Università degli Studi di Milano

Facoltà di Scienze Matematiche, Fisiche e Naturali

Scuola di Dottorato in Matematica

Matematica e Statistica per le Scienze Computazionali
(MaSSC)

Academic Year 2009-2010

On quasiconvex conditional maps

Duality results and applications to Finance

Autore: Dott. Marco MAGGIS

Matricola: R07605

Relatore: Prof. Marco FRITTELLI

Coordinatore: Prof. Vincenzo CAPASSO

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Introduction

A brief history on quasiconvex duality on vector spaces and our contribution in the conditional case

Quasiconvex analysis has important applications in several optimization problems in science, economics and in finance, where convexity may be lost due to absence of global risk aversion, as for example in Prospect Theory [56].

The first relevant mathematical findings on quasiconvex functions were provided by De Finetti [18], mostly motivated by Paretian ordinal utility. Since then many authors, as [13], [14], [26], [57], [69] and [71] - to mention just a few, contributed significantly to the subject. More recently, a Decision Theory complete duality involving quasiconvex real valued functions has been proposed by [10]: in this theory a key role is played by the uniqueness of the representation and in such a way a one to one relationship between the primal functional and his dual counterpart is provided. For a review of quasiconvex analysis and its application and for an exhaustive list of references on this topic we refer to Penot [70].

Our interest in quasiconvex analysis was triggered by the recent paper [11] on quasiconvex risk measures, where the authors show that it is reasonable to weaken the convexity axiom in the theory of convex risk measures, introduced in [31] and [35]. This allows to maintain a good control of the risk, if one also replaces cash additivity by cash subadditivity [25]. The choice of relax the axiom of cash additivity is one of the main topics nowadays, especially when markets present lack of liquidity. Maccheroni et al. [11] point out that loosing this property convexity is not anymore equivalent to the principle of diversification: 'diversification should not increase the risk '. The recent interest in quasiconvex static risk measures is also testified by a second paper [19] on this subject, that was inspired by [11].
Furthermore when passing to the dynamics of the risk the usual axioms of risk measures seem too restrictive and incompatible with time consistency: Kupper and Schachermayer [54] showed that the only law invariant time consistent convex risk measure turns out to be the entropic one.

A function $f : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ defined on a vector space L is quasiconvex if for all $c \in \mathbb{R}$ the lower level sets $\{X \in L \mid f(X) \leq c\}$ are convex. In a general setting, the dual representation of such functions was shown by Penot and Volle [71]. The following theorem, reformulated in order to be compared to our results, was proved by Volle [76], Th. 3.4. and its proof relies on a straightforward application of Hahn Banach Theorem.

Theorem ([76]). *Let L be a locally convex topological vector space, L' be its dual space and $f : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ be quasiconvex and lower semicontinuous. Then*

$$f(X) = \sup_{X' \in L'} R(X'(X), X') \tag{C.1}$$

where $R : \mathbb{R} \times L' \to \overline{\mathbb{R}}$ is defined by

$$R(t, X') := \inf_{\xi \in L} \left\{ f(\xi) \mid X'(\xi) \geq t \right\}.$$

The generality of this theorem rests on the very weak assumptions made on the domain of the function f, i.e. on the space L. On the other hand, the fact that only *real valued* maps are admitted considerably limits its potential applications, specially in a dynamic framework.

To the best of our knowledge, a *conditional* version of this representation was lacking in the literature. When $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space, many problems having dynamic features lead to the analysis of maps $\pi : L_t \to L_s$ between the subspaces $L_t \subseteq L^1(\Omega, \mathscr{F}_t, \mathbb{P})$ and $L_s \subseteq L^0(\Omega, \mathscr{F}_s, \mathbb{P})$, $0 \leq s < t$.

In the first chapter of this thesis we consider quasiconvex maps of this form and analyze their dual representation. We provide (see Theorem 1.2 for the exact statement) a conditional version of (C.1):

$$\pi(X) = ess \sup_{Q \in L_t^* \cap \mathscr{P}} R(E_Q[X|\mathscr{F}_s], Q), \tag{C.2}$$

where

$$R(Y, Q) := ess \inf_{\xi \in L_t} \{\pi(\xi) \mid E_Q[\xi|\mathscr{F}_s] \geq_Q Y\},\ Y \in L_s,$$

L_t^* is the order continuous dual space of L_t and $\mathscr{P} =: \left\{ \frac{dQ}{d\mathbb{P}} \mid Q << \mathbb{P} \right\}$.

Furthermore, we show that if the map π is quasiconvex, monotone and cash additive then it is convex and we easily derive from (C.2) the well known representation of a conditional risk measure [17].

The formula (C.2) is obtained under quite weak assumptions on the space L_t which allow us to consider maps π defined on the typical spaces used in the literature in this framework: $L^\infty(\Omega, \mathscr{F}_t, \mathbb{P})$, $L^p(\Omega, \mathscr{F}_t, \mathbb{P})$, the Orlicz spaces $L^\Psi(\Omega, \mathscr{F}_t, \mathbb{P})$. In Theorem 1.2 we assume that π is lower semicontinuous, with respect to the weak topology $\sigma(L_t, L_t^*)$. As shown in Proposition 1.2 this condition is equivalent to continuity from below, which is a natural requirement in this context. We also provide in Theorem 1.3 the dual representation under a strong upper semicontinuity assumption.

The proofs of our main Theorems 1.2 and 1.3 are not based on techniques similar to those applied in the quasiconvex real valued case [76], nor to those used for convex conditional maps [17]. Indeed, the so called scalarization of π via the real valued map $X \to E_{\mathbb{P}}[\pi(X)]$ does not work, since this scalarization preserves convexity but not quasiconvexity. The idea of our proof is to apply (C.1) to the real valued quasiconvex map $\pi_A : L_t \to \overline{\mathbb{R}}$ defined by $\pi_A(X) := ess\sup_{\omega\in A} \pi(X)(\omega)$, $A \in \mathscr{F}_s$, and to approximate $\pi(X)$ with

$$\pi^{\Gamma}(X) := \sum_{A\in\Gamma} \pi_A(X)\mathbf{1}_A,$$

where Γ is a finite partition of Ω of $\mathscr{F}_s$ measurable sets $A \in \Gamma$. As explained in Section 1.6.1, some delicate issues arise when one tries to apply this simple and natural idea to prove that:

$$ess \sup_{Q\in L_t^*\cap\mathscr{P}} ess \inf_{\xi\in L_t} \{\pi(\xi)|E_Q[\xi|\mathscr{F}_s] \geq_Q E_Q[X|\mathscr{F}_s]\}$$

$$= ess\inf_{\Gamma} ess \sup_{Q\in L_t^*\cap\mathscr{P}} ess \inf_{\xi\in L_t} \left\{\pi^{\Gamma}(\xi)|E_Q[\xi|\mathscr{F}_s] \geq_Q E_Q[X|\mathscr{F}_s]\right\} \tag{C.3}$$

The uniform approximation result here needed is stated in the key Lemma 1.8 and Section 1.6.3 is devoted to prove it.

The starting point of this Thesis: Stochastic Utilities and the Conditional Certainty Equivalent

In the last decade many methodologies for pricing in incomplete markets were build on expected utility maximization with respect to terminal wealth: classic examples of this approach are the notions of fair price [15], certainty equivalent [32] and indifference price [5], [16], [43].

These techniques were developed both in a static framework and in a dynamic context [22]. In the dynamic case however, the utility function represents preferences at a fixed time T, while the pricing occurs at any time between today and the expiration T (backward pricing). The martingale property of the indirect utility (the value function of the optimization problem [24]) is an automatic consequence of the dynamic programming principle.

This classic backward approach has recently been argued in [6], [42], [62], [63] and a novel forward theory has been proposed: the utility function is stochastic, time dependent and moves forward.

In this theory, the forward utility (which replaces the indirect utility of the classic case) is built through the underlying financial market and must satisfy some appropriate martingale conditions.

Our research is inspired by the theory just mentioned, but a different approach is here developed: our preliminary object will be a stochastic dynamic utility $u(x,t,\omega)$ - i.e. a stochastic field [52] - representing the evolution of the preferences of the agent (see Definition 2.1).

The definition of the Conditional Certainty Equivalent (CCE) that we propose and analyze (Definition 2.9), is the natural generalization to the dynamic and stochastic environment of the classical notion of the certainty equivalent, as given in [74]. The CCE, denoted by $C_{s,t}(\cdot)$, provides the time s value of an F_t measurable claim $(s \leq t)$ in terms only of the Stochastic Dynamic Utility (SDU) and the filtration.

The SDU that we consider does not require *a priori* the presence of a financial market; neither it will have any specific recursive structure, nor will necessarily be an indirect utility function based on optimal trading in the market. However appropriate conditions are required on the SDU in order to deduce interesting properties for the CCE.

The next step, which is left for future research, would be the investigation of the compatibility conditions between the value assigned by the CCE and existing prices when an underlying market indeed exists. Clearly, not all SDU are compatible with the market. One extreme case is when the SDU can be determined by the market and the initial preferences structure, as in the case of the forward utility theory.

When we first bumped into the notion of Conditional Certainty Equivalent we immediately realized that this was in general a non concave map: anyway it was a monotone and quasiconcave operator between vector lattices. For this reason a theory of duality involving quasiconcavity instead of concavity was necessary to start a rigorous study of this topic. Due to the particular structure of the CCE, we were soon able to provide a direct proof of the dual representation (see Section 2.5): we exploit directly the results of Maccheroni et al. [10], avoiding any intermediate approximation argument. In this way the reader can appreciate the value of the result -that confirms what have been obtained in Chapter 1- without getting crazy in a thick maze of technical lemmas.

However, in order to show the dual representation of the CCE we must first define it on appropriate vector lattices. A common approach is to restrict the view to bounded random variables, so that no further integrability conditions are requested. But as soon as we try to extend the scenario to unbounded random variables it immediately appears that the distortion provoked by utility function can be mastered only in *ad hoc* frameworks.

To this end we introduce in Section 2.4, in the spirit of [7], a generalized class of Orlicz spaces which are naturally associated to the SDU taken into account. We show with some examples that these spaces also play a fundamental role for time compatibility of the CCE, since $C_{s,t} : M^{\widehat{u}_t} \to M^{\widehat{u}_s}$, where $M^{\widehat{u}_t}$ is the generalized Orlicz space of $\mathscr{F}_t$ measurable random variables associated to $u(x,t,\omega)$.

Further comments

Chapter 2 appears as a short parenthesis in this work and can be read as a self contained discussion. But as a matter of fact this was the main reason that lead us in

our research: one of the simplest example of evaluation map, such it is the Certainty Equivalent, fails in general to be concave. Since the standard duality theory for concave maps fails we were forced to look for a generalization of the duality results provided by Penot and Volle.
For this reason we report here the original proof of the dual representation theorem for the CCE (Theorem 1.2), which gave us the motivation and the strength to look for the more general and involving one provided in Chapter 1.

A brand new point of view: the module approach

The concept of module over a ring of functions is not new in the overview of mathematical studies but appeared around fifties as in [37], [40], [41] and [68]. Hahn Banach type extension theorems were firstly provided for particular classes of rings and finally at the end of seventies (see for instance [9]) general ordered rings were considered, so that the case of L^0 was included. Anyway, until [28], no Hyperplane Separation Theorems were obtained. It is well known that many fundamental results in Mathematical Finance rely on it: for instance Arbitrage Theory and the duality results on risk measure or utility maximization.

In the series of three papers [27], [28] and [53] the authors brilliantly succeed in the hard task of giving an opportune and useful topological structure to L^0-modules and to extent those functional analysis theorems which are relevant for financial applications. Once a rigorous analytical background has been carefully built up, it is easy to develop it obtaining many interesting results. In Chapter 3 of this Thesis we are able to generalize the quasiconvex duality theory to this particular framework.

It is worth to notice that this effort to extend the results in Chapter 1 to L^0-modules, is not a mathematical itch. Whenever dealing with conditional financial applications - such as conditional risk measures - vector spaces present many draw-backs as it has been argued in Filipovic et al. [27]. In the paper Approaches to Conditional Risk, the authors compare the two possible points of view using vector spaces (as it is common in the present literature) or L^0- modules. The results obtained are crystalline and highlight how the second choice better suites the financial scopes.
The intuition hidden behind the use of modules is simple and natural: suppose a set $\mathscr{S}$ of time-T maturity contingent claims is fixed and an agent is computing the risk of a portfolio selection at an intermediate time $t < T$. A flow of information - described by $\mathscr{F}_t$ - will be available at that time t: as a consequence, all the $\mathscr{F}_t$-measurable random variables will be known. Thus the $\mathscr{F}_t$ measurable random variables will act as constants in the process of diversification of our portfolio, forcing us to consider the new set $\mathscr{S} \cdot L^0(\Omega, \mathscr{F}_t, \mathbb{P})$ as the domain of the risk measures. This product structure is exactly the one that appears when working with L^0-modules.

The main result of quasiconvex duality is given in Theorem 3.1 and Corollaries 3.1 and 3.2. Differently from Theorems 1.2 and 1.3 here the representation is obtained dropping the assumption of monotonicity, as it happened for real valued quasiconvex maps. The map $\pi : E \to \bar{L}^0(\mathscr{G})$ can be represented as

$$\pi(X) = \sup_{\mu \in \mathscr{L}(E, L^0(\mathscr{G}))} R(\mu(X), \mu),$$

where E is a L^0-module and $\mathscr{L}(E, L^0(\mathscr{G}))$ the module of continuous L^0-linear functionals over E.
A posteriori, adding the assumption of monotonicity, we can restrict the optimization problem over the set of positive and normalized functional, as we show in Theorem 3.2.

The proof of these results are plain applications of the Hyperplane Separation theorems and not in any way linked to some approximation or scalarization argument. If one carefully analyzes them then he would appreciate many similarities with the original demonstrations by Penot and Volle.
A remarkable upgrade compared to Chapter 1, which appears as the best evidence of the power an novelty brought by modules, is the strong uniqueness result for conditional risk measures (see Theorem 3.2 for the precise statement), which perfectly matches what had been obtained in [10] for the static case.
Under suitable conditions, $\rho : L^p_{\mathscr{G}}(\mathscr{F}) \to L^0(\mathscr{G})$ is a conditional quasiconvex risk measure *if and only if*

$$\rho(X) = \sup_{Q \in \mathscr{P}^q} R\left(E\left[-\frac{dQ}{d\mathbb{P}}X|\mathscr{G}\right], Q\right) \tag{C.4}$$

where R is unique in the class $\mathscr{M}^{prop}(L^0(\mathscr{G}) \times \mathscr{P}^q)$. In this sense, in agreement with [10], we may assert that there exists a complete quasiconvex duality between quasiconvex risk measures and $\mathscr{M}^{prop}(L^0(\mathscr{G}) \times \mathscr{P}^q)$.

Chapter 1
On the dual representation on vector spaces

Conditional maps are a characteristic feature of the Probabilistic environment. We may hazard that the 'red line'that distinguishes Probability from Analysis is the concept of Conditional Expectation, which is the simplest example of conditional map. The conditional expectation $E_{\mathbb{P}}[X|\mathscr{G}]$ filters a random variable X with the information provided by the sigma algebra $\mathscr{G}$, giving a sort of backward projection of X. When Probability crashes in Mathematical Finance and Economics a great number of questions arise: in fact any linear property -such those satisfied by the conditional expectation- crumbles under the heavy load of the risk aversion of the agents playing in the markets. This affects the properties of the conditional maps taken into account in Pricing Theory and Risk Management. A peculiar example can be found in [73] where a general theory of Nonlinear Expectations is developed relying on Backward Stochastic Differential Equations.
The current literature is rolling around four mainstreams about conditional maps: the discussion of the axioms, the right domain (usually vector spaces of random variables), the robustness of the method and the time consistency. In this Chapter we would like to make a tiny step forward on these themes: considering general vector spaces and quasiconvex conditional maps we will nevertheless obtain a robust representation which is a crucial prerequisite for discussing (in the future research) time consistency.

1.1 Conditional quasiconvex maps

The probability space $(\Omega,\mathscr{F},\mathbb{P})$ is fixed throughout this chapter and supposed to be non-atomic. $\mathscr{G}\subseteq\mathscr{F}$ is any sigma algebra contained in $\mathscr{F}$. As usual we denote with $L^0(\Omega,\mathscr{F},\mathbb{P})$ the space of $\mathscr{F}$ measurable random variables that are $\mathbb{P}$ a.s. finite and by $\bar{L}^0(\Omega,\mathscr{F},\mathbb{P})$ the space of extended random variables that take values in $\mathbb{R}\cup\{\infty\}$. We also define $L^0_+(\mathscr{F})=\{Y\in L^0_{\mathscr{F}}\mid Y\geq 0\}$ and $L^0_{++}(\mathscr{F})=\{Y\in L^0_{\mathscr{F}}\mid Y>0\}$. $E_Q[X]$ represents the expected value of a random variable X with respect to a given probability measure Q. For every set $A\in\mathscr{F}$ the indicator function $\mathbf{1}_A$ belongs to

$L^0(\Omega,\mathscr{F},\mathbb{P})$ and is valued 1 for $\mathbb{P}$-almost every $\omega\in A$ and 0 for $\mathbb{P}$-almost every $\omega\in A^C$.

The Lebesgue spaces,

$$L^p(\Omega,\mathscr{F},\mathbb{P})=\{X\in L^0(\Omega,\mathscr{F},\mathbb{P})\mid E_{\mathbb{P}}[|X|^p]<+\infty\}\quad p\in[0,\infty]$$

and the Orlicz spaces (see next Chapter for further details)

$$L^{\hat{u}}(\Omega,\mathscr{F},\mathbb{P})=\left\{X\in L^0(\Omega,\mathscr{F},\mathbb{P})|\quad\exists\alpha>0\quad E_{\mathbb{P}}[\hat{u}(\alpha X)]<\infty\right\}$$
$$M^{\Phi}(\Omega,\mathscr{F},\mathbb{P})=\left\{X\in L^0(\Omega,\mathscr{F},\mathbb{P})|E_{\mathbb{P}}[\Phi(\alpha X)]<\infty\quad\forall\alpha>0\right\}$$

will simply be denoted by $L^p/L^{\hat{u}}/M^{\hat{u}}$, unless it is necessary to specify the sigma algebra, in which case we write $L^p_{\mathscr{F}}/L^{\hat{u}}_{\mathscr{F}}/M^{\hat{u}}_{\mathscr{F}}$.
It may happen that given a TVS L we denote by L^* either the topological dual space of L or the order dual space (see [2] p. 327 for the exact definition). Topological/order dual spaces may coincide as for L^p spaces or Morse spaces M^{Φ}, but in general they can differ as for the Orlicz space L^{Φ} (for an opportune choice of Φ). Anyway we will specify case by case what we are intending by L^*.

In presence of an arbitrary measure μ, if confusion may arise, we will explicitly write $=_\mu$ (resp. $\geq_\mu$), meaning μ almost everywhere. Otherwise, all equalities/inequalities among random variables are meant to hold $\mathbb{P}$-a.s..

The essential ($\mathbb{P}$ almost surely) *supremum* $\mathrm{ess\,sup}_\lambda(X_\lambda)$ of an arbitrary family of random variables $X_\lambda\in L^0(\Omega,\mathscr{F},\mathbb{P})$ will be simply denoted by $\sup_\lambda(X_\lambda)$, and similarly for the essential *infimum*. The *supremum* $\sup_\lambda(X_\lambda)\in\bar{L}^0(\Omega,\mathscr{F},\mathbb{P})$ gives by definition the smallest extended random variable greater of any X_λ; similarly the *infimum* is the greatest extended random variable smaller of any X_λ. Both of them are unique up to a set of $\mathbb{P}$-measure equal to 0. The reader can look at [30] Section A.5 for an exhaustive list of properties. Here we only recall that $1_A\sup_\lambda(X_\lambda)=\sup_\lambda(1_AX_\lambda)$ for any $\mathscr{F}$ measurable set A.

$\vee$ (resp. $\wedge$) denotes the essential ($\mathbb{P}$ almost surely) *maximum* (resp. the essential *minimum*) between two random variables, which are the usual lattice operations. Hereafter the symbol $\hookrightarrow$ denotes inclusion and lattice embedding between two lattices; a lattice embedding is an isomorphism between two vector spaces that preserves the lattice operations.

We consider a lattice $L_{\mathscr{F}}:=L(\Omega,\mathscr{F},\mathbb{P})\subseteq L^0(\Omega,\mathscr{F},\mathbb{P})$ and a lattice $L_{\mathscr{G}}:=L(\Omega,\mathscr{G},\mathbb{P})\subseteq\bar{L}^0(\Omega,\mathscr{G},\mathbb{P})$ of $\mathscr{F}$ (resp. $\mathscr{G}$) measurable random variables.

Definition 1.1. A map $\pi:L_{\mathscr{F}}\to L_{\mathscr{G}}$ is said to be

(MON) monotone increasing if for every $X,Y\in L_{\mathscr{F}}$

$$X\leq Y\quad\Rightarrow\quad\pi(X)\leq\pi(Y)\ ;$$

(QCO) quasiconvex if for every $X,Y\in L_{\mathscr{F}}$, $\Lambda\in L^0_{\mathscr{G}}$ and $0\leq\Lambda\leq 1$

$$\pi(\Lambda X+(1-\Lambda)Y)\leq\pi(X)\vee\pi(Y)\ ;$$

(LSC) $\tau-$lower semicontinuous if the set $\{X \in L_{\mathscr{F}} \mid \pi(X) \leq Y\}$ is closed for every $Y \in L_{\mathscr{G}}$ with respect to a topology τ on $L_{\mathscr{F}}$.

(USC)* $\tau-$strong upper semicontinuous if the set $\{X \in L_{\mathscr{F}} \mid \pi(X) < Y\}$ is open for every $Y \in L_{\mathscr{G}}$ with respect to a topology τ on $L_{\mathscr{F}}$ and there exists at least one $\theta \in L_{\mathscr{F}}$ such that $\pi(\theta) < +\infty$.

Remark 1.1. On the condition (QCO)

As it happens for real valued maps, the definition of (QCO) is equivalent to the fact that all the lower level sets

$$\mathscr{A}(Y) = \{X \in L_{\mathscr{F}} \mid \pi(X) \leq Y\} \quad \forall Y \in L_{\mathscr{G}}$$

are conditionally convex i.e. for all $X_1, X_2 \in \mathscr{A}(Y)$ and any $\mathscr{G}$-measurable r.v. Λ, $0 \leq \Lambda \leq 1$, one has $\Lambda X_1 + (1-\Lambda)X_2 \in \mathscr{A}(Y)$.

Indeed let $\pi(X_i) \leq Y$, $i = 1,2$: thanks to (QCO)

$$\pi(\Lambda X_1 + (1-\Lambda)X_2) \leq \max\{\pi(X), \pi(Y)\} \leq Y$$

i.e. $\mathscr{A}(Y)$ is conditionally convex.

Viceversa set $Y = \max\{\pi(X_1), \pi(X_2)\}$ then $X_1, X_2 \in \mathscr{A}(Y)$ implies from convexity that $\Lambda X_1 + (1-\Lambda)X_2 \in \mathscr{A}(Y)$ and then $\pi(\Lambda X_1 + (1-\Lambda)X_2) \leq Y$.

Remark 1.2. On the condition (LSC)

The class of closed and convex sets is the same in any topology compatible with a given dual system (Grothendieck [38] Chapter 2, Section 15). We remind the reader that a topology τ is compatible with a dual system (E, E') if the topological dual space of E w.r.t. τ is E'. Therefore - assuming *a priori* (QCO) - if two topologies τ_1, τ_2 give rise to the same dual space, then the conditions τ_1-(LSC), τ_2 -(LSC), are equivalent. This simplifies the things up when dealing with nice spaces such as L^p spaces.

Remark 1.3. On the condition (USC)*

When $\mathscr{G} = \sigma(\Omega)$ is the trivial sigma algebra, the map π is real valued and (USC)* is equivalent to

$$\{X \in L_{\mathscr{F}} \mid \pi(X) \geq Y\} \text{ is closed for every } Y \in \mathbb{R}.$$

But in general this equivalence does not hold true: in fact

$$\{X \in L_{\mathscr{F}} \mid \pi(X) < Y\}^C = \{X \in L_{\mathscr{F}} \mid \mathbb{P}(\pi(X) \geq Y) > 0\} \supsetneq \{X \in L_{\mathscr{F}} \mid \pi(X) \geq Y\}$$

Anyway (USC)* implies that considering a net $\{X_\alpha\}$, $X_\alpha \xrightarrow{\tau} X$ then $\limsup_\alpha \pi(X_\alpha) \leq \pi(X)$. For sake of simplicity suppose that $\pi(X) < +\infty$: let $Y \in L_{\mathscr{G}}$, $\pi(X) < Y$ then X belongs to the open set $V = \{\xi \in L_{\mathscr{F}} \mid \pi(\xi) < Y\}$. If $X_\alpha \xrightarrow{\tau} X$ then there will exists α_0 such that for every $X_\beta \in V$ for every $\beta \geq \alpha_0$. This means that $\pi(X_\beta) < Y$ for every $\beta \geq \alpha_0$ and

$$\limsup_\alpha \pi(X_\alpha) \leq \sup_{\beta \geq \alpha_0} \pi(X_\beta) \leq Y \quad \forall Y > \pi(X).$$

Conversely it is easy to check that $X_\alpha \xrightarrow{\tau} X \Rightarrow \limsup_\alpha \pi(X_\alpha) \leq \pi(X)$ implies that the set $\{X \in L_{\mathscr{F}} \mid \pi(X) \geq Y\}$ is closed. We thus can conclude that the condition (USC)* is a stronger condition than the one usually given in the literature for upper semicontinuity. The reason why we choose this one is that it will be preserved by the map π_A.
Finally we are assuming that there exists at least one $\theta \in L_{\mathscr{F}}$ such $\pi(\theta) < +\infty$: otherwise the set $\{X \in L_{\mathscr{F}} \mid \pi(X) < Y\}$ is always empty (and then open) for every $Y \in L_{\mathscr{G}} \cap L^0_{\mathscr{G}}$.

Definition 1.2. A vector space $L_{\mathscr{F}} \subseteq L^0_{\mathscr{F}}$ satisfies the property $(1_{\mathscr{F}})$ if

$$X \in L_{\mathscr{F}} \text{ and } A \in \mathscr{F} \Longrightarrow X\mathbf{1}_A \in L_{\mathscr{F}}. \qquad (1_{\mathscr{F}})$$

Suppose that $L_{\mathscr{F}}$ (resp. $L_{\mathscr{G}}$) satisfies the property $(1_{\mathscr{F}})$ (resp $1_{\mathscr{G}}$).
A map $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ is said to be

(REG) regular if for every $X, Y \in L_{\mathscr{F}}$ and $A \in \mathscr{G}$

$$\pi(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = \pi(X)\mathbf{1}_A + \pi(Y)\mathbf{1}_{A^C}.$$

or equivalently if $\pi(X\mathbf{1}_A)\mathbf{1}_A = \pi(X)\mathbf{1}_A$.

Remark 1.4. The assumption (REG) is actually weaker than the assumption

$$\pi(X\mathbf{1}_A) = \pi(X)\mathbf{1}_A \quad \forall A \in \mathscr{G}. \qquad (1.1)$$

As shown in [17], (1.1) always implies (REG), and they are equivalent if and only if $\pi(0) = 0$.

It is well known that $\pi(0) = 0$ and conditional convexity implies (REG) (a simple proof can be found in [17] Proposition 2). However, such implication does not hold true any more if convexity is replaced by quasiconvexity. Obviously, (QCO) and (REG) does not imply conditional convexity, as shown by the map

$$X \to f^{-1}\left(E\left[f(X)|\mathscr{G}\right]\right)$$

when $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and convex on $\mathbb{R}$.

1.2 The case of real valued maps when $\mathscr{G} = \sigma(\Omega)$.

In this section we resume what has been already fully studied in the case $\mathscr{G}$ is the trivial sigma algebra and then $L_{\mathscr{G}}$ reduces to the extended real line $\overline{\mathbb{R}}$. We report also the proofs which matches those given by Penot and Volle, to help the understanding of the role played by Hahn Banach Separation Theorem. In this way the reader will be helped to appreciate the analogies between the following proofs and the generalizations to the modules framework in Chapter 3.

Here $L_{\mathscr{F}} = L$ can be every locally convex topological vector space and L^* denotes its topological dual space. Consider $\pi : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ satisfying (QCO) and define: $R : L^* \times \mathbb{R} \to \overline{\mathbb{R}}$ by

$$R(X^*,t) := \sup\{\pi(X) \mid X \in L \text{ such that } X^*(X) \geq t\}.$$

Theorem 1.1. *Let π as before*
(i) If π is (LSC) then:

$$\pi(X) = \sup_{X' \in L^*} R(X', X'(X)).$$

(ii) If π is (USC)* then:

$$\pi(X) = \max_{X' \in L^*} R(X', X'(X)),$$

Proof. (i)By definition, for any $X' \in L'$, $R(X'(X), X') \leq \pi(X)$ and therefore

$$\sup_{X' \in L'} R(X'(X), X') \leq \pi(X),\ X \in L.$$

Fix any $X \in L$ and take $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. Then X does not belong to the closed convex set $\{\xi \in L : \pi(\xi) \leq \pi(X) - \varepsilon\} := \mathscr{C}_\varepsilon$ (if $\pi(X) = +\infty$, replace the set $\mathscr{C}_\varepsilon$ with $\{\xi \in L : \pi(\xi) \leq M\}$, for any M). By the Hahn Banach theorem there exists a continuous linear functional that strongly separates X and $\mathscr{C}_\varepsilon$, i.e. there exists $\alpha \in \mathbb{R}$ and $X'_\varepsilon \in L'$ such that

$$X'_\varepsilon(X) > \alpha > X'_\varepsilon(\xi) \text{ for all } \xi \in \mathscr{C}_\varepsilon. \tag{1.2}$$

Hence:

$$\left\{\xi \in L : X'_\varepsilon(\xi) \geq X'_\varepsilon(X)\right\} \subseteq (\mathscr{C}_\varepsilon)^C = \{\xi \in L : \pi(\xi) > \pi(X) - \varepsilon\} \tag{1.3}$$

and

$$\begin{aligned}
\pi(X) \geq \sup_{X' \in L'} R(X'(X), X') &\geq R(X'_\varepsilon(X), X'_\varepsilon) \\
&= \inf\left\{\pi(\xi) \mid \xi \in L \text{ such that } X'_\varepsilon(\xi) \geq X'_\varepsilon(X)\right\} \\
&\geq \inf\{\pi(\xi) \mid \xi \in L \text{ satisfying } \pi(\xi) > \pi(X) - \varepsilon\} \geq \pi(X) - \varepsilon.
\end{aligned}$$

(ii)For any fixed $X \in L$, the set $\{\xi \in L : \pi(\xi) < \pi(X)\} := \mathscr{E}$ is convex open and $X \notin \mathscr{E}$. By the Hahn Banach theorem there exists a continuous linear functional that properly separates X and $\mathscr{E}$, i.e. there exists $\alpha \in \mathbb{R}$ and $X^* \in L^*$ such that: $X^*(X) > \alpha \geq X^*(\xi)$ for all $\xi \in \mathscr{E}$.
Hence: $\{\xi \in L : X^*(\xi) \geq X^*(X)\} \subseteq (\mathscr{E})^C = \{\xi \in L : \pi(\xi) \geq \pi(X)\}$ and

$$\begin{aligned}\pi(X) &\geq \sup_{Y^*\in L^*} R(Y^*,Y^*(X)) \geq R(X^*,X^*(X))\\ &= \inf\{\pi(\xi) \mid \xi\in L \text{ such that } X^*(\xi)\geq X^*(X)\}\\ &\geq \inf\left\{\pi(\xi) \mid \xi \in (\mathscr{E})^C\right\} \geq \pi(X).\end{aligned}$$

Proposition 1.1. *Suppose L is a lattice, $L^* = (L,\geq)^*$ is the order continuous dual space satisfying $L^* \hookrightarrow L^1$ and $(L,\sigma(L,L^*))$ is a locally convex TVS. If $f : L \to \overline{\mathbb{R}}$ is quasiconvex, $\sigma(L,L^*)$-lsc (resp usc) and monotone increasing then*

$$\pi(X) = \sup_{Q\in L^*_+ \mid Q(\mathbf{1})=1} R(Q(X),Q),$$

$$\textit{resp.}\ \ \pi(X) = \max_{Q\in L^*_+ \mid Q(\mathbf{1})=1} R(Q(X),Q).$$

Proof. We apply Theorem 1.1 to the locally convex TVS $(L,\sigma(L,L^*))$ and deduce:

$$\pi(X) = \sup_{Z\in L^*\subseteq L^1} R(Z(X),Z).$$

We now adopt the same notations of the proof of Theorem 1.1 and let $Z\in L$, $Z\geq 0$. Obviously if $\xi\in\mathscr{C}_\varepsilon$ then $\xi - nZ\in\mathscr{C}_\varepsilon$ for every $n\in\mathbb{N}$ and from (1.2) we deduce:

$$X'_\varepsilon(\xi-nZ) < \alpha < X'_\varepsilon(X) \quad\Rightarrow\quad X'_\varepsilon(Z) > \frac{X'_\varepsilon(\xi-X)}{n} \quad , \quad \forall n\in\mathbb{N}$$

i.e. $X'_\varepsilon \in L^*_+ \subseteq L^1$ and $X'_\varepsilon \neq 0$. Hence $X'_\varepsilon(\mathbf{1}) = E_{\mathbb{P}}[X'_\varepsilon] > 0$ and we may normalize X'_ε to $X'_\varepsilon / X'_\varepsilon(\mathbf{1})$.

1.3 Dual representation for an arbitrary $\mathscr{G}$

From now on $\mathscr{G}$ is any σ-algebra $\mathscr{G}\subset\mathscr{F}$.

1.3.1 Topological assumptions

Definition 1.3. We say that $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ is

(CFB) *continuous from below* if

$$X_n \uparrow X \quad \mathbb{P}\ a.s. \quad\Rightarrow\quad \pi(X_n)\uparrow\pi(X) \quad \mathbb{P}\ a.s.$$

In [8] it is proved the equivalence between: (CFB), order lsc and $\sigma(L_{\mathscr{F}},L^*_{\mathscr{F}})$-(LSC), for monotone convex real valued functions. In the next proposition we show that this equivalence holds true for monotone quasiconvex conditional maps, under the same assumption on the topology $\sigma(L_{\mathscr{F}},L^*_{\mathscr{F}})$ adopted in [8].

Definition 1.4 ([8]). Let $\{X_\alpha\} \subset L_{\mathscr{F}}$ be a net. A linear topology τ on the Riesz space $L_{\mathscr{F}}$ has the C-property if $X_\alpha \xrightarrow{\tau} X$ implies the existence of of a sequence $\{X_{\alpha_n}\}_n$ and a convex combination $Z_n \in conv(X_{\alpha_n}, \dots)$ such that $Z_n \xrightarrow{o} X$.

As explained in [8], the assumption that $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$ has the C-property is very weak and is satisfied in all cases of interest. When this is the case, in Theorem 1.2 the $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$-(LSC) condition can be replaced by (CFB), which is often easy to check.

Proposition 1.2. *Suppose that $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$ satisfies the C-property and that $L_{\mathscr{F}}$ is order complete. Given $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ satisfying (MON) and (QCO) we have:*
(i) π is $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$-(LSC) if and only if (ii) π is (CFB).

Proof. Recall that a sequence $\{X_n\} \subseteq L_{\mathscr{F}}$ order converge to $X \in L_{\mathscr{F}}$, $X_n \xrightarrow{o} X$, if there exists a sequence $\{Y_n\} \subseteq L_{\mathscr{F}}$ satisfying $Y_n \downarrow 0$ and $|X - X_n| \leq Y_n$.

(i)$\Rightarrow$ (ii): Consider $X_n \uparrow X$. Since $X_n \uparrow X$ implies $X_n \xrightarrow{o} X$, then for every order continuous $Z \in L^*_{\mathscr{F}}$ the convergence $Z(X_n) \to Z(X)$ holds. From $L^*_{\mathscr{F}} \hookrightarrow L^1_{\mathscr{F}}$

$$E_{\mathbb{P}}[ZX_n] \to E_{\mathbb{P}}[ZX] \quad \forall Z \in L^*_{\mathscr{F}}$$

and we deduce that $X_n \xrightarrow{\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})} X$.
(MON) implies $\pi(X_n) \uparrow$ and $p := \lim_n \pi(X_n) \leq \pi(X)$. The lower level set $\mathscr{A}_p = \{\xi \in L_{\mathscr{F}} \mid \pi(\xi) \leq p\}$ is $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$ closed and then $X \in \mathscr{A}_p$, i.e. $\pi(X) = p$.

(ii)$\Rightarrow$(i): First we prove that if $X_n \xrightarrow{o} X$ then $\pi(X) \leq \liminf_n \pi(X_n)$. Define $Z_n := (\inf_{k \geq n} X_k) \wedge X$ and note that $X - Y_n \leq X_n \leq X + Y_n$ implies

$$X \geq Z_n = \left(\inf_{k \geq n} X_k\right) \wedge X \geq \left(\inf_{k \geq n} (-Y_k) + X\right) \wedge X \uparrow X$$

i.e. $Z_n \uparrow X$. We actually have from (MON) $Z_n \leq X_n$ implies $\pi(Z_n) \leq \pi(X_n)$ and from (CFB) $\pi(X) = \lim_n \pi(Z_n) \leq \liminf_n \pi(X_n)$ which was our first claim.
For $Y \in L_{\mathscr{G}}$ consider $\mathscr{A}_Y = \{\xi \in L_{\mathscr{F}} \mid \pi(\xi) \leq Y\}$ and a net $\{X_\alpha\} \subseteq L_{\mathscr{F}}$ such that $X_\alpha \xrightarrow{\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})} X \in L_{\mathscr{F}}$. Since $L_{\mathscr{F}}$ satisfies the C-property, there exists $Y_n \in Conv(X_{\alpha_{n,\dots}})$ such $Y_n \xrightarrow{o} X$. The property (QCO) implies that $\mathscr{A}_Y$ is convex and then $\{Y_n\} \subseteq \mathscr{A}_Y$. Applying the first step we get

$$\pi(X) \leq \liminf_n \pi(Y_n) \leq Y \qquad \text{i.e. } X \in \mathscr{A}_Y$$

Standing assumptions on the spaces

(a) *$\mathscr{G} \subseteq \mathscr{F}$ and the lattice $L_{\mathscr{F}}$ (resp. $L_{\mathscr{G}}$) satisfies the property* $(1_{\mathscr{F}})$ (resp $1_{\mathscr{G}}$). *Both $L_{\mathscr{G}}$ and $L_{\mathscr{F}}$ contains the constants as a vector subspace.*
(b) *The order continuous dual of $(L_{\mathscr{F}}, \geq)$, denoted by $L^*_{\mathscr{F}} = (L_{\mathscr{F}}, \geq)^*$, is a lattice ([2], Th. 8.28) that satisfies $L^*_{\mathscr{F}} \hookrightarrow L^1_{\mathscr{F}}$ and property* $(1_{\mathscr{F}})$.
(c) *The space $L_{\mathscr{F}}$ endowed with the weak topology $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$ is a locally convex Riesz space.*

The condition (c) requires that the order continuous dual $L^*_{\mathscr{F}}$ is rich enough to separate the points of $L_{\mathscr{F}}$, so that $(L_{\mathscr{F}},\sigma(L_{\mathscr{F}},L^*_{\mathscr{F}}))$ becomes a locally convex TVS and Proposition 1.1 can be applied.

Remark 1.5. Many important classes of spaces satisfy these conditions, such as
- The L^p-spaces, $p\in[1,\infty]$: $L_{\mathscr{F}}=L^p_{\mathscr{F}}$, $L^*_{\mathscr{F}}=L^q_{\mathscr{F}}\hookrightarrow L^1_{\mathscr{F}}$.
- The Orlicz spaces L^{Ψ} for any Young function Ψ: $L_{\mathscr{F}}=L^{\Psi}_{\mathscr{F}}$, $L^*_{\mathscr{F}}=L^{\Psi^*}_{\mathscr{F}}\hookrightarrow L^1_{\mathscr{F}}$, where Ψ^* denotes the conjugate function of Ψ;
- The Morse subspace M^{Ψ} of the Orlicz space L^{Ψ}, for any continuous Young function Ψ: $L_{\mathscr{F}}=M^{\Psi}_{\mathscr{F}}$, $L^*_{\mathscr{F}}=L^{\Psi^*}_{\mathscr{F}}\hookrightarrow L^1_{\mathscr{F}}$.

1.3.2 Statements of the dual results

Set

$$\mathscr{P}=:\left\{\frac{dQ}{d\mathbb{P}}\mid Q<<\mathbb{P}\text{ and }Q\text{ probability}\right\}=\left\{\xi'\in L^1_+\mid E_{\mathbb{P}}[\xi']=1\right\}$$

From now on we will write with a slight abuse of notation $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$ instead of $\frac{dQ}{d\mathbb{P}}\in L^*_{\mathscr{F}}\cap\mathscr{P}$. Define $K:L_{\mathscr{F}}\times(L^*_{\mathscr{F}}\cap\mathscr{P})\to\overline{L}^0_{\mathscr{G}}$ and $R:L^0_{\mathscr{G}}\times L^*_{\mathscr{F}}$ as

$$K(X,Q):=\inf_{\xi\in L_{\mathscr{F}}}\{\pi(\xi)\mid E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]\} \tag{1.4}$$

$$R(Y,\xi'):=\inf_{\xi\in L_{\mathscr{F}}}\left\{\pi(\xi)\mid E_{\mathbb{P}}[\xi'\xi|\mathscr{G}]\geq Y\right\}. \tag{1.5}$$

K is well defined on $L_{\mathscr{F}}\times(L^*_{\mathscr{F}}\cap\mathscr{P})$. On the other hand the actual domain of R is not on the whole $L^0_{\mathscr{G}}\times L^*_{\mathscr{F}}$ but we must restrict to

$$\Sigma=\{(Y,\xi')\in L^0_{\mathscr{G}}\times L^*_{\mathscr{F}}\mid\exists\xi\in L_{\mathscr{F}}\text{ s.t. }E_{\mathbb{P}}[\xi'\xi|\mathscr{G}]\geq Y\}. \tag{1.6}$$

Obviously $(E_{\mathbb{P}}[\xi'X|\mathscr{G}],\xi')\in\Sigma$ for every $X\in L_{\mathscr{F}}$, $\xi'\in L^*_{\mathscr{F}}$. Notice that $K(X,Q)$ depends on X only through $E_Q[X|\mathscr{G}]$. Moreover $R(E_{\mathbb{P}}[\xi'X|\mathscr{G}],\xi')=R(E_{\mathbb{P}}[\lambda\xi'X|\mathscr{G}],\lambda\xi')$ for every $\lambda>0$. Thus we can consider $R(E_{\mathbb{P}}[\xi'X|\mathscr{G}],\xi')$, $\xi'\geq0$, $\xi'\neq0$, always defined on the normalized elements $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$.
It is easy to check that

$$E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}\xi\mid\mathscr{G}\right]\geq E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}X\mid\mathscr{G}\right]\Longleftrightarrow E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}],$$

and for $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$ we deduce

$$K(X,Q)=R\left(E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}X\mid\mathscr{G}\right],Q\right).$$

Remark 1.6. Since the order continuous functional on $L_{\mathscr{F}}$ are contained in L^1, then $Q(\xi) := E_Q[\xi]$ is well defined and finite for every $\xi \in L_{\mathscr{F}}$ and $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$. In particular this and $(1_{\mathscr{F}})$ imply that $E_Q[\xi|\mathscr{G}]$ is well defined. Moreover, since $L^*_{\mathscr{F}} \hookrightarrow L^1_{\mathscr{F}}$ *satisfies property* $(1_{\mathscr{F}})$ then $\frac{dQ}{d\mathbb{P}}1_A \in L^*_{\mathscr{F}}$ whenever $Q \in L^*_{\mathscr{F}}$ and $A \in \mathscr{F}$.

Theorem 1.2. *Suppose that* $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$ *satisfies the C-property and* $L_{\mathscr{F}}$ *is order complete If* $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ *is (MON), (QCO), (REG) and* $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$*-(LSC) then*

$$\pi(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} K(X,Q). \tag{1.7}$$

Theorem 1.3. *If* $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ *is (MON), (QCO), (REG) and* τ*-(USC)** *then*

$$\pi(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} K(X,Q). \tag{1.8}$$

Notice that in (1.7), (1.8) the *supremum* is taken over the set $L^*_{\mathscr{F}} \cap \mathscr{P}$. In the following corollary, proved in Section 1.6.2, we show that we can match the conditional convex dual representation, restricting our optimization problem over the set

$$\mathscr{P}_{\mathscr{G}} =: \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \in \mathscr{P} \text{ and } Q = \mathbb{P} \text{ on } \mathscr{G} \right\}.$$

Clearly, when $Q \in \mathscr{P}_{\mathscr{G}}$ then $\bar{L}^0(\Omega, \mathscr{G}, \mathbb{P}) = \bar{L}^0(\Omega, \mathscr{G}, Q)$ and comparison of $\mathscr{G}$ measurable random variables is understood to hold indifferently for $\mathbb{P}$ or Q almost surely.

Corollary 1.1. *Under the same hypothesis of Theorem 1.2 (resp. Theorem 1.3), suppose that for* $X \in L_{\mathscr{F}}$ *there exists* $\eta \in L_{\mathscr{F}}$ *and* $\delta > 0$ *such that* $\mathbb{P}(\pi(\eta) + \delta < \pi(X)) = 1$. *Then*

$$\pi(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}_{\mathscr{G}}} K(X,Q).$$

1.4 Possible applications

1.4.1 Examples of quasiconvex maps popping up from the financial world

As a further motivation for our findings, we give some examples of quasiconvex (quasiconcave) conditional maps arising in economics and finance. The first one is studied in detail in the second chapter: as explained in the introduction this was the main reason that moved us to this research and the complexity of the theme deserves much space to be dedicated. The analysis of Dynamic Risk Measures and Acceptability Indices was out of the scope of this thesis and for this reason we limit ourselves to give some simple concrete examples. For sure the questions arisen on the meaning of diversification will play a central role in the Math Finance academic world in the next few years.

Certainty Equivalent in dynamic settings

Consider a stochastic dynamic utility

$$u : \mathbb{R} \times [0,\infty) \times \Omega \to \mathbb{R}$$

We introduce the *Conditional Certainty Equivalent* (CCE) of a random variable $X \in L_t$, as the random variable $\pi(X) \in L_s$ solution of the equation:

$$u(\pi(X),s) = E_{\mathbb{P}}\left[u(X,t)|\mathscr{F}_s\right],$$

where L_t and L_s are appropriate lattices of random variables. Thus the CCE defines the *valuation* operator

$$\pi : L_t \to L_s,\ \pi(X) = u^{-1}\left(E_{\mathbb{P}}\left[u(X,t)|\mathscr{F}_s\right], s\right).$$

The CCE, as a map $\pi : L_t \to L_s$ is monotone, quasi concave, regular.

Dynamic Risk Measures

As already mentioned the dual representation of a conditional *convex* risk measure can be found in [17]. The findings of the present paper show the dual representation of conditional *quasiconvex* risk measures when cash additivity does not hold true.

For a better understanding we give a concrete example: consider $t \in [0,T]$ and a non empty convex set $C_T \in L^\infty(\Omega,\mathscr{F}_T,\mathbb{P})$ such that $C_T + L^\infty_+ \subseteq C_T$. The set C_T represents the future positions considered acceptable by the supervising agency. For all $m \in \mathbb{R}$ denote by $v_t(m,\omega)$ the price at time t of m euros at time T. The function $v_t(m,\cdot)$ will be in general $\mathscr{F}_t$ measurable as in the case of stochastic discount factor where $v_t(m,\omega) = D_t(\omega)m$. By adapting the definitions in the static framework of [3] and [11] we set:

$$\rho_{C_T,v_t}(X)(\omega) = ess \inf_{Y \in L^0_{\mathscr{F}_t}} \{v_t(Y,\omega) \mid X + Y \in C_T\}.$$

When v_t is linear, then ρ_{C,v_t} is a convex monetary dynamic risk measure, but the linearity of v_t may fail when zero coupon bonds with maturity T are illiquid. It seems anyway reasonable to assume that $v_t(\cdot,\omega)$ is increasing and upper semicontinuous and $v_t(0,\omega) = 0$, for $\mathbb{P}$ almost every $\omega \in \Omega$. In this case

$$\rho_{C_T,v_t}(X)(\omega) = v_t(ess \inf_{Y \in L^0_{\mathscr{F}_t}} \{Y \mid X + Y \in C_T\}, \omega) = v_t(\rho_{C_T}(X),\omega),$$

where $\rho_{C_T}(X)$ is the convex monetary dynamic risk measure induced by the set C_T. Thus in general ρ_{C_T,v_t} is neither convex nor cash additive, but it is quasiconvex and eventually cash subadditive (under further assumptions on v_t).

Acceptability Indices

As studied in [12] the index of acceptability is a map α from a space of random variables $L(\Omega,\mathscr{F},\mathbb{P})$ to $[0,+\infty)$ which measures the performance or quality of the random X which may be the terminal cash flow from a trading strategy. Associated with each level x of the index there is a collection of terminal cash flows $\mathscr{A}_x=\{X\in L|\alpha(X)\geq x\}$ that are acceptable at this level . The authors in [12] suggest four axioms as the stronghold for an acceptability index in the static case: quasiconcavity (i.e. the set $\mathscr{A}_x$ is convex for every $x\in[0,+\infty)$), monotonicity, scale invariance and the Fatou property. It appears natural to generalize these kind of indices to the conditional case and to this aim we propose a couple of basic examples:

i) Conditional Gain Loss Ratio: let $\mathscr{G}\subseteq\mathscr{F}$

$$CGLR(X|\mathscr{G})=\frac{E_{\mathbb{P}}[X|\mathscr{G}]}{E_{\mathbb{P}}[X^-|\mathscr{G}]}\mathbf{1}_{\{E_{\mathbb{P}}[X|\mathscr{G}]>0\}}.$$

This measure is clearly monotone, scale invariant, and well defined on $L^1(\Omega,\mathscr{F},\mathbb{P})$. It can be proved that it is continuous from below and quasiconcave.

ii) Conditional Coherent Risk-Adjusted Return on Capital: let $\mathscr{G}\in\mathscr{F}$ and suppose a coherent conditional risk measure $\rho:L(\Omega,\mathscr{F},\mathbb{P})\to L^0(\Omega,\mathscr{G},\mathbb{P})$ is given with $L(\Omega,\mathscr{F},\mathbb{P})\subseteq L^1(\Omega,\mathscr{F},\mathbb{P})$ is any vector space. We define

$$CRARoC(X|\mathscr{G})=\frac{E_{\mathbb{P}}[X|\mathscr{G}]}{\rho(X)}\mathbf{1}_{\{E_{\mathbb{P}}[X|\mathscr{G}]>0\}}.$$

We use the convention that $CRARoC(X|\mathscr{G})=+\infty$ on the $\mathscr{G}$-measurable set where $\rho(X)\leq 0$. Again $CRARoC(\cdot|\mathscr{G})$ is well defined on the space $L(\Omega,\mathscr{F},\mathbb{P})$ and takes values in the space of extended random variables; moreover is monotone, quasiconcave, scale invariant and continuous from below whenever ρ is continuous from above.

1.4.2 Back to the representation of convex risk measures

In the following Lemma and Corollary, proved in Section 1.5.2, we show that the (MON) property implies that the constraint $E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]$ may be restricted to $E_Q[\xi|\mathscr{G}]=_Q E_Q[X|\mathscr{G}]$ and that we may recover the dual representation of a dynamic risk measure. When $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}_{\mathscr{G}}$ the previous inequality/equality may be equivalently intended Q-a.s. or $\mathbb{P}$-a.s. and so we do not need any more to emphasize this in the notations.

Lemma 1.1. *Suppose that for every $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}_{\mathscr{G}}$ and $\xi\in L_{\mathscr{F}}$ we have $E_Q[\xi|\mathscr{G}]\in L_{\mathscr{F}}$. If $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}_{\mathscr{G}}$ and if $\pi:L_{\mathscr{F}}\to L_{\mathscr{G}}$ is (MON) and (REG) then*

$$K(X,Q)=\inf_{\xi\in L_{\mathscr{F}}}\{\pi(\xi)\mid E_Q[\xi|\mathscr{G}]=E_Q[X|\mathscr{G}]\}\,. \tag{1.9}$$

Proof. Let us denote with $r(X,Q)$ the right hand side of equation (3.20) and notice that $K(X,Q) \leq r(X,Q)$. By contradiction, suppose that $\mathbb{P}(A) > 0$ where $A =: \{K(X,Q) < r(X,Q)\}$. As shown in Lemma 1.4 iv), there exists a r.v. $\xi \in L_{\mathscr{F}}$ satisfying the following conditions

- $E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]$ and $Q(E_Q[\xi|\mathscr{G}] > E_Q[X|\mathscr{G}]) > 0$.
- $K(X,Q)(\omega) \leq \pi(\xi)(\omega) < r(X,Q)(\omega)$ for $\mathbb{P}$-almost every $\omega \in B \subseteq A$ and $\mathbb{P}(B) > 0$.

Set $Z =_Q E_Q[\xi - X|\mathscr{G}]$. By assumption, $Z \in L_{\mathscr{F}}$ and it satisfies $Z \geq_Q 0$ and, since $Q \in \mathscr{P}_{\mathscr{G}}$, $Z \geq 0$. Then, thanks to (MON), $\pi(\xi) \geq \pi(\xi - Z)$.
From $E_Q[\xi - Z|\mathscr{G}] =_Q E_Q[X|\mathscr{G}]$ we deduce:

$$K(X,Q)(\omega) \leq \pi(\xi)(\omega) < r(X,Q)(\omega) \leq \pi(\xi - Z)(\omega) \text{ for } \mathbb{P}\text{-a.e. } \omega \in B,$$

which is a contradiction.

Definition 1.5. The conditional Fenchel convex conjugate π^* of π is given, for $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}_{\mathscr{G}}$, by the extended valued $\mathscr{G}-$measurable random variable:

$$\pi^*(Q) = \sup_{\xi \in L_{\mathscr{F}}} \left\{ E_Q[\xi|\mathscr{G}] - \pi(\xi) \right\}.$$

A map $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ is said to be

(CAS) *cash invariant if for all $X \in L_{\mathscr{F}}$ and $\Lambda \in L_{\mathscr{G}}$*

$$\pi(X + \Lambda) = \pi(X) + \Lambda.$$

In the literature [36], [17], [29] a map $\rho : L_{\mathscr{F}} \to L_{\mathscr{G}}$ that is monotone (decreasing), convex, cash invariant and regular is called a *convex conditional (or dynamic) risk measure*. As a corollary of our main theorem, we deduce immediately the dual representation of a map $\rho(\cdot) =: \pi(-\cdot)$ satisfying (CAS), in terms of the Fenchel conjugate π^*, in agreement with [17]. Of course, this is of no surprise since the (CAS) and (QCO) properties imply convexity, but it supports the correctness of our dual representation.

Corollary 1.2. *Suppose that for every $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}_{\mathscr{G}}$ and $\xi \in L_{\mathscr{F}}$ we have $E_Q[\xi|\mathscr{G}] \in L_{\mathscr{F}}$.*
(i) If $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}_{\mathscr{G}}$ and if $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ is (MON), (REG) and (CAS) then

$$K(X,Q) = E_Q[X|\mathscr{G}] - \pi^*(Q). \tag{1.10}$$

(ii) Under the same assumptions of Theorem 1.2 and if π satisfies in addition (CAS) then

$$\pi(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}_{\mathscr{G}}} \left\{ E_Q[X|\mathscr{G}] - \pi^*(Q) \right\}.$$

so that $\rho(\cdot) = \pi(-\cdot)$ is a conditional convex risk measure and can be represented as

$$\rho(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}_{\mathscr{G}}} \{E_Q[-X|\mathscr{G}] - \rho^*(-Q)\}.$$

with $\rho^*(-Q)$ *given by*

$$\rho^*(-Q) = \sup_{\xi \in L_{\mathscr{F}}} \{E_Q[-\xi|\mathscr{G}] - \rho(\xi)\}.$$

Proof. The (CAS) property implies that for every $X \in L_{\mathscr{F}}$ and $\delta > 0$, $\mathbb{P}(\pi(X-2\delta)+\delta < \pi(X)) = 1$. So the hypothesis of Corollary 1.1 holds true and we only need to prove (3.23), since (ii) is a consequence of (i) and Corollary 1.1. Let $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}_{\mathscr{G}}$. Applying Lemma 1.1 we deduce:

$$\begin{aligned}
K(X,Q) &= \inf_{\xi \in L_{\mathscr{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathscr{G}] =_Q E_Q[X|\mathscr{G}]\} \\
&= E_Q[X|\mathscr{G}] + \inf_{\xi \in L_{\mathscr{F}}} \{\pi(\xi) - E_Q[X|\mathscr{G}] \mid E_Q[\xi|\mathscr{G}] =_Q E_Q[X|\mathscr{G}]\} \\
&= E_Q[X|\mathscr{G}] + \inf_{\xi \in L_{\mathscr{F}}} \{\pi(\xi) - E_Q[\xi|\mathscr{G}] \mid E_Q[\xi|\mathscr{G}] =_Q E_Q[X|\mathscr{G}]\} \\
&= E_Q[X|\mathscr{G}] - \sup_{\xi \in L_{\mathscr{F}}} \{E_Q[\xi|\mathscr{G}] - \pi(\xi) \mid E_Q[\xi|\mathscr{G}] =_Q E_Q[X|\mathscr{G}]\} \\
&= E_Q[X|\mathscr{G}] - \pi^*(Q),
\end{aligned}$$

where the last equality follows from $Q \in \mathscr{P}_{\mathscr{G}}$ and

$$\begin{aligned}
\pi^*(Q) &= \sup_{\xi \in L_{\mathscr{F}}} \{E_Q[\xi + E_Q[X-\xi|\mathscr{G}] \mid \mathscr{G}] - \pi(\xi + E_Q[X-\xi|\mathscr{G}])\} \\
&= \sup_{\eta \in L_{\mathscr{F}}} \{E_Q[\eta|\mathscr{G}] - \pi(\eta) \mid \eta = \xi + E_Q[X-\xi|\mathscr{G}]\} \\
&\leq \sup_{\xi \in L_{\mathscr{F}}} \{E_Q[\xi|\mathscr{G}] - \pi(\xi) \mid E_Q[\xi|\mathscr{G}] =_Q E_Q[X|\mathscr{G}]\} \leq \pi^*(Q).
\end{aligned}$$

1.5 Preliminaries

In the sequel of this section it is always assumed that $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ satisfies (REG).

1.5.1 *Properties of* $R(Y,\xi')$

We remind that Σ denotes the actual domain of R as given in (1.6). Given an arbitrary $(Y,\xi') \in \Sigma$, we have $R(Y,\xi') = \inf \mathscr{A}(Y,\xi')$ where

$$\mathscr{A}(Y,\xi') := \{\pi(\xi) \mid \xi \in L_{\mathscr{F}},\ E_{\mathbb{P}}[\xi'\xi|\mathscr{G}] \geq Y\}.$$

By convention $R(Y,\xi') = +\infty$ for every $(Y,\xi') \in (L^0_{\mathscr{G}} \times L^*_{\mathscr{F}}) \setminus \Sigma$

Lemma 1.2. *For every* $(Y,\xi')\in\Sigma$ *the set* $\mathscr{A}(Y,\xi')$ *is downward directed and therefore there exists a sequence* $\{\eta_m\}_{m=1}^{\infty}\in L_{\mathscr{F}}$ *such that* $E_{\mathbb{P}}[\xi'\eta_m|\mathscr{G}]\geq Y$ *and as* $m\uparrow\infty$*,* $\pi(\eta_m)\downarrow R(Y,\xi')$.

Proof. We have to prove that for every $\pi(\xi_1),\pi(\xi_2)\in\mathscr{A}(Y,\xi')$ there exists $\pi(\xi^*)\in\mathscr{A}(Y,\xi')$ such that $\pi(\xi^*)\leq\min\{\pi(\xi_1),\pi(\xi_2)\}$. Consider the $\mathscr{G}$-measurable set $G=\{\pi(\xi_1)\leq\pi(\xi_2)\}$ then

$$\min\{\pi(\xi_1),\pi(\xi_2)\}=\pi(\xi_1)\mathbf{1}_G+\pi(\xi_2)\mathbf{1}_{G^C}=\pi(\xi_1\mathbf{1}_G+\xi_2\mathbf{1}_{G^C})=\pi(\xi^*),$$

where $\xi^*=\xi_1\mathbf{1}_G+\xi_2\mathbf{1}_{G^C}$. Hence $E_{\mathbb{P}}[\xi'\xi^*|\mathscr{G}]=E_{\mathbb{P}}[\xi'\xi_1|\mathscr{G}]\mathbf{1}_G+E_{\mathbb{P}}[\xi'\xi_2|\mathscr{G}]\mathbf{1}_{G^C}\geq Y$ so that we can deduce $\pi(\xi^*)\in\mathscr{A}(Y,\xi')$.

Lemma 1.3. *Properties of* $R(Y,\xi')$.
i) $R(\cdot,\xi')$ *is monotone, for every* $\xi'\in L^*_{\mathscr{F}}$.
ii) $R(\lambda Y,\lambda\xi')=R(Y,\xi')$ *for any* $\lambda>0$, $Y\in L^0_{\mathscr{G}}$ *and* $\xi'\in L^*_{\mathscr{F}}$.
iii) For every $A\in\mathscr{G}$, $(Y,\xi')\in\Sigma$

$$R(Y,\xi')\mathbf{1}_A=\inf_{\xi\in L_{\mathscr{F}}}\left\{\pi(\xi)\mathbf{1}_A\mid E_{\mathbb{P}}[\xi'\xi|\mathscr{G}]\geq Y\right\}\tag{1.11}$$

$$=\inf_{\xi\in L_{\mathscr{F}}}\left\{\pi(\xi)\mathbf{1}_A\mid E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathscr{G}]\geq Y\mathbf{1}_A\right\}=R(Y\mathbf{1}_A,\xi')\mathbf{1}_A.\tag{1.12}$$

iv) $R(Y,\xi')$ *is jontly quasiconcave on* $L^0_{\mathscr{G}}\times L^*_{\mathscr{F}}$.
v) $\inf_{Y\in L^0_{\mathscr{G}}}R(Y,\xi_1')=\inf_{Y\in L^0_{\mathscr{G}}}R(Y,\xi_2')$ *for every* $\xi_1',\xi_2'\in L^*_{\mathscr{F}}$.
vi) For every $Y_1,Y_2\in L^0_{\mathscr{G}}$
(a) $R(Y_1,\xi')\wedge R(Y_2,\xi')=R(Y_1\wedge Y_2,\xi')$
(b) $R(Y_1,\xi')\vee R(Y_2,\xi')=R(Y_1\vee Y_2,\xi')$
vii) The map $R(\cdot,\xi')$ *is quasi-affine in the sense that for every* $Y_1,Y_2,\Lambda\in L^0_{\mathscr{G}}$ *and* $0\leq\Lambda\leq 1$, *we have*
$R(\Lambda Y_1+(1-\Lambda)Y_2,\xi')\geq R(Y_1,\xi')\wedge R(Y_2,\xi')$ *(quasiconcavity)*
$R(\Lambda Y_1+(1-\Lambda)Y_2,\xi')\leq R(Y_1,\xi')\vee R(Y_2,\xi')$ *(quasiconvexity).*

Proof. (i) and (ii) are trivial consequences of the definition.
(iii) By definition of the essential infimum one easily deduce (1.11). To prove (1.12), for every $\xi\in L_{\mathscr{F}}$ such that $E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathscr{G}]\geq Y\mathbf{1}_A$ we define the random variable $\eta=\xi\mathbf{1}_A+\zeta\mathbf{1}_{A^C}$ where $E_{\mathbb{P}}[\xi'\zeta|\mathscr{G}]\geq Y$. Then $E_{\mathbb{P}}[\xi'\eta|\mathscr{G}]\geq Y$ and we can conclude

$$\left\{\eta\mathbf{1}_A\mid\eta\in L_{\mathscr{F}},E_{\mathbb{P}}[\xi'\eta|\mathscr{G}]\geq Y\right\}=\left\{\xi\mathbf{1}_A\mid\xi\in L_{\mathscr{F}},E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathscr{G}]\geq Y\mathbf{1}_A\right\}$$

Hence from (1.11) and (REG):

$$\begin{aligned}\mathbf{1}_A R(Y,\xi')&=\inf_{\eta\in L_{\mathscr{F}}}\left\{\pi(\eta\mathbf{1}_A)\mathbf{1}_A\mid E_{\mathbb{P}}[\xi'\eta|\mathscr{G}]\geq Y\right\}\\&=\inf_{\xi\in L_{\mathscr{F}}}\left\{\pi(\xi\mathbf{1}_A)\mathbf{1}_A\mid E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathscr{G}]\geq Y\mathbf{1}_A\right\}\\&=\inf_{\xi\in L_{\mathscr{F}}}\left\{\pi(\xi)\mathbf{1}_A\mid E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathscr{G}]\geq Y\mathbf{1}_A\right\}.\end{aligned}$$

The second equality in (1.12) follows in a similar way since again

$$\left\{\eta\mathbf{1}_A \mid \eta\in L_{\mathscr{F}},\, E_{\mathbb{P}}[\xi'\eta|\mathscr{G}]\geq Y\right\}=\left\{\xi\mathbf{1}_A \mid \xi\in L_{\mathscr{F}},\, E_{\mathbb{P}}[\xi'\xi|\mathscr{G}]\geq Y\mathbf{1}_A\right\}$$

(iv) Consider $(Y_1,\xi_1'),(Y_2,\xi_2')\in L^0_{\mathscr{G}}\times L^*_{\mathscr{F}}$ and $\lambda\in(0,1)$. Define $(Y,\xi')=(\lambda Y_1+(1-\lambda)Y_2,\lambda\xi_1'+(1-\lambda)\xi_2')$ and notice that for every $A\in\mathscr{G}$ the set $\{\xi\in L_{\mathscr{F}}\mid E[\xi'\xi\mathbf{1}_A]\geq E[Y\mathbf{1}_A]\}$ is contained in

$$\{\xi\in L_{\mathscr{F}}\mid E[\xi_1'\xi\mathbf{1}_A]\geq E[Y_1\mathbf{1}_A]\}\cup\{\xi\in L_{\mathscr{F}}\mid E[\xi_2'\xi\mathbf{1}_A]\geq E[Y_2\mathbf{1}_A]\}.$$

Taking the intersection over all $A\in\mathscr{G}$ we get that $\{\xi\in L_{\mathscr{F}}\mid E[\xi'\xi|\mathscr{G}]\geq Y\}$ is included in

$$\{\xi\in L_{\mathscr{F}}\mid E[\xi_1'\xi|\mathscr{G}]\geq Y_1\}\cup\{\xi\in L_{\mathscr{F}}\mid E[\xi_2'\xi|\mathscr{G}]\geq Y_2\},$$

which implies $R(Y,\xi')\geq R(Y_1,\xi_1')\wedge R(Y_2,\xi_2')$.
(v) This is a generalization of Theorem 2 (H2) in [10]. In fact on one hand

$$R(Y,\xi')\geq\inf_{\xi\in L_{\mathscr{F}}}\pi(\xi)\quad\forall Y\in L^0_{\mathscr{F}}$$

implies

$$\inf_{Y\in L^0_{\mathscr{G}}}R(Y,\xi')\geq\inf_{\xi\in L_{\mathscr{F}}}\pi(\xi).$$

On the other

$$\pi(\xi)\geq R(E_{\mathbb{P}}[\xi\xi'|\mathscr{G}],\xi')\geq\inf_{Y\in L^0_{\mathscr{G}}}R(Y,\xi')\quad\forall\xi\in L_{\mathscr{F}}$$

implies

$$\inf_{Y\in L^0_{\mathscr{G}}}R(Y,\xi')\leq\inf_{\xi\in L_{\mathscr{F}}}\pi(\xi).$$

vi) a): Since $R(\cdot,\xi')$ is monotone, the inequalities $R(Y_1,\xi')\wedge R(Y_2,\xi')\geq R(Y_1\wedge Y_2,\xi')$ and $R(Y_1,\xi')\vee R(Y_2,\xi')\leq R(Y_1\vee Y_2,\xi')$ are always true.
To show the opposite inequalities, define the $\mathscr{G}$-measurable sets: $B:=\{R(Y_1,\xi')\leq R(Y_2,\xi')\}$ and $A:=\{Y_1\leq Y_2\}$ so that

$$\begin{aligned}
R(Y_1,\xi')\wedge R(Y_2,\xi')&=R(Y_1,\xi')\mathbf{1}_B+R(Y_2,\xi')\mathbf{1}_{B^C}\leq R(Y_1,\xi')\mathbf{1}_A+R(Y_2,\xi')\mathbf{1}_{A^C}\\
R(Y_1,\xi')\vee R(Y_2,\xi')&=R(Y_1,\xi')\mathbf{1}_{B^C}+R(Y_2,\xi')\mathbf{1}_B\geq R(Y_1,\xi')\mathbf{1}_{A^C}+R(Y_2,\xi')\mathbf{1}_A
\end{aligned}\tag{1.13}$$

Set: $D(A,Y)=\{\xi\mathbf{1}_A\mid\xi\in L_{\mathscr{F}},E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathscr{G}]\geq Y\mathbf{1}_A\}$ and check that

$$D(A,Y_1)+D(A^C,Y_2)=\left\{\xi\in L_{\mathscr{F}}\mid E_{\mathbb{P}}[\xi'\xi|\mathscr{G}]\geq Y_1\mathbf{1}_A+Y_2\mathbf{1}_{A^C}\right\}:=D$$

From (3.10) and using (1.12) we get:

$$\begin{aligned}
R(Y_1,\xi')\wedge R(Y_2,\xi') \leq\ & R(Y_1,\xi')\mathbf{1}_A + R(Y_2,\xi')\mathbf{1}_{A^C}\\
=\ & \inf_{\xi\mathbf{1}_A\in D(A,Y_1)}\{\pi(\xi\mathbf{1}_A)\mathbf{1}_A\} + \inf_{\eta\mathbf{1}_{A^C}\in D(A^C,Y_2)}\{\pi(\eta\mathbf{1}_{A^C})\mathbf{1}_{A^C}\}\\
=\ & \inf_{\substack{\xi\mathbf{1}_A\in D(A,Y_1)\\ \eta\mathbf{1}_{A^C}\in D(A^C,Y_2)}}\{\pi(\xi\mathbf{1}_A)\mathbf{1}_A + \pi(\eta\mathbf{1}_{A^C})\mathbf{1}_{A^C}\}\\
=\ & \inf_{(\xi\mathbf{1}_A+\eta\mathbf{1}_{A^C})\in D(A,Y_1)+D(A^C,Y_2)}\{\pi(\xi\mathbf{1}_A+\eta\mathbf{1}_{A^C})\}\\
=\ & \inf_{\xi\in D}\{\pi(\xi)\} = R(Y_1\mathbf{1}_A + Y_2\mathbf{1}_{A^C},\xi') = R(Y_1\wedge Y_2,\xi').
\end{aligned}$$

Simile modo: vi) $b)$.
(vii) Follows from point (vi) and (i).

1.5.2 Properties of $K(X,Q)$

For $\xi'\in L^*_{\mathscr{F}}\cap(L^1_{\mathscr{F}})_+$ and $X\in L_{\mathscr{F}}$

$$R(E_{\mathbb{P}}[\xi' X|\mathscr{G}],\xi') = \inf_{\xi\in L_{\mathscr{F}}}\left\{\pi(\xi)\mid E_{\mathbb{P}}[\xi'\xi|\mathscr{G}]\geq E_{\mathbb{P}}[\xi' X|\mathscr{G}]\right\} = K(X,\xi').$$

Notice that $K(X,\xi') = K(X,\lambda\xi')$ for every $\lambda>0$ and thus we can consider $K(X,\xi')$, $\xi'\neq 0$, always defined on the normalized elements $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$.
Moreover, it is easy to check that:

$$E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}\xi\mid\mathscr{G}\right]\geq E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}X\mid\mathscr{G}\right]\Longleftrightarrow E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}].$$

For $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$ we then set:

$$K(X,Q):=\inf_{\xi\in L_{\mathscr{F}}}\{\pi(\xi)\mid E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]\} = R\left(E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}X\mid\mathscr{G}\right],\frac{dQ}{d\mathbb{P}}\right).$$

Lemma 1.4. *Properties of $K(X,Q)$. Let $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$ and $X\in L_{\mathscr{F}}$.*
i) $K(\cdot,Q)$ is monotone and quasi affine.
ii) $K(X,\cdot)$ is scaling invariant: $K(X,\Lambda Q)=K(X,Q)$ for every $\Lambda\in(L^0_{\mathscr{G}})_+$.
iii) $K(X,Q)\mathbf{1}_A = \inf_{\xi\in L_{\mathscr{F}}}\{\pi(\xi)\mathbf{1}_A\mid E_Q[\xi\mathbf{1}_A|\mathscr{G}]\geq_Q E_Q[X\mathbf{1}_A|\mathscr{G}]\}$ for all $A\in\mathscr{G}$.
iv) There exists a sequence $\left\{\xi^Q_m\right\}_{m=1}^{\infty}\in L_{\mathscr{F}}$ such that

$$E_Q[\xi^Q_m|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]\quad\forall m\geq 1,\quad \pi(\xi^Q_m)\downarrow K(X,Q)\quad \text{as } m\uparrow\infty.$$

v) The set $\mathscr{K}=\left\{K(X,Q)\mid Q\in L^*_{\mathscr{F}}\cap\mathscr{P}\right\}$ is upward directed, i.e. for every $K(X,Q_1)$, $K(X,Q_2)\in\mathscr{K}$ there exists $K(X,\widehat{Q})\in\mathscr{K}$ such that $K(X,\widehat{Q})\geq K(X,Q_1)\vee K(X,Q_2)$.

vi) Let Q_1 and Q_2 be elements of $L^*_{\mathscr{F}} \cap \mathscr{P}$ and $B \in \mathscr{G}$. If $\frac{dQ_1}{d\mathbb{P}} \mathbf{1}_B = \frac{dQ_2}{d\mathbb{P}} \mathbf{1}_B$ then $K(X,Q_1)\mathbf{1}_B = K(X,Q_2)\mathbf{1}_B$.

Proof. The monotonicity property in (i), (ii) and (iii) are trivial; from Lemma 1.3 v) it follows that $K(\cdot,Q)$ is quasi affine; (iv) is an immediate consequence of Lemma 3.1.

(v) Define $F = \{K(X,Q_1) \geq K(X,Q_2)\}$ and let $\widehat{Q}$ given by $\frac{d\widehat{Q}}{d\mathbb{P}} := \mathbf{1}_F \frac{dQ_1}{d\mathbb{P}} + \mathbf{1}_{F^C} \frac{dQ_2}{d\mathbb{P}}$; up to a normalization factor (from property (ii)) we may suppose $\widehat{Q} \in L^*_{\mathscr{F}} \cap \mathscr{P}$. We need to show that

$$K(X,\widehat{Q}) = K(X,Q_1) \vee K(X,Q_2) = K(X,Q_1)\mathbf{1}_F + K(X,Q_2)\mathbf{1}_{F^C}.$$

From $E_{\widehat{Q}}[\xi|\mathscr{G}] =_{\widehat{Q}} E_{Q_1}[\xi|\mathscr{G}]\mathbf{1}_F + E_{Q_2}[\xi|\mathscr{G}]\mathbf{1}_{F^C}$ we get $E_{\widehat{Q}}[\xi|\mathscr{G}]\mathbf{1}_F =_{Q_1} E_{Q_1}[\xi|\mathscr{G}]\mathbf{1}_F$ and $E_{\widehat{Q}}[\xi|\mathscr{G}]\mathbf{1}_{F^C} =_{Q_2} E_{Q_2}[\xi|\mathscr{G}]\mathbf{1}_{F^C}$. In the second place, for $i = 1,2$, consider the sets

$$\widehat{A} = \{\xi \in L_{\mathscr{F}} \mid E_{\widehat{Q}}[\xi|\mathscr{G}] \geq_{\widehat{Q}} E_{\widehat{Q}}[X|\mathscr{G}]\} \quad A_i = \{\xi \in L_{\mathscr{F}} \mid E_{Q_i}[\xi|\mathscr{G}] \geq_{Q_i} E_{Q_i}[X|\mathscr{G}]\}.$$

For every $\xi \in A_1$ define $\eta = \xi\mathbf{1}_F + X\mathbf{1}_{F^C}$

$$Q_1 << \mathbb{P} \Rightarrow \quad \eta\mathbf{1}_F =_{Q_1} \xi\mathbf{1}_F \Rightarrow \quad E_{\widehat{Q}}[\eta|\mathscr{G}]\mathbf{1}_F \geq_{\widehat{Q}} E_{\widehat{Q}}[X|\mathscr{G}]\mathbf{1}_F$$
$$Q_2 << \mathbb{P} \Rightarrow \eta\mathbf{1}_{F^C} =_{Q_2} X\mathbf{1}_{F^C} \Rightarrow E_{\widehat{Q}}[\eta|\mathscr{G}]\mathbf{1}_{F^C} =_{\widehat{Q}} E_{\widehat{Q}}[X|\mathscr{G}]\mathbf{1}_{F^C}$$

Then $\eta \in \widehat{A}$ and $\pi(\xi)\mathbf{1}_F = \pi(\xi\mathbf{1}_F) - \pi(0)\mathbf{1}_{F^C} = \pi(\eta\mathbf{1}_F) - \pi(0)\mathbf{1}_{F^C} = \pi(\eta)\mathbf{1}_F$. Viceversa, for every $\eta \in \widehat{A}$ define $\xi = \eta\mathbf{1}_F + X\mathbf{1}_{F^C}$. Then $\xi \in A_1$ and again $\pi(\xi)\mathbf{1}_F = \pi(\eta)\mathbf{1}_F$. Hence

$$\inf_{\xi \in A_1} \pi(\xi)\mathbf{1}_F = \inf_{\eta \in \widehat{A}} \pi(\eta)\mathbf{1}_F.$$

In a similar way: $\inf_{\xi \in A_2} \pi(\xi)\mathbf{1}_{F^C} = \inf_{\eta \in \widehat{A}} \pi(\eta)\mathbf{1}_{F^C}$ and we can finally deduce $K(X,Q_1) \vee K(X,Q_2) = K(X,\widehat{Q})$.

(vi). By the same argument used in (v), it can be shown that $\inf_{\xi \in A_1} \pi(\xi)\mathbf{1}_B = \inf_{\xi \in A_2} \pi(\xi)\mathbf{1}_B$ and the thesis.

1.5.3 Properties of H(X) and an uniform approximation

For $X \in L_{\mathscr{F}}$ we set

$$H(X) := \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} K(X,Q) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} \inf_{\xi \in L_{\mathscr{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\}$$

and notice that for all $A \in \mathscr{G}$

$$H(X)\mathbf{1}_A = \sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}} \inf_{\xi\in L_{\mathscr{F}}} \{\pi(\xi)\mathbf{1}_A \mid E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\}.$$

In the following Lemma we show that H is a good candidate to reach the dual representation.

Lemma 1.5. *Properties of $H(X)$. Let $X \in L_{\mathscr{F}}$.*

i) H is (MON) and (QCO)

ii) $H(X\mathbf{1}_A)\mathbf{1}_A = H(X)\mathbf{1}_A$ for any $A \in \mathscr{G}$ i.e. H is (REG) .

iii) There exist a sequence $\{Q^k\}_{k\geq 1} \in L^*_{\mathscr{F}}$ and, for each $k \geq 1$, a sequence $\{\xi_m^{Q^k}\}_{m\geq 1} \in L_{\mathscr{F}}$ satisfying $E_{Q^k}[\xi_m^{Q^k} \mid \mathscr{G}] \geq_{Q^k} E_{Q^k}[X|\mathscr{G}]$ and

$$\pi(\xi_m^{Q^k}) \downarrow K(X,Q^k) \text{ as } m\uparrow\infty,\ K(X,Q^k)\uparrow H(X) \text{ as } k\uparrow\infty, \tag{1.14}$$

$$H(X) = \lim_{k\to\infty}\lim_{m\to\infty} \pi(\xi_m^{Q^k}). \tag{1.15}$$

Proof. i) (MON) and (QCO) follow from Lemma 1.4 (i); ii) follows applying the same argument used in equation (1.12); the other property is an immediate consequence of what proved in Lemma 1.4 and 3.1 regarding the properties of being downward directed and upward directed.

The following Proposition is an uniform approximation result which stands under stronger assumptions, that are satisfied, for example, by L^p spaces, $p \in [1,+\infty]$. We will not use this Proposition in the proof of Theorem 1.2, even though it can be useful for understanding the heuristic outline of its proof, as sketched in Section 1.6.1.

Proposition 1.3. *Suppose that $L^*_F \hookrightarrow L^1_{\mathscr{F}}$ is a Banach Lattice with the property: for any sequence $\{\eta_n\}_n \subseteq (L^*_F)_+$, $\eta_n\eta_m = 0$ for every $n \neq m$, there exists a sequence $\{\alpha_k\}_k \subset (0,+\infty)$ such that $\sum_n \alpha_n\eta_n \in (L^*_F)_+$. Then for every $\varepsilon > 0$ there exists $Q_\varepsilon \in L^*_F \cap \mathscr{P}$ such that*

$$H(X) - K(X,Q_\varepsilon) < \varepsilon \tag{1.16}$$

on the set $F^\infty = \{H(X) < +\infty\}$.

Proof. From Lemma 1.5, eq. (1.14), we know that there exists a sequence $Q_k \in L^*_{\mathscr{F}} \cap \mathscr{P}$ such that:

$$K(X,Q_k)\uparrow H(X), \text{ as } k\uparrow\infty.$$

Define for each $k \geq 1$ the sets

$$D_k =: \{\omega \in F^\infty \mid H(X)(\omega) - K(X,Q_k)(\omega) \leq \varepsilon\}$$

and note that

$$\mathbb{P}(F^\infty \setminus D_k) \downarrow 0 \text{ as } k\uparrow\infty. \tag{1.17}$$

Consider the disjoint family $\{F_k\}_{k\geq 1}$ of $\mathscr{G}-$measurable sets: $F_1 = D_1$, $\quad F_k = D_k \setminus D_{k-1}$, $k \geq 2$. By induction one easily shows that $\bigcup_{k=1}^n F_k = D_n$ for all $n \geq 1$. This and (1.17) imply that $\mathbb{P}\left(F^{\infty}\setminus\bigcup_{k=1}^{\infty}F_k\right)=0$. Consider the sequence $\left\{\frac{dQ_k}{d\mathbb{P}}\mathbf{1}_{F_k}\right\}$. From the assumption on $L^*_{\mathscr{F}}$ we may find a sequence $\{\alpha_k\}_k\subset(0,+\infty)$ such that $\frac{d\widetilde{Q}_{\varepsilon}}{d\mathbb{P}}=:\sum_{k=1}^{\infty}\alpha_k\frac{dQ_k}{d\mathbb{P}}\mathbf{1}_{F_k}\in L^*_{\mathscr{F}}\hookrightarrow L^1_{\mathscr{F}}$. Hence, $\widetilde{Q}_{\varepsilon}\in(L^*_{\mathscr{F}})_+\cap(L^1_{\mathscr{F}})_+$ and, since $\{F_k\}_{k\geq 1}$ are disjoint,

$$\frac{d\widetilde{Q}_{\varepsilon}}{d\mathbb{P}}\mathbf{1}_{F_k}=\alpha_k\frac{dQ_k}{d\mathbb{P}}\mathbf{1}_{F_k},\ \text{for any } k\geq 1.$$

Normalize $\widetilde{Q}_{\varepsilon}$ and denote with $Q_{\varepsilon}=\lambda\widetilde{Q}_{\varepsilon}\in L^*_{\mathscr{F}}\cap\mathscr{P}$ the element satisfying $\|\frac{dQ_{\varepsilon}}{d\mathbb{P}}\|_{L^1_F}=1$. Applying Lemma 1.4 (vi) we deduce that for any $k\geq 1$

$$K(X,Q_{\varepsilon})\mathbf{1}_{F_k}=K(X,\widetilde{Q}_{\varepsilon})\mathbf{1}_{F_k}=K(X,\alpha_kQ_k)\mathbf{1}_{F_k}=K(X,Q_k)\mathbf{1}_{F_k},$$

and

$$H(X)\mathbf{1}_{F_k}-K(X,Q_{\varepsilon})\mathbf{1}_{F_k}=H(X)\mathbf{1}_{F_k}-K(X,Q_k)\mathbf{1}_{F_k}\leq\varepsilon\mathbf{1}_{F_k}.$$

The condition (3.7) is then a consequence of equation (1.17).

1.5.4 On the map π_A

Consider the following

Definition 1.6. Given $\pi:L_{\mathscr{F}}\to L_{\mathscr{G}}$ we define for every $A\in\mathscr{G}$, the map

$$\pi_A:L_{\mathscr{F}}\to\overline{\mathbb{R}}\ \text{ by }\ \pi_A(X):=ess\sup_{\omega\in A}\pi(X)(\omega).$$

Proposition 1.4. *Under the same assumptions of Theorem 1.2 (resp. Theorem 1.3) and for any $A\in\mathscr{G}$*

$$\pi_A(X)=\sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}}\inf_{\xi\in L_{\mathscr{F}}}\left\{\pi_A(\xi)\mid E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]\right\}. \tag{1.18}$$

Proof. Notice that the map π_A inherits from π the properties (MON) and (QCO).

1) Under the assumptions of Theorem 1.2, applying Proposition 1.2 we get that π is (CFB) and this obviously implies that π_A is (CFB). Applying to π_A Proposition 1.2 , which holds also for real valued maps, we deduce that π_A is $\sigma(L_{\mathscr{F}},L^*_{\mathscr{F}})$-(LSC).

2) Under the assumptions of Theorem 1.3 we prove that π_A is τ-(USC) by showing that $\mathscr{B}_c:=\{\xi\in L_{\mathscr{F}}|\pi_A(\xi)<c\}$ is τ open, for any fixed $c\in\mathbb{R}$. W.l.o.g. $\mathscr{B}_c\neq\emptyset$. If we fix an arbitrary $\eta\in\mathscr{B}_c$, we may find $\delta>0$ such that $\pi_A(\eta)<c-\delta$. Define

$$\mathscr{B}:=\{\xi\in L_{\mathscr{F}}\mid\pi(\xi)<(c-\delta)\mathbf{1}_A+(\pi(\eta)+\delta)\mathbf{1}_{A^C}\}.$$

Since $(c-\delta)\mathbf{1}_A+(\pi(\eta)+\delta)\mathbf{1}_{A^C}\in L_{\mathscr{G}}$ and π is (USC) we deduce that $\mathscr{B}$ is τ open. Moreover $\pi_A(\xi)\leq c-\delta$ for every $\xi\in\mathscr{B}$, i.e. $\mathscr{B}\subseteq\mathscr{B}_c$, and $\eta\in\mathscr{B}$ since $\pi(\eta)<c-\delta$ on A and $\pi(\eta)<\pi(\eta)+\delta$ on A^C.

We can apply Proposition 1.1 and get the representation of π_A both in the (LSC) and (USC) case. Only notice that in case π_A is (USC) the sup can be replaced by a max. Moreover

$$\begin{aligned}\pi_A(X) &= \sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}} \inf_{\xi\in L_{\mathscr{F}}} \{\pi_A(\xi) \mid E_Q[\xi] \geq E_Q[X]\} \\ &\leq \sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}} \inf_{\xi\in L_{\mathscr{F}}} \{\pi_A(\xi) \mid E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\} \leq \pi_A(X).\end{aligned}$$

1.6 Proofs of the main results

We remind that a partition $\Gamma = \{A^\Gamma\}$ is a collection of measurable sets such that $\mathbb{P}(A^{\Gamma_1} \cap A^{\Gamma_2}) = 0$ and $\mathbb{P}(\cup_{A^\Gamma\in\Gamma} A^\Gamma) = 1$. Notations: in the following, we will only consider *finite* partitions $\Gamma = \{A^\Gamma\}$ of $\mathscr{G}$ measurable sets $A^\Gamma \in \Gamma$ and we set

$$\begin{aligned}\pi^\Gamma(X) &:= \sum_{A^\Gamma\in\Gamma} \pi_{A^\Gamma}(X)\mathbf{1}_{A^\Gamma}, \\ K^\Gamma(X,Q) &:= \inf_{\xi\in L_{\mathscr{F}}} \left\{\pi^\Gamma(\xi) \mid E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\right\} \\ H^\Gamma(X) &:= \sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}} K^\Gamma(X,Q)\end{aligned}$$

1.6.1 Outline of the proof

We anticipate an heuristic sketch of the proof of Theorem 1.2, pointing out the essential arguments involved in it and we defer to the following section the details and the rigorous statements.

The proof relies on the equivalence of the following conditions:

1. $\pi(X) = H(X)$.
2. $\forall\varepsilon > 0$, $\exists Q_\varepsilon \in L^*_{\mathscr{F}} \cap \mathscr{P}$ such that $\pi(X) - K(X,Q_\varepsilon) < \varepsilon$.
3. $\forall\varepsilon > 0$, $\exists Q_\varepsilon \in L^*_{\mathscr{F}} \cap \mathscr{P}$ such that

$$\{\xi \in L_{\mathscr{F}} \mid E_{Q_\varepsilon}[\xi|\mathscr{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X|\mathscr{G}]\} \subseteq \{\xi \in L_{\mathscr{F}} \mid \pi(\xi) > \pi(X) - \varepsilon\}. \quad (1.19)$$

Indeed, 1. $\Rightarrow$ 2. is a consequence of Proposition 1.3 (when it holds true); 2. $\Rightarrow$ 3. follows from the observation that $\pi(X) < K(X,Q_\varepsilon)+\varepsilon$ implies $\pi(X) < \pi(\xi)+\varepsilon$ for every ξ satisfying $E_{Q_\varepsilon}[\xi|\mathscr{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X|\mathscr{G}]$; 3. $\Rightarrow$ 1. is implied by the inequalities:

$$\begin{aligned}\pi(X) - \varepsilon &\leq \inf\{\pi(\xi) \mid \pi(\xi) > \pi(X) - \varepsilon\} \\ &\leq \inf_{\xi\in L_{\mathscr{F}}} \{\pi(\xi) \mid E_{Q_\varepsilon}[\xi|\mathscr{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X|\mathscr{G}]\} \leq H(X) \leq \pi(X).\end{aligned}$$

Unfortunately, we cannot prove Item 3. directly, relying on Hahn-Banach Theorem, as it happened in the real case (see the proof of Theorem 1.1, equation (1.3), in Appendix). Indeed, the complement of the set in the RHS of (1.19) is not any more a convex set - unless π is real valued - regardless of the continuity assumption made on π.

Also the method applied in the conditional convex case [17] can not be used here, since the map $X \to E_{\mathbb{P}}[\pi(X)]$ there adopted preserves convexity but not quasiconvexity.

The idea is then to apply an approximation argument and the choice of approximating $\pi(\cdot)$ by $\pi^{\Gamma}(\cdot)$, is forced by the need to preserve quasiconvexity.

I The first step is to prove (see Proposition 1.5) that: $H^{\Gamma}(X) = \pi^{\Gamma}(X)$. This is based on the representation of the *real valued* quasiconvex map π_A in Proposition 1.4. Therefore, the assumptions (LSC), (MON), (REG) and (QCO) on π are here all needed.

II Then it is a simple matter to deduce $\pi(X) = \inf_{\Gamma} \pi^{\Gamma}(X) = \inf_{\Gamma} H^{\Gamma}(X)$, where the inf is taken with respect to all finite partitions.

III As anticipated in (C.3), the last step, i.e. proving that $\inf_{\Gamma} H^{\Gamma}(X) = H(X)$, is more delicate. It can be shown easily that is possible to approximate $H(X)$ with $K(X,Q_{\varepsilon})$ on a set A_{ε} of probability arbitrarily close to 1. However, we need the following *uniform* approximation: For any $\varepsilon > 0$ there exists $Q_{\varepsilon} \in L^*_{\mathscr{F}} \cap \mathscr{P}$ such that for any finite partition Γ we have $H^{\Gamma}(X) - K^{\Gamma}(X,Q_{\varepsilon}) < \varepsilon$ on the same set A_{ε}. This key approximation result, based on Lemma 1.8, shows that the element Q_{ε} does not depend on the partition and allows us (see equation (1.26)) to conclude the proof .

1.6.2 Details

The following two lemmas are applications of measure theory

Lemma 1.6. *For every $Y \in L^0_{\mathscr{G}}$ there exists a sequence $\Gamma(n)$ of finite partitions such that $\sum_{\Gamma(n)} \left(\sup_{A^{\Gamma(n)}} Y\right) \mathbf{1}_{A^{\Gamma(n)}}$ converges in probability, and $\mathbb{P}$-a.s., to Y.*

Proof. Fix $\varepsilon, \delta > 0$ and consider the partitions $\Gamma(n) = \{A^n_0, A^n_1, \dots A^n_{n2^{n+1}+1}\}$ where

$$\begin{aligned} A^n_0 &= \{Y \in (-\infty, -n]\} \\ A^n_j &= \left\{Y \in \left(-n + \frac{j-1}{2^n}, -n + \frac{j}{2^n}\right]\right\} \quad \forall\, j = 1, \dots, n2^{n+1} \\ A^n_{n2^{n+1}+1} &= \{Y \in (n, +\infty)\} \end{aligned}$$

Since $\mathbb{P}(A^n_0 \cup A^n_{n2^{n+1}+1}) \to 0$ as $n \to \infty$, we consider N such that $\mathbb{P}(A^N_0 \cup A^N_{N2^N+1}) \leq 1 - \varepsilon$. Moreover we may find M such that $\frac{1}{2^M} < \delta$, and hence for $\Gamma = \Gamma(M \vee N)$ we have:

$$\mathbb{P}\left\{\omega\in\Omega \mid \sum_{A^{\Gamma}\in\Gamma}\big(\sup_{A^{\Gamma}}Y\big)\mathbf{1}_{A^{\Gamma}}(\omega)-Y(\omega)<\delta\right\}>1-\varepsilon. \tag{1.20}$$

Lemma 1.7. *For each $X\in L_{\mathscr{F}}$ and $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$*

$$\inf_{\Gamma}K^{\Gamma}(X,Q)=K(X,Q)$$

where the infimum *is taken with respect to all finite partitions Γ.*

Proof.

$$\begin{aligned}\inf_{\Gamma}K^{\Gamma}(X,Q)&=\inf_{\Gamma}\inf_{\xi\in L_{\mathscr{F}}}\left\{\pi^{\Gamma}(\xi)\mid E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]\right\}\\&=\inf_{\xi\in L_{\mathscr{F}}}\left\{\inf_{\Gamma}\pi^{\Gamma}(\xi)\mid E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]\right\}\\&=\inf_{\xi\in L_{\mathscr{F}}}\{\pi(\xi)\mid E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]\}=K(X,Q).\end{aligned} \tag{1.21}$$

where the first equality in (1.21) follows from the convergence shown in Lemma 1.6.

The following already mentioned key result is proved in the Appendix, for it needs a pretty long argument.

Lemma 1.8. *Let $X\in L_{\mathscr{F}}$ and let P and Q be arbitrary elements of $L^*_{\mathscr{F}}\cap\mathscr{P}$. Suppose that there exists $B\in\mathscr{G}$ satisfying: $K(X,P)\mathbf{1}_B>-\infty$, $\pi_B(X)<+\infty$ and*

$$K(X,Q)\mathbf{1}_B\leq K(X,P)\mathbf{1}_B+\varepsilon\mathbf{1}_B,$$

for some $\varepsilon\geq 0$. Then for every partition $\Gamma=\{B^C,\widetilde{\Gamma}\}$, where $\widetilde{\Gamma}$ is a partition of B, we have

$$K^{\Gamma}(X,Q)\mathbf{1}_B\leq K^{\Gamma}(X,P)\mathbf{1}_B+\varepsilon\mathbf{1}_B.$$

Since π^{Γ} assumes only a finite number of values, we may apply Proposition 1.4 and deduce the dual representation of π^{Γ}.

Proposition 1.5. *Suppose that the assumptions of Theorem 1.2 (resp. Theorem 1.3) hold true and Γ is a finite partition. Then:*

$$H^{\Gamma}(X)=\pi^{\Gamma}(X)\geq\pi(X) \tag{1.22}$$

and therefore

$$\inf_{\Gamma}H^{\Gamma}(X)=\pi(X).$$

Proof. First notice that $K^{\Gamma}(X,Q)\leq H^{\Gamma}(X)\leq\pi^{\Gamma}(X)$ for all $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$. Consider the sigma algebra $\mathscr{G}^{\Gamma}:=\sigma(\Gamma)\subseteq\mathscr{G}$, generated by the finite partition Γ. Hence from Proposition 1.4 we have for every $A^{\Gamma}\in\Gamma$

$$\pi_{A^\Gamma}(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} \inf_{\xi \in L_{\mathscr{F}}} \{\pi_{A^\Gamma}(\xi) \mid E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\}. \tag{1.23}$$

Moreover $H^\Gamma(X)$ is constant on A^Γ since it is $\mathscr{G}^\Gamma$-measurable as well. Using the fact that $\pi^\Gamma(\cdot)$ is constant on each A^Γ, for every $A^\Gamma \in \Gamma$ we then have:

$$\begin{aligned} H^\Gamma(X)\mathbf{1}_{A^\Gamma} &= \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} \inf_{\xi \in L_{\mathscr{F}}} \left\{\pi^\Gamma(\xi)\mathbf{1}_{A^\Gamma} \mid E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\right\} \\ &= \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} \inf_{\xi \in L_{\mathscr{F}}} \{\pi_{A^\Gamma}(\xi)\mathbf{1}_{A^\Gamma} \mid E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\} \\ &= \pi_{A^\Gamma}(X)\mathbf{1}_{A^\Gamma} = \pi^\Gamma(X)\mathbf{1}_{A^\Gamma} \end{aligned} \tag{1.24}$$

where the first equality in (1.24) follows from (1.23). The remaining statement is a consequence of (1.22) and Lemma 1.6

Proof (Proofs of Theorems 1.2 and 1.3). Obviously $\pi(X) \geq H(X)$, since X satisfies the constraints in the definition of $H(X)$.

Step 1. First we assume that π is uniformly bounded, i.e. there exists $c > 0$ such that for all $X \in L_{\mathscr{F}}$ $|\pi(X)| \leq c$. Then $H(X) > -\infty$.

From Lemma 1.5, eq. (1.14), we know that there exists a sequence $Q_k \in L^*_{\mathscr{F}} \cap \mathscr{P}$ such that:

$$K(X,Q_k) \uparrow H(X), \text{ as } k \uparrow \infty.$$

Therefore, for any $\varepsilon > 0$ we may find $Q_\varepsilon \in L^*_{\mathscr{F}} \cap \mathscr{P}$ and $A_\varepsilon \in \mathscr{G}$, $\mathbb{P}(A_\varepsilon) > 1-\varepsilon$ such that

$$H(X)\mathbf{1}_{A_\varepsilon} - K(X,Q_\varepsilon)\mathbf{1}_{A_\varepsilon} \leq \varepsilon \mathbf{1}_{A_\varepsilon}.$$

Since $H(X) \geq K(X,Q)\ \forall Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$,

$$(K(X,Q_\varepsilon)+\varepsilon)\mathbf{1}_{A_\varepsilon} \geq K(X,Q)\mathbf{1}_{A_\varepsilon}\ \forall Q \in L^*_{\mathscr{F}} \cap \mathscr{P}.$$

This is the basic inequality that enable us to apply Lemma 1.8, replacing there P with Q_ε and B with A_ε. Only notice that $\sup_\Omega \pi(X) \leq c$ and $K(X,Q) > -\infty$ for every $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$. This Lemma assures that for every partition Γ of Ω

$$(K^\Gamma(X,Q_\varepsilon)+\varepsilon)\mathbf{1}_{A_\varepsilon} \geq K^\Gamma(X,Q)\mathbf{1}_{A_\varepsilon}\ \forall Q \in L^*_{\mathscr{F}} \cap \mathscr{P}. \tag{1.25}$$

From the definition of *essential supremum* of a class of r.v. equation (1.25) implies that for every Γ

$$(K^\Gamma(X,Q_\varepsilon)+\varepsilon)\mathbf{1}_{A_\varepsilon} \geq \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} K^\Gamma(X,Q)\mathbf{1}_{A_\varepsilon} = H^\Gamma(X)\mathbf{1}_{A_\varepsilon}. \tag{1.26}$$

Since $\pi^\Gamma \leq c$, applying Proposition 1.5, equation (1.22), we get

$$(K^\Gamma(X,Q_\varepsilon)+\varepsilon)\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon}.$$

Taking the *infimum* over all possible partitions, as in Lemma 1.7, we deduce:

$$(K(X,Q_\varepsilon)+\varepsilon)\mathbf{1}_{A_\varepsilon}\geq \pi(X)\mathbf{1}_{A_\varepsilon}. \tag{1.27}$$

Hence, for any $\varepsilon>0$

$$(K(X,Q_\varepsilon)+\varepsilon)\mathbf{1}_{A_\varepsilon}\geq \pi(X)\mathbf{1}_{A_\varepsilon}\geq H(X)\mathbf{1}_{A_\varepsilon}\geq K(X,Q_\varepsilon)\mathbf{1}_{A_\varepsilon}$$

which implies $\pi(X)=H(X)$, since $\mathbb{P}(A_\varepsilon)\to 1$ as $\varepsilon\to 0$.

Step 2. Now we consider the case when π is not necessarily bounded. We define the new map $\psi(\cdot):=\arctan(\pi(\cdot))$ and notice that $\psi(X)$ is a $\mathscr{G}$-measurable r.v. satisfying $|\psi(X)|\leq\frac{\Pi}{2}$ for every $X\in L_{\mathscr{F}}$. Moreover ψ is (MON), (QCO) and $\psi(X\mathbf{1}_G)\mathbf{1}_G=\psi(X)\mathbf{1}_G$ for every $G\in\mathscr{G}$. In addition, ψ inherits the (LSC) (resp. the (USC)*) property from π. The first is a simple consequence of (CFB) of π. For the second we may apply Lemma 1.9 below.

ψ is surely uniformly bounded and by the above argument we may conclude

$$\psi(X)=H_\psi(X):=\sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}}K_\psi(X,Q)$$

where

$$K_\psi(X,Q):=\inf_{\xi\in L_{\mathscr{F}}}\{\psi(\xi)\mid E_Q[\xi|\mathscr{G}]\geq_Q E_Q[X|\mathscr{G}]\}.$$

Applying again Lemma 1.5, equation (1.14), there exists $Q^k\in L^*_{\mathscr{F}}$ such that

$$H_\psi(X)=\lim_k K_\psi(X,Q^k).$$

We will show below that

$$K_\psi(X,Q^k)=\arctan K(X,Q^k). \tag{1.28}$$

Admitting this, we have for $\mathbb{P}$-almost every $\omega\in\Omega$

$$\begin{aligned}\arctan(\pi(X)(\omega)) &= \psi(X)(\omega)=H_\psi(X)(\omega)=\lim_k K_\psi(X,Q^k)(\omega)\\ &=\lim_k\arctan K(X,Q^k)(\omega))=\arctan(\lim_k K(X,Q^k)(\omega)),\end{aligned}$$

where we used the continuity of the function arctan. This implies $\pi(X)=\lim_k K(X,Q^k)$ and we conclude:

$$\pi(X)=\lim_k K(X,Q^k)\leq H(X)\leq\pi(X).$$

It only remains to show (1.28). We prove that for every fixed $Q\in L^*_{\mathscr{F}}\cap\mathscr{P}$

$$K_\psi(X,Q)=\arctan(K(X,Q)).$$

Since π and ψ are regular, from Lemma 1.4 iv), there exist $\xi_h^Q\in L_{\mathscr{F}}$ and $\eta_h^Q\in L_{\mathscr{F}}$ such that

$$E_Q[\xi_h^Q|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}],\, E_Q[\eta_h^Q|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}],\, \forall h \geq 1, \tag{1.29}$$

$\psi(\xi_h^Q) \downarrow K_\psi(X,Q)$ and $\pi(\eta_h^Q) \downarrow K(X,Q)$, as $h \uparrow \infty$. From (1.29) and the definitions of $K(X,Q)$, $K_\psi(X,Q)$ and by the continuity and monotonicity of arctan we get:

$$\begin{aligned} K_\psi(X,Q) &\leq \lim_h \psi(\eta_h^Q) = \lim_h \arctan \pi(\eta_h^Q) = \arctan \lim_h \pi(\eta_h^Q) \\ &= \arctan K(X,Q) \leq \arctan \lim_h \pi(\xi_h^Q) = \lim_h \psi(\xi_h^Q) = K_\psi(X,Q). \end{aligned}$$

and this ends the proof of both Theorem 1.2 and 1.3.

Remark 1.7. Let $D \in \mathscr{F}$. If U is a neighborhood of $\xi \in L_{\mathscr{F}}$ then also the set

$$U\mathbf{1}_D + U\mathbf{1}_{D^C} =: \{Z = X1_D + Y1_{D^c} \mid X \in U, Y \in U\}$$

is a neighborhood of ξ. Indeed, since U is a neighborhood of ξ, there exists an open set V such that $\xi \in V \subseteq U$. Since $U \subseteq U\mathbf{1}_D + U\mathbf{1}_{D^C}$, we deduce that $\xi \in V \subseteq U\mathbf{1}_D + U\mathbf{1}_{D^C}$ and therefore ξ is in the interior of $U\mathbf{1}_D + U\mathbf{1}_{D^C}$.

Let Y be $\mathscr{G}$-measurable and define:

$$A := \{\xi \in L_{\mathscr{F}} \mid \pi(\xi) < \tan(Y)\} \quad B := \{\xi \in L_{\mathscr{F}} \mid \arctan(\pi(\xi)) < Y\},$$

where

$$\tan(x) = \begin{cases} -\infty & x \leq -\frac{\Pi}{2} \\ \tan(x) & -\frac{\Pi}{2} < x < \frac{\Pi}{2} \\ +\infty & x \geq \frac{\Pi}{2} \end{cases}$$

Notice that $A = \left\{\xi \in B \mid \pi(\xi) < \infty \text{ on } \left\{Y > \frac{\pi}{2}\right\}\right\} \subset B$ but the reverse inclusion does not hold true in general: in fact every $\xi_0 \in A$ satisfies $\pi(\xi_0) < +\infty$ on the set $\{Y > \frac{\Pi}{2}\}$ but it may happen that a $\xi_0 \in B$ brings to $\pi(\xi_0) = +\infty$ on $\{Y > \frac{\Pi}{2}\}$.

Lemma 1.9. *Suppose that π is regular and there exists $\theta \in L_{\mathscr{F}}$ such that $\pi(\theta) < +\infty$. For any $\mathscr{G}$-measurable random variable Y, if A is open then also B is open. As a consequence if the map π is (USC)* so it is the map* $\arctan \pi$.

Proof. We may assume $Y \geq -\frac{\pi}{2}$, otherwise $B = \emptyset$. Let $\xi \in B$, $\theta \in L_{\mathscr{F}}$ such that $\pi(\theta) < +\infty$. Define $\xi_0 := \xi 1_{\{Y \leq \frac{\pi}{2}\}} + \theta 1_{\{Y > \frac{\pi}{2}\}}$. Then $\xi_0 \in A$ (since π is regular and $\pi(\theta) < tg(Y)$). Since A is open, we may find a neighborhood U of 0 such that:

$$\xi_0 + U \subseteq A.$$

Define:

$$V := (\xi_0 + U)1_{\{Y \leq \frac{\pi}{2}\}} + (\xi + U)1_{\{Y > \frac{\pi}{2}\}} = \xi + U1_{\{Y \leq \frac{\pi}{2}\}} + U1_{\{Y > \frac{\pi}{2}\}}.$$

Then $\xi \in V$ and, by the previous remark, $U1_{\{Y \leq \frac{\pi}{2}\}} + U1_{\{Y > \frac{\pi}{2}\}}$ is a neighborhood of 0. Hence V is a neighborhood of ξ. To show that B is open it is then sufficient to show that $V \subseteq B$. Let $\eta \in V$. Then

$$\eta = \eta_1 1_{\left\{Y \leq \frac{\pi}{2}\right\}} + \eta_2 1_{\left\{Y > \frac{\pi}{2}\right\}},\ \eta_1 \in (\xi_0 + U),\ \eta_2 \in (\xi + U).$$

Since $\xi_0 + U \subseteq A$, $\eta_1 \in A$; therefore: $\pi(\eta_1) < tg(Y)$. Since π is regular and $\left\{Y \leq \frac{\pi}{2}\right\}$ is $\mathscr{G}$ measurable, $\pi(\eta) = \pi(\eta_1)$ on the set $\left\{Y \leq \frac{\pi}{2}\right\}$, which implies: $\pi(\eta) < tg(Y)$ on $\left\{Y \leq \frac{\pi}{2}\right\}$ and $\eta \in B$.

Remark 1.8. Consider $Q \in \mathscr{P}$ such that $Q \sim \mathbb{P}$ on $\mathscr{G}$ and define the new probability

$$\widetilde{Q}(F) := E_Q\left[\frac{d\mathbb{P}^{\mathscr{G}}}{dQ}\mathbf{1}_F\right] \quad \text{where} \quad \frac{d\mathbb{P}^{\mathscr{G}}}{dQ} =: E_Q\left[\frac{d\mathbb{P}}{dQ}\middle|\mathscr{G}\right],\ F \in \mathscr{F}.$$

Then $\widetilde{Q}(G) = \mathbb{P}(G)$ for all $G \in \mathscr{G}$, and so $\widetilde{Q} \in \mathscr{P}_{\mathscr{G}}$. Moreover, it is easy to check that for all $X \in L_{\mathscr{F}}$ and $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$ such that $Q \sim \mathbb{P}$ on $\mathscr{G}$ we have:

$$E_{\widetilde{Q}}[X|\mathscr{G}] = E_Q[X|\mathscr{G}] \tag{1.30}$$

which implies $K(X, \widetilde{Q}) = K(X, Q)$. To get (1.30) consider any $A \in \mathscr{G}$

$$\begin{aligned}
E_{\mathbb{P}}[E_{\tilde{Q}}[X|\mathscr{G}]\mathbf{1}_A] &= E_{\tilde{Q}}[E_{\tilde{Q}}[X|\mathscr{G}]\mathbf{1}_A] = E_{\tilde{Q}}[X\mathbf{1}_A] \\
&= E_Q\left[X\frac{d\mathbb{P}^{\mathscr{G}}}{dQ}\mathbf{1}_A\right] = E_Q\left[E_Q\left[X\frac{d\mathbb{P}^{\mathscr{G}}}{dQ}\mathbf{1}_A\middle|\mathscr{G}\right]\right] \\
&= E_Q\left[E_Q[X|\mathscr{G}]\frac{d\mathbb{P}^{\mathscr{G}}}{dQ}\mathbf{1}_A\right] = E_{\tilde{Q}}\left[E_Q[X|\mathscr{G}]\mathbf{1}_A\right] \\
&= E_{\mathbb{P}}\left[E_Q[X|\mathscr{G}]\mathbf{1}_A\right]
\end{aligned}$$

Proof (Proof of Corollary 1.1). Consider the probability $Q_\varepsilon \in L^*_{\mathscr{F}} \cap \mathscr{P}$ built up in Theorem 1.2, equation (1.27). We claim that Q_ε is equivalent to $\mathbb{P}$ on A_ε. By contradiction there exists $B \in \mathscr{G}$, $B \subseteq A_\varepsilon$, such that $\mathbb{P}(B) > 0$ but $Q_\varepsilon(B) = 0$. Consider $\eta \in L_{\mathscr{F}}$, $\delta > 0$ such that $\mathbb{P}(\pi(\eta) + \delta < \pi(X)) = 1$ and define $\xi = X\mathbf{1}_{B^C} + \eta\mathbf{1}_B$ so that $E_{Q_\varepsilon}[\xi|\mathscr{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X|\mathscr{G}]$. By regularity $\pi(\xi) = \pi(X)\mathbf{1}_{B^C} + \pi(\eta)\mathbf{1}_B$ which implies for $\mathbb{P}$-a.e. $\omega \in B$

$$\pi(\xi)(\omega) + \delta = \pi(\eta)(\omega) + \delta < \pi(X)(\omega) \leq K(X, Q_\varepsilon)(\omega) + \varepsilon \leq \pi(\xi)(\omega) + \varepsilon$$

which is impossible for $\varepsilon \leq \delta$. So $Q_\varepsilon \sim \mathbb{P}$ on A_ε for all small $\varepsilon \leq \delta$.
Consider $\widehat{Q}_\varepsilon$ such that $\frac{d\widehat{Q}_\varepsilon}{d\mathbb{P}} = \frac{dQ_\varepsilon}{d\mathbb{P}}\mathbf{1}_{A_\varepsilon} + \frac{d\mathbb{P}}{d\mathbb{P}}\mathbf{1}_{(A_\varepsilon)^C}$. Up to a normalization factor $\widehat{Q}_\varepsilon \in L^*_{\mathscr{F}} \cap \mathscr{P}$ and is equivalent to $\mathbb{P}$. Moreover from Lemma 1.4 (vi), $K(X, \widehat{Q}_\varepsilon)\mathbf{1}_{A_\varepsilon} = K(X, Q_\varepsilon)\mathbf{1}_{A_\varepsilon}$ and from Remark 1.8 we may define $\widetilde{Q}_\varepsilon \in \mathscr{P}_{\mathscr{G}}$ such that $K(X, \widetilde{Q}_\varepsilon)\mathbf{1}_{A_\varepsilon} = K(X, \widehat{Q}_\varepsilon)\mathbf{1}_{A_\varepsilon} = K(X, Q_\varepsilon)\mathbf{1}_{A_\varepsilon}$. From (1.27) we finally deduce: $K(X, \widetilde{Q}_\varepsilon)\mathbf{1}_{A_\varepsilon} + \varepsilon\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon}$, and the thesis then follows from $\widetilde{Q}_\varepsilon \in \mathscr{P}_{\mathscr{G}}$.

1.6.3 Proof of the key approximation Lemma 1.8

We will adopt the following notations: If Γ_1 and Γ_2 are two finite partitions of $\mathscr{G}$-measurable sets then $\Gamma_1 \cap \Gamma_2 := \{A_1 \cap A_2 \mid A_i \in \Gamma_i,\ i = 1,2\}$ is a finite partition finer than each Γ_1 and Γ_2.

Lemma 1.10 is the natural generalization of Lemma 3.1 to the approximated problem.

Lemma 1.10. *For every partition Γ, $X \in L_{\mathscr{F}}$ and $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$, the set*

$$\mathscr{A}^{\Gamma}_Q(X) \doteq \{\pi^{\Gamma}(\xi) \mid \xi \in L_{\mathscr{F}} \text{ and } E_Q[\xi|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\}$$

is downward directed. This implies that there exists exists a sequence $\left\{\eta^Q_m\right\}_{m=1}^{\infty} \in L_{\mathscr{F}}$ such that

$$E_Q[\eta^Q_m|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]\ \forall m \geq 1\,, \quad \pi^{\Gamma}(\eta^Q_m) \downarrow K^{\Gamma}(X,Q) \text{ as } m \uparrow \infty.$$

Proof. To show that the set $\mathscr{A}^{\Gamma}_Q(X)$ is downward directed we use the notations and the results in the proof of Lemma 3.1 and check that

$$\pi^{\Gamma}(\xi^*) = \pi^{\Gamma}(\xi_1 \mathbf{1}_G + \xi_2 \mathbf{1}_{G^C}) \leq \min\left\{\pi^{\Gamma}(\xi_1), \pi^{\Gamma}(\xi_2)\right\}.$$

Now we show that for any given sequence of partition there exists one sequence that works for all.

Lemma 1.11. *For any fixed, at most countable, family of partitions $\{\Gamma(h)\}_{h\geq 1}$ and $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$, there exists a sequence $\left\{\xi^Q_m\right\}_{m=1}^{\infty} \in L_{\mathscr{F}}$ such that*

$$\begin{aligned} E_Q[\xi^Q_m|\mathscr{G}] &\geq_Q E_Q[X|\mathscr{G}] && \text{for all } m \geq 1 \\ \pi(\xi^Q_m) &\downarrow K(X,Q) && \text{as } m \uparrow \infty \\ \text{and for all } h \quad \pi^{\Gamma(h)}(\xi^Q_m) &\downarrow K^{\Gamma(h)}(X,Q) && \text{as } m \uparrow \infty. \end{aligned}$$

Proof. Apply Lemma 3.1 and Lemma 1.10 and find $\{\varphi^0_m\}_m, \{\varphi^1_m\}_m, \dots, \{\varphi^h_m\}_m, \dots$ such that for every i and m we have $E_Q[\varphi^i_m \mid \mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]$ and

$$\begin{aligned} \pi(\varphi^0_m) &\downarrow K(X,Q) && \text{as } m \uparrow \infty \\ \text{and for all } h \quad \pi^{\Gamma(h)}(\varphi^h_m) &\downarrow K^{\Gamma(h)}(X,Q) && \text{as } m \uparrow \infty. \end{aligned}$$

For each $m \geq 1$ consider $\bigwedge_{i=0}^m \pi(\varphi^i_m)$: then there will exists a (non unique) finite partition of Ω, $\{F^i_m\}_{i=1}^m$ such that

$$\bigwedge_{i=0}^{m} \pi(\varphi^i_m) = \sum_{i=0}^{m} \pi(\varphi^i_m) \mathbf{1}_{F^i_m}.$$

Denote $\xi_m^Q =: \sum_{i=0}^m \varphi_m^i \mathbf{1}_{F_m^i}$ and notice that $\sum_{i=0}^m \pi(\varphi_m^i)\mathbf{1}_{F_m^i} \overset{(REG)}{=} \pi\left(\xi_m^Q\right)$ and $E_Q[\xi_m^Q|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}]$ for every m. Moreover $\pi(\xi_m^Q)$ is decreasing and $\pi(\xi_m^Q) \leq \pi(\varphi_m^0)$ implies $\pi(\xi_m^Q) \downarrow K(X,Q)$.
For every fixed h we have $\pi(\xi_m^Q) \leq \pi(\varphi_m^h)$ for all $h \leq m$ and hence:

$$\pi^{\Gamma(h)}(\xi_m^Q) \leq \pi^{\Gamma(h)}(\varphi_m^h) \text{ implies } \pi^{\Gamma(h)}(\xi_m^Q) \downarrow K^{\Gamma(h)}(X,Q) \text{ as } m \uparrow \infty.$$

Finally, we state the basic step used in the proof of Lemma 1.8.

Lemma 1.12. *Let $X \in L_{\mathscr{F}}$ and let P and Q be arbitrary elements of $L_{\mathscr{F}}^* \cap \mathscr{P}$. Suppose that there exists $B \in \mathscr{G}$ satisfying: $K(X,P)\mathbf{1}_B > -\infty$, $\pi_B(X) < +\infty$ and*

$$K(X,Q)\mathbf{1}_B \leq K(X,P)\mathbf{1}_B + \varepsilon \mathbf{1}_B,$$

for some $\varepsilon \geq 0$. Then for any $\delta > 0$ and any partition Γ_0 there exists $\Gamma \supseteq \Gamma_0$ for which

$$K^{\Gamma}(X,Q)\mathbf{1}_B \leq K^{\Gamma}(X,P)\mathbf{1}_B + \varepsilon \mathbf{1}_B + \delta \mathbf{1}_B$$

Proof. By our assumptions we have: $-\infty < K(X,P)\mathbf{1}_B \leq \pi_B(X) < +\infty$ and $K(X,Q)\mathbf{1}_B \leq \pi_B(X) < +\infty$. Fix $\delta > 0$ and the partition Γ_0. Suppose by contradiction that for any $\Gamma \supseteq \Gamma_0$ we have $\mathbb{P}(C) > 0$ where

$$C = \{\omega \in B \mid K^{\Gamma}(X,Q)(\omega) > K^{\Gamma}(X,P)(\omega) + \varepsilon + \delta\}. \tag{1.31}$$

Notice that C is the union of a finite number of elements in the partition Γ.

Consider that Lemma 1.4 guarantees the existence of $\left\{\xi_h^Q\right\}_{h=1}^{\infty} \in L_{\mathscr{F}}$ satisfying:

$$\pi(\xi_h^Q) \downarrow K(X,Q), \text{ as } h \uparrow \infty, \quad , \quad E_Q[\xi_h^Q|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}] \;\forall h \geq 1. \tag{1.32}$$

Moreover, for each partition Γ and $h \geq 1$ define:

$$D_h^{\Gamma} := \left\{\omega \in \Omega \mid \pi^{\Gamma}(\xi_h^Q)(\omega) - \pi(\xi_h^Q)(\omega) < \frac{\delta}{4}\right\} \in \mathscr{G},$$

and observe that $\pi^{\Gamma}(\xi_h^Q)$ decreases if we pass to finer partitions. From Lemma 1.6 equation (1.20), we deduce that for each $h \geq 1$ there exists a partition $\widetilde{\Gamma}(h)$ such that $\mathbb{P}\left(D_h^{\widetilde{\Gamma}(h)}\right) \geq 1 - \frac{1}{2^h}$. For every $h \geq 1$ define the new partition $\Gamma(h) = \left(\bigcap_{j=1}^{h} \widetilde{\Gamma}(h)\right) \cap \Gamma_0$ so that for all $h \geq 1$ we have: $\Gamma(h+1) \supseteq \Gamma(h) \supseteq \Gamma_0$, $\mathbb{P}\left(D_h^{\Gamma(h)}\right) \geq 1 - \frac{1}{2^h}$ and

$$\left(\pi(\xi_h^Q) + \frac{\delta}{4}\right)\mathbf{1}_{D_h^{\Gamma(h)}} \geq \left(\pi^{\Gamma(h)}(\xi_h^Q)\right)\mathbf{1}_{D_h^{\Gamma(h)}}, \;\forall h \geq 1. \tag{1.33}$$

Lemma 1.11 guarantees that for the fixed sequence of partitions $\{\Gamma(h)\}_{h\geq 1}$, there exists a sequence $\{\xi_m^P\}_{m=1}^{\infty} \in L_{\mathscr{F}}$, which does not depend on h, satisfying

$$E_P[\xi_m^P|\mathscr{G}] \geq_P E_P[X|\mathscr{G}] \;\forall m \geq 1, \tag{1.34}$$

$$\pi^{\Gamma(h)}(\xi_m^P) \;\downarrow\; K^{\Gamma(h)}(X,P), \text{ as } m \uparrow \infty, \quad \forall h \geq 1. \tag{1.35}$$

For each $m \geq 1$ and $\Gamma(h)$ define:

$$C_m^{\Gamma(h)} := \left\{ \omega \in C \mid \pi^{\Gamma(h)}(\xi_m^P)(\omega) - K^{\Gamma(h)}(X,P)(\omega) \leq \frac{\delta}{4} \right\} \in \mathscr{G}.$$

Since the expressions in the definition of $C_m^{\Gamma(h)}$ assume only a finite number of values, from (1.35) and from our assumptions, which imply that $K^{\Gamma(h)}(X,P) \geq K(X,P) > -\infty$ on B, we deduce that for each $\Gamma(h)$ there exists an index $m(\Gamma(h))$ such that: $\mathbb{P}\left(C \setminus C_{m(\Gamma(h))}^{\Gamma(h)}\right) = 0$ and

$$K^{\Gamma(h)}(X,P)\mathbf{1}_{C_{m(\Gamma(h))}^{\Gamma(h)}} \geq \left(\pi^{\Gamma(h)}(\xi_{m(\Gamma(h))}^P) - \frac{\delta}{4}\right)\mathbf{1}_{C_{m(\Gamma(h))}^{\Gamma(h)}}, \forall h \geq 1. \tag{1.36}$$

Set $E_h = D_h^{\Gamma(h)} \cap C_{m(\Gamma(h))}^{\Gamma(h)} \in \mathscr{G}$ and observe that

$$\mathbf{1}_{E_h} \to \mathbf{1}_C \quad \mathbb{P}-\text{a.s.} \tag{1.37}$$

From (1.33) and (1.36) we then deduce:

$$\left(\pi(\xi_h^Q) + \frac{\delta}{4}\right)\mathbf{1}_{E_h} \geq \left(\pi^{\Gamma(h)}(\xi_h^Q)\right)\mathbf{1}_{E_h}, \;\forall h \geq 1, \tag{1.38}$$

$$K^{\Gamma(h)}(X,P)\mathbf{1}_{E_h} \geq \left(\pi^{\Gamma(h)}(\xi_{m(\Gamma(h))}^P) - \frac{\delta}{4}\right)\mathbf{1}_{E_h}, \;\forall h \geq 1. \tag{1.39}$$

We then have for any $h \geq 1$

$$\pi(\xi_h^Q)\mathbf{1}_{E_h} + \frac{\delta}{4}\mathbf{1}_{E_h} \geq \left(\pi^{\Gamma(h)}(\xi_h^Q)\right)\mathbf{1}_{E_h} \tag{1.40}$$

$$\geq K^{\Gamma(h)}(X,Q)\mathbf{1}_{E_h} \tag{1.41}$$

$$\geq \left(K^{\Gamma(h)}(X,P) + \varepsilon + \delta\right)\mathbf{1}_{E_h} \tag{1.42}$$

$$\geq \left(\pi^{\Gamma(h)}(\xi_{m(\Gamma(h))}^P) - \frac{\delta}{4} + \varepsilon + \delta\right)\mathbf{1}_{E_h} \tag{1.43}$$

$$\geq \left(\pi(\xi_{m(\Gamma(h))}^P) + \varepsilon + \frac{3}{4}\delta\right)\mathbf{1}_{E_h}. \tag{1.44}$$

(in the above chain of inequalities, (1.40) follows from (1.38); (1.41) follows from (1.32) and the definition of $K^{\Gamma(h)}(X,Q)$; (1.42) follows from (1.31); (1.43) follows from (1.39); (1.44) follows from the definition of the maps $\pi_{A^{\Gamma(h)}}$).
Recalling (1.34) we then get, for each $h \geq 1$,

$$\pi(\xi_h^Q)\mathbf{1}_{E_h} \geq \left(\pi(\xi^P_{m(\Gamma(h))}) + \varepsilon + \frac{\delta}{2}\right)\mathbf{1}_{E_h} \geq \left(K(X,P) + \varepsilon + \frac{\delta}{2}\right)\mathbf{1}_{E_h} > -\infty. \quad (1.45)$$

From equation (1.32) and (1.37) we have $\pi(\xi_h^Q)\mathbf{1}_{E_h} \to K(X,Q)\mathbf{1}_C$ $\mathbb{P}$-a.s. as $h \uparrow \infty$ and so from (1.45)

$$\begin{aligned}\mathbf{1}_C K(X,Q) &= \lim_h \pi(\xi_h^Q)\mathbf{1}_{E_h} \geq \lim_h \mathbf{1}_{E_h}\left(K(X,P) + \varepsilon + \frac{\delta}{2}\right) \\ &= \mathbf{1}_C\left(K(X,P) + \varepsilon + \frac{\delta}{2}\right)\end{aligned}$$

which contradicts the assumption of the Lemma, since $C \subseteq B$ and $\mathbb{P}(C) > 0$.

Proof (Proof of Lemma 1.8). First notice that the assumptions of this Lemma are those of Lemma 1.12. Assume by contradiction that there exists $\Gamma_0 = \{B^C, \widetilde{\Gamma_0}\}$, where $\widetilde{\Gamma_0}$ is a partition of B, such that

$$\mathbb{P}(\omega \in B \mid K^{\Gamma_0}(X,Q)(\omega) > K^{\Gamma_0}(X,P)(\omega) + \varepsilon) > 0. \quad (1.46)$$

By our assumptions we have $K^{\Gamma_0}(X,P)\mathbf{1}_B \geq K(X,P)\mathbf{1}_B > -\infty$ and $K^{\Gamma_0}(X,Q)\mathbf{1}_B \leq \pi_B(X)\mathbf{1}_B < +\infty$. Since K^{Γ_0} is constant on every element $A^{\Gamma_0} \in \Gamma_0$, we denote with $K^{A^{\Gamma_0}}(X,Q)$ the value that the random variable $K^{\Gamma_0}(X,Q)$ assumes on A^{Γ_0}. From (1.46) we deduce that there exists $\widehat{A}^{\Gamma_0} \subseteq B$, $\widehat{A}^{\Gamma_0} \in \Gamma_0$, such that

$$+\infty > K^{\widehat{A}^{\Gamma_0}}(X,Q) > K^{\widehat{A}^{\Gamma_0}}(X,P) + \varepsilon > -\infty.$$

Let then $d > 0$ be defined by

$$d =: K^{\widehat{A}^{\Gamma_0}}(X,Q) - K^{\widehat{A}^{\Gamma_0}}(X,P) - \varepsilon. \quad (1.47)$$

Apply Lemma 1.12 with $\delta = \frac{d}{3}$: then there exists $\Gamma \supseteq \Gamma_0$ (w.l.o.g. $\Gamma = \{B^C, \widetilde{\Gamma}\}$ where $\widetilde{\Gamma} \supseteq \widetilde{\Gamma_0}$) such that

$$K^{\Gamma}(X,Q)\mathbf{1}_B \leq \left(K^{\Gamma}(X,P) + \varepsilon + \delta\right)\mathbf{1}_B. \quad (1.48)$$

Considering only the two partitions Γ and Γ_0, we may apply Lemma 1.11 and conclude that there exist two sequences $\{\xi_h^P\}_{h=1}^{\infty} \in L_{\mathscr{F}}$ and $\{\xi_h^Q\}_{h=1}^{\infty} \in L_{\mathscr{F}}$ satisfying as $h \uparrow \infty$:

$$E_P[\xi_h^P|\mathscr{G}] \geq_P E_P[X|\mathscr{G}], \;\; \pi^{\Gamma_0}(\xi_h^P) \downarrow K^{\Gamma_0}(X,P), \;\; \pi^{\Gamma}(\xi_h^P) \downarrow K^{\Gamma}(X,P) \quad (1.49)$$

$$E_Q[\xi_h^Q|\mathscr{G}] \geq_Q E_Q[X|\mathscr{G}], \;\; \pi^{\Gamma_0}(\xi_h^Q) \downarrow K^{\Gamma_0}(X,Q), \;\; \pi^{\Gamma}(\xi_h^Q) \downarrow K^{\Gamma}(X,Q) \quad (1.50)$$

Since $K^{\Gamma_0}(X,P)$ is constant and finite on $\widehat{A}^{\Gamma_0}$, from (1.49) we may find $h_1 \geq 1$ such that

$$\pi_{\widehat{A}^{\Gamma_0}}(\xi_h^P) - K^{\widehat{A}^{\Gamma_0}}(X,P) < \frac{d}{2}, \forall h \geq h_1. \quad (1.51)$$

From equation (1.47) and (3.3) we deduce that

$$\pi_{\widehat{A}^{\Gamma_0}}(\xi_h^P) < K^{\widehat{A}^{\Gamma_0}}(X,P) + \frac{d}{2} = K^{\widehat{A}^{\Gamma_0}}(X,Q) - \varepsilon - d + \frac{d}{2}, \; \forall h \geq h_1,$$

and therefore, knowing from (1.50) that $K^{\widehat{A}^{\Gamma_0}}(X,Q) \leq \pi_{\widehat{A}^{\Gamma_0}}(\xi_h^Q)$,

$$\pi_{\widehat{A}^{\Gamma_0}}(\xi_h^P) + \frac{d}{2} < \pi_{\widehat{A}^{\Gamma_0}}(\xi_h^Q) - \varepsilon \quad \forall h \geq h_1. \quad (1.52)$$

We now take into account all the sets $A^{\Gamma} \subseteq \widehat{A}^{\Gamma_0} \subseteq B$. For the convergence of $\pi_{A^{\Gamma}}(\xi_h^Q)$ we distinguish two cases. On those sets A^{Γ} for which $K^{A^{\Gamma}}(X,Q) > -\infty$ we may find, from (1.50), $\overline{h} \geq 1$ such that

$$\pi_{A^{\Gamma}}(\xi_h^Q) - K^{A^{\Gamma}}(X,Q) < \frac{\delta}{2} \; \forall h \geq \overline{h}.$$

Then using (1.48) and (1.49) we have

$$\pi_{A^{\Gamma}}(\xi_h^Q) < K^{A^{\Gamma}}(X,Q) + \frac{\delta}{2} \leq K^{A^{\Gamma}}(X,P) + \varepsilon + \delta + \frac{\delta}{2} \leq \pi_{A^{\Gamma}}(\xi_h^P) + \varepsilon + \delta + \frac{\delta}{2}$$

so that

$$\pi_{A^{\Gamma}}(\xi_h^Q) < \pi_{A^{\Gamma}}(\xi_h^P) + \varepsilon + \frac{3\delta}{2} \quad \forall h \geq \overline{h}.$$

On the other hand, on those sets A^{Γ} for which $K^{A^{\Gamma}}(X,Q) = -\infty$ the convergence (1.50) guarantees the existence of $\widehat{h} \geq 1$ for which we obtain again:

$$\pi_{A^{\Gamma}}(\xi_h^Q) < \pi_{A^{\Gamma}}(\xi_h^P) + \varepsilon + \frac{3\delta}{2} \quad \forall h \geq \widehat{h} \quad (1.53)$$

(notice that $K^{\Gamma}(X,P) \geq K(X,P)\mathbf{1}_B > -\infty$ and (1.49) imply that $\pi_{A^{\Gamma}}(\xi_h^P)$ converges to a finite value, for $A^{\Gamma} \subseteq B$).

Since the partition Γ is finite there exists $h_2 \geq 1$ such that equation (1.53) stands for every $A^{\Gamma} \subseteq \widehat{A}^{\Gamma_0}$ and for every $h \geq h_2$ and for our choice of $\delta = \frac{d}{3}$ (1.53) becomes

$$\pi_{A^{\Gamma}}(\xi_h^Q) < \pi_{A^{\Gamma}}(\xi_h^P) + \varepsilon + \frac{d}{2} \quad \forall h \geq h_2 \quad \forall A^{\Gamma} \subseteq \widehat{A}^{\Gamma_0}. \quad (1.54)$$

Fix $h^* > \max\{h_1, h_2\}$ and consider the value $\pi_{\widehat{A}^{\Gamma_0}}(\xi^Q_{h^*})$. Then among all $A^\Gamma \subseteq \widehat{A}^{\Gamma_0}$ we may find $B^\Gamma \subseteq \widehat{A}^{\Gamma_0}$ such that $\pi_{B^\Gamma}(\xi^Q_{h^*}) = \pi_{\widehat{A}^{\Gamma_0}}(\xi^Q_{h^*})$. Thus:

$$\pi_{\widehat{A}^{\Gamma_0}}(\xi^Q_{h^*}) = \pi_{B^\Gamma}(\xi^Q_{h^*}) \overset{(1.54)}{<} \pi_{B^\Gamma}(\xi^P_{h^*}) + \varepsilon + \frac{d}{2} \leq \pi_{\widehat{A}^{\Gamma_0}}(\xi^P_{h^*}) + \varepsilon + \frac{d}{2} \overset{(1.52)}{<} \pi_{\widehat{A}^{\Gamma_0}}(\xi^Q_{h^*}).$$

which is a contradiction.

1.7 A complete characterization of the map π

In this section we show that any conditional map π can be characterized *via* the dual representation (see Proposition 1.6): we introduce the class $\mathscr{R}^{cfb}$ of maps $S : \Sigma \to \overline{L}^0_{\mathscr{G}}$ such that $S(\cdot, \xi')$ is (MON), (CFB) and (REG) (i.e. $S(Y\mathbf{1}_A, Q)\mathbf{1}_A = S(Y, Q)\mathbf{1}_A \; \forall A \in \mathscr{G}$).

Remark 1.9. $S : \Sigma \to \overline{L}^0_{\mathscr{G}}$ such that $S(\cdot, \xi')$ is (MON) and (REG) is automatically (QCO) in the first component: let $Y_1, Y_2, \Lambda \in L^0_{\mathscr{G}}$, $0 \leq \Lambda \leq 1$ and define $B = \{Y_1 \leq Y_2\}$, $S(\cdot, Q) = S(\cdot)$.
$S(Y_1\mathbf{1}_B) \leq S(Y_2\mathbf{1}_B)$ and $S(Y_2\mathbf{1}_{B^C}) \leq S(Y_1\mathbf{1}_{B^C})$ so that from (MON) and (REG)

$$\begin{aligned} S(\Lambda Y_1 + (1-\Lambda)Y_2) &\leq S(Y_2\mathbf{1}_B + Y_1\mathbf{1}_{B^C}) \overset{(REG)}{=} S(Y_2)\mathbf{1}_B + S(Y_1)\mathbf{1}_{B^C} \\ &\leq S(Y_1) \vee S(Y_2). \end{aligned}$$

Notice that the class $\mathscr{R}^{cfb}$ is non-empty: for instance consider the map $R^+(\cdot, \xi')$ defined by

$$R^+(Y, \xi') = ess \sup_{Y' < Y} R(Y', \xi') \tag{1.55}$$

As shown in the next Lemma, R^+ inherits from R (MON), (REG) and is automatically (CFB). This function plays an important role in the proof of Proposition 1.7.
Proposition 1.6 is in the spirit of [10]: as a consequence of the dual representation the map π induces on R (resp. R^+) its characteristic properties and so does R (resp. R^+) on π.

Lemma 1.13. *If $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ is (REG) and (MON) then $R^+ \in \mathscr{R}$.*

Proof. Clearly $R^+(\cdot, Q)$ inherits from $R(\cdot, Q)$ the properties (REG) and (MON). From Remark 1.9 we then know that $R^+(\cdot, Q)$ is (QCO). We show that it is also (CFB). Let $Y_n \uparrow Y$. It is easy to check that (MON) of $R(\cdot, \xi')$ implies that the set $\{R(\eta, \xi') | \eta < Y\}$ is upward directed. Then for every $\varepsilon, \delta > 0$ we can find $\eta_\varepsilon < Y$ such that

$$\mathbb{P}(R^+(Y, \xi') - R(\eta_\varepsilon, \xi') < \varepsilon) > 1 - \delta \tag{1.56}$$

There exists an n_ε such that $\mathbb{P}(Y_n > \eta_\varepsilon) > 1 - \delta$ for every $n > n_\varepsilon$. Denote by $A_n = \{Y_n > \eta_\varepsilon\}$ so that from (REG) we have $R^+(Y_n, \xi')\mathbf{1}_{A_n} \geq R(\eta_\varepsilon, \xi')\mathbf{1}_{A_n}$. This last inequality together with equation (1.56) implies

$$\mathbb{P}(R^+(Y,\xi') - R^+(Y_n,\xi') < \varepsilon) > 1-2\delta \quad \forall n > n_\varepsilon$$

i.e. $R^+(Y_n,Q) \xrightarrow{\mathbb{P}} R^+(Y,Q)$. Since $R^+(Y_n,Q)\uparrow$ we conclude that $R^+(Y_n,Q)\uparrow R^+(Y,Q)$ $\mathbb{P}$-almost surely.

Proposition 1.6. *Consider a map $S:\Sigma\to L_{\mathscr{G}}$.*
(a) Let $\chi\subseteq L^*_{\mathscr{F}}$, $X\in L_{\mathscr{F}}$ and

$$\pi(X) = \sup_{\xi'\in\chi} S(E[X\xi'|\mathscr{G}],\xi').$$

(Recall that $(E[\xi'X|\mathscr{G}],\xi')\in\Sigma$ for every $X\in L_{\mathscr{F}}$, $\xi'\in L^*_{\mathscr{F}}$).
Then for every $A\in\mathscr{G}$, $(Y,\xi')\in\Sigma$, $\Lambda\in L_{\mathscr{G}}\cap L_{\mathscr{F}}$ and $X\in L_{\mathscr{F}}$
i) $S(Y\mathbf{1}_A,\xi')\mathbf{1}_A = S(Y,\xi')\mathbf{1}_A \Longrightarrow \pi$ (REG);
ii) $Y\mapsto S(Y,\xi')$ (MON) $\Longrightarrow \pi$ (MON);
iii) $Y\mapsto S(Y,\xi')$ is conditionally convex $\Longrightarrow \pi$ is conditionally convex;
iv) $Y\mapsto S(Y,\xi')$ (QCO) $\Longrightarrow \pi$ (QCO);
v) $S(\lambda Y,\xi')=\lambda S(Y,\xi') \Longrightarrow \pi(\lambda X)=\lambda\pi(X)$, $(\lambda>0)$;
vi) $S(\lambda Y,\xi')=S(Y,\xi') \Longrightarrow \pi(\lambda X)=\pi(X)$, $(\lambda>0)$;
vii) $Y\mapsto S(Y,\xi')$ (CFB) $\Longrightarrow \pi$ (CFB).
viii) $S(E[(X+\Lambda)\xi'|\mathscr{G}],\xi') = S(E[X\xi'|\mathscr{G}],\xi')+\Lambda \Longrightarrow \pi(X+\Lambda)=\pi(X)+\Lambda$.
ix) $S(E[(X+\Lambda)\xi'|\mathscr{G}],\xi') \geq S(E[X\xi'|\mathscr{G}],\xi')+\Lambda \Longrightarrow \pi(X+\Lambda)\geq\pi(X)+\Lambda$.

(b) When the map S is replaced by R defined in (1.5), all the above items - except (vii) - hold true replacing "$\Longrightarrow$" by "$\Longleftrightarrow$".

(c) When the map S is replaced by R^+ defined in (1.55), all the above items - except (iii) - hold true replacing "$\Longrightarrow$" by "$\Longleftrightarrow$".

Proof. (a) Items from (i) to (ix) are trivial. To make an example we show (iv): for every $\mathscr{G}$-measurable Λ, $0\leq\Lambda\leq 1$, and $X_1,X_2\in L_{\mathscr{F}}$, we have $E_{\mathbb{P}}[(\Lambda X_1+(1-\Lambda)X_2)\xi'|\mathscr{G}] = \Lambda E_{\mathbb{P}}[X_1\xi'|\mathscr{G}]+(1-\Lambda)E_{\mathbb{P}}[X_2\xi'|\mathscr{G}]$. Thus

$$\begin{aligned}
&S(\Lambda E_{\mathbb{P}}[X_1\xi'|\mathscr{G}]+(1-\Lambda)E_{\mathbb{P}}[X_2\xi'|\mathscr{G}],\xi')\\
&\leq \max\left\{S(E_{\mathbb{P}}[X_1\xi'|\mathscr{G}],\xi'),S(E_{\mathbb{P}}[X_2\xi'|\mathscr{G}],\xi')\right\}\\
&\leq \max\left\{\sup_{\xi'\in\chi} S(E_{\mathbb{P}}[X_1\xi'|\mathscr{G}],\xi'),\sup_{\xi'\in\chi} S(E_{\mathbb{P}}[X_2\xi'|\mathscr{G}],\xi')\right\}
\end{aligned}$$

thus

$$\begin{aligned}\pi(\Lambda X_1+(1-\Lambda)X_2) &= \sup_{Q\in\mathscr{P}} S\big(\Lambda E_Q[X|\mathscr{G}]+(1-\Lambda)E_Q[Y|\mathscr{G}],\mathbb{Q}\big)\\ &\le \max\left\{\sup_{Q\in\mathscr{P}} S(E_Q[X|\mathscr{G}],Q),\ \sup_{Q\in\mathscr{P}} S(E_Q[Y|\mathscr{G}],Q)\right\}\\ &= \pi(X_1)\vee\pi(X_2).\end{aligned}$$

(b): The ‘only if’ in (i) and (ii) follow from Lemma 1.3. Now we prove the remaining ‘only if’ conditions.

(iii): let $Y_1,Y_2,\Lambda\in L^0_{\mathscr{G}}$, $0\le\Lambda\le 1$ then

$$\begin{aligned}&R(\Lambda Y_1+(1-\Lambda)Y_2,\xi')\\ &=\inf_{\xi\in L_{\mathscr{F}}}\{\pi(\xi)\mid E[\xi\xi'|\mathscr{G}]\ge\Lambda Y_1+(1-\Lambda)Y_2\}\\ &=\inf_{\eta_1,\eta_2\in L_{\mathscr{F}}}\{\pi(\Lambda\eta_1+(1-\Lambda)\eta_2)\mid E[(\Lambda\eta_1+(1-\Lambda)\eta_2)\xi'|\mathscr{G}]\ge\Lambda Y_1+(1-\Lambda)Y_2\}\\ &\le\inf_{\eta_1,\eta_2\in L_{\mathscr{F}}}\{\pi(\Lambda\eta_1+(1-\Lambda)\eta_2)\mid E[\eta_1\xi'|\mathscr{G}]\ge Y_1\cap E[\eta_2\xi'|\mathscr{G}]\ge Y_2\}\\ &\le\Lambda R(Y_1,\xi')+(1-\Lambda)R(Y_2,\xi')\end{aligned}$$

(iv): follows from Remark 1.9 since R is (MON) and (REG).

(v):

$$\begin{aligned}R(\lambda Y,\xi') &= \inf_{\xi\in L_{\mathscr{F}}}\{\pi(\xi)\mid E[\lambda^{-1}\xi\xi'|\mathscr{G}]\ge Y\}\\ &=\inf_{\lambda\eta\in L_{\mathscr{F}}}\{\pi(\lambda\eta)\mid E[\eta\xi'|\mathscr{G}]\ge Y\}=\lambda R(Y,\xi')\end{aligned}$$

(vi): similar to (v).

(viii):

$$\begin{aligned}&R(E[(X+\Lambda)\xi'|\mathscr{G}],\xi')\\ &=\inf_{\xi\in L_{\mathscr{F}}}\left\{\pi(\xi)\mid E\left[(\xi-\Lambda)\xi'|\mathscr{G}\right]\ge E[X\xi'|\mathscr{G}]\right\}\\ &=\inf_{\eta+\Lambda\in L_{\mathscr{F}}}\left\{\pi(\eta+\Lambda)\mid E\left[\eta\xi'|\mathscr{G}\right]\ge E[X\xi'|\mathscr{G}]\right\}=R(E[X\xi'|\mathscr{G}],Q)+\Lambda\end{aligned}$$

(ix): similar to (viii).

(c): by definition R^+ inherits from R (MON) and (REG) so that we can also conclude by Remark 1.9 that the ‘only if ’in (i), (ii) and (iv) holds true.

(v): we know that $\pi(\lambda\cdot)=\lambda\pi(\cdot)$ implies $R(\lambda\cdot,\xi')=\lambda R(\cdot,\xi')$. By definition

$$\begin{aligned}R^+(Y,\xi') &= \sup_{Y'<Y}R(Y',\xi')=\sup_{Y'<Y}\frac{1}{\lambda}R(\lambda Y',\xi')\\ &=\frac{1}{\lambda}\sup_{Y'<\lambda Y}R(Y',\xi')=\frac{1}{\lambda}R^+(\lambda Y,\xi').\end{aligned}$$

(vi), (vii) and (ix) follows as in (v).

(vii): is proved in Lemma 3.10.

Proposition 1.7. *Suppose that $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$ satisfies the C-property and $L_{\mathscr{F}}$ is order complete. $\pi : L_{\mathscr{F}} \to L_{\mathscr{G}}$ is (MON), (QCO), (REG) and $\sigma(L_{\mathscr{F}}, L^*_{\mathscr{F}})$-(LSC)* ***if and only if*** *there exists $S \in \mathscr{R}^{cfb}$ such that*

$$\pi(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} S\left(E\left[\frac{dQ}{d\mathbb{P}} X | \mathscr{G}\right], Q\right). \tag{1.57}$$

Proof. The 'if 'follows from Proposition (1.6). For the 'only if 'we already know from Theorem 1.2 that

$$\pi(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} R\left(E\left[\frac{dQ}{d\mathbb{P}} X | \mathscr{G}\right], Q\right).$$

where R is defined in (1.5). For every $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$ we consider $R^+(\cdot, Q) \leq R(\cdot, Q)$ and denote $X^Q = E\left[\frac{dQ}{d\mathbb{P}} X | \mathscr{G}\right]$. We observe that

$$\begin{aligned}
\pi(X) &\geq \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} R^+(X^Q, Q) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} \sup_{Y' < X^Q} R(Y', Q) \\
&\overset{\delta>0}{\geq} \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} \sup_{X^Q - \delta < X^Q} R(X^Q - \delta, Q) \\
&= \sup_{\delta>0} \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} R(E[(X-\delta) \cdot dQ/d\mathbb{P} | \mathscr{G}], Q) = \sup_{\delta>0} \pi(X - \delta) \overset{(CFB)}{=} \pi(X)
\end{aligned}$$

and so for $R^+ \in \mathscr{R}^{cfb}_Q$ we have the representation

$$\pi(X) = \sup_{Q \in L^*_{\mathscr{F}} \cap \mathscr{P}} R^+(E_Q[X|\mathscr{G}], Q).$$

1.7.1 A hint for further research: on the uniqueness of the representation

In [10] the authors provide a complete duality for real valued quasiconvex functionals when the space $L_{\mathscr{F}}$ is an M-space (such as L^∞): the idea is to reach a one to one relationship between quasiconvex monotone functionals π and the function R of the dual representation. Obviously R will be unique only in an opportune class of maps satisfying certain properties. A similar result is obtained in [11] for the L^p spaces with $p \in [1, +\infty)$, which are not M-spaces.
Other later results can be found in the recent preprint by Drapeau and Kupper [19] where a slightly different duality is reached, gaining on the generality of the spaces.

Uniqueness is surely a more involving task to be proved for the conditional case and a complete proof need further investigation in the vector space case. Fortunately we are able in Chapter 3 to succeed it for the class of L^0-modules of L^p type, which is the counterpart of the findings presented in [11].

For what concerns vector spaces, we provide only a partial - not much rigorous - result when $\mathscr{G}$ is countably generated. For sake of simplicity we restrict our discussion to the space $L_{\mathscr{F}} = L^{\infty}_{\mathscr{F}}$ in order to exploit directly the uniqueness results in [10] section 5. The following argument can be adapted to the case of L^p, $p \in [1,+\infty)$, combining the results in [11] and [10].

Consider the following conditions

H1 $S(\cdot,Q)$ is increasing for every $Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$;
H2 $\inf_{Y\in L^0_{\mathscr{G}}} S(Y,Q_1) = \inf_{Y\in L^0_{\mathscr{G}}} S(Y,Q_2)$ for every $Q_1,Q_2 \in L^*_{\mathscr{F}} \cap \mathscr{P}$;
H3 $S(Y,Q)\mathbf{1}_A = S(Y\mathbf{1}_A,Q)\mathbf{1}_A = S(Y\mathbf{1}_A,Q\mathbf{1}_A)\mathbf{1}_A$;
H4 for every n, $S(\cdot,Q)\mathbf{1}_{A_n} = S^{A_n}(\cdot,Q)\mathbf{1}_{A_n}$, where $S^{A_n}(\cdot,Q)$ is jointly $\diamondsuit$-evenly quasiconcave on $\mathbb{R}\times Q \in L^*_{\mathscr{F}} \cap \mathscr{P}$;
H5 for every $X \in L_{\mathscr{F}}$

$$\sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}} S(E[XdQ/d\mathbb{P}|\mathscr{G}],\xi') = \sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}} S^{+}(E[XdQ/d\mathbb{P}|\mathscr{G}],\xi')$$

with S^+ as in (1.55).

Claim: let $\mathscr{G} = \sigma(\{A_n\}_{n\in\mathbb{N}})$ where $\{A_n\}_{n\in\mathbb{N}}$ is a partition of Ω and π satisfying the assumptions of Theorem 1.2. The function R is the unique in the class $\mathscr{M}^0_{qcx}$ of functions S satisfying H1, H2, H3, H4 and H5.

Idea of the proof. Surely from Lemma 1.3 $R \in \mathscr{M}^0_{qcx}$ (the last item is explained in the second part of the proof). By contradiction suppose that there exists $S \in \mathscr{M}^0_{qcx}$ such that

$$\pi(X) = \sup_{Q\in L^*_{\mathscr{F}}\cap\mathscr{P}} S\left(E\left[\frac{dQ}{d\mathbb{P}}X|\mathscr{G}\right],Q\right). \tag{1.58}$$

and $\mathbb{P}(S(Y,Q) \neq R(Y,Q)) > 0$ for some $(Y,Q) \in L^0_{\mathscr{G}} \times (L^*_{\mathscr{F}} \cap \mathscr{P})$. Hence we can find $A = A_n$ for some n such that $R\mathbf{1}_A \neq S\mathbf{1}_A$.
As previously mentioned π induces on π_A the properties (MON), (QCO), (CFB). The space $L^{\infty}_{\mathscr{F}}\mathbf{1}_A = \{\xi\mathbf{1}_A | \xi \in L^{\infty}_{\mathscr{F}}\}$ is an M-space so we may apply Theorem 5 in [10] on the map $\pi_A : L^{\infty}_{\mathscr{F}}\mathbf{1}_A \to \overline{R}$. Clearly the order dual $(L^{\infty}_{\mathscr{F}}\mathbf{1}_A)^* = L^1_{\mathscr{F}}\mathbf{1}_A$ and then we get

$$\pi_A(X) = \sup_{Q\in L^1_{\mathscr{F}}\cap\mathscr{P}} R^A\left(E\left[\frac{dQ}{d\mathbb{P}}X\mathbf{1}_A\right],Q\mathbf{1}_A\right) = \sup_{Q\in L^1_{\mathscr{F}}\cap\mathscr{P}} R^A_+\left(E\left[\frac{dQ}{d\mathbb{P}}X\mathbf{1}_A\right],Q\mathbf{1}_A\right) \tag{1.59}$$

$R^A : \mathbb{R}\times(L^*_{\mathscr{F}} \cap \mathscr{P}) \to \overline{\mathbb{R}}$ is given by

$$R^A(y, Q\mathbf{1}_A) = \inf_{\xi \in L_{\mathscr{F}}} \left\{ \pi_A(\xi) \mid E\left[\frac{dQ}{d\mathbb{P}}\xi\mathbf{1}_A\right] \geq y \right\}$$

and $R^A_+(t, Q\mathbf{1}_A) = \sup_{t'<t} R^A(t', Q\mathbf{1}_A)$. R^A is unique in the class $\mathscr{M}^0_{qcx}(A)$ of functions $S^A : \mathbb{R} \times (L^1_{\mathscr{F}} \cap \mathscr{P}) \to \overline{\mathbb{R}}$ such that S^A is increasing in the first argument in the first component, jointly $\diamondsuit$-evenly quasiconcave, $\inf_{t\in\mathbb{R}} S^A(t, Q_1\mathbf{1}_A) = \inf_{t\in\mathbb{R}} S^A(t, Q_2\mathbf{1}_A)$ for every $Q_1, Q_2 \in L^*_{\mathscr{F}} \cap \mathscr{P}$ and the second equality in (1.59) holds true.
Now notice that $R^A\mathbf{1}_A = R\mathbf{1}_A$ and from (1.58)

$$\pi_A(X)\mathbf{1}_A = \sup_{Q \in L^1_{\mathscr{F}} \cap \mathscr{P}} S\left(E\left[\frac{dQ}{d\mathbb{P}}X\mathbf{1}_A\right], Q\mathbf{1}_A\right)\mathbf{1}_A$$

hence from uniqueness $S\mathbf{1}_A = R^A\mathbf{1}_A = R\mathbf{1}_A$ which is absurd.

Chapter 2
An application to Finance and Economics: the Conditional Certainty Equivalent

2.1 An intuitive flavour of the problem

A non-atomic probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a right continuous filtration $\{\mathscr{F}_t\}_{t\geq 0}$ are fixed throughout this chapter. All the other notations are conformed to those in Chapter 1.

It is well known in Mathematical Finance literature that under opportune No-Arbitrage assumptions we can guarantee the existence of an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the price processes are martingales. Let us consider a replicable claim C, with time T maturity (i.e. $\mathscr{F}_T$ measurable). The Black and Scholes time-t value, is given by the formula

$$V_t(H) = \pi_{t,T}(C) = \frac{1}{\beta_t} E_{\mathbb{Q}}[\beta_T C \,|\, \mathscr{F}_t] \quad t < T \tag{2.1}$$

where $V_T(H) = C$, H is the replication strategy and β the discount stochastic factor.

In order to introduce the main purpose of this chapter we want to look to this formula from an utility point of view. Suppose that an investor's preferences are described by the stochastic field

$$u(x,t,\omega) = x\beta_t(\omega)\frac{d\mathbb{Q}_t}{d\mathbb{P}}(\omega)$$

where $\frac{d\mathbb{Q}_t}{d\mathbb{P}} = E_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}} \,|\, \mathscr{F}_t]$. If one consider a $\mathscr{F}_t$-measurable random variable X, then the solution Y_s of the equation

$$u(Y_s, s, \omega) = E_{\mathbb{P}}[u(X,t) \mid \mathscr{F}_s]$$

gives the the time-s equivalent random endowment of X with respect to the preferences induced by u. It is well known that a process Y turns out to be a $\mathbb{Q}$ martingale if and only if $E_{\mathbb{P}}[Y_t \frac{d\mathbb{Q}_t}{d\mathbb{P}} \mid \mathscr{F}_s] = Y_s \frac{d\mathbb{Q}_s}{d\mathbb{P}}$; applying this result to the equation

$$\beta_s Y_s \frac{d\mathbb{Q}_s}{d\mathbb{P}} = E_{\mathbb{P}}\left[\beta_t X \frac{d\mathbb{Q}_t}{d\mathbb{P}} \mid \mathscr{F}_s\right] \tag{2.2}$$

we get that the process $\{\beta_s Y_s\}_{0\leq s\leq t}$ is a $\mathbb{Q}$-martingale. Then

$$\beta_s Y_s = E_{\mathbb{Q}}[\beta_t X \mid \mathscr{F}_s]$$

i.e. whenever X is replicable Y_s is exactly the price $\pi_{s,t}(X)$ given by (2.1).
From this point of view Black and Scholes theory appears as a particular case of a general theory involving dynamic stochastic preferences, in which the linearity of the utility functions implies the complete absence of the investor's risk aversion.

Moreover the formula (2.2) highlights another troublesome feature arising when we work with stochastic fields: it concerns with the $\mathbb{P}$-integrability of $\beta_t X \frac{d\mathbb{Q}_t}{d\mathbb{P}}$, namely

$$E_{\mathbb{P}}\left[\beta_t |X| \frac{d\mathbb{Q}_t}{d\mathbb{P}}\right] < \infty \tag{2.3}$$

One may overcome it assuming that β is deterministic or satisfies some boundary conditions. Another approach could be introducing the right space of random variables for which condition (2.3) is naturally satisfied, without any further assumption on β. As we will show later Musielak-Orlicz spaces seem to fit perfectly to our aim: for each time t the utility $u(x,t,\omega)$ induces a generalized Young function $\hat{u}_t$ which defines a space $M^{\hat{u}_t}(\Omega, \mathscr{F}_t, \mathbb{P})$. Thus we are dealing with a time-indexed class of spaces for which the pricing functional $\pi_{s,t}$ is compatible with time consistency.

2.2 Definitions and first properties

Definition 2.1. A stochastic dynamic utility (SDU)

$$u : \mathbb{R}\times[0,\infty)\times\Omega \to \mathbb{R}\cup\{-\infty\}$$

satisfies the following conditions: for any $t \in [0,+\infty)$ there exists $A_t \in \mathscr{F}_t$ such that $\mathbb{P}(A_t) = 1$ and

(a) the effective domain, $\mathscr{D}(t) := \{x \in \mathbb{R} : u(x,t,\omega) > -\infty\}$ and the range $\mathscr{R}(t) := \{u(x,t,\omega) \mid x \in \mathscr{D}(t)\}$ do not depend on $\omega \in A_t$; moreover $0 \in int\mathscr{D}(t)$, $E_{\mathbb{P}}[u(0,t)] < +\infty$ and $\mathscr{R}(t) \subseteq \mathscr{R}(s)$;
(b) for all $\omega \in A_t$ and $t \in [0,+\infty)$ the function $x \to u(x,t,\omega)$ is strictly increasing on $\mathscr{D}(t)$ and increasing, concave and upper semicontinuous on $\mathbb{R}$.
(c) $\omega \to u(x,t,\cdot)$ is $\mathscr{F}_t-$measurable for all $(x,t) \in \mathscr{D}(t)\times[0,+\infty)$

The following assumption may turn out to be relevant in the sequel of the paper, even if not necessary for the definition of SDU.

(d) For any fixed $x \in \mathscr{D}(t)$, $u(x,t,\cdot) \leq u(x,s,\cdot)$ for every $s \leq t$.

Remark 2.1. We identify two SDU, $u \sim \widetilde{u}$, if for every $t \in [0,+\infty)$, the two domains are equal ($\mathscr{D}(t) = \widetilde{\mathscr{D}}(t)$) and there exists an $\mathscr{F}_t$-measurable set B_t such that $\mathbb{P}(B_t) = 1$ and $u(x,t,\omega) = \widetilde{u}(x,t,\omega)$ for every $(x,\omega) \in \mathscr{D}(t) \times B_t$.
In the sequel, we denote $u(x,t,\cdot)$ simply by $u(x,t)$, unless confusion may arise.

In order to define the conditional certainty equivalent we introduce the set

$$\mathscr{U}(t) = \{X \in L^0(\Omega,\mathscr{F}_t,\mathbb{P}) \,|\, u(X,t) \in L^1(\Omega,\mathscr{F},\mathbb{P})\}.$$

Lemma 2.1. *Let u be a SDU.*

i) (Inverse) Let $t \in [0,\infty)$ and $A_t \in \mathscr{F}_t$ as in Definition 2.1: the inverse function $u^{-1} : \mathscr{R}(t) \times [0,\infty) \times A_t \to \mathscr{D}(t)$

$$u^{-1}(u(x,t,\omega),t,\omega) = x \tag{2.4}$$

is well defined. For each $\omega \in A_t$, the function $u^{-1}(\cdot,t,\omega)$ is continuous and strictly increasing on $\mathscr{R}(t)$ and $u^{-1}(y,t,\cdot)$ is $\mathscr{F}_t-$ measurable for all $y \in \mathscr{R}(t)$.
ii) (Comparison) Fix any $t \in [0,\infty)$; if $X,Y \in \mathscr{U}(t)$ then $u(X,t) \le u(Y,t)$ if and only if $X \le Y$. The same holds if the inequalities are replaced by equalities.
iii) (Jensen) If $X \in L^1_{\mathscr{F}_t}$ and $u(X,s)$ is integrable, then, for all $s \le t$,

$$E_{\mathbb{P}}[u(X,s)|\mathscr{F}_s] \le u(E_{\mathbb{P}}[X|\mathscr{F}_s],s).$$

iv) (Extended Jensen) Suppose $u(x,s)$ is integrable for every $x \in \mathscr{D}(s)$. Let $X \in L^0_{\mathscr{F}_t}$, such that $u(X,s)^-$ is integrable. Then

$$E_{\mathbb{P}}[u(X,s)|\mathscr{F}_s] \le u(E_{\mathbb{P}}[X|\mathscr{F}_s],s). \tag{2.5}$$

where the conditional expectation is meant in an extended way.

Proof. i) Since both assumptions (a) and (b) hold on A_t, the existence of a continuous, increasing inverse function follows straightforwardly. From assumption (c) we can deduce that $u^{-1}(y,t,\cdot)$ is $\mathscr{F}_t$-measurable for all $y \in \mathscr{R}(t)$.
ii) Is also immediate since u is strictly increasing as a function of x.
iii) This property follows from the Theorem p.79 in [59].
iv) First we suppose that $u(0,s) = 0$. This implies that $u(X,s)\mathbf{1}_A = u(X\mathbf{1}_A,s)$ for every $A \in \mathscr{F}_t$. Recall that if $Y \in L^0_{\mathscr{F}_t}$ and $Y \ge 0$ then $E_{\mathbb{P}}[Y|\mathscr{F}_s] := \lim_n E_{\mathbb{P}}[Y\mathbf{1}_{Y\le n}|\mathscr{F}_s]$ is well defined.

First we show that $u(X,s)^-$ integrable implies $E_{\mathbb{P}}[X\mathbf{1}_{\{X<0\}}|\mathscr{F}_s] > -\infty$ and therefore both terms in (2.5) are well defined. From the equality $-u(X,s)\mathbf{1}_{\{X<0\}} = u(X,s)^-$ we get that $u(X,s)\mathbf{1}_{\{X<0\}}$ is integrable. From iii) we have that $u(0,s) \ge u(X\mathbf{1}_{\{0>X\ge -n\}},s) \ge u(-n,s)$ implies:

$$E_{\mathbb{P}}[u(X\mathbf{1}_{\{0>X\ge -n\}},s)|\mathscr{F}_s] \le u(E_{\mathbb{P}}[X\mathbf{1}_{\{0>X\ge -n\}}|\mathscr{F}_s],s). \tag{2.6}$$

By monotone convergence, from (2.6) we then get our claim:

$$-\infty < E_{\mathbb{P}}[u(X\mathbf{1}_{\{X<0\}},s)|\mathscr{F}_s] \leq u(E_{\mathbb{P}}[X\mathbf{1}_{\{X<0\}}|\mathscr{F}_s],s).$$

Applying iii) in the second inequality below we get:

$$E_{\mathbb{P}}[u(X,s)|\mathscr{F}_s] = \lim_n E_{\mathbb{P}}[u(X,s)\mathbf{1}_{\{0\leq u(X,s)\leq n\}}|\mathscr{F}_s]+E_{\mathbb{P}}[u(X,s)\mathbf{1}_{\{u(X,s)<0\}}|\mathscr{F}_s] \tag{2.7}$$

$$\leq \lim_n E_{\mathbb{P}}[u(X,s)\mathbf{1}_{\{0\leq X\leq n\}}|\mathscr{F}_s] = \lim_n E_{\mathbb{P}}[u(X\mathbf{1}_{\{0\leq X\leq n\}},s)|\mathscr{F}_s]$$

$$\leq \lim_n u(E_{\mathbb{P}}[X\mathbf{1}_{\{0\leq X\leq n\}}|\mathscr{F}_s],s) = u(E_{\mathbb{P}}[X^+|\mathscr{F}_s],s). \tag{2.8}$$

Notice that on the $\mathscr{F}_s$-measurable set $G^\infty := \{E_{\mathbb{P}}[X|\mathscr{F}_s] = +\infty\}$ the equation (2.5) is trivial. Since $E_{\mathbb{P}}[-X^-|\mathscr{F}_s] > -\infty$, it is clear that $E_{\mathbb{P}}[|X||\mathscr{F}_s] = +\infty$ on a set $A \in \mathscr{F}$ iff $E_{\mathbb{P}}[X|\mathscr{F}_s] = +\infty$ on the same set A. Therefore, by defining $G_n := \{\omega \in \Omega \setminus G^\infty \mid E_{\mathbb{P}}[|X| \mid \mathscr{F}_s](\omega) \leq n\}$, we have: $G_n \uparrow \Omega \setminus G^\infty$. Since each G_n is $\mathscr{F}_s$-measurable, the inequality (2.7)-(2.8) guarantees that

$$-E_{\mathbb{P}}[u(X\mathbf{1}_{G_n},s)^-|\mathscr{F}_s] \leq E_{\mathbb{P}}[u(X\mathbf{1}_{G_n},s)|\mathscr{F}_s] \leq u(E_{\mathbb{P}}[X^+\mathbf{1}_{G_n}|\mathscr{F}_s],s)$$
$$\leq u(E_{\mathbb{P}}[|X||\mathscr{F}_s],s)\mathbf{1}_{G_n} \leq u(n,s)$$

and therefore $u(X\mathbf{1}_{G_n},s)$ is integrable. Obviously, $X\mathbf{1}_{G_n}$ is also integrable and we may apply iii) (replacing X with $X\mathbf{1}_{G_n}$) and deduce

$$E_{\mathbb{P}}[u(X,s)|\mathscr{F}_s]\mathbf{1}_{G_n} = E_{\mathbb{P}}[u(X\mathbf{1}_{G_n},s)|\mathscr{F}_s] \leq u(E_{\mathbb{P}}[X\mathbf{1}_{G_n}|\mathscr{F}_s],s) = u(E_{\mathbb{P}}[X|\mathscr{F}_s],s)\mathbf{1}_{G_n}.$$

The thesis follows immediately by taking the limit as $n \to \infty$, since $G_n \uparrow \Omega \setminus G^\infty$.

For a general $u(x,s)$, apply the above argument to $v(x,s) =: u(x,s) - u(0,s)$.

A SDU allows us to define the backward conditional certainty equivalent, that represents the time-s-value of the time-t-claim X, for $0 \leq s \leq t < \infty$.

Definition 2.2. (Conditional Certainty Equivalent) Let u be a SDU. The backward Conditional Certainty Equivalent $C_{s,t}(X)$ of the random variable $X \in \mathscr{U}(t)$, is the random variable in $\mathscr{U}(s)$ solution of the equation:

$$u(C_{s,t}(X),s) = E_{\mathbb{P}}\left[u(X,t)|\mathscr{F}_s\right]. \tag{2.9}$$

Thus the CCE defines the *valuation* operator

$$C_{s,t} : \mathscr{U}(t) \to \mathscr{U}(s),\ C_{s,t}(X) = u^{-1}\left(E_{\mathbb{P}}\left[u(X,t)|\mathscr{F}_s\right]\right), s). \tag{2.10}$$

Observe that $E_{\mathbb{P}}\left[u(C_{s,t}(X),s)\right] = E_{\mathbb{P}}[u(X,t)]$ and so indeed $C_{s,t}(X) \in \mathscr{U}(s)$.

The definition is well posed

1. For any given $X \in \mathscr{U}(t)$, $E_{\mathbb{P}}\left[u(X,t)|\mathscr{F}_s\right] \in L^1(\Omega,\mathscr{F}_s,P)$.

2. Choose two arbitrary versions of the conditional expectation and of the SDU at time s, namely $\widetilde{E}_{\mathbb{P}}[u(X,t)|\mathscr{F}_s]$, $\widehat{E}_{\mathbb{P}}[u(X,t)|\mathscr{F}_s]$ and $\widetilde{u}(x,s)$, $\widehat{u}(x,s)$.
3. For all $\omega \in A_t$, $\widetilde{E}_{\mathbb{P}}[u(X,t)|\mathscr{F}_s](\omega) \in \mathscr{R}(t) \subseteq \mathscr{R}(s)$. We find a unique solution of $\widetilde{u}(\widetilde{C}_{s,t}(X),s) = \widetilde{E}_{\mathbb{P}}[u(X,t)|\mathscr{F}_s]$ defined as

$$\widetilde{C}_{s,t}(X)(\omega) = \widetilde{u}^{-1}(\widetilde{E}_{\mathbb{P}}[u(X,t)|\mathscr{F}_s](\omega), s, \omega) \quad \forall\, \omega \in A_t.$$

4. Repeat the previous argument for the second version and find $\widehat{C}_{s,t}(X)$ which differs from $\widetilde{C}_{s,t}(X)$ only on a $\mathbb{P}$-null set.

We could equivalently reformulate the definition of the CCE as follows:

Definition 2.3. The conditional certainty equivalent process is the only process $\{Y_s\}_{0\le s\le t}$ such that $Y_t \equiv X$ and the process $\{u(Y_s,s)\}_{0\le s\le t}$ is a martingale.

In the following proposition we show some elementary properties of the CCE, which have however very convenient interpretations. In i) we show the semigroup property of the valuation operator; iii) show the time consistency of the CCE: if the time-v-values of two time t claims are equal, then the two values should be equal at any previous time; iv) and v) are the key properties to obtain a dual representation of the map $C_{s,t}$ as shown in Chapter 1; property vi) shows that the expectation of the valuation operator is increasing, as a function of the *valuation time s* and the second issue expresses the risk aversion of the economic agent.

Proposition 2.1. *Let u be a SDU,* $0 \le s \le v \le t < \infty$ *and* $X, Y \in \mathscr{U}(t)$.

i) $C_{s,t}(X) = C_{s,v}(C_{v,t}(X))$.
ii) $C_{t,t}(X) = X$.
iii) If $C_{v,t}(X) \le C_{v,t}(Y)$ *then for all* $0 \le s \le v$ *we have:* $C_{s,t}(X) \le C_{s,t}(Y)$. *Therefore,* $X \le Y$ *implies that for all* $0 \le s \le t$ *we have:* $C_{s,t}(X) \le C_{s,t}(Y)$. *The same holds if the inequalities are replaced by equalities.*
iv) Regularity: for every $A \in F_s$ *we have*

$$C_{s,t}(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = C_{s,t}(X)\mathbf{1}_A + C_{s,t}(Y)\mathbf{1}_{A^C}$$

and then $C_{s,t}(X)\mathbf{1}_A = C_{s,t}(X\mathbf{1}_A)\mathbf{1}_A$.
v) Quasiconcavity: the upper level set $\{X \in \mathscr{U}_t \mid C_{s,t}(X) \ge Y\}$ *is conditionally convex for every* $Y \in L^0_{\mathscr{F}_s}$.
vi) Suppose u satisfies (d) and for every $t \in [0,+\infty)$, $u(x,t)$ *is integrable for every* $x \in \mathscr{D}(t)$. *Then* $C_{s,t}(X) \le E_{\mathbb{P}}[C_{v,t}(X)|\mathscr{F}_s]$ *and* $E_{\mathbb{P}}[C_{s,t}(X)] \le E_{\mathbb{P}}[C_{v,t}(X)]$. *Moreover* $C_{s,t}(X) \le E_{\mathbb{P}}[X|\mathscr{F}_s]$ *and therefore* $E_{\mathbb{P}}[C_{s,t}(X)] \le E_{\mathbb{P}}[X]$.

Proof. By definition:

$$u(C_{v,t}(X),v) \overset{(\cdot)}{=} E_{\mathbb{P}}[u(X,t)|\mathscr{F}_v],\ X \in \mathscr{U}(t)$$
$$u(C_{s,t}(X),s) \overset{(+)}{=} E_{\mathbb{P}}[u(X,t)|\mathscr{F}_s],\ X \in \mathscr{U}(t)$$
$$u(C_{s,v}(Z),s) \overset{(\times)}{=} E_{\mathbb{P}}[u(Z,v)|\mathscr{F}_s],\ Z \in \mathscr{U}(v)$$

i) Let $Z = C_{v,t}(X)$ and compute:

$$u(C_{s,v}(C_{v,t}(X)),s) = \ u(C_{s,v}(Z),s) \overset{(\times)}{=} \ E_{\mathbb{P}}[u(Z,v)|\mathscr{F}_s]$$
$$\overset{(\cdot)}{=} E_{\mathbb{P}}[E_{\mathbb{P}}[u(X,t)|\mathscr{F}_v]|\mathscr{F}_s] = E_{\mathbb{P}}[u(X,t)|\mathscr{F}_s] \overset{(+)}{=} u(C_{s,t}(X),s)$$

ii) Obvious, since $u(C_{t,t}(X),t) \overset{(\cdot)}{=} E_{\mathbb{P}}[u(X,t)|\mathscr{F}_t] \overset{(c)}{=} u(X,t)$.
iii)

$$\begin{aligned} u(C_{s,t}(X),s) &\overset{(+)}{=} E_{\mathbb{P}}[u(X,t)|\mathscr{F}_s] = E_{\mathbb{P}}[E_{\mathbb{P}}[u(X,t)|\mathscr{F}_v]|\mathscr{F}_s] \\ &\overset{(\cdot)}{=} E_{\mathbb{P}}[u(C_{v,t}(X),v)|\mathscr{F}_s] \leq E_{\mathbb{P}}[u(C_{v,t}(Y),v)|\mathscr{F}_s] \\ &\overset{(\cdot)}{=} E_{\mathbb{P}}[E_{\mathbb{P}}[u(Y,t)|\mathscr{F}_v]|\mathscr{F}_s] \overset{(+)}{=} u(C_{s,t}(Y),s). \end{aligned}$$

If $X \leq Y$ then $C_{t,t}(X) \leq C_{t,t}(Y)$ and the statement follows from what we just proved. The same for equalities.
iv) Consider every $A \in \mathscr{F}_s$ and notice that

$$\begin{aligned} C_{s,t}(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) &= u^{-1}(E_{\mathbb{P}}[u(X,t)\mathbf{1}_A + u(Y,t)\mathbf{1}_{A^C} \mid \mathscr{F}_s],s) \\ &= u^{-1}(E_{\mathbb{P}}[u(X,t) \mid \mathscr{F}_s]\mathbf{1}_A,s) + u^{-1}(E_{\mathbb{P}}[u(Y,t) \mid \mathscr{F}_s]\mathbf{1}_{A^C},s) \\ &= C_{s,t}(X)\mathbf{1}_A + C_{s,t}(Y)\mathbf{1}_{A^C} \end{aligned}$$

v) Fix an arbitrary $Y \in L^0_{\mathscr{F}_s}$ and consider the set $\mathscr{Y} = \{X \in \mathscr{U}_t \mid C_{s,t}(X) \geq Y\}$. Take $X_1, X_2 \in \mathscr{Y}$ and $\Lambda \in L^0_{\mathscr{F}_s}$, $0 \leq \Lambda \leq 1$:

$$E_{\mathbb{P}}[u(\Lambda X_1 + (1-\Lambda)X_2,t)|\mathscr{F}_s] \geq \Lambda E_{\mathbb{P}}[u(X_1,t)|\mathscr{F}_s] + (1-\Lambda)E_{\mathbb{P}}[u(X_2,t)|\mathscr{F}_s] \geq u(Y,s)$$

hence we get the thesis composing both sides with $u^{-1}(\cdot,s)$.
vi)

$$\begin{aligned} u(C_{s,t}(X),s) &\overset{(+)}{=} E_{\mathbb{P}}[u(X,t)|\mathscr{F}_s] = E_{\mathbb{P}}[E_{\mathbb{P}}[u(X,t)|\mathscr{F}_v]|\mathscr{F}_s] \\ &\overset{(\cdot)}{=} E_{\mathbb{P}}[u(C_{v,t}(X),v)|\mathscr{F}_s] \overset{(d)}{\leq} E_{\mathbb{P}}[u(C_{v,t}(X),s)|\mathscr{F}_s] \\ &\leq u(E_{\mathbb{P}}[C_{v,t}(X)|\mathscr{F}_s],s). \end{aligned}$$

We applied in the last inequality the extended Jensen inequality, since $(u(C_{v,t}(X),s))^-$ is integrable. The second property follows by taking $v = t$ and observing that $C_{t,t}(X) = X$.

Remark 2.2. Comparing the definition of SDU with the existing literature about forward performances ([6],[62],[63]), we may notice that the CCE does not rely on the existence of a market: this allows a higher level of generality and freedom in the choice of the preferences of the agent. We recall that an adapted process $U(x,t)$ is said to be a forward utility if

1. it is increasing and concave as a function of x for each t.
2. $U(x,0) = u_0(x) \in \mathbb{R}$
3. for all $T \geq t$ and each self-financing strategy represented by π, the associated discounted wealth X^π (see Section 2.3 for the rigorous definitions) satisfies

$$E_{\mathbb{P}}[U(X_T^\pi, T) \,|\, \mathscr{F}_t] \leq U(X_t^\pi, t)$$

4. for all $T \geq t$ there exists a self-financing strategy π^* such that X^{π^*} satisfies

$$E_{\mathbb{P}}[U(X_T^{\pi^*}, T) \,|\, \mathscr{F}_t] = U(X_t^{\pi^*}, t)$$

Surely if one take into account this stronger definition and tries to apply it for the computation of the CCE of these self-financing discounted portfolios X^π then only for the optimal strategy π_t^* we have that

$$C_{s,t}\left(X_t^{\pi^*}\right) = X_s^{\pi^*}$$

whereas in general

$$C_{s,t}\,(X_t^\pi) \leq X_s^\pi$$

This points out an economic interpretation of the CCE: given the final outcome of some risky position we backwardly build up a process which takes into account the agent's random risk-aversion. For replicable contingent claims it means that $X_s^\pi - C_{s,t}(X_t^\pi)$ measures the gap between the real value of the claim at time s, and the smallest amount for which the decision maker would willingly sell the claim if he had it. The gap will be deleted whenever we move through an optimal strategy.

The previous remark suggests the following

Definition 2.4. Let $0 \leq s \leq t < \infty$ and let u be a SDU. The *conditional risk premium* of the random variable $X \in \mathscr{U}(t)$ is the random variable $\rho_{s,t}(X) \in L^0(\Omega, \mathscr{F}_s, P; \mathscr{D})$ defined by:

$$\rho_{s,t}(X) := E_{\mathbb{P}}[X|\mathscr{F}_s] - C_{s,t}(X).$$

We now consider some properties of the dynamic stochastic utility u when it is computed on stochastic processes.

Proposition 2.2. *Let $\{S_t\}_{t\geq 0}$ be an $\{\mathscr{F}_t\}_{t\geq 0}-$ adapted process such that $S_t \in \mathscr{U}(t)$ and consider the process $\{V_t\}_{t\geq 0}$ defined by $V_t = u(S_t, t)$.*

i) $\{V_t\}_{t\geq 0}$ is a $(\{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})-$supermartingale (resp. submartingale, resp martingale) if and only if $C_{s,t}(S_t) \leq S_s$ (resp. $C_{s,t}(S_t) \geq S_s$, resp $C_{s,t}(S_t) = S_s$) for all $0 \leq s \leq t < \infty$.

Moreover if in addition u satisfies (d) and for every $t \in [0, +\infty)$, $u(x,t)$ is integrable for every $x \in \mathscr{D}(t)$ then

ii) If $\{S_t\}_{t\geq 0}$ is a $(\{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})-$supermartingale, then the process $\{V_t\}_{t\geq 0}$ defined by $V_t = u(S_t, t)$ is a $(\{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})-$supermartingale and thus $C_{s,t}(S_t) \leq S_s$ for all $0 \leq s \leq t < \infty$.

iii) If $C_{s,t}(S_t) = S_s$ *for all* $0 \leq s \leq t < \infty$ *then* $\{S_t\}_{t\geq 0}$ *is a* $(\{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})-$*submartingale.*

Proof. i) If $u(S_t,t)$ is a supermartingale, then

$$u(C_{s,t}(S_t),s) \overset{(2.9)}{=} E_{\mathbb{P}}\left[u(S_t,t)|\mathscr{F}_s\right] \leq u(S_s,s) \text{ for all } 0 \leq s \leq t$$

and therefore $C_{s,t}(S_t) \leq S_s$. Conversely if $C_{s,t}(S_t) \leq S_s$ then

$$E_{\mathbb{P}}\left[u(S_t,t)|\mathscr{F}_s\right] \overset{(2.9)}{=} u(C_{s,t}(S_t),s) \leq u(S_s,s)$$

and $u(S_t,t)$ is a supermartingale. Similarly, for the other cases.
ii) From extended Jensen we get:

$$E_{\mathbb{P}}\left[u(S_t,t)|\mathscr{F}_s\right] \overset{(d)}{\leq} E_{\mathbb{P}}\left[u(S_t,s)|\mathscr{F}_s\right] \leq u\left(E_{\mathbb{P}}\left[S_t|\mathscr{F}_s\right],s\right) \leq u\left(S_s,s\right).$$

iii) From Proposition 2.1 vi) we deduce: $S_s = C_{s,t}(S_t) \leq E_{\mathbb{P}}\left[S_t|\mathscr{F}_s\right]$.

Remark 2.3. When u satisfies (d) and for every $t \in [0,+\infty)$, $u(x,t)$ is integrable for every $x \in \mathscr{D}(t)$ and $\{S_t\}_{t\geq 0}$ is a $(\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})-$martingale, then $\{V_t\}_{t\geq 0}$ is a $(\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})-$ supermartingale, not necessarily a martingale.

2.3 A local formulation of the CCE

Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ be a filtered probability space where the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is generated by a d-dimensional brownian motion $W = \{(W_t^1,...,W_t^d)^\dagger\}_{t\geq 0}$, where $\dagger$ indicates the transposed of a matrix. For $i = 1,...,k$, the price of the i^{th} risky asset and the bond are described respectively by

$$dS_t^i = S_t^i\left(\mu_t^i dt + \sigma_t^i \cdot dW_t\right), \quad dB_t = r_t B_t dt$$

with $S_0^i > 0$, $B_0 = 1$; $\sigma_t = (\sigma_t^{j,i})$ is the $d\times k$ volatility matrix and $\cdot$ the usual vector product, which will be often omitted. Following Musiela and Zariphopoulou, we assume $\mu_t - r_t\mathbf{1} \in Lin(\sigma_t^\dagger)$, i.e. the linear space generated by the columns of $\sigma_t^\dagger$. Denote by $(\sigma_t^\dagger)^+$ the Moore-Penrose pseudo-inverse of the matrix $\sigma_t^\dagger$ and define $\lambda_t = (\sigma_t^\dagger)^+(\mu_t - \mathbf{1}r_t)$, which is the solution of the equation $\sigma_t^\dagger x = \mu_t - \mathbf{1}r_t$. The present value of the amounts invested in B_t, S_t^i are denoted by π_t^0, π_t^i, respectively. The present value of investment is then given by $X_t^\pi = \sum_{i=0}^k \pi_t^i$ and satisfies the SDE

$$dX_t^\pi = \sigma_t \pi_t (\lambda_t dt + dW_t)$$

where $\pi_t = (\pi_t^1,...,\pi_t^k)^\dagger$.

Let $U(x,t)$ be a dynamic stochastic utility of the form

$$U(x,t) = U(x,0) + \sum_{j=1}^{m} \int_0^t u^j(x,s) d\zeta_s^j = U(x,0) + \int_0^t u(x,s) \cdot d\zeta_s$$

$$d\zeta_t^j = a^j(\zeta_t,t)dt + \sum_{i=1}^{d} b^{i,j}(\zeta_t,t) dW_t^i = a^j(\zeta_t,t)dt + b^j(\zeta_t,t) \cdot dW_t$$

where every $u^j(x,t)$ belongs to $C^{2,1}(\mathbb{R} \times [0,T])$ and is a strictly increasing concave function of x. We denote by $\mathbf{b}_s$ the $d \times m$-matrix $(b^{i,j}(\zeta_s,s))$.

Proposition 2.3. *Suppose that for every $t > 0$,*

$$\int_0^t E_{\mathbb{P}}\left[(\mathbf{b}_s u(X_s^\pi,s))^2\right] ds < +\infty \text{ and } \int_0^t E_{\mathbb{P}}\left[(U_x(X_s^\pi,s)\sigma_s\pi_s)^2\right] ds < +\infty$$

The conditional certainty equivalent can be approximated as

$$C_{t,T}(X_T^\pi) = E_{\mathbb{P}}[X_T^\pi|\mathscr{F}_t] - \frac{1}{2}\alpha(X_t^\pi,t)(\sigma_t\pi_t)^2(T-t) - \beta(X_t^\pi,t)(T-t) + o(T-t)$$

where we have denoted respectively the coefficient of absolute risk aversion *and the* impatience factor *by*

$$\alpha(x,t) := -\frac{U_{xx}(x,t)}{U_x(x,t)}$$
$$\beta(x,t) := -\frac{u(x,t)\cdot a(\zeta_t,t) + \mathbf{b}_t u_x(x,t)\cdot \sigma_t\pi_t}{U_x(x,t)}$$

As a consequence the risk premium is given by

$$\rho_{t,T}(X_T^\pi) = +\frac{1}{2}\alpha(X_t^\pi,t)(\sigma_t\pi_t)^2(T-t) + \beta(X_t^\pi,t)(T-t) + o(T-t)$$

Proof. For simplicity we denote X_t^π by X_t. We apply the generalized Itô's formula (see [52], Chapter 2), so for every $v \in [t,T]$

$$U(X_v,v) = U(X_t,t) + \int_t^v u(X_s,s)\cdot d\zeta_s + \int_t^v U_x(X_s,s) dX_s$$
$$+ \frac{1}{2}\int_t^v U_{xx}(X_s,s)(\sigma_s\pi_s)^2 ds + \langle \int_t^v U_x(X_s,ds), X_v\rangle$$

Notice that in this case

$$\langle \int_t^v U_x(X_s,ds), X_v\rangle = \langle \int_t^v u_x(X_s,s)\cdot d\zeta_s, X_v\rangle = \sum_{j=1}^{m}\int_t^v u_x^j(X_s,s)\left(\sum_{i=1}^{d} b^{i,j}(\zeta_s,s)(\sigma_s\pi_s)^k\right) ds$$

and then we have

$$\begin{aligned} U(X_v,v) &= U(X_t,t) \\ &\quad + \int_t^v (u(X_s,s)\cdot a(\zeta_s,s) + \sigma_s\pi_s\lambda_s U_x(X_s,s) \\ &\quad + \frac{1}{2}U_{xx}(X_s,s)(\sigma_s\pi_s)^2 + \mathbf{b}_s u_x(X_s,s)\sigma_s\pi_s)ds \\ &\quad + \int_t^v (\mathbf{b}_s u(X_s,s) + U_x(X_s,s)\sigma_s\pi_s)\,dW_s \end{aligned}$$

From the assumption of the theorem, $I_t = \int_0^t (u(X_s,s)\cdot\mathbf{b}_s + U_x(X_s,s)\sigma_s\pi_s)\,dW_s$ is a martingale: so the conditional expectation is given by

$$\begin{aligned} E_{\mathbb{P}}[U(X_v,v)|\mathscr{F}_t] &= U(X_t,t) \\ &\quad + \int_t^v E_{\mathbb{P}}\left[ua + \sigma\pi\lambda U_x + \frac{1}{2}U_{xx}(\sigma\pi)^2 + \mathbf{b}u_x\sigma\pi \,\middle|\, \mathscr{F}_t\right] ds \end{aligned} \tag{2.11}$$

From the definition of CCE we have

$$U(C_{t,v}(X_v),t) = E_{\mathbb{P}}[U(X_v,v)|\mathscr{F}_t]$$

If we denote $\{Z_v\}_{v\in[t,T]}$ the stochastic process defined by $Z_v =: E_{\mathbb{P}}[U(X_v,v)|\mathscr{F}_t]$ then the stochastic differential

$$\begin{aligned} dC_{t,v}(X_v) &= dU^{-1}(Z_v,t) = \left(\frac{\partial(U(x,t))}{\partial y}\Big|_{x=U^{-1}(Z_v,t)}\right)^{-1} dZ_v \\ &= \frac{1}{U_x(C_{t,v}(X_v),t)} E_{\mathbb{P}}\left[ua + \sigma\pi\lambda U_x + \frac{1}{2}U_{xx}(\sigma\pi)^2 + \mathbf{b}u_x\sigma\pi \,\middle|\, \mathscr{F}_t\right] dv \end{aligned}$$

Hence, since $U^{-1}(Z_t,t) = X_t$

$$C_{t,T}(X_T) = X_t + \int_t^T E_{\mathbb{P}}[(\star) \mid \mathscr{F}_t]ds$$

where

$$(\star) = \frac{u(X_s,s)a(\zeta_s,s) + \sigma_s\pi_s\lambda_s U_x(X_s,s) + \frac{1}{2}U_{xx}(X_s,s)(\sigma_s\pi_s)^2 + \mathbf{b}_s u_x(X_s,s)\sigma_s\pi_s}{U_x(C_{t,s}(X_s),t)}$$

Notice that

$$E_{\mathbb{P}}[X_T|\mathscr{F}_t] = X_t + E_{\mathbb{P}}\left[\int_t^T \sigma_s\pi_s\lambda_s ds \mid \mathscr{F}_t\right] = X_t + \sigma_t\pi_t\lambda_t(T-t) + o(T-t)$$

$$\begin{aligned} C_{t,T}(X) &= X_t + \sigma_t \pi_t \lambda_t (T-t) + \frac{1}{2}\frac{U_{xx}(X_t,t)}{U_x(X_t,t)}(\sigma_t \pi_t)^2 (T-t) \\ &\quad + \frac{u(X_t,t)a(\zeta_t,t) + \mathbf{b}_t u_x(X_t,t)\sigma_t \pi_t}{U_x(X_t,t)}(T-t) + o(T-t) \\ &= E_{\mathbb{P}}[X_T|\mathscr{F}_t] - \frac{1}{2}\alpha(X_t,t)(\sigma_t\pi_t)^2(T-t) - \beta(X_t,t)(T-t) + o(T-t) \end{aligned}$$

Remark 2.4. If the utility $U(x,t)$ is deterministic (i.e. the matrix $\mathbf{b}_t \equiv \mathbf{0}$ for every $t \geq 0$) we deduce that

$$\beta(x,t) = -\frac{u(x,t)a(\zeta_t,t)}{U_x(x,t)} = -\frac{U_t(x,t)}{U_x(x,t)}$$

which is the usual definition of impatience factor.

2.4 The right framework for the CCE

Until now we have considered $C_{s,t}$ as a map defined on the set of random variables $\mathscr{U}(t)$ which is not in general a vector space. In order to show the dual representation of the CCE it is convenient to define it on a Banach lattice.

Orlicz spaces have become an important tool whenever we approach to the utility-maximization framework and we are dealing with unbounded random variables (see for instance [7] and [8]).
The question which naturally arise is: what happens if we consider a utility functions which has some explicit dependence on the randomness? May we actually define a class of “stochastic” Orlicz spaces?
Therefore we now introduce the general class of Musielak-Orlicz spaces induced by the stochastic dynamic utility taken into account.

2.4.1 Generalities on Musielak-Orlicz Spaces

Given a non-atomic probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a function $\Psi : \mathbb{R} \times \Omega \to \mathbb{R} \cup \{+\infty\}$, with $\mathscr{D} = \{x \in \mathbb{R} \mid \Psi(x,\omega) < +\infty\} \neq \emptyset$, we say that Ψ is a (generalized) Young function if $\Psi(x,\cdot)$ is $\mathscr{F}$-measurable and for $\mathbb{P}$ a.e. $\omega \in \Omega$

1. $\Psi(\cdot,\omega)$ is even and convex;
2. the effective domain $\mathscr{D}$ does not depend on ω and $0 \in int(\mathscr{D})$;
3. $\Psi(\infty,\omega) = +\infty$, $\Psi(0,\omega) = 0$.

Note that Ψ may jump to $+\infty$ outside of a bounded neighborhood of 0. In case Ψ is finite valued however, it is also continuous w.r.t. x by convexity. Whenever possible, we will suppress the explicit dependence of Ψ from ω.

The Musielak-Orlicz space L^Ψ, on $(\Omega,\mathscr{F},\mathbb{P})$ is then defined as

$$L^\Psi = \left\{X \in L^0 \mid \exists\, \alpha > 0\, E_{\mathbb{P}}[\Psi(\alpha X)] < +\infty\right\}.$$

endowed with the Luxemburg norm

$$N_\Psi(X) = \inf\left\{c > 0 \mid E_{\mathbb{P}}\left[\Psi\left(X \cdot c^{-1}\right)\right] \le 1\right\}.$$

Although there are no particular differences with Musielak work (see [67]), here we are dropping the hypothesis on Ψ to be finite (and so continuous). But since the domain $\mathscr{D}$ does not depend on ω we have that non continuous Ψs always induce the space $L^\infty(\Omega,\mathscr{F},\mathbb{P})$ and the Luxemburg norm is equivalent to the supremum norm.

It is known that (L^Ψ, N_Ψ) is a Banach space (Theorem 7.7 in [67]), and with the usual pointwise lattice operations, L^Ψ is a Banach lattice.

There is an important linear subspace of L^Ψ, which is also a Banach lattice

$$M^\Psi = \left\{X \in L^0 \mid E_{\mathbb{P}}\left[\Psi(\alpha X)\right] < +\infty\ \forall \alpha > 0\right\}.$$

In general, $M^\Psi \subsetneq L^\Psi$ and this can be easily seen when Ψ is non continuous since in this case $M^\Psi = \{0\}$, but there are also non trivial examples of the strict containment with finite-valued, continuous Young functions, that we will consider soon.

Other convenient assumptions on Ψ that we will use in the forthcoming discussion are

(int) $E_{\mathbb{P}}[\Psi(x)]$ is finite for every $x \in \mathscr{D}$;

(sub) there exists a Young function $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that $g(x) \le \Psi(x,\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$

(Δ_2) There exists $K \in \mathbb{R}$, $h \in L^1$ and $x_0 \in \mathbb{R}$ such that

$$\Psi(2x,\cdot) \le K\Psi(x,\cdot) + h(\cdot) \quad \text{for all } x > x_0,\ \mathbb{P}-a.s.$$

When Ψ satisfies (int) and the (Δ_2) condition (and it is henceforth finite-valued and continuous) the two spaces M^Ψ, L^Ψ coincide and L^Ψ can simply be written as $\{X \in L^0 \mid E_{\mathbb{P}}[\Psi(X)] < +\infty\}$ (see [67], Theorem 8.14). This is the case of the L^p spaces when Ψ does not depend on ω.

In [67] (Theorem 7.6) it is also shown that when Ψ is (int) and continuous on $\mathbb{R}$, then $M^\Psi = \overline{L^\infty}^\Psi$ with closure taken in the Luxemburg norm. When Ψ is continuous but grows too quickly, it may happen that $M^\Psi = \overline{L^\infty}^\Psi \subsetneq L^\Psi$. As a consequence, simple functions are not necessarily dense in L^Ψ.

If both (int) and (sub) hold, it is not difficult to prove that

$$L^\infty \hookrightarrow M^\Psi \hookrightarrow L^\Psi \hookrightarrow L^g \hookrightarrow L^1$$

with linear lattice embeddings (the inclusions).

As usual, the convex conjugate function Ψ^* of Ψ is defined as

$$\Psi^*(y,\omega) =: \sup_{x\in\mathbb{R}} \{xy - \Psi(x,\omega)\}$$

and it is also a Young function. The function Ψ^* in general does not satisfy (int), but a sufficient condition for it is that Ψ is (sub). The Musielak-Orlicz space L^{Ψ^*} will be endowed with the Orlicz (or dual) norm

$$\|X\|_{\Psi^*} = \sup\{E_{\mathbb{P}}[|Xf|] \mid f \in L^{\Psi} : E_{\mathbb{P}}[\Psi(f)] \leq 1\},$$

which is equivalent to the Luxemburg norm.

2.4.2 The Musielak-Orlicz space $L^{\widehat{u}}$ induced by an SDU

In the spirit of [7], we now build the time-dependent stochastic Orlicz space induced by the SDU $u(x,t,\omega)$. The even function $\widehat{u}: \mathbb{R}\times[0,+\infty)\times\Omega \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\widehat{u}(x,t,\omega) = u(0,t) - u(-|x|,t,\omega)$$

is a Young function and the induced Orlicz spaces are

$$L^{\widehat{u}_t} = \{X \in L^0_{\mathscr{F}_t} \mid \exists \alpha > 0\, E_{\mathbb{P}}[\widehat{u}(\alpha X,t)] < +\infty\}$$

$$M^{\widehat{u}_t} = \{X \in L^0_{\mathscr{F}_t} \mid E_{\mathbb{P}}[\widehat{u}(\alpha X,t)] < +\infty\ \forall \alpha > 0\}$$

endowed with the Luxemburg norm $N_{\widehat{u}_t}(\cdot)$.

Notice the following important fact:

$$M^{\widehat{u}_t} \subseteq \mathscr{U}(t).$$

Indeed, for any given $\lambda > 0$ and $X \in L^0_{\mathscr{F}_t}$ such that $E_{\mathbb{P}}[\widehat{u}(\lambda X,t)] < +\infty$ we have: $E_{\mathbb{P}}[u(\lambda X,t)] \geq E_{\mathbb{P}}[u(-\lambda|X|,t)] > -\infty$. On the other hand $u(x,t) - u(0,t) \leq \widehat{u}(x,t)$ so that $E_{\mathbb{P}}[u(\lambda X,t)] \leq E_{\mathbb{P}}[\widehat{u}(\lambda X,t) + u(0,t)] < +\infty$ and the claim follows. In particular this means that (int) implies $u(x,t)$ is integrable for every $x \in \mathscr{D}(t)$.

This argument highlights one relevant feature: every $X \in M^{\widehat{u}_t}$ belongs to the set $\mathscr{U}(t)$ so that the CCE is well defined on $M^{\widehat{u}_t}$. In the following examples also $C_{s,t}(X) \in M^{\widehat{u}_s}$ holds true, so that $C_{s,t}: M^{\widehat{u}_t} \to M^{\widehat{u}_s}$ and it make sense to study the time consistency of $C_{s,t}$.

2.4.3 Examples

Exponential random utilities

Let us consider $u: \mathbb{R}\times[0,\infty)\times\Omega \to \mathbb{R}$ defined by

$$u(x,t,\omega) = -e^{-\alpha_t(\omega)x + \beta_t(\omega)}$$

where $\alpha_t > 0$ and β_t are adapted stochastic processes.
In this example the CCE may be simply computed inverting the function $u(\cdot,t,\omega)$:

$$C_{s,t}(X) = -\frac{1}{\alpha_s}\ln\left\{E_{\mathbb{P}}[e^{-\alpha_t X+\beta_t}|\mathscr{F}_s]\right\} + \frac{\beta_s}{\alpha_s} \tag{2.12}$$

Notice the measurability requirement on the risk aversion process α_t, which is different from what can be found in some examples in the literature related to dynamic risk measures, as e.g. in [1], where the α_t in (2.12) is replaced by α_s.

Assumptions: We suppose that β_t belongs to $L^\infty(\mathscr{F}_t)$ for any $t > 0$ and that $e^{\alpha_t x} \in L^1_{\mathscr{F}_t}$ for every $x \in \mathbb{R}$.

These assumptions guarantee that (int) holds. In particular if $\alpha_t(\omega) \equiv \alpha \in \mathbb{R}$ and $\beta_t \equiv 0$ then $C_{s,t}(X) = -\rho_{s,t}(X)$, where $\rho_{s,t}$ is the dynamic entropic risk measure induced by the exponential utility. Unfortunately when the risk aversion coefficient is stochastic we have no chance that $C_{s,t}$ has any monetary property. On the other hand monotonicity and concavity keep standing. The first is due to Proposition 2.1, whereas the second is a straightforward application of Holder-conditional inequality. This means that in general $\rho_{s,t}(X) =: -C_{s,t}(X)$ satisfies all the usual assumptions of dynamic risk measures, only failing the cash additive property. We now show a sufficient condition by which $\rho_{s,t}(X)$ is at least cash subadditive, i.e. $\rho_{s,t}(X+Y) \geq \rho_{s,t}(X) - Y$ where $Y \in L^\infty_{\mathscr{F}_s}$ and $Y \geq 0$.

Proposition 2.4. *Under the previous assumptions, the functional*

$$\rho_{s,t}(X) = \frac{1}{\alpha_s}\ln\left\{E_{\mathbb{P}}[e^{-\alpha_t X+\beta_t}|\mathscr{F}_s]\right\} + \frac{\beta_s}{\alpha_s}$$

is cash subadditive if the process $\{\alpha_t\}_{t\geq 0}$ *is almost surely decreasing.*

Proof. For every $Y \in L^\infty_{\mathscr{F}_s}$ and $Y \geq 0$:

$$\begin{aligned}\rho_{s,t}(X+Y) &= \frac{1}{\alpha_s}\ln\left\{E_{\mathbb{P}}[e^{-\frac{\alpha_t}{\alpha_s}\alpha_s Y}e^{-\alpha_t X+\beta_t}|\mathscr{F}_s]\right\} - \frac{\beta_s}{\alpha_s}\\ &\geq \frac{1}{\alpha_s}\ln\left\{E_{\mathbb{P}}[e^{-\alpha_s Y}e^{-\alpha_t X+\beta_t}|\mathscr{F}_s]\right\} - \frac{\beta_s}{\alpha_s} = \rho_{s,t}(X) - Y.\end{aligned}$$

Proposition 2.5. *Under the previous assumptions*

$$M^{\hat{u}_t} \overset{i}{\hookrightarrow} L^{\hat{u}_t} \overset{j}{\hookrightarrow} L^p(\Omega,\mathscr{F},\mathbb{P}) \quad p \geq 2$$

where i, j *are isometric embeddings given by the set inclusions.*

Proof. The first inclusion is trivial since the two spaces are endowed with the same norm. Moreover $M^{\hat{u}_t}$ is a closed subspace of $L^{\hat{u}_t}$.
For the second inclusion we simply observe that since

$$\frac{d(u(x,t))}{dx}_{|x=0} = \alpha_t e^{A_t} > 0$$

for almost every $\omega \in \Omega$ then for every $p \geq 2$ and $\lambda > 0$

$$|x|^p \leq \hat{u}(\lambda x, t, \omega) \quad \forall x \in \mathbb{R}, \text{ for } \mathbb{P}-\text{a.e. } \omega \in \Omega$$

which implies

$$||X||_p \leq k N_{\hat{u}_t}(X) \tag{2.13}$$

Proposition 2.6. *Under the the previous assumptions*

$$C_{s,t} : M^{\widehat{u}_t} \to M^{\widehat{u}_s}$$

Proof. Let $\lambda \geq 1$, and since no confusion arises we denote by $u_t(x) \circeq u(x,t)$. Define $A = \{\ln E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t} | \mathscr{F}_s] \leq \beta_s\}$ and notice that

$$\begin{aligned}
\widehat{u}_s(\lambda C_{s,t}(X)) - u_s(0) &= -u_s\left(-\frac{\lambda}{\alpha_s}\left|-\ln E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t}|\mathscr{F}_s] + \beta_s\right|\right) \\
&= e^{\beta_s} \exp(\lambda | -\ln E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t}|\mathscr{F}_s] + \beta_s|) \\
&= e^{\beta_s} \exp(\lambda(\beta_s - \ln E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t}|\mathscr{F}_s]))\mathbf{1}_A \\
&+ e^{\beta_s} \exp(\lambda(\ln E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t}|\mathscr{F}_s] - \beta_s))\mathbf{1}_{A^C} \\
&= e^{\beta_s(1+\lambda)} E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t}|\mathscr{F}_s]^{-\lambda} \mathbf{1}_A + e^{\beta_s(1-\lambda)} E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t}|\mathscr{F}_s]^{\lambda} \mathbf{1}_{A^C}
\end{aligned}$$

Since on A we have $E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t}|\mathscr{F}_s] \leq e^{\beta_s}$ and in general $e^{\beta_s(1-\lambda)} \leq a \in \mathbb{R}_+$ then

$$\begin{aligned}
E\left[\widehat{u}_s(\lambda C_{s,t}(X))\right] &\leq E_{\mathbb{P}}[e^{\beta_s(1+\lambda-\lambda)}\mathbf{1}_A] + aE\left\{E_{\mathbb{P}}[e^{-\alpha_t X + \beta_t}|\mathscr{F}_s]^{\lambda}\mathbf{1}_{A^C}\right\} + E_{\mathbb{P}}[u_s(0)] \\
&\leq -E_{\mathbb{P}}[u_s(0)] + aE_{\mathbb{P}}\left[e^{\lambda(-\alpha_t X + \beta_t)}\right] + E_{\mathbb{P}}[u_s(0)] \\
&\leq +a\|(e^{(\lambda-1)\beta_t})\|_\infty E_{\mathbb{P}}\left[\widehat{u}_t(\lambda X) + e^{\beta_t}\right] \leq K E_{\mathbb{P}}\left[\widehat{u}_t(\lambda X)\right]
\end{aligned}$$

Notice that the second step is a simple application of Jensen's inequality, in fact: $E[Y|\mathscr{G}]^{\lambda} \leq E_{\mathbb{P}}[Y^{\lambda}|\mathscr{G}]\ \forall \lambda \geq 1$. Moreover we have that for $0 < \lambda < 1$ $E\left[\widehat{u}_s(\lambda C_{s,t}(X))\right] \leq E\left[\widehat{u}_s(C_{s,t}(X))\right] < \infty$ and then $C_{s,t}(X) \in M^{\widehat{u}_s}$.

Random-power utilities

Consider the utility function given by

$$u(x,t,\omega) = -\gamma_t(\omega)|x|^{p_t(\omega)}\mathbf{1}_{(-\infty,0)}$$

where γ_t, p_t are adapted stochastic processes satisfying $\gamma_t > 0$ and $p_t > 1$. We have $\widehat{u}(x,t) = \gamma_t |x|^{p_t}$. Here assumption (int) is troublesome but not needed for what follows. On the other hand the utility fails to be strictly increasing so that we won't have uniqueness of the solution for the equation defining the CCE, namely

$$-\gamma_s|C_{s,t}(X)|^{p_s}\mathbf{1}_{\{C_{s,t}(X)<0\}} = E_{\mathbb{P}}\left[-\gamma_t|X|^{p_t}\mathbf{1}_{\{X<0\}}|\mathscr{F}_s\right] \tag{2.14}$$

Notice that $C_{s,t}(X) = C_{s,t}(X^- + K\mathbf{1}_{X\geq 0})$ where K is any positive $\mathscr{F}_t$ r.v.; moreover if $G := \{E_{\mathbb{P}}[\gamma_t|X|^{p_t}\mathbf{1}_{\{X<0\}}|\mathscr{F}_s] > 0\}$ then $\mathbb{P}(G \setminus \{C_{s,t}(X) < 0\}) = 0$. If we decompose X as $X^+ - X^-$ we can conclude that

$$C_{s,t}(X) = -\frac{1}{\gamma_s}\left(E_{\mathbb{P}}[\gamma_t(X^-)^{p_t}|\mathscr{F}_s]\right)^{\frac{1}{p_s}} + K\mathbf{1}_{G^C}$$

it's the class of solutions of (2.14) where $K \in L^0_{\mathscr{F}_s}$ and $K > 0$. This is a natural consequence of the choice of a preference system in which the agent is indifferent among all the positive variables. If in particular $K \in M^{\hat{u}_s}$ then it is easy to check that $C_{s,t} : M^{\widehat{u}_t} \longrightarrow M^{\widehat{u}_s}$.

Stochastic transformations of static utilities

One may wonder what happens for an arbitrary SDU. Clearly the fact that $C_{s,t}$ is a map between the the two corresponding Orlicz spaces at time t and s is a key feature for the time-consistency. We take into account a particular class of SDU, which are a stochastic transformation of a standard utility function.

Let $V : \mathbb{R} \to \mathbb{R}$ a concave, strictly increasing function: take an adapted stochastic process, $\{\alpha_t\}_{t\geq 0}$, such that for every $t \geq 0$, $\alpha_t > 0$. Then $u(x,t,\omega) = V(\alpha_t(\omega)x)$ is a SDU and

$$C_{s,t}(X) = \frac{1}{\alpha_s}V^{-1}\left(E_{\mathbb{P}}[V(\alpha_t X) \mid \mathscr{F}_s]\right)$$

Proposition 2.7. *Let* $\Theta_t = \{X \in L^{\widehat{u}_t} \mid E_{\mathbb{P}}[u(-X^-,t)] > -\infty\} \supseteq M^{\widehat{u}_t}$. *Then*

$$C_{s,t} : \Theta_t \to \Theta_s$$

Moreover if $\widehat{u}(x,s)$ *satisfies the* (Δ_2) *condition, then*

$$C_{s,t} : M^{\widehat{u}_t} \to M^{\widehat{u}_s}.$$

Proof. Denote $\widehat{u}_t(x) = \widehat{u}(x,t)$; from Jensen inequality we have

$$\frac{1}{\alpha_s}V^{-1}\left(E_{\mathbb{P}}[V(\alpha_t X) \mid \mathscr{F}_s]\right) \leq \frac{1}{\alpha_s}E_{\mathbb{P}}[\alpha_t X \mid \mathscr{F}_s] \tag{2.15}$$

Define the $\mathscr{F}_s$ measurable sets

$$F = \{E_{\mathbb{P}}[V(\alpha_t X) \mid \mathscr{F}_s] \geq V(0)\}, \quad G = \{E_{\mathbb{P}}[\alpha_t X \mid \mathscr{F}_s] \geq 0\}$$

and deduce from equation (2.15) that

$$0 \le C_{s,t}(X)^+ = \frac{1}{\alpha_s} V^{-1}\left(E_{\mathbb{P}}[V(\alpha_t X) \mid \mathscr{F}_s]\right) \mathbf{1}_F \le \frac{1}{\alpha_s} E_{\mathbb{P}}[(\alpha_t X)\mathbf{1}_G \mid \mathscr{F}_s]$$

For every $X \in L^{\hat{u}_t}$ we may find a $\lambda > 0$ such that $E_{\mathbb{P}}[\hat{u}_t(\lambda X \mathbf{1}_G)] < +\infty$:

$$\begin{aligned} E_{\mathbb{P}}\left[\hat{u}_s\left(\frac{\lambda}{\alpha_s} V^{-1}\left(E_{\mathbb{P}}[V(\alpha_t X) \mid \mathscr{F}_s]\right)\mathbf{1}_F\right)\right] &\le E_{\mathbb{P}}\left[\hat{u}_s\left(\frac{\lambda}{\alpha_s} E_{\mathbb{P}}[(\alpha_t X)\mathbf{1}_G \mid \mathscr{F}_s]\right)\right] \\ = E_{\mathbb{P}}[V(0) - V(-E_{\mathbb{P}}[(\lambda \alpha_t X)\mathbf{1}_G \mid \mathscr{F}_s])] &\le E_{\mathbb{P}}[V(0) - V(-\lambda \alpha_t X \mathbf{1}_G)] \\ &\le E_{\mathbb{P}}[\hat{u}_t(\lambda X \mathbf{1}_G)]. \end{aligned}$$

Hence $X \in L^{\hat{u}_t}$ implies $C_{s,t}(X)^+ \in L^{\hat{u}_s}$.
Now let's consider a r.v. $X \in \Theta_t$: $-C_{s,t}(X)^- = \frac{1}{\alpha_s} V^{-1}\left(E_{\mathbb{P}}[V(\alpha_t X) \mid \mathscr{F}_s]\right)\mathbf{1}_{F^C}$.
We can conclude that

$$\begin{aligned} 0 \le E_{\mathbb{P}}[\widehat{u}_s(-C_{s,t}(X)^-)] &= E_{\mathbb{P}}\left[-V \circ V^{-1}\left(E_{\mathbb{P}}[V(\alpha_t X)\mathbf{1}_{F^C} | \mathscr{F}_s]\right) + V(0)\right] = \\ &= E_{\mathbb{P}}[-V(\alpha_t X)\mathbf{1}_{F^C} + V(0)] < +\infty \end{aligned}$$

where the last inequality follows from $X \in \Theta_t$, $\{X \ge 0\} \subseteq F$ and

$$V(\alpha_t X)\mathbf{1}_{F^C} = \left(V(\alpha_t X^+)\mathbf{1}_{\{X \ge 0\}} + V(-\alpha_t X^-)\mathbf{1}_{\{X<0\}}\right)\mathbf{1}_{F^C} = V(-\alpha_t X^-)\mathbf{1}_{\{X<0\} \cap F^C}$$

This shows that surely $C_{s,t}(X) \in \Theta_s$, if $X \in \Theta_t$.

2.5 Dual representation of CCE

In this section we prove a dual formula for the CCE, which is similar to the general result that can be found in [33]: due to the particular structure of the CCE the proof is simpler and more readable.

Consider the condition:

$$\text{there exists } X^* \in (L^{\widehat{u}_t})^* \text{ s.t. } E_{\mathbb{P}}[f^*(X^*, t)] < +\infty \tag{2.16}$$

where $f^*(x,t,\omega) = \sup_{y \in \mathbb{R}} \{xy + u(y,t,\omega)\}$.

As a consequence of Theorem 1 [75], we may deduce that if (2.16) holds, if $\widehat{u}(x,t)$ is (int) and $X \in L^{\widehat{u}_t}$ then $E_{\mathbb{P}}[u(\lambda X, t)] < +\infty$ for every $\lambda > 0$.

Remark 2.5. The condition (2.16) is quite weak: it is satisfied, for example, if $u(x,t,\omega) \le ax + b$ with $a, b \in \mathbb{R}$ since

$$f^*(-a,t,\omega) \le \sup_{y \in \mathbb{R}} \{(-a+a)y + b\} = b.$$

We now take into account $(L^{\Psi})^*$, the norm dual of L^{Ψ} and consider the following three cases which cover a pretty large class of possible Young functions.

1. $\Psi(\cdot,\omega)$ is (int) and discontinuous, i.e. $\mathscr{D} \subsetneq \mathbb{R}$.

In this case, $L^{\Psi}=L^{\infty}$ and from the Yosida-Hewitt decomposition for elements of $ba(\Omega,\mathscr{F},P)$ we have

$$ba=(L^{\infty})^{*}=L^{1}\oplus\mathscr{A}^{d},$$

where $\mathscr{A}^{d}$ consists of pure charges, i.e. purely finitely additive measures (which are not order continuous).

2. $\Psi(\cdot,\omega)$ is continuous, Ψ and Ψ^{*} are (int) and satisfy:

$$\frac{\Psi(x,\omega)}{x}\rightarrow+\infty\ \mathbb{P}-a.s,\text{ as }x\rightarrow\infty.$$

These conditions are not restrictive and hold as soon as Ψ is (int) and (sub) with $\lim_{x\rightarrow\infty}\frac{g(x)}{x}\rightarrow+\infty$. For such Young functions it can be easily deduced from Theorem 13.17 in [67] that $(M^{\Psi})^{*}=L^{\Psi^{*}}$: $\mu_{r}\in(M^{\Psi})^{*}$ can be identified with its density $\frac{d\mu_{r}}{dP}\in L^{\Psi^{*}}$ so that we will write its action on $X\in L^{\Psi}$ as $\mu_{r}(X)=E_{\mathbb{P}}[\mu_{r}X]$. Moreover $(M^{\Psi})^{*}$ is a band in the dual space $(L^{\Psi})^{*}$ (see [2] Section 8) so that we may decompose

$$(L^{\Psi})^{*}=(M^{\Psi})^{*}\oplus(M^{\Psi})^{\perp}$$

i.e. every $X^{*}\in(L^{\Psi})^{*}$ can be uniquely represented as $X^{*}=\mu_{r}+\mu_{s}$ where μ_{s} belongs to the annihilator of M^{Ψ} ($\mu_{s}(X)=0$ for every $X\in M^{\Psi}$) and $\mu_{r}\in(M^{\Psi})^{*}=L^{\Psi^{*}}$. Notice that every element $\mu_{r}\in(M^{\Psi})^{*}$ is clearly order continuous. Moreover it can be shown, applying an argument similar to the one used in Lemma 10 [7], that every $\mu_{s}\in(M^{\Psi})^{\perp}$ is not order continuous.

3. $\Psi(\cdot,\omega)$ is continuous and

$$0<a=ess\inf_{\omega\in\Omega}\lim_{x\rightarrow\infty}\frac{\Psi(x,\omega)}{x}\leq ess\sup_{\omega\in\Omega}\lim_{x\rightarrow\infty}\frac{\Psi(x,\omega)}{x}=b<+\infty$$

Here (int) automatically holds for both Ψ and Ψ^{*}. It follows that $L^{\Psi}=L^{1}$ and the L^{1}-norm is equivalent to the Luxemburg norm, so that $(L^{\Psi})^{*}=L^{\Psi^{*}}=L^{\infty}$.

Assumptions for the dual result

In this section $u(x,t,\omega)$ is a SDU, such that:

1. For all $t\geq0$, the induced Young function $\widehat{u}(x,t,\omega)$ belongs to one of the three classes mentioned above
2. The condition (2.16) holds true.

As shown above, under the assumption (1) the order dual space of $L^{\widehat{u}_{t}}$ is known and is contained in L^{1}. This will also allow us to apply Proposition 1.1. The second assumption implies that $E_{\mathbb{P}}[u(\cdot,t)]:L^{\widehat{u}_{t}}\rightarrow[-\infty,+\infty)$ is a well defined convex functional ([75]).
Thus we have $u(X^{+},t)\in L^{1}_{\mathscr{F}_{t}}$, but in general we do not have integrability for $u(-X^{-},t)$. This means that if $X\notin\Theta_{t}=\{X\in L^{\widehat{u}_{t}}\mid E_{\mathbb{P}}[u(-X^{-},t)]>-\infty\}$ we are

forced to consider the generalized conditional expectation

$$E_{\mathbb{P}}[u(X,t)\mid \mathscr{F}_s] := E_{\mathbb{P}}[u(X,t)^+ \mid \mathscr{F}_s] - \lim_n E_{\mathbb{P}}[u(X,t)^- \mathbf{1}_{\{-n\leq -u(X,s)^- <0\}} \mid \mathscr{F}_s],$$

which can be equivalently written as:

$$E_{\mathbb{P}}[u(X,t)\mid \mathscr{F}_s] = E_{\mathbb{P}}[u(X^+,t)\mathbf{1}_{\{X\geq 0\}} \mid \mathscr{F}_s] + \lim_n E_{\mathbb{P}}[u(-X^-,t)\mathbf{1}_{\{-n\leq X<0\}} \mid \mathscr{F}_s].$$

Therefore, $E_{\mathbb{P}}[u(X,t)\mid \mathscr{F}_s] \in \overline{L}^0_{\mathscr{F}_s}$ and $C_{s,t}(\cdot)$ is defined on the entire space $L^{\widehat{u}_t}$. We fix throughout this section $0 < s \leq t$ and define

$$\mathscr{P}_{\mathscr{F}_t} = \{X^* \in (L^{\widehat{u}_t^*})_+ \mid E_{\mathbb{P}}[X^*] = 1\} \subseteq \{Q << \mathbb{P} \mid Q \text{ probability}\}$$

$$U : L^{\widehat{u}_t} \to \overline{L}^0_{\mathscr{F}_s} \text{ given by } U(X) := E_{\mathbb{P}}[u(X,t)\mid \mathscr{F}_s]$$

The map U is concave and increasing and admits the dual representation stated in Lemma 2.2. From equation (2.18) we deduce the dual representation of $C_{s,t}(\cdot) = u^{-1}(U(\cdot),s)$ as follows.

Theorem 2.1. *Fix $s \leq t$. For every $X \in L^{\widehat{u}_t}$*

$$C_{s,t}(X) = \inf_{Q\in\mathscr{P}_{\mathscr{F}_t}} G(E_Q[X|\mathscr{F}_s],Q) \tag{2.17}$$

where for every $Y \in L^0_{\mathscr{F}_s}$,

$$G(Y,Q) = \sup_{\xi\in L^{\widehat{u}_t}} \{C_{s,t}(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q Y\}.$$

Moreover if $X \in M^{\widehat{u}_t}$ then the essential infimum *in (2.17) is actually a* minimum.

The proof is based on the following Lemma.

Lemma 2.2. *Let $s \leq t$. For every $X \in L^{\widehat{u}_t}$*

$$U(X) = \inf_{Q\in\mathscr{P}_{\mathscr{F}_t}} S(E_Q[X|\mathscr{F}_s],Q) \tag{2.18}$$

where $S(Y,Q) = \sup_{\xi\in L^{\widehat{u}_t}} \{U(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q Y\}$ for any $Y \in L^0_{\mathscr{F}_s}$.
Moreover if $X \in M^{\widehat{u}_t}$ then the essential infimum *in (2.18) is actually a* minimum.

Proof. Obviously $\forall Q \in \mathscr{P}_{\mathscr{F}_t}$

$$E_{\mathbb{P}}[u(X,t)\mid \mathscr{F}_s] \leq \sup_{\xi\in L^{\widehat{u}_t}} \{U(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\}$$

and then

$$E_{\mathbb{P}}[u(X,t)\mid \mathscr{F}_s] \leq \inf_{Q\in\mathscr{P}_{\mathscr{F}_t}} \sup_{\xi\in L^{\widehat{u}_t}} \{U(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\}. \tag{2.19}$$

Important remark: we have that $E(U(X)) = E(u(X,t))$; this means that

$$E(U(\cdot)) : L^{\widehat{u}_t} \to [-\infty, +\infty)$$

is a concave functional. From the monotone convergence theorem and Jensen inequality the functional $E(u(X,t))$ is continuous from above (i.e. $X_n \downarrow X \Rightarrow E(u(X_n,t)) \downarrow E(u(X,t))$). Applying Lemma 15 in [8], $E(U(X))$ is order u.s.c. and thus $\sigma(L^{\widehat{u}_t}, L^{\widehat{u}_t^*})$-u.s.c. (Proposition 24 [8]).
From Proposition 1.1 in Section 1.2:

$$\begin{aligned} E(U(X)) &= \inf_{Q \in \mathscr{P}_{\mathscr{F}_t}} \sup_{\xi \in L^{\widehat{u}_t}} \{E_{\mathbb{P}}[U(\xi)] \mid E_Q[\xi] = E_Q[X]\} \\ &\geq \inf_{Q \in \mathscr{P}_{\mathscr{F}_t}} \sup_{\xi \in L^{\widehat{u}_t}} \{E(U(\xi)) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\} \geq E(U(X)) \end{aligned}$$

i.e.

$$E(U(X)) = \inf_{Q \in \mathscr{P}_{\mathscr{F}_t}} \sup_{\xi \in L^{\widehat{u}_t}} \{E(U(\xi)) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\} \tag{2.20}$$

Surely the map U is regular (i.e. for every $A \in \mathscr{F}_s$, $U(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = U(X)\mathbf{1}_A + U(Y)\mathbf{1}_{A^C}$) and then the set $\mathscr{A} = \{U(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\}$ is upward directed. In fact given $\xi_1, \xi_2 \in \mathscr{A}$ we have

$$U(\xi_1) \vee U(\xi_2) = U(\xi_1)\mathbf{1}_F + U(\xi_2)\mathbf{1}_{F^C} = U(\xi_1\mathbf{1}_F + \xi_2\mathbf{1}_{F^C})$$

where $F = \{U(\xi_1) \geq U(\xi_2)\}$ and $E_Q[\xi_1\mathbf{1}_F + \xi_2\mathbf{1}_{F^C}|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]$. By this last property and the monotone convergence theorem we deduce

$$E_{\mathbb{P}}[S(E_Q[X|\mathscr{F}_s], Q)] = \sup_{\xi \in L^{\widehat{u}_t}} \{E_{\mathbb{P}}[U(\xi)] \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\}$$

Hence

$$\begin{aligned} E(U(X)) &= \inf_{Q \in \mathscr{P}_{\mathscr{F}_t}} \sup_{\xi \in L^{\widehat{u}_t}} \{E(U(\xi)) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\} \\ &= \inf_{Q \in \mathscr{P}_{\mathscr{F}_t}} E\left(\sup_{\xi \in L^{\widehat{u}_t}} \{U(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\}\right) \\ &\geq E\left(\inf_{Q \in \mathscr{P}_{\mathscr{F}_t}} \sup_{\xi \in L^{\widehat{u}_t}} \{U(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\}\right) \end{aligned}$$

This last chain of inequalities together with inequality (2.19) gives

$$U(X) = \inf_{Q \in \mathscr{P}_{\mathscr{F}_t}} \sup_{\xi \in L^{\widehat{u}_t}} \{U(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\} \quad \forall X \in L^{\widehat{u}_t} \tag{2.21}$$

Moreover from generalized Namioka-Klee theorem, the functional $E(u(\cdot)) : L^{\widehat{u}_t}$ is norm continuous on $int(\Theta_u) \supseteq M^{\widehat{u}_t}$ (see [8] Lemma 32) and then $E(U(X))$ as well since $E(U(X)) = E(u(X))$.
Again from Proposition 1.1 we have that:

$$
\begin{aligned}
E(U(X)) &= \min_{Q\in\mathscr{P}_{\mathscr{F}_t}} \sup_{\xi\in M^{\widehat{u}_t}} \{E(U(\xi)) \mid E_Q[\xi] = E_Q[X]\} \\
&= \sup_{\xi\in M^{\widehat{u}_t}} \{E_{\mathbb{P}}[U(\xi)] \mid E_{Q_{min}}[\xi] = E_{Q_{min}}[X]\} \\
&\geq \sup_{\xi\in M^{\widehat{u}_t}} \left\{E_{\mathbb{P}}[U(\xi)] \mid E_{Q_{min}}[\xi|\mathscr{F}_s] =_{Q_{\min}} E_{Q_{min}}[X|\mathscr{F}_s]\right\} \geq E_{\mathbb{P}}[(U(X))
\end{aligned}
$$

The remaining proof matches the previous case and then we get

$$
U(X) = \min_{Q\in\mathscr{P}_{\mathscr{F}_t}} \sup_{\xi\in M^{\widehat{u}_t}} \{U(\xi) \mid E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\} \quad \forall X \in M^{\widehat{u}_t} \tag{2.22}
$$

where the minimizer is exactly Q_{min}.

Proof (of Theorem 2.1). Since s,t are fixed throughout this proof we redefine $C_{s,t}(\cdot) = C(\cdot)$, $u(x,t) = u(x)$ and $u(x,s) = v(x)$. We show that for every fixed $Q \in \mathscr{P}_{\mathscr{F}_t}$, $v^{-1}S(E_Q[X|\mathscr{F}_s],Q) = G(E_Q[X|\mathscr{F}_s],Q)$.

Since C,U are regular, for every fixed $Q \in \mathscr{P}_{\mathscr{F}}$ the sets

$$
\{C(\xi) \mid \xi \in L^{\widehat{u}_t}, E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\},\ \{U(\xi) \mid \xi \in L^{\widehat{u}_t}, E_Q[\xi|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]\}
$$

are upward directed and then there exist ξ_h^Q, η_h^Q such that $E_Q[\xi_h^Q|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]$, $E_Q[\eta_h^Q|\mathscr{F}_s] =_Q E_Q[X|\mathscr{F}_s]$, for every $h > 0$, and

$$
C(\xi_h^Q) \uparrow G(E_Q[X|\mathscr{F}_s],Q), \quad U(\eta_h^Q) \uparrow S(E_Q[X|\mathscr{F}_s],Q) \quad \mathbb{P}-\text{a.s.}
$$

Thus since v^{-1} is continuous in the interior of its domain:

$$
\begin{aligned}
G(E_Q[X|\mathscr{F}_s],Q) &\geq \lim_h C(\eta_h^Q) = v^{-1} \lim_h U(\eta_h^Q) = v^{-1}S(E_Q[X|\mathscr{F}_s],Q) \\
&\geq v^{-1} \lim_h U(\xi_h^Q) = \lim_h C(\xi_h^Q) = G(E_Q[X|\mathscr{F}_s],Q)
\end{aligned}
$$

and this ends the first claim.

It's not hard to prove that the infimum is actually a limit (using the property of downward directness of the set as has been shown in Chapter 1 Lemma 1.4 (v)): therefore we deduce from the continuity of v^{-1} that

$$
\begin{aligned}
C(X) = v^{-1} \inf_{Q\in\mathscr{P}_{\mathscr{F}_t}} S(E_Q[X|\mathscr{F}_s],Q) &= \inf_{Q\in\mathscr{P}_{\mathscr{F}_t}} v^{-1}S(E_Q[X|\mathscr{F}_s],Q) \\
&= \inf_{Q\in\mathscr{P}_{\mathscr{F}_t}} G(E_Q[X|\mathscr{F}_s],Q)
\end{aligned}
$$

Chapter 3
Conditional quasiconvex maps: a L^0-module approach

This last Chapter -compared to Chapter 1- is not a mere generalization to a different framework. Our desire is to motivate future researchers to this new tool that shows huge potentiality in the financial and economic applications. Convex/quasiconvex conditional maps (see also [27]) is only one of these numerous applications. It was our surprise and pleasure to discover how $L^0(\mathscr{G})$-modules naturally fitted to our purposes and simplified most of the proofs.
Anyway there is a drawback that still urges to be answered: is there a way to combine modules with a time continuous financial problem? Is there a notion of time consistency in agreement with modules?

3.1 A short review on L^0 modules

The probability space $(\Omega,\mathscr{F},\mathbb{P})$ is fixed throughout this chapter and $\mathscr{G}\subseteq\mathscr{F}$ is any sigma algebra contained in $\mathscr{F}$. We denote with $L^0(\Omega,\mathscr{F},\mathbb{P})=L^0(\mathscr{F})$ (resp. $L^0(\mathscr{G})$) the space of $\mathscr{F}$ (resp. $\mathscr{G}$) measurable random variables that are $\mathbb{P}$ a.s. finite, whereas by $\bar{L}^0(\mathscr{F})$ the space of extended random variables which may take values in $\mathbb{R}\cup\{\infty\}$; this differs from the previous chapters, but this choice is needed not to mess the things up with the notations linked to the presence of modules. In general since $(\Omega,\mathbb{P})$ are fixed we will always omit them. We define $L^0_+(\mathscr{F})=\{Y\in L^0(\mathscr{F})\mid Y\geq 0\}$ and $L^0_{++}(\mathscr{F})=\{Y\in L^0(\mathscr{F})\mid Y>0\}$. We remind that all equalities/inequalities among random variables are meant to hold $\mathbb{P}$-a.s.. Since in this chapter the expected value $E_{\mathbb{P}}[\cdot]$ of random variables is mostly computed w.r.t. the reference probability $\mathbb{P}$, we will often omit $\mathbb{P}$ in the notation.
Moreover the essential ($\mathbb{P}$ almost surely) *supremum ess* $\sup_\lambda(X_\lambda)$ of an arbitrary family of random variables $X_\lambda\in L^0(\Omega,\mathscr{F},\mathbb{P})$ will be simply denoted by $\sup_\lambda(X_\lambda)$, and similarly for the essential *infimum*. $\vee$ (resp. $\wedge$) denotes the essential ($\mathbb{P}$ almost surely) *maximum* (resp. the essential *minimum*) between two random variables, which are the usual lattice operations.

We choose the framework introduced by Filipovic et al. and just recall here some definitions. To help the reader in finding further details we use the same notations as in [28] and [53].

$L^0(\mathscr{G})$ equipped with the order of the almost sure dominance is a lattice ordered ring: define for every $\varepsilon \in L^0_{++}(\mathscr{G})$ the ball $B_\varepsilon = \{Y \in L^0\mathscr{G} \mid |Y| \leq \varepsilon\}$ centered in $0 \in L^0(\mathscr{G})$, which gives the neighborhood basis of 0. A set $V \subset L^0(\mathscr{G})$ is a neighborhood of $Y \in L^0(\mathscr{G})$ if there exists $\varepsilon \in L^0_{++}(\mathscr{G})$ such that $Y + B_\varepsilon \subset V$. A set V is open if it is a neighborhood of all $Y \in V$. $(L^0(\mathscr{G}), |\cdot|)$ stands for $L^0(\mathscr{G})$ endowed with this topology: in this case the space looses the property of being a topological vector space. It is easy to see that a net converges in this topology, namely $Y_N \xrightarrow{|\cdot|} Y$ if for every $\varepsilon \in L^0_{++}(\mathscr{G})$ there exists $\overline{N}$ such that $|Y - Y_N| < \varepsilon$ for every $N > \overline{N}$.

From now on we suppose that $E \subseteq L^0(\mathscr{F})$.

Definition 3.1. A **topological** $L^0(\mathscr{G})$**-module** (E, τ) is an algebraic module E on the ring $L^0(\mathscr{G})$, endowed with a topology τ such that the operations

(i) $(E, \tau) \times (E, \tau) \to (E, \tau)$, $(X_1, X_2) \mapsto X_1 + X_2$,

(ii) $(L^0(\mathscr{G}), |\cdot|) \times (E, \tau) \to (E, \tau)$, $(\Gamma, X_2) \mapsto \Gamma X_2$

are continuous w.r.t. the corresponding product topology.
A set $\mathscr{C}$ is said to be $L^0_{\mathscr{G}}$-convex if for every $X_1, X_2 \in \mathscr{C}$ and $\Lambda \in L^(\mathscr{G})$, $0 \leq \Lambda \leq 1$, we have $\Lambda X_1 + (1 - \Lambda) X_2 \in \mathscr{C}$.
A topology τ on E is locally $L^0(\mathscr{G})$-convex if (E, τ) is a topological $L^0(\mathscr{G})$-module and there is a neighborhood base $\mathscr{U}$ of $0 \in E$ for which each $U \in \mathscr{U}$ is $L^0(\mathscr{G})$-convex, $L^0(\mathscr{G})$-absorbent and $L^0(\mathscr{G})$-balanced. In this case (E, τ) is a **locally** $L^0(\mathscr{G})$**-convex module**.

Definition 3.2. A function $\|\cdot\| : E \to L^0_+(\mathscr{G})$ is a $L^0(\mathscr{G})$-seminorm on E if

(i) $\|\Gamma X\| = |\Gamma| \|X\|$ for all $\Gamma \in L^0(\mathscr{G})$ and $X \in E$,

(ii) $\|X_1 + X_2\| \leq \|X_1\| + \|X_2\|$ for all $X_1, X_2 \in E$.

$\|\cdot\|$ becomes a $L^0(\mathscr{G})$-norm if in addition

(iii) $\|X\| = 0$ implies $X = 0$.

Any family $\mathscr{Z}$ of $L^0(\mathscr{G})$-seminorms on E induces a topology in the following way. For any finite $\mathscr{S} \subset \mathscr{Z}$ and $\varepsilon \in L^0_{++}(\mathscr{G})$ we define

$$U_{\mathscr{S},\varepsilon} := \Big\{X \in E \mid \sup_{\|\cdot\| \in \mathscr{S}} \|X\| \leq \varepsilon\Big\}$$
$$\mathscr{U} := \{U_{\mathscr{S},\varepsilon} \mid \mathscr{S} \subset \mathscr{Z} \text{ finite and } \varepsilon \in L^0_{++}(\mathscr{G})\}.$$

$\mathscr{U}$ gives the neighborhood base of 0 and then we induce a topology as for $L^0(\mathscr{G})$ obtaining a locally $L^0(\mathscr{G})$-convex module. In fact Filipovic et al. proved (Theorem 2.4 [28]) that a topological $L^0(\mathscr{G})$-convex module (E, τ) is locally $L^0(\mathscr{G})$-convex if and only if τ is induced by a family of $L^0(\mathscr{G})$-seminorms. When $\|\cdot\|$ is a norm we will always endow E with the topology induced by $\|\cdot\|$.

Definition 3.3 (Definition 2.7 [28]). A topological $L^0(\mathscr{G})$-module has the countable concatenation property if for every countable collection $\{U_n\}_n$ of neighborhoods of $0 \in E$ and for every countable partition $\{A_n\}_n \subseteq \mathscr{G}$ the set $\sum_n \mathbf{1}_{A_n} U_n$ is again a neighborhood of $0 \in E$.

This property is satisfied by $L^0(\mathscr{G})$-normed modules.

From now on we suppose that (E, τ) is a locally $L^0(\mathscr{G})$-convex module and we denote by $\mathscr{L}(E, L^0(\mathscr{G}))$ the $L^0(\mathscr{G})$-module of continuous $L^0(\mathscr{G})$-linear maps.
Recall that $\mu : E \to L^0(\mathscr{G})$ is $L^0(\mathscr{G})$-linear if

$$\mu(\alpha X_1 + \beta X_2) = \alpha \mu(X_1) + \beta \mu(X_2) \quad \forall \alpha, \beta \in L^0(\mathscr{G}) \text{ and } X_1, X_2 \in E.$$

In particular this implies $\mu(X_1 \mathbf{1}_A + X_2 \mathbf{1}_{A^C}) = \mu(X_1)\mathbf{1}_A + \mu(X_2)\mathbf{1}_{A^C}$ which corresponds to the property (REG) in Chapter 1. On the other hand $\mu : E \to L^0(\mathscr{G})$ is continuous if the counterimage of any open set (in the topology of almost sure dominance provided on $L^0(\mathscr{G})$) is an open set in τ.

Definition 3.4. A set $\mathscr{C}$ is said to be evenly $L^0(\mathscr{G})$-convex if for every $X \in E$ such that $\mathbf{1}_B\{X\} \cap \mathbf{1}_B \mathscr{C} = \emptyset$ for every $B \in \mathscr{G}$ with $\mathbb{P}(B) > 0$, there exists a $L^0(\mathscr{G})$-linear continuous functional $\mu : E \to L^0(\mathscr{G})$ such that

$$\mu(X) > \mu(\xi) \quad \forall \xi \in \mathscr{C}$$

Example 3.1. We now give an important class of $L^0(\mathscr{G})$-normed modules which plays a key role in the financial applications and is studied in detail in [53] Section 4.2.
The classical conditional expectation can be generalized to $E[\cdot|\mathscr{G}] : L^0_+(\mathscr{F}) \to \bar{L}^0_+(\mathscr{G})$ by

$$E[X|\mathscr{G}] =: \lim_{n \to +\infty} E[X \wedge n|\mathscr{G}]. \tag{3.1}$$

The basic properties of conditional expectation still hold true: for every $X, X_1, X_2 \in L^0_+(\mathscr{F})$ and $Y \in L^0(\mathscr{G})$

- $YE[X|\mathscr{G}] = E[YX|\mathscr{G}]$;
- $E[X_1 + X_2|\mathscr{G}] = E[X_1|\mathscr{G}] + E[X_2|\mathscr{G}]$;
- $E[X] = E[E[X|\mathscr{G}]]$.

For every $p \geq 1$ we introduce the algebraic L^0-module defined as

$$L^p_{\mathscr{G}}(\mathscr{F}) =: \{X \in L^0(\Omega, \mathscr{F}, \mathbb{P}) \mid \|X|\mathscr{G}\|_p \in L^0(\Omega, \mathscr{G}, \mathbb{P})\} \tag{3.2}$$

where $\|\cdot|\mathscr{G}\|_p$ is a $L^0(\mathscr{G})$-norm given by

$$\|X|\mathscr{G}\|_p =: \begin{cases} E[|X|^p|\mathscr{G}]^{\frac{1}{p}} & \text{if } p < +\infty \\ \inf\{Y \in \bar{L}^0(\mathscr{G}) \mid Y \geq |X|\} & \text{if } p = +\infty \end{cases} \tag{3.3}$$

We denote by τ_p the L^0-module topology induced by (3.3). We remind that $L^p_{\mathscr{G}}(\mathscr{F})$ has the product structure i.e.

$$L^p_{\mathscr{G}}(\mathscr{F}) = L^0(\mathscr{G})L^p(\mathscr{F}) = \{YX \mid Y \in L^0(\mathscr{G}),\ X \in L^p(\mathscr{F})\}$$

This last property allows the conditional expectation to be well defined for every $\tilde{X} \in L^p_{\mathscr{G}}(\mathscr{F})$: since $\tilde{X} = YX$ with $Y \in L^0(\mathscr{G})$ and $X \in L^p(\mathscr{F})$ then $E[\tilde{X}|\mathscr{G}] = YE[X|\mathscr{G}]$ is a finite valued random variable.
For $p \in [1,+\infty)$, any $L^0(\mathscr{G})$-linear continuous functional $\mu : L^p_{\mathscr{G}}(\mathscr{F}) \to L^0(\mathscr{G})$ can be identified with a random variable $Z \in L^q_{\mathscr{G}}(\mathscr{F})$ as $\mu(\cdot) = E[Z\cdot|\mathscr{G}]$ where $\frac{1}{p}+\frac{1}{q}=1$.

3.2 Quasiconvex duality on general L^0 modules

Definition 3.5. A map $\pi : E \to \bar{L}^0(\mathscr{G})$ is said to be

(MON) monotone: for every $X,Y \in E$, $X \le Y$ we have $\pi(X) \le \pi(Y)$;
(QCO) quasiconvex: for every $X,Y \in E$, $\Lambda \in L^0(\mathscr{G})$ and $0 \le \Lambda \le 1$

$$\pi(\Lambda X + (1-\Lambda)Y) \le \pi(X) \vee \pi(Y),$$

(or equivalently if the lower level sets $\{\xi \in L^p_{\mathscr{G}}(\mathscr{F}) | \pi(\xi) \le \eta\}$ are $L^0_{\mathscr{G}}$-convex for every $\eta \in L^0_{\mathscr{G}}$.)
(REG) regular if for every $X,Y \in E$ and $A \in \mathscr{G}$,

$$\pi(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = \pi(X)\mathbf{1}_A + \pi(Y)\mathbf{1}_{A^C};$$

(EVQ) evenly quasiconvex if the lower level sets $\{\xi \in E | \pi(\xi) \le \eta\}$ are evenly $L^0_{\mathscr{G}}$-convex for every $\eta \in L^0_{\mathscr{G}}$.
Finally the following optional assumptions will be important in the dual result
(PRO) there is at least a couple $X_1, X_2 \in E$ such that $\pi(X_1) < \pi(X_2) < +\infty$.
(TEC) if for some $Y \in L^0(\mathscr{G})$ $\{\xi \in L^p_{\mathscr{G}}(\mathscr{F}) \mid \pi(\xi) < Y\} = \emptyset$ then $\pi(\xi) \ge Y$ for every $\xi \in L^p_{\mathscr{G}}(\mathscr{F})$.

Remark 3.1. Remarks on the assumptions.

- Notice that surely an evenly $L^0(\mathscr{G})$-convex set is also $L^0(\mathscr{G})$-convex and then (EVQ) implies (QCO).
- (PRO) assure that the map π is in some sense a proper map. In fact we want to avoid that the map π is constant on some set $A \in \mathscr{G}$ i.e. $\pi(\xi_1)\mathbf{1}_A = \pi(\xi_2)\mathbf{1}_A$ for every $\xi_1, \xi_2 \in E$. If this is the case, it appears reasonable to split the measure space Ω in the two parts A, A^C and threat them separately, since on A the representation turns out to be trivial. This is anyway a pretty weak assumption.
- (TEC) is obviously satisfied if $\{\xi \in E \mid \pi(\xi) < Y\} \neq \emptyset$ for every $Y \in L^0(\mathscr{G})$, and in general by maps like $f(E[u(\cdot)|\mathscr{G}])$ where f, u are real function.
- As shown in Chapter 1 the dual representation is linked to the continuity properties of the map: it can be shown (see for instance Proof of Corollary 3.1 and 3.2) that (EVQ) is implied by (QCO) together with either

(LSC) lower semicontinuity i.e. the lower level sets $\{\xi \in E \mid \pi(\xi) \leq Y\}$ are closed for every $Y \in L^0(\mathscr{G})$)

or

(USC)* strong upper semicontinuity i.e. the strict lower level sets $\{\xi \in E \mid \pi(\xi) < Y\}$ are open for every $Y \in L^0(\mathscr{G})$.

This is basically consequence of Hahn Banach Separation Theorems for modules (see [28] Theorems 2.7/2.8).

3.2.1 Statements of the main results

This first Theorem matches the representation obtained by Maccheroni et al. in [10] for general topological spaces. Respect to the first chapter, the interesting feature here, is that in the module framework we are able to have a dual representation for evenly quasiconvex maps: as shown in the corollaries above this is a weaker condition that (QCO) plus (LSC) (resp. (USC)*) and is an important starting point to obtain a complete quasiconvex duality as in [10]. From now on we suppose that $F \subseteq \overline{L}^0(\mathscr{G})$ is a lattice of extended random variable, which represents the codomain of the map π.

Theorem 3.1. *Let E be a locally $L^0(\mathscr{G})$-convex module. If $\pi : E \to F$ is (REG), (EVQ) and (TEC) then*

$$\pi(X) = \sup_{\mu \in \mathscr{L}(E, L^0(\mathscr{G}))} R(\mu(X), \mu), \tag{3.4}$$

where

$$R(Y, \mu) := \inf_{\xi \in E} \{\pi(\xi) \mid \mu(\xi) \geq Y\}$$

If in addition E satisfies the countable concatenation property then (TEC) can be replaced by (PRO).

Corollary 3.1. *Let E be a locally $L^0(\mathscr{G})$-convex module satisfying the countable concatenation property. If $\pi : E \to F$ is (QCO), (REG), (TEC) and τ-(LSC) then*

$$\pi(X) = \sup_{\mu \in \mathscr{L}(E, L^0(\mathscr{G}))} R(\mu(X), \mu). \tag{3.5}$$

In alternative, since the concatenation property holds true (TEC) can be switched into (PRO).

Corollary 3.2. *Let E be a locally $L^0(\mathscr{G})$-convex module. If $\pi : E \to F$ is (QCO), (REG), (TEC) and τ-(USC)* then*

$$\pi(X) = \max_{\mu \in \mathscr{L}(E, L^0(\mathscr{G}))} R(\mu(X), \mu). \tag{3.6}$$

If in addition E satisfies the countable concatenation property then (TEC) can be replaced by (PRO).

In Theorem 3.1, π can be represented as a supremum but not as a maximum. The following Corollary shows that nevertheless we can find a $R(\mu(X),\mu)$ arbitrary close to $\pi(X)$.

Corollary 3.3. *Under the same assumption of Theorem 3.1 or Corollary 3.1, for every $\varepsilon > 0$ there exists $\mu_\varepsilon \in \mathscr{L}(E, L^0(\mathscr{G}))$ such that*

$$\pi(X) - R(\mu_\varepsilon(X), \mu_\varepsilon) < \varepsilon \text{ on the set } \{\pi(X) < +\infty\} \tag{3.7}$$

3.2.2 General properties of $R(Y,\mu)$

In this section $\pi : E \to F \subseteq \bar{L}^0(\mathscr{G})$ always satisfies (REG). Following the path traced in the first Chapter, we state and adapt the proofs to the module framework, of the foremost properties holding for the function $R(Y,\mu)$. Notice that R is not defined on the whole product space $L^0(\mathscr{G}) \times \mathscr{L}(E, L^0(\mathscr{G}))$ but its actual domain is given by

$$\Sigma = \{(Y,\mu) \in L^0_{\mathscr{G}} \times \mathscr{L}(E, L^0(\mathscr{G})) | \exists \xi \in E \text{ s.t. } \mu(\xi) \geq Y\}. \tag{3.8}$$

Lemma 3.1. *Let $\mu \in \mathscr{L}(E, L^0(\mathscr{G}))$ and $X \in E$.*
i) $R(\cdot,\mu)$ is monotone non decreasing.
ii) $R(\Lambda\mu(X), \Lambda\mu) = R(\mu(X),\mu)$ for every $\Lambda \in L^0(\mathscr{G})$.
iii) For every $Y \in L^0(\mathscr{G})$ and $\mu \in \mathscr{L}(E, L^0(\mathscr{G}))$, the set

$$\mathscr{A}_\mu(Y) \mathrel{\hat=} \{\pi(\xi) \,|\, \xi \in E,\ \mu(\xi) \geq Y\}$$

is downward directed in the sense that for every $\pi(X_1), \pi(X_2) \in \mathscr{A}_\mu(Y)$ there exists $\pi(X^*) \in \mathscr{A}_\mu(Y)$ such that $\pi(X^*) \leq \min\{\pi(X_1), \pi(X_2)\}$. Thus there exists a sequence $\{\xi^\mu_m\}_{m=1}^\infty \in E$ such that

$$\mu(\xi^\mu_m) \geq Y \quad \forall m \geq 1, \quad \pi(\xi^\mu_m) \downarrow R(Y,\mu) \quad \text{as } m \uparrow \infty.$$

In particular if for $\alpha \in L^0(\mathscr{G})$, $R(Y,\mu) < \alpha$ then there exists ξ such that $\mu(\xi) \geq Y$ and $\pi(\xi) < \alpha$.
iv) For every $A \in \mathscr{G}$, $(Y,\mu) \in \Sigma$

$$R(Y,\mu)\mathbf{1}_A = \inf_{\xi \in E} \{\pi(\xi)\mathbf{1}_A \mid Y\mathbf{1}_A \geq \mu(X\mathbf{1}_A)\} = R(Y\mathbf{1}_A, \mu)\mathbf{1}_A \tag{3.9}$$

v) For every $X_1, X_2 \in E$
(a) $R(\mu(X_1),\mu) \wedge R(\mu(X_2),\mu) = R(\mu(X_1) \wedge \mu(X_2),\mu)$
(b) $R(\mu(X_1),\mu) \vee R(\mu(X_2),\mu) = R(\mu(X_1) \vee \mu(X_2),\mu)$
vi) The map $R(\mu(X),\mu)$ is quasi-affine with respect to X in the sense that for every $X_1, X_2 \in E$, $\Lambda \in L^0(\mathscr{G})$ and $0 \leq \Lambda \leq 1$, we have

$R(\mu(\Lambda X_1 + (1-\Lambda)X_2), \mu) \geq R(\mu(X_1), \mu) \wedge R(\mu(X_2), \mu)$ *(quasiconcavity)*
$R(\mu(\Lambda X_1 + (1-\Lambda)X_2), \mu) \leq R(\mu(X_1), \mu) \vee R(\mu(X_2), \mu)$ *(quasiconvexity).*

vii) $\inf_{Y \in L^0(\mathscr{G})} R(Y, \mu_1) = \inf_{Y \in L^0(\mathscr{G})} R(Y, \mu_2)$ *for every* $\mu_1, \mu_2 \in \mathscr{L}(E, L^0(\mathscr{G}))$.

Proof. i) and ii) follow trivially from the definition. Most of the leftover items are proved in similar way than the properties in Lemma 1.3. We report here all of them for sake of completeness.

iii) Consider the $\mathscr{G}$-measurable set $G = \{\pi(X_1) \leq \pi(X_2)\}$ then

$$\min\{\pi(X_1), \pi(X_2)\} = \pi(X_1)\mathbf{1}_G + \pi(X_2)\mathbf{1}_{G^C} \stackrel{REG}{=} \pi(X_1\mathbf{1}_G + X_2\mathbf{1}_{G^C})$$

Since $\mu(X_1\mathbf{1}_G + X_2\mathbf{1}_G^C) = \mu(X_1)\mathbf{1}_G + \mu(X_2)\mathbf{1}_{G^C} \geq Y$ then $\pi(X_1\mathbf{1}_G + X_2\mathbf{1}_G^C) \in \mathscr{A}_\mu(Y)$. The existence of the sequence $\{\xi_m^\mu\}_{m=1}^\infty \in E$ such that $\pi(\xi_m^\mu) \downarrow R(Y, \mu)$ for $\mu(\xi_m^\mu) \geq Y$ is a well known consequence for downward directed sets. Now let $R(Y, \mu) < \alpha$: consider the sets $F_m = \{\pi(\xi_m^\mu) < \alpha\}$ and the partition of Ω given by $G_1 = F_1$ and $G_m = F_m \setminus G_{m-1}$. We have from the properties of the module E and (REG) that

$$\xi = \sum_{m=1}^{\infty} \xi_m^\mu \mathbf{1}_{G_m} \in E, \quad \mu(\xi) \geq Y \text{ and } \pi(\xi) < \alpha$$

iv) To prove the first equality in (1.12): for every $\xi \in E$ such that $\mu(\xi\mathbf{1}_A) \geq Y\mathbf{1}_A$ we define the random variable $\eta = \xi\mathbf{1}_A + \zeta\mathbf{1}_{A^C}$ with $\mu(\zeta\mathbf{1}_{A^C}) \geq Y\mathbf{1}_{A^C}$, which satisfies $\mu(\eta) \geq Y$. Therefore

$$\{\eta\mathbf{1}_A \mid \eta \in E,\, \mu(\eta) \geq Y\} = \{\xi\mathbf{1}_A \mid \xi \in E,\, \mu(\xi\mathbf{1}_A) \geq Y\mathbf{1}_A\}$$

Hence from from the properties of the *essinf* and (REG):

$$\begin{aligned}
\mathbf{1}_A R(Y, \mu) &= \inf_{\eta \in E} \{\pi(\eta\mathbf{1}_A)\mathbf{1}_A \mid \mu(\eta) \geq Y\} \\
&= \inf_{\xi \in E} \{\pi(\xi\mathbf{1}_A)\mathbf{1}_A \mid \mu(\xi\mathbf{1}_A) \geq Y\mathbf{1}_A\} \\
&= \inf_{\xi \in E} \{\pi(\xi)\mathbf{1}_A \mid \mu(\xi\mathbf{1}_A) \geq Y\mathbf{1}_A\}
\end{aligned}$$

and (1.12) follows. Similarly for the second equality.

v) a): Since $R(\cdot, \mu)$ is monotone, the inequalities $R(\mu(X_1), \mu) \wedge R(\mu(X_2), \mu) \geq R(\mu(X_1) \wedge \mu(X_2), \mu)$ and $R(\mu(X_1), \mu) \vee R(\mu(X_2), \mu) \leq R(\mu(X_1) \vee \mu(X_2), \mu)$ are always true.

To show the opposite inequalities, define the $\mathscr{G}$-measurable sets: $B := \{R(\mu(X_1), \mu) \leq R(\mu(X_2), \mu)\}$ and $A := \{\mu(X_1) \leq \mu(X_2)\}$ so that

$$\begin{aligned}
R(\mu(X_1), \mu) \wedge R(\mu(X_2), \mu) &= R(\mu(X_1), \mu)\mathbf{1}_B + R(\mu(X_2), \mu)\mathbf{1}_{B^C} \\
&\leq R(\mu(X)_1, \mu)\mathbf{1}_A + R(\mu(X_2), \mu)\mathbf{1}_{A^C} \qquad (3.10) \\
R(\mu(X_1), \mu) \vee R(\mu(X_2), \mu) &= R(\mu(X)_1, \mu)\mathbf{1}_{B^C} + R(\mu(X_2), \mu)\mathbf{1}_B \\
&\geq R(\mu(X)_1, \mu)\mathbf{1}_{A^C} + R(\mu(X_2), \mu)\mathbf{1}_A
\end{aligned}$$

Set: $D(A,X)=\{\xi\mathbf{1}_A \mid \xi\in E,\ \mu(\xi\mathbf{1}_A)\geq\mu(X\mathbf{1}_A)\}$ and check that

$$D(A,X_1)+D(A^C,X_2)=\{\xi\in E\mid \mu(\xi)\geq\mu(X_1\mathbf{1}_A+X_2\mathbf{1}_{A^C})\}:=D$$

From (3.10) and using (1.12) we get:

$$\begin{aligned}
R(\mu(X_1),\mu)\wedge R(\mu(X_2),\mu) \leq\ & R(\mu(X_1),\mu)\mathbf{1}_A+R(\mu(X_2),\mu)\mathbf{1}_{A^C}\\
=\ & \inf_{\xi\mathbf{1}_A\in D(A,X_1)}\{\pi(\xi\mathbf{1}_A)\}+\inf_{\eta\mathbf{1}_{A^C}\in D(A^C,X_2)}\{\pi(\eta\mathbf{1}_{A^C})\}\\
=\ & \inf_{\substack{\xi\mathbf{1}_A\in D(A,X_1)\\ \eta\mathbf{1}_{A^C}\in D(A^C,X_2)}}\{\pi(\xi\mathbf{1}_A)+\pi(\eta\mathbf{1}_{A^C})\}\\
=\ & \inf_{(\xi\mathbf{1}_A+\eta\mathbf{1}_{A^C})\in D(A,X_1)+D(A^C,X_2)}\{\pi(\xi\mathbf{1}_A+\eta\mathbf{1}_{A^C})\}\\
=\ & \inf_{\xi\in D}\{\pi(\xi)\}=R(\mu(X_1)\mathbf{1}_A+\mu(X_2)\mathbf{1}_{A^C},\mu)\\
=\ & R(\mu(X_1)\wedge\mu(X_2),\mu).
\end{aligned}$$

Simile modo: v) $b)$.
vi) From the monotonicity of $R(\cdot,\mu)$, $R(\mu(X_1)\wedge\mu(X_2),\mu)\leq R(\mu(\Lambda X_1+(1-\Lambda)X_2),\mu)$ (resp. $R(\mu(X_1)\vee\mu(X_2),\mu)\geq R(\mu(\Lambda X_1+(1-\Lambda)X_2),\mu)$) and then the thesis follows from iv).
(vii) Notice that

$$R(Y,\mu)\geq\inf_{\xi\in E}\pi(\xi)\quad\forall Y\in L^0_{\mathscr{F}}$$

implies

$$\inf_{Y\in L^0(\mathscr{G})}R(Y,\mu)\geq\inf_{\xi\in E}\pi(\xi).$$

On the other hand

$$\pi(\xi)\geq R(\mu(\xi),\mu)\geq\inf_{Y\in L^0(\mathscr{G})}R(Y,\mu)\quad\forall\xi\in E$$

implies

$$\inf_{Y\in L^0(\mathscr{G})}R(Y,\mu)\leq\inf_{\xi\in E}\pi(\xi).$$

3.2.3 Bridging the gap between convex and non convex maps

In this short section we would like to analyze how the Fenchel conjugate is related to the function R in the quasiconvex representation. The above simple result can be used in order to obtain a risk/economic interpretation of the role acted by R (see later Remark 3.3).

Consider $\pi:E\to F$ and $\mu\in E_+^\circ$ where

$$E_+^\circ =: \{\mu \in \mathscr{L}(E, L^0(\mathscr{G})) \mid \mu(X) \geq 0, \text{ for every } X \geq 0\}.$$

We define for $X \in E$ and $\mu \in E_+^\circ$

$$\begin{aligned} r(X,\mu) &:= \inf_{\xi \in E}\{\pi(\xi) \mid \mu(\xi) = \mu(X)\} \\ r^\star(\mu) &:= \sup_{\xi \in E}\{\mu(\xi) - r(\xi,\mu)\} \\ R^\star(\mu) &:= \sup_{\xi \in E}\{\mu(\xi) - R(\mu(\xi),\mu)\} \\ \pi^*(\mu) &:= \sup_{\xi \in E}\{\mu(\xi) - \pi(\xi)\} \end{aligned}$$

Proposition 3.1. *For an arbitrary π we have the following properties*

1. $r(X,\mu) \geq R(\mu(X),\mu) \geq \mu(X) - \pi^*(\mu)$*;*
2. $r^\star(\mu) = R^\star(Z) = \pi^*(\mu)$*.*

Proof. 1. For all $\xi \in E$ we have $\pi^*(\mu) = \sup_{\xi \in E}\{\mu(\xi) - \pi(\xi)\} \geq \mu(\xi) - \pi(\xi)$. Hence: $\mu(X) - \pi^*(\mu) \leq \mu(X) - \mu(\xi) + \pi(\xi) \leq \pi(\xi)$ for all $\xi \in E$ s.t. $\mu(\xi) \geq \mu(X)$. Therefore

$$\mu(X) - \pi^*(\mu) \leq \inf_{\xi \in E}\{\pi(\xi) \mid \mu(\xi) \geq \mu(X)\} = R(\mu(X),\mu) \leq r(X,\mu)$$

2. From 1. we have $\mu(\xi) - R(\mu(\xi),\mu) \leq \pi^*(\mu)$ and

$$r^\star(\mu) = \sup_{\xi \in E}\{\mu(\xi) - r(\xi,\mu)\} \leq \sup_{\xi \in E}\{\mu(\xi) - R(\mu(\xi),\mu)\} = R^\star(\mu) \leq \pi^*(\mu) \tag{3.11}$$

since $r(\xi,\mu) \leq \pi(\xi)$ we have

$$\mu(\xi) - r(\xi,\mu) \geq \mu(\xi) - \pi(\xi) \quad \Rightarrow \quad r^\star(\mu) \geq \pi^*(\mu)$$

and together with equation (3.11) we deduce

$$r^\star(\mu) \geq \pi^*(\mu) \geq R^\star(\mu) \geq r^\star(\mu).$$

3.2.4 Proofs

Proof (Proof of Theorem 3.1). Fix $X \in E$ and denote $G = \{\pi(X) < +\infty\}$; for every $\varepsilon \in L^0_{++}(\mathscr{G})$ consider the evenly convex set

$$\mathscr{C}_\varepsilon =: \{\xi \in E \mid \pi(\xi) \leq (\pi(X) - \varepsilon)\mathbf{1}_G + \varepsilon\mathbf{1}_{G^C}\}.$$

Step 1. If $\mathscr{C}_\varepsilon = \emptyset$ then by assumption (TEC) we have $\pi(\xi) \geq (\pi(X) - \varepsilon)\mathbf{1}_G + \varepsilon\mathbf{1}_{G^C}$ for every $\xi \in E$. In particular it follows that $R(\mu(X),\mu) \geq (\pi(X) - \varepsilon)\mathbf{1}_G +$

$\varepsilon \mathbf{1}_{G^C}$ for every $\mu \in \mathscr{L}(E, L^0(\mathscr{G}))$ and thus

$$\pi(X) \geq \sup_{\mu \in \mathscr{L}(E, L^0(\mathscr{G}))} R(\mu(X), \mu) \geq (\pi(X) - \varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C} \tag{3.12}$$

Step 2. Now suppose that $\mathscr{C}_\varepsilon \neq \emptyset$. For every $B \in \mathscr{G}$, $\mathbb{P}(B) > 0$ we have $\mathbf{1}_B\{X\} \cap \mathbf{1}_B \mathscr{C}_\varepsilon = \emptyset$: in fact if $\xi \mathbf{1}_B = X \mathbf{1}_B$ then by (REG) we get $\pi(\xi)\mathbf{1}_B = \pi(\xi \mathbf{1}_B)\mathbf{1}_B = \pi(X\mathbf{1}_B)\mathbf{1}_B = \pi(X)\mathbf{1}_B$. Since $\mathscr{C}_\varepsilon$ is evenly L^0-convex then we can find $\mu_\varepsilon \in \mathscr{L}(E, L^0(\mathscr{G}))$ such that

$$\mu_\varepsilon(X) > \mu_\varepsilon(\xi) \quad \forall \xi \in \mathscr{C}_\varepsilon. \tag{3.13}$$

Let now $A \in \mathscr{G}$ be an arbitrary element such that $\mathbb{P}(A) > 0$ and define

$$\mathscr{C}_\varepsilon^A =: \{\xi \in E | \pi(\xi)\mathbf{1}_A \leq (\pi(X) - \varepsilon)\mathbf{1}_{A\cap G} + \varepsilon \mathbf{1}_{A \cap G^C}\}.$$

We want to show that $\mu_\varepsilon(X) > \mu_\varepsilon(\xi)$ on A for every $\xi \in \mathscr{C}_\varepsilon^A$. Let $\xi \in \mathscr{C}_\varepsilon^A$, $\eta \in \mathscr{C}_\varepsilon$ and define $\bar{\xi} = \xi \mathbf{1}_A + \eta \mathbf{1}_{A^C}$ which surely will belong to $\mathscr{C}_\varepsilon$. Hence $\mu_\varepsilon(X) > \mu_\varepsilon(\bar{\xi})$ so that $\mu_\varepsilon(X\mathbf{1}_A) = \mu_\varepsilon(X)\mathbf{1}_A \geq \mu_\varepsilon(\bar{\xi})\mathbf{1}_A = \mu_\varepsilon(\xi \mathbf{1}_A)$ and $\mu_\varepsilon(X) > \mu_\varepsilon(\xi)$ on A. We then deduce that $\mathscr{C}_\varepsilon^A \subseteq \mathscr{D}_\varepsilon^A =: \{\xi \in E | \mu_\varepsilon(X) > \mu_\varepsilon(\xi) \text{ on } A\}$ for every $A \in \mathscr{G}$ which means that

$$\bigcap_{A \in \mathscr{G}} \left(\mathscr{D}_\varepsilon^A\right)^C \subseteq \bigcap_{A \in \mathscr{G}} \left(\mathscr{C}_\varepsilon^A\right)^C$$

By definition

$$\left(\mathscr{C}_\varepsilon^A\right)^C = \{\xi \in E \mid \exists B \subseteq A, \, \mathbb{P}(B) > 0 \text{ and } [\star]\}$$

where

$$[\star] \longleftrightarrow \left[\begin{array}{ll} \pi(\xi)(\omega) > \pi(X)(\omega) - \varepsilon(\omega) & \text{for a.e. } \omega \in B \cap G \\ \text{or} & \\ \pi(\xi)(\omega) > \varepsilon(\omega) & \text{for a.e. } \omega \in B \cap G^C \end{array} \right.$$

so that

$$\begin{aligned} \bigcap_{A \in \mathscr{G}} \left(\mathscr{C}_\varepsilon^A\right)^C &= \{\xi \in E \mid \forall A \in \mathscr{G}, \, \exists B \subseteq A, \, \mathbb{P}(B) > 0 \text{ and } [\star]\} \\ &= \{\xi \in E \mid \pi(\xi) > (\pi(X) - \varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C}\}. \end{aligned}$$

Indeed if $\xi \in E$ such that $\pi(\xi) > (\pi(X) - \varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C}$ then $\xi \in \bigcap_{A \in \mathscr{G}} \left(\mathscr{C}_\varepsilon^A\right)^C$. Viceversa let $\xi \in \bigcap_{A \in \mathscr{G}} \left(\mathscr{C}_\varepsilon^A\right)^C$: suppose that there exists a $D \in \mathscr{G}$, $\mathbb{P}(D) > 0$ and $\pi(\xi) \leq (\pi(X) - \varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C}$ on D. By definition of $(\mathscr{C}_\varepsilon^D)^C$ we can find $B \subseteq D$ such that $\pi(\xi) > \pi(X) - \varepsilon$ on $G \cap D$ or $\pi(\xi) > +\varepsilon$ on $D \cap G^C$ and this is clearly a contradiction. Hence $\bigcap_{A \in \mathscr{G}} \left(\mathscr{C}_\varepsilon^A\right)^C = \{\xi \in E | \pi(\xi) > (\pi(X) - \varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C}\}$. Matching the previous argument we can prove that $\bigcap_{A \in \mathscr{G}} \left(\mathscr{D}_\varepsilon^A\right)^C = \{\xi \in E | \mu_\varepsilon(X) \leq \mu_\varepsilon(\xi)\}$. We finally deduce that

$$\pi(X) \geq \sup_{\mu \in \mathscr{L}(E,L^0(\mathscr{G}))} R(\mu(X),\mu) \geq R(\mu_\varepsilon(X),\mu_\varepsilon) = \inf_{\xi \in E}\{\pi(\xi) \mid \mu_\varepsilon(X) \leq \mu_\varepsilon(\xi)\}$$
$$\geq \inf_{\xi \in E}\{\pi(\xi) \mid \pi(\xi) > (\pi(X)-\varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C}\} \geq (\pi(X)-\varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C}.$$

By equation (3.12) and this last sequence of inequalities we can assure that for every $\varepsilon \in L^0_{++}(\mathscr{G})$ $\pi(X) \geq \sup_{\mu \in \mathscr{L}(E,L^0(\mathscr{G}))} R(\mu(X),\mu) \geq (\pi(X)-\varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C}$. The thesis follows taking ε arbitrary small on G and arbitrary big on G^C.

Step 3. Now we pass to that the second part of the Theorem and assume that E have the concatenation property. We follow the notations of the first part of the proof and introduce the $\mathscr{G}$ measurable random variable $Y_\varepsilon =: (\pi(X)-\varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C}$ and the set

$$\mathscr{A} = \{A \in \mathscr{G} \mid \exists \xi \in E \text{ s.t. } \pi(\xi) \leq Y_\varepsilon \text{ on } A\}$$

For every $A,B \in \mathscr{A}$ we have that $A \cup B$. Consider the set $\{\mathbf{1}_A | A \in \mathscr{A}\}$: the set is upward directed since $\mathbf{1}_{A_1} \vee \mathbf{1}_{A_2} = \mathbf{1}_{A_1 \cup A_2}$ for every $A_1, A_2 \in \mathscr{A}$. Hence we can find a sequence $\mathbf{1}_{A_n} \uparrow \sup\{\mathbf{1}_A | A \in \mathscr{A}\} = \mathbf{1}_{A^{max}}$ where $A^{max} = \cup_n A_n \in \mathscr{G}$.
By definition for every A_n we can find ξ_n such that $\pi(\xi_n) \leq Y_\varepsilon$ on A_n. Now redefine the sequence of set $B_n = A_n \setminus B_{n-1}$, so that $\eta = \sum_n \xi_n \mathbf{1}_{B_n}$ has the property that $\pi(\eta) \leq Y_\varepsilon$ on A^{max} i.e. $A^{max} \in \mathscr{A}$.

As a consequence of the definition of $\mathscr{A}$ and since A^{max} is the maximal element in $\mathscr{A}$ we deduce that $\pi(\xi) > Y_\varepsilon$ on $(A^{\max})^C$ for every $\xi \in E$.
In particular it follows that $R(\mu(X),\mu) \geq Y_\varepsilon$ on $(A^{max})^C$ for every $\mu \in \mathscr{L}(E,L^0(\mathscr{G}))$ and thus

$$\pi(X) \geq \sup_{\mu \in \mathscr{L}(E,L^0(\mathscr{G}))} R(\mu(X),\mu) \geq (\pi(X)-\varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C} \text{ on } (A^{max})^C \tag{3.14}$$

We know by (PRO) that there exists a $\zeta_1, \zeta_2 \in E$ such that $\pi(\zeta_1) < \pi(\zeta_2) \in L^0(\mathscr{G})$. Introduce the evenly convex set

$$\mathscr{C}^1_\varepsilon =: \{\xi \in E \mid \pi(\xi) \leq Y_\varepsilon \mathbf{1}_{A^{max}} + \pi(\zeta_1)\mathbf{1}_{(A^{max})^C}\} \neq \emptyset.$$

Surely $\tilde{X} = X\mathbf{1}_{A^{max}} + \zeta_2 \mathbf{1}_{(A^{max})^C}$ has the property that $\mathbf{1}_B\{\tilde{X}\} \cap \mathbf{1}_B \mathscr{C}^1_\varepsilon = \emptyset$ for every $B \in \mathscr{G}$ so that we can find $\mu_\varepsilon \in \mathscr{L}(E,L^0(\mathscr{G}))$ such that

$$\mu_\varepsilon(\tilde{X}) > \mu_\varepsilon(\xi) \quad \forall \xi \in \mathscr{C}^1_\varepsilon. \tag{3.15}$$

Repeating the argument of Step 2 we get

$$\pi(\tilde{X}) \geq \sup_{\mu \in \mathscr{L}(E,L^0(\mathscr{G}))} R(\mu(\tilde{X}),\mu) \geq R(\mu_\varepsilon(\tilde{X}),\mu_\varepsilon) = \inf_{\xi \in E}\{\pi(\xi) \mid \mu_\varepsilon(\tilde{X}) \leq \mu_\varepsilon(\xi)\}$$
$$\geq \inf_{\xi \in E}\{\pi(\xi) \mid \pi(\xi) > Y_\varepsilon \mathbf{1}_{A^{max}} + \pi(\zeta_1)\mathbf{1}_{(A^{max})^C}\} \geq Y_\varepsilon \mathbf{1}_{A^{max}} + \pi(\zeta_1)\mathbf{1}_{(A^{max})^C}.$$

Restricting to the set A^{max} we deduce

$$\pi(X\mathbf{1}_{A^{max}})\mathbf{1}_{A^{max}} \geq \sup_{\mu\in\mathscr{L}(E,L^0(\mathscr{G}))} R(\mu(X\mathbf{1}_{A^{max}}),\mu)\mathbf{1}_{A^{max}} \geq Y_\varepsilon \mathbf{1}_{A^{max}}.$$

This last inequality together with equation (3.14) gives by (REG)

$$\pi(X) \geq R(\mu_\varepsilon(X),\mu_\varepsilon) \geq (\pi(X)-\varepsilon)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^C} \tag{3.16}$$

and the thesis follows taking again ε arbitrary small on G and arbitrary big on G^C.

Proof (Proof of Corollary 3.1). Assuming (TEC). We only have to show that the set $\mathscr{C}_\varepsilon$ - which is now closed - defined in the previous proof can be separated as in (3.13). For every $B\in\mathscr{G}$, $\mathbb{P}(B)>0$ we have already shown that $\mathbf{1}_B\{X\}\cap\mathbf{1}_B\mathscr{C}_\varepsilon=\emptyset$. We thus can apply the generalized Hahn Banach Separation Theorem (see [28] Theorem 2.8) and find $\mu_\varepsilon\in\mathscr{L}(E,L^0(\mathscr{G}))$ and $\delta\in L^0_{++}(\mathscr{G})$ so that

$$\mu_\varepsilon(X) > \mu_\varepsilon(\xi)+\delta \quad \forall \xi\in\mathscr{C}_\varepsilon. \tag{3.17}$$

Similarly when we assume (PRO).

Proof (Proof of Corollary 3.2). In order to obtain the representation in terms of a *maximum* we prove the claim directly. Fix $X\in E$ and consider the open convex set $\mathscr{C} =: \{\xi\in E \mid \pi(\xi)<\pi(X)\}$.
If $\mathscr{C}=\emptyset$ then by assumption (TEC) we have $\pi(\xi)\geq\pi(X)$ for every $\xi\in E$. In particular it follows that $R(\mu(X),\mu)\geq\pi(X)$ for every $\mu\in\mathscr{L}(E,L^0(\mathscr{G}))$ and thus the thesis follows since

$$\pi(X) \geq \sup_{\mu\in\mathscr{L}(E,L^0(\mathscr{G}))} R(\mu(X),\mu) \geq \pi(X) \tag{3.18}$$

Now suppose $\mathscr{C}\neq\emptyset$: notice that $\mathbf{1}_B\{X\}\cap\mathbf{1}_B\mathscr{C}=\emptyset$. We thus can apply the generalized Hahn Banach Separation Theorem (see [28] Theorem 2.7) and find $\mu_{\max}\in\mathscr{L}(E,L^0(\mathscr{G}))$ so that

$$\mu_{\max}(X) > \mu_{\max}(\xi) \quad \forall \xi\in\mathscr{C}.$$

Let now $A\in\mathscr{G}$ be an arbitrary element such that $\mathbb{P}(A)>0$: repeat the argument of the previous proof considering

$$\mathscr{C}^A =: \{\xi\in E|\pi(\xi)<\pi(X) \text{ on } A\}.$$

$$\mathscr{D}^A =: \{\xi\in E|\mu_{\max}(X\mathbf{1}_A)>\mu_{\max}(\xi\mathbf{1}_A) \text{ on } A\}$$

and find that

$$\{\xi\in E|\mu_{\max}(X)\leq\mu_{\max}(\xi)\}\subseteq\{\xi\in E|\pi(\xi)\geq\pi(X)\}$$

Again the thesis follows from the inequalities

$$\begin{aligned}\pi(X) &\geq \sup_{\mu\in\mathscr{L}(E,L^0(\mathscr{G}))} R(\mu(X),\mu) \geq \inf_{\xi\in E}\{\pi(\xi) \mid \mu_{\max}(X)\leq \mu_{\max}(\xi)\} \\ &\geq \inf_{\xi\in E}\{\pi(\xi) \mid \pi(\xi)\geq \pi(X)\} \geq \pi(X)\end{aligned}$$

When we assume (PRO) instead of (TEC) we just have to repeat the argument in the proof of Theorem 3.1.

Proof (Proof of Corollary 3.3). Follows directly from the last three lines of Step 2 (or Step 3) in the proof of Theorem 3.1.

3.3 Application to Risk Measures

In Section 1.4 we briefly discussed the application of quasiconvex analysis to the theory of Risk Measures. Now we would like to better detail this powerful tool in the module environment. It's important to notice that at the actual *status* of the research on this subject, not all of the following results can be adapted to the vector space case. Hopefully this will be developed in the future.
First of all we specify the definition of risk measure.

Definition 3.6. A quasiconvex (conditional) risk measure is a map $\rho : L^p_{\mathscr{G}}(\mathscr{F}) \to \bar{L}^0(\mathscr{G})$ satisfying
(MON)$'$ monotonicity: for every $X,Y \in L^p_{\mathscr{G}}(\mathscr{F})$, $X \leq Y$ we have $\rho(X)\geq\rho(Y)$;
(QCO) quasiconvexity: for every $X,Y \in L^p_{\mathscr{G}}(\mathscr{F})$, $\Lambda \in L^0(\mathscr{G})$ and $0\leq\Lambda\leq 1$

$$\rho(\Lambda X+(1-\Lambda)Y)\leq\rho(X)\vee\rho(Y),$$

(REG) regular if for every $X,Y \in L^p_{\mathscr{G}}(\mathscr{F})$ and $A\in\mathscr{G}$,

$$\rho(X\mathbf{1}_A+Y\mathbf{1}_{A^C})=\rho(X)\mathbf{1}_A+\rho(Y)\mathbf{1}_{A^C};$$

Recall that the principle of diversification states that 'diversification should not increase the risk ', i.e. the diversified position $\Lambda X+(1-\Lambda)Y$ is less risky than both the positions X and Y. Under cash additivity axiom convexity and quasiconvexity are equivalent, so that they both give the right interpretation of this principle. As already mentioned with an example in Section 1.4 (and vividly discussed by El Karoui and Ravanelli [25]) the lack of liquidity of the zero coupon bonds is the primary reason of the failure of cash additivity. Thus it is unavoidable to relax the convexity axiom to quasiconvexity in order to regain the best modeling of diversification.

3.3.1 *A characterization* via *the risk acceptance family*

In this subsection we assume for sake of simplicity that $\rho(0) \in L^0(\mathscr{G})$: in this way we do not loose any generality imposing $\rho(0) = 0$ (if not just define $\tilde{\rho}(\cdot) = \rho(\cdot) - \rho(0)$). We remind that if $\rho(0) = 0$ then (REG) turns out to be $\rho(X\mathbf{1}_A) = \rho(X)\mathbf{1}_A$. Given a risk measure one can always define for every $Y \in L^0(\mathscr{G})$ the risk acceptance set of level Y as

$$\mathscr{A}_\rho^Y = \{X \in L^p_{\mathscr{G}}(\mathscr{F}) \mid \rho(X) \leq Y\}.$$

This set represents the collection of financial positions whose risk is smaller of the fixed level Y and are strictly related to the Acceptability Indices [12]. Given a risk measure we can associate a family of risk acceptance sets, namely $\{\mathscr{A}_\rho^Y | Y \in L^0(\mathscr{G})\}$ which are called Risk Acceptance Family of the risk measure ρ as suggested in [19]. In general

Definition 3.7. A family $\mathbb{A} = \{\mathscr{A}^Y | Y \in L^0(\mathscr{G})\}$ of subsets $\mathscr{A}^Y \subset L^p_{\mathscr{G}}(\mathscr{F})$ is called risk acceptance family if
(i) convex: $\mathscr{A}^Y$ is $L^0(\mathscr{G})$-convex for every $Y \in L^0(\mathscr{G})$;
(ii) monotone:

- $X_1 \in \mathscr{A}^Y$ and $X_2 \in L^p_{\mathscr{G}}(\mathscr{F})$, $X_2 \geq X_1$ implies $X_2 \in \mathscr{A}^Y$;
- for any $Y' \leq Y$ we have $\mathscr{A}^{Y'} \subseteq \mathscr{A}^Y$;

(iii) regular: $X \in \mathscr{A}^Y$ then for every $G \in \mathscr{G}$ we have

$$\inf\{Y\mathbf{1}_G \in L^0(\mathscr{G}) \mid X \in \mathscr{A}^Y\} = \inf\{Y \in L^0(\mathscr{G}) \mid X\mathbf{1}_G \in \mathscr{A}^Y\}$$

(iv) right continuous: $\mathscr{A}^Y = \bigcap_{Y' > Y} \mathscr{A}^{Y'}$ for every $Y \in L^0(\mathscr{G})$.

These four properties allows to induce a one to one relationship between quasiconvex risk measures and risk acceptance families as we prove in the following

Proposition 3.2. *For any quasiconvex risk measure* $\rho : L^p_{\mathscr{G}}(\mathscr{F}) \to \bar{L}^0(\mathscr{G})$ *the family*

$$\mathbb{A}_\rho = \{\mathscr{A}_\rho^Y | Y \in L^0(\mathscr{G})\}$$

with $\mathscr{A}_\rho^Y = \{X \in L^p_{\mathscr{G}}(\mathscr{F}) \mid \rho(X) \leq Y\}$ *is a risk acceptance family.*
Viceversa *for every risk acceptance family* $\mathbb{A}$ *the map*

$$\rho_{\mathbb{A}}(X) = \inf\{Y \in L^0(\mathscr{G}) \mid X \in \mathscr{A}^Y\}$$

is a well defined quasiconvex risk measure $\rho_{\mathbb{A}} : L^p_{\mathscr{G}}(\mathscr{F}) \to \bar{L}^0(\mathscr{G})$ *such that* $\rho_{\mathbb{A}}(0) = 0$.
Moreover $\rho_{\mathbb{A}_\rho} = \rho$ *and* $\mathbb{A}_{\rho_{\mathbb{A}}} = \mathbb{A}$.

Proof. (MON)$'$ and (QCO) of ρ imply that $\mathscr{A}_\rho^Y$ is convex and monotone. Also notice that

$$\inf\{Y \in L^0(\mathscr{G}) \mid X\mathbf{1}_G \in \mathscr{A}^Y_\rho\} = \inf\{Y \in L^0(\mathscr{G}) \mid \rho(X\mathbf{1}_G) \le Y\} = \rho(X\mathbf{1}_G)$$
$$= \rho(X)\mathbf{1}_G = \inf\{Y\mathbf{1}_G \in L^0(\mathscr{G}) \mid \rho(X) \le Y\} = \inf\{Y \in L^0(\mathscr{G}) \mid X\mathbf{1}_G \in \mathscr{A}^Y_\rho\},$$

i.e. $\mathscr{A}^Y_\rho$ is regular.
Obviously $\mathscr{A}^Y_\rho \subset \bigcap_{Y'>Y} \mathscr{A}^{Y'}$ for any $Y \in L^0(\mathscr{G})$. If $X \in \bigcap_{Y'>Y} \mathscr{A}^{Y'}$ then $\rho(X) \le Y'$ for every $Y' > Y$ and hence $\rho(X) \le Y$ i.e. $\mathscr{A}^Y_\rho \supset \bigcap_{Y'>Y} \mathscr{A}^{Y'}$.

Viceversa: we first prove that $\rho_{\mathbb{A}}$ is (REG). For every $G \in \mathscr{G}$

$$\rho_{\mathbb{A}}(X\mathbf{1}_G) = \inf\{Y \in L^0(\mathscr{G}) \mid X\mathbf{1}_G \in \mathscr{A}^Y\} \overset{(iii)}{=} \inf\{Y\mathbf{1}_G \in L^0(\mathscr{G}) \mid X \in \mathscr{A}^Y\} = \rho_{\mathbb{A}}(X)\mathbf{1}_G$$

Now consider $X_1, X_2 \in L^p_{\mathscr{G}}(\mathscr{F})$, $X_1 \le X_2$. Let $G^C = \{\rho_{\mathbb{A}}(X_1) = +\infty\}$ so that $\rho_{\mathbb{A}}(X_1\mathbf{1}_{G^C}) \ge \rho_{\mathbb{A}}(X_2\mathbf{1}_{G^C})$. Otherwise consider the collection of Ys such that $X_1\mathbf{1}_G \in \mathscr{A}^Y$. Since $\mathscr{A}^Y$ is monotone we have that $X_2\mathbf{1}_G \in \mathscr{A}^Y$ if $X_1\mathbf{1}_G \in \mathscr{A}^Y$ and this implies that

$$\begin{aligned}\rho_{\mathbb{A}}(X_1)\mathbf{1}_G &= \inf\{Y\mathbf{1}_G \in L^0(\mathscr{G}) \mid X_1 \in \mathscr{A}^Y\} = \inf\{Y \in L^0(\mathscr{G}) \mid X_1\mathbf{1}_G \in \mathscr{A}^Y\}\\ &\ge \inf\{Y \in L^0(\mathscr{G}) \mid X_2\mathbf{1}_G \in \mathscr{A}^Y\} = \inf\{Y\mathbf{1}_G \in L^0(\mathscr{G}) \mid X_2 \in \mathscr{A}^Y\} = \rho(X_2)\mathbf{1}_G,\end{aligned}$$

i.e. $\rho_{\mathbb{A}}(X_1\mathbf{1}_G) \ge \rho_{\mathbb{A}}(X_2\mathbf{1}_G)$. And this shows that $\rho_{\mathbb{A}}(\cdot)$ is (MON)$'$.
Let $X_1, X_2 \in L^p_{\mathscr{G}}(\mathscr{F})$ and take any $\Lambda \in L^0(\mathscr{G})$, $0 \le \Lambda \le 1$. Define the set $B =: \{\rho_{\mathbb{A}}(X_1) \le \rho_{\mathbb{A}}(X_2)\}$. If $X_1\mathbf{1}_{B^C} + X_2\mathbf{1}_B \in \mathscr{A}^{Y'}$ for some $Y' \in L^0(\mathscr{G})$ then for sure $Y' \ge \rho_{\mathbb{A}}(X_1) \vee \rho_{\mathbb{A}}(X_2) \ge \rho(X_i)$ for $i = 1, 2$. Hence also $\rho(X_i) \in \mathscr{A}^{Y'}$ for $i = 1, 2$ and by convexity we have that $\Lambda X_1 + (1-\Lambda)X_2 \in \mathscr{A}^{Y'}$. Then $\rho_{\mathbb{A}}(\Lambda X_1 + (1-\Lambda)X_2) \le \rho_{\mathbb{A}}(X_1) \vee \rho_{\mathbb{A}}(X_2)$.
If $X_1\mathbf{1}_{B^C} + X_2\mathbf{1}_B \notin \mathscr{A}^{Y'}$ for every $Y' \in L^0(\mathscr{G})$ then from property (iii) we deduce that $\rho_{\mathbb{A}}(X_1) = \rho_{\mathbb{A}}(X_2) = +\infty$ and the thesis is trivial.

Now consider $B = \{\rho(X) = +\infty\}$: $\rho_{\mathbb{A}_\rho}(X) = \rho(X)$ follows from

$$\begin{aligned}\rho_{\mathbb{A}_\rho}(X)\mathbf{1}_B &= \inf\{Y\mathbf{1}_B \in L^0(\mathscr{G}) \mid \rho(X) \le Y\} = +\infty\mathbf{1}_B\\ \rho_{\mathbb{A}_\rho}(X)\mathbf{1}_{B^C} &= \inf\{Y\mathbf{1}_{B^C} \in L^0(\mathscr{G}) \mid \rho(X) \le Y\}\\ &= \inf\{Y \in L^0(\mathscr{G}) \mid \rho(X)\mathbf{1}_{B^C} \le Y\} = \rho(X)\mathbf{1}_{B^C}\end{aligned}$$

For the second claim notice that if $X \in \mathscr{A}^Y$ then $\rho_{\mathbb{A}}(X) \le Y$ which means that $X \in \mathscr{A}^Y_{\rho_{\mathbb{A}}}$. Conversely if $X \in \mathscr{A}^Y_{\rho_{\mathbb{A}}}$ then $\rho_{\mathbb{A}}(X) \le Y$ and by monotonicity this implies that $X \in \mathscr{A}^{Y'}$ for every $Y' > Y$. From the right continuity we take the intersection and get that $X \in \mathscr{A}^Y$.

3.3.2 Complete duality

This last Section is devoted to one of the most interesting result of this thesis: a complete quasiconvex duality between the risk measure ρ and the dual map R. We restrict the discussion to the particular case of $L^0(\mathscr{G})$-modules of $L^p_{\mathscr{G}}(\mathscr{F})$ type for one main reason: actually it is the only class of modules for which there is a full knowledge of the dual module $\mathscr{L}(E, L^0(\mathscr{G}))$. When analytical results will be available on modules of the Orlicz type (see [53] for the exact definition) or others the following proof will be easily adapted.

We transpose the definitions of Section 3.2, with some little differences of signs.

$$R(Y,Z) := \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{\rho(\xi) \mid E[-\xi Z|\mathscr{G}] \geq Y\} \tag{3.19}$$

is well defined on the domain

$$\Sigma = \{(Y,Z) \in L^0_{\mathscr{G}} \times L^q_{\mathscr{G}}(\mathscr{F}) | \exists \xi \in L^p_{\mathscr{G}}(\mathscr{F}) \text{ s.t. } E[-Z\xi|\mathscr{G}] \geq Y\}.$$

Let also introduce the following notations:

$$\begin{aligned}\mathscr{P}^q &=: \{Z \in L^q_{\mathscr{G}}(\mathscr{F}) \mid Z \geq 0,\ E[Z|\mathscr{G}] = 1\} \\ &= \left\{\frac{dQ}{d\mathbb{P}} \in L^q_{\mathscr{G}}(\mathscr{F}) \mid Q \text{ probability},\ E\left[\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] = 1\right\}\end{aligned}$$

and the class $\mathscr{M}(L^0(\mathscr{G}) \times \mathscr{P}^q)$ composed by maps $K : L^0(\mathscr{G}) \times \mathscr{P}^q \to \bar{L}^0(\mathscr{G})$ s.t.

- K is increasing in the first component.
- $K(Y\mathbf{1}_A, Q)\mathbf{1}_A = K(Y,Q)\mathbf{1}_A$ for every $A \in \mathscr{G}$ and $(Y, \frac{dQ}{d\mathbb{P}}) \in \Sigma$.
- $\inf_{Y \in L^0(\mathscr{G})} K(Y,Q) = \inf_{Y \in L^0(\mathscr{G})} K(Y,Q')$ for every $Q, Q' \in \mathscr{P}^q$.
- S is $\diamond$-evenly $L^0(\mathscr{G})$-quasiconcave: for every $(\bar{Y}, \bar{Q}) \in L^0(\mathscr{G}) \times \mathscr{P}^q$, $A \in \mathscr{G}$ and $\alpha \in L^0(\mathscr{G})$ such that $K(\bar{Y}, \bar{Q}) < \alpha$ on A, there exists $(\bar{S}, \bar{X}) \in L^0_{++}(\mathscr{G}) \times L^p_{\mathscr{G}}(\mathscr{F})$ with
$$\bar{Y}\bar{S} + E\left[\bar{X}\frac{d\bar{Q}}{d\mathbb{P}}|\mathscr{G}\right] < Y\bar{S} + E\left[\bar{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \text{ on } A$$
for every (Y,Q) such that $K(Y,Q) \geq \alpha$ on A.
- the set $\mathscr{K}(X) = \left\{K(E[X\frac{dQ}{d\mathbb{P}}|\mathscr{G}], Q) \mid Q \in \mathscr{P}^q\right\}$ is upward directed for every $X \in L^p_{\mathscr{G}}(\mathscr{F})$.

We will write with a slight abuse of notation $R(Y,Q)$ instead of $R\left(Y, \frac{dQ}{d\mathbb{P}}\right)$. The class $\mathscr{M}(L^0(\mathscr{G}) \times \mathscr{P}^q)$ is non empty in general as we show in the following Lemma.

Lemma 3.2. *The function R defined in (3.19) belongs to $\mathscr{M}(L^0(\mathscr{G}) \times \mathscr{P}^q)$*

Proof. First: R monotone in the first component follows from 3.1 i).
Second: $R(Y\mathbf{1}_A, Q)\mathbf{1}_A = R(Y,Q)\mathbf{1}_A$ follows from 3.1 iv).

Third: observe that $R(Y,Q) \geq \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \rho(\xi)$ for all $(Y,Q) \in L^0(\mathscr{G}) \times \mathscr{P}^q$ so that

$$\inf_{Y \in L^0(\mathscr{G})} R(Y,Q) \geq \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \rho(\xi).$$

Conversely notice that the set $\{\rho(\xi) | \xi \in L^p_{\mathscr{G}}(\mathscr{F})\}$ is downward directed and then there exists $\rho(\xi_n) \downarrow \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \rho(\xi)$. For every $Q \in \mathscr{P}^q$ we have

$$\rho(\xi_n) \geq R\left(E\left[-\xi_n \frac{dQ}{d\mathbb{P}} |\mathscr{G}\right], Q\right) \geq \inf_{Y \in L^0(\mathscr{G})} R(Y,Q)$$

so that

$$\inf_{Y \in L^0(\mathscr{G})} R(Y,Q) \leq \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \rho(\xi).$$

Fourth: for $\alpha \in L^0(\mathscr{G})$ and $A \in \mathscr{G}$ define $U^A_\alpha = \{(Y,Q) \in L^0(\mathscr{G}) \times \mathscr{P}^q | R(Y,Q) \geq \alpha \text{ on } A\}$, and suppose $\emptyset \neq U^A_\alpha \neq L^0(\mathscr{G}) \times \mathscr{P}^q$. Let $(\overline{Y}, \overline{Q}) \in L^0(\mathscr{G}) \times \mathscr{P}^q$ such that $R(\overline{Y}, \overline{Q}) < \alpha$ on A. From Lemma 3.1 (iii) there exists $\overline{X} \in L^p_{\mathscr{G}}(\mathscr{F})$ such that $E[-\overline{X} \frac{d\overline{Q}}{d\mathbb{P}} |\mathscr{G}] \geq \overline{Y}$ and $\rho(\overline{X}) < \alpha$ on A. Since $R(Y,Q) \geq \alpha$ on A for every $(Y,Q) \in U^A_\alpha$ then $E[-\overline{X} \frac{dQ}{d\mathbb{P}} |\mathscr{G}] < Y$ for every $(Y,Q) \in U_\alpha$ on A: otherwise we could define $B = \{\omega \in A \mid E[-\overline{X} \frac{dQ}{d\mathbb{P}} |\mathscr{G}] \geq Y\}$, $\mathbb{P}(B) > 0$ and then from Lemma 3.1 (iv) it must be that $R(Y\mathbf{1}_B, Q) < \alpha$ on the set B. Finally we can conclude that for every $(Y,Q) \in U^A_\alpha$

$$\overline{Y} + E\left[\overline{X} \frac{d\overline{Q}}{d\mathbb{P}} |\mathscr{G}\right] \leq 0 < Y + E\left[\overline{X} \frac{dQ}{d\mathbb{P}} |\mathscr{G}\right] \text{ on } A.$$

Fifth: $\mathscr{K} = \left\{R(E[X \frac{dQ}{d\mathbb{P}} |\mathscr{G}], Q) \mid Q \in \mathscr{P}^q\right\}$ is upward directed. Take $Q_1, Q_2 \in \mathscr{P}^q$ and define $F = \{R(E[X \frac{dQ_1}{d\mathbb{P}} |\mathscr{G}], Q_1) \geq R(E[X \frac{dQ_2}{d\mathbb{P}} |\mathscr{G}], Q_2)\}$ and let $\widehat{Q}$ given by

$$\frac{d\widehat{Q}}{d\mathbb{P}} := \mathbf{1}_F \frac{dQ_1}{d\mathbb{P}} + \mathbf{1}_{F^C} \frac{dQ_2}{d\mathbb{P}} \in \mathscr{P}^q.$$

It is easy to show, using an argument similar to the one in Lemma 1.4 that

$$R\left(E\left[X \frac{d\widehat{Q}}{d\mathbb{P}} |\mathscr{G}\right], \widehat{Q}\right) = R\left(E\left[X \frac{dQ_1}{d\mathbb{P}} |\mathscr{G}\right], Q_1\right) \vee R\left(E\left[X \frac{dQ_2}{d\mathbb{P}} |\mathscr{G}\right], Q_2\right).$$

Lemma 3.3. *Let $Q \in \mathscr{P}^q$ and ρ satisfying (MON)$'$, (REG) then*

$$R(Y,Q) = \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \left\{\rho(\xi) \mid E\left[-\xi \frac{dQ}{d\mathbb{P}} |\mathscr{G}\right] = Y\right\}. \tag{3.20}$$

Proof. For sake of simplicity denote by $\mu(\cdot) = E[\cdot \frac{dQ}{d\mathbb{P}} |\mathscr{G}]$ and $r(Y,\mu)$ the right hand side of equation (3.20). Notice that $R(Y,\mu) \leq r(Y,\mu)$. By contradiction, suppose that $\mathbb{P}(A) > 0$ where $A =: \{R(Y,\mu) < r(Y,\mu)\}$. From Lemma 3.1, there exists a r.v. $\xi \in L^p_{\mathscr{G}}(\mathscr{F})$ satisfying the following conditions

- $\mu(-\xi) \geq Y$ and $\mathbb{P}(\mu(-\xi) > Y) > 0$.
- $R(Y,\mu)(\omega) \leq \rho(\xi)(\omega) < r(Y,\mu)(\omega)$ for $\mathbb{P}$-almost every $\omega \in A$.

Set $Z = \mu(-\xi) - Y \in L^0(\mathscr{G}) \subseteq L^p_{\mathscr{G}}(\mathscr{F})$ and it satisfies $Z \geq 0$, $\mathbb{P}(Z > 0) > 0$. Then, thanks to (MON)$'$, $\rho(\xi) \geq \rho(\xi + Z)$. From $\mu(-(\xi + Z)) = Y$ we deduce:

$$R(Y,\mu)(\omega) \leq \rho(\xi)(\omega) < r(Y,\mu)(\omega) \leq \rho(\xi + Z)(\omega) \text{ for } \mathbb{P}\text{-a.e. } \omega \in A,$$

which is a contradiction.

Consider the class $\mathscr{M}^{prop}(L^0(\mathscr{G}) \times \mathscr{P}^q)$ composed by maps $K : L^0(\mathscr{G}) \times \mathscr{P}^q \to \bar{L}^0(\mathscr{G})$ such that $K \in \mathscr{M}(L^0(\mathscr{G}) \times \mathscr{P}^q)$ and there exist X_1, X_2 such that

$$\sup \mathscr{K}(X_1) < \sup \mathscr{K}(X_2) < +\infty.$$

Theorem 3.2. $\rho : L^p_{\mathscr{G}}(\mathscr{F}) \to L^0(\mathscr{G})$ *satisfies (MON)$'$, (REG), (EVQ) and (PRO) if and only if*

$$\rho(X) = \sup_{Q \in \mathscr{P}^q} R\left(E\left[-\frac{dQ}{d\mathbb{P}}X|\mathscr{G}\right], Q\right) \tag{3.21}$$

where

$$R(Y,Q) = \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \left\{\rho(\xi) \mid E\left[-\xi\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] = Y\right\}$$

is unique in the class $\mathscr{M}^{prop}(L^0(\mathscr{G}) \times \mathscr{P}^q)$.

Remark 3.2. Since $Q << \mathbb{P}$ we can observe

$$E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}\xi \mid \mathscr{G}\right] = E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}}X \mid \mathscr{G}\right] \Longleftrightarrow E_Q[\xi|\mathscr{G}] =_Q E_Q[X|\mathscr{G}],$$

so that we will write sometimes with a slight abuse of notation

$$R(E_Q[X|\mathscr{G}], Q) = \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{\rho(\xi) \mid E_Q[\xi|\mathscr{G}] =_Q E_Q[X|\mathscr{G}]\}$$

From this last proposition we can deduce the following important result which confirm what we have obtained in Chapter 1.

Proposition 3.3. *Suppose that ρ satisfies the same assumption of Theorem 3.2. Then the restriction $\widehat{\rho} := \rho\mathbf{1}_{L^p(\mathscr{F})}$ defined by $\widehat{\rho}(X) = \rho(X)$ for every $X \in L^p(\mathscr{F})$ is a quasiconvex risk measure that can be represented as*

$$\widehat{\rho}(X) = \sup_{Q \in \mathscr{P}^q} \inf_{\xi \in L^p(\mathscr{F})} \{\widehat{\rho}(\xi) \mid E_Q[-\xi|\mathscr{G}] =_Q E_Q[-X|\mathscr{G}]\}.$$

Proof. For every $X \in L^p(\mathscr{F})$, $Q \in \mathscr{P}^q$ we have

$$\widehat{\rho}(X) \geq \inf_{\xi \in L^p(\mathscr{F})} \{\widehat{\rho}(\xi) \mid E_Q[-\xi|\mathscr{G}] =_Q E_Q[-X|\mathscr{G}]\}$$
$$\geq \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{\rho(\xi) \mid E_Q[-\xi|\mathscr{G}] =_Q E_Q[-X|\mathscr{G}]\}$$

and hence the thesis.

It's a moot point in financial literature whether cash additivity (CAS) ($\rho(X+\Lambda) = \rho(X) - \Lambda$ for $\Lambda \in L^0(\mathscr{G})$) is a too much restrictive assumption or not. Surely adding (CAS) to a quasiconvex risk measure it automatically follows that ρ is convex. The following result is meant to confirm that the dual representation chosen for quasiconvex maps is indeed a good generalization of the convex case. Differently from Corollary 1.2 here there are no restrictive additional hypothesis and it becomes clear how a powerful tool the modules are in this kind of applications.

Corollary 3.4. *(i) If $Q \in \mathscr{P}^q$ and if ρ is (MON), (REG) and (CAS) then*

$$R(E_Q(-X|\mathscr{G}), Q) = E_Q(-X|\mathscr{G}) - \rho^*(-Q) \tag{3.22}$$

where

$$\rho^*(-Q) = \sup_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{E_Q[-\xi|\mathscr{G}] - \rho(\xi)\}. \tag{3.23}$$

(ii) Under the same assumptions of Proposition 3.2 and if ρ satisfies in addition (CAS) then

$$\rho(X) = \sup_{Q \in \mathscr{P}^q} \{E_Q(-X|\mathscr{G}) - \rho^*(-Q)\}.$$

Proof. Denote by $\mu(\cdot) =: E\left[\frac{dQ}{d\mathbb{P}} \cdot \mid \mathscr{G}\right]$; by definition of R

$$\begin{aligned} R(E_Q(-X|\mathscr{G}), Q) = & \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{\rho(\xi) \mid \mu(-\xi) = \mu(-X)\} \\ = & \ \mu(-X) + \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{\rho(\xi) - \mu(-X) \mid \mu(-\xi) = \mu(-X)\} \\ = & \ \mu(-X) + \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{\rho(\xi) - \mu(-\xi) \mid \mu(-\xi) = \mu(-X)\} \\ = & \ \mu(-X) - \sup_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{\rho(\xi) - \mu(-X) \mid \mu(-\xi) = \mu(-X)\} \\ = & \ \mu(-X) - \rho^*(-Q), \end{aligned}$$

where the last equality follows from

$$\begin{aligned} \rho^*(-Q) \overset{(CAS)}{=} & \sup_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \{\mu(-\xi - \mu(X-\xi)) - \rho(\xi + \mu(X-\xi))\} \\ = & \sup_{\eta \in L^p_{\mathscr{G}}(\mathscr{F})} \{\mu(-\eta) - \rho(\eta) \mid \eta = \xi + \mu(X-\xi)\} \\ \leq & \sup_{\eta \in L^p_{\mathscr{G}}(\mathscr{F})} \{\mu(-\eta) - \rho(\eta) \mid \mu(-\eta) = \mu(-X)\} \leq \rho^*(-Q). \end{aligned}$$

Remark 3.3. If we look at equation (3.21) in the light of Proposition 3.1 we could naively claim that the inequality

$$R\left(E\left[-\frac{dQ}{d\mathbb{P}}X|\mathscr{G}\right],Q\right) \geq E\left[-\frac{dQ}{d\mathbb{P}}X|\mathscr{G}\right] - \rho^*(-Q)$$

can be translated into : 'If the preferences of an agent are described by a quasiconvex - not convex - risk measure I can't recover the risk only taking a *supremum* of the Fenchel conjugate over all the possible probabilistic scenarios. I shall need to choose a more cautious and conservative penalty function.'

3.3.3 Proof of Theorem 3.2

We recall that $L^p_{\mathscr{G}}(\mathscr{F})$ is a normed module so that the concatenation property always holds true. During the whole proof we fix an arbitrary $X \in L^p_{\mathscr{G}}(\mathscr{F})$. We are assuming (PRO) and for this reason we refer to proof of Theorem 3.1 step 3 for the definitions and notations. There exists a $\zeta_1, \zeta_2 \in E$ such that $\rho(\zeta_1) < \rho(\zeta_2) \in L^0(\mathscr{G})$ and we recall that the evenly convex set

$$\mathscr{C}^1_\varepsilon =: \{\xi \in L^p_{\mathscr{G}}(\mathscr{F}) \mid \rho(\xi) \leq Y_\varepsilon \mathbf{1}_{A^{max}} + \rho(\zeta_1)\mathbf{1}_{(A^{max})^C}\} \neq \emptyset.$$

may be separated from $\tilde{X} = X\mathbf{1}_{A^{max}} + \zeta_2 \mathbf{1}_{(A^{max})^C}$ by $\mu_\varepsilon \in \mathscr{L}(L^p_{\mathscr{G}}(\mathscr{F}), L^0(\mathscr{G}))$ i.e.

$$\mu_\varepsilon(\tilde{X}) > \mu_\varepsilon(\xi) \quad \forall \xi \in \mathscr{C}^1_\varepsilon.$$

ONLY IF.

Let $\eta \in L^p_{\mathscr{G}}(\mathscr{F})$, $\eta \geq 0$. If $\xi \in \mathscr{C}^1_\varepsilon$ then (MON) implies $\xi + n\eta \in \mathscr{C}^1_\varepsilon$ for every $n \in \mathbb{N}$. In this case $\mu_\varepsilon(\cdot) = E[Z_\varepsilon \cdot |\mathscr{G}]$ for some $Z_\varepsilon \in L^q_{\mathscr{G}}(\mathscr{F})$ and from (3.17) we deduce:

$$E[Z_\varepsilon(\xi + n\eta)|\mathscr{G}] < E[Z_\varepsilon \tilde{X}|\mathscr{G}] \quad \Rightarrow \quad E[-Z_\varepsilon \eta|\mathscr{G}] > \frac{E[Z_\varepsilon(\xi - \tilde{X})|\mathscr{G}]}{n} \quad , \quad \forall n \in \mathbb{N}$$

i.e. $E[Z_\varepsilon \eta|\mathscr{G}] \leq 0$ for every $\eta \in L^p_{\mathscr{G}}(\mathscr{F})$, $\eta \geq 0$. In particular $Z_\varepsilon \leq 0$: only notice that $\mathbf{1}_{\{Z_\varepsilon > 0\}} \in L^p_{\mathscr{G}}(\mathscr{F})$ so that $E[Z_\varepsilon \mathbf{1}_{\{Z_\varepsilon > 0\}}] \leq 0$ if and only if $\mathbb{P}(\{Z_\varepsilon > 0\}) = 0$.
If there exists a $\mathscr{G}$-measurable set G, $\mathbb{P}(G) > 0$, on which $Z_\varepsilon = 0$, then we have a contradiction. In fact fix $\xi \in \mathscr{C}^1_\varepsilon$: from $E[Z_\varepsilon \xi|\mathscr{G}] < E[Z_\varepsilon \tilde{X}|\mathscr{G}]$ we can find a $\delta_\xi \in L^0_{++}(\mathscr{G})$ such that

$$E[Z_\varepsilon \xi|\mathscr{G}] + \delta_\xi < E[Z_\varepsilon \tilde{X}|\mathscr{G}] \quad \Rightarrow \quad \delta_\xi \mathbf{1}_G = E[Z_\varepsilon \mathbf{1}_G \xi|\mathscr{G}] + \delta_\xi \mathbf{1}_G \leq E[Z_\varepsilon \mathbf{1}_G \tilde{X}|\mathscr{G}] = 0.$$

which is absurd because $\mathbb{P}(\delta_\xi \mathbf{1}_G > 0) > 0$.
We deduce that $E[Z_\varepsilon \mathbf{1}_B] = E[E[Z_\varepsilon|\mathscr{G}]\mathbf{1}_B] < 0$ for every $B \in \mathscr{G}$ and then $\mathbb{P}(E[Z_\varepsilon|\mathscr{G}] <$

$0) = 1$. Hence we may normalize Z_ε to $\frac{Z_\varepsilon}{E[Z_\varepsilon|\mathscr{G}]} = \frac{dQ}{d\mathbb{P}} \in L^1(\mathscr{F})$.
From equation (3.16) in the proof of Theorem 3.1 we can deduce that

$$\begin{aligned}\rho(X) = \pi(-X) &= \sup_{Q\in\mathscr{P}^q} \inf_{\xi\in L^p_{\mathscr{G}}(\mathscr{F})} \left\{\pi(\xi) \mid E\left[\xi\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \geq E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right]\right\} \\ &= \sup_{Q\in\mathscr{P}^q} \inf_{\xi\in L^p_{\mathscr{G}}(\mathscr{F})} \left\{\rho(\xi) \mid E\left[-\xi\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \geq E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right]\right\}\end{aligned} \tag{3.24}$$

Applying Lemma 3.3 we can substitute $=$ in the constraint.

To complete the proof of the ‘only if ’statement we only need to show that $R \in \mathscr{M}^{prop}(L^0(\mathscr{G}) \times \mathscr{P}^q)$. By Lemma 3.2 we already know that $R \in \mathscr{M}(L^0(\mathscr{G}) \times \mathscr{P}^q)$ so that applying (PRO) and (3.24) we have that $R \in \mathscr{M}^{prop}(L^0(\mathscr{G}) \times \mathscr{P}^q)$.

IF.

We assume that $\rho(X) = \sup_{Q\in\mathscr{P}^q} R(E[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}], Q)$ holds for some $R \in \mathscr{M}^{prop}(L^0(\mathscr{G}) \times \mathscr{P}^q)$. Since R is monotone in the first component and $R(Y\mathbf{1}_A, Q)\mathbf{1}_A = R(Y,Q)\mathbf{1}_A$ for every $A \in \mathscr{G}$ we easily deduce that ρ is (MON) and (REG). Also ρ is clearly (PRO).
We need to show that ρ i (EVQ).
Let $V_\alpha = \{\xi \in L^p_{\mathscr{G}}(\mathscr{F}) | \rho(\xi) \leq \alpha\}$ where $\alpha \in L^0(\mathscr{G})$ and $\overline{X} \in L^p_{\mathscr{G}}(\mathscr{F})$ such that $\overline{X}\mathbf{1}_A \cap V_\alpha \mathbf{1}_A = \emptyset$. Hence $\rho(\overline{X}) = \sup_{Q\in\mathscr{P}^q} R(E[-\overline{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}], Q) > \alpha$.
Since the set $\{R(E[-\overline{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}], Q) | Q \in \mathscr{P}^q\}$ is upward directed we find

$$R\left(E\left[-\overline{X}\frac{dQ_m}{d\mathbb{P}}|\mathscr{G}\right], Q_m\right) \uparrow \rho(\overline{X}) \quad \text{as } m \uparrow +\infty.$$

Consider the sets $F_m = \{R(E[-\overline{X}\frac{dQ_m}{d\mathbb{P}}|\mathscr{G}], Q_m) > \alpha\}$ and the partition of Ω given by $G_1 = F_1$ and $G_m = F_m \setminus G_{m-1}$. We have from the properties of the module $L^q_{\mathscr{G}}(\mathscr{F})$ that

$$\frac{d\overline{Q}}{d\mathbb{P}} = \sum_{m=1}^{\infty} \frac{dQ_m}{d\mathbb{P}}\mathbf{1}_{G_m} \in L^q_{\mathscr{G}}(\mathscr{F})$$

and then $\overline{Q} \in \mathscr{P}^q$ with $R(E[-\overline{X}\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}], \overline{Q}) > \alpha$.
Let $X \in V_\alpha$: if there exists $A \in \mathscr{G}$ such that $E[X\frac{d\overline{Q}}{d\mathbb{P}}\mathbf{1}_A|\mathscr{G}] \leq E[\overline{X}\frac{d\overline{Q}}{d\mathbb{P}}\mathbf{1}_A|\mathscr{G}]$ on A then $\rho(X\mathbf{1}_A) \geq R(E[-X\frac{d\overline{Q}}{d\mathbb{P}}\mathbf{1}_A|\mathscr{G}], \overline{Q}) \geq R(E[-\overline{X}\frac{d\overline{Q}}{d\mathbb{P}}\mathbf{1}_A|\mathscr{G}], \overline{Q}) > \alpha$ on A. This implies $\rho(X) > \alpha$ on A which is a contradiction unless $\mathbb{P}(A) = 0$. Hence $E[X\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}] > E[\overline{X}\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}]$ for every $X \in V_\alpha$.

UNIQUENESS.

We show that for every $K \in \mathscr{M}^{prop}(L^0(\mathscr{G}) \times \mathscr{P}^q)$ such that

$$\rho(X) = \sup_{Q \in \mathscr{P}^q} K(E[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}], Q),$$

K must satisfy

$$K(Y,Q) = \inf_{\xi \in L^p_{\mathscr{G}}(\mathscr{F})} \left\{ \rho(\xi) \mid E\left[-\xi \frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \geq Y \right\}.$$

Define the set $\mathscr{A}(Y,Q) = \left\{ \xi \in L^p_{\mathscr{G}}(\mathscr{F}) \mid E\left[-\xi \frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \geq Y \right\}$.

Lemma 3.4. *For each* $(\overline{Y}, \overline{Q}) \in L^0(\mathscr{G}) \times \mathscr{P}^q$

$$K(\overline{Y}, \overline{Q}) = \sup_{Q \in \mathscr{P}^q} \inf_{X \in \mathscr{A}(\overline{Y}, \overline{Q})} K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right], Q\right) \tag{3.25}$$

Proof (Proof of the Lemma). To prove (3.25) we consider

$$\psi(Q, \overline{Q}, \overline{Y}) = \inf_{X \in \mathscr{A}(\overline{Y}, \overline{Q})} K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right], Q\right)$$

Notice that $E[-X\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}] \geq \overline{Y}$ for every $X \in \mathscr{A}(\overline{Y}, \overline{Q})$ implies

$$\psi(\overline{Q}, \overline{Q}, \overline{Y}) = \inf_{X \in \mathscr{A}(\overline{Y}, \overline{Q})} K\left(E\left[-X\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}\right], \overline{Q}\right) \geq K(\overline{Y}, \overline{Q})$$

On the other hand $E[\overline{Y}\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}] = \overline{Y}$ so that $-\overline{Y} \in \mathscr{A}(\overline{Y}, \overline{Q})$ and the the second inequality is actually an equality basically

$$\psi(\overline{Q}, \overline{Q}, \overline{Y}) \leq K\left(E\left[-(-\overline{Y})\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}\right], \overline{Q}\right) = K(\overline{Y}, \overline{Q}).$$

If we show that $\psi(Q, \overline{Q}, \overline{Y}) \leq \psi(\overline{Q}, \overline{Q}, \overline{Y})$ for every $Q \in \mathscr{P}^q$ then (3.25) is done. To this aim we define

$$\mathscr{C} = \left\{ A \in \mathscr{G} \mid E\left[X\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}\right] = E\left[X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \text{ on } A,\ \forall X \in L^p_{\mathscr{G}}(\mathscr{F}) \right\}$$
$$\mathscr{D} = \left\{ A \in \mathscr{G} \mid \exists X \in L^p_{\mathscr{G}}(\mathscr{F}) \text{ s.t. } E\left[X\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}\right] \lneqq E\left[X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \text{ on } A \right\}$$

For every $C \in \mathscr{C}$ we have for every $X \in L^p_{\mathscr{G}}(\mathscr{F})$

$$K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right],Q\right)\mathbf{1}_C = K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right]\mathbf{1}_C,Q\right)\mathbf{1}_C$$
$$= K\left(E\left[-X\frac{d\bar{Q}}{d\mathbb{P}}|\mathscr{G}\right]\mathbf{1}_C,\bar{Q}\right)\mathbf{1}_C = K\left(E\left[-X\frac{d\bar{Q}}{d\mathbb{P}}|\mathscr{G}\right],\bar{Q}\right)\mathbf{1}_C$$

which implies $\psi(Q,\bar{Q},\bar{Y})\mathbf{1}_C = \psi(\bar{Q},\bar{Q},\bar{Y})\mathbf{1}_C$.
For every $D \in \mathscr{D}$ there will exists $X \in L^p_{\mathscr{G}}(\mathscr{F})$ such that whether $E\left[-X\frac{d\bar{Q}}{d\mathbb{P}}|\mathscr{G}\right] > E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right]$ on D or $<$ on D. Let us define $Z = X - E\left[-X\frac{d\bar{Q}}{d\mathbb{P}}|\mathscr{G}\right]$. Surely $E\left[Z\frac{d\bar{Q}}{d\mathbb{P}}|\mathscr{G}\right] = 0$ but $E\left[Z\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \lessgtr 0$ on D. We may deduce that for every $\alpha \in L^0(\mathscr{G})$, $-\bar{Y} + \alpha Z \in \mathscr{A}(\bar{Y},\bar{Q})$ and also notice that any $Y \in L^0(\mathscr{G})$ can be written as $Y = E[(-\bar{Y} + \alpha_Y Z)\frac{dQ}{d\mathbb{P}}|\mathscr{G}]$ on the set D. Finally

$$\begin{aligned}\psi(Q,\bar{Q},\bar{Y})\mathbf{1}_D &\le \inf_{\alpha\in L^0(\mathscr{G})} K\left(E\left[-(-\bar{Y}+\alpha Z)\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right],Q\right)\mathbf{1}_D \\ &= \inf_{Y\in L^0(\mathscr{G})} K(Y\mathbf{1}_D,Q)\mathbf{1}_D = \inf_{Y\in L^0(\mathscr{G})} K\left(Y\mathbf{1}_D,\bar{Q}\right)\mathbf{1}_D \\ &= K\left(\bar{Y},\bar{Q}\right)\mathbf{1}_D\end{aligned}$$

Now we need to show that there exists a maximal element in both class $\mathscr{C}$ and $\mathscr{D}$. To this aim notice that if $A,B \in \mathscr{C}$ then $A\cup B$, $A\cap B$ belong to $\mathscr{C}$. Consider the set $\{\mathbf{1}_C | C \in \mathscr{C}\}$: the set is upward directed since $\mathbf{1}_{C_1} \vee \mathbf{1}_{C_2} = \mathbf{1}_{C_1\cup C_2}$ for every $C_1,C_2 \in \mathscr{C}$. Hence we can find a sequence $\mathbf{1}_{C_n} \uparrow \sup\{\mathbf{1}_C | C \in \mathscr{C}\} = \mathbf{1}_{C^{max}}$ where $C^{max} = \cup_n C_n \in \mathscr{G}$. Through a similar argument we can get a maximal element for $\mathscr{D}$, namely $D^{\max}$: notice that $\mathbb{P}(C^{\max}\cup D^{\max}) = 1$ so that we conclude that $\psi(Q,\bar{Q},\bar{Y}) \le \psi(\bar{Q},\bar{Q},\bar{Y}) = K(\bar{Y},\bar{Q})$ and the claim is proved.

Back to the proof of uniqueness. By the Lemma

$$\begin{aligned}K(\bar{Y},\bar{Q}) &= \sup_{Q\in\mathscr{P}^q}\inf_{X\in\mathscr{A}(\bar{Y},\bar{Q})} K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right],Q\right) \\ &\le \inf_{X\in\mathscr{A}(\bar{Y},\bar{Q})}\sup_{Q\in\mathscr{P}^q} K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right],Q\right) = \inf_{X\in\mathscr{A}(\bar{Y},\bar{Q})}\rho(X)\end{aligned}$$

We need to prove the reverse inequality and then we are done. Again we consider two classes of $\mathscr{G}$-measurable sets:

$$\begin{aligned}\mathscr{C} &= \left\{A\in\mathscr{G} \mid K(\bar{Y},\bar{Q})\mathbf{1}_A \ge K(Y,Q)\mathbf{1}_A \;\forall (Y,Q)\in L^0(\mathscr{G})\times\mathscr{P}^q\right\} \\ \mathscr{D} &= \left\{A\in\mathscr{G} \mid \exists (Y,Q)\in L^0(\mathscr{G})\times\mathscr{P}^q \text{ s.t. } K(\bar{Y},\bar{Q}) < K(Y,Q) \text{ on } A\right\}\end{aligned}$$

For every $C \in \mathscr{C}$ the reverse inequality is obviously true.
For every $D \in \mathscr{D}$ there exists some $(Q,Y) \in L^0(\mathscr{G}) \times \mathscr{P}^q$ such that $K(Y,Q) >$

$K(\overline{Y},\overline{Q})$ on D. This means that it can be easily build up a $\beta\in L^0(\mathscr{G})$ such that $\beta>K(\overline{Y},\overline{Q})$ on D and the set $U_\beta^D=\{(Y,Q)\in L^0(\mathscr{G})\times\mathscr{P}^q | K(Y,Q)\geq\beta \text{ on } D\}$ will be non empty. There exists $(\overline{S},\overline{X})\in L^0_{++}(\mathscr{G})\times L^p_{\mathscr{G}}(\mathscr{F})$ with

$$\overline{Y}\overline{S}+E\left[\overline{X}\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}\right]<Y\overline{S}+E\left[\overline{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right] \text{ on } D$$

for every $(Y,Q)\in U_\beta^D$.

All the following equalities and inequalities are meant to be holding $\mathbb{P}$ almost surely only on the set D. Set $\Lambda=-\overline{Y}-E[\frac{\overline{X}}{\overline{S}}\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}]$ and $\widehat{X}=\frac{\overline{X}}{\overline{S}}+\Lambda$, so that $E[\widehat{X}\frac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}]=-\overline{Y}$: for every $(Y,Q)\in U_\beta$

$$\begin{aligned} &\overline{Y}\overline{S}+E\left[\overline{X}\tfrac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}\right]<Y\overline{S}+E\left[\overline{X}\tfrac{dQ}{d\mathbb{P}}|\mathscr{G}\right]\\ \text{implies}\quad &\overline{Y}+E\left[\left(\tfrac{\overline{X}}{\overline{S}}+\Lambda\right)\tfrac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}\right]<Y+E\left[\left(\tfrac{\overline{X}}{\overline{S}}+\Lambda\right)\tfrac{dQ}{d\mathbb{P}}|\mathscr{G}\right]\\ \text{implies}\quad &\overline{Y}+E\left[\widehat{X}\tfrac{d\overline{Q}}{d\mathbb{P}}|\mathscr{G}\right]<Y+E\left[\widehat{X}\tfrac{dQ}{d\mathbb{P}}|\mathscr{G}\right]\end{aligned}$$

i.e. $Y+E\left[\widehat{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right]>0$ for every $(Y,Q)\in U_\beta$.

For every $Q\in\mathscr{P}^q$ define $Y_Q=E\left[-\widehat{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right]$. If there exists a $B\subseteq D\in\mathscr{G}$ such that $K(Y_Q,Q)\geq\beta$ on B then $Y_Q+E\left[\widehat{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right]>0$ on B.
In fact just take $(Y_1,Q_1)\in U_\beta^D$ and define $\tilde{Y}=Y_Q\mathbf{1}_B+Y_1\mathbf{1}_{B^C}$ and $\tilde{Q}\in\mathscr{P}^q$ such that

$$\frac{d\tilde{Q}}{d\mathbb{P}}=\frac{dQ}{d\mathbb{P}}\mathbf{1}_B+\frac{dQ_1}{d\mathbb{P}}\mathbf{1}_{B^C}$$

Thus $K(\tilde{Y},\tilde{Q})\geq\beta$ on D and $\tilde{Y}+E\left[\widehat{X}\frac{d\tilde{Q}}{d\mathbb{P}}|\mathscr{G}\right]>0$ on D, which implies $Y_Q+E\left[\widehat{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right]>0$ on B and this is absurd.

Hence $K(Y_Q,Q)<\beta$. Surely $\widehat{X}\in\mathscr{A}(\overline{Y},\overline{Q})$ and we can conclude that

$$\begin{aligned}K(\overline{Y},\overline{Q})\mathbf{1}_D&\leq\inf_{X\in\mathscr{A}(\overline{Y},\overline{Q})}\sup_{Q\in\mathscr{P}^q}K\left(E\left[-X\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right],Q\right)\mathbf{1}_D\\&\leq\sup_{Q\in\mathscr{P}^q}K\left(E\left[-\widehat{X}\frac{dQ}{d\mathbb{P}}|\mathscr{G}\right],Q\right)\mathbf{1}_D\leq\beta\mathbf{1}_D\end{aligned}$$

The equality follows since β can be taken near as much as we want to $K(\overline{Y},\overline{Q})$ and then we conclude that

$$K(\overline{Y},\overline{Q})=\inf_{X\in\mathscr{A}(\overline{Y},\overline{Q})}\rho(X).$$

Repeating the argument in Lemma 3.4 we can find a maximal element $D^{\max} \in \mathscr{D}$ and $C^{\max} \in \mathscr{C}$ and conclude from $\mathbb{P}(C^{max} \cup D^{max}) = 1$.

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