IDENTIFYING A SPACE DEPENDENT COEFFICIENT IN A REACTION-DIFFUSION EQUATION

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ABSTRACT. We consider a reaction-diffusion equation for the front motion u in which the reaction term is given by $c(x)g(u)$. We formulate a suitable inverse problem for the unknowns u and c, where u satisfies homogeneous Neumann boundary conditions and the additional condition is of integral type on the time interval $[0, T]$. Uniqueness of the solution is proved in the case of a linear g. Assuming g non linear, we show uniqueness for large T.

1. Introduction and formulation of the problem

Front propagation phenomena described by reaction-diffusion equations can be conveniently applied in many areas of sciences such as physics, biology, ecology and chemistry. According to the model, the reaction term can assume different forms. In particular, if we consider the front propagation in heterogeneous media, the reaction term may depend explicitly on the space variables. In this framework, an important case for applications is analysed in [11] and [15] where the authors consider the following nonlinear reaction-diffusion equation

$$u_t - D\Delta u + c(x)g(u) = 0. \tag{1.1}$$

Here D is the diffusion coefficient, c measures the reaction rate and the function g depends only on the state variable u, i.e., the front motion.

It is well known that the evolution of u depends on the interplay between D and c, even if in many concrete cases these functions are unknown or only partially known. Consequently, in applications the identification of the diffusion coefficient and/or the reaction term from additional data is an important issue. Let us focus our attention on the second one.

The unique determination of $c(x)$ from a final observation, when $g(u)$ is linear or is replaced by $g = g(x, t)$, has been studied, for instance, in [5], [6], [7], [8], [1], [2], [13]. In all these papers the authors assume either null initial conditions or Dirichlet boundary conditions. In [3] the case of g linear with general initial and boundary conditions is considered. Here the authors prove a result of uniqueness and continuous dependence, provided that c is a priori known in some suitable set. On the other hand, the problem of determining c in the nonlinear equation (1.1) with homogeneous boundary data from final overdetermination is still an open problem.

In our paper we consider equation (1.1) under the assumption of the physically meaningful case of homogeneous Neumann boundary conditions. Observe that in this case

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among the steady states we can have also the physically relevant constant ones. Here we investigate the problem of recovering $c(x)$ from a final integral overdetermination. Such type of additional data have been considered for example in [9] and [14] to identify the coefficient $c(x)$ or some sources independent of time in linear parabolic equations.

Main goal of our paper is to study the unique solvability of the inverse problem of determining the pair (u, c) in (1.1) for an initial-boundary problem with homogeneous Neumann boundary conditions from integral overdetermination. To our knowledge our result is completely new in the nonlinear case and it is based on the qualitative asymptotic behaviour of the solutions, on account of suitable assumptions on the nonlinearity.

More precisely, taking $D = 1$ for the sake of simplicity, we consider

Problem (P): Find $u : \Omega \times (0, T) \to \mathbb{R}$ and $c : \Omega \to \mathbb{R}$ such that

$$u_t - \Delta u + c(x)g(u) = 0 \qquad \text{in } \Omega_T, \tag{1.2}$$

$$u_{\mathbf{n}}(x, t) = 0 \qquad (x, t) \in \Gamma_T, \tag{1.3}$$

$$u(x, 0) = u_0(x) \qquad x \in \Omega, \tag{1.4}$$

$$\int_0^T u(x, s)ds = \varphi(x) \qquad x \in \Omega, \tag{1.5}$$

where Ω is a bounded domain of $\mathbf{R}^N$ with smooth boundary Γ, T is a positive constant, $\Omega_T := \Omega \times (0, T)$ and $\Gamma_T := \Gamma \times (0, T)$. Here $u_{\mathbf{n}}$ is the normal derivative of u on the boundary, where $\mathbf{n}$ is the normal vector to Γ pointing outward Ω.

Concerning general notations, from now on we denote by $C^m(\Omega)$ the space of all continuous functions whose partial derivatives up to the m-th order are continuous in Ω and by $C^{2l,l}(\Omega_T)$ the space of functions u such that $D_t^r D_x^s u \in C(\Omega_T)$ with r, s satisfying $2r + |s| \leq 2l$. Moreover, we indicate by $C^{m+\lambda}(\Omega)$ and $C^{2l+2\lambda,l+\lambda}(\Omega_T)$, $\lambda \in (0, 1)$, the Banach spaces of Hölder $C^m(\Omega)-$functions of exponent λ and of parabolic Hölder $C^{2l,l}(\Omega_T)-$functions of exponent λ, respectively. Similar notations are used for $C^m(\overline{\Omega})$, $C^{2l,l}(\overline{\Omega}_T)$, $C^{m+\lambda}(\overline{\Omega})$, $C^{2l+2\lambda,l+\lambda}(\overline{\Omega}_T)$, where $\overline{\Omega}$ and $\overline{\Omega}_T$ are the respective closures of Ω and Ω_T. Finally, we denote by $W^{k,p}(\Omega)$ $(p \in [1 + \infty], k \in \mathbf{N})$ the usual Sobolev space and by $W_p^{2l,l}(\Omega_T)$, $(p \in [1 + \infty], l \in \mathbf{N})$, the Banach space of functions $u \in L_p(\Omega_T)$ such that $D_t^r D_x^s u \in L_p(\Omega_T)$ with r, s satisfying $2r + |s| \leq 2l$.

The plan of the paper goes as follows. We are going to prove uniqueness of the solution (u, c) to our inverse problem in two different cases: g linear, e.g., $g(u) = u$, and g a nonlinear function. In Section 2 we consider the first case and we solve the problem without any restriction on the initial datum u_0 and for all times T. In Section 3 the nonlinear case is analysed. Here, in order to prove uniqueness, we need to restrict ourselves to a class of initial data that are *close* to an asymptotically stable steady state and to take T large enough. Finally, Section 4 contains some remarks.

2. The linear case

Here we study the problem for $g(u) = u$. Then problem (P) becomes

Problem (P_1): Find $u : \Omega \times (0,T) \to \mathbb{R}$ and $c : \Omega \to \mathbb{R}$ such that

$$u_t - \Delta u + c(x)u = 0 \qquad \text{in } \Omega_T, \tag{2.1}$$

$$u_{\mathbf{n}}(x,t) = 0 \qquad (x,t) \in \Gamma_T, \tag{2.2}$$

$$u(x,0) = u_0(x) \qquad x \in \Omega, \tag{2.3}$$

$$\int_0^T u(x,s)ds = \varphi(x) \qquad x \in \Omega. \tag{2.4}$$

In this section we assume that

$$\Gamma \in C^{2+\lambda}, \tag{2.5}$$

for some $\lambda \in (0,1)$. Concerning the initial datum, we suppose

$$u_0 \in C^{2+\lambda}(\overline{\Omega}), \tag{2.6}$$

$$(u_0)_{\mathbf{n}}(x) = 0 \quad x \in \Gamma, \tag{2.7}$$

$$0 < \alpha \le u_0(x) \le \beta \quad x \in \overline{\Omega}. \tag{2.8}$$

Moreover, let the following a priori assumptions on the unknown coefficient c hold

$$c \in C^{\lambda}(\overline{\Omega}), \tag{2.9}$$

$$0 < \mu \le c(x) \le \nu \quad x \in \overline{\Omega}. \tag{2.10}$$

Given c and u_0 satisfying the above conditions it is well known from classical results (cf., for example, [4]) that there exists a unique solution u to (2.1)-(2.3) such that $u \in C^{2+\lambda,1+\lambda/2}(\overline{\Omega}_T)$.

It is worth recalling here the following positivity lemma which is a consequence of the maximum principle for parabolic equations (see, for instance, [12, Lemma 2.2.1])

Lemma 2.1. *Let $z \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$ be such that*

$$z_t - \Delta z + k(x,t)z \ge 0 \qquad \text{in } \Omega_T, \tag{2.11}$$

$$z_{\mathbf{n}}(x,t) \ge 0 \qquad (x,t) \in \Gamma_T. \tag{2.12}$$

$$z(x,0) \ge 0 \qquad x \in \Omega \tag{2.13}$$

where k is a bounded function in Ω_T. Then $z \ge 0$ in $\overline{\Omega}_T$. Moreover, $z > 0$ in $\overline{\Omega} \times (0,T]$, unless it is identically zero.

On account of the assumptions we made in this section, from Lemma 2.1 one gets that any function u satisfying problem (P_1) is positive on $\overline{\Omega}_T$.

Let us go back to our inverse problem. Then we have

Theorem 2.2. *Assume* (2.5)-(2.10). *Then, problem* (2.1)-(2.4) *admits at most one solution $(u,c) \in C^{2+\lambda,1+\lambda/2}(\overline{\Omega}_T) \times C^{\lambda}(\overline{\Omega})$, for any $T > 0$.*

Proof. Suppose that problem (2.1)-(2.4) has two different solutions (u_i, c_i), $i = 1, 2$, such that $u_i \in C^{2+\lambda,1+\lambda/2}(\overline{\Omega}_T)$ and c_i satisfies conditions (2.9)-(2.10). Then we have, for $i = 1, 2$,

$$(u_i)_t - \Delta u_i + c_i(x)u_i = 0 \qquad \text{in } \Omega_T, \tag{2.14}$$

$$(u_i)_{\mathbf{n}}(x,t) = 0 \qquad (x,t) \in \Gamma_T, \tag{2.15}$$

$$u_i(x,0) = u_0(x) \qquad x \in \Omega, \tag{2.16}$$

$$\int_0^T u_i(x,s)ds = \varphi(x) \qquad x \in \Omega. \tag{2.17}$$

Setting

$$u = u_1 - u_2, \quad f = c_2 - c_1, \tag{2.18}$$

one obtains

$$u_t - \Delta u + c_1(x)u = f(x)u_2 \qquad \text{in } \Omega_T, \tag{2.19}$$

$$u_{\mathbf{n}}(x,t) = 0 \qquad (x,t) \in \Gamma_T, \tag{2.20}$$

$$u(x,0) = 0 \qquad x \in \Omega, \tag{2.21}$$

$$\int_0^T u(x,s)ds = 0 \qquad x \in \Omega. \tag{2.22}$$

We introduce the new unknown $v(x,t) = \int_0^t u(x,s)ds$ and we integrate equation (2.19) on $[0,t]$. Using (2.21) and (2.22) we obtain

$$v_t - \Delta v + c_1(x)v = f(x)h(x,t) \qquad \text{in } \Omega_T, \tag{2.23}$$

$$v_{\mathbf{n}}(x,t) = 0 \qquad (x,t) \in \Gamma_T, \tag{2.24}$$

$$v(x,0) = 0 \qquad x \in \Omega, \tag{2.25}$$

$$v(x,T) = 0 \qquad x \in \Omega, \tag{2.26}$$

where $h(x,t) = \int_0^t u_2(x,s)ds$. Since $u_2 > 0$ on $\overline{\Omega}_T$ we have that

$$h > 0, \qquad h_t > 0, \quad \text{on } \Omega_T. \tag{2.27}$$

To prove uniqueness we adapt a result obtained by Isakov in [6, Theorem 9.1.2] for the Dirichlet boundary problem to the case of homogeneous Neumann boundary conditions. For the sake of completeness we give an outline of the proof. By contradiction assume f different from zero in Ω. Denote by f^+ and f^- the positive and negative parts of f. If f^- (or f^+) are identically equal to zero on Ω then $hf > 0$ (or $hf < 0$). Hence, applying Lemma 2.1 to problem (2.23)-(2.25) we obtain that $v > 0$ ($v < 0$) on $\Omega \times (0,T]$, in contradiction with (2.26).

Let $\Omega^+ = \{x \in \Omega : f^+(x) > 0\}$ and $\Omega^- = \{x \in \Omega : f^-(x) > 0\}$. Due to the previous argument the two sets Ω^+ and Ω^- are nonempty and open in Ω, because of the continuity of f^+ and f^-. Let v^+ and v^- be the solutions to problem (2.23)-(2.25) with source hf^+ and hf^-, respectively. Clearly, by linearity, $v = v^+ - v^-$. Since $h > 0$ we have that $(hf)^+ = hf^+$ and $(hf)^- = hf^-$. Hence, by the positivity lemma , $v^+ > 0$ and $v^- > 0$ on $\Omega \times (0,T]$, unless they are identically zero. Furthermore, solving the problem for $w^+ = v^+_t$, we obtain:

$$w_t^+ - \Delta w^+ + c_1(x)w^+ = f^+(x)h_t(x,t) \qquad \text{in } \Omega_T, \tag{2.28}$$

$$w_{\mathbf{n}}^+(x,t) = 0 \qquad (x,t) \in \Gamma_T, \tag{2.29}$$

$$w^+(x,0) = f^+(x)h(x,0) \qquad x \in \Omega. \tag{2.30}$$

Since h, $h_t > 0$ again, by the positivity lemma, we get $w^+ = v_t^+ > 0$ on $\Omega \times (0,T]$. Similarly, we can prove that $v_t^- > 0$ on $\Omega \times (0,T]$. Observe that v^+ (and, analogously, v^-) has a positive maximum in $\overline{\Omega}_T$. This maximum is attained at a point $(x_0,T) \in \Omega \times \{T\}$. In fact, since $v_t^+ > 0$, the maximum is a point of $\overline{\Omega} \times \{T\}$. On the other hand, due to the Hopf Lemma, it cannot lie on $\partial\Omega \times \{T\}$ (recall that $v_{\mathbf{n}}^+(x,t) = 0$ on Γ_T). Following the same reasoning, we prove that v^- attains the maximum at a point $(x_1,T) \in \Omega \times \{T\}$. By (2.26), it holds $x_0 = x_1$ and consequently $x_0 \in \overline{\Omega^+} \cap \overline{\Omega^-}$. Hence, $f^+(x_0,T) = f^-(x_0,T) = 0$. Finally, observing that

$$\Delta v^+(x_0,T) = v_t^+(x_0,T) + c_1(x_0)v^+(x_0,T) > 0, \tag{2.31}$$

we get a contradiction because (x_0,T) is the maximum point for v^+. Hence, we conclude that $f = 0$ which gives $c_1 = c_2$. □

3. The nonlinear case

In this section we consider the inverse problem when g is nonlinear. A very important case in applications is that of reaction terms of the form $c(x)g(u) = c(x)F'(u)$, where F is a double-well potential. For example, in the case of the Allen-Cahn equation we have $F(u) = u^4/4 - u^2/2 + K$. In this context our analysis is based on the study of the inverse problem with initial data in the neighborhood of a positive asymptotically stable steady state solution. We will derive a uniqueness result for large times T using the asymptotic behaviour of solutions of problem (P).

More precisely we will assume that $g : \mathbb{R} \to \mathbb{R}$ is a smooth nonlinear function which vanishes at least in one point $z_0 \in \mathbb{R}^+$. Without loss of generality we might assume that $z_0 = 1$. Moreover, we make the following assumptions:

$$g \in C^2(\mathbb{R}), \quad g(1) = 0, \tag{3.1}$$

$$\exists\, \sigma_0 \in (0,1) \text{ such that } g'(z) > 0,\ \forall\, z \in [\sigma_0, 1]. \tag{3.2}$$

On account of the previous assumptions we deduce that

$$g(z) < 0, \quad \forall\, z \in [\sigma_0, 1). \tag{3.3}$$

Moreover, there exist some positive constants m, M and N such that

$$g(z) \le 0, \quad 0 < m \le g'(z) \le M, \quad |g''(z)| \le N, \quad \forall\, z \in [\sigma_0, 1]. \tag{3.4}$$

Then our problem reads

Problem (P_2): Find $u : \Omega \times (0,T) \to \mathbb{R}$ and $c : \Omega \to \mathbb{R}$ such that

$$u_t - \Delta u + c(x)g(u) = 0 \qquad \text{in } \Omega_T, \tag{3.5}$$

$$u_{\mathbf{n}}(x,t) = 0 \qquad (x,t) \in \Gamma_T, \tag{3.6}$$

$$u(x,0) = u_0(x) \qquad x \in \Omega, \tag{3.7}$$

$$\int_0^T u(x,s)ds = \varphi(x) \qquad x \in \Omega. \tag{3.8}$$

Hence we can weaken the regularity assumptions on Γ, u_0 and c. More precisely, we suppose, for $\lambda \in (0,1)$,

$$\Gamma \in C^{1+\lambda}, \tag{3.9}$$

$$u_0 \in W^{2,2}(\Omega), \tag{3.10}$$

$$c \in L^\infty(\Omega), \tag{3.11}$$

$$0 < \mu \le c(x) \le \nu \quad \text{a.e.} \quad x \in \Omega. \tag{3.12}$$

However, condition (2.8) on u_0 is now replaced by the stronger one

$$0 < \sigma_0 < 1 - 2\delta < u_0(x) < 1 - \delta < 1 \quad \text{with} \quad 0 < \delta < \frac{1-\sigma_0}{2}, \quad x \in \overline{\Omega}, \tag{3.13}$$

that is, we are choosing a set of initial data in a left neighborhood of the steady state $u = 1$. Under the above assumptions we can prove the following result

Lemma 3.1. *Assume* (3.1)-(3.2) *and* (3.9)-(3.13)*. Then the direct problem* (3.5)-(3.7) *admits a unique positive global solution* $u \in C^\lambda(\overline{\Omega}_T) \cap W_2^{2,1}(\Omega_T)$ *such that*

$$\sigma_0 < 1 - 2\delta < u(x,t) < 1, \quad (x,t) \in \overline{\Omega}_T. \tag{3.14}$$

Proof. On account of (3.1), (3.3) and (3.13), observe that $\hat{u} = 1 - 2\delta$ and $\tilde{u} = 1$ are respectively lower and upper solution to (3.5)-(3.7). Hence, from the results contained in [12, Lemma 2.3.6 and Theorem 2.5.2], any solution u satisfies the a priori estimate (3.14). Then, the existence, uniqueness and regularity of u follows by a standard procedure regularizing the semilinear problem with a family of linear homogeneous Neumann boundary problems and applying the regularity results of [10] . □

In the next lemma we establish some finer estimates for the solution u.

Lemma 3.2. *Assume* (3.1)-(3.2) *and* (3.9)-(3.13)*. Let* $u \in C^\lambda(\overline{\Omega}_T) \cap W_2^{2,1}(\Omega_T)$ *be the solution to problem* (3.5)-(3.7)*. Then it holds*

$$\sigma_0 < 1 - 2\delta e^{-\gamma_1 t} < u(x,t) < 1 - \delta e^{-\gamma_2 t} \quad (x,t) \in \overline{\Omega}_T, \tag{3.15}$$

where

$$\gamma_1 = \mu m, \quad \gamma_2 = \nu M. \tag{3.16}$$

Proof. Consider the functions

$$w_1(x,t) = 1 - 2\delta e^{-\gamma_1 t}, \quad w_2(x,t) = 1 - \delta e^{-\gamma_2 t} \quad (x,t) \in \Omega_T \tag{3.17}$$

and observe that they solve the following problems, respectively,

$$(w_1)_t - \Delta w_1 + \gamma_1 w_1 = \gamma_1 \quad \text{in } \Omega_T, \tag{3.18}$$

$$(w_1)_{\mathbf{n}}(x,t) = 0 \quad (x,t) \in \Gamma_T, \tag{3.19}$$

$$w_1(x,0) = 1 - 2\delta \quad x \in \Omega, \tag{3.20}$$

$$(w_2)_t - \Delta w_2 + \gamma_2 w_2 = \gamma_2 \quad \text{in } \Omega_T, \tag{3.21}$$

$$(w_2)_{\mathbf{n}}(x,t) = 0 \quad (x,t) \in \Gamma_T, \tag{3.22}$$

$$w_2(x,0) = 1 - \delta \quad x \in \Omega. \tag{3.23}$$

Setting

$$v_i = u - w_i, \quad i = 1, 2, \tag{3.24}$$

then we have, recalling that $g(1) = 0$,

$$(v_1)_t - \Delta v_1 + \gamma_1 v_1 = (u-1)\left(\gamma_1 - c(x)\frac{g(u) - g(1)}{u-1}\right) \quad \text{a.e. in } \Omega_T, \tag{3.25}$$

$$(v_1)_{\mathbf{n}}(x,t) = 0 \quad (x,t) \in \Gamma_T, \tag{3.26}$$

$$v_1(x,0) = u_0(x) - (1 - 2\delta) \quad x \in \Omega, \tag{3.27}$$

$$(v_2)_t - \Delta v_2 + \gamma_2 v_2 = (u-1)\left(\gamma_2 - c(x)\frac{g(u) - g(1)}{u-1}\right) \quad \text{a.e. in } \Omega_T, \tag{3.28}$$

$$(v_2)_{\mathbf{n}}(x,t) = 0 \quad (x,t) \in \Gamma_T, \tag{3.29}$$

$$v_2(x,0) = u_0(x) - (1 - \delta) \quad x \in \Omega. \tag{3.30}$$

We now prove that $v_1 > 0$ in Ω_T and $v_2 < 0$ in Ω_T. For, observe that by the mean value theorem there exists $\xi \in (u, 1)$ (and then $1 - 2\delta < \xi < 1$) such that

$$(u-1)\left(\gamma_1 - c(x)\frac{g(u) - g(1)}{u-1}\right) = (u-1)(\gamma_1 - c(x)g'(\xi)).$$

By (3.4), (2.10) and (3.13), if we choose $\gamma_1 = \mu m$ we deduce

$$(u-1)\left(\gamma_1 - c(x)\frac{g(u) - g(1)}{u-1}\right) = (u-1)(\gamma_1 - c(x)g'(\xi)) \geq 0 \quad \text{a.e. in } \Omega_T, \tag{3.31}$$

$$u_0(x) - (1 - 2\delta) > 0 \quad \text{in } \Omega, \tag{3.32}$$

and, analogously, choosing $\gamma_2 = \nu M$ one gets

$$(u-1)\left(\gamma_2 - c(x)\frac{g(u) - g(1)}{u-1}\right) \leq 0 \quad \text{a.e. in } \Omega_T, \tag{3.33}$$

$$u_0(x) - (1 - \delta) < 0 \quad \text{in } \Omega. \tag{3.34}$$

Hence, applying Lemma 2.1 to problems (3.25)-(3.27) and (3.28)-(3.30), we obtain

$$v_1 > 0, \quad v_2 < 0 \quad \text{in } \overline{\Omega}_T. \tag{3.35}$$

Recalling (3.17) and (3.24), we get (3.15). □

Going back to the inverse problem, we have

Theorem 3.3. *Assume* (3.1)-(3.2) *and* (3.9)-(3.13). *Then there exist* $T_0 > 0$ *and* $\delta_0 > 0$, *depending only on the a-priori constants, such that problem* (3.5)-(3.8) *admits at most one solution* $(u, c) \in C^\lambda(\overline{\Omega}_T) \cap W_2^{2,1}(\Omega_T) \times L^\infty(\Omega)$, *for* $\delta \in (0, \min(\frac{1-\sigma_0}{2}, \delta_0))$ *and* $T \geq T_0$.

Proof. Assume that problem (3.5)-(3.8) admits two different solutions (u_i, c_i), $i = 1, 2$, such that $u_i \in C^\lambda(\overline{\Omega}_T) \cap W_2^{2,1}(\Omega_T)$ and c_i satisfies conditions (3.11)-(3.12). Then we have, for $i = 1, 2$,

$$(u_i)_t - \Delta u_i + c_i(x) g(u_i) = 0 \qquad \text{a.e. in } \Omega_T, \tag{3.36}$$

$$(u_i)_{\mathbf{n}}(x, t) = 0 \qquad (x, t) \in \Gamma_T, \tag{3.37}$$

$$u_i(x, 0) = u_0(x) \qquad x \in \Omega, \tag{3.38}$$

$$\int_0^T u_i(x, s) ds = \varphi(x) \qquad x \in \Omega. \tag{3.39}$$

Setting

$$u = u_1 - u_2, \quad f = c_2 - c_1, \tag{3.40}$$

one obtains

$$u_t - \Delta u + c_1(x)(g(u_1) - g(u_2)) = f(x) g(u_2) \qquad \text{a.e. in } \Omega_T, \tag{3.41}$$

$$u_{\mathbf{n}}(x, t) = 0 \qquad (x, t) \in \Gamma_T, \tag{3.42}$$

$$u(x, 0) = 0 \qquad x \in \Omega, \tag{3.43}$$

$$\int_0^T u(x, s) ds = 0 \qquad x \in \Omega. \tag{3.44}$$

Integrating on $[0, T]$ both the hand-sides of (3.41) with respect to time, we get, for a.e. $x \in \Omega$,

$$\begin{aligned} &\int_0^T u_t(x, s) ds - \int_0^T \Delta u(x, s) ds + c_1(x) \int_0^T (g(u_1) - g(u_2))(x, s) ds \\ &= f(x) \int_0^T g(u_2)(x, s) ds. \end{aligned} \tag{3.45}$$

Using (3.43) and (3.44), we deduce, for a.e. $x \in \Omega$,

$$u(x, T) + c_1(x) \int_0^T (g(u_1) - g(u_2))(x, s) ds = f(x) \int_0^T g(u_2)(x, s) ds. \tag{3.46}$$

Since $g(u_2) < 0$ in $[\sigma_0, 1]$ then (3.46) is equivalent to

$$f(x) = B(x), \qquad \text{for a.e. } x \in \Omega \tag{3.47}$$

where

$$B(x) = \frac{H(x)}{K(x)}, \qquad \text{for a.e. } x \in \Omega \tag{3.48}$$

$$H(x) = u(x, T) + c_1(x) \int_0^T (g(u_1) - g(u_2))(x, s) ds, \qquad \text{for a.e. } x \in \Omega \tag{3.49}$$

$$K(x) = \int_0^T g(u_2)(x, s) ds, \qquad x \in \Omega \tag{3.50}$$

and u, u_i solve problem (3.41)-(3.44) and (3.36)-(3.39), respectively. Observe that

$$\|B\|^2_{L^2(\Omega)} = \int_\Omega \frac{H^2(x)}{K^2(x)}dx \tag{3.51}$$

Hence, we need first to get an estimate from below of the term $K^2(x) = \left(\int_0^T g(u_2)(x,s)ds\right)^2$. On account of (3.1), (3.4), (3.14) and (3.15), we get

$$-2\delta M e^{-\gamma_1 t} \le g(u_2) = g(u_2) - g(1) \le -m\delta e^{-\gamma_2 t} < 0. \tag{3.52}$$

Then we deduce

$$\int_0^T g(u_2)(x,s)ds \le -m\delta \int_0^T e^{-\gamma_2 s}ds = \frac{m\delta}{\gamma_2}(e^{-\gamma_2 T} - 1) < 0, \tag{3.53}$$

so that

$$K^2(x) \ge \frac{m^2\delta^2}{\gamma_2^2}(1 - e^{-\gamma_2 T})^2. \tag{3.54}$$

Hence, it follows

$$\|B\|^2_{L^2(\Omega)} \le \frac{\gamma_2^2}{m^2\delta^2(1 - e^{-\gamma_2 T})^2}\|H\|^2_{L^2(\Omega)}. \tag{3.55}$$

Recalling definition (3.49), thanks to (2.10), we obtain

$$\|H\|^2_{L^2(\Omega)} \le 2\left(\|u(T)\|^2_{L^2(\Omega)} + \nu^2\int_\Omega\left(\int_0^T (g(u_1) - g(u_2))(x,s)ds\right)^2 dx\right). \tag{3.56}$$

Consequently, in order to estimate $\|H\|^2_{L^2(\Omega)}$, we need first to evaluate $\|u(t)\|^2_{L^2(\Omega)}$ in terms of $\|f\|^2_{L^2(\Omega)}$. To this aim, let us multiply both the hand-sides of (3.41) by u and integrate on Ω. Then we find the energy identity (cf. (3.42))

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \int_\Omega c_1(x)(g(u_1) - g(u_2))(x,t)u(x,t)dx \\ = \int_\Omega f(x)g(u_2)(x,t)u(x,t)dx, \quad t \in [0,T].\end{aligned} \tag{3.57}$$

Observe that

$$mu^2 \le (g(u_1) - g(u_2))u \le Mu^2, \quad \text{in} \quad \Omega_T. \tag{3.58}$$

Hence, on account of (2.10) and (3.58), an application of the Young inequality to (3.57) gives, for any $t \in [0,T]$,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \mu m\|u(t)\|^2_{L^2(\Omega)} \le \frac{\varepsilon}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\int_\Omega f^2(x)g^2(u_2)(x,t)dx, \tag{3.59}$$

ε being a positive constant. Thanks to (3.52) we have

$$g^2(u_2) \le 4\delta^2M^2e^{-2\gamma_1 t}, \quad \text{in} \quad \Omega_T \tag{3.60}$$

Hence, choosing $\varepsilon = \mu m$ in (3.59) and using (3.60), then it holds

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \mu m\|u(t)\|^2_{L^2(\Omega)} \le \frac{4\delta^2M^2}{\mu m}e^{-2\gamma_1 t}\|f\|^2_{L^2(\Omega)}, \quad t \in [0,T]. \tag{3.61}$$

By an application of the Gronwall Lemma and recalling (3.43), we deduce

$$\|u(t)\|^2_{L^2(\Omega)} \leq \frac{4\delta^2 M^2}{\mu m}\|f\|^2_{L^2(\Omega)}\int_0^t e^{-\mu m(t-s)}e^{-2\gamma_1 s}ds, \quad t\in[0,T]. \tag{3.62}$$

Then we have (recall that $\gamma_1 = \mu m$)

$$\|u(t)\|^2_{L^2(\Omega)} \leq \frac{4\delta^2 M^2}{\mu^2 m^2}\left(e^{-\mu m t} - e^{-2\mu m t}\right)\|f\|^2_{L^2(\Omega)}, \quad t\in[0,T]. \tag{3.63}$$

On the other hand, thanks to (3.44), the following identity holds

$$\begin{aligned}&\int_0^T (g(u_1)-g(u_2))(x,s)ds = \\ &\int_0^T \left(\int_0^1 (g'(\sigma u_1(x,s)+(1-\sigma)u_2(x,s)) - g'(1))\,d\sigma\right)u(x,s)ds.\end{aligned} \tag{3.64}$$

Since both u_i satisfy (3.14) and (3.15) and $\sigma\in[0,1]$, then we also have

$$0 < 1 - \Big(\sigma u_1(x,s) + (1-\sigma)u_2(x,s)\Big) < 2\delta e^{-\gamma_1 s}, \quad \forall\,(x,s)\in\Omega_T, \quad \forall\,\sigma\in[0,1] \tag{3.65}$$

and

$$\sigma_0 < \sigma u_1(x,s) + (1-\sigma)u_2(x,s) < 1, \quad \forall\,(x,s)\in\Omega_T, \quad \forall\,\sigma\in[0,1]. \tag{3.66}$$

Moreover, recalling that g satisfies assumptions (3.4), we deduce

$$\begin{aligned}&\left|\int_0^T\left(\int_0^1 (g'(\sigma u_1(x,s)+(1-\sigma)u_2(x,s)) - g'(1))d\sigma\right)u(x,s)ds\right| \\ &\leq \int_0^T 2N\delta e^{-\gamma_1 s}|u(x,s)|ds\end{aligned} \tag{3.67}$$

and a combination with (3.64) gives

$$\int_\Omega\left(\int_0^T (g(u_1)-g(u_2))(x,s)ds\right)^2 dx \leq \int_\Omega\left(\int_0^T 2N\delta e^{-\gamma_1 s}|u(x,s)|ds\right)^2 dx. \tag{3.68}$$

Applying the Hölder inequality and changing the order of integration, then it holds

$$\begin{aligned}&\int_\Omega\left(\int_0^T (g(u_1)-g(u_2))(x,s)ds\right)^2 dx \\ &\leq 4N^2\delta^2\int_\Omega\left(\frac{1-e^{-2\gamma_1 T}}{2\gamma_1}\int_0^T |u^2(x,s)|ds\right)dx \\ &\leq 4N^2\delta^2\frac{1-e^{-2\gamma_1 T}}{2\gamma_1}\int_0^T \|u(s)\|^2_{L^2(\Omega)}ds.\end{aligned} \tag{3.69}$$

Now, using (3.69) in (3.56), we obtain

$$\|H\|^2_{L^2(\Omega)} \leq 2\left(\|u(T)\|^2_{L^2(\Omega)} + 4N^2\nu^2\delta^2\frac{1-e^{-2\gamma_1 T}}{2\gamma_1}\int_0^T \|u(s)\|^2_{L^2(\Omega)}ds\right), \tag{3.70}$$

from which, combining with (3.63), one deduces

$$\begin{aligned}\|H\|^2_{L^2(\Omega)} &\le \frac{8\delta^2M^2}{\mu^2m^2}\Bigg((e^{-\mu mT}-e^{-2\mu mT}) \\ &+ 4\nu^2\delta^2N^2\frac{1-e^{-2\gamma_1T}}{2\gamma_1}\int_0^T(e^{-\mu ms}-e^{-2\mu ms})ds)\Bigg)\|f\|^2_{L^2(\Omega)} \\ &\le \frac{8\delta^2M^2}{\mu^2m^2}\Bigg((e^{-\mu mT}-e^{-2\mu mT}) \\ &+ 4\nu^2\delta^2N^2\frac{1-e^{-2\mu mT}}{2\mu m}\left(\frac{1-e^{-\mu mT}}{\mu m}-\frac{1-e^{-2\mu mT}}{2\mu m}\right)\Bigg)\|f\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.71}$$

Recalling (3.55) we infer

$$\|B\|^2_{L^2(\Omega)} \le k(T)\Big(I_1(T)+I_2(T,\delta)\Big)\|f\|^2_{L^2(\Omega)}, \tag{3.72}$$

where we have that

$$k(T) = \frac{8\nu^2M^4}{\mu^2m^4(1-e^{-\nu MT})^2}, \tag{3.73}$$

$$I_1(T) = (e^{-\mu mT}-e^{-2\mu mT}), \tag{3.74}$$

$$I_2(T,\delta) = 4\nu^2\delta^2N^2\frac{1-e^{-2\mu mT}}{2\mu m}\left(\frac{1-e^{-\mu mT}}{\mu m}-\frac{1-e^{-2\mu mT}}{2\mu m}\right). \tag{3.75}$$

Observe that

$$k(T)>0,\quad I_1(T)\ge 0,\quad I_2(T,\delta)\ge 0,\quad \forall\, T>0,\quad \forall\, 0<\delta<\frac{1-\sigma_0}{2}, \tag{3.76}$$

$$\lim_{T\to+\infty} I_1(T)=0, \tag{3.77}$$

$$\lim_{\delta\to 0^+} I_2(T,\delta)=0,\quad \lim_{T\to+\infty} I_2(T,\delta)\text{ is bounded},\quad \forall\, 0<\delta<\frac{1-\sigma_0}{2}, \tag{3.78}$$

$$k(T)\text{ is monotone decreasing for } T\to+\infty. \tag{3.79}$$

Hence, recalling (3.47), there exist two positive constants T_0, δ_0 such that

$$\|f\|^2_{L^2(\Omega)} = \|B\|^2_{L^2(\Omega)} \le \frac{1}{2}\|f\|^2_{L^2(\Omega)},\quad \forall\, T\ge T_0,\quad 0<\delta\le\delta_0, \tag{3.80}$$

from which we deduce that $f=0$ a.e. in Ω. Finally, on account of (3.41)-(3.44), we conclude that $u=u_1-u_2=0$ in Ω. □

4. Remarks

Remark 4.1. Replacing assumption (3.2) with

$$\exists\,\sigma_0>1 \text{ such that } g'(z)>0,\ \forall z\in[1,\sigma_0] \tag{4.1}$$

and choosing a set of initial data in a right neighborhood of $u=1$

$$0<1+\delta\le u_0(x)\le 1+2\delta<\sigma_0,\ \text{ with }\quad 0<\delta<\frac{\sigma_0-1}{2},\quad x\in\overline{\Omega}, \tag{4.2}$$

then we can prove analogous results to the ones contained in Section 3.

Remark 4.2. The proofs of Theorems 2.2 and 3.3 can be easily adapted to the case of $\Omega = \Pi_{i=1}^{N}[0, L_i]$, with u subject to spatial periodic boundary conditions. This kind of boundary conditions are physically relevant in applications (cf., for instance, [15]). Observe that in this case among the steady states we can have also the constant ones, as for the homogeneous Neumann boundary conditions considered in the previous Sections.

Remark 4.3. The choice of additional information (1.5) is consistent with the fact that we have to identify a function c depending on the spatial variables. Other relevant additional conditions are, for instance,

$$u(x,T) = \varphi(x), \quad x \in \Omega, \tag{4.3}$$

or

$$\int_{T_1}^{T} u(x,s)ds = \varphi(x) \quad x \in \Omega, \quad 0 < T_1 < T. \tag{4.4}$$

However, these cases seem to be more problematic and will be likely object of future investigations.

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