - 1)\, dx - 4\pi \|u_n\|_2^2 \right] - \left[\int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)\, dx - 4\pi \|u\|_2^2 \right] \right| \ ,$$

we have to prove that

$$\mathcal{I}_n \to 0 \quad \text{as } n \to +\infty \ . \tag{5.12}$$

To this aim we can estimate for any $n \geq 1$

$$\mathcal{I}_n \leq \mathcal{I}_n^1 + \mathcal{I}^2 + \mathcal{I}_n^3 + \mathcal{I}_n^4$$

where

$$\mathcal{I}_n^1(R) := \int_{\mathbb{R}^2 \setminus B_R} (e^{4\pi u_n^2} - 1)\, dx - 4\pi \int_{\mathbb{R}^2 \setminus B_R} u_n^2\, dx \quad \forall n \geq 1 \ ,$$

$$\mathcal{I}^2(R) := \int_{\mathbb{R}^2 \setminus B_R} (e^{4\pi u^2} - 1)\, dx + 4\pi \int_{\mathbb{R}^2 \setminus B_R} u^2\, dx$$

$$\mathcal{I}_n^3(R) := \left| \int_{B_R} (e^{4\pi u_n^2} - 1)\, dx - \int_{B_R} (e^{4\pi u^2} - 1)\, dx \right| \quad \forall n \geq 1 \ ,$$

$$\mathcal{I}_n^4(R) := 4\pi \left| \int_{B_R} u_n^2\, dx - \int_{B_R} u^2\, dx \right| \quad \forall n \geq 1 \ ,$$

with $R > 1$ to be chosen.

From (5.11) it follows that

$$\mathcal{I}_n^1(R) \leq \frac{4\pi^2}{2} \int_{\mathbb{R}^2 \setminus B_R} u_n^4\, dx + \frac{2\pi}{R} e^{2(1+M)} \quad \forall n \geq 1 \ .$$

Since the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^2) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^2)$ is compact for any $p \in (2, +\infty)$, in particular we have that $u_n \to u$ in $L^4(\mathbb{R}^2)$ and for any $\varepsilon > 0$ there exists $R > 1$ such that

$$\frac{4\pi^2}{2} \int_{\mathbb{R}^2 \setminus B_R} u_n^4 \le \frac{\varepsilon}{3} \quad \forall n \ge 1$$

and moreover

$$\frac{2\pi}{R} e^{2(1+M)} \le \frac{\varepsilon}{3}, \quad \mathcal{I}^2(R) \le \frac{\varepsilon}{3} \, .$$

Hence for any $\varepsilon > 0$ there exists $R > 1$ such that

$$\mathcal{I}_n \le \varepsilon + \mathcal{I}_n^3(R) + \mathcal{I}_n^4(R) \qquad \forall n \ge 1$$

and passing to the limit as $n \to +\infty$ we get

$$\lim_{n \to +\infty} \mathcal{I}_n \le \varepsilon \qquad \forall \varepsilon > 0 \, . \tag{5.13}$$

In fact, since (5.10) holds, we have $\mathcal{I}_n^3 \to 0$ as $n \to +\infty$ and, since $u_n \to u$ in $L^2(B_R)$, we have also that $\mathcal{I}_n^4 \to 0$ as $n \to +\infty$.

Now (5.12) follows directly from (5.13) letting $\varepsilon \to 0^+$. □

We recall that in [10] the authors proved the following result

Lemma 5.11. *Assume* (f_0) *and* (f_1). *Let* $\{u_n\}_n \subset H^1_{rad}(\mathbb{R}^2)$ *be a sequence satisfying* (i) *and* (ii) *of Lemma 5.10. Then*

$$\int_{\mathbb{R}^2} F(u_n) \, dx \to \int_{\mathbb{R}^2} F(u) \, dx$$

where $u \in H^1_{rad}(\mathbb{R}^2)$ *is the weak limit of* $\{u_n\}_n$ *in* $H^1(\mathbb{R}^2)$.

For the convenience of the reader, we give here a proof of this result.

Proof. As in Step 2 of the proof of Lemma 5.10, the idea is to apply the *compactness lemma of Strauss* (see Theorem A.I in [15]). Arguing as in Step 1 of the proof of Lemma 5.10, we have the existence of $\alpha > 4\pi$ such that

$$\sup_n \int_{\mathbb{R}^2} (e^{\alpha u_n^2} - 1) \, dx < +\infty \, .$$

As before we have that

$$\lim_{|s| \to +\infty} \frac{F(s)}{e^{\alpha s^2} - 1} = 0 \, ,$$

since (f_0) holds, and $F(u_n) \to F(u)$ a.e. in $\mathbb{R}^2$. But moreover from (f_1) it follows that

$$\lim_{s \to 0} \frac{F(s)}{e^{\alpha s^2} - 1} = 0$$

and $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n, as a direct consequence of the *radial lemma* (5.8). Therefore, using the *compactness lemma of Strauss*, we can conclude that $F(u_n) \to F(u)$ in $L^1(\mathbb{R}^2)$. □

We now prove that the infimum A is strictly positive, but before we point out that whenever we deal with a minimizing sequence for A, that is a sequence $\{u_n\}_n \subset H^1(\mathbb{R}^2) \setminus \{0\}$ such that

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\, dx \quad \overset{n\to+\infty}{\longrightarrow} \quad A$$

and

$$\int_{\mathbb{R}^2} G(u_n)\, dx = 0 \qquad \forall n \geq 1\ ,$$

without loss of generality we may assume that $\{u_n\}_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$ and that $\|u_n\|_2 = 1$. In fact if $\{u_n\}_n \in H^1(\mathbb{R}^2) \setminus \{0\}$ is a minimizing sequence for A then the sequence $\{u_n^*\}_n \subset H^1(\mathbb{R}^2) \setminus \{0\}$, where u_n^* is the spherically symmetric decreasing rearrangement of u_n, is a minimizing sequence too. Furthermore letting

$$v_n(x) := u_n(x\|u\|_2) \qquad \text{for a.e. } x \in \mathbb{R}^2$$

for any $n \geq 1$, we have that

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\ , \quad \int_{\mathbb{R}^2} G(v_n)\, dx = \frac{1}{\|u_n\|_2^2}\int_{\mathbb{R}^2} G(u_n)\, dx = 0$$

and $\|v_n\|_2 = 1$.

Lemma 5.12. *Assume either f is of the form (5.3) with $0 < \lambda < 1$ or assume (f_0) and (f_1). Then $A > 0$.*

Proof. In the case that we assume (f_0) and (f_1), since Lemma 5.11 holds, we can argue as in the proof of [10], Lemma 5.3 to conclude that $A > 0$. Therefore we only consider the case when

$$f(s) := \lambda s e^{4\pi s^2} \qquad \forall s \in \mathbb{R}$$

with $0 < \lambda < 1$. Obviously $A \geq 0$ and we argue by contradiction assuming that $A = 0$. Let $\{u_n\}_n \in H^1(\mathbb{R}^2) \setminus \{0\}$ be a minimizing sequence for A, namely

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\, dx \quad \overset{n\to+\infty}{\longrightarrow} \quad 0\ ,$$

$$\int_{\mathbb{R}^2} G(u_n)\, dx = 0 \qquad \forall n \geq 1$$

and, without loss of generality, we may assume that $\{u_n\}_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$ and that $\|u_n\|_2 = 1$. Let $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ be the weak limit of $\{u_n\}_n$ in $H^1(\mathbb{R}^2)$, then from Lemma 5.10, it follows that

$$\int_{\mathbb{R}^2} F(u_n)\, dx - \frac{\lambda}{2}\|u_n\|_2^2 \quad \overset{n\to+\infty}{\longrightarrow} \quad \int_{\mathbb{R}^2} F(u)\, dx - \frac{\lambda}{2}\|u\|_2^2\ .$$

Since

$$0 = \int_{\mathbb{R}^2} G(u_n)\, dx = \int_{\mathbb{R}^2} F(u_n)\, dx - \frac{1}{2}\|u_n\|_2^2 = \int_{\mathbb{R}^2} F(u_n)\, dx - \frac{1}{2}\ ,$$

we have that
$$\int_{\mathbb{R}^2} F(u_n)\,dx = \frac{1}{2}$$
and thus
$$\int_{\mathbb{R}^2} F(u)\,dx - \frac{\lambda}{2}\|u\|_2^2 = \frac{1}{2}(1-\lambda) > 0$$
from which it follows that $u \neq 0$. On the other hand, the weak convergence $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ implies that
$$0 = \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\,dx \geq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \geq 0\ ,$$
namely
$$\int_{\mathbb{R}^2} |\nabla u|^2\,dx = 0\ ,$$
and thus $u \equiv 0$ which leads to a contradiction. □

We introduce the set $\mathcal{P}$ of non-trivial functions satisfying the Pohozaev identity
$$\mathcal{P} := \left\{ u \in H^1(\mathbb{R}^2) \setminus \{0\} \mid \int_{\mathbb{R}^2} G(u)\,dx = 0 \right\}$$
and we can notice that
$$A = \inf_{u\in\mathcal{P}} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx\ .$$
Since $A > 0$, arguing as in the proof of [40], Lemma 4.1 we obtain the following result

Lemma 5.13. *Assume either f is of the form (5.3) with $0 < \lambda < 1$ or assume (f_0) and (f_1). Then for any $\gamma \in \Gamma$*
$$\gamma([0,\,1]) \cap \mathcal{P} \neq 0\ .$$

This lemma leads to the following relation between the infimum A and the mountain pass level c

Lemma 5.14. *Assume either f is of the form (5.3) with $0 < \lambda < 1$ or assume (f_0) and (f_1). Then the infimum A satisfies the inequality $A \leq c$.*

Proof. Let $\gamma \in \Gamma$ and let $t_0 \in (0,\,1]$ be such that $\gamma(t_0) \in \mathcal{P}$, the existence of such a t_0 is guaranteed by Lemma 5.13. Since $\gamma(t_0) \in \mathcal{P}$, we have
$$I(\gamma(t_0)) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx$$
and thus
$$\max_{t\in[0,\,1]} I(\gamma(t)) \geq I(\gamma(t_0)) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \geq A\ .$$
From the arbitrary choice of $\gamma \in \Gamma$ it follows that
$$\max_{t\in[0,\,1]} I(\gamma(t)) \geq A \qquad \forall\gamma \in \Gamma$$
and this leads to the desired inequality. □

5.3. Estimate of the mountain pass level c

In order to get an upper bound for the mountain pass level c we will show the existence of $u \in H^1(\mathbb{R}^2)$ such that

$$\max_{t\geq 0} I(tu) < \frac{1}{2} \,. \tag{5.14}$$

Firstly we consider the case when f is as in (5.3) with $0 < \lambda < 1$. To obtain the existence of $u \in H^1(\mathbb{R}^2)$ which satisfies the inequality (5.14), the fact that

$$\lim_{|s|\to+\infty} \frac{sf(s)}{e^{4\pi s^2}} = +\infty \tag{5.15}$$

plays an important role. In particular we can notice from (5.15) it follows that fixed

$$\beta_0 > \frac{1}{\pi} \tag{5.16}$$

there exists $\overline{s} = \overline{s}(\beta_0) > 0$ such that

$$sf(s) \geq \beta_0 e^{4\pi s^2} \quad \forall |s| \geq \overline{s} \,. \tag{5.17}$$

We consider the modified *Moser's sequence* introduced in [28]:

$$\overline{\omega}_n(x) := \frac{1}{\sqrt{2}\pi} \begin{cases} (\log n)^{\frac{1}{2}} & 0 \leq |x| \leq \frac{1}{n} \,, \\ \frac{\log \frac{1}{|x|}}{(\log n)^{\frac{1}{2}}} & \frac{1}{n} \leq |x| \leq 1 \,, \\ 0 & |x| \geq 1 \,. \end{cases}$$

We can notice that $\overline{\omega}_n \in H_0^1(B_1) \subset H^1(\mathbb{R}^2)$, $\|\nabla \overline{\omega}_n\|_2 = 1$ and

$$\|\overline{\omega}_n\|_2^2 = \mathcal{O}\left(\frac{1}{\log n}\right)$$

as $n \to +\infty$. We then define

$$\omega_n := \frac{\overline{\omega}_n}{\|\overline{\omega}_n\|_{H^1}} \,.$$

Lemma 5.15. *Assume f is of the form (5.3) with $0 < \lambda < 1$. Then there exists $n \geq 1$ such that*

$$\max_{t\geq 0} I(t\omega_n) < \frac{1}{2} \,.$$

Proof. We argue by contradiction assuming that

$$\max_{t\geq 0} I(t\omega_n) \geq \frac{1}{2} \qquad \forall n \geq 1 \,.$$

For any $n \geq 1$, let $t_n > 0$ be such that

$$I(t_n\omega_n) = \max_{t\geq 0} I(t\omega_n) \geq \frac{1}{2} \,,$$

then we can estimate

$$\frac{1}{2} \leq I(t_n\omega_n) = \frac{1}{2}t_n^2\|\omega_n\|_{H^1}^2 - \int_{\mathbb{R}^2} F(t_n\omega_n)\,dx \leq \frac{1}{2}t_n^2$$

and

$$t_n^2 \geq 1 \qquad \forall n \geq 1\ . \tag{5.18}$$

At $t = t_n$ we have

$$0 = \frac{d}{dt}I(t\omega_n)\Big|_{t=t_n} = t_n - \int_{\mathbb{R}^2} f(t_n\omega_n)\omega_n\,dx\ ,$$

which implies that

$$t_n^2 = \int_{\mathbb{R}^2} f(t_n\omega_n)t_n\omega_n\,dx\ . \tag{5.19}$$

We claim that $\{t_n\}_n \subset \mathbb{R}$ is bounded. In fact, since

$$t_n\omega_n = \frac{t_n}{\|\overline{\omega}_n\|_{H^1}}\frac{1}{\sqrt{2\pi}}\sqrt{\log n} \to +\infty \qquad \text{a.e. in } B_{\frac{1}{n}}\ ,$$

from (5.17), it follows that at least for $n \geq 1$ sufficiently large

$$t_n^2 \geq \int_{B_{\frac{1}{n}}} f(t_n\omega_n)t_n\omega_n\,dx \geq \beta_0\int_{B_{\frac{1}{n}}} e^{4\pi(t_n\omega_n)^2}\,dx = \frac{\pi}{n^2}\beta_0 e^{2\frac{t_n^2}{\|\overline{\omega}_n\|_{H^1}^2}\log n}\ . \tag{5.20}$$

Consequently

$$1 \geq \pi\beta_0 e^{2\frac{t_n^2}{\|\overline{\omega}_n\|_{H^1}^2}\log n - 2\log t_n - 2\log n}$$

for $n \geq 1$ sufficiently large and $\{t_n\}_n$ must be bounded.

We claim that

$$t_n^2 \to 1$$

as $n \to +\infty$. Arguing by contradiction, since (5.18) holds, we have to assume that

$$\lim_{n\to+\infty} t_n^2 > 1\ .$$

Recalling (5.20), for $n \geq 1$ sufficiently large we have

$$t_n^2 \geq \pi\beta_0 e^{2\log n\left(\frac{t_n^2}{\|\overline{\omega}_n\|_{H^1}^2} - 1\right)}$$

and letting $n \to +\infty$ we get a contradiction with the boundedness of the sequence $\{t_n\}_n$.

In order to estimate (5.19) more precisely, we define the sets

$$A_n := \{x \in B_1 \mid t_n\omega_n(x) \geq \overline{s}\}\ , \quad C_n := B_1 \setminus A_n$$

where $\overline{s} > 0$ is given in (5.17). With (5.19) and (5.17) we can estimate

$$t_n^2 \geq \int_{B_1} f(t_n\omega_n)t_n\omega_n\,dx \geq \beta_0\int_{B_1} e^{4\pi t_n^2\omega_n^2}\,dx + \int_{C_n} f(t_n\omega_n)t_n\omega_n\,dx - \beta_0\int_{C_n} e^{4\pi t_n^2\omega_n^2}\,dx$$

for any $n \geq 1$. Since $\omega_n \to 0$ a.e. in B_1, from the definition of C_n we obtain that the characteristic functions

$$\chi_{C_n} \to 1 \quad \text{a.e. in } B_1 \ ,$$

and the Lebesgue dominated convergence theorem implies that

$$\int_{C_n} f(t_n\omega_n)t_n\omega_n \, dx \to 0 \ ,$$

$$\int_{C_n} e^{4\pi t_n^2\omega_n^2} \, dx \to \pi$$

as $n \to +\infty$. If we prove that

$$\lim_{n\to+\infty} \int_{B_1\setminus B_{\frac{1}{n}}} e^{4\pi t_n^2\omega_n^2} \, dx \geq 2\pi \tag{5.21}$$

then

$$1 = \lim_{n\to+\infty} t_n^2 \geq \pi\beta_0$$

which is in contradiction with (5.16). To end the proof it remains only to prove that inequality (5.21) holds. As a consequence of (5.18)

$$\int_{B_1\setminus B_{\frac{1}{n}}} e^{4\pi t_n^2\omega_n^2} \, dx \geq \int_{B_1\setminus B_{\frac{1}{n}}} e^{4\pi\omega_n^2} \, dx = 2\pi \int_{\frac{1}{n}}^1 e^{\frac{2}{\|\overline{\omega}_n\|^2_{H^1}} \frac{1}{\log n} \log^2\left(\frac{1}{s}\right)} s \, ds$$

and if we make the change of variable

$$\tau = \frac{\log \frac{1}{s}}{\|\overline{\omega}_n\|_{H^1} \log n}$$

then we obtain the following estimate

$$\int_{B_1\setminus B_{\frac{1}{n}}} e^{4\pi t_n^2\omega_n^2} \, dx \geq 2\pi\|\overline{\omega}_n\|_{H^1} \log n \int_0^{\frac{1}{\|\overline{\omega}_n\|_{H^1}}} e^{2\log n(\tau^2 - \|\overline{\omega}_n\|_{H^1}\tau)} \, d\tau \ .$$

Now it suffices to notice that

$$\tau^2 - \|\overline{\omega}_n\|_{H^1}\tau \geq \begin{cases} -\|\overline{\omega}_n\|_{H^1}\tau & 0 \leq \tau \leq \frac{1}{2\|\overline{\omega}_n\|_{H^1}} \\ \left(\frac{2}{\|\overline{\omega}_n\|_{H^1}} - \|\overline{\omega}_n\|_{H^1}\right)\left(\tau - \frac{1}{\|\overline{\omega}_n\|_{H^1}}\right) + \frac{1}{\|\overline{\omega}_n\|^2_{H^1}} - 1 & \frac{1}{2\|\overline{\omega}_n\|_{H^1}} \leq \tau \leq \frac{1}{\|\overline{\omega}_n\|_{H^1}} \end{cases}$$

to conclude that (5.21) holds. □

Now we consider the case when (f_2) and (f_3) holds. In this case, as a consequence of (f_3), we have that for any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that

$$sf(s) \geq (\beta_0 - \varepsilon)e^{4\pi s^2} \qquad \forall |s| \geq s_\varepsilon \ .$$

Let $r > 0$ be such that
$$\beta_0 > \frac{1}{r^2\pi} \, ,$$
we consider the modified *Moser's sequence* introduced in [31]:
$$\overline{M}_n(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{\frac{1}{2}} & 0 \le |x| \le \frac{r}{n} \, , \\ \frac{\log \frac{r}{|x|}}{(\log n)^{\frac{1}{2}}} & \frac{r}{n} \le |x| \le r \, , \\ 0 & |x| \ge r \, . \end{cases}$$
We can notice that $\overline{M}_n \in H_0^1(B_r) \subset H^1(\mathbb{R}^2)$, $\|\nabla \overline{M}_n\|_2 = 1$ and
$$\|\overline{M}_n\|_2^2 = \mathcal{O}\left(\frac{1}{\log n}\right)$$
as $n \to +\infty$. We then define
$$M_n := \frac{\overline{M}_n}{\|\overline{M}_n\|_{H^1}}$$
and arguing as before (see also [31], Lemma 4.4) we have the following result

Lemma 5.16. *Assume (f_2) and (f_3). Then there exists $n \in \mathbb{N}$ such that*
$$\max_{t \ge 0} I(tM_n) < \frac{1}{2} \, .$$

Lemma 5.15 and Lemma 5.16 give indeed more precise informations about the mountain pass level c both in the case when f is as in (5.3) with $0 < \lambda < 1$ and in the case when (f_2) and (f_3). In fact, from these lemmas we get the existence of $u \in H^1(\mathbb{R}^2)$ such that
$$\max_{t \ge 0} I(tu) < \frac{1}{2} \, .$$
Let $\bar{t} > 0$ be such that $I(\bar{t}u) < 0$ and let $u_0 := \bar{t}u$. If we consider the path
$$\overline{\gamma} := t \cdot \bar{t}u \qquad \forall t \in [0, 1]$$
then $\overline{\gamma} \in \Gamma := \left\{\gamma \in \mathcal{C}([0, 1], H^1(\mathbb{R}^2)) \mid \gamma(0) = 0, \gamma(1) = u_0 \in H^1(\mathbb{R}^2)\right\}$ and we have
$$c \le \max_{t \in [0, 1]} I(\overline{\gamma}(t)) \le \max_{t \ge 0} I(tu) < \frac{1}{2} \, . \tag{5.22}$$

5.4. The infimum A is attained

In this Section we will prove Proposition 5.5. We can notice that, either in the case when f is of the form (5.3) with $0 < \lambda < 1$ or in the case when (f_0), (f_1), (f_2) and (f_3) hold, if the infimum A is attained then the minimizer $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$ is a solution of problem (5.2), under a suitable change of scale. In fact, if $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$ is such that
$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx = A$$

and

$$\int_{\mathbb{R}^2} G(u)\,dx = 0$$

then there exists a Lagrange multiplier $\theta \in \mathbb{R}$, namely

$$\frac{1}{2}\int_{\mathbb{R}^2} \nabla u \cdot \nabla v\,dx = \theta \int_{\mathbb{R}^2} g(u)v\,dx \qquad \forall v \in H^1(\mathbb{R}^2)\ .$$

Since it is easy to see that $\theta > 0$, we can set

$$u_\theta(x) := u\left(\frac{x}{\sqrt{\theta}}\right) \tag{5.23}$$

for a.e. $x \in \mathbb{R}^2$. We have that u_θ is a non-trivial solution of problem (5.2) and hence

$$m \le I(u_\theta)\ .$$

Moreover

$$\int_{\mathbb{R}^2} |\nabla u_\theta|^2\,dx = \int_{\mathbb{R}^2} |\nabla u|^2\,dx = A\ , \quad \int_{\mathbb{R}^2} G(u_\theta)\,dx = \theta \int_{\mathbb{R}^2} G(u)\,dx = 0\ ,$$

from which we get $I(u_\theta) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_\theta|^2\,dx = A$ and thus $m \le A$.

Therefore to prove Proposition 5.5, it remains to show that the infimum A is achieved. The proof in the case that we assume (f_0), (f_1), (f_2) and (f_3) can be easily reduced to the proof of [10], Theorem 1.4. It suffices to notice that from Lemma 5.14 and from inequality (5.22), it follows that

$$A < \frac{1}{2}$$

and thus we are in the same framework of the proof of [10], Theorem 1.4.

Proof of Proposition 5.5 in the case $f(s) := \lambda s e^{4\pi s^2}$ $\forall s \in \mathbb{R}$ with $0 < \lambda < 1$. From Lemma 5.14 and from inequality (5.22), it follows that

$$A < \frac{1}{2}\ .$$

Let $\{u_n\}_n \in H^1(\mathbb{R}^2) \setminus \{0\}$ be a minimizing sequence for A:

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\,dx \quad \overset{n\to+\infty}{\longrightarrow} \quad A\ ,$$

$$\int_{\mathbb{R}^2} G(u_n)\,dx = 0 \qquad \forall n \ge 1\ . \tag{5.24}$$

Without loss of generality we may assume that $\{u_n\}_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^2)\setminus\{0\}$ and that $\|u_n\|_2 = 1$. We will prove that the weak limit $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ of $\{u_n\}_n$ in $H^1(\mathbb{R}^2)$ is a minimizer for A.

Since

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla u_n|^2\,dx = 2A < 1\ ,$$

from Lemma 5.10 it follows that

$$\int_{\mathbb{R}^2} F(u_n)\,dx - \frac{\lambda}{2}\|u_n\|_2^2 \quad \overset{n\to+\infty}{\longrightarrow} \quad \int_{\mathbb{R}^2} F(u)\,dx - \frac{\lambda}{2}\|u\|_2^2 \,.$$

Furthermore, (5.24) leads to

$$\int_{\mathbb{R}^2} F(u_n)\,dx = \frac{1}{2}\,.$$

Therefore

$$\int_{\mathbb{R}^2} F(u)\,dx - \frac{\lambda}{2}\|u\|_2^2 = \frac{1}{2}(1-\lambda) > 0$$

which in particular implies that $u \neq 0$.

From the weak convergence $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, we get

$$A = \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 \geq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,.$$

Hence, to conclude, it suffices to prove that

$$\int_{\mathbb{R}^2} G(u)\,dx = 0\,.$$

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, we have

$$\|u\|_2^2 \leq \liminf_{n\to+\infty} \|u_n\|_2^2 = 1$$

and thus

$$\begin{aligned}\int_{\mathbb{R}^2} G(u)\,dx &= \int_{\mathbb{R}^2} F(u)\,dx - \frac{1}{2}\|u\|_2^2 = \int_{\mathbb{R}^2} F(u)\,dx - \frac{\lambda}{2}\|u\|_2^2 + \frac{1}{2}(\lambda-1)\|u\|_2^2 = \\ &= \frac{1}{2}(1-\lambda) + \frac{1}{2}(\lambda-1)\|u\|_2^2 = \frac{1}{2}(1-\lambda)(1-\|u\|_2^2) \geq 0\,.\end{aligned}$$

If we argue by contradiction assuming that

$$\int_{\mathbb{R}^2} G(u)\,dx \neq 0$$

then we have necessarily

$$\int_{\mathbb{R}^2} G(u)\,dx > 0\,. \tag{5.25}$$

Let

$$h(t) := \int_{\mathbb{R}^2} G(tu)\,dx = \int_{\mathbb{R}^2} F(tu)\,dx - \frac{t^2}{2}\|u\|_2^2 \quad \forall t > 0\,.$$

We can notice that for any $t \in (0,\,1)$ we have

$$\begin{aligned}\int_{\mathbb{R}^2} F(tu)\,dx &= \frac{\lambda}{8\pi}\int_{\mathbb{R}^2} (e^{4\pi t^2 u^2} - 1)\,dx = \frac{\lambda}{2}t^2\|u\|_2^2 + \frac{\lambda}{8\pi}\sum_{j=2}^{+\infty} \frac{(4\pi)^j}{j!} t^{2j} \int_{\mathbb{R}^2} u^{2j}\,dx \leq \\ &\leq \frac{\lambda}{2}t^2\|u\|_2^2 + t^4\frac{\lambda}{8\pi}\sum_{j=2}^{+\infty}\frac{(4\pi)^j}{j!}\int_{\mathbb{R}^2} u^{2j}\,dx = \\ &= \frac{\lambda}{2}t^2\|u\|_2^2 + t^4\frac{\lambda}{8\pi}\int_{\mathbb{R}^2}(e^{4\pi u^2} - 1)\,dx\,.\end{aligned}$$

Hence for any $t \in (0, 1)$

$$h(t) \le \frac{1}{2}(\lambda - 1)t^2\|u\|_2^2 + t^4\frac{\lambda}{8\pi}\int_{\mathbb{R}^2}(e^{4\pi u^2} - 1)\,dx\ ,$$

from which we deduce that $h(t) < 0$ for $t > 0$ sufficiently small. But $h(1) > 0$, as a consequence of (5.25), and thus there exists $t_0 \in (0, 1)$ such that $h(t_0) = 0$, that is

$$\int_{\mathbb{R}^2} G(t_0 u)\,dx = 0\ .$$

Therefore

$$A \le \frac{1}{2}\int_{\mathbb{R}^2}|\nabla(t_0 u)|^2\,dx = \frac{1}{2}t_0^2\int_{\mathbb{R}^2}|\nabla u|^2\,dx \le t_0^2 A < A$$

which is a contradiction. □

5.5. Proofs of Theorem 5.1 and Theorem 5.3

In order to prove Theorem 5.1 and Theorem 5.3 we can notice that, both in the case when f is of the form (5.3) with $0 < \lambda < 1$ and in the case when (f_0), (f_1), (f_2) and (f_3) hold, from Proposition 5.5 we have $m \le A$. Moreover, Lemma 5.14 tells us that $A \le c$ and hence

$$m \le c\ .$$

It remains only to show that

$$m \ge c \tag{5.26}$$

to conclude that the mountain pass level c gives the ground state level.

In [40] the authors proved the following result

Theorem 5.17 ([40], Lemma 2.1)**.** *Assume (g_0), (g_1), (g_2) and (g_3) as in Theorem 5.4. Then for any solution u of (5.1) there exists a path $\gamma \in \Gamma$ such that $u \in \gamma([0, 1])$ and*

$$\max_{t\in[0,1]} I(\gamma(t)) = m\ .$$

It is easy to see that the proof of this theorem works also under our assumptions and this leads to (5.26).

Indeed, we can notice that in this way we proved that

$$m = A = c\ .$$

Hence if $u \in H^1(\mathbb{R}^2)$ is a minimizer for A and we define u_θ as in (5.23) then u_θ is a ground state solution of problem (5.2). This gives a proof of Proposition 5.6.

CHAPTER 6

A biharmonic equation in $\mathbb{R}^4$: the subcritical case

In this Chapter we consider a biharmonic equation of the form

$$\begin{cases} \Delta^2 u + V(|x|)u = f(u) & \text{in } \mathbb{R}^4 \\ u \in H^2(\mathbb{R}^4) \end{cases} \tag{6.1}$$

where the condition $u \in H^2(\mathbb{R}^4)$ expresses explicitely that the biharmonic equation is to be satisfied in the weak sense. Assuming that the potential V satisfies some symmetry conditions and is bounded away from zero and that the nonlinearity f is odd and has subcritical exponential growth (in the sense of the Adams-type inequality (3.1), see also Theorem 2.1), we prove a multiplicity result. More precisely we prove the existence of infinitely many nonradial sign-changing solutions and infinitely many radial solutions in $H^2(\mathbb{R}^4)$. The main difficulty is the lack of compactness due to the unboundedness of the domain $\mathbb{R}^4$ and in this respect the symmetries of the problem play an important role.

In order to obtain the existence of infinitely many nonradial sign-changing and radial solutions for the biharmonic problem (6.1), we make the following assumptions on the potential V and the nonlinearity f:

(V_1) $V \in \mathcal{C}(\mathbb{R}^4, \mathbb{R})$ is bounded from below by a positive constant V_0,

$$V(x) \geq V_0 > 0 \qquad \forall x \in \mathbb{R}^4\ ;$$

(V_2) V is spherically symmetric with respect to $x \in \mathbb{R}^4$,

$$V(x) = V(|x|) \qquad \forall x \in \mathbb{R}^4\ ;$$

(f_1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ has subcritical exponential growth, i.e.

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0 \quad \forall \alpha > 0\,;$$

(f_2) $f(s) = o(|s|)$ as $|s| \to 0$;

(f_3) f is odd.

We can notice that, as a consequence of assumption (f_3), nonzero solutions of (6.1) occour in antipodal pairs, namely if u is a solution of (6.1) then $-u$ is a solution of (6.1) too.

Furthermore, setting $F(s) := \int_0^s f(t)\,dt$, we will assume that:

(F_1) $\exists \mu > 2$ such that

$$\mu F(s) \leq s f(s) \qquad \forall s \in \mathbb{R}\ ;$$

(F_2) $\exists \overline{s} > 0$ such that $\inf\limits_{|s| \geq \overline{s}} F(s) > 0\,.$

Remark 6.1. *(F_1) and (F_2) implies the Ambrosetti-Rabinowitz condition, namely*

$$(A-R) \quad \exists \mu > 2 \quad \text{ such that } \quad 0 < \mu F(s) \leq s f(s) \quad \forall s \geq \overline{s}\ .$$

As we will see during the proof, we need the stronger condition (F_1) to obtain the Palais-Smale condition.

Example 6.2. *The function $f(s) := s(e^{\gamma|s|} - 1)$ $\forall s \in \mathbb{R}$, where $\gamma > 0$, satisfies conditions (f_1), (f_2), (f_3), (F_1) and (F_2), for a proof see Proposition 6.15.*

We can now state our main result:

Theorem 6.3. *Assume that (V_1), (V_2), (f_1), (f_2), (f_3), (F_1) and (F_2) hold. Then there exists an unbounded sequence $\{\pm u_k\}_{k\in\mathbb{N}}$ of sign-changing solutions of (6.1) which are not radial. There also exists an unbounded sequence $\{\pm u_k\}_{k\in\mathbb{N}}$ of radial solutions of (6.1).*

Here and below the unboundedness of the sequences $\{u_k\}_{k\in\mathbb{N}}$ of solutions has to be understood as follows:

$$\int_{\mathbb{R}^4} \left[(\Delta u_k)^2 + V(|x|)u_k^2\right] dx \to +\infty \quad \text{as } k \to +\infty\ .$$

We point out that, since problem (6.1) is invariant under rotations, it is natural to look for radially symmetric solutions. Therefore it seems to be more interesting the multiplicity of nonradial solutions of (6.1). Concerning the unbounded sequence $\{u_k\}_{k\in\mathbb{N}}$ of sign-changing solutions of (6.1) which are *not radial*, we can notice that the *orbit of* u_k

$$\mathcal{O}(4) * u_k := \{u \circ g \ : \ g \in \mathcal{O}(4)\} \subseteq H^2(\mathbb{R}^4)$$

is diffeomorphic to the quotient space $\mathcal{O}(4)/\mathcal{Z}(u_k)$, where

$$\mathcal{Z}(u_k) := \left\{g \in \mathcal{O}(4) \ : \ u_k(gx) = u_k(x)\ \forall x \in \mathbb{R}^4\right\} \subseteq \mathcal{O}(4)$$

is the *isotropy group of* u_k. Since u_k is not radial, it is possible that for some $g \in \mathcal{O}(4)$

$$u(gx) = u(x) \quad \forall x \in \mathbb{R}^4\ ,$$

but this cannot happen for all $g \in \mathcal{O}(4)$, namely $\mathcal{Z}(u_k) \subsetneq \mathcal{O}(4)$. Therefore

$$\dim \mathcal{O}(4) * u_k = \dim \mathcal{O}(4)/\mathcal{Z}(u_k) \geq 1$$

and to each u_k corresponds a nontrivial $\mathcal{O}(4)$-orbit of solutions to (6.1). Furthermore the orbits $\mathcal{O}(4) * u_{k_1}$ and $\mathcal{O}(4) * u_{k_2}$ with $k_1 \neq k_2$ and k_1, k_2 sufficiently large are disjoint because

$$\begin{aligned} &\int_{\mathbb{R}^4} \left[(\Delta\,(u_k \circ g)\,)^2 + V(|x|)(u_k \circ g)^2 \right] dx = \\ = \quad &\int_{\mathbb{R}^4} \left[(\Delta u_k)^2 + V(|x|) u_k^2 \right] dx \to +\infty \quad \text{as } k \to +\infty\ . \end{aligned}$$

It will be clear during the proof that it is possible to obtain an unbounded sequence of nonradial sign-changing solutions of (6.1) without requiring the potential V to be spherically symmetric with respect to $x \in \mathbb{R}^4$. In fact we may replace the assumption (V_2) on the potential V with the following weaker assumptions:

(V_2') V is spherically symmetric with respect to $x_1, x_2 \in \mathbb{R}^2$,

$$V(x) = V(|x_1|, |x_2|) \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2\ ;$$

(V_2'') $V(|x_1|, |x_2|) = V(|x_2|, |x_1|) \quad \forall x_1, x_2 \in \mathbb{R}^2$.

Theorem 6.4. *Assume that (V_1), (V_2'), (V_2''), (f_1), (f_2), (f_3), (F_1) and (F_2) hold. Then there exists an unbounded sequence $\{\pm u_k\}_{k\in\mathbb{N}}$ of sign-changing solutions of*

$$\begin{cases} \Delta^2 u + V(|x_1|, |x_2|) u = f(u) & \text{in } \mathbb{R}^4 \\ u \in H^2(\mathbb{R}^4) \end{cases} \tag{6.2}$$

which are not radial.

Furthermore, requiring only the potential V to be spherically symmetric with respect to $x_1, x_2 \in \mathbb{R}^2$, it is possible to obtain an unbounded sequence of solutions of (6.2).

Theorem 6.5. *Assume that (V_1), (V_2'), (f_1), (f_2), (f_3), (F_1) and (F_2) hold. Then (6.2) possesses an unbounded sequence $\{\pm u_k\}_{k\in\mathbb{N}}$ of solutions.*

To prove these theorems we will follow a variational approach. Let X be the subspace of $H^2(\mathbb{R}^4)$ defined as

$$X := \left\{ u \in H^2(\mathbb{R}^4) \;\Big|\; \int_{\mathbb{R}^4} \left[(\Delta u)^2 + V(x) u^2 \right] dx < +\infty \right\}\ .$$

By (V_1), it follows that X is a Hilbert space endowed with the scalar product

$$\langle u, v\rangle := \int_{\mathbb{R}^4} \Delta u \Delta v\, dx + \int_{\mathbb{R}^4} V(x) uv\, dx \qquad u, v \in X$$

to which corresponds the norm $\|u\| := \sqrt{\langle u, u\rangle}$. Applying an interpolation inequality, it is easy to see that the embedding $X \hookrightarrow H^2(\mathbb{R}^4)$ is continuous.

The solutions of (6.1) are critical points of the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^4} F(u)\,dx \qquad \forall u \in X$$

which is well defined and differentiable on X, namely $I \in \mathcal{C}^1(X, \mathbb{R})$. The difficulty in working in this variational framework is the lack of compactness, infact I fails to satisfy the Palais-Smale condition in X. However, to gain compactness, we shall exploit the symmetries of the problem imposing the invariance with respect to a group G acting on X.

Let X_G be the space of fixed points in X with respect to the action of the group G:

$$X_G := \left\{u \in X \mid u(gx) = u(x) \quad \forall g \in G \;\text{ and a.e. } x \in \mathbb{R}^4\right\} \subseteq X\ .$$

We will prove the following result:

Proposition 6.6. *Let G be a group acting on X via orthogonal maps such that:*

(G_1) *$I : X \to \mathbb{R}$ is G-invariant;*

(G_2) *X_G is compactly embedded in $L^p(\mathbb{R}^4)$ for any $p \in (4, +\infty)$;*

(G_3) *dim $X_G = +\infty$.*

Then I has an unbounded sequence of critical points lying on X_G.

To prove Proposition 6.6, we will show that the problem reduces to the study of the multiplicity of critical points of the restriction $I|_{X_G}$ which behaves like a mountain pass and satisfies the Palais-Smale condition. More precisely $I|_{X_G}$ satisfies the assumptions of a generalized mountain pass theorem due to A. Ambrosetti and P. H. Rabinowitz [59] which gives the multiplicity of critical points.

This Chapter is organized as follows. In Section 6.1, we will prove Proposition 6.6. In Section 6.2, we will show the existence of a group G, which satisfies the assumptions of Proposition 6.6, following an approach introduced by T. Bartsch and M. Willem in [13] (see also [12]). This approach allows to obtain additional informations on the nodal structure of the critical points. The existence of such a group together with Proposition 6.6 will allow us to conclude that the main theorem, Theorem 6.3, holds. We will also explain how to adapt these arguments to prove Theorem 6.4 and Theorem 6.5. As shown in Section 6.3, these arguments can also be adapted to obtain similar results for biharmonic problems of the form

$$\begin{cases} \Delta^2 u - \text{div}\,(U(x)\nabla u) + V(x)u = f(u) & \text{in } \mathbb{R}^4 \\ u \in H^2(\mathbb{R}^4) \end{cases}$$

under suitable assumptions on U, $V : \mathbb{R}^4 \to \mathbb{R}$.

6.1. Mountain pass structure and Palais-Smale condition

We consider the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^4} F(u)\, dx \qquad \forall u \in X\ .$$

We can notice that, as a consequence of (f_1) and (f_2), fixed $\alpha > 0$ and $q > 0$, we have the existence of two constants c_1, $c_2 > 0$ such that

$$|f(s)| \leq c_1|s| + c_2|s|^q(e^{\alpha s^2} - 1) \qquad \forall s \in \mathbb{R}\ ;$$

therefore, from $(A - R)$, it follows the existence of two constants $\bar{c}_1$, $\bar{c}_2 > 0$ such that:

$$|F(s)| \leq \bar{c}_1|s|^2 + \bar{c}_2|s|^{q+1}(e^{\alpha s^2} - 1) \qquad \forall s \in \mathbb{R}\ .$$

This together with Remark 3.8 implies that the functional I is well defined on X. Using standard arguments, it is easy to see that $I \in \mathcal{C}^1(X, \mathbb{R})$,

$$I'(u)v = \langle u, v\rangle - \int_{\mathbb{R}^4} f(u)v\, dx \qquad \forall u, v \in X$$

and the critical points of I are solutions of problem (6.1).

The main aim of this Section is the proof Proposition 6.6. Thus let G be a group acting on X via orthogonal maps satisfying (G_1), (G_2) and (G_3). Firsly we can notice that, as a consequence of the principle of symmetric criticality [55], any critical point of the restriction $I|_{X_G}$ is a critical point of I too. Therefore the proof of Proposition 6.6 reduces to show that $I|_{X_G}$ has an unbounded sequence of critical points. To do this we apply the following generalized mountain pass theorem.

Theorem 6.7 ([59], Theorem 9.12)**.** *Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space over $\mathbb{R}$ and let $I \in \mathcal{C}^1(E, \mathbb{R})$ be an even functional such that $I(0) = 0$. We assume that:*

(I_1) *$\exists \rho, \gamma > 0$ such that $I|_{B_\rho\setminus\{0\}} > 0$ and $I|_{\partial B_\rho} \geq \gamma > 0$ where*

$$B_\rho := \{u \in E \mid \|u\| \leq \rho\} \subset E\ ;$$

(I_2) *for any finite dimensional subspace $\tilde{E} \subset E$ the set $\{u \in \tilde{E} \mid I(u) \geq 0\}$ is bounded;*

(I_3) *the Palais-Smale condition holds.*

Then I possesses an unbounded sequence of critical values $c_k \to +\infty$ as $k \to +\infty$.

As mentioned above $I \in C^1(X, \mathbb{R})$, moreover I is even and $I(0) = 0$. We have to show that the functional $I|_{X_G}$ satysfies the remaining hypotheses of Theorem 6.7 for $E = X_G$ which is infinite dimensional by assumption (G_3).

Lemma 6.8. *Assume (f_1), (f_2) and $(A - R)$. Then $I|_{X_G}$ satisfies (I_1).*

Proof. As a consequence of (f_1), (f_2) and $(A-R)$, fixed $\alpha > 0$ and $q > 1$, we have that for any $0 < \varepsilon < 1$ there exists a constant $C(\alpha,\, q,\, \varepsilon) > 0$ such that

$$|F(s)| \leq \varepsilon |s|^2 + C(\alpha,\, q,\, \varepsilon)|s|^q (e^{\alpha s^2} - 1) \qquad \forall s \in \mathbb{R}\ . \tag{6.3}$$

Thus in particular if we fix $\alpha = 1$ and $q > 3$ then for any $0 < \varepsilon < 1$ we have

$$\int_{\mathbb{R}^4} F(u)\,dx \leq \varepsilon \|u\|_2^2 + C(q,\, \varepsilon) \int_{\mathbb{R}^4} |u|^q (e^{u^2} - 1)\,dx \qquad \forall u \in X_G\ .$$

Now, recalling that the embedding $X_G \hookrightarrow H^2(\mathbb{R}^4)$ is continuous, namely there exists a constant $\overline{C} > 0$ such that

$$\|u\|_{H^2} \leq \overline{C}\|u\| \qquad \forall u \in X_G\ ,$$

we have that if $\|u\| \leq \frac{1}{\overline{C}}$ then $\|u\|_{H^2} \leq 1$. Applying Lemma 3.9

$$\int_{\mathbb{R}^4} F(u)\,dx \leq \varepsilon \|u\|_2^2 + C_1(q,\, \varepsilon)\|u\|^q \qquad \forall u \in X_G,\ \|u\| \leq \frac{1}{\overline{C}}$$

and without loss of generality we may assume that $[C_1(q,\, \varepsilon)]^{\frac{1}{q-2}} > \overline{C}$.

So for any $u \in X_G$ with $\|u\| \leq \frac{1}{\overline{C}}$ we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \varepsilon \|u\|_2^2 - C_1(q,\, \varepsilon)\|u\|^q \geq \left(\frac{1}{2} - \frac{\varepsilon}{V_0}\right)\|u\|^2 - C_1(q,\, \varepsilon)\|u\|^q = \\ &= \|u\|^2 \left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C_1(q,\, \varepsilon)\|u\|^{q-2}\right)\ . \end{aligned} \tag{6.4}$$

Now we choose $0 < \varepsilon < 1$ as follows

$$0 < \varepsilon < \min\left\{1,\ \left(\frac{1}{2} - \frac{1}{2^{q-2}}\right) V_0\right\}$$

and we set

$$\rho := \frac{1}{2[C_1(q,\, \varepsilon)]^{\frac{1}{q-2}}}\ .$$

Since $\rho < \frac{1}{\overline{C}}$, (6.4) holds for any $u \in X_G$ with $\|u\| = \rho$ and we have

$$I(u) \geq \rho^2 \left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C_1(q,\, \varepsilon)\rho^{q-2}\right) = \rho^2 \left(\frac{1}{2} - \frac{\varepsilon}{V_0} - \frac{1}{2^{q-2}}\right) \qquad \forall u \in X_G,\ \|u\| = \rho.$$

Setting

$$\gamma := \rho^2 \left(\frac{1}{2} - \frac{\varepsilon}{V_0} - \frac{1}{2^{q-2}}\right) > 0\ ,$$

we get

$$I(u) \geq \gamma \qquad \forall u \in X_G,\ \|u\| = \rho\ .$$

In conclusion, if $\rho_1 \leq \rho$ then applying (6.4) we have that for any $u \in X_G$ with $\|u\| = \rho_1$

$$I(u) \geq \rho_1^2 \left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C_1(q,\, \varepsilon)\rho_1^{q-2}\right) \geq \rho_1^2 \left(\frac{1}{2} - \frac{\varepsilon}{V_0} - C_1(q,\, \varepsilon)\rho^{q-2}\right) = \rho_1^2 \frac{\gamma}{\rho^2} > 0$$

and this means that

$$I(u) > 0 \qquad \forall u \in X_G \setminus \{0\}, \|u\| \leq \rho \ .$$

□

Lemma 6.9. *Assume (f_2) and $(A - R)$. Then $I|_{X_G}$ satisfies (I_2).*

Proof. As a consequence of (f_2) and $(A - R)$ there exist C_1, $C_2 > 0$ such that

$$F(s) \geq C_1 |s|^\mu - C_2 |s|^2 \qquad \forall s \in \mathbb{R} \ .$$

Therefore for any $u \in X_G$ we have that

$$I(u) \leq \frac{1}{2}\|u\|^2 + C_2\|u\|_2^2 - C_1\|u\|_\mu^\mu \leq \left(\frac{1}{2} + \frac{C_2}{V_0}\right)\|u\|^2 - C_1\|u\|_\mu^\mu \ . \tag{6.5}$$

Let $\tilde{E}$ be a finite dimensional subspace of X_G. Since all norms in $\tilde{E}$ are equivalent, there exists a constant $C > 0$ such that for any $u \in \tilde{E}$ we have $\|u\|_\mu \geq C\|u\|$. Thus

$$I(u) \leq \left(\frac{1}{2} + \frac{C_2}{V_0}\right)\|u\|^2 - \tilde{C}_1\|u\|^\mu \qquad \forall u \in \tilde{E}$$

and in particular for any $u \in \tilde{E}$ with $\|u\| = R$

$$I(u) \leq \left(\frac{1}{2} + \frac{C_2}{V_0}\right) R^2 - \tilde{C}_1 R^\mu \ .$$

This means that for $R > 0$ sufficiently large

$$I(u) < 0 \qquad \forall u \in \tilde{E},\ \|u\| > R$$

and the set $\{u \in \tilde{E} \mid I(u) \geq 0\}$ is bounded. □

Lemma 6.10. *Assume (f_1), (f_2) and (F_1). Then $I|_{X_G}$ satisfies (I_3).*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset X_G$ be a Palais-Smale sequence, that is $|I(u_n)| \leq C_1$ $\forall n \in \mathbb{N}$ and $I'(u_n) \to 0$ as $n \to +\infty$.

Firstly we prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_G. For any u, $v \in X_G$

$$I'(u)v = \langle u,\, v\rangle - \int_{\mathbb{R}^4} f(u)v\, dx$$

therefore, for any $n \in \mathbb{N}$

$$I(u_n) - \frac{1}{\mu}\, I'(u_n)u_n = \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - \int_{\mathbb{R}^4}\left(F(u_n) - \frac{1}{\mu}\, f(u_n)u_n\right) dx \ .$$

As a consequence of (F_1) we have that

$$\int_{\mathbb{R}^4}\left(F(u_n) - \frac{1}{\mu}\, f(u_n)u_n\right) dx \leq 0$$

and so we obtain

$$I(u_n) - \frac{1}{\mu} I'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2 \qquad \forall n \in \mathbb{N} \; . \tag{6.6}$$

On the other hand

$$I(u_n) - \frac{1}{\mu} I'(u_n)u_n \leq |I(u_n)| + \frac{1}{\mu} |I'(u_n)u_n| \leq C_1 + \frac{C_2}{\mu}\|u_n\| \qquad \forall n \in \mathbb{N} \; . \tag{6.7}$$

From (6.6) and (6.7) it follows that

$$0 \leq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2 \leq C_1 + \frac{C_2}{\mu}\|u_n\| \qquad \forall n \in \mathbb{N}$$

which means that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_G.

It remains to prove that $\{u_n\}_{n\in\mathbb{N}}$ converges up to subsequences. Since, by assumption, (G_2) holds and since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_G, we have that $u_n \to u$ in $L^p(\mathbb{R}^4)$ for any $p \in (4, +\infty)$. Here and below, up to the end of the proof, convergence has to be understood up to the passage to a subsequence.

Fix $n \in \mathbb{N}$. Since

$$[I'(u_n) - I'(u)](u - u_n) = \|u - u_n\|^2 - \int_{\mathbb{R}^4} [f(u_n) - f(u)](u - u_n)\, dx \; ,$$

then

$$\|u - u_n\|^2 = [I'(u_n) - I'(u)](u - u_n) + \int_{\mathbb{R}^4} [f(u_n) - f(u)](u - u_n)\, dx \; .$$

As $\{u_n\}_{n\in\mathbb{N}}$ is a bounded Palais-Smale sequence we have that

$$[I'(u_n) - I'(u)](u - u_n) \to 0$$

when $n \to +\infty$. If we show that for any $0 < \varepsilon < 1$ there exist constants $C_3 > 0$ and $C_4(\varepsilon) > 0$ such that

$$E_n := \int_{\mathbb{R}^4} [f(u_n) - f(u)](u - u_n)\, dx \leq C_3\varepsilon + C_4(\varepsilon)\|u - u_n\|_p \qquad \forall n \in \mathbb{N} \tag{6.8}$$

for some $p > 4$, then it follows that $\|u - u_n\|^2 \to 0$ as $n \to +\infty$ that is what we wanted to prove.

Therefore to end the proof we have to show that (6.8) holds. At this aim we can notice that as a consequence of (f_1) and (f_2), fixed $\alpha > 0$ and $q > 0$, for any $0 < \varepsilon < 1$ there exists a constant $C(\alpha,\, q,\, \varepsilon) > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C(\alpha,\, q,\, \varepsilon)\, |s|^q(e^{\alpha s^2 - 1}) \qquad \forall s \in \mathbb{R} \; .$$

Let $\alpha > 0$ and $q > 0$ to be choosen during the proof. Then for any $0 < \varepsilon < 1$ we have

$$\begin{aligned} E_n \;\; &\leq \;\; \int_{\mathbb{R}^4} \Big[|f(u_n)| + |f(u)|\Big]\, |u - u_n|\, dx \leq \\ &\leq \;\; \int_{\mathbb{R}^4} \Big[\varepsilon(|u_n| + |u|) + \\ &\qquad + C(\alpha,\, q,\, \varepsilon)\left(|u_n|^q(e^{\alpha u_n^2} - 1) + |u|^q(e^{\alpha u^2} - 1)\right)\Big]\, |u - u_n|\, dx = \\ &= \;\; \varepsilon\, E_{1,\,n} + C(\alpha,\, q,\, \varepsilon)\, E_{2,\,n} \end{aligned}$$

where we have set

$$\begin{aligned} E_{1,n} &:= \int_{\mathbb{R}^4} \Big(|u_n| + |u|\Big)\, |u - u_n|\, dx \qquad \forall n \in \mathbb{N}\,, \\ E_{2,n} &:= \int_{\mathbb{R}^4} \Big(|u_n|^q (e^{\alpha u_n^2} - 1) + |u|^q (e^{\alpha u^2} - 1)\Big)\, |u - u_n|\, dx \qquad \forall n \in \mathbb{N}\,. \end{aligned}$$

We estimate $E_{1,n}$ as follows:

$$E_{1,n} \leq \int_{\mathbb{R}^4} (|u_n|^2 + |u|^2)\, dx \leq 2(\|u_n\|_2^2 + \|u\|_2^2) \leq \frac{2}{V_0}(\|u_n\|^2 + \|u\|^2) \leq C_3 \quad \forall n \in \mathbb{N}.$$

To estimate $E_{2,n}$ we apply Hölder's inequality with $\frac{4}{5} + \frac{1}{5} = 1$ obtaining that

$$\begin{aligned} E_{2,n} &\leq \Bigg[\left(\int_{\mathbb{R}^4} |u_n|^{\frac{5}{4}q} (e^{\alpha u_n^2} - 1)^{\frac{5}{4}}\, dx \right)^{\frac{4}{5}} + \\ &\qquad + \left(\int_{\mathbb{R}^4} |u|^{\frac{5}{4}q} (e^{\alpha u^2} - 1)^{\frac{5}{4}}\, dx \right)^{\frac{4}{5}} \Bigg] \|u - u_n\|_5 = \\ &= \Big[(E_{4,n})^{\frac{4}{5}} + (E_{5,n})^{\frac{4}{5}} \Big]\, \|u - u_n\|_5 \end{aligned}$$

where we have set

$$E_{4,n} := \int_{\mathbb{R}^4} |u_n|^{\frac{5}{4}q} (e^{\alpha u_n^2} - 1)^{\frac{5}{4}}\, dx \qquad E_{5,n} := \int_{\mathbb{R}^4} |u|^{\frac{5}{4}q} (e^{\alpha u^2} - 1)^{\frac{5}{4}}\, dx \quad \forall n \in \mathbb{N}.$$

Now, it suffices to prove that $E_{4,n}$ and $E_{5,n}$ are bounded by a constant independent of n to conclude that (6.8) holds with $p = 5$. As $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X there exists a constant $M > 0$ such that $\|u_n\|_{H^2} \leq M$ $\forall n \in \mathbb{N}$ and $\|u\|_{H^2} \leq M$. Thus, choosing $\alpha < \frac{64\pi^2}{5M^2}$ and $q \geq \frac{4}{5}$, we can apply Lemma 3.10 obtaining the desired estimate for $E_{4,n}$ and $E_{5,n}$. This completes the proof of Lemma 6.10. $\square$

In conclusion $I|_{X_G}$ satisfies the assumptions of Theorem 6.7 and possesses a sequence $\{c_k\}_{k\in\mathbb{N}}$ of critical values such that $c_k \to +\infty$ as $k \to +\infty$. The associated sequence of critical points $\{u_k\}_{k\in\mathbb{N}}$ lies in X_G and is unbounded. Infact, reasoning as in (6.5), we get

$$c_k = I(u_k) \leq \left(\frac{1}{2} + \frac{C_2}{V_0} \right) \|u_k\|^2$$

from which it follows that $\|u_k\| \to +\infty$ as $k \to +\infty$.

6.2. Exploiting symmetries

We have to construct a group acting on X which satisfies the assumptions of Proposition 6.6, namely a subgroup $G \subseteq \mathcal{O}(4)$ acting on X and satisfying (G_1), (G_2) and (G_3). As already mentioned, at this aim we will follow an idea of T. Bartsch and M. Willem ([13], see also [12]).

Let H be the subgroup of $\mathcal{O}(4)$ defined as

$$H := \mathcal{O}(2) \times \mathcal{O}(2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ : \ a,\, b \in \mathcal{O}(2) \right\} \subset \mathcal{O}(4)$$

and consider

$$\tau := \begin{pmatrix} 0 & i_2 \\ i_2 & 0 \end{pmatrix} \in \mathcal{O}(4)$$

where i_2 denotes the identity matrix in $\mathbb{R}^2$. We can notice that $\tau^{-1} = \tau$ and τ is in the normalizer of H in $\mathcal{O}(4)$, namely $\tau H = H\tau$. We define $G :=< H \cup \{\tau\} >$, an element $g \in G$ can be written uniquely in the form

$$g = h \quad \text{or} \quad g = h\tau$$

with $h \in H$. We consider the action of G on X defined by

$$\begin{aligned} h * u(x) &:= u(h^{-1}x) && \text{for a.e. } x \in \mathbb{R}^4,\, \forall h \in H\ , \\ h\tau * u(x) &:= -u(\tau h^{-1}x) && \text{for a.e. } x \in \mathbb{R}^4,\, \forall h \in H \end{aligned}$$

for any $u \in X$. It is easy to see that this indeed defines an action of G on X, namely $i_4 * g = g$ and $(g_1 g_2) * u = g_1 * (g_2 * u)$ for $g_1,\, g_2 \in G$, $u \in X$, and that this action is continuous.

Remark 6.11. *A special case of the action of G over X is the following:*

$$\tau * u(x) = -u(\tau x) \qquad \textit{for a.e. } x \in \mathbb{R}^4\ .$$

So in particular if $u \in X_G$ and $x \in \mathbb{R}^4$ with $\tau x = hx$ for some $h \in H$ then

$$-u(x) = u(\tau x) = u(\tau h^{-1} \tau x) = u(h^{-1}x) = u(x)$$

and $u(x) = 0$. Therefore any $u \in X_G$ must necessarily be zero on the set

$$\{x \in \mathbb{R}^4 \mid \tau x = hx \ \textit{ for some } h \in H\}\ .$$

Since I is even according to (f_3), I is G-invariant. In fact if we assume that (V_2) holds then the potential V is spherically symmetric and in particular V is G-invariant

$$V(gx) = V(|gx|) = V(|x|) = V(x) \qquad \forall g \in G \subset \mathcal{O}(4),\, \forall x \in \mathbb{R}^4\ .$$

Also under the assumptions (V_2') and (V_2'') on the potential V we have that

$$V(hx) = V(|ax_1|,\, |bx_2|) = V(|x_1|,\, |x_2|) = V(x)\ ,$$

$$V(h\tau x) = V(|ax_2|,\, |bx_1|) = V(|x_2|,\, |x_1|) = V(|x_1|,\, |x_2|) = V(x)\ ,$$

$$\forall h = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in H,\ \forall x_1,\, x_2 \in \mathbb{R}^2$$

and V is G-invariant. Therefore condition (G_1) is satified. The compactness condition (G_2) is a consequence of a result due to E. Hebey and M. Vaugon (see [39], Corollary 4) which generalize a well known result of P. L. Lions (see [48], Théorème III.1). For the convenience of the reader we report here below the part of this more general result that we will use. If $x \in \mathbb{R}^4$ then we write $x = (x_1, x_2)$ where $x_1, x_2 \in \mathbb{R}^2$ with respect to the splitting $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Let $W^{1,4}_H(\mathbb{R}^4)$ the subspace of $W^{1,4}(\mathbb{R}^4)$ consisting of all $u \in W^{1,4}(\mathbb{R}^4)$ radially symmetric with respect to $x_i \in \mathbb{R}^2$ for $i \in \{1, 2\}$

$$W^{1,4}_H(\mathbb{R}^4) := \{u \in W^{1,4}(\mathbb{R}^4) \mid h * u = u \quad \forall h \in H\} .$$

$W^{1,4}_H(\mathbb{R}^4)$ is nothing but the space of fixed points in $W^{1,4}(\mathbb{R}^4)$ with respect to the action of H.

Theorem 6.12 ([39], Corollary 4)**.** *For any $p \in (4, +\infty)$ the embedding*

$$W^{1,4}_H(\mathbb{R}^4) \hookrightarrow L^p(\mathbb{R}^4)$$

is compact, i.e. $W^{1,4}_H(\mathbb{R}^4) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4)$.

Now, as a consequence of Theorem 6.12, the hypothesis (G_2) of Proposition 6.6 easily follows

$$X_G \hookrightarrow W^{1,4}_H(\mathbb{R}^4) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4) \qquad \forall p \in (4, +\infty) .$$

Obviously we have also that G satisfies hypothesis (G_3). Therefore, from Proposition 6.6, we obtain the existence of an unbounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset X_G$ of critical points for the functional I. These critical points are not radial, infact by construction

$$u_k(x) = -u_k(\tau x) \qquad \text{for a.e. } x \in \mathbb{R}^4, \forall k \in \mathbb{N}$$

and, furthermore, are sign-changing (see Remark 6.11 above). This ends the proof of Theorem 6.4 and of the first part of Theorem 6.3.

We can notice that it is easy to adapt the previous arguments to obtain a proof of Theorem 6.5. In fact we can apply again Proposition 6.6 with the action of H defined by

$$h * u(x) := u(h^{-1}x) \qquad \text{for a.e. } x \in \mathbb{R}^4, \forall h \in H .$$

Since the potential V is spherically symmetric with respect to $x_1, x_2 \in \mathbb{R}^2$ according to (V_2'), I is H-invariant and condition (G_1) is satisfied. Furthermore, from Theorem 6.12 we have

$$X_H \hookrightarrow W^{1,4}_H(\mathbb{R}^4) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4) \qquad \forall p \in (4, +\infty)$$

and also condition (G_2) is satisfied.

To conclude the proof of Theorem 6.3, it remains to prove the existence of an unbounded sequence of critical points of I which are radial. At this aim it suffices to notice that the orthogonal group $\mathcal{O}(4)$ satisfies the assumptions of Proposition 6.6 with respect to the action defined by

$$g * u(x) := u(h^{-1}x) \qquad \text{for a.e. } x \in \mathbb{R}^4, \forall g \in \mathcal{O}(4) .$$

Infact, since the potential V is spherically symmetric according to (V_2), condition (G_1) is satisfied and we have the following result of P. L. Lions [48] which states that $\mathcal{O}(4)$ satisfies (G_2). Let $H^2_{\mathrm{rad}}(\mathbb{R}^4)$ be the subspace of $H^2(\mathbb{R}^4)$ consisting of all $u \in H^2(\mathbb{R}^4)$ which are radially symmetric. $H^2_{\mathrm{rad}}(\mathbb{R}^4)$ is nothing but the space of fixed points in $H^2(\mathbb{R}^4)$ with respect to the action of $\mathcal{O}(4)$.

Theorem 6.13 ([48], Théorème II.1)**.** *For any $p \in (2, +\infty)$ the embedding*

$$H^2_{rad}(\mathbb{R}^4) \hookrightarrow L^p(\mathbb{R}^4)$$

is compact, i.e. $H^2_{rad}(\mathbb{R}^4) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4)$.

Therefore, applying again Proposition 6.6 with $G = \mathcal{O}(4)$ we obtain an unbounded sequence of critical points for the functional I lying on $H^2_{\mathrm{rad}}(\mathbb{R}^4)$.

6.3. Final remarks

Nothing needs to be modified in the previous arguments, in order to obtain similar results for equations of the form

$$\begin{cases} \Delta^2 u - \mathrm{div}\ (U(x)\nabla u) + V(x)u = f(u) & \text{in } \mathbb{R}^4 \\ u \in H^2(\mathbb{R}^4) \end{cases} \tag{6.9}$$

where $U,\, V : \mathbb{R}^4 \to \mathbb{R}$ are continuous functions bounded away from zero and which satisfy some suitable symmetry conditions. In fact, assume (V_1) and

(U_1) $U \in \mathcal{C}(\mathbb{R}^4, \mathbb{R})$ is such that $U(x) \geq U_0 > 0 \quad \forall x \in \mathbb{R}^4$.

Then the functional space for a variational treatment of problem (6.9) is the subspace $\tilde{X}$ of $H^2(\mathbb{R}^4)$ defined as

$$\tilde{X} := \left\{ u \in H^2(\mathbb{R}^4) \;\Big|\; \int_{\mathbb{R}^4} \left[(\Delta u)^2 + U(x)|\nabla u|^2 + V(x)u^2\right]\, dx < +\infty \right\} \subseteq X\ ,$$

and, by (V_1) and (U_1), it follows that $\tilde{X}$ is a Hilber space endowed with the scalar product

$$\langle u,\, v\rangle_{\tilde{X}} := \int_{\mathbb{R}^4} \Delta u \Delta v\, dx + \int_{\mathbb{R}^4} U(x)\nabla u \cdot \nabla v\, dx + \int_{\mathbb{R}^4} V(x)uv\, dx \quad u,\, v \in \tilde{X}$$

to which corresponds the norm $\|u\|_{\tilde{X}} := \sqrt{\langle u,\, u\rangle_{\tilde{X}}}$. Moreover the embedding $\tilde{X} \hookrightarrow H^2(\mathbb{R}^4)$ is continuous. The associated energy functional is

$$I(u) := \frac{1}{2}\|u\|^2_{\tilde{X}} - \int_{\mathbb{R}^4} F(u)\, dx \quad \forall u \in \tilde{X}$$

and the following result holds

Theorem 6.14. *Assume (U_1), (V_1), (f_1), (f_2), (f_3), (F_1) and (F_2).*

(*i*) *Assume* (V_2) *and*

(U_2) *U is spherically symmetric with respect to* $x \in \mathbb{R}^4$, $U(x) = U(|x|)$ *for any* $x \in \mathbb{R}^4$.

Then there exists an unbounded sequence $\{\pm u_k\}_{k\in\mathbb{N}}$ *of sign-changing solutions of (6.9) which are not radial. There also exists an unbounded sequence* $\{\pm u_k\}_{k\in\mathbb{N}}$ *of radial solutions of (6.9).*

(*ii*) *Assume* (V_2'), (V_2'') *and*

(U_2') *U is spherically symmetric with respect to* $x_1,\, x_2 \in \mathbb{R}^2$, $U(x) = U(|x_1|,\, |x_2|)$ *for any* $x = (x_1,\, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

(U_2'') $U(|x_1|,\, |x_2|) = U(|x_2|,\, |x_1|)$ *fro any* $x_1,\, x_2 \in \mathbb{R}^2$.

Then there exists an unbounded sequence $\{\pm u_k\}_{k\in\mathbb{N}}$ *of sign-changing solutions of (6.9) which are not radial.*

(*iii*) *Assume* (V_2') *and* (U_2'). *Then (6.9) possesses an unbounded sequence* $\{\pm u_k\}_{k\in\mathbb{N}}$ *of solutions.*

We end this Section with the Example 6.2 mentioned in the introduction of this Chapter.

Proposition 6.15. *Let* $\gamma > 0$. *The function* $f(s) := s(e^{\gamma|s|} - 1)$ $\forall s \in \mathbb{R}$ *satisfies the assumptions* (f_1), (f_2), (f_3), (F_1) *and* (F_2).

Proof. It is easy to see that (f_1), (f_2) and (f_3) are satisfied. Since f is odd, we have that F is even and

$$F(s) = \frac{1}{\gamma} s e^{\gamma s} - \frac{1}{\gamma^2}(e^{\gamma s} - 1) - \frac{s^2}{2} \quad \forall s > 0\ .$$

Exploiting the symmetry of F, to prove that (F_1) and (F_2) hold it suffices to show that

$$g(s) := sf(s) - \mu F(s) \geq 0 \quad \forall s > 0 \quad \text{for some } \mu > 2\ , \tag{6.10}$$

and

$$\inf_{s > \overline{s}} F(s) > 0 \quad \text{for some } \overline{s} > 0\ . \tag{6.11}$$

But (6.11) is a direct consequence of

$$\lim_{s\to+\infty} F(s) = +\infty\ ,$$

and hence it remains only to prove (6.10). We can notice that $g(0) = 0$ and for any $s > 0$

$$g'(s) = \gamma s^2 e^{\gamma s} - (\mu - 2)s(e^{\gamma s} - 1) \geq \gamma s^2 e^{\gamma s} - \gamma(\mu - 2)s^2 e^{\gamma s} = (3 - \mu)\gamma s^2 e^{\gamma s} \geq 0$$

provided that $\mu \leq 3$. Therefore, choosing $2 < \mu \leq 3$ we have that $g(s) \geq 0$ for any $s > 0$ and (6.10) holds. $\square$

CHAPTER 7

A biharmonic equation in $\mathbb{R}^4$: the critical case

In this Chapter we give sufficient conditions for the existence of solutions of a biharmonic equation of the form

$$\begin{cases} \Delta^2 u + V(x)u = f(u) & \text{in } \mathbb{R}^4 \\ u \in H^2(\mathbb{R}^4) \end{cases} \tag{7.1}$$

where V is a continuos positive potential bounded away from zero and the nonlinearity $f(s)$ behaves like $e^{\alpha_0 s^2}$ at infinity for some $\alpha_0 > 0$.

In order to overcome the lack of compactness due to the unboundedness of the domain $\mathbb{R}^4$, we require some additional assumptions on V. In the case when the potential V is large at infinity we obtain the existence of a nontrivial solution, while requiring the potential V to be spherically symmetric we obtain the existence of a nontrivial *radial* solution. In both cases, the main difficulty is the loss of compactness due to the critical exponential growth of the nonlinear term f.

For easy reference, we introduce now the conditions on the non linear term f appearing in (7.1) which will be assumed in all theorems of this Chapter:

(f_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous and has critical growth with $\alpha_0 > 0$, i.e.

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0 \,, \\ +\infty & \text{if } \alpha < \alpha_0 \,; \end{cases}$$

(f_1) there exists $\mu > 2$ such that $0 < \mu F(s) = \mu \int_0^s f(t)\, dt \le s f(s)$ for any $s \in \mathbb{R} \setminus \{0\}$;

(f_2) there exist s_0, $M_0 > 0$ such that $0 < F(s) \le M_0 |f(s)|$ for any $|s| \ge s_0$;

(f_3) $\displaystyle \lim_{s\to+\infty} \frac{s f(s)}{e^{\alpha_0 s^2}} \ge \beta_0 > 0.$

We will treat two different problems distinguished by the behaviour of the potential V. The first result is concerned with the case when the potential $V : \mathbb{R}^4 \to \mathbb{R}$ is bounded from below by a positive constant and unbounded at infinity, namely V satisfies the following conditions:

(V_0) $V : \mathbb{R}^4 \to \mathbb{R}$ is continuous and $V(x) \geq V_0 > 0$ for any $x \in \mathbb{R}^4$;

(V_1) either $\lim\limits_{|x|\to+\infty} V(x) = +\infty$ or $\frac{1}{V} \in L^1(\mathbb{R}^4)$.

The natural space for a variational treatment of the biharmonic problem (7.1) is the subspace E of $H^2(\mathbb{R}^4)$ defined as

$$E := \left\{ u \in H^2(\mathbb{R}^4) \mid \int_{\mathbb{R}^4} V(x) u^2 \, dx < +\infty \right\} . \tag{7.2}$$

From (V_0) it follows that E is a Hilbert space endowed with the inner product

$$\langle u, v \rangle := \int_{\mathbb{R}^4} \Delta u \Delta v \, dx + \int_{\mathbb{R}^4} V(x) uv \, dx \qquad \forall u, v \in E$$

to which corresponds the norm $\|u\| := \sqrt{\langle u, u\rangle}$.

Condition (V_0) implies that the embedding $E \hookrightarrow H^2(\mathbb{R}^4)$ is continuos and thus the embedding $E \hookrightarrow L^p(\mathbb{R}^4)$ is also continuous for any $p \in [2, +\infty)$. Moreover, exploiting condition (V_1) and using standard arguments (see [26]), it is easy to prove that the embeddings

$$E \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4) \qquad \forall p \in [2, +\infty)$$

are compact (see Proposition 7.7).

Thus, in the case when (V_0) and (V_1) hold, the lack of compactness due to the unboundedness of the domain $\mathbb{R}^4$ is recovered. In this case we obtain the following existence result.

Theorem 7.1. *Assume that the potential V satisfies (V_0), (V_1). Assume that the nonlinearity f satisfies (f_0) – (f_3) and*

(f_4) *f is odd, namely $f(-s) = -f(s)$ for any $s \in \mathbb{R}$, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, namely $f(s_1) \leq f(s_2)$ for any $0 < s_1 \leq s_2$.*

Then problem (7.1) has a nontrivial solution.

Example 7.2. *Let $\alpha_0 > 0$. The functions $f(s) = s(e^{\alpha_0 s^2} - 1)$ and $f(s) = sign(s)(e^{\alpha s^2} - 1)$ satisfy conditions (f_0) – (f_4), for a proof see Proposition 7.17 and Proposition 7.18.*

If we require only the potential V to be bounded from below by a positive constant, namely (V_0) holds, then we still have the continuous embedding $E \hookrightarrow L^p(\mathbb{R}^4)$ for any $p \in [2, +\infty)$ but these embeddings are not compact. However in the case when the potential V is spherically symmetric, namely

(V_2) $V(x) = V(|x|)$ for any $x \in \mathbb{R}^4$,

then the lack of compactness due to the unboundedness of the domain $\mathbb{R}^4$ can be overcome by exploiting the spherical symmetry of the problem, obtaining the existence of a nontrivial radial solution.

Theorem 7.3. *Assume that the potential V satisfies (V_0), (V_2) and that the nonlinearity f satisfies (f_0) – (f_3). Then problem (7.1) has a nontrivial radial solution.*

We point out that if the potential V is constant, i.e.

$$V(x) = V_0 > 0 \quad \forall x \in \mathbb{R}^4 ,$$

then (V_0) and (V_2) hold. Hence as a particular case we have the following

Corollary 7.4. *Let $V_0 > 0$ and assume that f satisfies $(f_0) - (f_3)$. Then the biharmonic problem*

$$\begin{cases} \Delta^2 u + V_0 u = f(u) & \text{in } \mathbb{R}^4 \\ u \in H^2(\mathbb{R}^4) \end{cases}$$

has a nontrivial radial solution.

This Chapter is organized as follows. In Section 7.1 we study the geometric properties of the functional associated to a variational approach to problem (7.1) and in particular we prove that this functional has a mountain pass structure. This leads to consider the minimax level given by the *mountain pass theorem* of Ambrosetti and Rabinowitz [11]. In order to overcome the difficulties caused by the lack of compactness due to the critical growth of the nonlinearity f, in Section 7.2 we introduce tests functions connected with the sharp Adams' inequality (see Theorem 2.1 or inequality (3.1) for easy reference) that will enable us to obtain an upper estimate for the mountain pass level. In this estimate assumption (f_3) will be crucial; we recall that an analogue of (f_3) for bounded domains was introduced in [28] to obtain an existence result for elliptic equations with nonlinearities in the critical growth range in bounded domains of $\mathbb{R}^2$. Finally in Section 7.3 we prove Theorem 7.1 and in Section 7.4 we prove Theorem 7.3.

We end this Chapter showing, in Section 7.5, how to adapt these arguments in order to obtain similar results for the following problem

$$\begin{cases} \Delta^2 u - \operatorname{div}\,(U(x)\nabla u) + V(x)u = f(u) & \text{in } \mathbb{R}^4 \\ u \in H^2(\mathbb{R}^4) \end{cases}$$

under suitable assumptions on U, $V : \mathbb{R}^4 \to \mathbb{R}$.

7.1. Variational approach

Let $(E, \langle \cdot, \cdot \rangle)$ be the Hilbert space introduced in (7.2) and assume (V_0). The natural functional corresponding to a variational approach of problem (7.1) is

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^4} F(u)\,dx \qquad \forall u \in E .$$

If the nonlinear term f satisfies (f_0) and (f_1), then $f(s) = o(s)$ as $s \to 0$, and for fixed $\alpha > \alpha_0$, $q \geq 1$ and for any $\varepsilon > 0$ we have

$$|f(s)| \leq \varepsilon |s| + C(\alpha, q, \varepsilon)|s|^{q-1}(e^{\alpha s^2} - 1) \qquad \forall s \in \mathbb{R} \tag{7.3}$$

where $C(\alpha, q, \varepsilon) > 0$. Consequently, from (f_1) we obtain easily that for fixed $\alpha > \alpha_0$ and for any $\varepsilon > 0$

$$|F(s)| \leq \varepsilon |s|^2 + C(\alpha, q, \varepsilon)|s|^q(e^{\alpha s^2} - 1) \qquad \forall s \in \mathbb{R} \ . \tag{7.4}$$

From (7.4) and Remark 3.8 it follows that $F(u) \in L^1(\mathbb{R}^4)$ for any $u \in E$ and thus $I: E \to \mathbb{R}$ is well defined. Furthermore, using (7.3) and standard arguments (see [15], Theorem A.VI), it is easy to prove that $I \in \mathcal{C}^1(E, \mathbb{R})$,

$$I'(u)v = \langle u, v\rangle - \int_{\mathbb{R}^4} f(u)v\,dx \qquad \forall u, v \in E\,.$$

The next lemma concerns the behaviour of I near $u = 0$.

Lemma 7.5. *Assume that the potential V satisfies (V_0) and that the nonlinearity f satisfies (f_0) and (f_1). Then there exist ϱ, $a > 0$ such that*

$$I(u) \geq a > 0$$

for any $u \in E$ with $\|u\| = \varrho$.

Proof. Fix $\alpha > \alpha_0$ and $q \geq 3$. Using (7.4), we have for any $\varepsilon > 0$

$$\int_{\mathbb{R}^4} F(u)\,dx \leq \varepsilon\|u\|_2^2 + C(\alpha, q, \varepsilon)\int_{\mathbb{R}^4} |u|^q(e^{\alpha u^2} - 1)\,dx \quad \forall u \in E\,.$$

Recalling that the embedding $E \hookrightarrow H^2(\mathbb{R}^4)$ is continuous, namely there exists a constant $\overline{C} > 0$ such that

$$\|u\|_{H^2} \leq \overline{C}\|u\| \quad \forall u \in E\,,$$

we have that if $\|u\| \leq \frac{1}{\tilde{C}\sqrt{\alpha}}$ then $\|u\|_{H^2} \leq \frac{1}{\sqrt{\alpha}}$ and applying Lemma 3.9

$$\int_{\mathbb{R}^4} F(u)\,dx \leq \varepsilon\|u\|_2^2 + \overline{C}(\alpha, q, \varepsilon)\|u\|^q \quad \forall u \in E,\, \|u\| \leq \frac{1}{\tilde{C}\sqrt{\alpha}}\,.$$

Therefore, for arbitrarily fixed $\varepsilon > 0$, we have for any $u \in E$ with $\|u\| \leq \frac{1}{\tilde{C}\sqrt{\alpha}}$

$$I(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon\|u\|_2^2 - \overline{C}(\alpha, q, \varepsilon)\|u\|^q \geq \left(\frac{1}{2} - \frac{\varepsilon}{V_0}\right)\|u\|^2 - \overline{C}(\alpha, q, \varepsilon)\|u\|^q\,.$$

Let

$$g(s) := \left(\frac{1}{2} - \frac{\varepsilon}{V_0}\right)s^2 - \overline{C}(\alpha, q, \varepsilon)s^q\,,$$

to complete the proof it suffices to choose $\varepsilon > 0$ so small that g achieves its maximum in $0 < \varrho \leq \frac{1}{\overline{C}\sqrt{\alpha}}$ and set $a := g(\varrho)$. $\square$

The next lemma concerns with the behaviour of I at infinity.

Lemma 7.6. *Assume that the potential V satisfies (V_0) and that the nonlinearity f satisfies (f_1) and (f_2). Then for any $u \in E$*

$$I(tu) \to -\infty$$

as $|t| \to +\infty$.

Proof. As a consequence of (f_1) and (f_2), there exist C_1, $C_2 > 0$ such that

$$F(s) \geq C_1|s|^\mu - C_2|s|^2 \quad \forall s \in \mathbb{R} \ .$$

Therefore for any $u \in E$ we have

$$I(tu) \leq \frac{1}{2}t^2\|u\|^2 + C_2t^2\|u\|_2^2 - C_1|t|^\mu\|u\|_\mu^\mu \leq \left(\frac{1}{2} + \frac{C_2}{V_0}\right)t^2\|u\|^2 - C_1|t|^\mu\|u\|_\mu^\mu \quad \forall t \in \mathbb{R},$$

and letting $|t| \to +\infty$, we obtain that $I(tu) \to -\infty$. □

Under the assumptions (V_0) and (V_1) on the potential V, we have the following compact embeddings of the functional space E.

Proposition 7.7. *Assume* (V_0)*.*

(i) If

$$\lim_{|s|\to+\infty} V(x) = +\infty \tag{7.5}$$

then the embedding $E \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4)$ *is compact for any* $p \in [2, +\infty)$*.*

(ii) If $\frac{1}{V} \in L^1(\mathbb{R}^4)$ *then the embedding* $E \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4)$ *is compact for any* $p \in [1, +\infty)$*.*

Proof of (i). Recalling that the embedding $E \hookrightarrow H^2(\mathbb{R}^4) \hookrightarrow L^p(\mathbb{R}^4)$ is continuos for any $p \in [2, +\infty)$, it suffices to prove that $u_n \to 0$ in $L^2(\mathbb{R}^4)$ whenever $u_n \rightharpoonup 0$ in E.

Let $\{u_n\}_n \subset E$ be such that $u_n \rightharpoonup 0$ in E, then $\{u_n\}_n$ is bounded in E and there exists a constant $C > 0$ such that

$$\|u_n\|^2 \leq C \qquad \forall n \geq 1 \ .$$

Arbitrarily fixed $\varepsilon > 0$, we have to show the existence of $n_\varepsilon \geq 1$ such that

$$\|u_n\|_2^2 \leq \varepsilon \qquad \forall n \geq 1 \ .$$

To this aim, we write

$$\|u_n\|_2^2 = \int_{B_R} u_n^2 \, dx + \int_{\mathbb{R}^4 \setminus B_R} u_n^2 \, dx \quad \forall n \geq 1$$

with $R > 0$ to be choosen during the proof.

For any $R > 0$, we have that $u_n \to 0$ in $L^2(B_R)$ and hence, to conclude, it suffices to find an $R > 0$ such that

$$\int_{\mathbb{R}^4 \setminus B_R} u_n^2 \, dx \leq \frac{\varepsilon}{2} \qquad \forall n \geq n_{\varepsilon,R}$$

for some $n_{\varepsilon,R} \geq 1$. From (7.5), it follows that there exists $R > 0$ such that

$$V(x) \geq \frac{2C}{\varepsilon} \qquad \forall x \in \mathbb{R}^4 \setminus B_R \ ,$$

consequently

$$\int_{\mathbb{R}^4 \setminus B_R} u_n^2 \, dx \leq \frac{\varepsilon}{2C} \int_{\mathbb{R}^4 \setminus B_R} V(x) u_n^2 \, dx \leq \frac{\varepsilon}{2} \qquad \forall n \geq 1 \ .$$

□

Proof of (ii). Let $\{u_n\}_n \subset E$ be such that $u_n \rightharpoonup 0$ in E and let $C > 0$ be such that

$$\|u_n\| \leq C \qquad \forall n \geq 1 \ .$$

Arguing as in the proof of (i), arbitrarily fixed $\varepsilon > 0$, it suffices to prove that there exists $R > 0$ such that

$$\int_{\mathbb{R}^4 \setminus B_R} |u_n| \, dx \leq \frac{\varepsilon}{2} \qquad \forall n \geq n_{\varepsilon, R}$$

for some $n_{\varepsilon, R} \geq 1$.

Since $\frac{1}{V} \in L^1(\mathbb{R}^4)$, we have the existence of $R > 0$ such that

$$\int_{\mathbb{R}^4 \setminus B_R} \frac{1}{V(x)} \, dx \leq \left(\frac{\varepsilon}{2C}\right)^2 \ ,$$

hence, for any $n \geq 1$

$$\begin{aligned} \int_{\mathbb{R}^4 \setminus B_R} |u_n| \, dx &= \int_{\mathbb{R}^4 \setminus B_R} \frac{1}{\sqrt{V(x)}} \sqrt{V(x)} |u_n| \, dx \leq \\ &\leq \left(\int_{\mathbb{R}^4 \setminus B_R} \frac{1}{V(x)} \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^4 \setminus B_R} V(x) u_n^2 \, dx\right)^{\frac{1}{2}} \leq \frac{\varepsilon}{2} \ . \end{aligned}$$

□

7.2. Estimate of the mountain pass level

In this Section we will assume that the potential V satisfies (V_0) and that the nonlinearity f satisfies (f_3). Furthermore, in order that the functional I has a mountain pass geometry, we will also assume that (f_0), (f_1) and (f_2) hold.

Let

$$\Gamma := \{\gamma \in \mathcal{C}([0, 1], E) \mid \gamma(0) = 0, \, \gamma(1) = e\}$$

with $e \in E$ to be chosen, our aim is to obtain a precise upper estimate for the mountain pass level

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)) \ .$$

In order to do this we will show the existence of $u_0 \in E_{\mathrm{rad}}$ such that

$$\max_{t \geq 0} I(t u_0) < \frac{16\pi^2}{\alpha_0} \tag{7.6}$$

and we can notice that this is indeed a maximum in view of Lemma 7.5 and Lemma 7.6. Let $t_0 > 0$ be such that

$$I(t_0 u_0) = \max_{t \geq 0} I(t u_0)$$

and let $t_1 > t_0$ be sufficiently large, so that $I(t_1 u_0) < 0$. Choosing $e := t_1 u_0$, then we have

$$c \leq c_{\mathrm{rad}} < \frac{16\pi^2}{\alpha_0} \ ,$$

where

$$c_{\text{rad}} := \inf_{\gamma \in \Gamma_{\text{rad}}} \max_{t \in [0,\,1]} I(\gamma(t))$$

and

$$\Gamma_{\text{rad}} := \{\gamma \in \mathcal{C}([0,\,1],\, E_{\text{rad}}) \mid \gamma(0) = 0,\, \gamma(1) = e\} \subset \Gamma\ .$$

In fact, defining the path $\gamma_0 \in \Gamma_{\text{rad}}$ as

$$\gamma_0 := t \cdot t_1 u_0 \quad \forall t \in [0,\,1]\ ,$$

we get

$$c_{\text{rad}} \leq \max_{t\in[0,1]} I(\gamma_0(t)) = I(t_0 u_0) < \frac{16\pi^2}{\alpha_0}\ .$$

We point out that, to obtain the existence of $u_0 \in E$ satisfying (7.6), assumption (f_3) will play a crucial role. In particular, (f_3) implies that for any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that

$$sf(s) \geq (\beta_0 - \varepsilon) e^{\alpha_0 s^2} \qquad \forall s \geq s_\varepsilon\ . \tag{7.7}$$

Let $r > 1$ be such that

$$\beta_0 > \frac{64}{\alpha_0 r^4}\ , \tag{7.8}$$

we introduce the sequence $\{\overline{\omega}_n\}_n$, where $\overline{\omega}_n$ is defined for any $n \geq 1$ as

$$\overline{\omega}_n(x) := \begin{cases} \sqrt{\frac{\log n}{32\pi^2}} - \frac{|x|^2}{\sqrt{8\pi^2 \frac{r^4}{n} \log n}} + \frac{1}{\sqrt{8\pi^2 \log n}} & 0 \leq |x| \leq \frac{r}{\sqrt[4]{n}}\ , \\ \frac{1}{\sqrt{2\pi^2 \log n}} \log \frac{r}{|x|} & \frac{r}{\sqrt[4]{n}} < |x| \leq r\ , \\ \eta_n & |x| \geq r\ . \end{cases}$$

Here η_n is a smooth compactly supported function satisfying for some $R > r$ independent of n

$$\eta_n|_{\partial B_r} = \eta_n|_{\partial B_R} = 0\ ,$$
$$\frac{\partial \eta_n}{\partial \nu}\Big|_{\partial B_r} = \frac{1}{\sqrt{2\pi^2 \log n}}\ , \quad \frac{\partial \eta_n}{\partial \nu}\Big|_{\partial B_R} = 0$$

and η_n, $|\nabla \eta_n|$, $\Delta \eta_n$ are all $\mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right)$. For any $n \geq 1$, we have that $\overline{\omega}_n \in E$ and easy computations show that

$$\|\overline{\omega}_n\|_2^2 = \mathcal{O}\left(\frac{1}{\log n}\right)\ , \quad \|\nabla \overline{\omega}_n\|_2^2 = \mathcal{O}\left(\frac{1}{\log n}\right)\ , \quad \|\Delta \overline{\omega}_n\|_2^2 = 1 + \mathcal{O}\left(\frac{1}{\log n}\right)\ .$$

Furthermore $\|\overline{\omega}_n\| \to 1$ as $n \to +\infty$. For any $n \geq 1$ we set

$$\omega_n := \frac{\overline{\omega}_n}{\|\overline{\omega}_n\|}\ ,$$

so that $\omega_n \in E$ and $\|\omega_n\| = 1$.

A version of the next Lemma concerning the case of bounded domains in $\mathbb{R}^2$ can be found in [28] (see also [31] for the whole space $\mathbb{R}^2$) and our proof follows the same type of arguments.

Lemma 7.8. *Assume that the potential* V *satisfies* (V_0) *and that the nonlinearity* f *satisfies* $(f_0)-(f_3)$*. Then there exists* $n \geq 1$ *such that*

$$\max_{t\geq 0} I(t\omega_n) < \frac{16\pi^2}{\alpha_0}$$

and

$$c \leq c_{rad} < \frac{16\pi^2}{\alpha_0}\ .$$

Proof. We argue by contradiction assuming that

$$\max_{t\geq 0} I(t\omega_n) \geq \frac{16\pi^2}{\alpha_0} \qquad \forall n \geq 1\ .$$

For any $n \geq 1$, let $t_n > 0$ be such that

$$I(t_n\omega_n) = \max_{t\geq 0} I(t\omega_n) \geq \frac{16\pi^2}{\alpha_0}\ ,$$

then, since (f_1) holds and $\|\omega_n\| = 1$, we can estimate

$$\frac{16\pi^2}{\alpha_0} \leq I(t_n\omega_n) = \frac{1}{2}t_n^2\|\omega_n\|^2 - \int_{\mathbb{R}^4} F(t_n\omega_n)\,dx \leq \frac{1}{2}t_n^2\ .$$

Hence

$$t_n^2 \geq \frac{32\pi^2}{\alpha_0} \qquad \forall n \geq 1\ . \tag{7.9}$$

At $t = t_n$ we have

$$0 = \frac{d}{dt}I(t\omega_n)\Big|_{t=t_n} = t_n - \int_{\mathbb{R}^4} f(t_n\omega_n)\omega_n\,dx$$

which implies that

$$t_n^2 = \int_{\mathbb{R}^4} f(t_n\omega_n)t_n\omega_n\,dx \qquad \forall n \geq 1\ . \tag{7.10}$$

We claim that $\{t_n\}_n \subset \mathbb{R}$ is bounded. In fact, since

$$\frac{t_n}{\|\overline{\omega}_n\|}\sqrt{\frac{\log n}{32\pi^2}} \to +\infty \qquad \text{as } n \to +\infty$$

and

$$t_n\omega_n \geq \frac{t_n}{\|\overline{\omega}_n\|}\sqrt{\frac{\log n}{32\pi^2}} \qquad \text{in } B_{\frac{r}{\sqrt[4]{n}}}\ ,$$

from (7.7) it follows that for n sufficiently large

$$\begin{aligned} t_n^2 &\geq \int_{B_{\frac{1}{\sqrt[4]{n}}}} f(t_n\omega_n)t_n\omega_n\,dx \geq (\beta_0-\varepsilon)\int_{B_{\frac{1}{\sqrt[4]{n}}}} e^{\alpha_0 t_n^2\omega_n^2}\,dx \geq \\ &\geq (\beta_0-\varepsilon)\int_{B_{\frac{1}{\sqrt[4]{n}}}} e^{\alpha_0\frac{t_n^2}{\|\overline{\omega}_n\|^2}\frac{\log n}{32\pi^2}}\,dx = 2\pi^2(\beta_0-\varepsilon)e^{\alpha_0\frac{t_n^2}{\|\overline{\omega}_n\|^2}\frac{\log n}{32\pi^2}-\log n}\ . \end{aligned} \tag{7.11}$$

Consequently

$$1 \geq 2\pi^2(\beta_0 - \varepsilon) e^{\alpha_0 \frac{t_n^2}{\|\overline{\omega}_n\|^2} \frac{\log n}{32\pi^2} - \log n - \log t_n^2}$$

for $n \geq 1$ sufficiently large and $\{t_n\}_n$ must be bounded.

We claim that

$$\lim_{n\to+\infty} t_n^2 = \frac{32\pi^2}{\alpha_0} \,.$$

Arguing by contradiction, since (7.9) holds, we necessarily have

$$\lim_{n\to+\infty} t_n^2 > \frac{32\pi^2}{\alpha_0}$$

Recalling (7.11), for $n \geq 1$ sufficiently large we have

$$t_n^2 \geq 2\pi^2(\beta_0 - \varepsilon) e^{\log n \left(\alpha_0 \frac{t_n^2}{\|\overline{\omega}_n\|^2} \frac{1}{32\pi^2} - 1 \right)}$$

and letting $n \to +\infty$ we get a contradition with the boundedness of the sequence $\{t_n\}_n$.

In order to estimate (7.10) more precisely, we define the sets

$$A_n := \{x \in B_r \mid t_n\omega_n \geq s_\varepsilon\} \,, \quad C_n := B_r \setminus A_n$$

where $s_\varepsilon > 0$ is given in (7.7). Using (7.10) and (7.7), we can estimate

$$\begin{aligned} t_n^2 &\geq \int_{B_r} f(t_n\omega_n) t_n\omega_n \, dx \geq \\ &\geq (\beta_0 - \varepsilon) \int_{B_r} e^{\alpha_0 t_n^2 \omega_n^2} \, dx + \int_{C_n} f(t_n\omega_n) t_n\omega_n \, dx - (\beta_0 - \varepsilon) \int_{C_n} e^{\alpha_0 t_n^2 \omega_n^2} \, dx \end{aligned}$$

for any $n \geq 1$. Since $\omega_n \to 0$ a.e in B_r, from the definition of C_n we obtain that the characteristic functions

$$\chi_{C_n} \to 1 \quad \text{a.e. in } B_r \,,$$

and the Lebesgue dominated convergence theorem implies that

$$\int_{C_n} f(t_n\omega_n) t_n\omega_n \, dx \to 0 \,,$$

$$\int_{C_n} e^{\alpha_0 t_n^2 \omega_n^2} \, dx \to \frac{\pi^2}{2} r^4$$

as $n \to +\infty$. If we prove that

$$\lim_{n\to+\infty} \int_{B_r} e^{\alpha_0 t_n^2 \omega_n^2} \, dx \geq \pi^2 r^4 \tag{7.12}$$

then

$$\frac{32\pi^2}{\alpha_0} = \lim_{n\to+\infty} t_n^2 \geq (\beta_0 - \varepsilon) \frac{\pi^2}{2} r^4 \,.$$

Since the choice of $\varepsilon > 0$ is arbitrary, we can let $\varepsilon \to 0$ obtaining that

$$\frac{32\pi^2}{\alpha_0} \geq \beta_0 \frac{\pi^2}{2} r^4$$

which is in contradiction with (7.8). Therefore to end the proof it remains only to show that (7.12) holds. From (7.9) it follows that

$$\int_{B_r} e^{\alpha_0 t_n^2 \omega_n^2}\, dx \geq \int_{B_r \setminus B_{\frac{r}{\sqrt[4]{n}}}} e^{32\pi^2 \omega_n^2}\, dx = 2\pi^2 \int_{\frac{r}{\sqrt[4]{n}}}^{r} e^{\frac{16}{\|\overline{\omega}_n\|^2} \frac{1}{\log n} \log^2 \frac{r}{s}} s^3\, ds$$

and, making the change of variable

$$\tau = \frac{1}{\|\overline{\omega}_n\| \log n} \log \frac{r}{s}\ ,$$

we obtain that

$$\int_{B_r} e^{\alpha_0 t_n^2 \omega_n^2}\, dx \geq 2\pi^2 r^4 \|\overline{\omega}_n\| \log n \int_0^{\frac{1}{4\|\overline{\omega}_n\|}} e^{\log n (16\tau^2 - 4\|\overline{\omega}_n\|\tau)}\, d\tau\ .$$

Now, using the following estimates from below

$$16\tau^2 - 4\|\overline{\omega}_n\|\tau \geq \begin{cases} -4\|\overline{\omega}_n\|\tau & 0 \leq \tau \leq \frac{1}{8\|\overline{\omega}_n\|}\ , \\ \left(\frac{2}{\|\overline{\omega}_n\|} - \|\overline{\omega}_n\|\right)\left(4\tau - \frac{1}{\|\overline{\omega}_n\|}\right) + \frac{1}{\|\overline{\omega}_n\|^2} - 1 & \frac{1}{8\|\overline{\omega}_n\|} \leq \tau \leq \frac{1}{4\|\overline{\omega}_n\|}\ , \end{cases}$$

it easy to see that (7.12) holds. □

7.3. Proof of Theorem 7.1

In this Section we assume that the potential V is bounded from below by a positive constant and is large at infinity, namely that V satifies (V_0) and (V_1). Furthermore we assume that the nonlinear term f satifies $(f_0) - (f_4)$.

To prove Theorem 7.1 we will follow the ideas introduced in [28] to treat elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range. In particular we will apply the well known *mountain-pass theorem* of Ambrosetti and Rabinowitz

Theorem 7.9 ([11]). *Let E be a Hilbert space and let $I \in \mathcal{C}^1(E, \mathbb{R})$ be a functional such that $I(0) = 0$. We assume that:*

(I_1) there exist ϱ, $a > 0$ such that $I(u) \geq a > 0$ for any $u \in E$ with $\|u\| = \varrho$,

(I_2) there exists $e \in E$ with $\|e\| > \varrho$ and $I(e) < 0$.

If I satisfies the Palais-Smale condition at the level c, where c is the mountain pass level

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,\,1]} I(\gamma(t))$$

and

$$\Gamma := \{\gamma \in \mathcal{C}([0,\,1],\, E) \mid \gamma(0) = 0,\ \gamma(1) = e\}\ ,$$

then c is a critical value of I.

We recall that I satisfies the Palais-Smale condition at a level $b \in \mathbb{R}$, $(PS)_b$ for short, if any sequence $\{u_n\}_n \subset E$ such that

$$I(u_n) \to b \tag{7.13}$$

and

$$\left| \langle u_n,\, v\rangle - \int_{\mathbb{R}^4} f(u_n) v\, dx \right| \le \varepsilon_n \|v\| \quad \forall v \in E\ , \tag{7.14}$$

where $\varepsilon_n \to 0$ as $n \to +\infty$, admits a strongly convergent subsequence. We will say that a sequence $\{u_n\}_n \subset E$ satisfying (7.13) and (7.14) is a $(PS)_b$*-sequence* or (PS)*-sequence*.

We have already proved in Section 7.1 that $I : E \to \mathbb{R}$, under our assumptions, is a $\mathcal{C}^1$-functional which behaves like a mountain pass, namely I satisfies (I_1) and (I_2) (see Lemma 7.5 and Lemma 7.6). Therefore, in order to apply the *mountain-pass theorem* of Ambrosetti and Rabinowitz, we have to show that the functional I satisfies $(PS)_c$. Since we deal with a critical nonlinearity, the functional I satisfies the Palais-Smale condition only at certain levels, and the main difficulty is to guarantee that the mountain pass level c is inside the Palais-Smale region. We know from Lemma 7.8 that $c < \frac{16\pi^2}{\alpha_0}$ and thus if we prove that I satisfies $(PS)_b$ for any $-\infty < b < \frac{16\pi^2}{\alpha_0}$ then we can conclude that $(PS)_c$ holds. The rest of this Section is therefore devoted to the proof of the following

Proposition 7.10. *Assume that the potential V satisfies (V_0), (V_1). Furthermore assume $(f_0) - (f_2)$ and (f_4). Then I satifies $(PS)_b$ for all*

$$b \in \left(-\infty,\, \frac{16\pi^2}{\alpha_0}\right)\ .$$

We first study some properties of (PS)-sequences that will be useful in the proof of Proposition 7.10.

Lemma 7.11. *Assume (V_0) and (f_1). If $\{u_n\}_n \subset E$ is a (PS)-sequence then for any $n \ge 1$*

$$\|u_n\| \le C\,, \quad \int_{\mathbb{R}^4} f(u_n) u_n\, dx \le C \quad \text{and} \quad \int_{\mathbb{R}^4} F(u_n)\, dx \le C$$

where $C > 0$ is a constant independent of n.

Proof. Let $\{u_n\}_n \subset E$ be a (PS)-sequence. Since (7.13) implies that $\{I(u_n)\}_n \subset \mathbb{R}$ is bounded, there exists a constant $C > 0$ such that

$$\frac{1}{2}\|u\|^2 \le C + \int_{\mathbb{R}^4} F(u_n)\, dx \qquad \forall n \ge 1\ .$$

From (f_1) it follows that

$$\int_{\mathbb{R}^4} F(u_n)\, dx \le \frac{1}{\mu} \int_{\mathbb{R}^4} f(u_n) u_n\, dx\ , \tag{7.15}$$

and, using (7.14) with $v = u_n$, we obtain

$$\int_{\mathbb{R}^4} f(u_n) u_n\, dx \le \varepsilon_n \|u_n\| + \|u_n\|^2 \quad \forall n \ge 1\ . \tag{7.16}$$

Therefore

$$\frac{1}{2}\|u_n\|^2 \le C + \frac{\varepsilon_n}{\mu}\|u_n\| + \frac{1}{\mu}\|u_n\|^2 \quad \forall n \ge 1$$

and, since $\mu > 2$,

$$0 \le \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 \le C + \frac{\varepsilon_n}{\mu}\|u_n\| \quad \forall n \ge 1$$

from which we deduce that $\{u_n\}_n$ must be bounded in E.

Using the boundedness of $\{u_n\}_n$ in E together with inequalities (7.15) and (7.16) we obtain the desired inequalities. □

Assuming also (V_1), we can notice that from Lemma 7.11 it follows that, given a (PS)-sequence $\{u_n\}_n \subset E$, we can always consider a subsequence denoted again by $\{u_n\}_n$ such that

$$\begin{array}{lcll} u_n \rightharpoonup u & \text{in} & E & \\ u_n \to u & \text{in} & L^p(\mathbb{R}^4) & \forall p \in [2, +\infty) \\ u_n \to u & \text{a.e. in} & \mathbb{R}^4 & \end{array}$$

where $u \in E$ and the strong convergence in $L^p(\mathbb{R}^4)$ for any $p \in [2, +\infty)$ is a consequence of the fact that under the assumptions (V_0) and (V_1) on the potential V the embedding

$$E \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4)$$

is compact for any $p \in [2, +\infty)$, see Proposition 7.7.

Assumption (f_4) together with (f_1) will enable us to reduce the following convergence result to the radial case.

Lemma 7.12. *Assume that the potential V satisfies (V_0) and (V_1). Furthermore assume $(f_0) - (f_2)$ and (f_4). If $\{u_n\}_n \subset E$ is a (PS)-sequence and $u \in E$ is its weak limit, then*

$$\int_{\mathbb{R}^4} F(u_n)\,dx \to \int_{\mathbb{R}^4} F(u)\,dx$$

up to subsequences.

Proof. Let $\{u_n\}_n \subset E$ be a (PS)-sequence and let $u \in E$ be its weak limit. We consider the sequence $\{u_n^*\}_n$ where $v*$ denotes the spherically symmetric decreasing rearrangement of $v \in E$. From (f_1) and (f_4) it follows that

$$\int_{\mathbb{R}^4} F(u_n)\,dx = \int_{\mathbb{R}^4} F(u_n^*)\,dx \quad \forall n \ge 1\ , \qquad \int_{\mathbb{R}^4} F(u)\,dx = \int_{\mathbb{R}^4} F(u^*)\,dx\ ,$$

and to end the proof it suffices to show that

$$\int_{\mathbb{R}^4} F(u_n^*)\,dx \to \int_{\mathbb{R}^4} F(u^*)\,dx\ . \tag{7.17}$$

Before proceeding with the proof we remark some properties of the sequence $\{u_n^*\}_n$. We have that $u_n \to u$ in $L^2(\mathbb{R}^4)$ and thus $|u_n| \to |u|$ in $L^2(\mathbb{R}^4)$; since the rearrangement is non-expansive on $L^2(\mathbb{R}^4)$ (see [45], Chapter 3, Section 3.4), namely

$$\|u_n^* - u^*\|_2 \leq \||u_n| - |u|\|_2 \qquad \forall n \geq 1 ,$$

we deduce that

$$u_n^* \to u^* \quad \text{ in } L^2(\mathbb{R}^4) .$$

Furthermore, since

$$\|\nabla u_n^*\|_2 \leq \|\nabla u_n\|_2 ,$$

we have that $\{u_n^*\}_n$ is bounded in $H_0^1(\mathbb{R}^4)$.
We divide the proof of (7.17) into several steps.

STEP 1. We claim that

$$f(u_n^*) \to f(u^*) \quad \text{ in } L^1(B_R) \tag{7.18}$$

for any $R > 0$. Fixed $R > 0$, to prove (7.18), the idea is to apply [28], Lemma 2.1. We can notice that, since $u_n^* \to u^*$ in $L^2(\mathbb{R}^4)$, we have that $u_n^* \to u^*$ in $L^1(B_R)$. Furthermore, from (f_4) it follows that

$$\int_{B_R} f(u_n^*)u_n^*\, dx \leq \int_{\mathbb{R}^4} f(u_n^*)u_n^*\, dx = \int_{\mathbb{R}^4} f(u_n)u_n\, dx \quad \forall n \geq 1 , \tag{7.19}$$

and Lemma 7.11 leads to

$$\int_{B_R} f(u_n^*)u_n^*\, dx \leq C \quad \forall n \geq 1 .$$

This last inequality together with the boundedness of the domain $B_R \subset \mathbb{R}^4$ ensures that $f(u_n^*)$ and $f(u^*)$ are in $L^1(B_R)$; hence the assumptions of [28], Lemma 2.1 are satisfied and the claim (7.18) follows.

STEP 2. From (7.18), we deduce that for any $R > 0$

$$\int_{B_R} F(u_n^*)\, dx \to \int_{B_R} F(u^*)\, dx . \tag{7.20}$$

Indeed (f_2) leads to

$$0 < F(u_n^*) \leq M_0|f(u_n^*)| \quad \text{ a.e. in } \{x \in \mathbb{R}^4 \mid u_n^* \geq s_0\}$$

and (f_1) leads to

$$0 \leq F(u_n^*) \leq \frac{s_0}{\mu}|f(u_n^*)| \quad \text{ a.e. in } \{x \in \mathbb{R}^4 \mid u_n^* < s_0\} ,$$

and hence, applying the generalized Lebesgue dominated convergence theorem (see [60], Chapter 4, Theorem 4.17), we can conclude that (7.20) holds for any $R > 0$.

Step 3. For any $\varepsilon > 0$ there exists $R > 1$ such that

$$\left| \int_{\mathbb{R}^4 \setminus B_R} [F(u_n^*) - F(u^*)] \, dx \right| \le \varepsilon \quad \forall n \ge 1 \,. \tag{7.21}$$

Let $R > 1$ arbitrarily fixed. Since (7.4) holds, we have the existence of C_1, $C_2 > 0$ such that

$$\int_{\mathbb{R}^4 \setminus B_R} F(u_n^*) \, dx \le C_1 \int_{\mathbb{R}^4 \setminus B_R} (u_n^*)^2 \, dx + C_2 \int_{\mathbb{R}^4 \setminus B_R} u_n^* (e^{\alpha (u_n^*)^2} - 1) \, dx \quad \forall n \ge 1 \,.$$

Using the power series expansion of the exponential function and estimating the single terms with the *radial lemma* (3.8), we get for any $n \ge 1$

$$\begin{aligned} \int_{\mathbb{R}^4 \setminus B_R} u_n^* (e^{\alpha (u_n^*)^2} - 1) \, dx &= \sum_{j=1}^{+\infty} \frac{\alpha^j}{j!} \int_{\mathbb{R}^4 \setminus B_R} (u_n^*)^{2j+1} \, dx \le \\ &\le 2\pi^2 \sum_{j=1}^{+\infty} \frac{\alpha^j}{j!} \left(\frac{1}{2\pi^2} \right)^{j+\frac{1}{2}} \|u_n\|_{H_1}^{2j+1} \frac{R^{\frac{3}{2}-3j}}{3j - \frac{3}{2}} \le \\ &\le \frac{\sqrt{2\pi^2}}{R} \|u_n\|_{H_1} \sum_{j=1}^{+\infty} \frac{1}{j!} \left(\frac{\alpha \|u_n\|_{H^1}^2}{2\pi^2} \right)^j \le \\ &\le \frac{\sqrt{2\pi^2}}{R} \|u_n\|_{H^1} e^{\frac{\alpha}{2\pi^2} \|u_n\|_{H^1}^2} \,. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^4 \setminus B_R} F(u_n^*) \, dx \le C_1 \int_{\mathbb{R}^4 \setminus B_R} (u_n^*)^2 \, dx + \frac{C_3}{R} \quad \forall n \ge 1 \,,$$

where $C_3 > 0$ is a constant independent of n and R, provided that $R > 1$. Since $u_n^* \to u^*$ in $L^2(\mathbb{R}^4)$, for any $\varepsilon > 0$ there exists $R > 1$ such that

$$C_1 \int_{\mathbb{R}^4 \setminus B_R} (u_n^*)^2 \, dx \le \frac{\varepsilon}{3}$$

and moreover

$$\frac{C_3}{R} \le \frac{\varepsilon}{3} \,, \qquad \int_{\mathbb{R}^4 \setminus B_R} F(u^*) \, dx \le \frac{\varepsilon}{3} \,.$$

In conclusion, for any $\varepsilon > 0$ we have the existence of $R > 1$ such that

$$\left| \int_{\mathbb{R}^4 \setminus B_R} [F(u_n^*) - F(u^*)] \, dx \right| \le \int_{\mathbb{R}^4 \setminus B_R} F(u_n^*) \, dx + \int_{\mathbb{R}^4 \setminus B_R} F(u^*) \, dx \le \varepsilon \quad \forall n \ge 1 \,,$$

which is (7.21).

Step 4. Combining (7.20) and (7.21) we get

$$\lim_{n \to +\infty} \left| \int_{\mathbb{R}^4} [F(u_n^*) - F(u^*)] \, dx \right| \le \varepsilon \quad \forall \varepsilon > 0$$

and letting $\varepsilon \to 0$ we obtain (7.17). $\square$

Arguing as in Step 1 of the proof of Lemma 7.12, it is easy to see that given a (PS)-sequence $\{u_n\}_n \subset E$

$$f(u_n) \to f(u) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^4) .$$

We can now give a proof of Proposition 7.10.

Proof of Proposition 7.10. Fix $-\infty < b < \frac{16\pi^2}{\alpha_0}$. Let $\{u_n\}_n \subset E$ be a $(PS)_b$-sequence and let $u \in E$ be its weak limit, we have to prove that $u_n \to u$ in E. Here and below the convergence has to be understood up to subsequences.

We deduce, from (7.13) and Lemma 7.12, that

$$\lim_{n\to+\infty} \|u_n\|^2 = 2\left(b + \int_{\mathbb{R}^4} F(u)\,dx\right) \tag{7.22}$$

and, from (7.14), that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^4} f(u_n)u_n\,dx = \lim_{n\to+\infty} \|u_n\|^2 . \tag{7.23}$$

Combining (7.13), (7.23) and (f_1), we have that $b \geq 0$.

Recalling that $u_n \rightharpoonup u$ in E and $f(u_n) \to f(u)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^4)$, from (7.14) we also deduce that u is a weak solution of (7.1), namely

$$\langle u,\, v\rangle - \int_{\mathbb{R}^4} f(u)v\,dx = 0 \quad \forall v \in \mathcal{C}_0^\infty(\mathbb{R}^4) ,$$

and this in particular implies that

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^4} f(u)u\,dx - \int_{\mathbb{R}^4} F(u)\,dx \geq 0 .$$

Now we distinguish three cases.

Case 1: $b = 0$. Using (7.22) with $b = 0$

$$0 \leq I(u) \leq \frac{1}{2}\liminf_{n\to+\infty} \|u_n\|^2 - \int_{\mathbb{R}^4} F(u)\,dx = 0 ,$$

consequently $I(u) = 0$. This together with (7.22) implies that $\|u_n\| \to \|u\|$ as $n \to +\infty$ and thus $u_n \to u$ in E.

Case 2: $b \neq 0$ and $u = 0$. We will prove that this case cannot happen. We show below that

$$\mathcal{I}_n := \int_{\mathbb{R}^4} f(u_n)u_n\,dx \to 0 \tag{7.24}$$

as $n \to +\infty$. Using (7.14) with $v = u_n$ and the boundedness of $\{u_n\}_n$ in E, we get

$$\|u_n\|^2 \leq C\varepsilon_n + \int_{\mathbb{R}^4} f(u_n)u_n\,dx = C\varepsilon_n + \mathcal{I}_n \quad \forall n \geq 1 ,$$

from which we deduce that $\|u_n\|^2 \to 0$ as $n \to +\infty$. On the other hand, from (7.22) it follows that $\|u_n\|^2 \to 2b \neq 0$ as $n \to +\infty$ which is a contradiction.

Therefore it remains to prove that (7.24) holds. Let $\alpha > \alpha_0$ and $q \geq 1$, using (7.3) and again the boundedness of $\{u_n\}_n$ in E, for any $\varepsilon > 0$ we have

$$\mathcal{I}_n \leq C_1\varepsilon + C(\alpha,\, q,\, \varepsilon) \int_{\mathbb{R}^4} |u_n|^q (e^{\alpha u_n^2} - 1)\, dx \quad \forall n \geq 1$$

and the proof of (7.24) reduces to show the existence of $\alpha > \alpha_0$ and $q \geq 1$ such that

$$\mathcal{J}_n := \int_{\mathbb{R}^4} |u_n|^q (e^{\alpha u_n^2} - 1)\, dx \to 0$$

as $n \to +\infty$. We estimate $\mathcal{J}_n$ applying Hölder's inequality with $1 < p,\, p' < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ to get

$$\mathcal{J}_n \leq C_\beta \|u_n\|_{qp'}^q \left(\int_{\mathbb{R}^4} (e^{\alpha\beta u_n^2} - 1)\, dx \right)^{\frac{1}{p}} \quad \forall n \geq 1$$

for any $\beta > p$.

We can notice that with a suitable choice of $\alpha > \alpha_0$ and $\beta > p > 1$ we have that

$$\sup_n \int_{\mathbb{R}^4} (e^{\alpha\beta u_n^2} - 1)\, dx < +\infty\ . \tag{7.25}$$

In fact, in view of the Adams-type inequality (3.1) (see also Theorem 2.1), we have that (7.25) holds provided that

$$\alpha\beta \|u_n\|_{H^2}^2 \leq 32\pi^2$$

for any n sufficiently large. Since $u_n \to 0$ in $L^2(\mathbb{R}^4)$ and $\|\Delta u_n\|_2 \leq C$ for any $n \geq 1$, using the interpolation inequality

$$\|\nabla u_n\|_2^2 \leq \tilde{C} \|\Delta u_n\|_2 \|u_n\|_2 \quad \forall n \geq 1\ ,$$

where $\tilde{C} > 0$ is a constant independent of n, we can notice that

$$\lim_{n\to+\infty} \|u_n\|_{H^2}^2 = \lim_n \|\Delta u_n\|_2^2 \leq \lim_{n\to+\infty} \|u_n\|^2\ .$$

Furthermore, recalling that $b < \frac{16\pi^2}{\alpha_0}$, from (7.13) we deduce that

$$\lim_{n\to+\infty} \|u_n\|_{H^2}^2 \leq 2b < \frac{32\pi^2}{\alpha_0}$$

and there exists $\varepsilon > 0$ such that

$$\|u_n\|_{H^2}^2 \leq 2b + \varepsilon < \frac{32\pi^2}{\alpha_0} \quad \forall n \geq n_\varepsilon$$

with $n_\varepsilon > 0$ sufficiently large. Therefore choosing $\alpha > \alpha_0$ sufficiently close to α_0 and $p > 1$ sufficiently close to 1, so that also $\beta > p$ can be choosen sufficiently close to 1, we have that

$$\|u_n\|_{H^2}^2 \leq \frac{32\pi^2}{\alpha\beta} \quad \forall n \geq n_\varepsilon$$

and (7.25) holds.

Finally, since $qp' > q$, choosing $q \geq 2$ we have that $u_n \to 0$ in $L^{qp'}(\mathbb{R}^4)$ and $\mathcal{J}_n \to 0$ as $n \to +\infty$.

Case 3: $b \neq 0$ and $u \neq 0$. We claim that $I(u) = b$. If this is the case, from (7.22) we deduce that

$$\lim_{n\to+\infty} \|u_n\|^2 = 2\left(I(u) + \int_{\mathbb{R}^4} F(u)\, dx\right) = \|u\|^2$$

which implies that $u_n \to u$ in E. Therefore we have to prove that $I(u) = b$.

We argue by contradiction assuming that $I(u) \neq b$. As a consequence of (7.22) and the weak convergence $u_n \rightharpoonup u$ in E, we have

$$I(u) \leq \frac{1}{2} \liminf_{n\to+\infty} \|u_n\|^2 - \int_{\mathbb{R}^4} F(u)\, dx = b\ .$$

Consequently, we have necessarily that $I(u) < b$ and thus

$$\|u\|^2 < 2\left(b + \int_{\mathbb{R}^4} F(u)\, dx\right)\ .$$

We can notice that if we prove that $u_n \to u$ in E then we get a contradiction with (7.22) and we can conclude that $I(u) = b$.

To this aim we let

$$v := \frac{u}{\sqrt{2\left(b + \int_{\mathbb{R}^4} F(u)\, dx\right)}} \qquad \text{and} \qquad v_n := \frac{u_n}{\|u_n\|} \quad \forall n \geq 1\ , \tag{7.26}$$

so that $v_n \rightharpoonup v$ in E, $\|v_n\| = 1$ for any $n \geq 1$ and $\|v\| < 1$. Recalling that the embedding $E \hookrightarrow L^2(\mathbb{R}^4)$ is compact, we can apply Lemma 3.4 obtaining that

$$\sup_n \int_{\mathbb{R}^4} (e^{pv_n^2} - 1)\, dx < +\infty \qquad \forall p \in \left(0,\ \frac{32\pi^2}{1 - \|v\|^2}\right)\ . \tag{7.27}$$

Since $u_n \rightharpoonup u$ in E, in order to prove that $u_n \to u$ in E, it suffices to show that

$$\langle u_n,\, u_n - u\rangle \to 0$$

as $n \to +\infty$, and this is indeed the case if

$$\mathcal{I}_n := \int_{\mathbb{R}^4} f(u_n)(u_n - u)\, dx \to 0 \tag{7.28}$$

as $n \to +\infty$. Arguing as in Case 2, we can reduce the proof of (7.28) to show the existence of $\alpha > \alpha_0$ and $q \geq 1$ such that

$$\mathcal{J}_n := \int_{\mathbb{R}^4} |u_n|^{q-1}|u_n - u|(e^{\alpha u_n^2} - 1)\, dx \to 0$$

as $n \to +\infty$. Applying Hölder's inequality with $1 < p,\, p' < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, we get

$$\mathcal{J}_n \leq C_\beta \|(|u_n|^{q-1}|u_n - u|)\|_{p'} \left(\int_{\mathbb{R}^4} (e^{\alpha\beta u_n^2} - 1)\, dx\right)^{\frac{1}{p}}$$

for any $\beta > p$.

From (2.19), it follows that

$$\sup_n \int_{\mathbb{R}^4} (e^{\alpha\beta u_n^2} - 1)\, dx = \sup_n \int_{\mathbb{R}^4} (e^{\alpha\beta\|u_n\|^2 v_n^2} - 1)\, dx < +\infty$$

provided that $\alpha > \alpha_0$, $p > 1$ and $\beta > p$ can be chosen so that

$$\alpha\beta\|u_n\|^2 < \frac{32\pi^2}{1 - \|v\|^2} = 32\pi^2 \frac{b + \int_{\mathbb{R}^4} F(u)\, dx}{b - I(u)} \tag{7.29}$$

at least for any $n \geq 1$ sufficiently large. From (7.13), we deduce that if

$$2\alpha\beta < \frac{32\pi^2}{b - I(u)} \tag{7.30}$$

then (7.29) holds. Since $I(u) \geq 0$ and $b < \frac{16\pi^2}{\alpha_0}$, there exists $\alpha > \alpha_0$ such that

$$\alpha < \frac{16\pi^2}{b - I(u)}$$

and choosing $p > 1$ sufficiently close to 1, so that also $\beta > p$ can be chosen sufficiently close to 1, we obtain (7.30).

Therefore with this choice of $\alpha > \alpha_0$ and $\beta > p > 1$, we have

$$\mathcal{J}_n \leq \overline{C}_\beta \|(|u_n|^{q-1}|u_n - u|)\|_{p'} \leq \overline{C}_\beta \|u_n\|^{q-1}_{(q-1)p'\sigma'} \|u_n - u\|_{p'\sigma}$$

as a consequence of Hölder's inequality with $1 < \sigma,\, \sigma' < +\infty$ and $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$. Now, since $(q-1)p'\sigma' > q - 1$ and $p'\sigma > \sigma$, choosing $q \geq 3$ and $\sigma \geq 2$ we have that $\{u_n\}_n$ is bounded in $L^{(q-1)p'\sigma'}(\mathbb{R}^4)$ and $u_n \to u$ in $L^{p'\sigma}(\mathbb{R}^4)$, hence $\mathcal{J}_n \to 0$ as $n \to +\infty$. □

7.4. Proof of Theorem 7.3

In this Section we assume that the potential V is spherically symmetric and bounded from below by a positive constant, namely that (V_0) and (V_2) hold. Furthermore we assume that the nonlinearity f satisfies $(f_0) - (f_3)$.

As proved in Section 7.1, we recall that $I \in \mathcal{C}^1(E, \mathbb{R})$. Moreover Lemma 7.5 and Lemma 7.6 hold, consequently I has a mountain pass geometry.

Since (V_2) holds, as a consequence of the *principle of symmetric criticality* of Palais [55], any critical point of the restriction $I|_{E_{\mathrm{rad}}}$ is a critical point of I too and the proof of Theorem 7.3 reduces to show that $I|_{E_{\mathrm{rad}}}$ satisfies the assumptions of the *mountain pass theorem* of Ambrosetti and Rabinowitz, i.e. Theorem 7.9. More precisely we have to prove that $I|_{E_{\mathrm{rad}}}$ satisfies the Palais-Smale condition at the mountain pass level

$$c_{\mathrm{rad}} := \inf_{\gamma \in \Gamma_{\mathrm{rad}}} \max_{t \in [0,\,1]} I(\gamma(t))$$

with

$$\Gamma_{\mathrm{rad}} := \{\gamma \in \mathcal{C}([0,\,1],\, E_{\mathrm{rad}}) \mid \gamma(0) = 0,\, \gamma(1) = e\}\ .$$

Since from Lemma 7.8 we know that $c_{\mathrm{rad}} < \frac{16\pi^2}{\alpha_0}$, to complete the proof of Theorem 7.3 we have to show that $(-\infty,\, \frac{16\pi^2}{\alpha_0})$ is a Palais-Smale region for $I|_{E_{\mathrm{rad}}}$, namely

Proposition 7.13. *Assume that the potential V satisfies (V_0). Furthermore assume (f_0)–(f_2). Then $I|_{E_{rad}}$ satisfies $(PS)_b$ for all*

$$b \in \left(-\infty,\, \frac{16\pi^2}{\alpha_0}\right) .$$

First, we can notice that Lemma 7.11 holds and thus any (PS)-sequence $\{u_n\}_n \subset E_{\mathrm{rad}}$ for $I|_{E_{\mathrm{rad}}}$ satisfies

$$\|u_n\| \le C\,, \quad \int_{\mathbb{R}^4} f(u_n)u_n\,dx \le C \quad \text{and} \quad \int_{\mathbb{R}^4} F(u_n)\,dx \le C \quad \forall n \ge 1 \tag{7.31}$$

where $C > 0$ is a constant independent of n. Consequently, given a (PS)-sequence $\{u_n\}_n \subset E_{\mathrm{rad}}$ for $I|_{E_{\mathrm{rad}}}$, without loss of generality, we may always assume the existence of $u \in E_{\mathrm{rad}}$ such that

$$\begin{array}{lcl} u_n \rightharpoonup u & \text{in} & E\,, \\ u_n \to u & \text{in} & L^p(\mathbb{R}^4) \quad \forall p \in (2,\, +\infty)\,, \\ u_n \to u & \text{a.e. in} & \mathbb{R}^4\,. \end{array}$$

Here the strong convergence in $L^p(\mathbb{R}^4)$ for any $p \in (2,\, +\infty)$ is given by the compact embeddings

$$E_{\mathrm{rad}} \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4) \quad \forall p \in (2,\, +\infty)\,.$$

Lemma 7.14. *Assume that the potential V satisfies (V_0). Furthermore assume (f_0)–(f_2). If $\{u_n\}_n \subset E_{rad}$ is a (PS)-sequence for $I|_{E_{rad}}$ and $u \in E_{rad}$ is its weak limit, then*

$$\int_{\mathbb{R}^4} F(u_n)\,dx \to \int_{\mathbb{R}^4} F(u)\,dx$$

up to subsequences.

Proof. Arguing as in Step 1 and Step 2 of Lemma 7.12, it is easy to see that $f(u_n) \to f(u)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^4)$ and consequently for any $R > 0$

$$\int_{B_R} F(u_n)\,dx \to \int_{B_R} F(u)\,dx\,.$$

Therefore the proof is complete if we show that for any $0 < \varepsilon < 1$ there exists $R > 1$ such that

$$\int_{\mathbb{R}^4\setminus B_R} F(u_n)\,dx \le C\varepsilon \quad \text{and} \quad \int_{\mathbb{R}^4\setminus B_R} F(u)\,dx \le C\varepsilon \quad \forall n \ge 1 \tag{7.32}$$

where $C > 0$ is constant independent of ε and R.

Let $0 < \varepsilon < 1$. Fixed $\alpha > \alpha_0$, for any $R > 1$ we have

$$\int_{\mathbb{R}^4\setminus B_R} F(u_n)\,dx \le \varepsilon\|u_n\|_2^2 + C(\alpha,\,\varepsilon)\int_{\mathbb{R}^4\setminus B_R} |u_n|^2(e^{\alpha u_n^2} - 1)\,dx \quad \forall n \ge 1\,,$$

as a consequence of (7.4). Using the power series expansion of the exponential function and the *radial lemma* (3.8) we get

$$\int_{\mathbb{R}^4\setminus B_R} |u_n|^2(e^{\alpha u_n^2}-1)\,dx \le \frac{\|u_n\|_{H^1}^2}{R}\,e^{\frac{\alpha\|u_n\|_{H^1}^2}{2\pi^2}} \quad \forall n\ge 1\ ,$$

and from (7.31) it follows that for any $R>1$

$$\int_{\mathbb{R}^4\setminus B_R} F(u_n)\,dx \le C_1\varepsilon + \frac{C_2(\alpha,\,\varepsilon)}{R} \quad \forall n\ge 1\ .$$

Without loss of generality we may always assume that $C_2(\alpha,\,\varepsilon)>1$, and choosing

$$R := \frac{C_2(\alpha,\,\varepsilon)}{\varepsilon} > 1$$

we have

$$\int_{\mathbb{R}^4\setminus B_R} F(u_n)\,dx \le (C_1+1)\varepsilon \quad \forall n\ge 1\ .$$

Similarly

$$\int_{\mathbb{R}^4\setminus B_R} F(u)\,dx \le (C_1+1)\varepsilon$$

and we get (7.32). □

As a by-product of this proof we have that given a (PS)-sequence $\{u_n\}_n\subset E_{\mathrm{rad}}$ for $I|_{E_{\mathrm{rad}}}$

$$f(u_n)\to f(u) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^4)\ .$$

Proof of Proposition 7.13. Fixed $-\infty<b<\frac{16\pi^2}{\alpha_0}$, let $\{u_n\}_n\subset E_{\mathrm{rad}}$ be a $(PS)_b$-sequence for $I|_{E_{\mathrm{rad}}}$, namely $I(u_n)\to b$ as $n\to+\infty$ and

$$\left|\langle u_n,\,v\rangle - \int_{\mathbb{R}^4} f(u_n)v\,dx\right| \le \varepsilon_n\|v\| \quad \forall v\in E_{\mathrm{rad}} \tag{7.33}$$

where $\varepsilon_n\to 0$ as $n\to+\infty$.

Arguing as in the proof of Proposition 7.10, it is easy to see that

$$\lim_{nto+\infty}\|u_n\|^2 = \lim_{n\to+\infty}\int_{\mathbb{R}^4} f(u_n)u_n\,dx = 2\left(b+\int_{\mathbb{R}^4}F(u)\,dx\right)\ , \tag{7.34}$$

furthermore $b\ge 0$ and $I(u)\ge 0$.

Since in the case when $b=0$ the same reasoning as in the proof of Proposition 7.10 leads us to conclude that $u_n\to u$ in E, we will focus our attention only in the remaining two cases.

Case 1: $b\ne 0$ and $u=0$. Our aim is to prove that $u_n\to 0$ in E, in this way we get a contradiction with (7.34) and we can conclude that this case cannot happen. Using (7.33) with $v=u_n$ and the boundedness of $\{u_n\}_n$ in E, the proof reduces to show that

$$\mathcal{I}_n := \int_{\mathbb{R}^4} f(u_n)u_n\,dx \to 0 \tag{7.35}$$

as $n \to +\infty$ then $u_n \to 0$ in E.

Let $\alpha > \alpha_0$ and $q \geq 1$. Using (7.3) and again the boundedness of $\{u_n\}_n$ in E, for any $\varepsilon > 0$ we have

$$\mathcal{I}_n \leq C_1\varepsilon + C(\alpha,\, q,\, \varepsilon) \int_{\mathbb{R}^4} |u_n|^q (e^{\alpha u_n^2} - 1)\, dx \quad \forall n \geq 1$$

and if we prove the existence of $\alpha > \alpha_0$ and $q \geq 1$ such that

$$\mathcal{J}_n := \int_{\mathbb{R}^4} |u_n|^q (e^{\alpha u_n^2} - 1)\, dx \to 0$$

as $n \to +\infty$ then the proof is complete. Applying Hölder's inequality with $1 < p,\, p' < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, we get

$$\mathcal{J}_n \leq C_\beta \|u_n\|_{qp'}^q \left(\int_{\mathbb{R}^4} (e^{\alpha\beta u_n^2} - 1)\, dx \right)^{\frac{1}{p}} \quad \forall n \geq 1$$

for any $\beta > p$.

We can notice that with a suitable choice of $\alpha > \alpha_0$ and $\beta > p > 1$ we have that

$$\sup_n \int_{\mathbb{R}^4} (e^{\alpha\beta u_n^2} - 1)\, dx < +\infty\,. \tag{7.36}$$

In fact, since

$$\lim_{n\to+\infty} \|\Delta u_n\|_2^2 \leq \lim_{n\to+\infty} \|u_n\|^2 = 2b < \frac{32\pi^2}{\alpha_0}\,,$$

there exist $\delta,\, \sigma > 0$ such that

$$\|\Delta u_n\|_2^2 \leq 2b + \delta < \frac{32\pi^2}{\alpha_0}(1-\sigma) \quad \forall n \geq \overline{n}$$

with $\overline{n} \geq 1$ sufficiently large. Consequently we can find $\tau > 0$ satisfying

$$\|u_n\|_{H^2,\tau}^2 \leq \frac{32\pi^2}{\alpha_0}(1-\sigma) \quad \forall n \geq \overline{n}$$

and, from (3.2), it follows that (7.36) holds provided that

$$\alpha\beta\frac{32\pi^2}{\alpha_0}(1-\sigma) \leq 32\pi^2\,.$$

Now it suffices to notice that this is indeed the case if we choose $\alpha > \alpha_0$ sufficiently close to α_0 and $p > 1$ sufficiently close to 1, so that we can choose $\beta > p > 1$ sufficiently close to 1 too.

Finally, since $qp' > q$, choosing $q > 2$ we have that $u_n \to 0$ in $L^{qp'}(\mathbb{R}^4)$ and $\mathcal{J}_n \to 0$ as $n \to +\infty$.

Case 2: $b \neq 0$ and $u \neq 0$. Note that $I(u) = b$ implies that $u_n \to u$ in E, therefore our aim is to show that $I(u) = b$. To do this, we argue by contradiction as in the proof of Proposition 7.10, assuming that $I(u) \neq b$ and defining v and the sequence $\{v_n\}_n \subset E_{\mathrm{rad}}$ as in (7.26). In this way $v_n \to v$ in E, $\|v_n\| = 1$ for any $n \geq 1$ and $0 < \|v\| < 1$, and applying Lemma 3.6 we obtain that

$$\sup_n \int_{\mathbb{R}^4} (e^{pv_n^2} - 1)\, dx < +\infty \quad \forall p \in \left(0,\, \frac{32\pi^2}{1 - \|v\|^2}\right) . \tag{7.37}$$

We can notice that if we prove that $u_n \to u$ in E then we get a contradiction with (7.34) and we can conclude that $I(u) = b$. Since $u_n \rightharpoonup u$ in E, it suffices to prove that

$$\langle u_n,\, u_n - u\rangle \to 0$$

as $n \to +\infty$, and this is indeed the case if

$$\mathcal{I}_n := \int_{\mathbb{R}^4} f(u_n)(u_n - u)\, dx \to 0 \tag{7.38}$$

as $n \to +\infty$. Arguing as in Case 1, we can reduce the proof of (7.38) to show the existence of $\alpha > \alpha_0$ and $q \geq 1$ such that

$$\mathcal{J}_n := \int_{\mathbb{R}^4} |u_n|^{q-1}|u_n - u|(e^{\alpha u_n^2} - 1)\, dx \to 0$$

as $n \to +\infty$. Applying twice Hölder's inequality with $1 < p,\, p',\, \sigma,\, \sigma' < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, we get for any $\beta > p$

$$\mathcal{J}_n \leq C_\beta \|u_n\|^{q-1}_{(q-1)p'\sigma'} \|u_n - u\|_{p'\sigma} \left(\int_{\mathbb{R}^4} (e^{\alpha\beta u_n^2} - 1)\, dx\right)^{\frac{1}{p}} .$$

From (7.37), it follows that

$$\sup_n \int_{\mathbb{R}^4} (e^{\alpha\beta u_n^2} - 1)\, dx = \sup_n \int_{\mathbb{R}^4} (e^{\alpha\beta \|u_n\|^2 v_n^2} - 1)\, dx < +\infty ,$$

provided that $\alpha > \alpha_0$ is sufficiently close to α_0 and $p > 1$ is sufficiently close to 1, so that we can choose $\beta > p > 1$ sufficiently close to 1 too.

Since $(q-1)p'\sigma' > q - 1$ and $p'\sigma > \sigma$, choosing $q \geq 3$ and $\sigma > 2$ we have that $\{u_n\}_n$ is bounded in $L^{(q-1)p'\sigma'}(\mathbb{R}^4)$ and $u_n \to u$ in $L^{p'\sigma}(\mathbb{R}^4)$, hence $\mathcal{J}_n \to 0$ as $n \to +\infty$. □

7.5. Final remarks

The arguments of this Chapter can be easily adapted to obtain existence result for equations of the form

$$\begin{cases} \Delta^2 u - \operatorname{div}\,(U(x)\nabla u) + V(x)u = f(u) & \text{in } \mathbb{R}^4 \\ u \in H^2(\mathbb{R}^4) \end{cases} \tag{7.39}$$

under suitable assumptions on U, $V : \mathbb{R}^4 \to \mathbb{R}$. More precisely, we will assume (V_0) and

(U_0) $U : \mathbb{R}^4 \to \mathbb{R}$ is continuous and $U(x) \geq U_0 > 0$ for any $x \in \mathbb{R}^4$.

The following result holds

Theorem 7.15. *Assume (U_0), (V_0) and $(f_1) - (f_3)$.*

(i) If the potential V satisfies (V_1) and the nonlinearity f satisfies (f_4) then problem (7.39) has a nontrivial solution.

(ii) If U, V are spherically symmetric, i.e.

$$U(x) = U(|x|), \quad V(x) = V(|x|) \qquad \forall x \in \mathbb{R}^4,$$

then problem (7.39) has a nontrivial radial solution.

We can notice that the functional space for a variational approach of problem (7.39) is the subspace $\tilde{E}$ of $H^2(\mathbb{R}^4)$

$$\tilde{E} := \left\{ u \in H^2(\mathbb{R}^4) \;\middle|\; \int_{\mathbb{R}^4} \left[U(x)|\nabla u|^2 + V(x)u^2\right] \, dx < +\infty \right\} \subseteq E,$$

which is a Hilbert space endowed with the inner product

$$\langle u,\, v\rangle_{\tilde{E}} := \int_{\mathbb{R}^4} \Delta u \Delta v \, dx + \int_{\mathbb{R}^4} U(x) \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^4} V(x) uv \, dx \quad \forall u,\, v \in \tilde{E}.$$

We will denote by $\|\cdot\|_{\tilde{E}}$ the corresponding norm, i.e. $\|u\|_{\tilde{E}} := \sqrt{\langle u,\, u\rangle_{\tilde{E}}}$ for any $u \in \tilde{E}$.

As a consequence of (U_0) and (V_0), it is easy to see that the embedding $\tilde{E} \hookrightarrow H^2(\mathbb{R}^4)$ is continuous and, from (V_1) it follows that the embedding

$$\tilde{E} \hookrightarrow\hookrightarrow L^p(\mathbb{R}^4)$$

is compact for any $p \in [2,\, +\infty)$.

The energy functional associated to problem (7.39) is

$$I(u) := \frac{1}{2}\|u\|_{\tilde{E}}^2 - \int_{\mathbb{R}^4} F(u) \, dx \qquad \forall u \in \tilde{E},$$

and, in order to adapt the arguments in the previous sections to this functional, it suffices only to specify how to recover the steps in which we applied the Adams-type inequality (3.1) (see also Theorem 2.1) and the Lions-type concentration-compactness Lemmas 3.4 and 3.6.

To this aim, we recall the following version of Adams' inequality (see (3.3))

$$\sup_{u \in H^2(\mathbb{R}^4),\, \|u\|_{H^2,\, U_0,\, V_0} \leq 1} \int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1) \, dx < +\infty \tag{7.40}$$

where

$$\|u\|_{H^2,\, U_0,\, V_0}^2 := \|\Delta u\|_2^2 + U_0 \|\nabla u\|_2^2 + V_0 \|u\|_2^2 \quad \forall u \in H^2(\mathbb{R}^4)\,.$$

It is straightforward to notice that

$$\|u\|_{H^2,\, U_0,\, V_0} \leq \|u\|_{\tilde{E}} \qquad \forall u \in \tilde{E},$$

and hence, to obtain (7.25) and (7.36), we can apply (7.40) instead of (3.1). Finally, the following Lions-type concentration-compactness result holds

Lemma 7.16. *Let $\{u_n\}_n \subset \tilde{E}$ be such that $\|u_n\|_{\tilde{E}} = 1$ for any $n \geq 1$ and let $u \in \tilde{E}$ be the weak limit of $\{u_n\}_n$ in $\tilde{E}$. If $\|u\|_{\tilde{E}} < 1$ then*

$$\sup_n \int_{\mathbb{R}^4} (e^{pu_n^2} - 1)\, dx < +\infty \qquad \forall p \in \left(0,\, \frac{32\pi^2}{1 - \|u\|_{\tilde{E}}}\right).$$

Proof. Since

$$\|u_n - u\|_{H^2, U_0, V_0} \leq \|u_n - u\|_{\tilde{E}} \qquad \forall n \geq 1,$$

the proof follows using the same arguments as in Lemma 3.2 and the modified version (7.40) of Adams' inequality. □

We end this Section with the Example 7.2 mentioned in the introduction of this Chapter.

Proposition 7.17. *Let $\alpha_0 > 0$. The function $f(s) = s(e^{\alpha_0 s^2} - 1) \quad \forall s \in \mathbb{R}$ satisfies the assumptions $(f_0) - (f_4)$.*

Proof. Obviously f satisfies (f_0), (f_3) and (f_4). In order to show that f satisfies also (f_1), (f_2), since f is odd and F is even, it suffices to prove that

$$0 < \mu F(s) \leq s f(s) \quad \forall s > 0, \qquad \text{for some } \mu > 2 \tag{7.41}$$

and that there exist s_0, $M_0 > 0$ such that

$$F(s) \leq M_0 f(s) \qquad \forall s \geq s_0. \tag{7.42}$$

We have that

$$F(s) = \frac{1}{2\alpha_0}(e^{\alpha_0 s^2} - \alpha s^2 - 1) = \frac{1}{2\alpha_0} \sum_{j=2}^{+\infty} \frac{\alpha_0^j}{j!} s^{2j} > 0 \qquad \forall s > 0,$$

consequently, for $0 < \mu < 4$ we have

$$\mu F(s) = \frac{\mu}{2\alpha_0} s^2 \sum_{j=2}^{+\infty} \frac{\alpha_0^j}{j!} s^{2(j-1)} \leq \frac{\mu}{4} s^2 \sum_{j=1}^{+\infty} \frac{\alpha_0^j}{j!} s^{2j} < s^2(e^{\alpha_0 s^2} - 1) = s f(s) \qquad \forall s > 0$$

and (7.41) holds provided that $2 < \mu < 4$.

Finally, for any $s \geq 1$ we have

$$F(s) \leq \frac{1}{2\alpha_0}(e^{\alpha_0 s^2} - 1) \leq \frac{1}{2\alpha_0}|s|(e^{\alpha_0 s^2} - 1) = \frac{1}{2\alpha_0} f(s),$$

which is nothing but (7.42) with $s_0 = \frac{1}{2\alpha_0}$ and $M_0 = 1$. □

The following example has been introduced in [32].

Proposition 7.18. *Let $\alpha_0 > 0$. The function $f(s) = sign\,(s)(e^{\alpha_0 s^2} - 1) \quad \forall s \in \mathbb{R}$ satisfies the assumptions $(f_0) - (f_4)$.*

Proof. As before, it suffices only to show that (7.41) and (7.42) holds. Let

$$\tilde{F}(s) = \frac{1}{s}(e^{\alpha_0 s^2} - \alpha_0 s^2 - 1) \qquad \forall s > 0,$$

then, using the power series expansion of the exponential function, we obtain that

$$\frac{3}{2}\alpha_0 f(s) \leq \tilde{F}'(s) \qquad \forall s > 0.$$

Consequently

$$F(s) \leq \frac{2}{3\alpha_0}\tilde{F}(s) \qquad \forall s > 0. \tag{7.43}$$

Using again the power series expansion of the exponential function,

$$\frac{2}{\alpha_0}\tilde{F}(s) \leq s f(s) \qquad \forall s > 0$$

which leads to

$$\mu F(s) \leq \mu \frac{2}{3\alpha_0}\tilde{F}(s) \leq \frac{\mu}{3} s f(s) \qquad \forall s > 0, \quad \forall \mu > 0.$$

Therefore, choosing $2 < \mu < 3$ we obtain (7.41) and hence (f_1).

Moreover, (f_2) follows easily by

$$\tilde{F}(s) \leq |f(s)| \qquad \forall s \geq 1.$$

In fact, this last inequality together with (7.43) gives

$$F(s) \leq \frac{2}{3\alpha_0}|f(s)| \qquad \forall s \geq 1.$$

□

Bibliography

[1] Shinji Adachi and Kazunaga Tanaka, *Trudinger type inequalities in* $\mathbb{R}^N$ *and their best exponents*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 2051–2057.

[2] David R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math. (2) **128** (1988), no. 2, 385–398.

[3] Adimurthi, *Positive solutions of the semilinear Dirichlet problem with critical growth in the unit disc in* $\mathbb{R}^2$, Proc. Indian Acad. Sci. Math. Sci. **99** (1989), no. 1, 49–73.

[4] ———, *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the* n*-Laplacian*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), no. 3, 393–413.

[5] Adimurthi, P. N. Srikanth, and Shyam L. Yadava, *Phenomena of critical exponent in* $\mathbb{R}^2$, Proc. Roy. Soc. Edinburgh Sect. A **119** (1991), no. 1-2, 19–25.

[6] Adimurthi and Shyam L. Yadava, *Multiplicity results for semilinear elliptic equations in a bounded domain of* $\mathbb{R}^2$ *involving critical exponents*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), no. 4, 481–504.

[7] Adimurthi and Yunyan Yang, *An interpolation of Hardy inequality and Trundinger-Moser inequality in* $\mathbb{R}^N$ *and its applications*, Int. Math. Res. Not. IMRN (2010), no. 13, 2394–2426.

[8] Claudianor O. Alves and João M. do Ó, *Positive solutions of a fourth-order semilinear problem involving critical growth*, Adv. Nonlinear Stud. **2** (2002), no. 4, 437–458.

[9] Claudianor O. Alves, João M. do Ó, and Olímpio H. Miyagaki, *Nontrivial solutions for a class of semilinear biharmonic problems involving critical exponents*, Nonlinear Anal. **46** (2001), no. 1, Ser. A: Theory Methods, 121–133.

[10] Claudianor O. Alves, Marcelo Montenegro, and Marco A. S. Souto, *Existence of a ground state solution for a nonlinear scalar field equation with critical growth*, Calc. Var. Partial Differential Equations, http://dx.doi.org/10.1007/s00526–011–0422–y.

[11] Antonio Ambrosetti and Paul H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.

[12] Thomas Bartsch, Matthias Schneider, and Tobias Weth, *Multiple solutions of a critical polyharmonic equation*, J. Reine Angew. Math. **571** (2004), 131–143.

[13] Thomas Bartsch and Michel Willem, *Infinitely many nonradial solutions of a Euclidean scalar field equation*, J. Funct. Anal. **117** (1993), no. 2, 447–460.

[14] Henri Berestycki, Thierry Gallouët, and Otared Kavian, *Équations de champs scalaires euclidiens non linéaires dans le plan*, C. R. Acad. Sci. Paris Sér. I Math. **297** (1983), no. 5, 307–310.

[15] Henri Berestycki and Pierre L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345.

[16] Haïm Brezis and Frank Merle, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Comm. Partial Differential Equations **16** (1991), no. 8-9, 1223–1253.

[17] Haïm Brezis and Louis Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[18] Marta Calanchi, Bernhard Ruf, and Zhitao Zhang, *Elliptic equations in* $\mathbb{R}^2$ *with one-sided exponential growth*, Commun. Contemp. Math. **6** (2004), no. 6, 947–971.

[19] D. M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations **17** (1992), no. 3-4, 407–435.

[20] Lennart Carleson and Sun-Yung A. Chang, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. (2) **110** (1986), no. 2, 113–127.

[21] Daniele Cassani, Bernhard Ruf, and Cristina Tarsi, *Best constants for Moser type inequalities in Zygmund spaces*, Mat. Contemp. **36** (2009), 79–90.

[22] ______, *Best constants in a borderline case of second-order Moser type inequalities*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), no. 1, 73–93.

[23] Jan Chabrowski and João M. do Ó, *On some fourth-order semilinear elliptic problems in* $\mathbb{R}^N$, Nonlinear Anal. **49** (2002), no. 6, Ser. A: Theory Methods, 861–884.

[24] Giuseppe Chiti, *Orlicz norms of the solutions of a class of elliptic equations*, Boll. Un. Mat. Ital. A (5) **16** (1979), no. 1, 178–185.

[25] Andrea Cianchi, *Moser-Trudinger inequalities without boundary conditions and isoperimetric problems*, Indiana Univ. Math. J. **54** (2005), no. 3, 669–705.

[26] David G. Costa, *On a class of elliptic systems in* $\mathbf{R}^N$, Electron. J. Differential Equations (1994), No. 07, approx. 14 pp. (electronic).

[27] Djairo G. de Figueiredo, João M. do Ó, and Bernhard Ruf, *On an inequality by N. Trudinger and J. Moser and related elliptic equations*, Comm. Pure Appl. Math. **55** (2002), no. 2, 135–152.

[28] Djairo G. de Figueiredo, Olímpio H. Miyagaki, and Bernhard Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (1995), no. 2, 139–153.

[29] Manuel del Pino, Monica Musso, and Bernhard Ruf, *New solutions for Trudinger-Moser critical equations in* $\mathbb{R}^2$, J. Funct. Anal. **258** (2010), no. 2, 421–457.

[30] João M. do Ó, N*-Laplacian equations in* $\mathbb{R}^N$ *with critical growth*, Abstr. Appl. Anal. **2** (1997), no. 3-4, 301–315.

[31] João M. do Ó, Everaldo Medeiros, and Uberlandio Severo, *A nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. **345** (2008), no. 1, 286–304.

[32] João M. do Ó and Bernhard Ruf, *On a Schrödinger equation with periodic potential and critical growth in* $\mathbb{R}^2$, NoDEA Nonlinear Differential Equations Appl. **13** (2006), no. 2, 167–192.

[33] David E. Edmunds, Donato Fortunato, and Enrico Jannelli, *Fourth-order nonlinear elliptic equations with critical growth*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **83** (1989), 115–119 (1990).

[34] ______, *Critical exponents, critical dimensions and the biharmonic operator*, Arch. Rational Mech. Anal. **112** (1990), no. 3, 269–289.

[35] Martin Flucher, *Extremal functions for the Trudinger-Moser inequality in* 2 *dimensions*, Comment. Math. Helv. **67** (1992), no. 3, 471–497.

[36] Luigi Fontana, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helv. **68** (1993), no. 3, 415–454.

[37] Filippo Gazzola, Hans-Christoph Grunau, and Guido Sweers, *Optimal Sobolev and Hardy-Rellich constants under Navier boundary conditions*, Ann. Mat. Pura Appl. (4) **189** (2010), no. 3, 475–486.

[38] Godfrey Harold Hardy, John Edensor Littlewood, and George Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988, Reprint of the 1952 edition.

[39] Emmanuel Hebey and Michel Vaugon, *Sobolev spaces in the presence of symmetries*, J. Math. Pures Appl. (9) **76** (1997), no. 10, 859–881.

[40] Louis Jeanjean and Kazunaga Tanaka, *A remark on least energy solutions in* $\mathbf{R}^N$, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2399–2408 (electronic).

[41] Otared Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 13, Springer-Verlag, Paris, 1993.

[42] S. Kesavan, *Symmetrization & applications*, Series in Analysis, vol. 3, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[43] Hideo Kozono, Tokushi Sato, and Hidemitsu Wadade, *Upper bound of the best constant of a Trudinger-Moser inequality and its application to a Gagliardo-Nirenberg inequality*, Indiana Univ. Math. J. **55** (2006), no. 6, 1951–1974.

[44] Yuxiang Li and Bernhard Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^n$, Indiana Univ. Math. J. **57** (2008), no. 1, 451–480.

[45] Elliott H. Lieb and Michael Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[46] Chang-Shou Lin and Juncheng Wei, *Locating the peaks of solutions via the maximum principle. II. A local version of the method of moving planes*, Comm. Pure Appl. Math. **56** (2003), no. 6, 784–809.

[47] Kai-Ching Lin, *Extremal functions for Moser's inequality*, Trans. Amer. Math. Soc. **348** (1996), no. 7, 2663–2671.

[48] Pierre-Louis Lions, *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal. **49** (1982), no. 3, 315–334.

[49] Guozhen Lu and Yunyan Yang, *Adams' inequalities for bi-Laplacian and extremal functions in dimension four*, Adv. Math. **220** (2009), no. 4, 1135–1170.

[50] Jürgen Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

[51] Dimitri Mugnai, *Four nontrivial solutions for subcritical exponential equations*, Calc. Var. Partial Differential Equations **32** (2008), no. 4, 481–497.

[52] Takayoshi Ogawa and Tohru Ozawa, *Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem*, J. Math. Anal. Appl. **155** (1991), no. 2, 531–540.

[53] Richard O'Neil, *Convolution operators and* $L(p, q)$ *spaces*, Duke Math. J. **30** (1963), 129–142.

[54] Tohru Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal. **127** (1995), no. 2, 259–269.

[55] Richard S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), no. 1, 19–30.

[56] Stanislav I. Pohozaev, *The Sobolev embedding in the case* $pl = n$, Proc. Tech. Sci. Conf. on Adv. Sci. Research 1964-1965, Mathematics Section, Moskov. Ènerget. Inst. Moscow (1965), 158–170.

[57] Patrizia Pucci and James Serrin, *A general variational identity*, Indiana Univ. Math. J. **35** (1986), no. 3, 681–703.

[58] ———, *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl. (9) **69** (1990), no. 1, 55–83.

[59] Paul H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

[60] Halsey L. Royden, *Real analysis*, third ed., Macmillan Publishing Company, New York, 1988.

[61] Bernhard Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^2$, J. Funct. Anal. **219** (2005), no. 2, 340–367.

[62] Bernhard Ruf and Federica Sani, *Ground states for elliptic equations in* $\mathbb{R}^2$ *with exponential critical growth*, submitted.

[63] ———, *Sharp Adams-type inequalities in* $\mathbb{R}^n$, Trans. Amer. Math. Soc. to appear.

[64] Federica Sani, *A biharmonic equation in* $\mathbb{R}^4$ *involving nonlinearities with critical exponential growth*, Commun. Pure Appl. Anal. to appear.

[65] ———, *A biharmonic equation in* $\mathbb{R}^4$ *involving nonlinearities with subcritical exponential growth*, Adv. Nonlinear Stud. **11** (2011), no. 4, 889–904.

[66] Michael Struwe, *Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces*, Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), no. 5, 425–464.

[67] Giorgio Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), no. 4, 697–718.

[68] Cristina Tarsi, *Adams' inequality and limiting Sobolev embeddings into Zygmund spaces*, preprint.

[69] Guido Trombetti and Juan Luis Vázquez, *A symmetrization result for elliptic equations with lower-order terms*, Ann. Fac. Sci. Toulouse Math. (5) **7** (1985), no. 2, 137–150.

[70] Neil S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[71] Viktor Iosifovich Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dokl. Akad. Nauk SSSR **138** (1961), 805–808.