A decomposition theorem for fuzzy set–valued random variables and a characterization of fuzzy random translation

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Abstract

Let X be a fuzzy set–valued random variable (FRV), and Θ_X the family of all fuzzy sets B for which the Hukuhara difference $X \ominus_H B$ exists $\mathbb{P}$–almost surely. In this paper, we prove that X can be decomposed as $X(\omega) = C \oplus Y(\omega)$ where the equality holds for $\mathbb{P}$–almost every $\omega \in \Omega$, C is the unique deterministic fuzzy set that minimizes $\mathbb{E}[d_2(X, B)^2]$ as B is varying in Θ_X, and Y is a centered FRV (i.e. its generalized Steiner point is the origin). This decomposition allows us to characterize all FRV translation (i.e. $X(\omega) = M \oplus \mathbb{I}_{\xi(\omega)}$ for some deterministic fuzzy convex set M and some random element in $\mathbb{R}^d$). In particular, X is an FRV translation if and only if the Aumann expectation $\mathbb{E}X$ is equal to C up to a translation.

This result includes the well–known case of Gaussian fuzzy random variable for which $X = \mathbb{E}X \oplus \xi$ with ξ being a Gaussian element in $\mathbb{R}^d$, and the fuzzy Brownian motion B_t that can be written as $B_t = \mathbb{I}_{\xi_t}$ where ξ_t is a Brownian process in $\mathbb{R}^d$.

Keywords: Fuzzy random variable; fuzzy random translation; Gaussian fuzzy random set; Aumann expectation; Hukuhara difference; decomposition theorem;

Introduction

It is widely known (e.g. [5, Theorem 6.1.7]) that a Gaussian fuzzy random variable decomposes according to

$$X = \mathbb{E}X \oplus \mathbb{I}_\xi, \tag{1}$$

where $\mathbb{E}X$ is the expectation of X in the Aumann sense, ξ is a Gaussian random element in $\mathbb{R}^d$ with $\mathbb{E}\xi = 0$ and $\mathbb{I}_A : \mathbb{R}^d \to \{0, 1\}$ denotes the indicator function of any $A \subseteq \mathbb{R}^d$

$$\mathbb{I}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

(for the sake of simplicity, whenever $A = \{a\}$ is a singleton we shall write $\mathbb{I}_a$ instead of $\mathbb{I}_{\{a\}}$). Plain speaking, a Gaussian FRV X is just a deterministic fuzzy set (its expected value $\mathbb{E}X$) up to a Gaussian translation ξ which carries out all the randomness of X. In this view, Equation (1) entails a "loss in complexity"for the randomness of the Gaussian FRV X according to which the underlying probability structure can be defined just only on $\mathbb{R}^d$ and no longer on $\mathbb{F}_{kc}$. Clearly, such loss of complexity occurs in the more general case of an FRV X that is a random translation of a deterministic fuzzy set M; i.e. $X = M \oplus \mathbb{I}_\xi$ with ξ being a random element on $\mathbb{R}^d$. In this paper we shall provide a characterization for random translations by means of a suitable decomposition theorem that holds for any FRV. In particular, given a centered FRV X, we shall define the family Θ_X of all deterministic $B \in \mathbb{F}_{kc}$ for which the Hukuhara difference $X \ominus_H B$ exists almost surely. We shall show that this set is not empty, convex and closed in $(\mathbb{F}_{kc}, d_2)$, where d_2 corresponds the L^2 metric in the space of support functions. Further,

$$C = \underset{U \in \Theta_X}{\arg\min}\, \mathbb{E}(d_2(X, U)^2)$$

is unique and there exists an FRV Y such that $X(\omega) = C \oplus Y(\omega)$; in some sense, C and Y are the deterministic part (with respect to $\oplus$) and the random part of X respectively.
Noting that, the Aumann expectation $\mathbb{E}X$ is the (unique) Frèchet expectation with respect to d_2, i.e.

$$\mathbb{E}X = \underset{U \in \mathbb{F}_{kc}}{\arg\min}\, \mathbb{E}(d_2(X, U)^2),$$

we obtain immediately the characterization announced above: an FRV X is a random translation of C (i.e. $Y(\omega)$ is almost surely a singleton) if and only if $\mathbb{E}X$ is equal to C. We want to point out how Frèchet expectation is often presented as the property of the mean value for which $\mathbb{E}X$ minimizes the variance of X.

The paper is organized as follow. Section 1 introduces necessary notations and literature results. Section 2 studies properties of the Hukuhara set Θ_X whilst Section 3 presents the decomposition theorem for FRV and a characterization of FRV translation.

1 Preliminaries

Here we refer mainly to [5]. Denote by $\mathbb{K}_{kc}$ the class of non–empty compact convex subsets of $\mathbb{R}^d$, endowed with the Hausdorff metric

$$\delta_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\},$$

and the operations

$$A+B = \{a+b : a \in A,\ b \in B\}, \qquad \lambda \cdot A = \lambda A = \{\lambda a : a \in A\} \ \text{ with } \lambda > 0.$$

For a non–empty closed convex set $A \subset \mathbb{R}^d$ the *support function* $s_A : S^{d-1} \to \mathbb{R}$ is defined by

$$s_A(x) = \sup\{\langle x, a\rangle : a \in A\}, \qquad \text{for } x \in S^{d-1},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$ and $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^d$. The *Steiner point* of $A \in \mathbb{K}_{kc}$ is defined by

$$\mathbf{ste}(A) = \frac{1}{v_d} \int_{S^{d-1}} s_A(x)\, x \ \mathrm{d}\lambda(x)$$

where $x \in S^{d-1}$ varies over the unit vectors of $\mathbb{R}^d$, λ is the Lebesgue measure on S^{d-1}, and v_d is the volume of the unit ball of $\mathbb{R}^d$.

A *fuzzy set* is a map $\nu : \mathbb{R}^d \to [0, 1]$. Let $\mathbb{F}_{kc}$ denote the family of all fuzzy sets, which satisfy the following conditions.

1. Each ν is an upper semicontinuous function, i.e. for each $\alpha \in (0, 1]$, the cut set $\nu_\alpha = \{x \in \mathbb{R}^d : \nu(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}^d$.
2. The cut set $\nu_1 = \{x \in \mathbb{R}^d : \nu(x) = 1\} \neq \emptyset$.
3. The support set $\nu_0 = \overline{\{x \in \mathbb{R}^d : \nu(x) > 0\}}$ of ν is compact; hence every ν_α is compact for $\alpha \in (0, 1]$.
4. For any $\alpha \in [0, 1]$, ν_α is a convex subset of $\mathbb{R}^d$.

For any $\nu \in \mathbb{F}_{kc}$ define the *support function* of ν as follows:

$$s_\nu(x, \alpha) = \begin{cases} s_{\nu_\alpha}(x) & \text{if } \alpha > 0, \\ s_{\nu_0}(x) & \text{if } \alpha = 0, \end{cases}$$

for $(x, \alpha) \in S^{d-1} \times [0, 1]$ and where $s_K(x) = \sup\{\langle x, a\rangle : a \in K\}$, for $x \in S^{d-1}$. Let us endow $\mathbb{F}_{kc}$ with the operations

$$(\nu^1 \oplus \nu^2)_\alpha = \nu^1_\alpha + \nu^2_\alpha, \qquad (\lambda \odot \nu^1)_\alpha = \lambda \cdot \nu^1_\alpha, \quad \text{with } \lambda > 0$$

so that $\mathbb{F}_{kc}$ is a convex cone, and with the metrics

$$\delta^\infty_H(\nu^1, \nu^2) = \sup\{\alpha \in [0, 1] : \delta_H(\nu^1_\alpha, \nu^2_\alpha)\},$$

$$d_2(\nu^1, \nu^2) = \left(\int_0^1 \int_{S^{d-1}} |s_{\nu^1}(\alpha, u) - s_{\nu^2}(\alpha, u)|^2 \ \mathrm{d}\alpha \ \mathrm{d}u \right)^{\frac{1}{2}}.$$

It is known that $(\mathbb{F}_{kc}, \delta^\infty_H)$ is a complete metric space while $(\mathbb{F}_{kc}, d_2)$ is not (cf. [3, Chapter 7]). The *generalized Steiner point* of $A \in \mathbb{F}_{kc}$ is defined by

$$\mathbf{Ste}(A) = \int_{[0,1]} \mathbf{ste}(A_\alpha) \ \mathrm{d}\alpha,$$

where $\mathrm{d}\alpha$ is the Lebesgue measure on $[0, 1]$. In other words, $\mathbf{Ste}(A)$ may be seen as a weighted average of steiner points of the level sets of A. The following properties are satisfied (cf. [8]).

1. For any $A \in \mathbb{F}_{kc}$, $\mathbf{Ste}(A) \in A_0$.

2. For any $A, B \in \mathbb{F}_{kc}$, $\mathbf{Ste}(A \oplus B) = \mathbf{Ste}(A) + \mathbf{Ste}(B)$.
3. $\mathbf{Ste} : \mathbb{F}_{kc} \to \mathbb{R}^d$ is continuous.

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space. A *fuzzy set–valued random variable* (FRV) is a function $X : \Omega \to \mathbb{F}_{kc}$, such that $X_\alpha : \omega \mapsto X(\omega)_\alpha$ are random compact convex sets for every $\alpha \in (0, 1]$ (i.e. X_α is a $\mathbb{K}_{kc}$–valued function measurable with respect to the δ_H–Borel σ–algebra). As a consequence of continuity of $\mathbf{Ste}(\cdot)$, if X is an FRV, then $\mathbf{Ste}(X)$ is a random element in $\mathbb{R}^d$.

An FRV X is *integrably bounded* and we shall write $X \in L^1[\Omega; \mathbb{F}_{kc}]$, if $\mathbb{E}(\sup_{x \in X_0} \|x\|) < +\infty$. The (Aumann) *expected value* of $X \in L^1[\Omega; \mathbb{F}_{kc}]$, denoted by $\mathbb{E}[X]$, is a fuzzy set such that, for every $\alpha \in [0, 1]$,

$$(\mathbb{E}[X])_\alpha = \int_\Omega X_\alpha \, \mathrm{d}\mu = \{\mathbb{E}(f) : f \in L^1[\Omega; \mathbb{R}^d], f \in X_\alpha \ \mu - \text{a.e.}\}.$$

It should be pointed out that, whenever $\mathbb{E}[(\sup_{x \in X_0} \|x\|)^2] < +\infty$ (we shall write $X \in L^2[\Omega; \mathbb{F}_{kc}]$), the expected value in the Aumann's sense is a Frèchet expectation with respect to d_2, i.e.

$$\mathbb{E}X = \arg\min_{U \in \mathbb{F}_{kc}} \mathbb{E}(d_2(X, U)^2),$$

see for example [7].

Embedding Theorems. It is known that the support function allows us to embed the space of fuzzy sets onto suitable Banach spaces preserving the metrics δ_H^∞ and d_2.
On the one hand, $(\mathbb{F}_{kc}, d_2)$ is trivially embeddable in the Hilbert space of square integrable functions $L^2([0, 1] \times S^{d-1})$ by means of the mapping $s : \mathbb{F}_{kc} \to L^2([0, 1] \times S^{d-1})$ where $s(\nu) = s_\nu$.
On the other hand, let $C(S^{d-1})$ denote the Banach space of all continuous functions v on S^{d-1} with respect to the norm $\|v\|_C = \sup_{x \in S^{d-1}} |v(x)|$. Let $\overline{C}([0, 1], C(S^{d-1}))$ be the set of all functions $f : [0, 1] \to C(S^{d-1})$ such that f is bounded, left continuous with respect to $\alpha \in (0, 1]$, right continuous at 0, and f has right limit for any $\alpha \in (0, 1)$. Then we have that $\overline{C}([0, 1], C(S^{d-1}))$ is a Banach space with the norm $\|f\|_{\overline{C}} = \sup_{\alpha \in [0,1]} \|f(\alpha)\|_C$, and the following embedding theorem holds.

Proposition 1 ([4] and the references therein.) *There exists a function $j : \mathbb{F}_{kc} \to \overline{C}([0, 1], C(S^{d-1}))$ such that:*

1. *$j(r\nu^1 + t\nu^2) = rj(\nu^1) + tj(\nu^2)$, $\nu^1, \nu^2 \in \mathbb{F}_{kc}$ and $r, t \geq 0$.*
2. *j is an isometric mapping, i.e.*

$$\delta_H^\infty(\nu^1, \nu^2) = \|j(\nu^1) - j(\nu^2)\|_{\overline{C}}, \quad \nu^1, \nu^2 \in \mathbb{F}_{kc},$$

3. *$j(\mathbb{F}_{kc})$ is a closed subset in $\overline{C}([0, 1], C(S^{d-1}))$.*

As a matter of fact, we can define an injection $j : \mathbb{F}_{kc} \to \overline{C}([0, 1], C(S^{d-1}))$ by $j(\nu) = s_\nu$, i.e. $j(\nu)(x, \alpha) = s_\nu(x, \alpha)$ for every $(x, \alpha) \in S^{d-1} \times [0, 1]$, and this mapping j satisfies above theorem. For simplification, let $\overline{\mathbf{C}} := \overline{C}([0, 1], C(S^{d-1}))$.

From Proposition 1 it follows that every FRV X can be regarded as a random element of $\overline{\mathbf{C}}$ by considering $j(X) = s_X : \Omega \to \overline{\mathbf{C}}$, where $s_X(\cdot)(\omega) = s_{X(\omega)}(\cdot)$. Moreover, if $X \in L^1[\Omega; \mathbb{F}_{kc}]$, for any $(x, \alpha) \in \mathbb{R}^d \times [0,1]$, $s_{X(\cdot)}(x, \alpha) \in L^1[\Omega; \mathbb{R}]$ and

$$\mathbb{E}[s_X(x,\alpha)] = s_{\mathbb{E}X}(x,\alpha). \tag{2}$$

It is known that the support function for fuzzy sets $\nu \in \mathbb{F}_{kc}$ can be defined equivalently on the closed unit ball $B(0,1) = \{x \in \mathbb{R}^d : \|x\| \leq 1\} \subset \mathbb{R}^d$ instead of the unit sphere S^{d-1} by

$$\begin{array}{rccl} s^*_\nu : & B(0,1) & \to & \mathbb{R} \\ & x & \mapsto & s^*_\nu(x) = \max\{\langle x, y\rangle : y \in \mathbb{R}^d, \nu(y) \geq \|x\|\}. \end{array}$$

In particular, the following relationship between support function definitions hold

$$\forall (x,\alpha) \in S^{d-1} \times [0,1], \qquad s_\nu(x,\alpha) = \begin{cases} s^*_\nu(\alpha x), & \text{if } \alpha \neq 0; \\ \sup_{y \in \nu_0} \langle y, x\rangle, & \text{if } \alpha = 0. \end{cases}$$

$$\forall x \in B(0,1), \qquad s^*_\nu(x) = \begin{cases} \|x\|\, s_\nu\left(\frac{x}{\|x\|}, \|x\|\right), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

In [1], the author prove that a function $f : B(0,1) \to \mathbb{R}$ is a support function of some fuzzy set $\nu \in \mathbb{F}_{kc}$ if and only if the following six properties are satisfied:

(Property.1) f is upper semicontinuous, i.e.,

$$f(x) = \limsup_{y \to x} f(y), \qquad \forall x \in B(0,1).$$

(Property.2) f is positively semihomogeneous, i.e.,

$$\lambda f(x) \leq f(\lambda x), \qquad \forall \lambda \in (0,1], \forall x \in B(0,1).$$

(Property.3) f is quasiadditive, i.e.,

$$\|x\| f\left(\lambda \frac{x}{\|x\|}\right) \leq \|x_1\| f\left(\lambda \frac{x_1}{\|x_1\|}\right) + \|x_2\| f\left(\lambda \frac{x_2}{\|x_2\|}\right),$$

for every $\lambda \in (0,1]$, and $x, x_1, x_2 \in \mathbb{R}^d \setminus \{0\}$, with $x = x_1 + x_2$.

(Property.4) f is normal, i.e.,

$$f(x) + f(-x) \geq 0, \qquad \forall x \in B(0,1).$$

(Property.5) $f(\cdot)/\|\cdot\|$ is bounded, i.e.,

$$\sup\{f(x)/\|x\| : x \in B(0,1) \setminus \{0\}\} < \infty.$$

(Property.6) $f(0) = 0$.

2 Hukuhara set

In this section we shall define the *Hukuhara set* associated to an FRV X, namely Θ_X. We shall provide some properties of Θ_X most of which will turn out to be useful in the next section where a decomposition theorem for fuzzy random variables will be set.

Let K be in $\mathbb{F}_{kc}$ such that $\mathbf{Ste}(K) = 0$ and consider

$$\theta_K = \{B \in \mathbb{F}_{kc} : \mathbf{Ste}(B) = 0 \text{ and } \exists A \in \mathbb{F}_{kc} \text{ s.t. } B \oplus A = K\};$$

i.e. the family of those centered convex compact fuzzy sets B for which the Hukuhara difference $K \ominus_H B$ does exist. Note that θ_K is not empty, since $\mathbb{I}_0, K \in \theta_K$ and $\{\lambda \odot K\}_{\lambda \in [0,1]} \subseteq \theta_K$. Clearly, if $B \in \theta_K$ and A is the Hukuhara difference between K and B, then $A \in \theta_K$.

Proposition 2 *θ_K is a closed subset in $(\mathbb{F}_{kc}, \delta_H^\infty)$.*

Proof. Let $\{B_n\} \subset \theta_K$ be a convergent sequence with limit $B \in \mathbb{F}_{kc}$ with respect to δ_H^∞, we have to prove that $B \in \theta_K$. Equivalently, we have to prove that there exists $A \in \mathbb{F}_{kc}$ such that $B \oplus A = X$. For each $n = 1, 2, \dots$ there exist $A_n \in \mathbb{F}_{kc}$ such that $B_n \oplus A_n = K$. Thus, the idea is to prove that $\{A_n\}_{n=1}^\infty$ converges, w.r.t. δ_H^∞, to some $A \in \mathbb{F}_{kc}$ such that $B \oplus A = X$. To do this, let us consider the following chains of equalities

$$\begin{aligned}
\delta_H^\infty(A_m, A_n) &= \|s_{A_m} - s_{A_n}\|_{\overline{C}} \\
&= \|(s_{A_m} + s_{B_m}) - (s_{A_n} + s_{B_n}) + s_{B_n} - s_{B_m}\|_{\overline{C}} \\
&= \|s_K - s_K + s_{B_n} - s_{B_m}\|_{\overline{C}} \\
&= \|s_{B_n} - s_{B_m}\|_{\overline{C}} = \delta_H^\infty(B_n, B_m) \to 0, \quad \text{for } n, m \to \infty
\end{aligned}$$

where we use the isometry $A \mapsto s_A$ (first and last equalities) and the fact that B_n, B_m belong to θ_K (third equality). Above limit implies that $\{A_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathbb{F}_{kc}, \delta_H^\infty)$ that is a complete metric space (e.g. [5, Theorem 5.1.6]), and then there exists A in $\mathbb{F}_{kc}$ such that $A_n \to A$. As a consequence, $B_n \oplus A_n \to B \oplus A$ for $n \to \infty$ combined with

$$0 = \delta_H^\infty(B_n \oplus A_n, X),$$

guarantees that $B \oplus A = X$ and hence $B \in \theta_X$; that is the thesis. ■

In what follows we shall need the next lemma according to which a fuzzy set can be defined starting from its α-cuts.

Lemma 3 (See [3, Proposition 6.1.7, p.39]) *If $\{C_\alpha\}_{\alpha \in [0,1]}$ satisfies*

(a) C_α is a non empty compact convex subset of $\mathbb{R}^d$, for every $\alpha \in [0, 1]$;

(b) $C_\beta \subseteq C_\alpha$ for $0 \le \alpha \le \beta \le 1$;

(c) $C_\alpha = \bigcap_{i=1}^\infty C_{\alpha_i}$ for all sequence $\{\alpha_i\}_{i \in \mathbb{R}}$ converging from above to α, i.e. $\alpha_i \uparrow \alpha$ in $[0, 1]$;

then the function

$$\nu(x) = \begin{cases} 0, & \text{if } x \notin C_0, \\ \sup\{\alpha \in [0,1] : x \in C_\alpha\}, & \text{if } x \in C_0, \end{cases}$$

is an element of $\mathbb{F}_{kc}$ *with* $\nu_\alpha = C_\alpha$ *for any* $\alpha \in (0,1]$ *and*

$$\nu_0 = \overline{\bigcup_{\alpha\in(0,1]} C_\alpha} \subseteq C_0.$$

Let X be an FRV. For the sake of simplicity and without loss of generality, let us suppose that $\mathbf{Ste}(X) = 0$; otherwise one can always considered its associated centered FRV $\widetilde{X} = X - \mathbb{I}_{\mathbf{Ste}(X)}$. Next theorem defines the *Hukuhara set* Θ_X associated to X, and provides some topological and vectorial properties of Θ_X.

Proposition 4 *The subset* $E = \{B \in \theta_X\} := \{\omega \in \Omega : B \in \theta_{X(\omega)}\}$ *is measurable in* $(\Omega, \mathfrak{F})$*. Moreover, if* $\Theta_X = \{B \in \mathbb{F}_{kc} : \mathbb{P}(B \in \theta_X) = 1\}$*, then the following statements hold.*

- **(i)** Θ_X *is non–empty.*
- **(ii)** $B \in \Theta_X$ *if and only if there exist an FRV* A *such that* $B \oplus A = X$*,* $\mathbb{P}$*–a.s.. If* $X \in L^2[\Omega; \mathbb{F}_{kc}]$*, then* A *is in* $L^2[\Omega; \mathbb{F}_{kc}]$ *too.*
- **(iii)** Θ_X *is a convex subset in* $(\mathbb{F}_{kc}, \oplus)$*. As a consequence, if* $B \in \Theta_X$*, then* $\{\lambda B\}_{\lambda\in[0,1]} \subseteq \Theta_X$*.*
- **(iv)** Θ_X *is a closed subset of* $(\mathbb{F}_{kc}, \delta_H^\infty)$*.*
- **(v)** Θ_X *is a closed subset of* $(\mathbb{F}_{kc}, d_2)$*.*

Proof of Proposition 4. Using the definition of $\theta_{X(\omega)}$ and the characterization of element in $\mathbb{F}_{kc}$ via the support functions, we get the following chains of equalities.

$$\begin{aligned}
E &= \{\omega \in \Omega : \mathbf{Ste}(B) = 0 \text{ and } \exists A_\omega \in \mathbb{F}_{kc},\ B \oplus A_\omega = X(\omega)\} \\
&= \{\omega \in \Omega : \mathbf{Ste}(B) = 0\} \cap \{\omega \in \Omega : \exists A_\omega \in \mathbb{F}_{kc},\ \text{s.t.}\quad s_B + s_{A_\omega} = s_{X(\omega)}\} \\
&= \Omega \cap \{\omega \in \Omega : f_\omega := s^*_{X(\omega)} - s^*_B \text{ is the support function of some } A_\omega \in \mathbb{F}_{kc}\} \\
&= \{\omega \in \Omega :\ f_\omega \text{ satisfies Properties 1–6}\} \\
&= E_1 \cap \ldots \cap E_6,
\end{aligned}$$

where $E_i = \{\omega \in \Omega : f_\omega \text{ satisfies Property } i\}$ for $i = 1, \ldots, 6$. If $E_1, \ldots, E_6$ are measurable events, then E is measurable too. To show this note that each E_i $(i = 1, \ldots, 6)$ can be written as $E_i = \{\omega : g_i(\omega) \leq 0\}$ where

$$\begin{aligned}
g_1 &= \sup\{|\limsup_{y\to x} f_\omega(y) - f_\omega(x)| : x \in B(0,1)\}, \\
g_2 &= \sup\{\lambda f_\omega(x) - f_\omega(\lambda x) : \lambda \in (0,1], x \in B(0,1)\}, \\
g_3 &= \sup\Big\{\|x\| f_\omega\left(\lambda\frac{x}{\|x\|}\right) - \|x_1\| f_\omega\left(\lambda\frac{x_1}{\|x_1\|}\right) - \|x_2\| f_\omega\left(\lambda\frac{x_2}{\|x_2\|}\right) \\
&\qquad\qquad : \lambda \in (0,1], x, x_1, x_2 \in \mathbb{R}^d \setminus \{0\},\ \text{with } x = x_1 + x_2\Big\}, \\
g_4 &= -\sup\{f_\omega(x) + f_\omega(-x) : x \in B(0,1)\}, \\
g_5 &= \sup\left\{\frac{|f_\omega(x)|}{\|x\|} : x \in B(0,1) \setminus \{0\}\right\}, \\
g_6 &= |f_\omega(0)|.
\end{aligned}$$

Clearly $\omega \mapsto g_i(\omega)$ are measurable maps and hence E is a measurable event in the σ–algebra $\mathfrak{F}$.

ITEM (i). Surely $\mathbb{I}_0$ belongs to Θ_X, hence Θ_X is not empty.

ITEM (ii). The sufficiency is trivial, let us prove the necessity. Let $M := E^c = \{\omega \in \Omega : B \notin \theta_{X(\omega)}\}$, by hypothesis $\mathbb{P}(M) = 0$. For every $\omega \in \Omega \setminus M$, there exists $A_\omega \in \mathbb{F}_{kc}$ such that $B \oplus A_\omega = X(\omega)$. Let us consider the map

$$\begin{array}{rccl} A: & \Omega & \to & \mathbb{F}_{kc} \\ & \omega & \mapsto & A(\omega) = \left\{ \begin{array}{ll} A_\omega, & \omega \in \Omega \setminus M, \\ \mathbb{I}_0, & \omega \in M. \end{array} \right. \end{array}$$

Since $s_A = s_X - s_B$ a.s., s_A is measurable. Hence, the map A defined above, is the FRV we are looking for. With the same arguments one shows that $A \in L^2[\Omega; \mathbb{F}_{kc}]$ if X is so.

ITEM (iii). Consider $B_1, B_2 \in \Theta_X$. From above part we know that there exist two FRV A_1, A_2 with values in $\mathbb{F}_{kc}$ such that $B_1 \oplus A_1 = X$ and $B_2 \oplus A_2 = X$. For any $\lambda \in [0, 1]$, the following hold

$$\lambda(B_1 \oplus A_1) = \lambda X, \qquad (1-\lambda)(B_2 \oplus A_2) = (1-\lambda)X,$$

from which we get

$$\lambda B_1 \oplus (1-\lambda)B_2 \oplus A = X,$$

with $A = \lambda A_1 \oplus (1-\lambda)A_2$. Hence $\lambda B_1 \oplus (1-\lambda)B_2 \in \Theta_X$.
To prove the last part consider $B \in \Theta_X$, then $\lambda B = \lambda B \oplus (1-\lambda)\mathbb{I}_0 \in \Theta_X$.

ITEM (iv). Consider a sequence $\{B_n\}_{n=1}^\infty \subset \Theta_X$ converging to $B \in \mathbb{F}_{kc}$ in $(\mathbb{F}_{kc}, \delta_H^\infty)$, i.e.

$$\delta_H^\infty(B, B_n) \to 0, \text{ as } n \to \infty.$$

We have to prove that $B \in \Theta_X$. Using the same arguments in (ii), for any $n \in \mathbb{N}$ and for every $\omega \in \Omega \setminus M_n = \{\omega \in \Omega : B_n \notin \theta_{X(\omega)}\}$, there exist $A_{\omega,n}$ such that $B_n \oplus A_{\omega,n} = X(\omega)$ and

$$\delta_H^\infty(A_{\omega,m}, A_{\omega,n}) = \delta_H^\infty(B_m, B_n) \to 0, \text{ as } n \to \infty.$$

Thus, the completeness of $(\mathbb{F}_{kc}, \delta_H^\infty)$ guarantees that, for every $\omega \in \Omega \setminus \bigcup_n M_n$, $\{A_{\omega,n}\}_{n \in \mathbb{N}}$ converges w.r.t. δ_H^∞ to some $A_\omega \in \mathbb{F}_{kc}$. Further, for every $\omega \in \Omega \setminus M$ and $n \in \mathbb{N}$ the following inequalities hold

$$\begin{aligned} 0 \le \delta_H^\infty(X(\omega), B \oplus A_\omega) &\le \delta_H^\infty(X(\omega), B_n \oplus A_{\omega,n}) + \delta_H^\infty(B_n \oplus A_{\omega,n}, B \oplus A_\omega) \\ &\le 0 + \delta_H^\infty(B_n, B) + \delta_H^\infty(A_{\omega,n}, A_\omega) \to 0 \end{aligned}$$

where, for the first addend, we use the fact that $X(\omega) = B_n \oplus A_{\omega,n}$. Thus we get the thesis.

ITEM (v). Let us consider a sequence $\{B_n\}_{n=1}^\infty \subset \Theta_X$ converging to $B \in \mathbb{F}_{kc}$ in $(\mathbb{F}_{kc}, d_2)$, i.e.

$$d_2(B, B_n) \to 0, \text{ as } n \to \infty.$$

We have to prove that $B \in \Theta_X$. In this case, $(\mathbb{F}_{kc}, d_2)$ is not complete and, hence, we can not repeat all arguments in (iv). In particular, for any

$n \in \mathbb{N}$ and for every $\omega \in \Omega \setminus M_n = \{\omega \in \Omega : B_n \notin \theta_{X(\omega)}\}$, there exist $A_{\omega,n}$ such that $B_n \oplus A_{\omega,n} = X(\omega)$ and, using the same arguments in (ii),

$$d_2(A_{\omega,m}, A_{\omega,n}) = \left(\int_0^1 \int_{S^{d-1}} |s_{A_{\omega,m}}(\alpha, u) - s_{A_{\omega,n}}(\alpha, u)|^2 \, \mathrm{d}\alpha \, \mathrm{d}u\right)^{\frac{1}{2}} = d_2(B_m, B_n) \to 0,$$

as $n \to \infty$ and where $\mathrm{d}\alpha$ and $\mathrm{d}u$ denote the Lebesgue measure on $[0,1]$ and the normalized Lebesgue measure on S^{d-1} respectively. Thus $\{s_{A_{\omega,n}}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space $L^2(\overline{C})$ and it admits limit in $L^2(\overline{C})$, namely f_ω. Since

$$\|s_{A_{\omega,n}} - (s_{X(\omega)} - s_B)\|_{L^2} = \|(s_{A_{\omega,n}} - s_{X(\omega)}) + s_B\|_{L^2} = \|s_B - s_{B_n}\|_{L^2} \to 0,$$

it holds

$$s_{A_{\omega,n}} \xrightarrow{L^2} f_\omega = s_{X(\omega)} - s_B.$$

Note that, f_ω is not necessarily the support function of some element in $\mathbb{F}_{kc}$. In other words, $\{A_{\omega,n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non–complete space $(\mathbb{F}_{kc}, d_2)$, but under the embedding j (cf. Proposition 1) we have that the sequence $\{j(A_{\omega,n})\}_{n\in\mathbb{N}} = \{s_{A_{\omega,n}}\}_{n\in\mathbb{N}}$ is a Cauchy sequence that admits limit in the Hilbert space $L^2(\overline{C})$. But, in general, this limit is not the image under j of some element of $\mathbb{F}_{kc}$. We claim, and we shall prove in what follows, that there exists $A_\omega \in \mathbb{F}_{kc}$ such that $s_{A_\omega} = f_\omega = s_{X(\omega)} - s_B$, i.e. $B \oplus A_\omega = X(\omega)$ that is the thesis.

In fact, let us consider the family $\{C_\alpha\}_{\alpha\in[0,1]}$ of subsets of $\mathbb{R}^d$ defined by

$$C_\alpha = \{y \in \mathbb{R}^d : \langle y, u\rangle \le f_\omega(\alpha, u), \forall u \in S^{d-1}\}, \quad \alpha \in [0,1].$$

In what follows, we shall prove that the family $\{C_\alpha\}_{\alpha\in[0,1]}$ satisfies (a), (b), (c) from Lemma 3, and it defines uniquely a fuzzy set ν whose support function is, clearly, f_ω. Thus the fuzzy set ν defined in Lemma 3 is just the A_ω in $\mathbb{F}_{kc}$ we are looking for.

(a). Let $\alpha \in [0,1]$.

C_α *is non–empty*: since $B_\alpha \subseteq (X(\omega))_\alpha$, then for every $u \in S^{d-1}$

$$f_\omega(\alpha, u) = s_{X(\omega)}(\alpha, u) - s_B(\alpha, u) \ge 0 = \langle 0, u\rangle, \tag{3}$$

i.e. $0 \in C_\alpha$.

C_α *is convex*: let $\lambda \in [0,1]$ and $y_1, y_2 \in C_\alpha$, for every $u \in S^{d-1}$

$$\langle \lambda y_1 + (1-\lambda)y_2, u\rangle \le \lambda f_\omega(\alpha, u) + (1-\lambda) f_\omega(\alpha, u) = f_\omega(\alpha, u)$$

i.e. $\lambda y_1 + (1-\lambda)y_2 \in C_\alpha$.

C_α *is compact*: we have to prove that it is a bounded closed subset of $\mathbb{R}^d$. Note that $\{0\} \subseteq B_\alpha \subseteq (X(\omega))_\alpha$, then $s_{X(\omega)}(\alpha, u) \ge s_B(\alpha, u) \ge 0$ for each $u \in S^{d-1}$ and $s_{X(\omega)}(\alpha, u) \ge s_{X(\omega)}(\alpha, u) - s_B(\alpha, u) = f_\omega(\alpha, u)$. This implies that $\langle y, u\rangle$ is bounded for every $u \in S^{d-1}$ and hence that $C_\alpha \subseteq \mathbb{R}^d$ is bounded. On the other hand, let $\{y_n\} \subset C_\alpha$ be convergent to $y \in \mathbb{R}^d$, then, for every $n \in \mathbb{N}$ and $u \in S^{d-1}$,

$$\langle y_n, u\rangle \le f_\omega(\alpha, u),$$

and passing to the limit we obtain the same inequality for y and for every $u \in S^{d-1}$; i.e. $y \in C_\alpha$. This fact allows us to conclude that C_α is closed

and hence compact.
(b). Let $0 \leq \alpha \leq \beta \leq 1$. Note that, for every $n \in \mathbb{N}$ and $u \in S^{d-1}$, $s_{A_{n,\omega}}(\beta, u) \leq s_{A_{n,\omega}}(\alpha, u)$. Let $n \to \infty$, then $f_\omega(\beta, u) \leq f_\omega(\alpha, u)$ for every $u \in S^{d-1}$; i.e., for every $u \in S^{d-1}$ and $n \in \mathbb{N}$, $s_{A_{n,\omega}}$ and f_ω are non–increasing functions with respect to α. Now, let us consider $y \in C_\beta$, then for every $u \in S^{d-1}$, $\langle y, u\rangle \leq f_\omega(\beta, u) \leq f_\omega(\alpha, u)$; i.e. $y \in C_\alpha$ and $C_\beta \subseteq C_\alpha$.
(c). Let $\{\alpha_i\}_{i\in\mathbb{N}} \subset [0, 1]$ such that $\alpha_i \uparrow \alpha$ as i tends to infinity, that is $\alpha_i \leq \alpha_{i+1}$ and $\alpha_i \to \alpha$ as $i \to \infty$. Because of $\alpha_i \leq \alpha$ and (b), we have $C_\alpha \subseteq C_{\alpha_i}$ and $C_\alpha \subseteq \bigcap_{i\in\mathbb{N}} C_{\alpha_i}$. It remains to show the opposite inclusion. To do this let $y \in \bigcap_{i\in\mathbb{N}} C_{\alpha_i}$, i.e. $y \in C_{\alpha_i}$ for all $i \in \mathbb{N}$ or

$$\langle y, u\rangle \leq f_\omega(\alpha_i, u), \qquad \text{for every } i \in \mathbb{N}, u \in S^{d-1}. \tag{4}$$

Note that, for every $u \in S^{d-1}$, $f_\omega(\cdot, u)$ is left–continuous with respect to α because it is the difference of two left–continuous functions (cf. Equation (3)). Hence, for the arbitrariness of i in (4), we can pass to the limit as i tends to infinity obtaining $\langle y, u\rangle \leq f_\omega(\alpha, u)$; i.e. $y \in C_\alpha$. ■

3 Hukuhara decomposition

Let us recall again the wide–known decomposition for Gaussian FRV X

$$X = \mathbb{E}X \oplus \mathbb{I}_\xi \tag{1}$$

where $\mathbb{E}X$ is the Aumann expectation of X, and ξ is a Gaussian random element in $\mathbb{R}^d$ with $\mathbb{E}\xi = 0$. In some sense, Equation (1) reduces the complexity of FRV X that is equal to its expected value $\mathbb{E}X$ (a deterministic fuzzy set) up to a random Gaussian translation ξ. In [2], the author showed another case of "reduction of complexity"; a Brownian fuzzy set–valued process is reduced to be a Brownian process in $\mathbb{R}^d$. In both cases, the randomness defined on $\mathbb{F}$ can be simpler defined on $\mathbb{R}^d$.
Now, our question becomes the following one. Under what conditions we can establish that a "reduction of complexity", like the former, occurs for fuzzy set–valued random process. In other words: *Can a fuzzy process, whose randomness is given only by vectors, be characterized in some way?* In this section we propose a positive answer to the above question and, to do this, we shall focus mainly on a decomposition theorem for FRV. In particular, we shall prove that such a X can be decomposed as the sum of a deterministic convex fuzzy set and a random FRV in a unique way. This decomposition allows us to characterize, by means of the Aumann expected value, the FRV that is a random translation of a deterministic fuzzy set.

Definition 5 An FRV X is a *translation* if there exists $M \in \mathbb{F}_{kc}$ with $\mathbf{Ste}(M) = 0$ such that

$$X(\omega) = M \oplus \mathbb{I}_{\mathbf{Ste}(X)}.$$

In other words, the randomness of a translation X depends only on the specific location in space and does not depend on its shape. Note that, according to (1), every Gaussian FRV X is an FRV translation with $M \oplus \mathbb{I}_{\mathbb{E}(\mathbf{Ste}(X))} = \mathbb{E}X$. Another sufficient condition for X to be an FRV translation is given by Proposition 6, while a necessary and sufficient condition will be given in Theorem 9. The latest is based upon the decomposition established in Theorem 7. Note that Theorem 7 is interesting by itself since allows us to decompose any FRV X as the sum of a deterministic fuzzy set $H_X^{\perp}$ and an FRV Y that represents the randomness part of X.

Proposition 6 *Let X be an FRV such that $\mathbb{E}X = \mathbb{I}_c$ where $c \in \mathbb{R}^d$. Then $X = \mathbb{I}_\xi$ $\mathbb{P}$–a.s. for some random element ξ in $\mathbb{R}^d$. (Clearly X is an FRV translation.)*

Proof. Thesis can be reached using similar arguments in [2, Theorem 8], or, whenever $X \in L^2[\Omega; \mathbb{F}_{kc}]$, as corollary of the next Theorem 7 and Theorem 9. ■

Clearly, the vice versa of Proposition 6 does not hold, for example in the case of Gaussian FRV. In order to characterize translation FRV, we need the following decomposition theorem.

Theorem 7 *Let $X \in L^2[\Omega; \mathbb{F}_{kc}]$ with $\mathbf{Ste}(X) = 0$. Thus there exists $H_X^{\perp} \in \mathbb{F}_{kc}$ with $\mathbf{Ste}(H_X^{\perp}) = 0$ and $Y \in L^2[\Omega; \mathbb{F}_{kc}]$ such that X decomposes according to*

$$X(\omega) = H_X^{\perp} \oplus Y(\omega), \tag{5}$$

for $\mathbb{P}$–almost all $\omega \in \Omega$. In particular, $H_X^{\perp}$ is the unique element in $\mathbb{F}_{kc}$ that satisfies (5) and minimizes $\mathbb{E}[(d_2(X,C))^2]$; i.e., there exists a unique $H_X^{\perp} \in \Theta_X$ such that

$$H_X^{\perp} := \underset{B \in \Theta_X}{arg\,min}\, \mathbb{E}[(d_2(X,B))^2]. \tag{6}$$

Hence Y is the unique (except on a $\mathbb{P}$–negligible set) FRV such that its support function is given by $s_Y = s_X - s_{H_X^{\perp}}$. Moreover, $H_X^{\perp}$ is a maximal element in Θ_X with respect to the level–wise set inclusion; that is, if $C \in \Theta_X$ with $(H_X^{\perp})_\alpha \subseteq C_\alpha$ for any $\alpha \in [0,1]$, then $H_X^{\perp} = C$.

Proof. Since Θ_X collects all the element of $\mathbb{F}_{kc}$ for which (5) holds, we have to prove that there exists a unique element in Θ_X that minimizes the map $B \in \Theta_X \to \mathbb{E}[(d_2(X,B))^2]$.

At first note that Θ_X can be seen as a subset of $L^2[\Omega; \mathbb{F}_{kc}]$; in fact, for each $B \in \Theta_X$ the constant map $\omega \mapsto B$ is an element of $L^2[\Omega; \mathbb{F}_{kc}]$ since

$$\mathbb{E}[(\sup_{b \in B_0} \|b\|)^2] = (\sup_{b \in B_0} \|b\|)^2 < +\infty.$$

Moreover, Θ_X is closed in $L^2[\Omega; \mathbb{F}_{kc}]$ as a consequence of

$$\mathbb{E}[(d_2(A,B))^2] = (d_2(A,B))^2,$$

for any couples $A, B \in \mathbb{F}_{kc}$, and thanks to the fact that Θ_X is closed in $(\mathbb{F}_{kc}, d_2)$, see Proposition 4.

Thus the minimization problem is equivalent to prove that there exists a unique projection of X onto Θ_X that is a closed convex subset of $L^2[\Omega; \mathbb{F}_{kc}]$ endowed with the metric $\Delta_2(\cdot,\cdot) := \mathbb{E}[(d_2(\cdot,\cdot))^2]$; hence, there exists a

unique element $H_X^\perp \in \Theta_X$ that realizes the required minimum (6).
As a consequence of $H_X^\perp \in \Theta_X$ and of (ii) in Proposition 4, the FRV Y is defined through its support function $s_Y = s_X - s_{H_X^\perp}$.
Finally, let C be as in the thesis; thus inclusions $(H_X^\perp)_\alpha \subseteq C_\alpha \subseteq X_\alpha$ imply $s_X - s_C \leq s_X - s_{H_X^\perp}$. Then, by definition of $H_X^\perp$ and d_2, necessarily $C = H_X^\perp$ holds. ■
The chosen notation wants to recall that, as the proof showed, $H_X^\perp$ is obtained through a projection theorem of the selected FRV X on its Hukuhara set Θ_X. Further, we want to stress out that the suffix X does not mean that $H_X^\perp$ is random. In fact, it does not depend on ω but rather it is a deterministic element of $\mathbb{F}_{kc}$ (that is a constant element in $L^2[\Omega; \mathbb{F}_{kc}]$) that depends on the whole map $\omega \mapsto X(\omega)$.

The following theorems provide necessary and sufficient condition for an FRV to be a translation.

Theorem 8 *Let X be an FRV translation, and $\widetilde{X} = X \oplus \mathbb{I}_{-\mathbf{Ste}(X)}$. Then*

$$X = H_{\widetilde{X}}^\perp \oplus \mathbb{I}_{\mathbf{Ste}(X)}, \qquad \mathbb{P} - a.s. \tag{7}$$

Proof. By hypothesis $X = M \oplus \mathbb{I}_{\mathbf{Ste}(X)}$ for some $M \in \mathbb{F}_{kc}$ with $\mathbf{Ste}(M) = 0$. Clearly, $\widetilde{X} = X \oplus \mathbb{I}_{-\xi} = M$, $M \in \Theta_{\widetilde{X}}$ and $\mathbb{E}[(d_2(M, \widetilde{X})^2)] = 0$; that is, $M = H_{\widetilde{X}}^\perp$. ■

Theorem 9 *Let $X \in L^2[\Omega; \mathbb{F}_{kc}]$. X is an FRV translation if and only if $H_{\widetilde{X}}^\perp$ satisfies*

$$\mathbb{E}X = H_{\widetilde{X}}^\perp \oplus \mathbb{I}_{\mathbb{E}(\mathbf{Ste}(X))} \tag{8}$$

with $\mathbb{E}X$ being the Aumann expectation; i.e. $H_{\widetilde{X}}^\perp$ is $\mathbb{E}X$ up to a translation.

Proof. For the "only if"part it is sufficient to compute the expectation in Equation (7) to get Equation (8).
Consider the "if"part. For the sake of simplicity, let us assume that $\mathbf{Ste}(X) = 0$, a straightforward argument extend the result in the more general case of an FRV with non–void $\mathbf{Ste}(X)$. Then, in term of support functions, Equation (5) becomes

$$s_X = s_{H_X^\perp} + s_Y = s_{\mathbb{E}X} + s_Y,$$

where we use the fact that $H_X^\perp = \mathbb{E}X$. Computing expectation of both sides and using relation (2), we get that $s_Y = 0$. Hence $Y = \mathbb{I}_\xi$ a.s. for some random element ξ in $\mathbb{R}^d$ (cf. [2]). ■

Remark 10 In view of Theorem 7 and Theorem 9, if $X \in L^2[\Omega; \mathbb{F}_{kc}]$, then Proposition 6 holds. In fact, suppose that $\mathbb{E}X = \mathbb{I}_c$ for some $c \in \mathbb{R}^d$, and compute expectation of both sides in Equation (5)

$$\mathbb{I}_c = \mathbb{E}X = H_X^\perp \oplus \mathbb{E}Y.$$

Hence, for any $\alpha \in [0, 1]$, $(H_X^\perp)_\alpha$ is a subset of $\{c\}$ up to a translation, that is $(H_X^\perp)_\alpha$ is a singleton as well as $(\mathbb{E}Y)_\alpha$. Then $H_X^\perp = \mathbb{I}_{c'}$ for some $c' \in \mathbb{R}^d$, i.e. $H_X^\perp$ is equal to $\mathbb{E}X$ up to a translation and, by Theorem 9,

X is an FRV translation that implies $Y = \mathbb{I}_\xi$ for some random element in $\mathbb{R}^d$. Finally, rewriting Equation (5) in this particular case, we obtain

$$X = H_X^\perp \oplus Y = \mathbb{I}_{c'} \oplus \mathbb{I}_\xi = \mathbb{I}_{\xi'},$$

that is the thesis of Proposition 6. $\square$

Moreover, the following results hold.

Corollary 11 *Let $X \in L^2[\Omega; \mathbb{F}_{kc}]$ with $\mathbf{Ste}(X) = 0$ and $\mathbb{E}X = H_X^\perp$. Thus X is almost surely deterministic and equal to $H_X^\perp$.*

Corollary 12 *Let $X \in L^2[\Omega; \mathbb{F}_{kc}]$, $D \in \mathbb{F}_{kc}$ and $X' = X \oplus D$ with $\mathbf{Ste}(X) = \mathbf{Ste}(D) = 0$ (hence $\mathbf{Ste}(X')$ is the origin too). Then $H_{X'}^\perp = H_X^\perp \oplus D$.*

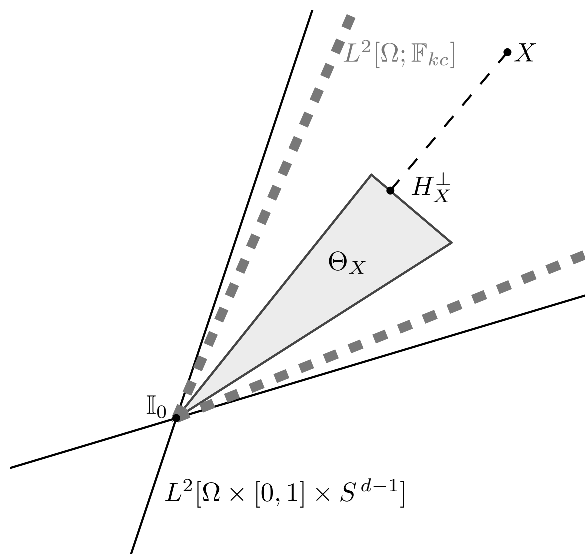

Figure 1: A qualitative graphical interpretation of some results of Section 2 and Section 3. In particular, Θ_X is represented as a closed convex subset of $\mathbb{F}_{kc}$ containing the origin and such that, for any $B \in \Theta_X$ and $\lambda \in [0,1]$, $\lambda B \in \Theta_X$. Hence, $H_X^\perp$ is the projection of X on Θ_X, as a subset of $L^2[\Omega; \mathbb{F}_{kc}]$, with respect to the metric $\mathbb{E}[d_2(\cdot,\cdot)^2]$. Moreover, here we stress out the uniqueness of this projection because cone $L^2[\Omega; \mathbb{F}_{kc}]$ is embeddable in the Hilbert space $L^2[\Omega \times [0,1] \times S^{d-1}]$ through the map $X \mapsto j(X)$ that preserves the metric. Finally, the following inclusions/embeddings are qualitatively represented: $\Theta_X \subseteq \mathbb{F}_{kc} \hookrightarrow L^2[\Omega; \mathbb{F}_{kc}] \hookrightarrow L^2[\Omega \times [0,1] \times S^{d-1}]$.

Remark 13 shows an example of an X in $L^2[\Omega; \mathbb{F}_{kc}]$ with $\mathbf{Ste}(X) = 0$ for which $\mathbb{E}(X) \neq H_X^\perp$ and for which $H_X^\perp$ is not necessarily $\mathbb{I}_0$; i.e., in terms of Theorem 9, X is not a translation but its deterministic part $H_X^\perp$ in the decomposition (5) is not just reduced to the origin.

Remark 13 Let $\mathbb{R}^d = \mathbb{R}$, $(\Omega = [0,1], \mathcal{B}_{[0,1]}, \mu)$ where μ is the Lebesgue measure and X be the FRV defined by $X := \mathbb{I}_{[\omega,\omega]}$, for any $\omega \in [0,1]$. Clearly $X \in L^2[\Omega; \mathbb{F}_{kc}]$ and $\mathbf{Ste}(X) = 0$. Moreover,

$$f_m(\omega) := \min X_1(\omega) = -\omega \quad \text{ and } \quad f_M(\omega) := \max X_1(\omega) = \omega$$

are integrable selections of the 1–level RaCS X_1; any other integrable selection f of X_1 satisfies

$$f_m(\omega) \leq f(\omega) \leq f_M(\omega), \quad \text{ for each } \omega \in [0,1].$$

Then

$$-\frac{1}{2} = \mathbb{E}f_m \leq \mathbb{E}f \leq \mathbb{E}f_M = \frac{1}{2},$$

and, by the convexity of Aumann expectation and because $X_1 = X_\alpha$ for any $\alpha \in [0,1]$, $\mathbb{E}X_1 = [-\frac{1}{2}, \frac{1}{2}] = \mathbb{E}X_\alpha$, that is $\mathbb{E}X = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}$.
We shall prove that $\mathbb{E}X \notin \Theta_X$ and hence, by Theorem 8, X is not an FRV translation. In fact, note that

$$X \ominus_H \mathbb{E}X = \mathbb{I}_{[-\omega,\omega]} \ominus_H \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]} = \begin{cases} \mathbb{I}_{[-\omega+\frac{1}{2},\omega-\frac{1}{2}]}, & \omega > \frac{1}{2}, \\ \mathbb{I}_0, & \omega = \frac{1}{2}, \\ \text{it does not exist}, & \omega < \frac{1}{2}, \end{cases}$$

implies

$$\mathbb{P}(\mathbb{E}X \in \theta_X) = \mathbb{P}(\text{there exists } X \ominus_H \mathbb{E}X) = \mathbb{P}\left(\omega > \frac{1}{2}\right) = \frac{1}{2},$$

and hence $\mathbb{E}X \notin \Theta_X$.
Actually we can show that $\Theta_X = \{\mathbb{I}_0\}$ and hence $H_X^\perp = \mathbb{I}_0$. In fact, by absurd let $B \in \Theta_X$ with $B \neq \mathbb{I}_0$, then there exists $\alpha \in [0,1]$ such that $B_\alpha = [a,b]$ with $a < b$ and there exists $X_\alpha \ominus_H B_\alpha$, here $\ominus_H$ is considered as the Hukuhara difference for subsets in $\mathbb{R}$. On the other hand

$$[-\omega,\omega] \ominus_H [a,b] = \begin{cases} [-\omega - a, \omega - b], & \omega - b > -\omega - a, \\ \{-\frac{b+a}{2}\}, & \omega = \frac{b-a}{2}, \\ \text{it does not exist}, & \omega < \frac{b-a}{2}, \end{cases}$$

and, as consequence,

$$\mathbb{P}([-\omega,\omega] \ominus_H [a,b] \text{ does not exist}) = \mu\left[\left(-\infty, \frac{b-a}{2}\right) \cap [0,1]\right] > 0$$

where the last inequality is due to the fact that, by hypothesis, $b - a > 0$. This is an absurd since $B \in \Theta_X$ by hypothesis. Thus $\Theta_X = \{\mathbb{I}_0\} \neq \mathbb{E}X = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}$.
Finally, in order to produce a more general example, let us consider

$$X = \mathbb{I}_{[-\omega,\omega]} \oplus \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]} = \mathbb{I}_{[-\omega-\frac{1}{2},\omega+\frac{1}{2}]}$$

so that, from Corollary 12, we immediately obtain that

$$\mathbb{I}_{[-1,1]} = \mathbb{E}X \neq H_X^\perp = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]} \neq \mathbb{I}_0.$$

□

4 Conclusion

In this paper, we have proven that any square integrable FRV can be decomposed as $X = H_X^{\perp} \oplus Y$, where $H_X^{\perp}$ is a unique deterministic fuzzy convex compact set (i.e. in $\mathbb{F}_{kc}$) and Y is an element of $L^2[\Omega; \mathbb{F}_{kc}]$. This decomposition leads us to characterize FRV translations for which $H_X^{\perp} = \mathbb{E}X$, where the expectation is in the Aumann sense.
This fact is important, for example, in view of Proposition 6 that allows us to simplify the complexity of an FRV process $\{X_t\}_{t\geq 0}$ for which the following $\mathbb{E}X_t = \mathbb{I}_c$ holds at any time t. In fact, since X_t is a translation at each t, it can be interpreted simpler as a random element on $\mathbb{R}^d$.
In general, working with a centered X in $L^2[\Omega; \mathbb{F}_{kc}]$ one may distinguish different cases:

- the trivial one, for which $\mathbb{E}X = H_X^{\perp}$ and hence X is a translation.
- the case for which $\mathbb{E}X \notin \Theta_X$ but $H_X^{\perp} = \mathbb{I}_0$; i.e. X is a "pure"or "reduced"FRV.
- the case for which $\mathbb{E}X \notin \Theta_X$ and $H_X^{\perp} \neq \mathbb{I}_0$. In this case, one can take advantages from decomposition $H_X^{\perp} \oplus Y$ splitting the deterministic case from the random one.

The procedure to construct $H_X^{\perp}$ starting from a squared integrable FRV X, can be viewed as a particular case of a general situation described, for example, in [6, p.174–175]. There the author plainly illustrates a "linearisation approach"in order to define expectation for random sets. The scheme can be easily extended to the FRV case. In particular, the computation of the expectation of a random element X is associated to a minimization problem (like in the case of Fréchet expectation). Some strictly related troubles may arise from this kind minimization problem: even though a solution to the minimization problem exists (some imposed constraints may cause to minimize over an empty set) it may be not unique or, there exists a solution in a larger space (than the one where X lies), but this solution does not correspond to any element in the work–space. Note that, in the present paper, we implicity or explicitly encountered and solved the same problems (existence, uniqueness and "identity"). We get over all these hitches thanks to the closeness and convexity properties of Θ_X as a subset of $L^2[\Omega; \mathbb{F}_{kc}] \hookrightarrow L^2[\Omega \times [0,1] \times S^{d-1}]$.
In spite of the fact that $H_X^{\perp}$ is defined, according to the above scheme, as an expected value of the square integrable FRV X, it is simple to show that it does not satisfy different basic properties of a "reasonable"expectation, cf. [6, p.190].

We would like to do another remark; decomposition theorem proposed in Section 3 could not be compared with a regression problem as stated, for example in [9]. In that paper, the authors look for the best linear approximation function of a given square integrable FRV Y by another square integrable FRV X, studying the minimization problem

$$\inf_{a\in\mathbb{R}, B\in\mathbb{F}_{kc}} \mathbb{E}[d_2(Y, aX \oplus B)^2].$$

On the other hand, since the regression problem is a linearisation problem, one can think to approach the same problem using the "Hukuhara reduced"FRV $Y \ominus_H H_Y^\perp$ and $X \ominus_H H_X^\perp$ instead of the original centered ($\mathbf{Ste}Y = \mathbf{Ste}X = 0$) square integrable FRV Y and X, or, equivalently to suppose that $H_X^\perp = H_Y^\perp = \mathbb{I}_0$.

Future works may consider the possibility to relax some hypothesis; for example, replacing $\mathbb{R}^d$ with an Hilbert or a Banach space (problems may arise considering the embedding j and hence the closure of the Hukuhara set Θ_X), or dropping convexity hypothesis and, hence, stating a decomposition theorem for a random element of $\mathbb{F}_k$ and not of $\mathbb{F}_{kc}$. Finally, note that we restricted our studies to the well–posedness of a such $H_X^\perp$ and it can be interesting to compute explicitly this fuzzy set, though even in particular cases.

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