A Note on Fuzzy Set–Valued Brownian Motion

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September 28, 2011

Abstract

In this paper, we prove that a fuzzy set–valued Brownian motion B_t, as defined in [2], can be handle by an $\mathbb{R}^d$–valued Wiener process b_t, in the sense that $B_t = \mathbb{I}_{b_t}$; i.e. it is actually the indicator function of a Wiener process.

1 Introduction

Stochastic (fuzzy) set–valued evolution is a relevant topic that was studied largely by different authors (e.g. [2, 3, 4] and references therein). The following question was stated by Molchanov in [4, Open Problem 1.24, p.316]:

> Define a set–valued analogue of the Wiener process and the corresponding stochastic integral.

In [2], the authors tackle the proposed problem defining a fuzzy set–valued Brownian motion in $\mathbb{F}_{kc}$, the family of convex fuzzy subsets of $\mathbb{R}^d$ with compact support. In the sequel we shall prove that such a process is equivalent to consider simply a Wiener process in $\mathbb{R}^d$. This is based upon the fact that the Brownian motion is a zero–mean Gaussian (fuzzy set–valued) process.
In fact, it is widely known (cf. [3, Theorem 6.1.7]) that a Gaussian random fuzzy set decomposes according to

$$X = \mathbb{E}X \oplus \mathbb{I}_\xi, \tag{1}$$

where $\mathbb{E}X$ is in the Aumann sense, ξ is a Gaussian random element in $\mathbb{R}^d$ with $\mathbb{E}\xi = 0$ and $\mathbb{I}_A : \mathbb{R}^d \to \{0, 1\}$ denotes the indicator function of any $A \subseteq \mathbb{R}^d$

$$\mathbb{I}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

(for the sake of simplicity, whenever $A = \{a\}$ is a singleton we shall write $\mathbb{I}_a$ instead of $\mathbb{I}_{\{a\}}$). Equation (1) means that X is just its expected value $\mathbb{E}X$ up to a random Gaussian translation ξ. In some sense, $\mathbb{E}X$ represents the "deterministic" part of X whilst ξ represents its random part. It is also known (cf. [4, Proposition 1.30, p.161]) that a zero–mean random set is actually a random element in $\mathbb{R}^d$ with zero–mean. Such a result can be easily extended to the fuzzy case and, jointly to decomposition (1), implies

$$X = \mathbb{I}_0 \oplus \mathbb{I}_\xi = \mathbb{I}_\xi.$$

Roughly speaking, the definition of Brownian motion in [2] for random fuzzy sets drives down the complexity of the chosen (fuzzy) framework. In fact, a Gaussian fuzzy random set with zero–mean is reduced to be a random Gaussian element in $\mathbb{R}^d$.
In this paper we shall provide an alternative proof of the last fact using selections.

The paper is organized as follow. Section 2 is devoted to preliminaries such as random (fuzzy) sets, embedding theorems and Brownian motion for fuzzy sets (according to [2]). In Section 3 we prove the main result of the paper, whilst in Section 4 we provide a proof to the statement "zero–mean random set is a random element in $\mathbb{R}^d$ with zero–mean".

2 Preliminaries

Here we refer mainly to [3]. Denote by $\mathbb{K}_{kc}$ the class of non–empty compact convex subsets of $\mathbb{R}^d$, endowed with the Hausdorff metric

$$\delta_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\},$$

and the operations

$$A + B = \{a + b : a \in A,\ b \in B\}, \qquad \lambda \cdot A = \lambda A = \{\lambda a : a \in A\}.$$

A *fuzzy set* is a map $\nu : \mathbb{R}^d \to [0, 1]$. Let $\mathbb{F}_{kc}$ denote the family of all fuzzy sets, which satisfy the following conditions.

1. Each ν is an upper semicontinuous function, i.e. for each $\alpha \in (0, 1]$, the cut set $\nu_\alpha = \{x \in \mathbb{R}^d : \nu(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}^d$.
2. The cut set $\nu_1 = \{x \in \mathbb{R}^d : \nu(x) = 1\} \neq \emptyset$.
3. The support set $\nu_{0+} = \overline{\{x \in \mathbb{R}^d : \nu(x) > 0\}}$ of ν is compact; hence every ν_α is compact for $\alpha \in (0, 1]$.
4. For any $\alpha \in [0, 1]$, ν_α is a convex subset of $\mathbb{R}^d$.

Let us endow $\mathbb{F}_{kc}$ with the metric

$$\delta_H^\infty(\nu^1, \nu^2) = \sup\{\alpha \in [0, 1] : \delta_H(\nu^1_\alpha, \nu^2_\alpha)\}.$$

and the operations

$$(\nu^1 \oplus \nu^2)_\alpha = \nu^1_\alpha + \nu^2_\alpha, \qquad (\lambda \odot \nu^1)_\alpha = \lambda \cdot \nu^1_\alpha.$$

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space. A *fuzzy set–valued random variable* (FRV) is a function $X : \Omega \to \mathbb{F}_{kc}$, such that $X_\alpha : \omega \mapsto X(\omega)_\alpha$ are random compact convex sets for every $\alpha \in (0, 1]$ (i.e. X_α is a $\mathbb{K}_{kc}$–valued function measurable with respect to the δ_H–Borel σ–algebra).

An FRV X is *integrably bounded* and we shall write $X \in L^1[\Omega, \mathfrak{F}, \mu; \mathbb{F}_{kc}] = L^1[\Omega; \mathbb{F}_{kc}]$, if $\|X_{0+}\|_H := \delta_H(X_{0+}, \{0\}) \in L^1[\Omega; \mathbb{R}]$.

The *expected value* of an FRV X, denoted by $\mathbb{E}[X]$, is a fuzzy set such that, for every $\alpha \in (0,1]$,

$$(\mathbb{E}[X])_\alpha = \left(\int_\Omega X_\alpha \mathrm{d}\mu\right) = \{\mathbb{E}(f) : f \in L^1[\Omega;\mathbb{R}^d], f \in X_\alpha \ \mu - \text{a.e.}\}.$$

Embedding Theorem. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$. For any $\nu \in \mathbb{F}_{kc}$ define the *support function* of ν as follows:

$$h_\nu(x,\alpha) = \begin{cases} h_{\nu_\alpha}(x) & \text{if } \alpha > 0, \\ h_{\nu_{0+}}(x) & \text{if } \alpha = 0, \end{cases}$$

for $(x,\alpha) \in S^{d-1} \times [0,1]$ and where $h_K(x) = \sup\{\langle x, a\rangle : a \in K\}$, for $x \in S^{d-1}$.

It is known that support function satisfies the following properties:

1. for any $\nu^1, \nu^2 \in \mathbb{F}_{kc}$, $h_{\nu^1 \oplus \nu^2}(\cdot,\cdot) = h_{\nu^1}(\cdot,\cdot) + h_{\nu^2}(\cdot,\cdot)$,
2. for any $(x,\alpha) \in \mathbb{R}^d \times [0,1]$, $h_{X(\cdot)}(x,\alpha) \in L^1[\Omega;\mathbb{R}]$, $\mathbb{E}[h_X(x,\alpha)] = h_{\mathbb{E}[X]}(x,\alpha)$.

Let $C(S^{d-1})$ denote the Banach space of all continuous functions v on S^{d-1} with respect to the norm $\|v\|_C = \sup_{x\in S^{d-1}} |v(x)|$. Let $\overline{C}([0,1], C(S^{d-1}))$ be the set of all functions $f : [0,1] \to C(S^{d-1})$ such that f is bounded, left continuous with respect to $\alpha \in (0,1]$, right continuous at 0, and f has right limit for any $\alpha \in (0,1)$. Then we have that $\overline{C}([0,1], C(S^{d-1}))$ is a Banach space with the norm $\|f\|_{\overline{C}} = \sup_{\alpha\in[0,1]} \|f(\alpha)\|_C$, and the following embedding theorem holds.

Proposition 1 *([2] and the references therein.)* There exists a function $j : \mathbb{F}_{kc} \to \overline{C}([0,1], C(S^{d-1}))$ such that:

1. j is an isometric mapping, i.e.

$$\delta_H^\infty(\nu^1,\nu^2) = \|j(\nu^1) - j(\nu^2)\|_{\overline{C}}, \quad \nu^1,\nu^2 \in \mathbb{F}_{kc},$$

2. $j(r\nu^1 + t\nu^2) = rj(\nu^1) + tj(\nu^2)$, $\nu^1,\nu^2 \in \mathbb{F}_{kc}$ and $r,t \geq 0$.
3. $j(\mathbb{F}_{kc})$ is a closed subset in $\overline{C}([0,1], C(S^{d-1}))$.

As a matter of fact, we can define an injection $j : \mathbb{F}_{kc} \to \overline{C}([0,1], C(S^{d-1}))$ by $j(\nu) = h_\nu$, i.e. $j(\nu)(x,\alpha) = h_\nu(x,\alpha)$ for every $(x,\alpha) \in S^{d-1} \times [0,1]$, and this mapping j satisfies above theorem. For simplification, let $\overline{\mathbf{C}} := \overline{C}([0,1], C(S^{d-1}))$.

From Proposition 1 it follows that every FRV X can be regarded as a random element of $\overline{\mathbf{C}}$ by considering $j(X) = h_X : \Omega \to \overline{\mathbf{C}}$, where $h_X(\omega) = h_{X(\omega)}$.

Fuzzy set–valued Brownian motion. For the results in this subsection we refer to [2] or we shall specify if otherwise.

Definition 2 *[6]* A FRV $X : \Omega \to \mathbb{F}_{kc}$ is *Gaussian* if h_X is a Gaussian random element of $\overline{\mathbf{C}}$.

A random element h_X taking values in $\overline{\mathbf{C}}$ is Gaussian if and only if, for any $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \overline{\mathbf{C}}^*$, the real vector–valued random variable $(f_1(h_X), f_2(h_X), \dots, f_n(h_X))$ is Gaussian, where $\overline{\mathbf{C}}^*$ is the conjugate space of $\overline{\mathbf{C}}$ (i.e. the set of all continuous linear functionals on $\overline{\mathbf{C}}$).
It follows from the properties of h_X and elements in $\overline{\mathbf{C}}^*$ that $X + Y$ is Gaussian if X and Y are Gaussian FRV. Also λX is Gaussian whenever X is Gaussian and $\lambda \in \mathbb{R}$.

Proposition 3 *[3, Theorem 6.1.7]* A FRV X is Gaussian if and only if X is representable in the form

$$X = \mathbb{E}[X] \oplus \mathbb{I}_\xi,$$

where ξ is a Gaussian random element of $\mathbb{R}^d$ with zero mean.

Definition 4 Assume that $\{\mathfrak{F}_t : t \geq 0\}$ is a σ–filtration satisfying the usual condition (complete and right continuous). $\{X_t : t \geq 0\}$ is called an adaptive fuzzy set–valued stochastic process if for any $t \in \mathbb{R}_+$, X_t is an $\mathfrak{F}_t$–measurable FRV. An adaptive fuzzy set–valued stochastic process $\{X_t : t \geq 0\}$ is called Gaussian if, for any $t \in \mathbb{R}_+$, X_t is Gaussian.

An adaptive fuzzy set–valued stochastic process $X = \{X_t : t \geq 0\}$ is Gaussian if and only if $\{(f_1(h_{X_t}), \dots, f_n(h_{X_t})) : t \geq 0\}$ is a real vector–valued Gaussian process, for any $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \overline{\mathbf{C}}^*$. Further, the following theorem holds.

Definition 5 An adaptive fuzzy set–valued stochastic process $\{B_t : t \in \mathbb{R}_+\}$ is called a fuzzy set–valued Brownian motion if and only if $\{h_{B_t} : t \in \mathbb{R}_+\}$ is a Brownian motion in $\overline{\mathbf{C}}$.

Proposition 6 Assume that a fuzzy set–valued stochastic process $\{B_t : t \geq 0\}$ satisfies $B_0 = \mathbb{I}_0$. Then $\{B_t : t \geq 0\}$ is a fuzzy set–valued Brownian motion if and only if it is a Gaussian process and

1. $\mathbb{E}[f_i(h_{B_t})] = 0$, for any $t \geq 0$, $f_i \in \overline{\mathbf{C}}^*$, $i = 1, \dots, n$,
2. $\mathbb{E}[f_i(h_{B_t}) f_i(h_{B_s})] = t \wedge s$, for any $s, t \geq 0$, $f_i \in \overline{\mathbf{C}}^*$, $i = 1, \dots, n$,
3. $\mathbb{E}[f_i(h_{B_t}) f_j(h_{B_s})] = 0$, for any $s, t \geq 0$, $f_i, f_j \in \overline{\mathbf{C}}^*$, $i \neq j$, $i, j = 1, \dots, n$.

In [2, Theorem 4.3 and Theorem 4.4] the authors provide also some properties of a fuzzy set–valued Brownian motion that are very similar to those of the real case.

Proposition 7 Let $\{B_t : t \geq 0\}$ be a fuzzy set–valued Brownian motion. The following hold.

1. $\{B_{t+t_0}\}_{t \geq 0}$ is a fuzzy set–valued Brownian motion for any $t_0 \geq 0$.
2. $\{\nu \oplus B_t\}_{t \geq 0}$ is a fuzzy set-valued Brownian motion for any fuzzy set $\nu \in \mathbb{F}_k$.
3. $\{\frac{1}{\sqrt{\lambda}} B_{\lambda t}\}_{t \geq 0}$ is a fuzzy set-valued Brownian motion for any $\lambda > 0$.
4. $\{t B_{\frac{1}{\sqrt{t}}}\}_{t \geq 0}$ is a fuzzy set-valued Brownian motion.
5. If $\mathfrak{F}_t = \sigma\{B_s : s \leq t\}$, then $\{B_t, \mathfrak{F}_t\}_{t \geq 0}$ is a fuzzy set–valued martingale.

3 A FRV Brownian motion is a Wiener process in $\mathbb{R}^d$

This section is devoted to prove Theorem 8: the main result of this paper.

Theorem 8 A fuzzy set–valued process $\{B_t : t \geq 0\}$ is a Brownian motion, if and only if,

$$B_t = \mathbb{I}_{b_t}, \qquad \mu\text{–a.e.}$$

where $\{b_t : t \geq 0\}$ is a Wiener process in $\mathbb{R}^d$.

According to Definition 5 a fuzzy set–valued Brownian motion B_t is a process taking values in $\mathbb{F}$ (that is a functional space over $\mathbb{R}^d$). On the other hand, the previous result provides a way to handle a fuzzy set–valued Brownian motion simply using a random vector of $\mathbb{R}^d$. In other words, we observe a "complexity reduction", i.e. from $\mathbb{F}$ to $\mathbb{R}^d$.
Moreover, in view of Theorem 8, Property 2 in Proposition 7 is true if and only if $\nu = \mathbb{I}_0$, whilst the remain properties in Proposition 7 still hold due to the same properties of the driving Wiener process b_t in $\mathbb{R}^d$.

Actually the "complexity reduction" stated in Theorem 8 is strictly related to the characterization of Gaussian FRV (cf. Proposition 3), to Property 1 of Proposition 6, and to the following result obtained for random closed sets.

Proposition 9 Let X be in $L^1[\Omega; \mathbb{K}]$ and let $a \in \mathbb{R}^d$. $\int_\Omega X d\mu = \{a\}$ if and only if there exists a $x \in L^1[\Omega; \mathbb{R}^d]$ such that $X = \{x\}$ μ–a.e. and $\int_\Omega x d\mu = a$.

Corollary 10 Let X be in $L^1[\Omega; \mathbb{K}]$. $\int_\Omega X d\mu = \{0\}$ if and only if there exists a $x \in L^1[\Omega; \mathbb{R}^d]$ such that $X = \{x\}$ μ-a.e. and $\int_\Omega x d\mu = 0$.

Although Proposition 9 and Corollary 10 are proved by Molchanov in [4, Proposition 1.30, p.161], we shall propose in Appendix 4 alternative proofs via selections avoiding the use of the support function as Molchanov did.

Lemma 11 For each $(x, \alpha) \in \mathbb{R}^d \times [0, 1]$, the following map belongs to $\overline{\mathbf{C}}^*$

$$\begin{array}{rccl} \varphi_{x,\alpha} : & \overline{\mathbf{C}} & \to & \mathbb{R} \\ & s & \mapsto & \varphi_{x,\alpha}(s) = s(x, \alpha). \end{array}$$

Proof. Map $\varphi_{x,\alpha}$ is linear since, for any s_1, s_2 in $\overline{\mathbf{C}}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, the following chain of equalities hold.

$$\begin{aligned} \varphi_{x,\alpha}(\lambda_1 s_1 + \lambda_2 s_2) =& [(\lambda_1 s_1 + \lambda_2 s_2)(\alpha)](x) = [\lambda_1 s_1(\alpha) + \lambda_2 s_2(\alpha)](x) \\ =& \lambda_1 s_1(\alpha, x) + \lambda_2 s_2(\alpha, x) = \lambda_1 \varphi_{x,\alpha}(s_1) + \lambda_2 \varphi_{x,\alpha}(s_2). \end{aligned}$$

For the continuity, let us consider any $s \in \overline{\mathbf{C}}$. For each $\varepsilon > 0$ and $h \in \overline{\mathbf{C}}$ such that $\|h\|_{\overline{C}} < \varepsilon$, the following relations complete the proof.

$$|\varphi_{x,\alpha}(s + h) - \varphi_{x,\alpha}(s)| = |\varphi_{x,\alpha}(h)| = |h(\alpha, x)| \leq \|h\|_{\overline{C}} < \varepsilon.$$

∎

Proof of Theorem 8. The "if"part is trivial.

In order to prove the "only if"part let us consider the fuzzy set–valued Brownian motion $\{B_t : t \geq 0\}$.
STEP 1. According to Proposition 6 and Proposition 3, for any $t \geq 0$ and $f \in \overline{\mathbf{C}}^*$, it satisfies

$$0 = \mathbb{E}[f(h_{B_t})] = \mathbb{E}[f(h_{\mathbb{E}[B_t]\oplus \mathbb{I}_{\xi_t}})].$$

where ξ_t is an Gaussian random element of $\mathbb{R}^d$ with $\mathbb{E}\xi_t = 0$. By the fact that, for any $\nu^1, \nu^2 \in \mathbb{F}_c$, $h_{\nu^1 \oplus \nu^2} = h_{\nu^1} + h_{\nu^2}$ (cf. Proposition 1), using the linearity of the expected value and of f, we get

$$\begin{aligned} 0 &= \mathbb{E}[f(h_{\mathbb{E}[B_t]})] + \mathbb{E}[f(h_{\mathbb{I}_{\xi_t}})] = f(h_{\mathbb{E}[B_t]}) + f(\mathbb{E}[h_{\mathbb{I}_{\xi_t}}]) \\ &= f(h_{\mathbb{E}[B_t]}) + f(h_{\mathbb{I}_{\mathbb{E}[\xi_t]}}) = f(h_{\mathbb{E}[B_t]}), \end{aligned} \tag{2}$$

for any $t \geq 0$ and $f \in \overline{\mathbf{C}}^*$, where for the last two equalities we use $h_{\mathbb{E}X} = \mathbb{E}h_X$ and the fact that ξ_t is zero mean.
Clearly $h_{\mathbb{E}[B_t]} \equiv 0$. On the contrary, there will exists an $\alpha \in [0, 1]$ such that $h_{\mathbb{E}[B_t]}(\alpha) \not\equiv 0$; i.e. there exists an $\alpha \in [0, 1]$ and $x \in \mathbb{R}^d$ such that $h_{\mathbb{E}[B_t]}(\alpha, x) \neq 0$. Let us consider the map defined by $\varphi_{x,\alpha}(s) = s(x, \alpha)$. It is an element of $\overline{\mathbf{C}}^*$ (cf. Lemma 11). Then $\varphi_{x,\alpha}(h_{\mathbb{E}[B_t]}) \neq 0$ contradicts Equation (2).
As a consequence, $\mathbb{E}[B_t] = \mathbb{I}_0$ for each $t \geq 0$; i.e.

$$\mathbb{E}[(B_t)_\alpha] = \{0\}, \tag{3}$$

for each $t \geq 0$ and $\alpha \in (0, 1]$.
STEP 2. Combining Corollary 10 with Equation (3) we obtain that, for each $t \geq 0$ and $\alpha \in (0, 1]$, $(B_t)_\alpha$ is actually μ–a.e. a random singleton with null mean value; i.e. $(B_t)_\alpha = \{b_t\}$ μ–a.e. with b_t being a random element of $\mathbb{R}^d$ such that $\mathbb{E}b_t = 0$. By definition of α–level sets for fuzzy set, $(B_t)_\alpha \supset (B_t)_\beta$ for any $0 \leq \alpha \leq \beta \leq 1$, and then $B_t = \mathbb{I}_{b_t}$ μ–a.e..
Since $\{B_t\}_{t\geq 0}$ is a fuzzy set–valued Brownian motion, $\{b_t\}_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$, and this fact concludes the proof. ■

Note that Proof of Theorem 8 only uses the fact that $\{B_t\}$ is a Gaussian process for which any finite distribution, at any time t, has null expectation.

We want to point out that, although one can associate a fuzzy set–valued Brownian motion at any Brownian motion in $\overline{\mathbf{C}}$ (using the embedding in Proposition 1), in general, the contrary is not possible. This is due to the embedding properties. In fact, $j(\mathbb{F}_{kc})$ is a proper subset of $\overline{C}([0, 1], C(S^{d-1}))$.
As a consequence, a Gaussian element in $\overline{C}([0, 1], C(S^{d-1}))$ can assume different values (even "negative"), whilst this could not happen in $\mathbb{F}_{kc}$ since, the embedding j could not carry back all the possible "fluctuations"of gaussian element.

In this view, a definition of fuzzy set–valued Brownian motion, that take care completely the complexity of the (fuzzy) set–valued framework, has to take into account the above arguments and must pay attention to the possibly degeneracy.

4 Proof of Proposition 9

In [4, Proposition 1.30, p.161] Molchanov proposed a proof of Proposition 9. It involves the support function of a set. Here we propose a different approach, via random sets selections, that is interesting by itself, and that leads to the same result.

For the sake of generality, here we shall consider $\mathfrak{X}$ to be a separable Banach space with $\mathcal{B}_{\mathfrak{X}}$ its borel σ–algebra and $(\Omega, \mathfrak{F})$ to be a measurable space endowed with a positive finite measure μ (till now $\mathfrak{X}$ was $\mathbb{R}^d$ and μ a probability measure).
In order to prove Proposition 9 we need the following two lemmas. Roughly speaking, the former says that any non–null vector in $\mathfrak{X}$ can be separated from zero using a suitable countable family of elements of $\mathfrak{X}^*$. The second lemma says that, for any couple of different (on some set of positive measure) integrable random elements in $\mathfrak{X}$, there exists an element of $\mathfrak{X}^*$ that separates (on a set of positive measure) these two random elements of $\mathfrak{X}$.

Lemma 12 There exists $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathfrak{X}^*$ such that whenever $x \in \mathfrak{X} \setminus \{0\}$ there exists $n \in \mathbb{N}$ for which $\phi_n(x) \neq 0$.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset of $\mathfrak{X}$. As a consequence of the Hahn-Banach Theorem (cf. [1, Corollary II.3.14, p. 65]) there exists $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathfrak{X}^*$ such that $\phi_n(x_n) = \|x_n\|_{\mathfrak{X}}$ and $\|\phi_n\|_{\mathfrak{X}^*} = 1$ for all $n \in \mathbb{N}$. Then

$$-\|y\|_{\mathfrak{X}} \le \phi_n(y) \le \|y\|_{\mathfrak{X}}, \qquad \forall y \in \mathfrak{X} \setminus \{0\}, \forall n \in \mathbb{N}. \tag{4}$$

Let $x \in \mathfrak{X} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $\|x - x_n\|_{\mathfrak{X}} \le \frac{\|x_n\|_{\mathfrak{X}}}{2}$. By (4) we have

$$\phi_n(x) = \phi_n(x_n) + \phi_n(x - x_n) \ge \|x_n\|_{\mathfrak{X}} - \|x - x_n\|_{\mathfrak{X}} \ge \frac{\|x_n\|_{\mathfrak{X}}}{2} > 0$$

i.e. $\phi_n(x) > 0$ that concludes the proof. ■

Lemma 13 Let $x_1,\ x_2 \in L^1[\Omega; \mathfrak{X}]$ and $A = \{\omega \in \Omega : x_1(\omega) \neq x_2(\omega)\}$ with $\mu(A) > 0$. Then there exists $\varphi \in \mathfrak{X}^*$ such that

$$A_\varphi = \{\omega \in \Omega : \varphi[x_1(\omega)] > \varphi[x_2(\omega)]\}$$

has positive measure (i.e. $\mu(A_\varphi) > 0$).

Proof. Let $x = (x_1 - x_2)$ then $A = \{\omega \in \Omega : x(\omega) \neq 0\}$ and let $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathfrak{X}^*$ as in Lemma 12. We claim that there exists $n \in \mathbb{N}$ such that $\mu(A_{\phi_n}) + \mu(A_{-\phi_n}) > 0$. By contradiction, if $A_n = A_{\phi_n} \cup A_{-\phi_n}$, we have

$$\mu(A_n) \le \mu(A_{\phi_n}) + \mu(A_{-\phi_n}) = 0, \qquad \forall n \in \mathbb{N}.$$

Now we prove that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$: let $\omega \in A$ then $x(\omega) \neq 0$ and, by hypothesis, there exists $n \in \mathbb{N}$ such that $\phi_n(x(\omega)) \neq 0$. Hence $\phi_n(x(\omega)) > 0$ or $\phi_n(x(\omega)) < 0$ i.e. $\omega \in A_n$ and thus $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$.
This means that $\mu(A) \le \mu(\bigcup_{n\in\mathbb{N}} A_n) = 0$ that contradicts hypothesis ($\mu(A) > 0$) and concludes the proof. ■
Proof of Proposition 9. The "if" part is trivial. Vice versa, let us suppose that $\int_\Omega x d\mu = a$ holds for all $x \in S_X$, where integral is in the

Bochner sense. Let us recall that a Bochner integrable map is also Pettis integrable and by definition (see [7, 5]) we have

$$\int_\Omega \phi(x)d\mu = \phi(a), \qquad \forall \phi \in \mathfrak{X}^*,\ \forall x \in S_X. \tag{5}$$

Now, by contradiction, let us suppose that x_1, x_2 are distinct elements of S_X i.e. $A = \{\omega \in \Omega : x_1(\omega) \neq x_2(\omega)\}$ has positive measure. Then, by Lemma 13, there exists $\varphi \in \mathfrak{X}^*$ such that $A_\varphi = \{\omega \in \Omega : \varphi[x_1(\omega)] > \varphi[x_2(\omega)]\}$ has positive measure. Let us consider $x_\varphi = \mathbb{I}_{A_\varphi} x_1 + \mathbb{I}_{A_\varphi^C} x_2$. Clearly x_φ is a selection of X (i.e. $x_\varphi \in S_X$), and

$$\begin{aligned} \int_\Omega \varphi(x_\varphi)d\mu &= \int_{A_\varphi} \varphi(x_1)d\mu + \int_{A_\varphi^C} \varphi(x_2)d\mu \\ &> \int_{A_\varphi} \varphi(x_2)d\mu + \int_{A_\varphi^C} \varphi(x_2)d\mu = \varphi(a) \end{aligned}$$

which contradicts Pettis integrability (5). ■

5 Conclusion

We proved that a fuzzy set–valued Brownian motion is actually a degenerated process. In particular, it can actually be handle by a wiener process in the understanding space. This simplification is due mainly both to the well–known Gaussian degeneracy and to the "null"expectation.
Moreover, we provided an alternative proof to Proposition 9: an integrable set–valued map, which integral is a singleton, is almost everywhere an integrable singleton–valued map

We think that used hypothesis can be relaxed in different ways in order to get generalizations. For example, the space $\mathbb{R}^d$ can be replaced with a more general one. In this case, the difficulty lies in the fact that one have to redefine fuzzy set–valued Brownian motion in the new space as well as to use a different embedding theorem.

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