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Ciclo XXIII

On some axiomatic extensions of the monoidal t-norm based logic MTL: an analysis in the propositional and in the first-order case

Settori scientifico-disciplinari: MAT/01, MAT/02

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Contents

Chapter 1

Introduction

The scientific area this thesis belongs to is many-valued logics: this means logics in which, from the semantical point of view, we have "intermediate" truth-values, between 0 and 1 (which in turns are designated to represent, respectively, the "false" and the "true").

The classical logic (propositional, for simplicity) is based on the fact that every statement is true or false: this is reflected by the excluded middle law

$$\varphi \vee \neg\varphi,$$

that is a theorem of this logic. However, there are many reasons that suggest to reject this law: for example, intuitionistic logic does not satisfy it, since this logic reflects a "constructive" conception of mathematics (see [Hey71, Tro69]).

More in general, this formula cannot hold in every logic that "has more than two truth-values". In fact the excluded middle law is not a theorem, in the logics studied in this thesis: however, as pointed out in section 4.6, it is possible to study various interesting weakenings of this axiom.

As we have already said, the topic of this thesis will concerns many-valued logics: there are many different methods to generalize classical logic in this way. We will refer to the hierarchy of logics firstly introduced in [Háj98b] and then extended in [EG01]: in the course of the years the hierarchy has been considerably expanded (see [CEG$^+$09, CH10] for a survey). A more informal presentation, about these logics can be found in [Got08] and [Háj06b]: an historical overview, about many-valued logics, is given in [Háj98b, chapter 10]. Other useful references are [Got01], [MPN99] and [Bel02].

The thesis is structured as follows: there are two parts and one appendix.

In the appendix all the necessaries notions of universal algebra are presented: for reader's convenience some useful textbooks are also listed.

Part I is devoted to introduce the general notions of many-valued logics and their semantics. There are four chapters: chapter 2 presents triangular norms and their residua (the book [KMP00] is a reference monograph), beginning to "justify" them as a semantics for some logical connectives. In chapter 3 almost all the logics studied in this thesis are presented, from the syntactical point of view; moreover, some (minor) new results are presented. The algebraic semantics of our logics is introduced in chapter 4: Finally, in chapter 5 we will introduce the first-order version (syntax, semantics, completeness and incompleteness results) for the axiomatic extensions of the Monoidal t-norm based logic MTL (informally, the logic of all left-continuous t-norms and their residua): an interesting thing to point out is the fact that in these logics the predicates are interpreted (from the semantical point of view) as fuzzy relations (see chapter 5).

In part II we dedicate to specific topics that have been developed during the PhD career.

There are five chapters: each of them is devoted to a particular problem about propositional or first-order logics. In particular, the first four chapters are based on papers [ABM09a], [BM09], [BM10], [Bia10].

In chapter 6 a "modal like" semantics is introduced, for the logic BL ([Háj98b]), mimicking temporal logics: in this temporal semantics the logic of every instant of time is Łukasiewicz with finitely or infinitely many truth-values. A completeness theorem is showed, with respect to particular classes of temporal flows. The reason to introduce this alternative semantics is to furnish an alternative way to analyze and to understand what the logic BL is (more in general, in my opinion, there is not necessarily "a" way to characterize this logic, but many: hence the various perspectives furnished by the alternative semantics can be useful to enlarge "the picture" of BL): in particular with this semantics is showed the strong connection of BL with Łukasiewicz logic.

However, there are also the first-order versions of MTL and its extensions: in particular these logics are much less studied with respect to their propositional versions and hence there are much more open problems. Even about the definitions of the semantics, there is some criticism: in particular about the concept of safe model (see chapter 5 for the definitions). In fact, in first-order case soundness and completeness are defined by restricting to safe models. But one can ask the following question: does the soundness continue to holds if we work with models in which the truth value of a provable formula is defined, but that are not necessarily safe ? This property is called supersoundness and was firstly introduced by Petr Hájek and other researchers, for the axiomatic extensions of (the first-order version of) BL.

In chapter 7, hence, we focus our attention on supersound logics, property originally introduced in [HS01]: we will extend this analysis to various axiomatic extension of MTL$\forall$, by using the MacNeille completions of MTL-chains (as developed in [Lv08, van10]) and other techniques.

In the MTL hierarchy, there are also some logics that are not "t-norm based" (a logic is t-norm based if it holds a completeness theorem with respect to some class of standard MTL-algebras, see chapter 4), for example the $n-$contractive extensions (i.e. satisfying $\varphi^n \to \varphi^{n+1}$) of BL: even if they are not (for $n > 1$) t-norm based, these logics enjoy interesting properties. For example, as shown in chapter 7, an axiomatic extension L of BL$\forall$ is supersound if and only if L is $n-$contractive, for some positive integer n.

For these reasons, in chapter 8 four families of $n-$contractive axiomatic extensions of BL are studied, in the propositional and in the first-order cases: we will analyze completeness, computational (and arithmetical, in the first-order case) complexity and also amalgamation and interpolation properties.

As it is well known, there are left-continuous but not continuous t-norms. In [BEG99, Boi98] it is shown that (see chapter 2) a t-norm has a residuum if and only if it is left-continuous: hence this class of t-norms is particularly important, since t-norms and residua are useful to give a semantics for the conjunction and implication connectives (for the logics that are t-norm based). The first example of left-continuous t-norm was the one of Nilpotent Minimum, introduced in [Fod95]. This t-norm induces a standard MTL-algebra that we will denote with $[0,1]_{NM}$; the logic associated to this algebra was introduced in [EG01] and was called Nilpotent Minimum (NM). This logic has some interesting properties, both in the propositional and in the first-order cases.

Hence chapters 9 and 10 are devoted to Nilpotent Minimum logic. In chapter 9, on the lines of [BPZ07, BCF07], we will analyze the sets of first-order tautologies of certain NM-chains (that are subalgebras of $[0,1]_{NM}$). We will study the decidability (or the undecidability) of these sets as well as the monadic fragments. We will also compare these results with the ones of [BPZ07, BCF07]. Chapter 10, instead, is devoted to analyze some logical and algebraic properties of the propositional version of NM.

Finally, in chapter 11 we will conclude with a discussion about the open problems presented in the preceding chapters.

We conclude the introduction with some remarks on applications of many-valued logics. Even if in this thesis we have discussed mainly mathematical problems, there are many examples of uses of many-valued logics to solve some practical problems.

An example is given by the "expert systems" to assist the diagnosis in internal medicine. In particular the system CADIAG-2 has been designed and implemented at the Medical University of Vienna: as pointed out in [CR10] CADIAG-2's knowledge base contains more than 20.000 rules expressing relationships between medical entities, i.e., patient's symptoms, signs, laboratory test results, clinical findings and diagnoses. In particular in [CR10] a particular fragment (with an additional involutive negation connective) of monadic first-order Gödel logic is used to formalize the rules of CADIAG-2.

A (classical) satisfiability check of the resulting formulas allowed the detection of some errors in the rules of the system. Another useful paper about this topic is [CV10].

These are examples of how the applications can benefit from a theoretical insight.

Generally speaking, there is a very extensive literature about fuzzy sets (introduced by L. A. Zadeh in [Zad65]), fuzzy systems, t-norms and residua in "applicative contexts": for example the scientific journal "Fuzzy Sets and Systems" has been created to deal with these topics. Some useful monographs about fuzzy sets are, for example, [DP80] and [Nov89].

Finally, one can ask if there is some "foundational" study about fuzzy set theory, in a mathematical sense, using many-valued logics (in particular, first-order axiomatic extensions of MTL). The answer is positive: for example in papers [HH01] (see also [Han03]) and [HH03] an axiomatic fuzzy set theory is presented, developed as a theory of the first-order version of BL (with an additional connective Δ).

Part I

General background and minor results

Chapter 2

Triangular Norms

Triangular norms are two variables functions whose behavior is particularly adequate to play the role of semantical interpretation of a conjunction, in a many-valued logic. Historically, triangular norms were introduced in the context of probabilistic metric spaces: see the monograph [SS05] for further details about this topic. A reference book concerning the themes of this section is [KMP00] and most of the results will be taken from it.

Definition 2.0.1. *A triangular norm (t-norm for short) is a function*

$$t : [0,1] \times [0,1] \to [0,1]$$

that satisfy the following properties, for every $x, y, z \in [0,1]$.

(t1) $$t(t(x,y),z) = t(x,t(y,z))$$

(t2) $$t(x,y) = t(y,z)$$

(t3) $$\textit{If } x \leq y \textit{ then } t(x,z) \leq t(y,z)$$

(t4) $$t(x,1) = x.$$

Note that an immediate consequence of (t3), (t4) is that

(t5) $$t(x,0) = 0.$$

In the rest of the section, in place of t we will use the symbol $*$ with the infix notation.

Remark 2.0.1. *From the definition 2.0.1 it is easy to see that $*$ behaves like the classical conjunction, over $\{0,1\}$. Moreover also the conditions (t1)-(t4) are quite reasonable for an operation that interprets a logical conjunction: this justifies the previous statement: "...whose behavior is particularly adequate to play the role of semantical interpretation of a conjunction". Note even that, thanks to (t4),(t5), all t-norms coincide on the boundary of $[0,1] \times [0,1]$.*

2.1 Continuous t-norms

As pointed out in [KMP00], a real function of two variables e.g. with domain $[0,1] \times [0,1]$ may be continuous in each variable without being continuous on $[0,1] \times [0,1]$.
As regards to t-norms, thanks to monotonicity we have

Proposition 2.1.1 ([KMP00, proposition 1.19]). *A t-norm is* continuous *if and only if it is continuous in both its arguments, as a function over reals.*

Moreover, thanks to the commutativity, we have that a t-norm $*$ is continuous if and only if its first component it is (i.e. $\cdot * y$ is continuous, for every $y \in [0,1]$).
We now list three examples of continuous t-norms

(Łukasiewicz t-norm) $$x *_{\mathrm{L}} y = \max(0, x + y - 1)$$

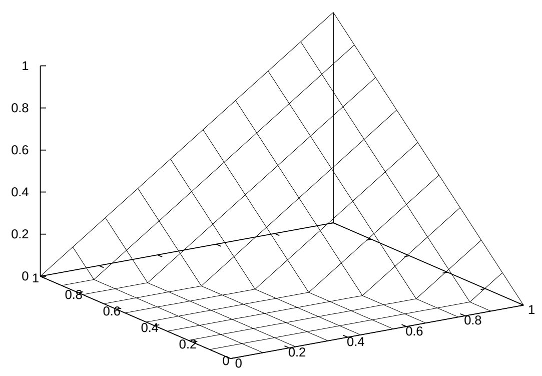


Figure 2.1: Łukasiewicz t-norm.

(Gödel t-norm) $$x *_{\mathrm{G}} y = \min(x, y)$$

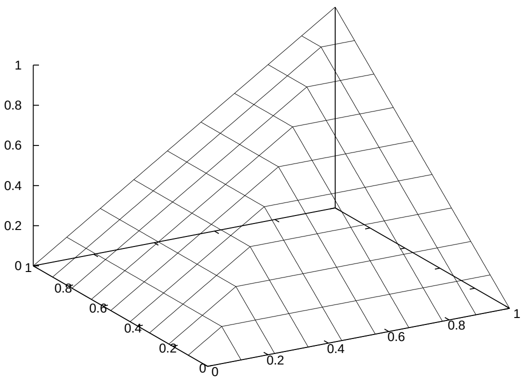


Figure 2.2: Gödel t-norm.

(Product t-norm) $$x *_{\Pi} y = x \cdot y$$

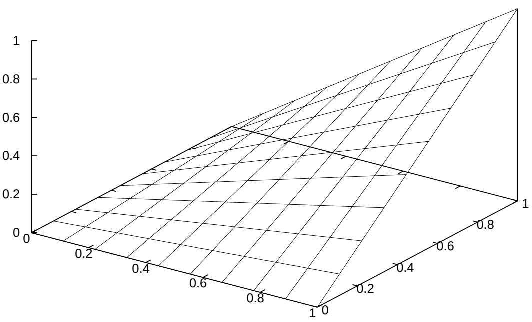


Figure 2.3: Product t-norm.

These three t-norms are very important, since any other continuous t-norm is constructed using them:

Remark 2.1.1 ([Háj98b, remark 2.1.15]). *Observe that, for each continuous t-norm, the set E of all its idempotents (i.e. the elements such that $x * x = x$) is a closed subset of $[0, 1]$ and hence its complement is a union of a set $\mathcal{I}_{OPEN}(E)$ of countably many non-overlapping open intervals. Let $[a, b] \in \mathcal{I}(E)$ iff $(a, b) \in \mathcal{I}_{OPEN}(E)$ (the corresponding closed intervals, contact intervals of E). For $I \in \mathcal{I}(E)$ let $(*|I)$ be the restriction of $*$ to I^2.*

The following theorem characterizes all continuous t-norms.

Theorem 2.1.1 ([Háj98b, theorem 2.1.16]). *If $*, E, \mathcal{I}(E)$ are as above, then*

1. *for each $I \in \mathcal{I}(E), (*|I)$ is isomorphic either to product t-norm or to Lukasiewicz t-norm.*
2. *if $x, y \in [0, 1]$ are such that there is no $I \in \mathcal{I}(E)$ with $x, y \in I$, then $x * y = \min(x, y)$ (i.e. it is isomorphic to Gödel t-norm).*

2.2 Non continuous t-norms

Not all the t-norms are continuous: an example of left-continuous (but non continuous) t-norm, presented in [Fod95], is the following

$$\text{(Nilpotent Minimum t-norm)} \qquad x *_{\mathrm{NM}} y = \begin{cases} 0 & \text{if } x \leq 1 - y \\ \min(x, y) & \text{otherwise.} \end{cases}$$

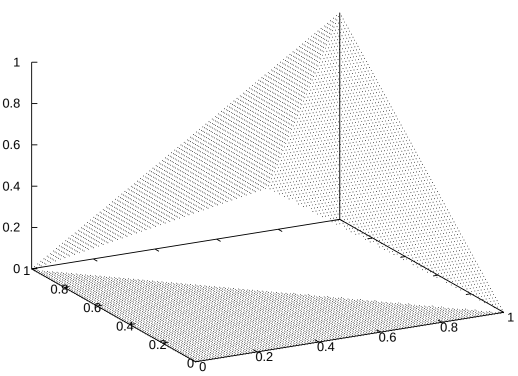


Figure 2.4: Nilpotent Minimum t-norm.

Definition 2.2.1. *Let $*, *'$ be two t-norms: we say that $*$ is smaller than $*'$ (written $* < *'$) if, for every $x, y \in [0, 1]$ we have that $x * y < x *' y$.*

Another example, that represents the "smallest" t-norm that can be constructed is

(Drastic Product t-norm) $$x *_{\mathrm{D}} y = \begin{cases} 0 & \text{if } x, y \in [0,1)^2 \\ \min(x,y) & \text{otherwise.} \end{cases}$$

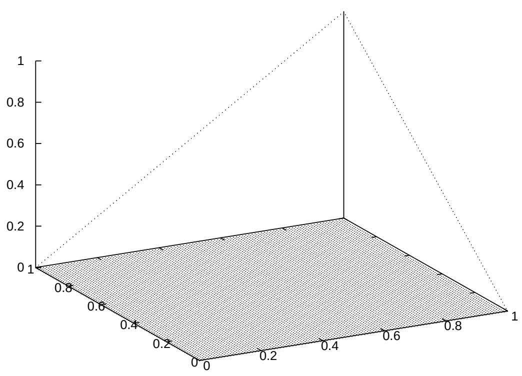


Figure 2.5: Drastic Product t-norm.

More in general, we have

Lemma 2.2.1 ([SS60, SS63])**.** *Let $*$ be a t-norm. Then $*_D \leq * \leq *_G$.*

A particular class of discontinuous t-norms in which we will be particularly interested is given by *left-continuous* t-norms.

Definition 2.2.2. *A t-norm $*$ is called lower semicontinuous if for each point $\langle x_0, y_0 \rangle \in [0,1]^2$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$x * y > x_0 * y_0 - \varepsilon \quad \textit{whenever} \quad \langle x, y \rangle \in (x_0 - \delta, x_0) \times (y_0 - \delta, y_0).$$

Proposition 2.2.1 ([KMP00, proposition 1.22])**.** *A t-norm $*$ is lower semicontinuous if and only if it is* left-continuous *in each component, i.e. for all $x_0, y_0 \in [0,1]$ and for all sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ we have*

$$\sup\{x_n * y_0\} = \sup\{x_n | n \in \mathbb{N}\} * y_0,$$
$$\sup\{x_0 * y_n\} = x_0 * \sup\{y_n | n \in \mathbb{N}\}.$$

Note that $*_{NM}$ is left continuous, whilst $*_D$ do not. However, it is possible (as pointed out in [Jen02]; se also [Wan07]) to modify $*_D$ to obtain a family of left-continuous t-norm:

$$\text{(Revised drastic Product t-norm)} \quad x *^a_{\text{RDP}} y = \begin{cases} 0 & \text{if } x, y \in [0, a)^2 \\ \min(x, y) & \text{otherwise.} \end{cases}$$

With $a \in (0, 1)$.

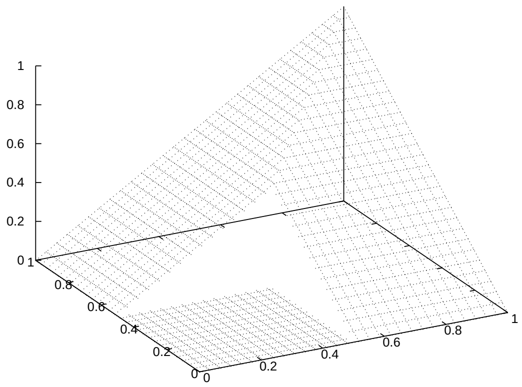


Figure 2.6: Revised Drastic Product t-norm: $a = \frac{1}{2}$.

2.3 Residuum

In the beginning of the chapter we claimed that t-norms will be useful to interpret semantically the conjunction, in many-valued logics. Another important connective, in a logic, is certainly the implication: the residuum associated to a t-norm furnishes an interpretation for it.

Definition 2.3.1. *Let $*$ be a t-norm. For every $x, y, z \in [0, 1]$ we define its* residuum *as an operation $\Rightarrow$ that satisfies*

$$\text{(R)} \qquad z * x \leq y \quad \textit{iff} \quad z \leq x \Rightarrow y.$$

A question that immediately rises is "if and when" this residuum there exists, for a t-norm.
An easy check shows that if this operation there exists, then it is unique. The rest of the answer is

Proposition 2.3.1 ([Boi98, BEG99])**.** *Let $*$ be a t-norm. The pair $\langle *, \Rightarrow \rangle$ satisfies the residuation condition (R) if and only if $*$ is left-continuous.*

Remark 2.3.1. *It is easy to check that for every left-continuous t-norm* $*$*, the only operation* $\Rightarrow$ *that satisfies (R) is*

$$x \Rightarrow y = \max\{z : z * x \le y\}.$$

Note that, over $\{0,1\}$*,* $\Rightarrow$ *behaves like the "classical" implication. Moreover, an easy check shows that* $\Rightarrow$ *is non-increasing in the first argument and non-decreasing in the second one: this is adequate for the semantics of an implication connective.*

It follows that the left-continuity plays a central role for the existence of the residuum: this is one of the reasons that render interesting the study of the class of left-continuous t-norms.
Note that, for example, $*_D$ does not admit residuum.
We now list the residuum associated to the five left-continuous t-norms previously introduced

(Łukasiewicz residuum) $$x \Rightarrow_{\text{L}} y = \min(1, 1 - x + y)$$

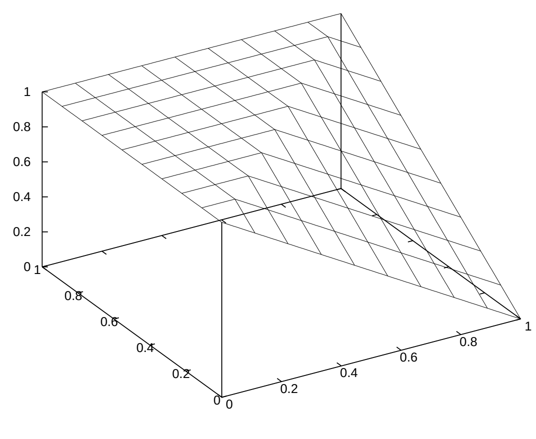


Figure 2.7: Łukasiewicz residuum.

(Gödel residuum) $$x \Rightarrow_{\text{G}} y = \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise.} \end{cases}$$

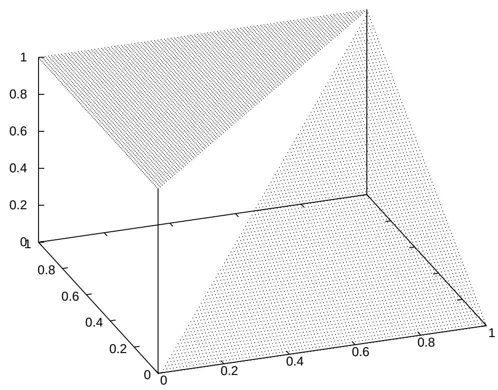


Figure 2.8: Gödel residuum.

(Product residuum) $$x \Rightarrow_{\Pi} y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

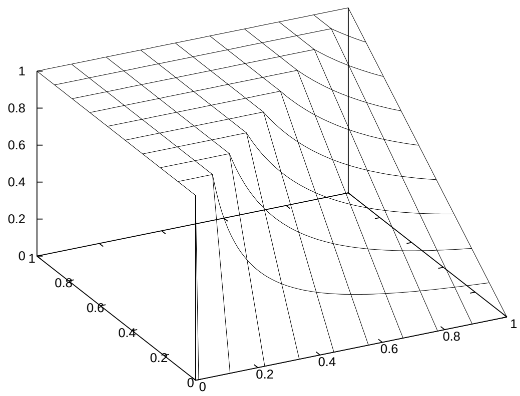


Figure 2.9: Product residuum.

(Nilpotent Minimum residuum) $$x \Rightarrow_{NM} y = \begin{cases} 1 & \text{if } x \leq y \\ \max(1-x, y) & \text{otherwise.} \end{cases}$$

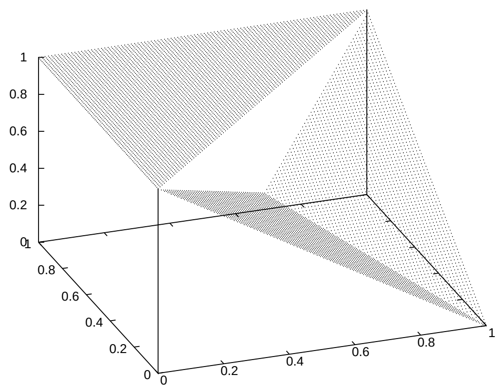


Figure 2.10: Nilpotent Minimum residuum.

(RDP residuum) $$x \Rightarrow^{a}_{\text{RDP}} y = \begin{cases} 1 & \text{if } x \leq y \\ a & \text{if } y < x \leq a \\ y & \text{if } x > a,\ x > y. \end{cases}$$

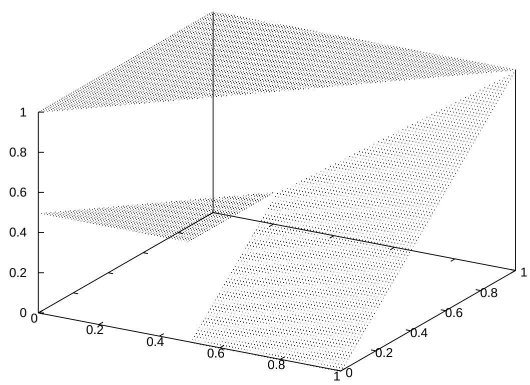


Figure 2.11: RDP residuum, $a = \frac{1}{2}$.

Note that the only residuum continuous is the one associated to Łukasiewicz's t-norm.

An important result, concerning the residuum of continuous t-norms is

Theorem 2.3.1 (see for example [BEG99]). *Let $*$ be a left continuous t-norm. The pair $\langle *, \Rightarrow\rangle$ satisfies the condition*

$$\text{(div)} \qquad x * (x \Rightarrow y) = min(x, y),$$

for all $x, y \in [0, 1]$, if and only if the t-norm $*$ is continuous.

This equation will be in strict connection with the axiomatization of Basic Logic (BL), introduced chapter 3.

Given a left-continuous t-norm $*$, we can define a negation, $\sim_*$, by pseudo-complementation: $\sim_* x := x \Rightarrow 0$, for every $x \in [0, 1]$. An easy check shows that $\sim_*$ is an order-reversing mapping (i.e. if $x \leq y$, then $\sim_* x \geq \sim_* y$, for every $x, y \in [0, 1]$) with $\sim_* 1 = 0$ and $\sim_* 0 = 1$. As regards to the previous examples of left-continuous t-norms, we have:

(Łukasiewicz negation) $\qquad \sim_{\text{Ł}} x = 1 - x$

(Gödel negation) $\qquad \sim_{\text{G}} x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$

(Product negation) $\qquad \sim_{\Pi} x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$

(Nilpotent Minimum negation) $\qquad \sim_{\text{NM}} x = 1 - x$

(RDP negation) $\qquad \sim^a_{\text{RDP}} x = \begin{cases} 1 & \text{if } x = 0 \\ a & \text{if } 0 < x \leq a \\ 0 & \text{if } x > a \end{cases}$

As can be seen the only continuous negation functions are the ones associated to Łukasiewicz and NM t-norms: moreover they are involutive, i.e. $\sim_* (\sim_* x) = x$, for every $x \in [0, 1]$ and with $* \in \{\text{Ł, NM}\}$. Instead, $\sim_{\text{G}}, \sim_{\Pi}$ are discontinuous and are the "the smallest" negations that can be costructed by pseudocomplementation.

Chapter 3

Monoidal t-norm based logic and some axiomatic extensions

3.1 Monoidal t-norm based logic

Monoidal t-norm based logic was introduced in [EG01], essentially to cope with the tautologies of left-continuous t-norms and their residua (see chapters 2 and 4). Initially this logic was called Quasi Basic Logic (QBL, see [EG99]) since it can be seen as a weakening of Basic Logic BL (we will define it in section 3.2). The name MTL takes inspiration from the fact that it is an extension of Hohle's monoidal logic ([Höh95])[1]: it is obtained from this last one by adding the prelinearity axiom, i.e. $(\varphi \to \psi) \vee (\psi \to \varphi)$.

MTL is based over connectives $\{\&, \wedge, \to, \bot\}$ (the first three are binary, whilst the last one is 0-ary). The notion of formula is defined inductively starting from the fact that all propositional variables (we will denote their set with VAR) and $\bot$ are formulas. The set of all formulas will be called $FORM$.

Useful derived connectives are the following

$$
\begin{array}{lr}
\text{(negation)} & \neg\varphi := \varphi \to \bot \\
\text{(disjunction)} & \varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\
\text{(top)} & \top := \neg\bot \\
\text{(equivalence)} & \varphi \leftrightarrow \psi := (\varphi \to \psi)\&(\psi \to \varphi)
\end{array}
$$

With the notation φ^n we will indicate $\varphi\underbrace{\&\dots\&}_{n \text{ times}}\varphi$.

[1]This logic is called FL_{ew}, in [GJKO07].

MTL is axiomatized as follows:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$(\varphi \& \psi) \to \varphi$
(A3)	$(\varphi \& \psi) \to (\psi \& \varphi)$
(A4)	$(\varphi \wedge \psi) \to \varphi$
(A5)	$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(A6)	$(\varphi \& (\varphi \to \psi)) \to (\psi \wedge \varphi)$
(A7a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A7b)	$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\bot \to \varphi$

Some comments about the axioms

- (A1) indicates that the implication is transitive.
- (A2) says that the conjunction of two formulas implies one of them.
- (A3) says that & is commutative.
- (A4) and (A5) are the analogous of (A2) and (A3) for $\wedge$.
- (A6) indicates the relation between $\wedge$ and &.
- (A7a) and (A7b) are the syntactical version of the semantical property of residuation, that we have already seen in the context of t-norms and that will be treated, in the next chapter, in section 4.1.
- (A8) is a sort of "proof by cases".
- (A9) is the "ex falso quodlibet" axiom: from a contradiction, we can derive everything.

As inference rule we have modus ponens:

(MP) $$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

A proof of a formula φ is a finite sequence $\{\varphi_1, \dots, \varphi_n = \varphi\}$ of formulas where every φ_i is an axiom or it is obtained from preceding elements of the sequence with modus ponens.

A theory is a set of formulas. Given a theory T and a formula φ, the notion $T \vdash \varphi$ means that φ is provable from the axioms of the logic and the formulas of T.

Another interesting inference rule is given by modus tollens

(MT) $$\frac{\neg\psi \quad \varphi \to \psi}{\neg\varphi}$$

However this rule is derivable in MTL, in fact

Lemma 3.1.1.
$$\{\neg\psi, \varphi \to \psi\} \vdash_{MTL} \neg\varphi.$$

Proof. An easy consequence of the fact that (see [EG01, proposition 1])

$$\vdash_{MTL} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi).$$

□

Concerning the deduction theorem, the "classical" form does not hold, for MTL. However it holds the following local form

Theorem 3.1.1 ([EG99, EG01]). *Let T, φ, ψ be a theory and two formulas. It holds that*

$$T \cup \{\psi\} \vdash_{MTL} \varphi \quad \textit{iff there exists } n \in \mathbb{N}^+ \textit{ s.t.} \quad T \vdash_{MTL} \psi^n \to \varphi.$$

The term *local* is due to the fact that the value of n depends from the formulas.
Now, if we define

Definition 3.1.1. *A logic L is called axiomatic extension of MTL if it is obtained (from the axiomatic point of view) by adding other axioms to (A1)-(A9).*

The previous local deduction theorem can be extended to every axiomatic extension of MTL.

Theorem 3.1.2 ([Cin04]). *Let L be an axiomatic extension of MTL and T, φ, ψ be a theory and two formulas. It holds that*

$$T \cup \{\psi\} \vdash_{L} \varphi \quad \textit{iff there exists } n \in \mathbb{N}^+ \textit{ s.t.} \quad T \vdash_{L} \psi^n \to \varphi.$$

3.2 Basic Logic and its extensions

Between the axiomatic extensions of MTL, one of the most important is certainly the Basic Logic (BL). Historically it was introduced to create a logical calculus that was complete with respect to a class of algebras determined (in a sense that will be explained later) by continuous t-norms and their residua. This result was firstly conjectured in the monograph [Háj98b] and leaved as open problem in [Háj98a]: the (positive) answer was finally given in [CEGT00].

From the axiomatic point of view, BL is obtained - as pointed out in [EG99, EG01] by adding to (A1)-(A9) the axiom

$$(\psi \wedge \varphi) \to (\varphi \& (\varphi \to \psi)),$$

that is, the other side of the axiom A6. This means that

$$\vdash_{BL} (\psi \wedge \varphi) \leftrightarrow (\varphi \& (\varphi \to \psi)).$$

This axiom is called *divisibility*. As can be seen, the connective $\wedge$ is definable in term of $\{\&, \to\}$: in fact, BL was originally defined over the connectives $\{\&, \to, \bot\}$ with the following axioms[2]:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A2) $(\varphi \& \psi) \to \varphi$

(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$

(A4) $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$

(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$

(A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

(A7) $\bot \to \varphi.$

The inference rule and the local deduction theorem are the same of MTL (and clearly the modus tollens is a derivable rule).

3.2.1 Some axiomatic extensions of BL

We will analyze axiomatic extensions of Basic Logic: n-contractive logics, Łukasiewicz logic, finite valued Łukasiewicz logics, Gödel and Product logic.

n-contractive BL-logics

A family of axiomatic extensions of BL is given by the logics that satisfy the axiom

(C_n) $\varphi^n \to \varphi^{n+1}$

for some $n \in \mathbb{N}$, $n > 0$.

This formula is said n-contraction (or n-potence): this axiom, coming from the wider framework of substructural logics, was introduced in the paper [CEG08] and systematically studied for wide varieties of MTL-algebras in [HNP07].

A deep analysis of n-contractive axiomatic extensions of BL will be done in chapter 8.

[2]However, as pointed out in [Cin05], this set of axioms is redundant: in fact, the axiom (A3) can be derived from the others.

Łukasiewicz logic

Łukasiewicz infinite valued logic (Ł) is one of the earliest studied many-valued logics: it was introduced, by the Polish logician Jan Łukasiewicz, for the first time in (for reader's convenience we cite an English translation) [Bor70, Investigations into the sentential calculus].
The original axiomatization was

(Ł1) $\varphi \to (\psi \to \varphi)$

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

As can be noted, the only connectives used are $\{\neg, \to\}$.

In [Háj98b] it is proved that it can be equivalently axiomatized, from BL, by adding

(involution) $\neg\neg\varphi \to \varphi.$

Finite valued Łukasiewicz logics

Finite valued Łukasiewicz logics were introduced in the thirties by Jan Łukasiewicz ([Bor70, Investigations into the sentential calculus]). Here we will present an axiomatization similar[3] to the one introduced in [Gri77]: in chapter 8 we will describe an alternative one. For $n \in \mathbb{N}$, $n > 1$ with $Ł_n$ we mean the logic obtained from BL, by adding

(C_n) $\varphi^n \to \varphi^{n+1}$

and, if $n > 2$

$(ndiv)$ $n(\varphi^j \veebar (\neg\varphi \& \neg\varphi^{j-1}))$

where

$$\varphi \veebar \psi := \neg(\neg\varphi \& \neg\psi),$$

$1 < j < n$ and j does not divide n. Finally, by $n\varphi$, we mean $\varphi \underbrace{\veebar \cdots \veebar}_{n \text{ times}} \varphi$.

[3] In [Gri77] the n-contraction is defined in a different way, from our: we have modified consequently Grigolia's axioms.

Gödel logic

Gödel logic (G) was introduced in 1932 in [Göd01] and formally axiomatized in [Dum59] in 1959[4]. As shown in [Háj98b] it can be equivalently axiomatized as BL with

(idempotency) $$\varphi \to (\varphi \& \varphi).$$

Moreover in [Háj02, lemma 1] it is proved that if we add the previous axiom to MTL then we obtain again Gödel logic.

Moreover G can be also obtained from intuitionistic logic by adding

(prelinearity) $$(\varphi \to \psi) \vee (\psi \to \varphi).$$

Finally, as pointed out in [Háj98b], G satisfy the classical deduction theorem, i.e.

Theorem 3.2.1 ([Háj98b, theorem 4.2.10])**.** *Let T, φ, ψ be a theory and two formulas. It holds that*

$$T \cup \{\psi\} \vdash_G \varphi \quad \textit{iff} \quad T \vdash_G \psi \to \varphi.$$

Product logic

Product logic (Π) was introduced in [EGH96] as an axiomatic extension of BL with (see also the presentation given in [Háj98b, chapter 4])

(Π1) $$\neg\neg\chi \to (((\varphi \& \chi) \to (\psi \& \chi)) \to (\varphi \to \psi))$$

(Π2) $$\neg(\varphi \wedge \neg\varphi)$$

More recently C. Noguera proved in [Nog06] that these axioms can be substituted with the (in my opinion, more intuitive) axiom

(precancellativity) $$\neg\varphi \vee ((\varphi \to (\varphi \& \psi)) \to \psi)$$

However, the original axiomatization keep an advantage with respect to the last one. In fact, if we add (Π2) to BL we obtain another logic, named strict basic logic (SBL) and it is immediate to see that Π extends SBL.

3.3 Some proper axiomatic extensions of MTL

With "proper" axiomatic extension of MTL we mean a logic that is not an extension of BL.

[4]To be precise Gödel limited himself to present a finite totally ordered Heyting algebra with its operations. In fact, the aim of the paper [Göd01] was only to show that intuitionistic logic cannot be seen as a many-valued logic with a finite number of truth values. Various authors attribute this logic (calling it Gödel-Dummett or LC) - more correctly - to Dummett. However, for brevity and to maintain the terminology of [Háj98b] (and of many other papers), we will continue to call this logic G.

3.3.1 Involutive monoidal t-norm logic

Involutive Monoidal t-norm logic, IMTL, was introduced in [EG01] and it is obtained, from MTL, by adding

$$\neg\neg\varphi \to \varphi$$

As can be seen IMTL is a "weakening" of Ł, obtained by removing the divisibility axiom.

One can ask if IMTL and Ł are equivalent. The (negative) answer was given in [EG01]: in fact the formula (Ł4) is not a theorem of IMTL.

3.3.2 Product monoidal t-norm basic logic

The logic ΠMTL was introduced in [Háj02] as an axiomatic extension of MTL with

(Π1) $\neg\neg\chi \to (((\varphi\&\chi) \to (\psi\&\chi)) \to (\varphi \to \psi))$

(Π2) $\neg(\varphi \wedge \neg\varphi)$

Analogously to what happens for product logic, these axioms can be replaced by

(precancellativity) $\neg\varphi \vee ((\varphi \to (\varphi\&\psi)) \to \psi)$

3.3.3 Strict monoidal t-norm based logic

Strict monoidal t-norm based logic, SMTL ([EGGM02]), is obtained from MTL by adding the axiom

(Π2) $\neg(\varphi \wedge \neg\varphi)$

As can be noted this is a weakening of ΠMTL, obtained by removing the axiom Π1.

3.3.4 Weak cancellative monoidal t-norm based logic

WCMTL ([MNH06]) is obtained from MTL by adding the axiom

(wcmtl) $\neg(\varphi\&\psi) \vee ((\varphi \to (\varphi\&\psi)) \to \psi)$

Note that this axiom is a weakening of precancallativity: for this reason the logic has taken that name.

3.3.5 Nilpotent Minimum Logic

Nilpotent Minimum logic was introduced in [EG01] to create a logical calculus that was complete with respect to the algebraic structure determined by Nilpotent Minimum t-norm.

From the axiomatic point of view it is obtained, from MTL, by adding

(wnm)	$\neg(\varphi\&\psi) \vee ((\varphi \wedge \psi) \to (\varphi\&\psi))$
(involution)	$\neg\neg\varphi \to \varphi$

Moreover, as showed in [EGCN03] it is possible to axiomatize NM by using only the connectives $\{\to, \bot\}$. In this thesis we will maintain the axiomatization of [EG01], but this alternative axiomatization can be useful, for example, to simplify an eventual proof by structural induction: in fact we can restrict to $\to$ and $\bot$, as induction steps. Concerning the deduction theorem, for Nilpotent Minimum logic we obtain the following form (compare this theorem with theorem 3.1.2)

Theorem 3.3.1 ([ABM07, proposition 4.3])**.** *Let T, φ, ψ be a theory and two formulas. It holds that*

$$T \cup \{\psi\} \vdash_{NM} \varphi \quad \textit{iff} \quad T \vdash_{NM} \psi^2 \to \varphi.$$

Moreover, by using theorem 3.1.2 it is easy to show that theorem 3.3.1 holds in every 2-contractive axiomatic extension L of MTL (i.e. $\vdash_L \varphi^2 \to \varphi^3$).

Other logical properties of NM will be studied in chapter 10.

3.3.6 Weak nilpotent minimum logic

Weak Nilpotent Minimum logic (WNM) was introduced in [EG01] as a weakening of Nilpotent Minimum logic. In fact it is obtained, from NM, by removing the involution axiom, i.e. it is obtained from MTL, by adding

(wnm)	$\neg(\varphi\&\psi) \vee ((\varphi \wedge \psi) \to (\varphi\&\psi))$

Moreover, as previously noticed, theorem 3.3.1 holds also for WNM.

3.3.7 Revised drastic product

RDP logic was introduced, in [Wan07] to find a logical calculi complete with respect to (the standard MTL-algebra induced by) Revised drastic product t-norm. It is obtained, from MTL, by adding

(rdp)	$\neg\neg\varphi \vee (\varphi \to \neg\varphi)$

Chapter 4

Algebraic semantics, residuated lattices and related structures

In this chapter we will introduce various algebraic structures that will be useful to represent the semantics of the logics previously introduced. It is important to remark that many of these algebras were initially introduced and studied for different reasons from the logical ones.

4.1 Residuated lattices and algebraic semantics

Definition 4.1.1. *A residuated lattice is an algebraic structure of the form* $\mathcal{A} = \langle A, *, \Rightarrow, \sqcap, \sqcup, 0, 1\rangle$ *such that*

- $\langle A, \sqcap, \sqcup, 0, 1\rangle$ *is a bounded lattice, where* 0 *is the bottom and* 1 *the top element.*
- $\langle A, *, 1\rangle$ *is a commutative monoid.*
- $\langle *, \Rightarrow\rangle$ *forms a residuated pair, i.e.*

$$z * x \leq y \quad \textit{iff} \quad z \leq x \Rightarrow y.$$

Remark 4.1.1. *An easy check shows that, in every residuated lattice*

$$x \Rightarrow y = \max\{z : z * x \leq y\}.$$

Compare this with remark 2.3.1. Note also that, in every totally ordered residuated lattice, we have $\sqcap = \min$ *and* $\sqcup = \max$.

We now begin to introduce various classes of residuated lattices that will represent the semantics of the logic presented in previous chapter

4.1.1 MTL-algebras

Definition 4.1.2. *An MTL algebra is a residuated lattice that satisfies*

(pl) $$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$$

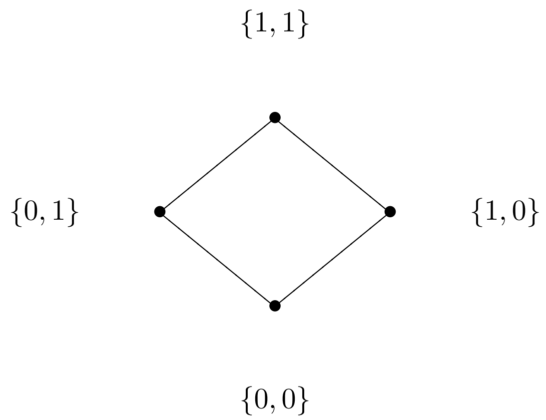
An MTL-algebra is said to be standard *if its lattice reduct is* $\langle [0,1], \min, \max \rangle$.

It is easy to check that a left-continuous t-norm $*$ with its residuum $\Rightarrow$ induces naturally a standard MTL-algebra $\langle [0,1], *, \Rightarrow, \min, \max, 0, 1 \rangle$. Moreover

Proposition 4.1.1 ([JM02])**.** *The class of standard MTL-algebras coincides with the one of MTL-algebras induced by left-continuous t-norms and their residua.*

Note that the equation (pl) holds true in every totally ordered residuated lattice $\mathcal{A}$: in fact $\max(x \Rightarrow y, y \Rightarrow x) = 1$, for every $x, y \in A$.

However (pl) does not imply the linearity: for example the poset $\langle \{0,1\} \times \{0,1\}, \leq \rangle$, where $\leq$ is the lexicographical order, i.e. $\langle 0,0 \rangle < \langle 0,1 \rangle, \langle 1,0 \rangle < \langle 1,1 \rangle$



can be extended to a residuated lattice by setting the operations pointwise (recall that the behavior of $\{*, \Rightarrow, \sqcap \sqcup\}$ is fixed, over 0 and 1).

Other useful properties of MTL algebras are

Lemma 4.1.1 ([Háj98b, 2.3.4])**.** *In every MTL algebra the following hold in each* x, y, z

(4.1) $x * (x \Rightarrow y) \leq y$ *and* $x \leq (y \Rightarrow (x * y))$

(4.2) $x \leq y$ *implies* $x * z \leq y * z, (z \Rightarrow x) \leq (z \Rightarrow y), (y \Rightarrow z) \leq (x \Rightarrow z)$

(4.3) $x \leq y$ *iff* $x \Rightarrow y = 1$

(4.4) $(x \sqcup y) * z = (x * z) \sqcup (y * z)$

(4.5) $x \sqcup y = ((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x)$.

Compare (4.5) with the definition of $\vee$, for MTL logic.

Remark 4.1.2. *Thanks to properties (4.1), (4.2) and the fact that* $*$ *is a monoidal operation with top element* 1*, it follows that* $\{*, \Rightarrow\}$ *are good candidates for the role of the semantics of* $\{\&, \rightarrow\}$.

Finally, given an MTL-algebra and one of its elements, say x, with the notation $\sim x$ we will indicate $x \Rightarrow 0$.

4.2 MTL-algebras as semantics of axiomatic extensions of MTL

MTL-algebras can be used as a semantics for the logics introduced in chapter 3.

Definition 4.2.1. *Let* $\mathcal{A}$ *be an MTL-algebra. An* $\mathcal{A}$*-evaluation over variables is a function* $v : VAR \rightarrow A$.

Every $\mathcal{A}$*-evaluation over variables* v *can be extended, uniquely, to an* $\mathcal{A}$*-evaluation over MTL-formulas* $v : FORM \rightarrow A$ *in the following (inductive) way.*

Let φ, ψ *be MTL-formulas, then*

$$\begin{aligned} v(\bot) &= 0 \\ v(\varphi \wedge \psi) &= v(\varphi) \sqcap v(\psi) \\ v(\varphi \& \psi) &= v(\varphi) * v(\psi) \\ v(\varphi \rightarrow \psi) &= v(\varphi) \Rightarrow v(\psi). \end{aligned}$$

We now introduce the concepts of satisfiability and model

Definition 4.2.2. *Let* $\mathcal{A}$ *be an MTL-algebra,* φ *be a formula and* T *a theory in the language of MTL.*

- *We say that* φ *is* satisfiable *if there is an* $\mathcal{A}$*-evaluation* v *such that* $v(\varphi) = 1$.
- *We say that* φ *is a* tautology *of* $\mathcal{A}$, $\mathcal{A} \models \varphi$*, if* $v(\varphi) = 1$ *for every* $\mathcal{A}$*-evaluation. Another terminology is that* $\mathcal{A}$ *is a* model *for* φ*. If* K *is a class of MTL-algebras, with* $K \models \varphi$ *we mean that* $\mathcal{A} \models \varphi$ *for every* $\mathcal{A} \in K$.
- *With* $\mathcal{A} \models T$ *we mean that* $\mathcal{A} \models \psi$ *for every* $\psi \in T$*: the same notion is extended to a class of MTL-algebras* K *in the obvious way.*
- *With* $T \models_{\mathcal{A}} \varphi$ *we mean that, for every* $\mathcal{A}$*-evaluation* v*, if* $v(\psi) = 1$ *for every* $\psi \in T$*, then* $v(\varphi) = 1$*: the same notion is extended to a class of MTL-algebras* K *in the obvious way.*

4.3 Some classes of MTL-algebras

Let L be an axiomatic extension of MTL: an MTL-algebra $\mathcal{A}$ is said to be an L-algebra if all the axioms of L are tautologies for $\mathcal{A}$.

In this section we will present various classes of MTL algebras, associated to the logics introduced in chapter 3.

4.3.1 BL-algebras

A particular subclass of MTL algebras is the one of BL algebras. From the chronological point of view, however, BL-algebras were introduced first, in the monograph [Háj98b].

Definition 4.3.1. *A BL algebra is an MTL algebra satisfying the equation*

(div) $$x \sqcap y = x * (x \Rightarrow y)$$

For every x, y.

Compare the previous equation with the divisibility axiom, in section 3.2. Finally, concerning standard BL-algebras

Proposition 4.3.1 ([CEGT00])**.** *The class of standard BL-algebras coincides with the one of BL-algebras induced by continuous t-norms and their residua.*

4.3.2 MV-algebras

MV-algebras were initially introduced by Chang in [Cha58] as a semantics for Łukasiewicz infinite valued logic[1]. More recently in [FRT84] another class of algebras, complete with respect to Ł, was presented: Wajsberg algebras. In the same paper was showed the equivalence between MV and Wajsberg algebras (their operations are inter-definable). In this chapter we will follow the style of [Háj98b], by presenting MV-algebras as a class of BL-algebras. For an historical overview, see [Cig07]; a reference monograph for this topic is [CDM99].

Definition 4.3.2. *An MV-algebra is a BL-algebra satisfying the following equation*

(inv) $$\sim\sim x = x$$

For every x.

[1] It is curious to note that, as Professor Chang tells by himself in [Cha98], the origin of MV-algebras is due to his difficulty to follow a presentation by Professor J. Barkley Rosser, who used formulas in polish notation, and the consequent idea to search for a simpler proof.

It is well known that, up to isomorphisms, there is only one standard MV-algebra:

$$\langle [0,1], *, \Rightarrow, \min, \max, 0, 1 \rangle ,$$

where $*, \Rightarrow$ are Łukasiewicz's t-norm and residuum.

4.3.3 MV$_n$-algebras

In this section we will briefly recall some results about MV_n-algebras: our main references will be [Gri77] and [CDM99]. These structures represent the semantics for the logics Ł$_n$. If we define

$$x \oplus y :=\sim (\sim x * \sim y)$$

and, with nx, we mean $x \underbrace{\oplus \cdots \oplus}_{n \text{ times}} x$, then

Definition 4.3.3. *An MV_n-algebra is an MV-algebra satisfying*

(c_n) $$x^n = x^{n+1}$$

and, if $n > 2$

(Ndiv) $$n(x^j \oplus (\sim x * \sim x^{j-1})) = 1$$

Where $1 < j < n$ and j does not divide n.

Consider the following algebra:

$$\mathbf{L_n} = \left\langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, *, \Rightarrow, \sqcap, \sqcup, 0, 1 \right\rangle$$

Where $1 < n < \omega$, $* := max(0, x+y-1)$, $\Rightarrow := min(1, 1-x+y)$, $\sqcap := min(x,y)$, $\sqcup := max(x,y)$.

We have that:

Proposition 4.3.2 ([Gri77])**.** $\mathbf{L_n}$ *is an MV_n-algebra.*

Theorem 4.3.1 ([Gri77])**.** *Given a formula φ (in the language of L_n), it holds that:*

$$\vdash_{L_n} \varphi \quad \textit{iff} \quad \models_{\mathbf{L_n}} \varphi$$

Corollary 4.3.1. *For $1 < n < \omega$ it holds that each MV_n-chain has at most $n+1$ elements.*

Proposition 4.3.3 ([Gri77])**.** *Given $1 < m, n < \omega$ it holds that:*

- *If m divides n, then $\mathbf{L_m}$ is isomorphic to a subalgebra of $\mathbf{L_n}$.*

- $\mathbf{L_m}$ *is an* MV_n*-algebra if and only if* m *divides* n.

Theorem 4.3.2 ([Gri77]). *Any* MV_n*-algebra* $\mathcal{A}$ *is isomorphic to a subdirect product of the algebras* $\mathbf{L_m}$*, where* $m \leq n$ *and* m *divides* n.

Corollary 4.3.2. *Each subdirectly irreducible* MV_n*-algebra* $\mathcal{A}$ *is isomorphic to some* $\mathbf{L_m}$*, where* m *divides* n.

Theorem 4.3.3 ([Gri77]). *Any finite* MV_n*-algebra* $\mathcal{A}$ *is isomorphic to the direct product of the algebras* $\mathbf{L_m}$*, where* $m \leq n$ *and* m *divides* n.

Corollary 4.3.3. *Each* MV_n*-chain* $\mathcal{A}$ *is isomorphic to an algebra* $\mathbf{L_m}$*, where* $m \leq n$ *and* m *divides* n.

A general result of universal algebra in the following: for a proof see, for example, [MMT87, theorem 4.99] or [BS81, theorem 10.16]. Remember that a variety is locally finite if every algebra of its that is finitely generated is finite.

Theorem 4.3.4. *Every variety generated by a finite set of finite algebras is locally finite.*

Corollary 4.3.4. *The variety of* MV_n*-algebras is locally finite.*

4.3.4 Gödel-algebras

Gödel algebras can be seen either as Heyting algebras satisfying (pl) or

Definition 4.3.4. *A G-algebra is a BL-algebra satisfying the following equation*

(id) $$x * x = x$$

For every x.

Moreover, in [Háj02] it is showed that the class of MTL-algebras satisfying (id) coincides with the one of G-algebras.
Also in this case, there is only one standard algebra: the one induced by Gödel t-norm (and its residuum).

4.3.5 Product-algebras

Product algebras were introduced in [EGH96], as a class of BL-algebras.

Definition 4.3.5. *A Product-algebra is a BL-algebra satisfying the following equations*

(p1) $$(\sim\sim z) \Rightarrow (((x * z) \Rightarrow (y * z)) \Rightarrow (x \Rightarrow y)) = 1$$

(p2) $$\sim (x \sqcap \sim x) = 1$$

For every x.

As pointed out in [Nog06], the previous two equations can be substituted by

(canc) $$(\sim x) \sqcup ((x \Rightarrow (x * y)) \Rightarrow y)$$

Analogously to what happens for the previous two classes of algebras, the only (up to isomorphisms) standard product algebra is induced by the homonymous t-norm.

4.3.6 Nilpotent Minimum algebras

In chapter 2 we saw that Nilpotent Minimum t-norm was the first example of left-continuous t-norm.
In [EG01] it is defined a class of algebras, named NM-algebras.

Definition 4.3.6. *An NM-algebra is an MTL-algebra that satisfies*

(wnm) $$\sim (x * y) \sqcup ((x \sqcap y) \Rightarrow (x * y)) = 1$$
(inv) $$\sim\sim x = x.$$

There is a useful characterization, given in [Gis03], concerning all NM-chains. In every of them it holds that

$$x * y = \begin{cases} 0 & \text{if } x \leq n(y) \\ \min(x, y) & \text{Otherwise.} \end{cases}$$
$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max(n(x), y) & \text{Otherwise.} \end{cases}$$

Where n is a strong negation function, i.e. $n : A \to A$ is an order-reversing mapping ($x < y$ implies $n(x) > n(y)$) such that $n(0) = 1$ and $n(n(x)) = x$, for each $x \in A$. Observe that $n(x) = x \Rightarrow 0$, for each $x \in A$.

A negation fixpoint is an element $x \in A$ such that $n(x) = x$: note that if this element exists then it must be unique (otherwise n fails to be order-reversing). A positive element is an $x \in A$ such that $x > n(x)$; the definition of negative element is the dual (substitute $>$ with $<$).

Concerning the finite chains, in [Gis03] it is showed that two finite NM-chains with the same cardinality are isomorphic (see the remarks after [Gis03, Proposition 2]): for this reason we will denote them with NM_n, n being an integer greater that 1.

We now give some examples of infinite NM-chains that will be useful in the following: for all of them the order is given by $\leq_{\mathbb{R}}$ and $n(x) = 1 - x$.

- $NM_\infty = \left\langle \{\frac{1}{n} :\ n \in \mathbb{N}^+\} \cup \{1 - \frac{1}{n} :\ n \in \mathbb{N}^+\}, *, \Rightarrow, \min, \max, 0, 1 \right\rangle$
- $NM_\infty^- = \left\langle \{\{\frac{1}{n} :\ n \in \mathbb{N}^+\} \cup \{1 - \frac{1}{n} :\ n \in \mathbb{N}^+\}\} \setminus \{\frac{1}{2}\}, *, \Rightarrow, \min, \max, 0, 1 \right\rangle$

- $NM'_\infty = \left\langle \{\frac{1}{2} - \frac{1}{2n} :\ n \in \mathbb{N}^+\} \cup \{\frac{1}{2} + \frac{1}{2n} :\ n \in \mathbb{N}^+\} \cup \{\frac{1}{2}\}, *, \Rightarrow, \min, \max, 0, 1 \right\rangle$
- $NM'^-_\infty = \left\langle \{\frac{1}{2} - \frac{1}{2n} :\ n \in \mathbb{N}^+\} \cup \{\frac{1}{2} + \frac{1}{2n}\} :\ n \in \mathbb{N}^+\}, *, \Rightarrow, \min, \max, 0, 1 \right\rangle$
- $[0,1]_{NM} = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$

As we already seen in chapter 2, in this last case $*$ is called Nilpotent Minimum t-norm [Fod95]. Note that the first four chains of the list and every finite NM-chain[2] are all subalgebras of $[0,1]_{NM}$.

Another useful characterization is the following:

Theorem 4.3.5 ([Gis03, Theorem 2])**.**

1. *An NM-chain is a model of*

$$(S_n(x_0,\dots,x_n)) \qquad \bigwedge_{i<n}((x_i \to x_{i+1}) \to x_{i+1}) \to \bigvee_{i<n+1} x_i$$

if and only if it has less than $2n+2$ elements.

2. *A nontrivial NM-chain is a model of*

$$(BP(x)) \qquad \neg(\neg x^2)^2 \leftrightarrow (\neg(\neg x)^2)^2 = 1$$

if and only if it does not contain the negation fixpoint.

Proposition 4.3.4. *Let $\mathcal{A}$ be an NM-chain, v be an $\mathcal{A}$-evaluation and φ a formula with variables $x_1,\dots,x_n$.*

It holds that

$$v(\varphi) = \begin{cases} 1 & or \\ v(x_i) & or \\ n(v(x_j)) & or \\ 0 \end{cases}$$

With $i,j \in \{1,\dots,n\}$.

Proof. An easy induction over the number of variables of φ. □

4.4 Hoops and ordinal sums

Hoops were introduced and studied, as algebraic structures, in an unpublished manuscript by J. R. Buchi and T. Owens. More recently, in his Phd thesis [Fer92], the author studied various classes of hoops more in detail: the results were further expanded in [BF00].

[2] Since two finite NM-chains with the same cardinality are isomorphic, then we can consider NM_n as defined over the set $\{0, \frac{1}{n-1}, \dots, \frac{n-1}{n-1}\}$ and $n(x) = 1 - x$.

Definition 4.4.1 ([BF00]). *A* hoop *is a structure* $\mathbf{A} = \langle A, *, \Rightarrow, 1\rangle$ *such that* $\langle A, *, 1\rangle$ *is a commutative monoid, and* $\Rightarrow$ *is a binary operation such that*

$$x \Rightarrow x = 1, \quad x \Rightarrow (y \Rightarrow z) = (x*y) \Rightarrow z \quad \text{and} \quad x*(x \Rightarrow y) = y*(y \Rightarrow x).$$

In any hoop, the operation $\Rightarrow$ induces a partial order $\leq$ defined by $x \leq y$ iff $x \Rightarrow y = 1$. Moreover, hoops are precisely the partially ordered commutative integral residuated monoids (pocrims) in which the meet operation $\sqcap$ is definable by $x \sqcap y = x * (x \Rightarrow y)$. Finally, hoops satisfy the following divisibility condition:

(div) If $x \leq y$, then there is an element z such that $z * y = x$.

Definition 4.4.2 ([AFM07],[EGHM03]). *A hoop is said to be* basic *iff it satisfies the identity*

(lin) $$(x \Rightarrow y) \Rightarrow z \leq ((y \Rightarrow x) \Rightarrow z) \Rightarrow z.$$

A bounded hoop *is a hoop with an additional constant* 0 *satisfying the equation* $0 \leq x$.
A BL algebra *is a bounded basic hoop. The variety of BL-algebras will be denoted by* $\mathcal{BL}$.

The variety of basic hoops (BL-algebras respectively) is generated by the class of totally ordered hoops (BL-algebras respectively). In any basic hoop or BL-algebra, the lattice operations are definable by $x \sqcap y = x * (x \Rightarrow y)$ and $x \sqcup y = ((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x)$.

We now present a useful construction that will be necessary in the following chapters. It was initially introduced in [Fer92] and further generalized in [AM03], as follows.

Definition 4.4.3 ([AM03]). *Let* $\langle I, \leq\rangle$ *be a totally ordered set with minimum* i_0. *For all* $i \in I$, *let* $\mathbf{A}_i$ *be a hoop such that for* $i \neq j$, $A_i \cap A_j = \{1\}$, *and assume that* $\mathbf{A}_{i_0}$ *is bounded. Then* $\bigoplus_{i\in I} \mathbf{A}_i$ *(the* ordinal sum *of the family* $(\mathbf{A}_i)_{i\in I}$*) is the structure whose base set is* $\bigcup_{i\in I} A_i$*, whose bottom is the minimum of* $\mathbf{A}_{i_0}$*, whose top is* 1*, and whose operations are*

$$x \Rightarrow y = \begin{cases} x \Rightarrow^{\mathbf{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

$$x * y = \begin{cases} x *^{\mathbf{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\},\ y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\},\ x \in A_j) \end{cases}$$

When defining the ordinal sum $\bigoplus_{i\in I} \mathbf{A}_i$ we will tacitly assume that whenever the condition $A_i \cap A_j = \{1\}$ is not satisfied for all $i, j \in I$ with $i \neq j$, we will replace the $\mathbf{A}_i$ by isomorphic copies satisfying such condition. Moreover if $I = \{i_0, ..., i_n\}$ with $i_0 < i_1 < ... < i_n$, we write $\mathbf{A}_{i_0} \oplus ... \oplus \mathbf{A}_{i_n}$ instead of $\bigoplus_{i\in I} \mathbf{A}_i$.
For more informations about hoops and many-valued logics, we suggest papers [AFM07] and [EGHM03].

4.5 Completeness

Having introduced logics and algebraic structures, we can study the relations between them, in term of sets of theorems and of tautologies.

Definition 4.5.1. *Let L be an axiomatic extension of MTL and K be a class of MTL-algebras. We say that L is strongly complete (respectively: finitely strongly complete, complete) with respect to K if for every set T of formulas (respectively, for every finite set T of formulas, for $T = \emptyset$) and for every formula φ we have*

$$T \vdash_L \varphi \quad \textit{iff} \quad T \models_K \varphi.$$

A general characterization of completeness properties is

Theorem 4.5.1 ([CEG+09])**.** *Let L be an axiomatic extension of MTL and K be a class of L-algebras. Then*

- *L is strongly complete with respect to K if and only if every countable L-chain is embeddable in some member of K.*
- *L is finitely strongly complete with respect to K if and only if every countable L-chain is partially embeddable into K.*

A first general result, concerning the completeness, is the following

Theorem 4.5.2. *Let L be an axiomatic extension of MTL, T a theory and φ a formula (in the language of MTL). Then*

$$T \vdash_L \varphi \quad \textit{iff} \quad T \models_C \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi$$

Where C is the class of L-chains and $\mathbb{L}$ the one of all L-algebras.

4.5.1 Standard completeness

Given an axiomatic extension L of MTL, it can be interesting to study the completeness with respect to different classes of L-algebras (for a systematic analysis in this sense, we suggest the paper [CEG+09]). A first example is given by the class of standard L-algebras: as we have seen previously this means a class of algebras induced by particular left-continuous t-norms and residua. This type of completeness takes the name of *standard completeness.*

We begin with BL and its extensions

Theorem 4.5.3 ([CEGT00]).

- *BL is finitely strongly complete with respect to the class of standard BL-algebras. Hence BL is the logic of continuous t-norms and their residua.*
- *SBL is finitely strongly complete with respect to the class of standard SBL-algebras.*

This result cannot be improved: as showed in [EGGM02] strong (standard) completeness does not hold for both the logics.

Concerning Łukasiewicz logic

Theorem 4.5.4 (see for example [Háj98b]). *Łukasiewicz logic is finitely strongly complete with respect to* $[0,1]_Ł$.

The completeness w.r.t. $[0,1]_{\text{Ł}}$ was initially proved in [Cha59]. Analogously to BL and SBL, this theorem cannot be generalized to infinite theories: for an explicit counterexample, see for example [Háj98b, remark 3.2.14].
For product logic, we have

Theorem 4.5.5 ([Háj98b]). *Product logic is finitely strongly complete with respect to* $[0,1]_\Pi$.

Also in this case, the theorem cannot be extended to infinite theories: see [Háj98b, corollary 4.1.18]

For Gödel logic, instead, we have a better result:

Theorem 4.5.6 ([Háj98b]). *Gödel logic is strongly complete with respect to* $[0,1]_L$.

We now move to MTL and its previously introduced (proper) extensions.

Theorem 4.5.7 ([JM02]). *MTL is strongly complete with respect to the class of standard MTL-algebras. Hence MTL is the logic of left-continuous t-norms and their residua.*

More recently, an alternative proof of the previous theorem has been presented in [Hor07a].

Theorem 4.5.8 ([EG01, EGGM02, Wan07]). *Let* $L \in \{NM, WNM, SMTL, IMTL, RDP\}$. *Then L enjoys the strong standard completeness.*

However, not all the results are so good

Theorem 4.5.9 ([Hor05, Hor07b, MNH06]). *For* $L \in \{\Pi MTL, WCMTL\}$, *L enjoys the finite strong standard completeness. However strong standard completeness does not hold for both of them.*

In chapters 7 and 8 will be studied completeness properties with respect to other classes of MTL-algebras (for example, complete chains).

We conclude with the following (well known) result, concerning finite-valued Łukasiewicz logics: this is an easy consequence of proposition 4.3.3, corollary 4.3.3 and [CEG+09, theorem 3.5])

Theorem 4.5.10. *L_n has the strong completeness with respect to $\mathbf{L_n}$.*

4.6 A weak excluded middle law for MTL

Consider the formula

$$\neg((\neg\varphi)^2) \vee \neg(\varphi^2).$$

It holds that

Theorem 4.6.1.

$$\vdash_{MTL} \neg((\neg\varphi)^2) \vee \neg(\varphi^2).$$

Proof. Thanks to chain completeness theorem ([EG01]), this is equivalent to show that, in every MTL-chain $\mathcal{A}$, it holds that $x^2 = 0$ or $(\sim x)^2 = 0$, for every $x \in \mathcal{A}$.

By contradiction, consider an MTL-chain $\mathcal{B}$ in which there is an element x such that $x^2 > 0$ and $(\sim x)^2 > 0$. We have three cases.

If $x = \sim x$, then $0 = x * \sim x = x^2$, in contrast with the fact that $x^2 > 0$. If $x < \sim x$, then $0 = x * \sim x \geq x^2$ and, again, we have a contradiction. If $x > \sim x$, then $0 = x * \sim x \geq (\sim x)^2$, but $(\sim x)^2$ must be greater than 0. □

In the variety of Gödel algebras, since every element is idempotent, then $\sim x \sqcup \sim\sim x = \sim (x^2) \sqcup \sim ((\sim x)^2)$. More in general, over Gödel logic the following formula is a theorem

$$\neg\varphi \vee \neg\neg\varphi.$$

It is called, in the context of intermediate logics (i.e. axiomatic extensions of intuitionistic logic), weak excluded middle or Jankov's axiom. It is interesting to point out that Jankov's axiom is not a theorem of MTL: for example, $\nvdash_{\text{Ł}} \neg\varphi \vee \neg\neg\varphi$.

Moreover, since $\wedge, \vee$ satisfy de Morgan's laws and $\vdash_{\text{MTL}} \neg\varphi \leftrightarrow \neg\neg\neg\varphi$ (see [EG01]), then

$$\vdash_{\text{MTL}} (\neg\varphi \vee \neg\neg\varphi) \leftrightarrow \neg(\varphi \wedge \neg\varphi).$$

As we have already previously noted, by adding $\neg(\varphi\wedge\neg\varphi)$ to MTL's axioms, we obtain the logic SMTL [EGGM02]: the correspondent variety is given by all the MTL-algebras that does not have non-trivial zero divisors (i.e. elements of the form $0 < x, y < 1$ such that $x * y = 0$).

In [Nog06] other forms of weak excluded middle are studied. In particular the formula

$$(\mathrm{S}_n) \qquad\qquad \varphi \vee \neg(\varphi^{n-1}).$$

Note that S_2 is the classical excluded middle law. An interesting result of [Nog06] is that in the variety correspondent to the logic S_nMTL (axiomatized as MTL plus (S_n)) all the chains are n-contractive (i.e. the equation $x^n = x^{n-1}$ holds) and simple.

4.7 A distance function over Nilpotent Minimum algebras (and logic)

Consider the following operations, over an NM-algebra:

$$\begin{aligned} x \oplus y &:= n(n(x) * n(y)) \\ x \ominus y &:= x * n(y) \\ d(x, y) &:= (x \ominus y) \oplus (y \ominus x). \end{aligned}$$

In particular, over $[0, 1]_{NM}$, the semantics associated to them is the following:

$$x \oplus y = \begin{cases} 1 & \text{if } x + y \geq 1 \\ \max(x, y) & \text{otherwise.} \end{cases}$$

$$x \ominus y = \begin{cases} 0 & \text{if } x \leq y \\ \min(x, 1 - y) & \text{otherwise.} \end{cases}$$

$$d(x, y) = \max((x \ominus y), (y \ominus x)).$$

It can be argued that $\oplus, \ominus$ are the sum and the difference in a sort of NM-arithmetic; note that both the functions, over $[0, 1]_{NM}$, are not continuous. As regards to d, we have the following

Lemma 4.7.1. *For every NM-algebra $\mathcal{A}$ and $x, y, z \in A$, the following equations hold:*

$$\begin{aligned} d(x, y) &\geq 0 \\ d(x, y) = 0 \quad &\mathit{iff} \quad x = y \\ d(x, y) &= d(y, x) \\ d(x, z) &\leq d(x, y) \oplus d(y, z). \end{aligned}$$

Proof. It is enough to show that these equations (and conditions) hold over $[0, 1]_{NM}$: for the first three the check is immediate. For the last one it is an exercise to prove that for every $x, y, z \in A$, $x \ominus z \leq (x \ominus y) \oplus (y \ominus z)$. □

The previous result justify the candidature of d as a distance function over an NM-algebra (note that, in [Ban03], a similar distance function is defined in the context of Post algebras. Compare also our d with the one defined in [CDM99, definition 1.2.4]). One can ask which is the "meaning" of this operation: an answer can be the following.

Proposition 4.7.1. *For every NM-algebra $\mathcal{A}$ and $x, y \in A$, it holds that*

$$d(x, y) = n(x \Leftrightarrow y).$$

Where $x \Leftrightarrow y := (x \Rightarrow y) * (y \Rightarrow x)$.

Proof. It is not difficult to see that, over $[0,1]_{NM}$, $(x \Rightarrow y) * (y \Rightarrow x) = \min((x \Rightarrow y), (y \Rightarrow x))$ and $x \Rightarrow y = n(x) \oplus y$. Since $n(x) = 1 - x$, then an easy check shows the result. □

Theorem 4.7.1. *Defining*

$$\begin{aligned} \varphi \veebar \psi &:= \neg(\neg\varphi \& \neg\psi) \\ \varphi \boxminus \psi &:= \varphi \& \neg\psi \\ \mathbf{d}(\varphi, \psi) &:= (\varphi \boxminus \psi) \veebar (\psi \boxminus \varphi). \end{aligned}$$

We have

$$\begin{aligned} &\vdash_{NM} \neg\mathbf{d}(\varphi, \varphi) \\ &\vdash_{NM} \mathbf{d}(\varphi, \psi) \leftrightarrow \mathbf{d}(\psi, \varphi) \\ &\vdash_{NM} \mathbf{d}(\varphi, \chi) \to (\mathbf{d}(\varphi, \psi) \veebar \mathbf{d}(\psi, \chi)) \\ &\vdash_{NM} \mathbf{d}(\bot, \top) \\ &\vdash_{NM} \mathbf{d}(\varphi, \psi) \leftrightarrow \neg(\varphi \leftrightarrow \psi) \\ &\vdash_{NM} \neg\mathbf{d}(\varphi, \psi) \leftrightarrow (\varphi \leftrightarrow \psi). \end{aligned}$$

Proof. An easy check. □

From the previous facts we can argue that (the algebraic interpretation of) $\mathbf{d}$ express a "distance of truth" between two formulas, in the sense that it indicates to what extent they are not (logically) equivalent.

Chapter 5

First-order logics

In this chapter we will introduce one of the concepts that will play a central role in this thesis: first-order axiomatic extensions of MTL. In fact, chapters 7, 9 and part of chapter 8 will be devoted to study various aspects of this topic.

In first-order logic, thanks to the richness of the language, we can express properties (using predicates) about individuals (represented using functions symbols, constants or variables). Moreover, since these predicates are interpreted over MTL-chains, then we can also express "properties" that are only "partially verified", thanks to the truth-values between 0 and 1. A particular case is the one of predicates of arity zero: in fact, as we will see, they are interpreted, from the semantical point of view, as truth values. This means that, in first-order (many-valued) logics, the role of predicates of arity zero is similar to the one of variables in propositional case: in fact the interpretation of these predicates (can) vary with the (first-order) model used.

5.1 Syntax

A first-order language is a pair $\langle \mathbf{P}, \mathbf{F}, ar \rangle$: $\mathbf{P}$ is called set of predicates symbols, $\mathbf{F}$ set of functions symbols and $ar : \mathbf{P} \cup \mathbf{F} \to \mathbb{N}$ associate the arity to every predicate and function symbol. The predicates of arity zero are also called *truth constants* and functions of arity zero are called *constants* .

Remark 5.1.1.

- *In this thesis we will restrict to countable languages, i.e.* $\mathbf{P}, \mathbf{F}$ *are countable.*
- *We will assume that our language does not contains equality. In fact, in many papers, in place of the set* $\mathbf{F}$*, it is used the set* $\mathbf{C}$ *of constants (see also [Háj98b, remark 5.1.5 (2)]): we have introduced them to maintain the (more general) notation of [CH10]. For a treatment*

of crisp and fuzzy equality, we suggest the papers [CH10, CEG+09, Háj00].

The notions of term and formulas are defined like in classical case.

Let L be an axiomatic extension of MTL: its first-order version, L$\forall$, is axiomatized as follows

- The axioms resulting from the axioms of L by the substitution of the propositional variables by the first-order formulas
- The following axioms:

($\forall$1)	$(\forall x)\varphi(x) \to \varphi(x/t)($ t substitutable for x in $\varphi(x))$
($\exists$1)	$\varphi(x/t) \to (\exists x)\varphi(x)($ t substitutable for x in $\varphi(x))$
($\forall$2)	$(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
($\exists$2)	$(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
($\forall$3)	$(\forall x)(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$ (x not free in ν)

The rules of L$\forall$ are:

(Modus Ponens) $$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

(Generalization) $$\frac{\varphi}{\forall x \varphi}.$$

An interesting discussion concerning the origin of these five axioms, in relation to some works by E. Rasiowa and A. Mostowski, is presented in [Háj06a].

Remark 5.1.2. *For some axiomatic extensions of MTL$\forall$, this set of five axioms (schemata) for quantifiers is redundant: for example Ł$\forall$ can be axiomatized using only the first two, whilst in NM$\forall$ (and in every axiomatic extension of MTL$\forall$ with an involutive negation) the axioms* ($\exists$1), ($\exists$2) *can be removed. See [CH10] for details.*

5.2 Semantics

As regards to semantics, we need to restrict to L-chains[1]: given an L-chain $\mathbf{A}$, an $\mathbf{A}$-interpretation (or $\mathbf{A}$-model) is a structure $\mathbf{M} = \langle M, \{m_f\}_{f\in\mathbf{F}}, \{r_P\}_{P\in\mathbf{P}}\rangle$, where

- M is a non-empty set.
- for each $f \in \mathbf{F}$ of arity $ar(f) > 0$, $m_f : M^{ar(f)} \to M$.

[1]If we do not use chains, then the soundness can fail, for some logics. In fact in [EGHM03, example 5.4] it is showed a Gödel-algebra in which ($\forall$3) is not a tautology.

- for each $P \in \mathbf{P}$ of arity $ar(P) > 0$, $r_P : M^{ar(P)} \to A$ (i.e. r_P is a fuzzy relation of arity $ar(P)$).
- for each $f \in \mathbf{F}$ ($P \in \mathbf{P}$) of arity 0, $m_f \in M$ ($r_P \in A$).

For each evaluation over variables $v : Var \to M$, the truth value of a formula φ ($\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}$) is defined inductively as follows:

- $\|P(x, \dots, f(t_1, \dots, t_n), \dots)\|^{\mathbf{A}}_{\mathbf{M},v} = r_P(v(x), \dots, m_f(\|t_1\|^{\mathbf{A}}_{\mathbf{M},v}, \dots, \|t_n\|^{\mathbf{A}}_{\mathbf{M},v}), \dots)$
- The truth value commutes with the connectives of L∀, i.e.

$$\|\varphi \to \psi\|^{\mathbf{A}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} \Rightarrow \|\psi\|^{\mathbf{A}}_{\mathbf{M},v}$$
$$\|\varphi \& \psi\|^{\mathbf{A}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} * \|\psi\|^{\mathbf{A}}_{\mathbf{M},v}$$
$$\|\bot\|^{\mathbf{A}}_{\mathbf{M},v} = 0; \;\; \|\top\|^{\mathbf{A}}_{\mathbf{M},v} = 1$$
$$\|\varphi \wedge \psi\|^{\mathbf{A}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} \sqcap \|\psi\|^{\mathbf{A}}_{\mathbf{M},v}$$

- $\|(\forall x)\varphi\|^{\mathbf{A}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\mathbf{A}}_{M,v'} : \; v'(y) = v(y)$ for all variables except for $x\}$
- $\|(\exists x)\varphi\|^{\mathbf{A}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\mathbf{A}}_{M,v'} : \; v'(y) = v(y)$ for all variables except for $x\}$

if these inf and sup exist in $\mathbf{A}$, otherwise the truth value is undefined.

A model $\mathbf{M}$ is called $\mathbf{A}$-safe if all inf e sup necessary to define the truth value of each formula exist in $\mathbf{A}$. In this case, the truth value of a formula φ over an $\mathbf{A}$-safe model is

$$\|\varphi\|^{\mathbf{A}}_{\mathbf{M}} = \inf\{\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} : \; v : \; Var \to M\}$$

Note that if $\mathbf{A}$ is a standard algebra or has a lattice-reduct that is a complete lattice, then every $\mathbf{A}$-model is safe; obviously every finite $\mathbf{A}$-model (M finite) is safe.

5.3 Completeness and incompleteness results

The notions of completeness are defined analogously to propositional case, with the difference that, with the notation $\models_{\mathbf{A}} \varphi$, we mean that $\|\varphi\|^{\mathbf{A}}_{\mathbf{M}} = 1$, for every safe $\mathbf{A}$-interpretation $\mathbf{M}$.

In particular, an axiomatic extension of MTL∀, L∀, is said to be sound, with respect to a class K of L-chains if

$$L\forall \vdash \varphi \quad \text{implies} \quad K \models \varphi$$

for every φ.

One can ask if the restriction to safe models can be dropped: we call supersound a logic if the previous condition holds for every model in which the truth-value of φ is defined (but not necessarily safe). The question is not trivial: in fact many logics are not supersound.

This problem will be treated in detail in chapter 7.

Now, analogously to what happens in propositional case, for the classes of L-chains (L being an axiomatic extension of MTL) we have good results.

Theorem 5.3.1 ([EG01]). *Let $L\forall$ be an axiomatic extension of $MTL\forall$, T be a theory and φ be a formula. If we call $\mathbb{L}$ the class of L-chains, then we obtain*

$$T \vdash_{L\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi$$

For some axiomatic extensions of MTL, this result was showed much before: for example, concerning Ł∀, the proof was originally given in [CB63, theorem 1].

Given an axiomatic extension of MTL, we can study the completeness with respect to more restricted classes of chains. An example is given by standard algebras: as we will see, the results will be much worse that in propositional case.

Theorem 5.3.2 ([MO02, EG01, EGGM02, TT84, MS03]). *Let $L \in \{MTL, WNM, NM, G, RDP, IMTL, SMTL\}$ and let $\mathbb{L}$ be the class of standard L-algebras. Then*

$$T \vdash_{L\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi,$$

for every theory T and formula φ.

For some logics we have weaker results

Theorem 5.3.3 ([MNH06, CEG+09]). *Let $L = WCMTL$ and let $\mathbb{L}$ be the class of standard L-algebras. Then, for every finite theory T and formula φ*

$$T \vdash_{L\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}} \varphi.$$

The previous result, however cannot be extended to arbitrary theories.

Finally, for $L = \Pi MTL$ the previous result does not hold. It remains an open problem if ΠMTL enjoys the standard completeness.

Before introducing the other incompleteness results, we recall briefly some notions of computability theory about the arithmetical hierarchy: for a more general treatment we suggest [Odi92] and [HP98].

Definition 5.3.1. *Given $W \subseteq \mathbb{N}$, we say that W is:*

- Σ_1 *if there exists a recursive binary relation (i.e. a relation whose characteristic function is recursive) $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $W = \{x : \exists y R(x, y)\}$. Another terminology is that W is* recursively enumerable.

- Σ_2 *if there exists a recursive ternary relation (i.e. a relation whose characteristic function is recursive)* $R \subseteq \mathbb{N}^3$ *such that* $W = \{z : \exists x \forall y R(z, x, y)\}$.

- Σ_n $(n > 2)$ *if there exists a recursive* $(n+1)$*-ary relation (i.e. a relation whose characteristic function is recursive)* $R \subseteq \mathbb{N}^{n+1}$ *such that* $W = \{z : \underbrace{\exists x \forall y \exists t \dots}_{n} R(\underbrace{z, x, y, t \dots}_{n+1})\}$.

- Π_1 *if there exists a recursive binary relation* $R \subseteq \mathbb{N} \times \mathbb{N}$ *such that* $W = \{x : \forall y R(x, y)\}$.

- Π_2 *if there exists a recursive ternary relation (i.e. a relation whose characteristic function is recursive)* $R \subseteq \mathbb{N}^3$ *such that* $W = \{z : \forall x \exists y R(z, x, y)\}$.

- Π_n $(n > 2)$ *if there exists a recursive* $(n+1)$*-ary relation (i.e. a relation whose characteristic function is recursive)* $R \subseteq \mathbb{N}^{n+1}$ *such that* $W = \{z : \underbrace{\forall x \exists y \forall t \dots}_{n} R(\underbrace{z, x, y, t \dots}_{n+1})\}$.

- Δ_0 *if* W *is* Σ_1 *and* Π_1. *In this case* W *is called* recursive.

- Σ_n*-hard* (Π_n*-hard) if each* Σ_n (Π_n) *set* W' *is* recursively reducible *to* W, *i.e. there exists a recursive function* $f : \mathbb{N} \to \mathbb{N}$ *such that* $x \in W'$ *if and only if* $f(x) \in W$ *(equivalently* $W' = \{x : f(x) \in W\}$*).*

- Σ_n*-complete* (Π_n*-complete) if it is* Σ_n (Π_n) *and* Σ_n*-hard* (Π_n*-hard).*

- *Non-arithmetical if there is no* n *such that* W *is* Σ_n.

As it can be seen, this is a measure of the undecidability of a set: clearly, if two sets of natural numbers V, W do not have the same arithmetical complexity, then $V \neq W$.

Now, since there are many ways of "coding" first-order formulas with natural numbers, then we can use the previous machinery to study the complexity of sets of formulas. We will speak about the arithmetical complexity of a set of formulas W, by meaning the arithmetical complexity of $cod(W)$, where cod is an opportune coding.

Definition 5.3.2. *Let* $L\forall$ *be an axiomatic extension of* $MTL\forall$. *We define the following sets of formulas*

- $P_{L\forall}$ *as the set of theorems of* $L\forall$.

- $genTAUT_{L\forall}$ *as the set of first-order tautologies in the class of L-chains.*

- *$stdTAUT_{L\forall}$ as the set of first-order tautologies in the class of standard L-chains.*

We begin with a general result:

Theorem 5.3.4 ([MN10, corollary 3.17])**.** *Let $L\forall$ be an axiomatic extension of $MTL\forall$: then $P_{L\forall} = genTAUT_{L\forall}$ is Σ_1-complete.*

From this theorem it is immediate to see that all these logics are undecidable: however, this last result was originally proved in [MO02].

Theorem 5.3.5.

- *$stdTAUT_{BL\forall}$ is Π_2-hard ([Háj01]) and, moreover, not arithmetical ([Mon01]): hence $BL\forall$ is not standard complete.*
- *$stdTAUT_{Ł\forall}$ is not recursively enumerable ([Sca62]), in fact, it is Π_2-complete ([Rag81]). It follows that $Ł\forall$ is not standard complete.*
- *$stdTAUT_{\Pi\forall}$ is Π_2-hard ([Háj01]) and, moreover, not arithmetical ([Mon01]): hence $\Pi\forall$ is not standard complete.*
- *$stdTAUT_{SBL\forall}$ is not arithmetical ([Mon05b]): hence $SBL\forall$ is not standard complete.*

One can argue that this is a quite complicate way to prove the incompleteness of a logic: moreover it does not show explicitly a "true but unprovable formula".

For BL∀ and SBL∀ a counterexample is given by the formula

$$(\forall x)(\varphi(x)\&\nu) \leftrightarrow ((\forall x)\varphi(x)\&\nu)$$

Concerning BL∀ the problem of the validity of this formula was left open in [Háj98b] and solved in [EG01]. The failure of this formula over SBL∀ was pointed out in [CH10].

Another counterexample, for these two logics, is the following.

Counterexample 5.3.1. *Consider the algebra $\mathbb{R}^+\mathbf{L_2}$, if $L = BL\forall$ and $\mathbf{2}\oplus\mathbb{R}^+\mathbf{L_2}$, if $L = SBL\forall$ (see chapter 8): direct inspection shows that $\mathbb{R}^+\mathbf{L_2}$ is a BL-algebra and $\mathbf{2}\oplus\mathbb{R}^+\mathbf{L_2}$ is an SBL-algebra. Fix now an integer $k > 1$ and let $I = \{i \in \mathbb{R}^+ :\ i > k\}$ and c_i be the coatom of the ith component. If we take $y = c_k$ it is not difficult to see that $\inf_{i\in I}(c_i * y) = y$, but $\inf\{c_i : i \in I\} * y = y * y < y$ (in particular $y * y$ is the zero of the kth component).*

To conclude the proof, consider the formula $(\forall x)(P(x)\&\nu) \leftrightarrow ((\forall x)P(x)\&\nu)$, where ν is a predicate of ariety zero.

For both the chains, construct a model $\mathbf{M}$ such that $M = I$, ν is interpreted in c_k and $r_P(m) = c_m$, for $m \in M$. Direct inspection shows that this is a countermodel, for the cited formula, in both chains. It follows that $(\forall x)(\varphi(x)\&\nu) \leftrightarrow ((\forall x)\varphi(x)\&\nu)$ cannot be a theorem of $BL\forall$ or $SBL\forall$.

However, this formula is a tautology in Ł$\forall$ and $\Pi\forall$.

Concerning $\Pi\forall$, a true (over $[0, 1]_\Pi$) but unprovable (in $\Pi\forall$) formula was presented in [Háj04].

As regards to Ł$\forall$, instead, finding a counterexample (to standard completeness) was leaved, in [Háj04], as an open problem. Recently the problem was solved in [Háj10]: we do not show the formula, since it is quite complicated (it contains the axioms of Robinson arithmetic).

Part II

Specific topics

Chapter 6

A temporal semantics for basic logic

The results of this chapter are taken from paper [ABM09a].

The completeness theorem proved in [CEGT00] shows that BL is the logic of all continuous t-norms and their residua. This result, however, does not directly yield any meaningful interpretation of the truth values in BL *per se*. In an attempt to address this issue, in this thesis we have introduced a complete temporal semantics for BL. Specifically, we have showed that BL formulas can be interpreted as modal formulas over a flow of time, where the logic of each instant is Łukasiewicz, with a finite or infinite number of truth values. As a main result, we have obtained validity with respect to all flows of times that are non-branching to the future, and completeness with respect to all finite linear flows of time, or to an appropriate single infinite linear flow of time. It may be argued that this reduces the problem of establishing a meaningful interpretation of the truth values in BL logic to the analogous problem for Łukasiewicz logic.

6.1 Introduction and statements of the results

Many-valued propositional logics generalise classical logic through the addition of new truth values between absolute falsity and absolute truth. This wholeheartedly semantical approach can be traced back to one of the founders of many-valued logics, the Polish logician and philosopher Jan Łukasiewicz [Bor70, *Two valued logic*]:

> By *logic* I mean the science of logical values. Conceived in this way, logic has it own subject-matter of research, with which no other discipline is concerned. Logic is not a science of propositions, since that belongs to grammar; it is not a science of

> judgements or convictions, since that belongs to psychology; it is not a science of contents expressed by propositions, since that, according to the content involved, is the concern of the various detailed disciplines; it is not a science of "objects in general", since that belongs to ontology. Logic is the science of objects of specific kind, namely of *logical values*.

At first, Łukasiewicz introduced a three valued logic; shortly thereafter, he extended it to accommodate infinitely many truth values ranging in the real unit interval $[0, 1]$. Today, this *Łukasiewicz logic* is the subject of intensive investigation, cf. [CDM99, and references therein].

It is well known that Łukasiewicz himself was vexed by the problem of giving a meaningful interpretation of the additional truth values he had introduced. This crucial issue has been addressed time and again ever since, and a few competing solutions are by now available. Here we only mention Mundici's approach through Ulam-Rényi games [CDM99, Ch. 5], and Giles' semantics in the style of Lorenzen and Hintikka [Gil75] (see also [Fer08] for further background and references). In a parallel line of development, truth values in the real unit interval $[0, 1]$ have also been proposed as a model of degrees of truth; see the monographs [Got01, Háj98b] for historical details. From this point of view, Łukasiewicz logic fits into the much wider framework of triangular-norm based logics developed in [Háj98b] and also presented in preceding chapters. The latter logics are $[0, 1]$-valued truth-functional propositional systems, in which the conjunction is interpreted by a continuous triangular norm, and the implication by its associated residuum (please see chapters 3, 4, 2 for definitions). Hájek's *Basic Fuzzy Logic* BL provides a complete axiomatization of the theorems common to all such systems [CEGT00]. Accordingly, BL admits a wide spectrum of schematic extensions other than Łukasiewicz. Their classification currently appears out of reach — there are uncountably many distinct such extensions [DEGM05]. In the light of such phenomena, the task of finding a meaningful interpretation of truth values in BL might appear, *prima facie*, formidable; for an attempt in the context of feedback coding, see [CM07]. In this chapter we show that the problem can be reduced to its analogue for Łukasiewicz logic. Specifically, we shall prove that BL formulas can be interpreted as modal formulas over appropriate flows of time, where the logic of each instant is Łukasiewicz with a finite or infinite number of truth values. Provided only Łukasiewicz logic is taken as understood, this temporal semantics affords an understanding of BL in modal-like terms (Compare the analogous result given in [AGM08] for Gödel logic, one of the best-known extensions of BL).

We now turn to the promised temporal semantics for BL. Throughout, we set $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write $\odot$ for the Łukasiewicz conjunction of $x, y \in [0, 1]$, that is, $x \odot y = \max(0, x + y - 1)$. We further write $\Rightarrow$ for the Łukasiewicz implication $x \Rightarrow y = \min(1, 1 - x + y)$. Following tradition, we also use $\oplus$ to denote the Łukasiewicz disjunction $x \oplus y = \min(1, x + y)$. For each integer $t > 0$, we set $Ł_t = \{\frac{i}{t} \mid i \in \mathbb{N}\,,\, 0 \leq i \leq t\} \subseteq [0,1]$. Further, we set $Ł_0 = [0,1]$.

A *temporal flow* is a pair $\langle T, L \rangle$, where T is a poset and $L : T \to \mathbb{N}$; and a *temporal assignment* over variables is a function $v : VAR \times T \to [0,1]$ such that, for each $t, t' \in T$ and each $x_i \in VAR$, the following hold.

(T1) $v(x_i, t) \in Ł_{L(t)}$.

(T2) If $t \leq t'$ then $v(x_i, t) \leq v(x_i, t')$.

(T3) If $t \neq t'$ and $v(x_i, t), v(x_i, t') \in (0,1)$, then $t <> t'$ (meaning that t and t' are incomparable).

To extend v to a temporal assignment $v : FORM \times T \to [0,1]$ over formulas, we stipulate the following conditions, for all φ, $\psi \in FORM$, and all $t \in T$.

(T4) $v(\bot, t) = 0$.

(T5) $v(\varphi \& \psi, t) = \max(0, v(\varphi, t) + v(\psi, t) - 1) = v(\varphi, t) \odot v(\psi, t)$.

(T6)

$$v(\varphi \to \psi, t) = \begin{cases} 1 & \text{if } v(\varphi, t') \leq v(\psi, t') \text{ for all } t' \geq t\,; \\ v(\varphi, t) \Rightarrow v(\psi, t) & \text{if } v(\psi, t) < v(\varphi, t) < 1 \text{ and} \\ & v(\psi, t') = 1 \text{ for all } t' > t\,; \\ v(\psi, t) & \text{otherwise,} \end{cases}$$

where $t' \in T$.

We now say that a temporal flow $\langle T, L \rangle$ *models* a formula φ (or that φ *holds* or is *valid* in $\langle T, L \rangle$) if $v(\varphi, t) = 1$ for all temporal assignments v and all $t \in T$; in symbols, $\langle T, L \rangle \models \varphi$. A class K of temporal flows is *sound* (for BL) if, for any formula φ, $\vdash_{BL} \varphi$ implies $\langle T, L \rangle \models \varphi$ for all $\langle T, L \rangle \in K$.

Not all temporal flows are sound for BL, as the following example shows.

Example 6.1.1. *Let $\langle T, L \rangle$ be any temporal flow such that T contains the poset $V = \{t_1, t_2, t_3\}$, with $t_1 < t_2, t_1 < t_3$ and $t_2 <> t_3$. Let v be a temporal assignment such that $v(x_1, t_1) = v(x_2, t_1) = v(x_3, t_1) = 0$, $v(x_1, t_2) = 0$, $v(x_2, t_2) = 1 = v(x_3, t_2)$, $v(x_1, t_3) = 1 = v(x_3, t_3)$, $v(x_2, t_3) = 0$. Direct inspection shows that $v(((x_1 \to x_2) \to x_3) \to (((x_2 \to x_1) \to x_3) \to x_3), t_1) = 0$, that is, Axiom (A6) does not hold in $\langle T, L \rangle$. Similarly, one checks that the* prelinearity law

(PL) $$(\varphi \to \psi) \vee (\psi \to \varphi)$$

fails in $\langle T, L \rangle$ at t_1 for an appropriate temporal assignment, while it is known to follow from the axioms of BL, see [Háj98b, Lemma 2.2.10].

In light of the preceding example, we focus attention on those posets that do not contain as a subposet the configuration V in Example 6.1.1. These are known as *root systems*, see e.g. [Dar95, Definition 9.4]. Thus, a root system is a poset T such that for each $x \in T$ the *upper set* $\uparrow x = \{y \in T \mid x \leq y\}$ of each element is totally ordered (is a *chain*). Because of the connection with the prelinearity law, we say a temporal flow $\langle T, L\rangle$ is *prelinear* if T is a root system, i.e. if the future of each instant does not branch. Further, we call $\langle T, L\rangle$ a (*finite*) *linear* temporal flow if T is a (finite) totally ordered set. Finally, we say $\langle T, L\rangle$ is *finite-valued* if 0 is not in the range of L. We can now state our main result.

Theorem 6.1.1. *For any formula $\varphi \in FORM$, the following hold.*

1. (Soundness.) *If $\vdash_{BL} \varphi$ then $\langle T, L\rangle \models \varphi$ for all prelinear temporal flows $\langle T, L\rangle$.*

2. (Completeness.) *If $\langle T, L\rangle \models \varphi$ for all finite, finite-valued linear temporal flows, then $\vdash_{BL} \varphi$.*

We prove Theorem 6.1.1 in Section 6.2. The proof reduces both statements to analogous facts that are known to hold for the algebraic semantics of BL; the necessary background is provided in Subsection 6.2.1. We collect additional results of some interest in Section 6.3. In Subsection 6.3.1, we show that BL is complete with respect to the single temporal flow consisting of a given instant ("today") having an infinite countable linear past, such that, at each instant, the logic is infinite-valued Łukasiewicz logic. In Subsection 6.3.2, we obtain characterisations of several important extensions of BL (including Łukasiewicz and Gödel logic) via appropriate classes of temporal flows. To mention one example, in this context Gödel logic may be seen as the logic of those prelinear temporal flows such that the logic at each instant is classical (Proposition 6.3.3). A notable exception here is Product logic, the extension of BL by the *precancellativity* axiom scheme

$$\neg\varphi \vee ((\varphi \rightarrow (\varphi \& \psi)) \rightarrow \psi)\,.$$

Product logic cannot be characterised by a class of temporal flows as defined in this chapter. This is because we are stipulating the logic at each instant to be (finite- or infinite-valued) Łukasiewicz logic. Thus, certain schematic extensions of BL (equivalently, subvarieties of BL-algebras) are beyond the expressive power of the temporal semantics introduced here. Since subvarieties of BL-algebras are known to be generated by linearly ordered BL-algebras, it is of course possible to elaborate our present definition of temporal flow further so as to encompass all schematic extensions of BL. However, it appears that further research will be needed in order to strike a defensible balance between expressive power and naturalness.

6.2 Proof of Theorem 6.1.1

6.2.1 Background results

Theorem 6.2.1 (Algebraic completeness theorem). *For any formula $\varphi \in FORM$, $\vdash_{BL} \varphi$ if and only if $C \models \varphi$ for all finite BL-chains C.*

Proof. In [CEGT00, Theorem 5.2] it is proved that BL is complete with respect to all BL-algebras whose lattice reduct is [0,1]. Completeness with respect to all finite BL-chains is proved in [Mon05a, Theorem 2]. $\square$

Remark 6.2.1. *In each BL-algebra the operations $\sqcap, \sqcup, 1$ are definable from $*$, $\Rightarrow$, and 0; indeed, we gave definitions for the corresponding logical connectives in Section 6.1. In the sequel we shall therefore feel free to use the shorter signature $\langle A, *, \Rightarrow, 0\rangle$.*

Lemma 6.2.1. *Any non-trivial finite BL-chain is isomorphic to the ordinal sum $\oplus_{i=1}^{u} Ł_{d_i}$, for some integers $u, d_i \geq 1$.*

Proof. This is not hard to prove directly; alternatively, see [AFM07, Corollary 1.12]. $\square$

6.2.2 Lemmas

Lemma 6.2.2. *For each prelinear temporal flow $\langle T, L\rangle$, temporal assignment v, and formula $\varphi \in FORM$, the following hold for each $t, t' \in T$.*

(T1′) $v(\varphi, t) \in Ł_{L(t)}$.

(T2′) *If $t \leq t'$ then $v(\varphi, t) \leq v(\varphi, t')$.*

(T3′) *If $t \neq t'$ and $v(\varphi, t), v(\varphi, t') \in (0,1)$, then $t <> t'$.*

Proof. (T1′) trivially follows form the definitions. For the remaining claims the proof proceeds by structural induction. If φ is a variable or $\perp$ there is nothing to prove.

Case 1. $\varphi = \psi \& \chi$.

We have $v(\psi\&\chi, t) = \max(0, v(\psi, t) + v(\chi, t) - 1)$, by (T5). By the induction hypothesis, $v(\psi, t) \leq v(\psi, t')$ and $v(\chi, t) \leq v(\chi, t')$. Then $v(\psi, t) + v(\chi, t) - 1 \leq v(\psi, t') + v(\chi, t') - 1$ and (T2′) holds.

To prove (T3′), suppose first $v(\varphi, t) \in (0, 1)$. We *claim* $v(\varphi, s) = 0$, for each $s < t$, and $v(\varphi, r) = 1$, for each $r > t$. Assume first $v(\psi, t), v(\chi, t) \in (0, 1)$. Running a second induction, we may assume $v(\psi, s) = v(\chi, s) = 0$ for $s < t$, and $v(\psi, r) = v(\chi, r) = 1$ for $r > t$. Then the analogous statements hold for $v(\varphi, s)$ and $v(\varphi, r)$, respectively, by (T5). Similarly, a trivial induction proves that if $v(\psi, t) = 1, v(\chi, t) \in (0, 1)$, then $v(\varphi, s) = 0$ and $v(\varphi, r) = 1$. The claim is *settled.* Now (T3′) follows at once.

Case 2. $\varphi = \psi \to \chi$.

We first prove (T2′). If $v(\varphi, t) = 0$, the statement holds trivially. If $v(\varphi, t) = 1$, then for all $t' \geq t$ we have $v(\psi, t') \leq v(\chi, t')$, and hence $v(\varphi, t') = 1$. Let us now assume $0 < v(\psi \to \chi, t) < 1$. If t is a maximal element, then there is nothing to prove. Assume t is not maximal. The value of $v(\psi \to \chi, t)$ coincides either with $v(\chi, t)$ or with $1 - v(\psi, t) + v(\chi, t)$. In both cases we have $v(\chi, t) < 1$. By an easy induction it follows that $v(\chi, t') = 1$ for all $t' > t$. Thus, $v(\psi \to \chi, t') = 1$ for all $t' > t$, and (T2′) is proved.

We next prove (T3′). Suppose $0 < v(\psi \to \chi, t) < 1$. As shown in the proof of (T2′), $v(\psi \to \chi, t') = 1$, for all $t' > t$; thus, it remains to prove that $v(\psi \to \chi, t') = 0$ for all $t' < t$. If $0 < v(\psi, t) < 1$ and $0 \leq v(\chi, t) < 1$, then we have $v(\psi, t') = v(\chi, t') = 0$ for all $t' < t$. Note that $v(\psi, t) > v(\chi, t)$, for otherwise we would have $v(\psi \to \chi, t) = 1$. Then $v(\psi \to \chi, t') = 0$, as was to be shown. The remaining case is $v(\psi, t) = 1$ and $0 < v(\chi, t) < 1$. By the induction hypothesis, $v(\chi, t') = 0$ for $t' < t$. Then (T6) implies $v(\psi \to \chi, t') = 0$ for all $t' < t$, and (T3′) holds. □

Lemma 6.2.3. *For each temporal flow $\langle T, L\rangle$, temporal assignment $v\colon FORM \times T \to [0,1]$, formula $\varphi \in FORM$, and instant $t \in T$, the value of $v(\varphi, t)$ only depends on the values that v assigns to the (principal) subformulas of φ at the set of instants $\{s \in T \mid t \leq s\}$.*

Proof. By inspection of (T4–6). □

Lemma 6.2.4. *Let $\langle T, L\rangle$ be a prelinear temporal flow, and let $\varphi \in FORM$. If $\langle T, L\rangle \not\models \varphi$ then there exists a chain C with minimum and maximum, and a function $L_C\colon C \to \mathbb{N}$, such that $\langle C, L_C\rangle \not\models \varphi$.*

Proof. Assume there is a temporal assignment v and an instant $t \in T$ such that $v(\varphi, t) < 1$. Consider the upper set $\uparrow t = \{s \in T \mid t \leq s\}$ of t in T. Then $\uparrow t$ is a chain with minimum t, because T is prelinear. If $\uparrow t$ has a maximum then let $C = \uparrow t$; otherwise let $C = \uparrow t \cup \{m\}$, for $m \notin T$ a new element such that $s < m$ for all $s \in \uparrow t$, that is, $m = \max C$. Let $\langle C, L_C\rangle$ be the temporal flow with $L_C(s) = L(s)$ for all $s \in \uparrow t$ and $L_C(m) = 1$. Let v' be the function $v'\colon VAR \times T \to [0,1]$ coinciding with v for each $s \in \uparrow t$, and such that $v'(x_i, m) = 0$ if $v(x_i, s) = 0$ for every $s \in \uparrow t$, while $v'(x_i, m) = 1$ otherwise. It is clear that v' is a temporal assignment over $\langle C, L_C\rangle$, as v' trivially satisfies (T1–3). It remains to show that $v'(\varphi, t) = v(\varphi, t) < 1$. If $C = \uparrow t$ then the statement follows at once from Lemma 6.2.3, as the set of instants which $v(\varphi, t)$ depends upon is precisely $\uparrow t$. In case $C = \uparrow t \cup \{m\}$, we run an induction to show that $v'(\psi, s) = v(\psi, s)$ for all $s \in C \setminus \{m\}$. The only non-trivial case is $\psi = \vartheta \to \chi$. Fix $s \neq m$ in C.

If $v(\vartheta, r) \leq v(\chi, r)$ for all $r \geq s$, then, by induction hypothesis, $v'(\vartheta, r) \leq v'(\chi, r)$ for all $r \geq s$ with $r \neq m$. If $v(\vartheta, r) = v(\chi, r) = 0$ for all $r \in \uparrow t$, then

$v'(\vartheta, m) = v'(\chi, m) = 0$. If, on the other hand $v(\vartheta, r') > 0$ for some $r' > s$, then $v'(\vartheta, m) = v'(\chi, m) = 1$. In both cases $v(\vartheta \to \chi, s) = 1 = v'(\vartheta \to \chi, s)$.

We now consider the case $v(\chi, s) < v(\vartheta, s) < 1$ and $v(\chi, r) = 1$ for all $r > s$. The last case of (T6) will then immediately follow from our standing inductive hypothesis. By induction hypothesis it holds that $v'(\chi, s) < v'(\vartheta, s) < 1$ and $v'(\chi, r) = 1$ for all $m > r > s$. Since $\uparrow t$ does not have maximum, then such r does exist. Moreover $v(\chi, r) = 1$ for some $r > s$ implies $v'(\chi, m) = 1$, and hence $v(\vartheta \to \chi, s) = v(\vartheta, s) \Rightarrow v(\chi, s) = v'(\vartheta \to \chi, s)$. We conclude v' coincides with v for all $s \in C \setminus \{m\}$ and hence $v'(\varphi, t) < 1$. □

Lemma 6.2.5. *Let $\langle C, L\rangle$ be a temporal flow such that C is a totally ordered set with maximum, and let φ be a formula. If $\langle C, L\rangle \not\models \varphi$ then $\not\vdash_{BL} \varphi$.*

Proof. Let $v\colon FORM \times C \to [0,1]$ be a temporal assignment such that $v(\varphi, t) < 1$ for some $t \in C$. By Lemma 6.2.4, it is safe to assume that t is the minimum of C. We construct a finite BL-chain B and an assignment $w\colon FORM \to B$ such that $w(\varphi)$ is not the top of B. By Theorem 6.2.1 this entails that $\not\vdash_{BL} \varphi$. For each subformula ψ of φ set

$$U_\psi = \{s \in C \mid v(\psi, s) \neq 1\} \ .$$

By (T2$'$), each U_ψ is a downward-closed subset of C, possibly empty. When partially ordered by reverse inclusion, the collection $U = \{\emptyset, C\} \cup \{U_\psi \mid \psi \text{ subformula of } \varphi\}$ forms a chain, that we display as $C = U_0 < \cdots < U_u = \emptyset$ for some integer $u \geq 1$. For each $i \in \{0, \ldots, u\}$, if U_i has a maximum m_i, then we let $B_i = Ł_{L(m_i)}$, otherwise we let $B_i = Ł_1$. Notice that if $v(\psi, s) \in (0,1)$ for some subformula ψ of φ, then $s = m_i$ for the unique i such that $U_i = U_\psi$. Further, observe that $U_0 = C$ has maximum. Let B be the BL-chain $\bigoplus_{i=0}^{u} B_i$. For each subformula ψ of φ, let k be the uniquely determined integer such that $U_\psi = U_k$. Set $w(\psi) \in B_k$ to the value $v(\psi, m_k)$, if U_k is not empty and has a maximum, to 1 if $U_k = \emptyset$, to 0 otherwise. Since $v(\varphi, t) < 1$ then $U_\varphi \neq \emptyset$ and hence $w(\varphi) \notin B_u$, that is, $w(\varphi)$ is not the top element of B, and we are done. There remains to show that w so defined is an assignment.

Let $w'\colon VAR \to B$ be the assignment $w'(x) = w(x)$ for each variable x. We have to show that for each subformula ψ of φ we have $w'(\psi) = w(\psi)$. We proceed by induction on subformulas. On variables there is nothing to prove. If $\perp$ occurs as a subformula of φ, we clearly have $U_\perp = C$ and $w(\perp) = v(\perp, m_0) = 0$, hence $w'(\perp) = w(\perp) = 0$.

Assume now that $\vartheta\&\chi$ occurs as a subformula of φ and let h, k be the uniquely determined integers such that $w'(\vartheta) \in B_h$ and $w'(\chi) \in B_k$. If $h \neq k$, say $h < k$ without loss of generality, then $w'(\vartheta\&\chi) = \min\{w'(\vartheta), w'(\chi)\} = w'(\vartheta)$, and by induction, $w'(\vartheta\&\chi) = w(\vartheta)$. On the other hand, $U_h \supsetneq U_k$ and hence $v(\chi, t) = 1$ for all $t \in U_h \setminus U_k$. Then $v(\vartheta\&\chi, t) = v(\vartheta, t)$ for all $t \in U_h \setminus U_k$, moreover $v(\vartheta\&\chi, t) = 1$ for all $t \in C \setminus U_h$. Hence $U_h =$

$U_\vartheta = U_{\vartheta\&\chi}$ and $w(\vartheta\&\chi) = w(\vartheta) \in B_h$, as was to be proved. If $h = k$ then $w'(\vartheta\&\chi) = w'(\vartheta) \odot w'(\chi) \in B_h$. On the other hand, $U_h = U_k$ and hence, by (T5), $U_{\vartheta\&\chi} = U_h$ and, in case U_h has maximum, $w(\vartheta\&\chi) = v(\vartheta\&\chi, m_h) = w(\vartheta) \odot w(\chi) \in B_h$; in case U_h does not have a maximum, then $w(\vartheta\&\chi) = 0 = w(\vartheta) \odot w(\chi) \in B_h$; in both cases, by induction $w(\vartheta\&\chi) = w'(\vartheta) \odot w'(\chi) = w'(\vartheta\&\chi)$, and the case $\vartheta\&\chi$ is a subformula of φ is settled.

Assume finally that $\vartheta \to \chi$ occurs as a subformula of φ. let again h, k be the uniquely determined integers such that $w'(\vartheta) \in B_h$ and $w'(\chi) \in B_k$. By induction $w(\vartheta) = w'(\vartheta)$ and $w(\chi) = w'(\chi)$. Notice that $w'(\vartheta \to \chi)$ evaluates to the top element of B if and only if $h < k$ or $h = k$ and $w'(\vartheta) \leq w'(\chi) \in B_h$. In the first case, $U_h \supsetneq U_k$ and hence, $v(\vartheta, s) < v(\chi, s) = 1$ for all $s \in U_h \setminus U_k$. In the second case $v(\vartheta, s) \leq v(\chi, s)$ for all $s \in U_h$. In both cases, by (T2$'$) and (T3$'$) it follows that $v(\vartheta, t) \leq v(\chi, t)$ for all $t \in C$. Hence $U_{\vartheta\to\chi} = \emptyset$, and $w(\vartheta \to \chi) = 1$, and this case is settled. If $h = k$ and $1 > w'(\vartheta) > w'(\chi) \in B_h$, then $w'(\vartheta \to \chi) = w'(\vartheta) \Rightarrow w'(\chi)$. By induction we have that $1 > w(\vartheta) > w(\chi) \in B_h$, hence U_h has maximum m_h and $1 > v(\vartheta, m_h) > v(\chi, m_h)$. Hence $U_{\vartheta\to\chi} = U_h$ and $w(\vartheta \to \chi) = v(\vartheta, m_h) \Rightarrow v(\chi, m_h) = w(\vartheta) \Rightarrow w(\chi)$, and we are done. If $h > k$ then $w'(\vartheta \to \chi) = w'(\chi)$ and $U_h \subsetneq U_k$. By induction $w(\vartheta) = w'(\vartheta) \in B_h$, while $w(\chi) = w'(\chi) \in B_k$. Hence, $1 = v(\vartheta, s) > v(\chi, s)$ for all $s \in U_k \setminus U_h$, and $1 = v(\vartheta, s) = v(\chi, s)$ for all $s \in C \setminus U_k$. By (T2$'$) and (T3$'$) this implies $v(\vartheta, t) \geq v(\chi, t)$ for all $t \in C$. Hence $v(\vartheta, t) \Rightarrow v(\chi, t) = v(\chi, t)$ for all $t \in C$, that is $w(\vartheta \to \chi) = w(\chi)$, and the proof is complete. □

6.2.3 End of proof of Theorem 6.1.1

1. (Soundness.) Lemma 6.2.4 and Lemma 6.2.5 show that if a formula φ is such that $\langle T, L\rangle \not\models \varphi$ for some prelinear temporal flow $\langle T, L\rangle$, then $\not\vdash_{BL} \varphi$.

2. (Completeness.) Suppose $\not\vdash_{BL} \varphi$. Then by Theorem 6.2.1 there is a finite BL-chain C and an assignment $w\colon FORM \to C$ such that $w(\varphi) \neq 1$, i.e. $w(\varphi)$ is not the top of C. To avoid trivialities, assume C is a non-trivial chain. By Lemma 6.2.1, we may safely assume $C = \bigoplus_{i=1}^{u} C_i$, where $C_i = \text{Ł}_{d_i}$, and $u, d_i \geq 1$ are integers. We now define a finite, finite-valued linear temporal flow $\langle T, L\rangle$. First, T is the finite ordered set $t_1 > t_2 > \cdots > t_u$. Then, we set $L(t_i) = d_i$, for each $i = 1, \ldots, u$. The assignment $w\colon VAR \to [0, 1]$ to variables uniquely extends to a temporal assignment $v\colon VAR \times T \to [0, 1]$, as follows. If $w(x_i) = 1$, set $v(x_i, t_j) = 1$ for all $j = 1, \ldots, u$. Otherwise, suppose $w(x_i) \in C_j$. Then we set $v(x_i, t_j) = w(x_i)$, $v(x_i, t_k) = 1$ if $t_k > t_j$, and $v(x_i, t_k) = 0$ if $t_j > t_k$. We write again v for the unique extension of v to formulas. To complete the proof, one runs a structural induction on φ and checks directly from the definitions that $v(\varphi, t_u) < 1$. In particular, one can show that if $w(\varphi) = 1$, then $v(\varphi, t_j) = 1$ for all $j = 1, \ldots, u$; and if $w(\varphi) \in C_j$, then $v(\varphi, t_j) = w(\varphi)$, $v(\varphi, t_k) = 1$

if $t_k > t_j$, and $v(\varphi, t_k) = 0$ if $t_j > t_k$. This requires a lengthy but routine verification. We only explicitly deal with the case of implication.

Suppose $\varphi = \psi \to \chi$, and assume inductively that the claim holds for ψ and χ.

First assume $w(\varphi) = 1$. This implies $w(\psi) \leq w(\chi)$ and hence we have two cases. In the first case, $w(\psi), w(\chi) \in C_j$; then $v(\psi, t_j) = w(\psi)$ and $v(\chi, t_j) = w(\chi)$. Applying the induction hypothesis and (T6), the claim follows. In the second case, $w(\psi) \in C_j$ and either $w(\chi) = 1$, or $w(\chi) \in C_i$ with $i > j$. In either case, applying the inductive hypothesis we obtain $v(\psi, t_j) \leq v(\chi, t_j)$ for all $j = 1, \ldots, u$, and hence $v(\varphi, t_j) = 1$ for all $j = 1, \ldots, u$, by (T6).

Next assume $w(\varphi) < 1$: we must have $w(\psi) > w(\chi)$ and we have two cases.

In the first case, $w(\psi), w(\chi) \in C_j$; then $v(\psi, t_j) = w(\psi)$ and $v(\chi, t_j) = w(\chi)$. Applying the induction hypothesis and the second case of (T6), the claim follows. In the second case, $w(\chi) \in C_j$ and either $w(\psi) = 1$, or $w(\psi) \in C_i$ with $i > j$: clearly we have $w(\varphi) = w(\chi)$. From the induction hypothesis and (T6) it follows that $v(\varphi, t_j) = v(\chi, t_j)$ for all $j = 1, \ldots, u$.

From the above we infer $v(\varphi, t_u) < 1$ whenever $w(\varphi) < 1$.

The proof is complete.

6.3 Further results

6.3.1 Completeness with respect to a single temporal flow

Write $\mathbb{Z}^- = \{z \in \mathbb{Z} \mid z \leq 0\}$ for the ordered set of negative integers with zero.

Proposition 6.3.1. *BL is complete with respect to the linear temporal flow* $\langle \mathbb{Z}^-, L \rangle$*, where* $L(z) = 0$ *for all* $z \in \mathbb{Z}^-$.

Proof. Suppose a formula φ is such $\not\vdash_{BL} \varphi$. The proof of 2 in Subsection 6.2.3 shows that there is a finite, linear, finite-valued temporal flow $\langle T, L' \rangle$ and a temporal assignment $v \colon FORM \times T \to [0,1]$ such that $v(\varphi, t) < 1$ at some $t \in T$. Since Ł$_k$ embeds as an MV-algebra into $[0,1]$ for each integer $k \geq 1$, a straightforward argument shows that $\langle T, L' \rangle$ embeds into $\langle \mathbb{Z}^-, L \rangle$ in such a way that v extends to a temporal assignment $v' \colon FORM \times \mathbb{Z}^- \to [0,1]$ satisfying $v'(\varphi, z) < 1$ at the integer z that corresponds to t under the embedding. Therefore $\langle \mathbb{Z}^-, L \rangle \not\models \varphi$, as was to be shown. $\square$

6.3.2 Extensions of BL as the logics of classes of temporal flows

As we have seen in chapter 3, some notable axiomatic extensions of BL are *Łukasiewicz*, *Gödel*, *Strict-negation Basic Logic* (SBL, for short), and classical (Boolean) logic: we now show how to characterise these extensions by appropriate classes of temporal flows.

Łukasiewicz logic. A temporal flow $\langle T, L\rangle$ is *trivially ordered* if the partial order relation $\leq$ on T is just $\{(t,t) \mid t \in T\}$.

Proposition 6.3.2. *1. Łukasiewicz logic is sound and complete with respect to the class of all trivially ordered temporal flows.*

2. If a temporal flow is sound for Łukasiewicz logic, then it is trivially ordered.

Proof. Let $\langle T, \leq\rangle$ be any poset, $\langle\langle T, \leq\rangle, L\rangle$ any temporal flow, and φ any formula.

We *claim* that $\langle\langle T, \leq\rangle, L\rangle \models \neg\neg\varphi \to \varphi$ if and only if $\leq$ coincides with $\Delta_T = \{(t,t) \mid t \in T\}$. Assume $\leq$ coincides with Δ_T. Then each $t \in T$ is maximal. Hence, for any temporal assignment v and formula φ we have $v(\varphi \to \bot, t) = 1 - v(\varphi, t)$, and $v((\varphi \to \bot) \to \bot, t) = 1 - (1 - v(\varphi, t))$. We conclude $v(\neg\neg\varphi \to \varphi, t) = 1$ and hence $\langle\langle T, \leq_T\rangle, L\rangle \models \neg\neg\varphi \to \varphi$. Conversely, suppose $\leq$ does not coincide with Δ_T. Then $\leq$, being a partial order relation, must properly include Δ_T. Therefore, there are $t \neq t'$ with $t \leq t'$, so that t is not maximal in T. Fix $a \in (0,1)$ and consider any temporal assignment such that $v(x_1, t) = a$. Then $v(x_1 \to \bot, t) = v(\bot, t)$ and $v((x_1 \to \bot) \to \bot, t) = v(\bot \to \bot, t) = 1$. We conclude $v(\neg\neg x_1 \to x_1, t) = v(x_1, t) = a < 1$, and thus $\langle\langle T, \leq_T\rangle, L\rangle \not\models \neg\neg x_1 \to x_1$. The claim is *settled.*

1. Soundness. Suppose φ is provable in Łukasiewicz logic; thus, it is derivable via (MP) from (A1–A7) and the double negation law. By Theorem 6.1.1 and the claim, we know that (A1–A7) and the double negation law hold in any trivially ordered temporal flow. Since (MP) preserves validity over all prelinear temporal flows (again by Theorem 6.1.1), φ holds in all trivially ordered temporal flows. Completeness. Suppose φ holds in all trivially ordered temporal flows. Then φ holds in all single-instant temporal flows, that is, in particular, holds in the MV-algebra $[0,1]$. By [CDM99, 2.5.3], Łukasiewicz logic proves φ.

2. If a temporal flow is sound for Łukasiewicz logic, then it satisfies the double negation law, and thus it is trivially ordered by the claim. $\square$

Let us observe additional completeness theorems for Łukasiewicz logic may be obtained translating known facts into the language of temporal flows. We provide two examples in the following remark, omitting details.

Remark 6.3.1. *1. Łukasiewicz logic is complete with respect to the single-instant temporal flow $\langle\{t\}, L\rangle$, where $L(t) = 0$. This is* Chang's completeness theorem *[CDM99, 2.5.3].*

2. *Łukasiewicz logic is complete with respect to the class of all finite, trivially ordered, finite-valued temporal flows. This is the* finite model property *for Łukasiewicz logic [CDM99, 8.1.2].*

Gödel logic. A temporal flow $\langle T, L\rangle$ is *locally Boolean* if L takes value 1 at each $t \in T$.

Proposition 6.3.3. *1. Gödel logic is sound and complete with respect to the class of all locally Boolean prelinear temporal flows.*

2. If a temporal flow is sound for Gödel logic, then it is prelinear and locally Boolean.

Proof. Let $\langle T, L\rangle$ be any temporal flow, and φ any formula.

We *claim* that $\langle T, L\rangle \models \varphi \to (\varphi \& \varphi)$ if and only if $\langle T, L\rangle$ is locally Boolean. Observe that the range of L is $\{1\}$ if and only if for each $t \in T$ and for any assignment v we have $v(\varphi, t) \in \{0, 1\}$ if and only if $v(\varphi \& \varphi, t) = v(\varphi, t)$ if and only if $v(\varphi \to (\varphi \& \varphi), t) = v(\varphi \to \varphi, t) = 1$. The claim is *settled.*

The result now follows from the claim as in the case of Łukasiewicz logic, *mutatis mutandis.*

□

Remark 6.3.2. *1. Gödel logic is complete with respect to any infinite linear locally Boolean temporal flow. This follows at once from the next item, upon noting that each finite Gödel chain (=BL-chain that is a Gödel algebra) embeds as a subalgebra into any infinite Gödel chain.*

2. Gödel logic is complete with respect to the class of all finite linear locally Boolean temporal flows. This is the finite model property *for Gödel logic, which was already proved in Dummett's classical investigation [Dum59].*

Strict-negation Basic Logic. A temporal flow $\langle T, L\rangle$ is *eventually Boolean* if any maximal element $m \in T$ satisfies $L(m) = 1$.

Proposition 6.3.4. *1. SBL is sound and complete with respect to the class of all eventually Boolean prelinear temporal flows.*

2. If a temporal flow is sound for SBL, then it is prelinear and eventually Boolean.

Proof. Let $\langle T, L\rangle$ be any temporal flow, and φ any formula.

We *claim* that $\langle T, L\rangle \models \neg(\varphi \wedge \neg\varphi)$ if and only if $\langle T, L\rangle$ is eventually Boolean. Assume first that $\max T \subseteq L^{-1}(1)$, and consider any assignment

v. If $t \notin \max T$, then either $v(\varphi \to \bot, t) = v(\bot, t)$ or $v(\varphi, t) = v(\bot, t)$, and hence $v((\varphi \wedge \neg\varphi) \to \bot, t) = v(\bot \to \bot, t) = 1$. If $t \in \max T$ then $v(\varphi \to \bot, t) = 1 - v(\varphi, t)$ and $v(\varphi, t) \in \{0, 1\}$. Hence $v(\varphi \wedge \neg\varphi, t) = \min\{0, 1\} = 0$ and $v((\varphi \wedge \neg\varphi) \to \bot, t) = 1$. We conclude that for all assignments v and all instants $t \in T$, we have $\langle T, L\rangle \models \neg(\varphi\wedge\neg\varphi)$. Conversely, suppose there exists a maximal element $t \in T$ such that $L(t) \neq 1$. Consider any assignment v such that $v(x_1, t) = a \in (0, 1)$. Then $v(x_1 \to \bot, t) = 1 - v(x_1, t)$ and hence $v(x_1 \wedge \neg x_1, t) = \min\{a, 1 - a\}$. It follows that $v((x_1 \wedge \neg x_1) \to \bot, t) = 1 - \min\{a, 1 - a\} < 1$ and hence $\langle T, L\rangle \not\models \neg(x_1 \wedge \neg x_1)$. The claim is *settled*.

The result now follows from the claim as in the case of Łukasiewicz logic, *mutatis mutandis* (completeness: using the fact that SBL is complete with respect to evaluations into finite SBL-chains, see [Mon05a, Theorem 5], we can proceed analogously to Section 6.2.3, point 2).

□

Chapter 7

Supersound many-valued logics and Dedekind-MacNeille completions

In this chapter we present the results published in [BM09].

In [HPS00] the authors have introduced the concept of supersound logic, proving that first-order Gödel logic enjoys this property, whilst first-order Łukasiewicz and product logics do not; in [HS01] this result has been improved showing that, among the logics given by continuous t-norms, Gödel logic is the only one that is supersound.

We have generalized the previous results. Two conditions have been presented: the first one implies the supersoundness and the second one non-supersoundness. To develop these results we have used, in addition to other machinery, the techniques of completions of MTL-chains developed in [Lv08, van10]. We list some of the main results. The first-order versions of MTL, SMTL, IMTL, WNM, NM, RDP are supersound; the first-order version of an axiomatic extension of BL is supersound if and only it is n-contractive. Concerning the negative results, we have that the first-order versions of ΠMTL, WCMTL and of each non-n-contractive axiomatic extension of BL are not supersound.

7.1 Supersound Logics

7.1.1 Some general results

Definition 7.1.1 ([CEG^{+}09])**.** *Let* $\mathbf{A}, \mathbf{B}$ *be two algebras of the same type with (defined) lattice operations. We say that an embedding* $f : \mathbf{A} \to \mathbf{B}$ *is a*

σ-embedding if $f(\sup C) = \sup f[C]$ (whenever $\sup C$ exists) and $f(\inf D) = \inf f[D]$ (whenever $\inf D$ exists) for each countable $C, D \subseteq A$.

Definition 7.1.2. *Let L be an axiomatic extension of MTL: we say that L enjoys the σ-embedding property, with respect to standard L-algebras, if each countable L-chain can be σ-embedded in some standard L-chain.*

Clearly we have that L enjoys this property if and only if L$\forall$ does. This property has a connection with the first-order strong standard completeness of a logic (strong completeness with respect to the class of standard algebras):

Theorem 7.1.1 ([CEG+09, theorem 5.10])**.** *Let* L *be an axiomatic extension of* MTL*. If* L *enjoys the σ-embedding property with respect to the class of standard* L*-algebras, then* L$\forall$ *has the strong standard completeness.*

7.1.2 Supersound logics

Normally, for each axiomatic extensions L$\forall$ of MTL$\forall$, we have a soundness theorem: if a formula is provable, then it is true in all safe interpretations over all L-chains.

An interesting question is if it is true that each provable formula is true over all interpretations over chains where the truth value of this formula is defined: note that these interpretations can be unsafe, in general. Formally,

Definition 7.1.3. *An axiomatic extension L of MTL$\forall$ is supersound if each provable formula φ is true in each $\mathbf{A}$-interpretation ($\mathbf{A}$ being* any *L-chain) in which the truth value of φ is defined.*

The answer to this question is not obvious, since in [HPS00] it is showed that first-order Łukasiewicz and product logic are not supersound; for first-order Gödel logic, instead, the answer is positive, but, as proved in [HS01], this is the only logic based over continuous t-norms that enjoys this property.

A first result is the following:

Theorem 7.1.2 (Weak supersoundness theorem)**.** *Let* L$\forall$ *be an axiomatic extension of* MTL$\forall$ *that enjoys the σ-embedding property with respect to standard* L*-algebras. Then* L$\forall$ *is supersound with respect to* countable L-*chains.*

Proof. Suppose not: then let φ be a formula such that L$\forall \vdash \varphi$, but $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} < 1$, for some v, where $\mathbf{A}$ is a countable L-chain and $\mathbf{M}$ is an $\mathbf{A}$-interpretation in which the value of φ is defined, but in general it is not a safe model.

From the hypothesis, there exists a σ-embedding $\phi : \mathbf{A} \to \mathbf{B}$, for some standard L-algebra $\mathbf{B}$. Starting from $\mathbf{M} = \langle M, \{m_c\}_{c\in\mathbf{C}}, \{r_P\}_{P\in\mathbf{P}}\rangle$, we construct a $\mathbf{B}$-interpretation $\mathbf{M}' = \langle M, \{m_c\}_{c\in\mathbf{C}}, \{r'_P\}_{P\in\mathbf{P}}\rangle$, where each $r'_P : M^{arity(P)} \to B$ is such that $r'_P = \phi(r_P)$.

Since the embedding ϕ preserves all inf and sup, we have that $1 > \phi(\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}) = \|\varphi\|^{\mathbf{B}}_{\mathbf{M}',v}$, but Ł∀ enjoys the strong standard completeness and hence Ł∀ $\nvdash \varphi$. □

Now, observing the results in [EG01, Háj98b, EGGM02, MO02, MS03, Wan07]:

Corollary 7.1.1. *For* $\mathrm{L} \in \{\mathrm{MTL, IMTL, SMTL, RDP, NM, WNM, G}\}$ *the previous theorem holds.*

7.2 Conditions for the supersoundness

Clearly, theorem 7.1.2 does not hold for *all* L-chains: to obtain a result of this type we need to use other techniques.

We will prove the following fact: let L be an axiomatic extension of MTL, and suppose that every L-chain can be embedded into a complete L-chain by an embedding preserving all existing suprema and infima. Then, L∀ is supersound and both L and L∀ are strongly complete with respect to the class of complete L-chains.

As regards to completions, we start from some results of the paper [van10] (see also [Lv08]):

Let $\mathbf{A} = \langle A, * \Rightarrow, \sqcap, \sqcup, 0, 1\rangle$ be an MTL-chain and consider the algebra $\mathbf{A}^c = \left\langle A^c = \{X \subseteq A :\ X^{\ell u} = X\}, \circ, \Rightarrow_\circ, \cup, \cap, A, \{1\}\right\rangle$, where X^ℓ, X^u denote, respectively, the sets of lower bounds and upper bounds of X. The operations are defined as follows:

$$X \circ Y = (X^\ell * Y^\ell)^u = \{x * y :\ x \in X^\ell, y \in Y^\ell\}^u$$
$$X \Rightarrow_\circ Y = \bigcap\{Z \in A^c :\ Z \circ X \supseteq Y\}$$

We can be more specific with respect to the nature of the elements of A^c:

Lemma 7.2.1. *For each* $X \in A^c$ *it holds that:*

1. *X is upward closed.*

2. *either X has a minimum or X does not have infimum.*

Proof. Direct inspection. □

It follows that:

Lemma 7.2.2 ([van10, Lv08])**.** *For each* MTL*-chain* $\mathbf{A}$ *it holds that*

- $\mathbf{A}^c$ *is a complete* MTL *chain with the order given by* $\supseteq$*.*

- *the map* $\phi : \mathbf{A} \to \mathbf{A}^c$ *such that* $\phi(a) = \{a\}^u$ *is an embedding that preserves all* inf *and* sup*.*

Let L be an axiomatic extension of MTL. We say that L has the *complete embedding property*, CEP, if each L-chain can be embedded in a complete L-chain preserving all inf and sup.

Theorem 7.2.1. *Let* L *be an axiomatic extension of* MTL *that enjoys the CEP, then*

1. L *enjoys the strong completeness with respect to the class of complete* L*-chains.*
2. L$\forall$ *enjoys the strong completeness with respect to the class of complete* L*-chains.*
3. L$\forall$ *is supersound.*

Proof. 1. Suppose that $T \nvdash_{\mathrm{L}} \varphi$: from the strong chain completeness theorem (see [EG01] and [Háj98b, theorem 2.4.3]) there exists an L-chain $\mathbf{A}$ and an $\mathbf{A}$-assignment v such that $v(\psi) = 1$, for all $\psi \in T$, but $v(\varphi) < 1$. Now, by the hypothesis, $\mathbf{A}$ embeds in a complete chain $\mathbf{B}$ with an embedding ϕ and hence $\phi(v(\varphi)) < 1$ (and obviously $\phi(v(\psi)) = 1$, for all $\psi \in T$).

2. Suppose that $T \nvdash_{\mathrm{L}\forall} \varphi$: from the strong chain completeness theorem (see [EG01] and [Háj98b, lemma 5.2.7 and the following results]) there exists an L-chain $\mathbf{A}$ and an $\mathbf{A}$-model $\mathbf{M}$ such that $\|\psi\|^{\mathbf{A}}_{\mathbf{M}} = 1$, for each $\psi \in T$, but $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} < 1$, for some evaluation v. Now, by the hypothesis, $\mathbf{A}$ embeds in a complete chain $\mathbf{B}$ with an embedding ϕ that preserves all inf and sup. Hence we can construct a $\mathbf{B}$-model $\mathbf{M}'$, as in the proof of theorem 7.1.2. Then $\phi(\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}) = \|\varphi\|^{\mathbf{B}}_{\mathbf{M}',v} < 1$ and $\phi(\|\psi\|^{\mathbf{A}}_{\mathbf{M}}) = \|\psi\|^{\mathbf{B}}_{\mathbf{M}'} = 1$, for each $\psi \in T$.

3. An easy adaptation of the proof of theorem 7.1.2.

□

7.2.1 A condition for non-supersoundness

From [HPS00, HS01] we know that not all many-valued logics are supersound. A sufficient condition for non-supersoundness is given by the following theorems.

Theorem 7.2.2. *Suppose that a variety* $\mathcal{V}$ *of MTL-algebras satisfies the following condition:*

(*) *For every chain* $\mathbf{A} \in \mathcal{V}$ *and for every* $a, b, c \in \mathbf{A}$ *with* $a \leq b \leq c$*, if* $b * c = b$*, then* $a * c = a$*.*

Let L be the logic corresponding to $\mathcal{V}$ and let $L\forall$ be its first-order extension. Then $(\forall x)(\exists y)(P(x) \to (P(y)\&C)) \vdash (\exists y)(P(y) \to (P(y)\&C))$, where P is a unary predicate and C is a predicate of arity zero.

Proof. Let $(\mathbf{L}, \mathbf{M}, v)$ be a safe interpretation such that $\|(\forall x)(\exists y)(P(x) \to (P(y)\&C))\|^{\mathbf{L}}_{\mathbf{M},v} = 1$. Let $c = \|C\|^{\mathbf{L}}_{\mathbf{M},v}$ and let $b = \|(\exists x)P(x)\|^{\mathbf{L}}_{\mathbf{M},v}$. We claim that $b \leq c$ and that $b * c = b$.

For all $m \in \mathbf{M}$, $\bigvee_{m' \in \mathbf{M}} \|(P(m) \to (P(m')\&C))\|^{\mathbf{L}}_{\mathbf{M},v} = 1$ (this follows from the fact that, following the proof of [Háj98b, theorems 5.1.14, 5.1.18], it can be showed that $MTL\forall \vdash (\exists x)(\nu \to \varphi(x)) \to (\nu \to (\exists x)\varphi(x))$ and $MTL\forall \vdash (\exists x)(\varphi(x)\&\nu) \leftrightarrow ((\exists x)\varphi(x)\&\nu)$, where ν does not contains x freely), and hence $\|P(m)\|^{\mathbf{L}}_{\mathbf{M},v} \Rightarrow (b * c) = 1$. Hence $\|P(m)\|^{\mathbf{L}}_{\mathbf{M},v} \leq b * c$, and taking the supremum we get $b \leq b * c$ and finally $b = b * c$. By condition (*), for every $m \in \mathbf{M}$, $c * \|P(m)\|^{\mathbf{L}}_{\mathbf{M},v} = \|P(m)\|^{\mathbf{L}}_{\mathbf{M},v}$, and $\|(\exists y)(P(y) \to (P(y)\&C))\|^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{m \in M}(\|P(m)\|^{\mathbf{L}}_{\mathbf{M},v} \Rightarrow (c * \|P(m)\|^{\mathbf{L}}_{\mathbf{M},v})) = 1$. □

Before continuing, recall that Chang's MV-algebra ([Cha58]) is defined as

$$\mathbf{C}_\infty = \langle \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}, *, \Rightarrow, \sqcap, \sqcup, b_0, a_0 \rangle .$$

Where for each $n, m \in \mathbb{N}$, it holds that $b_n < a_m$, and, if $n < m$, then $a_m < a_n$, $b_n < b_m$; moreover $a_0 = 1$, $b_0 = 0$ (the top and the bottom element).

The operation $*$ is defined as follows, for each $n, m \in \mathbb{N}$:

$$b_n * b_m = b_0,\ b_n * a_m = b_{\max(0, n-m)},\ a_n * a_m = a_{n+m}.$$

We can now state

Theorem 7.2.3. *Suppose that a variety $\mathcal{V}$ of MTL-algebras contains either an infinite product chain or the Chang MV-algebra. Then the consequence relation*

$$(\forall x)(\exists y)(P(x) \to (P(y)\&C)) \vdash (\exists y)(P(y) \to (P(y)\&C))$$

can be invalidated in a (non safe) interpretation in which the truth values of both formulas $(\forall x)(\exists y)(P(x) \to (P(y)\&C))$ and $(\exists y)(P(y) \to (P(y)\&C))$ are defined.

Proof. In the case of Chang's algebra $\mathbf{C}_\infty$: let ε be the atom of the algebra. Take M to be the set of natural numbers, and let for $m \in M$, $\|P(m)\|^{\mathbf{L}}_{\mathbf{M},v} = (m+2)\varepsilon$ and $\|C\|^{\mathbf{L}}_{\mathbf{M},v} = 1 - \varepsilon$. Then $\|(P(m+1)\&C)\|^{\mathbf{L}}_{\mathbf{M},v} = (m+2)\varepsilon$ and $\|(P(m) \to (P(m+1)\&C))\|^{\mathbf{L}}_{\mathbf{M},v} = 1$. Hence, $\|P(m) \to (P(m)\&C)\|^{\mathbf{L}}_{\mathbf{M},v} = 1 - \varepsilon$. Therefore, $\|(\exists y)(P(y) \to (P(y)\&C))\|^{\mathbf{L}}_{\mathbf{M},v} = 1 - \varepsilon < 1$.

In case of a product algebra, any infinite product chain generates the whole variety of product algebras (see [CT00]). Thus $\mathcal{V}$ contains the product algebra on $[0,1]^*$, a non-standard extension of the standard product algebra on $[0,1]$. Now let M be the set of natural numbers and for $m \in M$, let $\|P(m)\|^{\mathbf{L}}_{\mathbf{M},v} = \frac{1}{2^{\frac{1}{m+1}}}$ and $\|C\|^{\mathbf{L}}_{\mathbf{M},v} = 1 - \varepsilon$, where ε is a positive infinitesimal in $[0,1]^*$. Then $\|P(m)\|^{\mathbf{L}}_{\mathbf{M},v} = \frac{1}{2^{\frac{1}{m+1}}} \leq \frac{1}{2^{\frac{1}{m+2}}}(1-\varepsilon)$, and $\|(P(m) \to (P(m+1)\&C))\|^{\mathbf{L}}_{\mathbf{M},v} = 1$. On the other hand $\|P(m) \to (P(m)\&C))\|^{\mathbf{L}}_{\mathbf{M},v} = 1-\varepsilon < 1$. Therefore, $\|(\exists y)(P(y) \to (P(y)\&C))\|^{\mathbf{L}}_{\mathbf{M},v} = 1-\varepsilon < 1$.

This ends the proof. □

Corollary 7.2.1. *Let L be an axiomatic extension of MTL and let $\mathcal{V}_L$ be the corresponding variety. If $\mathcal{V}_L$ satisfies condition (*) and contains either Chang's algebra or an infinite product chain, then $L\forall$ is not supersound.*

7.3 Applications to many-valued logics

In this section, we will apply the previous conditions to the most important many-valued logics.

A little remark: except for the logic RDP, the results of the following proposition have been showed, independently, in [van10, Lv08].

Proposition 7.3.1. *Let* L $\in$ {MTL, SMTL, IMTL, G, WNM, NM, RDP}, *then* L *enjoys the CEP.*

Proof. Let $\mathbf{A}$ be an L-chain: from lemma 7.2.2 we know that $\mathbf{A}^c$ is an MTL-chain. To prove the theorem we have to show that $\mathbf{A}^c$ is an L-chain.

MTL: Lemma 7.2.2.

SMTL: We have to check that for each $X \in A^c$, $((X \Rightarrow_\circ A) \cup X) \Rightarrow_\circ A = \{1\}$: the claim is obvious if $X \in \{A, \{1\}\}$.

Suppose now that $\{1\} \subset X \subset A$: we have to show that $(X \Rightarrow_\circ A) \cup X = A$, that means $(X \Rightarrow_\circ A) = A$. It suffices to prove that, for each $Z \in A^c$, $X \circ Z = A$ if and only if $Z = A$. For the non-trivial direction, suppose that $Z \subset A$. Then $Z^\ell \supset \{0\}$ and hence, since A has no zero divisors, $X^\ell * Z^\ell \supset \{0\}$. It follows that $X \circ Z \subset A$.

IMTL: It must hold that, for all $X \in A^c$, $((X \Rightarrow_\circ A) \Rightarrow_\circ A) \Rightarrow_\circ X = \{1\}$ i.e. $((X \Rightarrow_\circ A) \Rightarrow_\circ A) \supseteq X$: the claim is obvious if $X \in \{A, \{1\}\}$.

Suppose now that $\{1\} \subset X \subset A$.

- If X has a minimum, then (noting that $X \Rightarrow_\circ A = \bigcap\{Z \in A^c : Z^\ell * X^\ell = \{0\}\}$) it is not difficult to see that the biggest $V = Z^\ell$ such that $V * X^\ell = \{0\}$ is[1] $V = \{\sim \max(X^\ell)\}^\ell$ and hence $X \Rightarrow_\circ A = \{\sim \max(X^\ell)\}^{\ell u} = V^u$. It remains to show

[1]In what that follows $\sim x$ indicates $x \Rightarrow 0$.

that $V^u \Rightarrow_\circ A \supseteq X$, but this easily follows from the fact that, for each $x \in A$, $x = \sim\sim x$ and hence the biggest Z^ℓ such that $Z^\ell * \{\sim \max(X^\ell)\}^\ell = \{0\}$ is $(\sim \max(\{\sim \max(X^\ell)\}^\ell))^\ell = X^\ell$.

- The last case is when X does not have infimum: suppose that $((X \Rightarrow_\circ A) \Rightarrow_\circ A) \subset X$. This means that $\bigcap\{Z \in A^c : Z^\ell * (X \Rightarrow_\circ A)^\ell = \{0\}\} \subset X$ and hence the biggest Z^ℓ such that $Z^\ell * (X \Rightarrow_\circ A)^\ell = \{0\}$ contains X^ℓ. Given $z \in Z^\ell \setminus X^\ell$ it holds that $\sim z < \sim x$, for all $x \in X^\ell$: this follows from the fact that, for all $x, y \in A$, $\sim\sim x = x$ and hence, if $x < y$ then $\sim x > \sim y$. From these facts we have that $\sim z \in (X \Rightarrow_\circ A)^\ell$: moreover, since X^ℓ does not have supremum, then $Z^\ell \setminus X^\ell$ does not have infimum, otherwise $X \neq X^{\ell u}$. It follows that it exists $z' \in Z^\ell \setminus X^\ell$ with $z' < z$. By the involutiveness of negation, we have that $\sim z' \in (X \Rightarrow_\circ A)^\ell$, $\sim z' > \sim z$ and hence $Z^\ell * (X \Rightarrow_\circ A)^\ell \supset \{0\}$ and $Z \circ (X \Rightarrow_\circ A) \subset A$: this is in contradiction with the hypothesis.

G: We have to check that, for all $X \in A^c$, it holds that $X \circ X = X$. This is an easy consequence of the definition of $\circ$ and the fact that, for all $x, y \in A$, $x * y = \min(x, y)$.

WNM: The following equality must hold, for each $X, Y \in A^c$: $((X \circ Y) \Rightarrow_\circ A) \cap ((X \cup Y) \Rightarrow_\circ (X \circ Y)) = \{1\}$.

If $X \circ Y = A$, then $(X \circ Y) \Rightarrow_\circ A = \{1\}$.

Suppose that $X \circ Y \subset A$ and, without loss of generality, that $X \subseteq Y$: we must show that $(Y \Rightarrow_\circ (X \circ Y)) = \{1\}$, that means $Y \supseteq X \circ Y$. Since for each $x, y \in A$ it holds that $x * y = 0$ or $x * y = \min(x, y)$, then, noting that $Y^\ell \subseteq X^\ell$ and $X^\ell * Y^\ell \supset \{0\}$ we have that:

- If Y has a minimum, say m, then there exists an $x \in X^\ell$, $x \geq m$ such that $x * m \neq 0$ and hence $x * m = m$. From the hypothesis over $*$ it follows that $X \circ Y = \{X^\ell * Y^\ell\}^u = \{m\}^u = Y$.
- If Y does not have infimum, then Y^ℓ does not have supremum (otherwise we have that $Y \neq Y^{\ell u}$) and there are two cases.

  If $X = Y$, then from the hypothesis it must exist $x, y \in X^\ell$ such that $x * y \neq 0$ and hence, for each $z \geq \max(x, y)$ it holds that $z * z = z$.

  If $X \subset Y$, then $Y^\ell \subset X^\ell$ and, from the hypothesis, it exist $y \in Y^\ell \setminus \{0\}$, $x \in X^\ell \setminus Y^\ell$ such that $x * y = y$. It follows that, for each $z \in X^\ell$ such that $z \geq x$, $z * y' = y'$, for each $y' \in Y^\ell$, $y' \geq y$. In both cases, clearly $X \circ Y = Y$.

NM: An easy consequence of the results for IMTL and WNM.

RDP: In this case we have to show that, for all $X \in A^c$, $((X \Rightarrow_\circ A) \Rightarrow_\circ A) \cap (X \Rightarrow_\circ (X \Rightarrow_\circ A)) = \{1\}$: if $X \in \{A, \{1\}\}$, then it is easy to see that the claim holds.

Suppose now that $\{1\} \subset X \subset A$.

By contradiction, suppose that the equation does not hold, i.e. it exists $X \in A^c$ such that $X \Rightarrow_\circ A \subset A$ and $X \subset X \Rightarrow_\circ A$: equivalently, it holds that $\{0\} \subset (X \Rightarrow_\circ A)^\ell \subset X^\ell$. Recall that, for each $x \in A$, it holds that $\sim x = 0$ or $x \leq \sim x$: now, for each $x \in X^\ell$, if $\sim x = 0$, then $X^\ell * (X \Rightarrow_\circ A)^\ell \supset \{0\}$ and hence $X \circ (X \Rightarrow_\circ A) \subset A$, a contradiction. The only possibility is then that $x \leq \sim x$, for each $x \in X^\ell$: from the fact that $\sim$ is an order reversing mapping (i.e. $x \leq y$ implies $\sim y \leq \sim x$), we will now show that for each $x \in X^\ell \setminus (X \Rightarrow_\circ A)^\ell$ and each $y \in X^\ell$, $x * y = 0$. If $y \leq x$, then $\sim y \geq \sim x \geq x$, that implies $x * y = 0$; if $y > x$, then, since $y \leq \sim y$ it holds that $0 = y * y = x * y$. From these facts it must happen that $x \in (\bigcap\{Z \in A^c : \ Z^\ell * X^\ell = \{0\}\})^\ell = (X \Rightarrow_\circ A)^\ell$, but this is in contradiction with the hypothesis.

□

7.3.1 Supersound extensions of BL

In this section we study the following problem. We want to characterize those axiomatic extensions L of BL such that $L\forall$ is supersound. We will prove that the extensions L such that $L\forall$ is supersound are precisely those in which, for some positive natural number n, the axiom $\varphi^n \to \varphi^{n+1}$ is derivable: such extensions are said n-potent. Since any axiomatic extension of BL is complete with respect to the class of chains in which it is valid, any axiomatic extension of BL plus $\varphi^n \to \varphi^{n+1}$ will be complete with respect to the class of n-potent BL-chains (A BL-algebra is said to be *n-potent* if it satisfies the equation $x^n = x^{n+1}$). We will start this section with a representation theorem for BL-chains.

Definition 7.3.1. *Let I be a totally ordered set of indexes with minimum i_0, and for all $i \in I$, let $\mathbf{H}_i$ be the reduct of a totally ordered MV-algebra in the language $\{\Rightarrow, *, 1\}$, or the negative cone of a totally ordered abelian group $\mathbf{G}$, where $1 = 1_i$ is the neutral element of $\mathbf{G}$, $*_i$ is the restriction of multiplication of $\mathbf{G}$ to its negative cone and $x \Rightarrow_i y = (x^{-1} *_i y) \wedge 1$. Assume further that $\mathbf{H}_{i_0}$ is the reduct of an MV-algebra, and that, for $i \neq j$, $\mathbf{H}_i \cap \mathbf{H}_j = \{1_i\} = \{1_j\}$. Then the* ordinal sum *$\bigoplus_{i \in I} \mathbf{H}_i$ is defined as follows:*

(a) The universe of $\bigoplus_{i \in I} \mathbf{H}_i$ is $\bigcup_{i \in I} \mathbf{H}_i$; 1 is the common neutral element of all $\mathbf{H}_i$, and $0 = \min(\mathbf{H}_{i_0})$.

(b) Multiplication is defined by

$$x * y = \begin{cases} x *_i y & \textit{if } x, y \in \mathbf{H}_i,\ i \in I \\ y & \textit{if } y \in \mathbf{H}_i \setminus \{1\},\ x \in \mathbf{H}_j,\ i < j \\ x & \textit{if } x \in \mathbf{H}_i \setminus \{1\},\ y \in \mathbf{H}_j,\ i < j \end{cases}$$

(c) Implication is defined by

$$x \Rightarrow y = \begin{cases} x \Rightarrow_i y & \textit{if } x, y \in \mathbf{H}_i,\ i \in I \\ y & \textit{if } y \in \mathbf{H}_i \setminus \{1\},\ x \in \mathbf{H}_j,\ \textit{with } i < j \\ 1 & \textit{otherwise.} \end{cases}$$

The algebras $\mathbf{H}_i$ *will be called the components of* $\bigoplus_{i \in I} \mathbf{H}_i$.

Note that the order in an ordinal sum is given by $x \leq y$ iff $x \Rightarrow y = 1$ and can be equivalently described as follows: (1) 1 is the top and 0 is the bottom; (2) the restriction of order to each component is the original order in that component; (3) if $i < j$, then every element of $\mathbf{H}_i \setminus \{1\}$ precedes every element of $\mathbf{H}_j$. It turns out that $\bigoplus_{i \in I} \mathbf{H}_i$ is in any case a BL-chain.

In [AM03] the converse is showed:

Proposition 7.3.2. *Every totally ordered BL-algebra* $\mathbf{H}$ *can be represented as an ordinal sum* $\bigoplus_{i \in I} \mathbf{H}_i$ *of totally ordered reducts of MV-algebras and of negative cones of totally ordered abelian groups.*

Theorem 7.3.1. *Let* $\mathbf{B}$ *be any* n*-potent BL-chain. Then* $\mathbf{B}$ *and its MacNeille completion generate the same variety.*

Proof. To begin with, $\mathbf{B}$ is the ordinal sum of n-potent components. Since no negative cone of a totally ordered group is n-potent, such components must be (reducts of) MV-chains with $n + 1$ elements at most. Thus, we can represent $\mathbf{B}$ as $\mathbf{B} = \bigoplus_{i \in I} \mathbf{H}_i$ where each $\mathbf{H}_i$ is a finite MV-chain.

Remark: the lemmas from 7.3.1 to 7.3.4 are part of the proof.

Lemma 7.3.1. *If* I *is complete (as an ordered set), then* $\mathbf{B}$ *itself is complete.*

Proof. Let $\emptyset \neq X \subseteq \mathbf{B}$. Without loss of generality, we may assume that $0, 1 \notin X$ (if $1 \in X$, then $1 = \sup(X)$ and if 1 is not the only element of X, then 1 is certainly not the infimum of X, and dually for 0). Let $I_X = \{i \in I : X \cap \mathbf{H}_i \neq \emptyset\}$. If I_X has a maximum j, then $\sup(X) = \max(X) = \max(X \cap \mathbf{H}_j)$, which exists because $X \cap \mathbf{H}_i$ is a finite set. If I_X has no maximum, then it has a supremum, j say. Since j is not $\max(I_X)$, $X \cap \mathbf{H}_j = \emptyset$ and $\min(\mathbf{H}_j) = \sup(X)$. As regards to $\inf(X)$, if I_X has a minimum h, then $\inf(X) = \min(X) = \min(X \cap \mathbf{H}_h)$; otherwise, $\inf(X) = \max(\mathbf{H}_h \setminus \{1\})$ (that is $\inf(X)$ is the coatom of $\mathbf{H}_h$). This concludes the proof of lemma 7.3.1. $\square$

By lemma 7.3.1, if I is complete, then $\mathbf{B}$ is in turn complete and it is the MacNeille completion of itself, and theorem 7.3.1 is proved. In particular, this condition holds if I is finite.

Now suppose that I is not complete. Let J be the MacNeille completion of I as a lattice. In particular, we have that: (1) I is meet dense and join dense in J, that is, every element of J is the infimum of a subset of I and the supremum of a subset of I; (2) every non-empty subset of J has an infimum and (3) every bounded non-empty subset of J has a supremum. (Note that I has a minimum i_0 but not necessarily a maximum; this determines an asymmetry between conditions (2) and (3)). Let $\mathbf{2}$ be the two-element MV-chain.

For every $j \in J$, we define

$$\mathbf{M}_j = \begin{cases} \mathbf{H}_j & \text{if } j \in I \\ \mathbf{2} & \text{otherwise.} \end{cases}$$

Next, we define $\mathbf{M} = \bigoplus_{j \in J} \mathbf{M}_j$. Clearly, $\mathbf{M}$ is an n-potent BL-chain. By the argument used in the proof of lemma 7.3.1, we obtain:

Lemma 7.3.2. $\mathbf{M}$ *is complete.*

Continuing with the proof of theorem 7.3.1, we prove that:

Lemma 7.3.3. $\mathbf{M}$ *is the MacNeille completion of* $\mathbf{B}$.

Proof. It suffices to show that $\mathbf{B}$ is both join dense and meet dense in $\mathbf{M}$ (this property characterizes MacNeille completions up to isomorphism). We have to prove that if $b \in \mathbf{M}$, then there are $X, Y \subseteq \mathbf{B}$ such that $b = \sup(X) = \inf(Y)$. The claim is obvious if $b \in \mathbf{B}$. Thus, suppose $b \notin \mathbf{B}$. Then, by the definition of $\mathbf{M}$, it must be $b = \min(\mathbf{M}_j)$ for some $j \in J \backslash I$. In this case, we have: $b = \sup(\bigcup_{i \in I, i<j} \mathbf{M}_i) = \inf(\bigcup_{i \in I, j<i} \mathbf{M}_i)$ and it suffices to take $X = \bigcup_{i \in I, i<j} \mathbf{M}_i$ and $Y = \bigcup_{i \in I, j<i} \mathbf{M}_i$. □

Lemma 7.3.4. *Every finitely generated subalgebra* $\mathbf{A}$ *of* $\mathbf{M}$ *is isomorphic to a subalgebra of* $\mathbf{B}$.

Proof. Let $a_1, \ldots, a_n$ be the generators of $\mathbf{A}$ (without loss of generality, we may assume that $1 \neq a_i$ for $i = 1, \ldots, n$). Suppose, without loss of generality, $a_1, \ldots, a_k \in \mathbf{B}$ and $a_{k+1}, \ldots, a_n \notin \mathbf{B}$. Then, for $i = k+1, \ldots, n$, $a_i = \min(\mathbf{M}_{h_i})$ where $h_i \in J \backslash I$. Let $i_1, \ldots, i_k$ be such that for $j = 1, \ldots, k$, $a_j \in \mathbf{H}_{i_j}$. Since $h_{k+1}, \ldots, h_n$ are limit points of I, for $i = k+1, \ldots, n$ we can find $s_i \in I$ such one of the following conditions hold:

(a) $s_i < h_i$ and there are no i_j, s_m with $j = 1, \ldots, k$ and with $m = k+1, \ldots, n$, and $m \neq i$ such that either $s_i \leq i_j \leq h_i$, or $s_i \leq s_m \leq h_i$.

(b) $h_i < s_i$ and there are no i_j, s_m with $j = 1, \ldots, k$ and with $m = k+1, \ldots, n$, and $m \neq i$ such that either $h_i \leq i_j \leq s_i$, or $h_i \leq s_m \leq s_i$.

Hence the function mapping $i_1, \dots, i_k$ into itself and h_i into $s_i : i = k+1, \dots, n$ is an order isomorphism from $\{i_1, \dots, i_k, h_{k+1}, \dots, h_n\}$ into $\{i_1, \dots, i_k, s_{k+1}, \dots, s_n\}$. Consider the subalgebra $\mathbf{C}$ of $\mathbf{M}$ generated by $a_1, \dots, a_k$ and $\min(\mathbf{M}_{s_{k+1}}), \dots, \min(\mathbf{M}_{s_n})$. It is readily seen that $\mathbf{C}$ is isomorphic to $\mathbf{A}$ (via the isomorphism mapping each $a_i : i = 1, \dots, k$ into itself and $a_{k+1}, \dots, a_n$ into $\min(\mathbf{M}_{s_{k+1}}), \dots, \min(\mathbf{M}_{s_n})$ respectively). Moreover, $a_1, \dots, a_k, \min(\mathbf{M}_{s_{k+1}}), \dots, \min(\mathbf{M}_{s_n})$ are all in $\mathbf{B}$, and hence $\mathbf{C}$ is a subalgebra of $\mathbf{B}$ isomorphic to $\mathbf{A}$. □

We conclude the proof of theorem 7.3.1. It is left to prove that $\mathbf{B}$ and its MacNeille completion $\mathbf{M}$ generate the same variety. Clearly each equation that holds in $\mathbf{M}$, holds in $\mathbf{B}$. From lemma 7.3.4, if an equation fails in $\mathbf{M}$, then it fails in $\mathbf{B}$: it follows that $\mathbf{M}$ and $\mathbf{B}$ generates the same variety. □

Theorem 7.3.2. *If L is an axiomatic extension of BL and for some n, the axiom schema $\varphi^n \to \varphi^{n+1}$ is derivable in L, then L enjoys the CEP and hence $L\forall$ is supersound with respect to the class of all models over L-chains.*

Proof. Every L-chain embeds into a complete L-chain by an embedding which preserves existing infima and suprema. This means that every (possibly unsafe) interpretation extends to a safe interpretation in which the existing evaluations of formulas are preserved. This concludes the proof. □

We are going to prove that if L does not prove the schema $\varphi^n \to \varphi^{n+1}$ for any n, then $L\forall$ is not supersound. Let $\mathcal{V}_L$ be the variety of all L-algebras.

Lemma 7.3.5. *Either $\mathcal{V}_L$ contains the variety of product algebras or it contains the variety generated by Chang's algebra.*

Proof. We distinguish two cases:

(1) For some k, $\mathcal{V}_L$ satisfies the equation $(\sim\sim x)^{k+1} = (\sim\sim x)^k$. Then, since for all x, $\sim\sim x$ belongs to the first component, we have that the first component of every L-chain is finite. On the other hand, for every $n \geq k$, $\mathcal{V}_L$ contains a non-n-potent chain, $\mathbf{B}_n$, say. Let $b_n \in \mathbf{B}_n$ be such that $b_n^{n+1} < b_n^n$. Note that b_n does not belong to the first component of $\mathbf{B}_n$. Now take an ultraproduct $\mathbf{B}$ of all $\mathbf{B}_n$ modulo a non-principal ultrafilter. By the ultraproduct theorem (see for example [CK90]), $\mathbf{B}$ is an L-chain, and in $\mathbf{B}$ there is an element b such that $b^{n+1} < b^n$ for every n. Now take the subalgebra $\mathbf{C}$ of $\mathbf{B}$ generated by b. Since all powers of b are in the same component, together with 1 they constitute a subreduct of $\mathbf{B}$ isomorphic to the negative cone of the integers $\mathbf{Z}$. Hence, $\mathbf{C}$ consists of 0 plus all powers of b and it is isomorphic to an infinite product chain. Since every infinite product chain generates the whole variety of product algebras, $\mathcal{V}_L$ contains the variety of product algebras.

	MTL	SMTL	IMTL	WNM	NM	RDP	G	L_n
CEP	yes	yes	yes	yes	yes	yes	yes	yes
SS	yes	yes	yes	yes	yes	yes	yes	yes
SCC	yes	yes	yes	yes	yes	yes	yes	yes
SCF	yes	yes	yes	yes	yes	yes	yes	yes

Table 7.1: Positive results.

(2) For every k, $\mathcal{V}_L$ does not satisfy the equation $(\sim\sim x)^{k+1} = (\sim\sim x)^k$. Then, for every k, there is an L-chain $\mathbf{B}_k$, whose first component is not k-potent. Let $\mathbf{B}$ be an ultraproduct of all the first components of the algebras $\mathbf{B}_k$ modulo a non-principal ultrafilter. Once again, by the ultraproduct theorem, $\mathbf{B}$ is an L-chain, and in $\mathbf{B}$ there is an element b such that $b^{n+1} < b^n$ for every n. Hence, the subalgebra $\mathbf{C}$ of $\mathbf{B}$ generated by b is Chang's algebra, and $\mathcal{V}_L$ contains the variety generated by Chang's algebra.

Thanks to corollary 7.2.1, in order to conclude the proof that $L\forall$ is not supersound, it suffices to prove that $\mathcal{V}_L$ satisfies condition (*) of theorem 7.2.2: for every chain $\mathbf{A} \in \mathcal{V}_L$, and for every $a \leq b \leq c \in \mathbf{A}$, if $b * c = b$, then $a * c = a$. Now the claim is trivial if $a = b$ or $c = 1$, and if $a < b$ (with $c \neq 1$), then a and c belong to different components. Therefore, $a * c = a$. □

Corollary 7.3.1. *Let* L $\in$ {MTL, SMTL, IMTL, G, WNM, NM, RDP, L_n}*, where* L_n *is an* n*-potent axiomatic extension of BL. Then*

- L$\forall$ *is supersound (SS).*
- L *is strongly complete with respect to the class of complete L-chains (SCC).*
- L$\forall$ *is strongly complete with respect to the class of complete L-chains (SCF).*

7.3.2 Negative results

In this section we investigate the negative results about supersoundness and CEP.

Theorem 7.3.3. *Let N be any axiomatic extension of BL that, for each n, it is not n-potent (see section 7.3.1). For $L \in \{\Pi MTL, WCMTL, N\}$, $L\forall$ is not supersound. In particular, a fortiori $Ł\forall, \Pi\forall, BL\forall, SBL\forall$ are not supersound.*

Proof. From corollary 7.2.1 it remains to check that $\mathcal{V}_L$ contains either Chang's algebra or an infinite product chain and satisfies (*) of theorem 7.2.2: for $L = N$, this has been proved in lemma 7.3.5.

For the other cases, an easy computation shows that in each ΠMTL (WCMTL) chain, given $a \le b \le c$, it holds $b * c = b$ if and only if $c = 1$ or $b = 0$: in both cases, clearly, $a * c = a$. Finally, the variety of ΠMTL (WCMTL) algebras contains an infinite product chain. □

Corollary 7.3.2. *For $L \in \{Ł, \Pi, BL, SBL, \Pi MTL, WCMTL, N\}$, L does not enjoy the CEP.*

To conclude, we show two examples regarding the failure of the CEP: the first one concerns Łukasiewicz logic, the second one ΠMTL and WCMTL logics.

Counterexample 7.3.1. *Consider Chang's* MV*-algebra. It is not difficult to see that the universe of $(\mathbf{C}_\infty)^c$ is the set $\{\{x\}^u :\ x \in \mathbf{C}_\infty\} \cup \{a_n :\ n \in \mathbb{N}\}$. Now we show that the equation $X \cup Y = X \circ (X \Rightarrow_\circ Y)$ does not hold:*

Take $X = \{a_n :\ n \in \mathbb{N}\}$ and $Y = \{b_i\}^u$, where $i \in \mathbb{N}^+$ and consider $X \Rightarrow_\circ Y = \bigcap\{Z \in A^c :\ (Z^\ell * X^\ell)^u \supseteq Y\}$: direct inspection shows that the biggest Z^ℓ that satisfies this condition is $X^\ell = \{b_n :\ n \in \mathbb{N}\}$ (suppose not, then $Z^\ell = \{a_k\}^\ell$ for some k: it follows that $a_k * X^\ell = X^\ell = Z^\ell * X^\ell$ and hence $(Z^\ell * X^\ell)^u = X \subset Y$). From these results it holds that $X \Rightarrow_\circ Y = X$ and $X \circ (X \Rightarrow_\circ Y) = X \circ X = A \supset Y = X \cup Y$.

Counterexample 7.3.2. *Consider the following algebraic structure: $\mathbf{A} = \langle A = \{\langle x, y\rangle :$
$x, y \in (0,1] \cap \mathbb{Q}\} \cup \{\langle 0, 1\rangle\}, *, \Rightarrow, \sqcap, \sqcup, \langle 0, 1\rangle, \langle 1, 1\rangle\rangle$. This algebra has a lexicographic order $\preceq$ (i.e. $\langle x, y\rangle \prec \langle x', y'\rangle$ when $x < x'$ or $x = x'$ and $y < y'$) and the operations are defined as follows:*

$$\langle s, r\rangle * \langle t, v\rangle := \begin{cases} \langle 0, 1\rangle & \text{if } s \cdot t = 0 \\ \langle s \cdot t, r \cdot v\rangle & \text{otherwise} \end{cases}$$

$$\langle s, r\rangle \Rightarrow \langle t, v\rangle := \begin{cases} \langle 1, 1\rangle & \text{if } \langle s, r\rangle \preceq \langle t, v\rangle \\ \langle \frac{t}{s}, \frac{v}{r}\rangle & \text{if } s > t \text{ and } r \ge v \text{ or } t = s \text{ and } r > v \\ \langle \frac{t}{s}, 1\rangle & \text{otherwise} \end{cases}$$

Direct inspection shows that $\mathbf{A}$ is a ΠMTL-chain (and hence a WCMTL-chain): we show that it exist $X, Y, Z \in A^c$ with $X \ne A$, $Z \ne Y$ such that $X \circ Y = X \circ Z$.

Let $X, Y, Z \in A^c$ be such that $X = \{\langle x, y\rangle \in A :\ \langle x, y\rangle \succ \langle \alpha, 1\rangle, \alpha$ being an irrational in $(0,1]\}$, $Y = \{\langle p, u\rangle\}^u$, $Z = \{\langle p, v\rangle\}^u$, with $u < v$ and $p, u, v \in (0,1) \cap \mathbb{Q}$: we prove that $X^\ell * Y^\ell$ and $X^\ell * Z^\ell$ have the same set of upper bounds, i.e. $X \circ Y = X \circ Z$. Consider the set $A_c = \{\langle p \cdot x, y\rangle :\ \langle x, y\rangle \in X\}$ (note that *this set has no infimum, since the same holds for X): given $\langle p \cdot x, y\rangle \in A_c$, for each $\langle z, 1\rangle \in X^\ell$ it holds that $\langle p \cdot z, u\rangle \prec \langle p \cdot z, v\rangle \prec \langle p \cdot x, y\rangle$ and hence $A_c \subset X \circ Y$, $A_c \subset X \circ Z$. For each $\langle x, y\rangle \in A$, if $\langle x', y'\rangle \prec \langle x, y\rangle \prec \langle x'', y''\rangle$ with $\langle x', y'\rangle, \langle x'', y''\rangle \in A_c$, then $\langle x, y\rangle \in X \circ Y, \langle x, y\rangle \in X \circ Z$: moreover, since $A_c^- = \{\langle p \cdot z, t\rangle : \langle z, t\rangle \in X^\ell\}$ and X^ℓ does not have supremum, it follows that there cannot be elements of A strictly between A_c^- and A_c.*

Finally note that each element greater than $\langle p, 1\rangle$ is an upper bound of $X^\ell * Y^\ell$ and $X^\ell * Z^\ell$; conversely, each $\langle x, y\rangle$ such that $\langle x', y'\rangle \preceq \langle x, y\rangle \preceq \langle x'', y''\rangle$ with $\langle x', y'\rangle, \langle x'', y''\rangle$ $\in A_c^-$ never belongs to $X \circ Y$ or $X \circ Z$, since X^ℓ does not have supremum. From these facts we have that $X \circ Y = X \circ Z$ and $\mathbf{A}^c$ cannot be a ΠMTL-chain. Moreover, since $X \circ Y \neq A$, we have that $\mathbf{A}^c$ cannot even be a WCMTL-chain.

7.3.3 Incompleteness results

In the sequel, N denotes an arbitrary axiomatic extension of BL such that for every n there is a formula φ such that $N \not\vdash \varphi^n \to \varphi^{n+1}$. Note that N may be any of Ł, Π, BL or SBL.

Theorem 7.3.4. *Let L be any of N, ΠMTL or $WCMTL$. Then L enjoys neither SCC nor SCF (see corollary 7.3.1).*

Proof. Let $\Gamma = \{p_0 \to p_1^2, p_1 \to p_2^2, \ldots, p_n \to p_{n+1}^2, \ldots\}$ and let $\Delta = \{p_n \to q : n \in \omega\}$.

Remark: the following two lemmas are part of the proof.

Lemma 7.3.6. *Let L be any of N, ΠMTL or $WCMTL$. Then in every complete L-chain $\mathbf{A}$ we have $\Gamma \cup \Delta \models_{\mathbf{A}} (p_0 \to p_0^2) \vee (p_0 \to (p_0 \& q))$.*

Proof. Let v be a valuation such that $v(p_0)$ is not an idempotent, and such that $v(\phi) = 1$ for all $\phi \in \Gamma \cup \Delta$. Let $\alpha = \sup\{v(p_n) : n \in \omega\}$. Then since $v(p_{n+1}^2) \geq v(p_n)$ we have $\alpha^2 = \sup\{v(p_{n+1})^2 : n \in \omega\} \geq \sup\{v(p_n) : n \in \omega\} = \alpha$, and α is an idempotent. Moreover $v(q) \geq \alpha$. For ΠMTL or $WCMTL$, this implies that either $\alpha = 0$ or $\alpha = 1$. Now $\alpha = 0$ would imply that $v(p_0) = 0$, which is excluded, as $v(p_0)$ is not an idempotent. Hence, $\alpha = v(q) = 1$ and $v(p_0) = v(p_0 \& q)$. For N, we have that α is an idempotent such that $v(p_0) \leq \alpha \leq v(q)$, and again $v(p_0) = v(p_0 \& q)$. □

Lemma 7.3.7. *In any of N, ΠMTL, $WCMTL$, $(p_0 \to p_0^2) \vee (p_0 \to (p_0 \& q))$ is not derivable from $\Gamma \cup \Delta$.*

Proof. Let L be any of these logics and let $\mathcal{V}_L$ be the corresponding variety. Then $\mathcal{V}_L$ contains either Chang's algebra or all product chains. Suppose that $\mathcal{V}_L$ contains Chang's algebra. Then, it contains the variety generated by Chang's algebra, and hence it contains all perfect MV-algebras. Let $[0, 1]^*$ be an ultrapower of $[0, 1]$ with respect to a non-principal ultrafilter,

	ΠMTL	WCMTL	BL	SBL	Ł	Π	N
CEP	no	no	no	no	no	no	no
SS	no	no	no	no	no	no	no
SCC	no	no	no	no	no	no	no
SCF	no	no	no	no	no	no	no

Table 7.2: Negative results.

equipped with MV-operations and with product. Let $\varepsilon \in [0,1]^*$ be a positive infinitesimal and let $\mathbf{A}$ be the MV-subalgebra of $[0,1]^*$ generated by all elements of the form $\{\alpha\varepsilon\colon\ \alpha \in \mathbb{R},\ \alpha > 0\} \cup \{\alpha\varepsilon^2\colon\ \alpha \in \mathbb{R},\ \alpha > 0\}$. Then, $\mathbf{A}$ is a perfect MV-algebra, and hence it is in $\mathcal{V}_L$. Let $v(p_n) = 1 - \frac{\varepsilon}{2^n}$ and let $v(q) = 1 - \varepsilon^2$. Then, $v(p_{n+1}^2) = 1 - 2\frac{\varepsilon}{2^{n+1}} = 1 - \frac{\varepsilon}{2^n} = v(p_n)$. Moreover $v(p_n) = 1-\frac{\varepsilon}{2^n} \leq 1-\varepsilon^2 = v(q)$. Hence, for every $\phi \in \Gamma\cup\Delta$, we have $v(\phi) = 1$. Moreover, $v((p_0 \to p_0^2) \vee (q \to (q\&p_0))) = 1-\varepsilon \vee (\varepsilon^2+1-\varepsilon^2-\varepsilon) = 1-\varepsilon < 1$.

Next, suppose that $\mathcal{V}_L$ contains all product chains. Let $[0,1]^*$ be a non-standard extension of the standard product algebra. Define inductively $v(p_0) = \frac{1}{2}$ and $v(p_{n+1}) = \sqrt{v(p_n)}$. Moreover, let $v(q) = 1 - \varepsilon$. Then, $v(p_{n+1}^2) = v(p_n) \leq v(q)$, and hence for all $\phi \in \Gamma \cup \Delta$ we have $v(\phi) = 1$. On the other hand, $v((p_0 \to p_0^2) \vee (p_0 \to (p_0\&q))) = \frac{1}{2} \vee 1 - \varepsilon = 1 - \varepsilon < 1$. □

Summing-up, for any complete L-chain $\mathbf{A}$, $\Gamma \cup \Delta \models_{\mathbf{A}} (p_0 \to p_0^2) \vee (p_0 \to (p_0\&q))$, but $\Gamma \cup \Delta \not\vdash_L (p_0 \to p_0^2) \vee (p_0 \to (p_0\&q))$. Hence, L has not the SCC. As regards to SCF, the proof is analogous to propositional case, substituting the variables with 0-ary predicates. □

7.4 Conclusions

In this chapter we have showed positive and negative conditions for the supersoundness property: the CEP is a sufficient condition for supersoundness, whilst corollary 7.2.1 presents a condition of non-supersoundness. As an application, we have obtained several results for some many-valued logics, generalizing the previous works [HPS00, HS01].

Concerning the CEP, a general question needs to be analyzed:

Problem 7.4.1. *In theorem 7.2.1 we have showed that if L enjoys the CEP, then $L\forall$ is supersound. Is the converse true ?*

It is interesting to note that, between the logics that enjoy the CEP, those ones that are given by left-continuous t-norms satisfy the σ-embedding property with respect to standard algebras. This originates the following problem:

Problem 7.4.2. *Let L be an axiomatic extension of MTL whose algebraic semantics has at least one standard L-algebra. Suppose that L enjoys the CEP: does this property imply the σ-embedding property with respect to standard L-algebras ?*

The answer, however, is negative.

Counterexample 7.4.1. *Consider the logic BL_3: it is axiomatized as BL, plus the 2-potence.*

Direct inspection shows that the standard Gödel algebra belongs to the variety of BL_3-algebras. In particular this is the only standard algebra in the variety: from proposition 7.3.2 and the proof of theorem 7.3.1 we easily see that each BL_3-chain is an ordinal sum of (reduct of) MV-chains of at most three elements. Hence, if we take the ordinal sum MV-chains (of at most three elements) different from **2***, then the resulting algebra is not dense. Moreover, thanks to theorem 7.3.2 we have that BL_3 enjoys the CEP.*

Finally, from [CEG$^+$09, theorem 3.5] we have that the σ-embedding property with respect to standard algebras implies the propositional strong standard completeness. However $\varphi \to \varphi^2$ is a tautology of standard Gödel algebra, but not a theorem of BL_3.

Chapter 8

n-contractive BL-logics

This chapter is based on paper [BM10].

We have studied four families of n-contractive axiomatic extensions of BL and their corresponding varieties: BL^n, SBL^n, BL_n and SBL_n. Concerning BL^n we have that every BL^n-chain is isomorphic to an ordinal sum of MV-chains of at most $n+1$ elements, whilst every BL_n-chain is isomorphic to an ordinal sum of MV_n-chains (for SBL^n and SBL_n a similar property holds, with the difference that the first component must be the two elements boolean algebra); all these varieties are locally finite. In particular, we have studied generic and k-generic algebras, completeness and computational complexity results, amalgamation and interpolation properties. Finally, we have analyzed the first-order versions of these logics, from the point of view of completeness and arithmetical complexity.

8.1 Introduction

The aim of this chapter is to investigate some n-contractive BL-logics and their algebras. The n-contraction law, coming from the wider framework of substructural logics, was introduced to the fuzzy logics setting in the paper [CEG08] and systematically studied for wide varieties of MTL-algebras in [HNP07]. We will concentrate our attention on four n-contractive BL-logics, namely, BL^n, that is, BL plus n-contraction, SBL^n, that is, BL^n plus the strict negation axiom, BL_n, whose algebraic semantics is the variety generated by of all ordinal sums of (isomorphic copies of) the $n+1$-element MV-chain $\mathbf{L}_n$, and SBL_n, that is, BL_n plus the strict negation axiom. The motivation of this topic is twofold: first of all, it is interesting to study many-valued logics with a weak form of contraction. Indeed, the divisibility principle $(\phi\&(\phi \to \psi)) \to (\psi\&(\psi \to \phi))$, which is one of the basic axioms of BL, can be derived using contraction, although it does not imply contraction, and hence it can be regarded as a weak form of contraction; n-contraction is also a weak form of contraction, and the strict negation principle $\neg(\neg\phi \wedge \neg\neg\phi)$ is in turn equivalent to contraction for negated formulas, that is $\neg\phi \to (\neg\phi \& \neg\phi)$.

Yet another motivation for the study of n-contractive BL-logics is that, as shown by Busaniche and Cabrer in [BC09], a variety of BL-algebras is dual canonical iff it is a variety of n-contractive BL-algebras. A similar result was also proved in chapter 7 (see also [BM09]): let $\mathcal{V}$ be a variety of BL-algebras. The following are equivalent:

(1) $\mathcal{V}$ is a variety of n-contractive BL-algebras.

(2) the MacNeille completion of any chain in $\mathcal{V}$ is in $\mathcal{V}$.

As a consequence, we have the following: let $\mathcal{V}$ be any variety of n-contractive BL-algebras, i.e., a subvariety of $\mathcal{BL}^n$, let L be its corresponding logic (axiomatized over BL by all formulas $\phi \leftrightarrow \psi$ such that $\phi = \psi$ is a defining equation of $\mathcal{V}$). Then, its first order extension L$\forall$ is strongly complete with respect to the class of complete (with respect to the order) chains in $\mathcal{V}$. In other words, we have strong completeness with respect to the most natural semantics for first-order many-valued logics, that is, complete many-valued chains in the corresponding variety.

Of course, if $n > 1$, then the standard semantics on the real interval $[0,1]$, as well as every densely ordered set (with maximum 1 and minimum 0), is not appropriate for these logics. Indeed, every n-contractive BL-chain $\mathbf{A}$ is the ordinal sum of finite MV-chains, and then: (a) if at least one of these chains has more than two elements, then $\mathbf{A}$ cannot be densely ordered and hence it cannot have $[0,1]$ (and not even $[0,1] \cap \mathbb{Q}$) as lattice reduct (if x is a coatom of an MV-component with more than two elements, then there is no element in the open interval (x^2, x)); (b) if all MV-components have two elements only, then $\mathbf{A}$ is a Gödel algebra, that is, a 1-contractive BL-algebra.

As pointed out by a referee, the logics studied here are axiomatic extensions of the logics $\Omega(\mathrm{S}_n\mathrm{MTL})$ studied in [HNP07]. In that paper it was proved that the corresponding varieties of MTL-algebras enjoy the Finite Embeddability Property although they fail in general to be locally finite. In contrast, as we have already pointed out, in the present chapter we will show that all of their varieties are locally finite. Moreover, the logics $\Omega(\mathrm{S}_n\mathrm{MTL})$ have strong standard completeness, while, as previously noticed, their BL counterparts do not even have standard algebras in their semantics.

Note that if $n > 2$, $\mathcal{BL}_n$ is a proper subvariety of $\mathcal{BL}^n$. For instance, if $m < n$ and m does not divide n, then $\mathbf{L}_m \in \mathcal{BL}^n \setminus \mathcal{BL}_n$. The variety $\mathcal{BL}_n$ is axiomatized over $\mathcal{BL}^n$ by all axioms of the form

$$(\mathrm{div}_{n,m}) \qquad ((x \to x^n) \leftrightarrow x^{m-1})^n \leq x^n,$$

where m does not divide n. (The axiom says that in each component there is no element x whose negation relative to the component it belongs to is equal to x^{m-1}). An alternative axiomatization can be provided by means of a generalization of Łukasiewicz sum $\oplus$, and will be introduced in the next section.

We will discuss the following topics:

(1) Models generating the whole variety (generic models) and models whose generated variety contains the class of k-generated algebras of the variety (k-generic models). In particular, we will prove that $\mathcal{BL}^n$ is not complete with respect to a single chain in $\mathcal{BL}^n$ (called BL^n-chain in the sequel), whereas $\mathcal{SBL}^n$, $\mathcal{BL}_n$ and $\mathcal{SBL}_n$ are strongly complete with respect to a single countable chain in $\mathcal{SBL}^n$, $\mathcal{BL}_n$ and $\mathcal{SBL}_n$ respectively. Moreover, for each of these varieties, we will build a countable algebra which generates it. Then, after noting that every variety of n-contractive BL-algebras has the finite model property, we will compute upper bounds for the minimum cardinality of a k-generic model, for both $\mathcal{BL}^n$ and $\mathcal{BL}_n$.

(2) n-contractive BL-logics, completeness and complexity. We will investigate the complexity of the logics BL^n, SBL^n, BL$_n$ and SBL$_n$, and we will prove the predictable fact that the set of positively satisfiable and of satisfiable formulas is NP-complete for all logics and that the 1-tautologicity and the positive tautologicity problem for all logics are both Co-NP complete.

For these logics we will also exhibit natural deterministic algorithms of complexity exponential in the number of variables of the given formula. Finally, for L $\in$ {SBL^n, BL$_n$, SBL$_n$} we will also construct an L-chain $\mathbf{C}$ such that, for every L-chain $\mathbf{A}$, L is strongly complete with respect to $\mathbf{A}$ iff it contains $\mathbf{C}$ as subalgebra.

(3) Amalgamation and interpolation. We prove that $\mathcal{BL}_n$ and $\mathcal{SBL}_n$ have the amalgamation property. It follows that the corresponding logics BL$_n$ and SBL$_n$ have the deductive interpolation property. For $n > 2$, BL$_n$ and SBL$_n$ do not have Craig's interpolation property, (incidentally, they do not even have Beth's definability property, cf [Mon06]), but they have a weak form of interpolation: if $\vdash_{BL_n} \phi^n \to \psi$ ($\vdash_{SBL_n} \phi^n \to \psi$ respectively), then there is a formula γ in the variables common to ϕ and ψ such that such that $\vdash_{BL_n} \phi^n \to \gamma$ and $\vdash_{BL_n} \gamma^n \to \psi$ ($\vdash_{SBL_n} \phi^n \to \gamma$ and $\vdash_{SBL_n} \gamma \to \psi$ respectively). On the contrary, $\mathcal{BL}^n$ and $\mathcal{SBL}^n$ do not have the amalgamation property, and hence BL^n and SBL^n do not even have the deductive interpolation property.

(4) First-order n-contractive BL-logics, problems of arithmetical complexity and complexity with respect to chains. First of all, due to a general theorem of Hájek, we have strong completeness of BL$^n\forall$, SBL$^n\forall$, BL$_n\forall$ and SBL$_n\forall$ with respect to the class of all BL^n-chains (SBL^n-chains, BL$_n$-chains and SBL$_n$-chains respectively). Hence the set of formulas valid in all BL^n-chains (SBL^n-chains, BL$_n$-chains and SBL$_n$-chains respectively) is Σ_1-complete. By a similar argument, the set of formulas which are pos-

itively satisfiable in some BL^n-chain (SBL^n-chain, BL_n-chain and SBL_n-chain respectively) is Π_1-complete. Finally, finite consequence relation is also Σ_1-complete, and hence the set of 1-satisfiable sentences in BL^n-chains (SBL^n-chains, BL_n-chains and SBL_n-chains respectively) is Π_1 complete (ϕ is 1-satisfiable iff $\phi \not\vdash 0$). To the contrary, the set of first-order formulas valid in every finite BL^n-chain (SBL^n-chain, BL_n-chain, SBL_n-chain respectively) is Π_2-complete.

Then we note that, by [BM09], the MacNeille completion of a chain in a variety of BL^n-algebras is a chain in the variety itself. Since the MacNeille completion of a residuated lattice preserves the existing suprema and infima, we have strong completeness of $BL^n\forall$ ($BL_n\forall$ respectively) with respect to the class of all complete BL^n chains (of all complete BL_n-chains respectively).

Finally, we prove that there is no BL_n-chain $\mathbf{A}$ such that $BL_n\forall$ is strongly complete with respect to $\{\mathbf{A}\}$. The same is true of $BL^n\forall$, whilst we have a positive result for $SBL^n\forall$ and for $SBL_n\forall$.

8.2 Preliminaries

All the logics in this chapter are *algebraizable* in the sense of [BP89]. In our case we have an even stronger property, which will be illustrated in a moment.

To every logic L we deal with, we associate a quasivariety (in our case, a variety) $\mathcal{L}$, called *the equivalent algebraic semantics of* L, whose language has an n-ary operation symbol for every n-ary connective of L and a constant symbol for every propositional constant of L. Usually, in abstract algebraic logic, a connective and its corresponding operation are denoted in the same way, and we will follow this usage in this chapter. Hence, L formulas are identified with terms of $\mathcal{L}$. A *valuation* of L into an algebra $\mathbf{A} \in \mathcal{L}$ is a homomorphism from the algebra of L-formulas into $\mathbf{A}$. In particular the logics studied in this chapter have an a constant 1 which plays a crucial role in defining the concept of semantic consequence. Let Γ be a set of formulas of L and ϕ be a formula of L, and let $\mathcal{K}$ be a class of algebras in the language of L. We say that ϕ is a *semantic consequence of* Γ in $\mathcal{K}$ (denoted by $\Gamma \models_{\mathcal{K}} \phi$) if for each valuation v into an algebra $\mathbf{A} \in \mathcal{K}$, if $v(\psi) = 1$ for all $\psi \in \Gamma$, then $v(\phi) = 1$. Moreover, $\vdash_L$ will denote logical consequence in L. We say that L is *strongly complete* with respect to $\mathcal{K}$, if for every set Γ of formulas and for every formula ϕ one has $\Gamma \vdash_L \phi$ iff $\Gamma \models_{\mathcal{K}} \phi$. We say that L is *finitely strongly complete* (*complete* respectively) with respect to $\mathcal{K}$ if the above condition holds for finite Γ (for $\Gamma = \emptyset$ respectively).

Given a set Σ of equations and an equation γ in the language of $\mathcal{L}$-algebras, the equation γ *is a semantic consequence of* Σ *in* $\mathcal{K}$, denoted by $\Sigma \models_{\mathcal{K}} \gamma$, is defined as usual in model theory. The next definition is a strengthening of the usual definition of algebraizable logic.

Definition 8.2.1. *We say that L is* algebraizable *and that $\mathcal{L}$ is its* equivalent algebraic semantics *iff there are maps* $^\circ$ *and* * *from formulas of L into equations of $\mathcal{L}$ and viceversa such that letting, for every set Γ of formulas of L and for every set Σ of equations of $\mathcal{L}$, $\Gamma^\circ = \{\psi^\circ : \psi \in \Gamma\}$ and $\Sigma^* = \{\delta^* : \Delta \in \Sigma\}$, for every L-formula ϕ and for every equation γ in the language of $\mathcal{L}$ the following conditions hold:*

1. $\{\phi\} \vdash_L (\phi^\circ)^*$, $\{(\phi^\circ)^*\} \vdash_L \phi$, $\{\gamma\} \models_{\mathcal{L}} (\gamma^*)^\circ$ *and* $\{(\gamma^*)^\circ\} \models_{\mathcal{L}} \gamma$.

2. $\Gamma \vdash_L \phi$ *iff* $\Gamma \models_{\mathcal{L}} \phi$ *iff* $\Gamma^\circ \models_{\mathcal{L}} \phi^\circ$.

3. $\Sigma \models_{\mathcal{L}} \gamma$ *iff* $\Sigma^* \models_{\mathcal{L}} \gamma^*$.

Remark 8.2.1. *In the usual definition of algebraizable logic, ϕ° and ϕ^* are assumed to be sets of equations (of formulas respectively) and not just equations or formulas. Thus the logics we are interested in are algebraizable in a stronger sense. Moreover, for these logics, for every formula ϕ, ϕ° is the equation $\phi = 1$, and for every equation γ of the form $t = s$ in the language of $\mathcal{L}$-algebras, γ^* is the formula $t \leftrightarrow s$.*

Although there are very interesting fuzzy logics which are weaker than BL, like e.g. MTL or Uninorm logic UL ([EG01], [MM07]), in this chapter we will investigate n-contractive extensions of BL. Thus for simplicity, several general concepts which might be extended to core fuzzy logics in the sense of [Cin04] will be treated here only for extensions of the logic BL described below.

For the axiomatization of BL, we refer to chapter 3. We inductively define, for every formula ϕ and for every natural number m, the formula ϕ^m as follows. $\phi^0 = 1$; $\phi^1 = \phi$; for $n > 0$, we define $\phi^{n+1} = (\phi^n)\&\phi$.

We recall that notable schematic extensions of BL are:

1. SBL, axiomatized over BL by $\neg(\phi \wedge \neg\phi)$.

2. Gödel logic G, axiomatized over BL by $\phi \to (\phi\&\phi)$.

3. Product logic Π, axiomatized over BL by $\neg\phi \vee ((\phi \to (\psi\&\phi)) \to \psi)$.

4. Łukasiewicz logic Ł, axiomatized over BL by $(\neg\neg\phi) \to \phi$.

5. Classical logic, axiomatized over BL by $\phi \vee \neg\phi$.

6. Given an axiomatic extension L of BL, we denote by L^n the logic whose axioms are those of L plus

   ($n$contr) $\qquad\qquad \phi^n \to \phi^{n+1}.$

7. Given an axiomatic extension L of BL, we denote by L_n the logic whose axioms are those of L plus

   ($n$contr) $\phi^n \to \phi^{n+1}$ and

   ($\mathrm{div}_{n,m}$) $(\phi^{m-1} \leftrightarrow (\phi \to \phi^n))^n \to \phi^n$

   for every $m < n$ such that m does not divide n.

Remark 8.2.2. *Following the presentation of [ABM09b], let* $\phi \oplus \psi = ((\phi \to (\phi\&\psi)) \to \psi) \vee ((\psi \to (\phi\&\psi)) \to \phi)$*, and let* $k\phi$ *be inductively defined by* $0\phi = 0$*;* $(k+1)\phi = (k\phi) \oplus \phi$*. Then the axiom (*$\mathrm{div}_{n,m}$*) may be replaced by*

($\mathrm{div}'_{n,m}$) $$(m\phi^{m-1})^{n+1} \leftrightarrow (n+1)\phi^m.$$

In Łukasiewicz logic, $\phi \oplus \psi$ *is equivalent to* $\neg(\neg\phi\&\neg\psi)$ *and the axioms (*n*contr) and (*$\mathrm{div}'_{n,m}$*) such that* $m < n$ *and* m *does not divide* n *are used in [CDM99] (they are equivalent to those initially introduced by Grigolia in [Gri77]) to axiomatize (the algebraic semantics of)* n*-valued Łukasiewicz logic.*

BL is algebraizable and its equivalent algebraic semantics is constituted by the variety of BL-algebras, cf [Háj98b]. Moreover, as shown in [Háj98b], BL is strongly complete with respect to the class of totally ordered BL-algebras (also called BL-chains). Finally, every schematic extension of BL is also algebraizable, and its equivalent algebraic semantics is a subvariety of the variety of BL-algebras. In particular:

1. The equivalent algebraic semantics of SBL is the variety $\mathcal{SBL}$ of *SBL-algebras*, i.e., of BL-algebras satisfying the equation $(\neg x) \wedge (\neg\neg x) = 0$.

2. The equivalent algebraic semantics of Gödel logic is constituted by the variety $\mathcal{G}$ of *Gödel algebras*, that is, of all BL-algebras satisfying the equation $x^2 = x$.

3. The equivalent algebraic semantics of product logic is the variety $\mathcal{P}$ of *product algebras*, that is, of all BL-algebras satisfying the equation $\neg x \vee ((x \to (x\&y)) \to y) = 1$.

4. The equivalent algebraic semantics of Łukasiewicz logic is the variety $\mathcal{MV}$ of *MV-algebras*, that is, of BL-algebras satisfying the equation $\neg\neg x = x$. MV-algebras may also be defined as *bounded Wajsberg hoops*, that is, as bounded hoops satisfying the equation $((x \to y) \to y) = ((y \to x) \to x)$.

5. The equivalent algebraic semantics of BL^n is constituted by the variety $\mathcal{BL}^n$ of n-contractive BL-algebras (also called BL^n-algebras), that is, of BL-algebras satisfying the equation

($n$contr) $$x^n = x^{n+1}.$$

More generally, if L is an axiomatic extension of BL and $\mathcal{L}$ is its equivalent algebraic semantics, then the equivalent algebraic semantics of L^n is the variety $\mathcal{L}^n$ constituted by all algebras of $\mathcal{L}$ satisfying the equation ($n$contr). Note that $\mathcal{G}^n$ is just $\mathcal{G}$ and $\mathcal{P}^n$ is just the variety of Boolean algebras. Moreover the equivalent algebraic semantics of Ł^n is the variety generated by the set of MV-chains with cardinality $\leq n+1$.

6. The equivalent algebraic semantics of BL_n is constituted by the variety $\mathcal{BL}_n$ of BL_n-algebras that is, of BL-algebras satisfying the equations ($n$contr) and

   ($\mathrm{div}_{n,m}$) $$(x^{m-1} \leftrightarrow (x \to x^n))^n \leq x^n$$

   for all $m < n$ such that m does not divide n.
   More generally, if L is an axiomatic extension of BL and $\mathcal{L}$ is its equivalent algebraic semantics, then the equivalent algebraic semantics of L_n is the variety $\mathcal{L}_n$ constituted by all algebras of $\mathcal{L}$ satisfying the equations ($n$contr) and ($\mathrm{div}_{n,m}$) for all $m < n$ such that m does not divide n.

Note that $\mathcal{G}_n = \mathcal{G}$ and $\mathcal{P}_n$ is the variety of Boolean algebras. Moreover $\mathcal{MV}_n$ is the variety generated by the (unique up to isomorphism) $n+1$-element MV-chain.

We recall that BL-algebras can be characterized as those bounded hoops which are isomorphic to a subdirect product of linearly ordered bounded hoops. Moreover, a Wajsberg hoop is basic, cf [Fer92] and [AFM07]. Furthermore, the variety of MV-algebras is generated as a quasivariety by the algebra $[0,1]_{MV} = ([0,1], \&, \to, 0, 1)$ where $x \& y = \max\{x+y-1, 0\}$ and $x \to y = \min\{1-x+y, 1\}$. Hence, every quasiequation which is true in $[0,1]_{MV}$ is true in every MV-algebra. Finally, every MV-chain embeds into an ultraproduct of $[0,1]_{MV}$, and hence every universal formula which holds in $[0,1]_{MV}$ holds in every Wajsberg chain. We will tacitly use these facts in the sequel. With $\mathbf{L}_n$ we denote the subalgebra of $[0,1]_{MV}$ with domain $\left\{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right\}$. Note that every MV-chain with cardinality $n+1$ is isomorphic to $\mathbf{L}_n$.

In [AM03], the following is proved:

Theorem 8.2.1. *Every linearly ordered BL-algebra* $\mathbf{A}$ *is the ordinal sum of an indexed family* $(\mathbf{W}_i : i \in I)$ *of linearly ordered Wajsberg hoops, where* I *is a linearly ordered set with minimum* i_0*, and* $\mathbf{W}_{i_0}$ *is bounded.*

In the sequel, the Wajsberg hoops $\mathbf{W}_i$ in Theorem 8.2.1 will be called *the Wajsberg components of* $\mathbf{A}$, and $\mathbf{W}_{i_0}$ is called *the first component*. Using the fact that the $\mathbf{W}_i$ are closed under hoop operations, it is easy to prove (cf [AM03]) that with reference to Theorem 8.2.1, the subalgebras of $\mathbf{A} = \bigoplus_{i\in I} \mathbf{W}_i$ are those of the form $\mathbf{B} = \bigoplus_{i\in I} \mathbf{U}_i$, where for $i \in I$, $\mathbf{U}_i$ is a subhoop of $\mathbf{W}_i$ (possibly trivial if $i \neq i_0$), and $\mathbf{U}_{i_0}$ is a Wajsberg subalgebra of $\mathbf{W}_{i_0}$.

Moreover, in [AM03] it is shown that a BL-chain is:

- An SBL-chain if its first component is isomorphic to $\mathbf{L}_1$.
- A Gödel chain if all its components are isomorphic to $\mathbf{L}_1$.
- A product chain if it is isomorphic to $\mathbf{L}_1$ or it has only two components, the first one isomorphic to $\mathbf{L}_1$ and the second one without minimum.
- An MV-chain if it has just one component.

We conclude this section reviewing some properties of n-contractive BL-algebras.

Proposition 8.2.1. *(1) Every n-contractive Wajsberg chain $\mathbf{A}$ is isomorphic to $\mathbf{L}_m$ for some $m \leq n$.*

(2) Every n-contractive BL-chain is the ordinal sum of an ordered family of algebras of the form $\mathbf{L}_m$ where $m \leq n$.

Proof. (1) Recall that in any MV-chain the following conditions hold: (a) there are exactly two idempotent elements, namely its minimum and its maximum; (b) $ab = ac$ implies that either $b = c$ or $ab = 0$, and (c) $b \to a = a$ iff either $b = 1$ or $a = 1$. These facts clearly hold in $[0, 1]_{MV}$ and hence they hold in all MV-chains.

Now claim (1) is easy if every element of $\mathbf{A}$ is an idempotent, because in this case $\mathbf{A} = \mathbf{L}_1$. Otherwise, let $m \leq n$ be the maximum natural number for which there is an element x such that $x^{m-1} > x^m$, and let $a \in A$ be such that $a^{m-1} > a^m$. Note that $m \geq 2$, because we are assuming that there are non-idempotent elements. We claim that a is the coatom of $\mathbf{A}$ and that $A = \{1, a, a^2, ..., a^m\}$. Suppose, by way of contradiction, that there is $b \in A$ with $a < b < 1$. Then by the divisibility condition, $a = b\&(b \to a)$, and by (c), $b \to a > a$. Thus letting $c = \max\{b \to a, b\}$, we have $c^2 \geq a$, and $c^{2(m-1)} \geq a^{m-1}$. Hence, $c^{2(m-1)} > c^{2m-1}$, otherwise $c^{2(m-1)}$ would be an idempotent less than 1, and hence it would be equal to the bottom. Since $2m-1 > m$, this contradicts the maximality of m. Now we claim that for all $h < m$, there is no b with $a^{h+1} < b < a^h$. Indeed, $a^{h+1} < b < a^h$ together with the residuation property would imply $a \leq a^h \to a^{h+1} \leq$

$b \to a^{h+1} < 1$, and $a = a^h \to a^{h+1} = b \to a^{h+1}$, as a is a coatom. Finally, by residuation and divisibility we would get $ab = a^{h+1} = aa^h$, against property (b).

(2) An n-contractive BL-chain is the ordinal sum of an ordered family of Wajsberg-chains which are clearly n-contractive, and the claim follows from (1).

□

Notation. In the sequel, we write $x \uparrow y$ for $(x \to y) \to y$ and $d_{n,m}(x)$ for $\left(x^{m-1} \leftrightarrow (x \to x^n)\right)^n$. Finally, in a BL-chain we write $x \ll y$ as an abbreviation for: $x < y$ and either $y = 1$ or x and y do not belong to the same component. The following lemma is rather straightforward.

Lemma 8.2.1. *(a) In any BL-chain, $x \uparrow y = 1$ iff either $y = 1$ or $y \ll x$.*

(b) In any BL^n-chain, if $x < 1$, then x^n is the minimum of the Wajsberg component x belongs to, and if $x = 1$, then $x^n = 1$.

(c) Let $x < 1$ be an element of a Wajsberg component $\mathbf{W}$ of a BL^n-chain. Then, $d_{n,m}(x) = 1$ iff $\mathbf{L}_m$ embeds into $\mathbf{W}$ via a (unique) embedding h, and x is the image of $\frac{m-1}{m}$ under h. If any of the above conditions is not satisfied, then $d_{n,m}(x)$ is the minimum of the component x belongs to.

Proof. We only prove (c), the other claims being easy. It follows from the definition of ordinal sum that $x^n \in W$, and that if $x < 1$, then x^n is the minimum of $\mathbf{W}$. Hence, $x \to x^n$ is the negation of x relative to $\mathbf{W}$. Now it is a well-known fact about MV-algebras ([CDM99], cf also [AM03]) that the equation $x^{m-1} \leftrightarrow \neg x = 1$ has a solution in an MV-chain $\mathbf{W}$ iff $\mathbf{L}_m$ embeds into $\mathbf{W}$, and since the unique solution of that equation in $\mathbf{L}_m$ is $\frac{m-1}{m}$, the claim follows. □

Theorem 8.2.2. *The variety $\mathcal{BL}_n$ generated by all ordinal sums of subalgebras of $\mathbf{L}_n$ is axiomatized by ($n$contr) plus all equations of the form (div$_{n,m}$) for all $0 < m < n$ such that m does not divide n.*

Proof. Let $\mathbf{A}$ be a chain in $\mathcal{BL}_n$. Then the equation ($n$contr) holds in $\mathbf{A}$. If $x = 1$, then x satisfies (div$_{n,m}$). Now suppose $x < 1$. Let $\mathbf{W}$ be the unique component x belongs to. Thus, $\mathbf{W}$ is a subalgebra of $\mathbf{L}_n$. If m does not divide n, then $\mathbf{L}_m$ does not embed in $\mathbf{W}$, and by Lemma 8.2.1, $d_{n,m}(x)$ is the minimum of $\mathbf{W}$, and (div$_{n,m}$) is satisfied.

Conversely, suppose that $\mathbf{A}$ satisfies the equations ($n$contr) and (div$_{n,m}$) for all m such that m does not divide n. Then by Proposition 8.2.1, $\mathbf{A}$ is the ordinal sum of components of the form $\mathbf{L}_m$ with $m \leq n$. Suppose that some component $\mathbf{W}$ is not isomorphic to a subalgebra of $\mathbf{L}_n$. Then $\mathbf{W}$ is isomorphic to $\mathbf{L}_m$ for some $m < n$ such that m does not divide n. Then, if x is the isomorphic image of $\frac{m-1}{m}$, we have $d_{n,m}(x) = 1$ and x^n is the minimum of $\mathbf{W}$. Hence ($\mathrm{div}_{n,m}$) is not satisfied, a contradiction. □

8.3 Generic and k-generic models

Let $\mathcal{V}$ be a variety. An algebra $\mathbf{A} \in \mathcal{V}$ is said to be *generic* for $\mathcal{V}$ if it generates $\mathcal{V}$, and *strongly generic* for $\mathcal{V}$ if it generates $\mathcal{V}$ as a quasivariety. An algebra $\mathbf{A} \in \mathcal{V}$ is said to be *k-generic* (*k-strongly generic* respectively) for $\mathcal{V}$ if the variety (the quasivariety respectively) generated by $\mathbf{A}$ contains all k-generated algebras of $\mathcal{V}$. In other words, $\mathbf{A}$ is k-generic (k-strongly generic) if every equation (quasiequation) in k variables at most which is valid in $\mathbf{A}$ is valid in $\mathcal{V}$.

In this section we investigate the (strongly) generic and the k-(strongly) generic models of $\mathcal{BL}^n$, $\mathcal{SBL}^n$, $\mathcal{BL}_n$ and $\mathcal{SBL}_n$.

Definition 8.3.1. *A variety $\mathcal{V}$ is said to be* locally finite *if any finitely generated algebra in $\mathcal{V}$ is finite.*

In [HMN06] it is proved that every locally finite subvariety of $\mathcal{BL}$ is n-contractive, for some n. In [BF00] it is shown that the variety of n-contractive hoops is locally finite. Since local finiteness is not affected by the adding of a constant and is preserved under taking subvarieties, we have:

Theorem 8.3.1. *A variety of BL-algebras is locally finite iff it is n-contractive, for some n.*

Thanks to previous theorem and [CEG+09, theorem 3.8], we easily obtain the following.

Corollary 8.3.1. *Any variety $\mathcal{V}$ of n-contractive BL-algebras is generated as a quasivariety by its finite chains.*

We now investigate the problem of constructing BL^n-algebras (SBL^n-algebras, BL$_n$-algebras, SBL$_n$-algebras respectively) which generate the variety $\mathcal{BL}^n$ ($\mathcal{SBL}^n$, $\mathcal{BL}_n$ and $\mathcal{SBL}_n$ respectively). To begin with, we prove that if $n > 2$, then $\mathcal{BL}^n$ cannot be generated by a single chain.

Lemma 8.3.1. *Consider, for $m \leq n$, the equation*

$$(\neg\neg div_{n,m}) \qquad d_{n,m}(\neg\neg x) \leq (\neg\neg x)^n.$$

Then for any BL^n-chain $\mathbf{A}$, ($\neg\neg div_{n,m}$) is valid in $\mathbf{A}$ iff $\mathbf{L}_m$ does not embed into the first component of $\mathbf{A}$.

Proof. For every $x \in A$, $\neg\neg x$ belongs to the first component, $\mathbf{W}$, of $\mathbf{A}$, and if $\mathbf{L}_m$ does not embed into $\mathbf{W}$, then $d_{n,m}(\neg\neg x) = 0$ and the equation ($\neg\neg\mathrm{div}_{n,m}$) is satisfied. Otherwise, if $\mathbf{L}_m$ embeds into $\mathbf{W}$, then (the isomorphic copy of) $\frac{m-1}{m}$ is a counterexample to ($\neg\neg\mathrm{div}_{n,m}$), because $d_{n,m}\left(\neg\neg\frac{m-1}{m}\right) = 1 > 0 = \left(\neg\neg\frac{m-1}{m}\right)^n$. □

Theorem 8.3.2. *If $n > 2$, then there is no generic BL^n-chain.*

Proof. Let $m < n$ be such that m does not divide n. Then the equations $(\neg\neg\mathrm{div}_{n,n})$ and $(\neg\neg\mathrm{div}_{n,m})$ are not valid in $\mathcal{BL}^n$ (they can be invalidated in $\mathbf{L}_n$ and in $\mathbf{L}_m$ respectively). Moreover, they cannot be invalidated in the *same* BL^n-chain $\mathbf{A}$: let $\mathbf{W}$ be the first component of $\mathbf{A}$. If $(\neg\neg\mathrm{div}_{n,n})$ is not valid in $\mathbf{A}$, then $\mathbf{W}$ must be an isomorphic copy of $\mathbf{L}_n$ and if $(\neg\neg\mathrm{div}_{n,m})$ is not valid in $\mathbf{A}$, then $\mathbf{L}_m$ must embed into $\mathbf{W}$. Since m does not divide n, $\mathbf{L}_m$ is not a subalgebra of $\mathbf{L}_n$, and the above conditions are incompatible. $\square$

As is well known, the free BL^n-algebra on countably many generators is generic for $\mathcal{BL}^n$. However, the free BL^n-algebra does not have an easy description. In the next lines we present an easy construction of a countable BL^n-algebra (not a chain) generating $\mathcal{BL}^n$. Let $r(m)$ denote the remainder of the division of m by n. Define, for $m \in \omega$ and for $h = 1, ..., n$, a Wajsberg hoop $\mathbf{W}^h_m$ as follows: if $m > 0$, then $\mathbf{W}^h_m = \mathbf{L}_{r(m)+1}$; if $m = 0$, then $\mathbf{W}^h_m = \mathbf{L}_h$. Now let for $h = 1, ..., n$, $\mathbf{B}^h_\infty = \bigoplus_{m\in\omega} \mathbf{W}^h_m$.

Theorem 8.3.3. *For every finite BL^n-chain $\mathbf{A}$ there is an h such that $\mathbf{A}$ embeds into $\mathbf{B}^h_\infty$. Hence, $\mathcal{BL}^n$ is generated as a quasivariety by the set $\left\{\mathbf{B}^h_\infty : h = 1, ..., n\right\}$, and it is generated as a variety by $\prod_{h=1}^n \mathbf{B}^h_\infty$. In particular, $\prod_{h=1}^n \mathbf{B}^h_\infty$ is generic for $\mathcal{BL}^n$.*

Proof. Let $\mathbf{D}$ be a finite BL^n-chain. Up to isomorphism, we may assume $\mathbf{D} = \bigoplus_{i=0}^r \mathbf{L}_{k_i}$ with $1 \le k_i \le n$. Then, the first components of $\mathbf{D}$ and of $\mathbf{B}^{k_0}_\infty$ are isomorphic. Moreover, for every $m \in \omega$ and for every h with $1 \le h \le n$, there is an $r > m$ such that $\mathbf{W}^{k_0}_r$ is isomorphic to $\mathbf{L}_h$. Hence, we can find natural numbers $0 = m_0 < m_1 < ... < m_r$ such that for $i = 0, ..., r$, $\mathbf{W}^{k_0}_{m_i}$ is isomorphic to $\mathbf{L}_{k_i}$. It follows that $\mathbf{D} = \bigoplus_{i=0}^r \mathbf{L}_{k_i}$ embeds into $\mathbf{B}^{k_0}_\infty = \bigoplus_{m\in\omega} \mathbf{W}^{k_0}_m$ and the claim is proved. $\square$

We now turn to the problem of characterizing generic BL_n-chains, that is, those BL_n-chains which generate the variety $\mathcal{BL}_n$. Since $\mathcal{BL}_n$ is generated as a quasivariety by its finite chains, a sufficient condition in order that a BL_n-chain generates the variety $\mathcal{BL}_n$ is that every finite BL_n-chain embeds in it. We will se that this condition is also necessary. We quote the following result from [AM03].

Proposition 8.3.1. *(cf [AM03]). Let $\bigoplus_{i\in I} \mathbf{W}_i$ be a BL-chain, where the $\mathbf{W}_i$ are totally ordered Wajsberg hoops, I is a totally ordered set with minimum i_0 and $\mathbf{W}_{i_0}$ is bounded, and let $\mathbf{U}_1, ..., \mathbf{U}_n$ be totally ordered Wajsberg hoops where $\mathbf{U}_1$ is bounded. Then $\mathbf{U}_1 \oplus ... \oplus \mathbf{U}_n$ is a subalgebra of $\bigoplus_{i\in I} \mathbf{W}_i$ iff $\mathbf{U}_1$ is a subalgebra of $\mathbf{W}_{i_0}$ and there are $i_0 < i_2 < ... < i_n$ in I such that for $h = 2, ..., n$, $\mathbf{U}_h$ is a subalgebra of $\mathbf{W}_{i_h}$.*

Corollary 8.3.2. *Let $\bigoplus_{i\in I} \mathbf{W}_i$ be as in Proposition 8.3.1, and assume in addition that every $\mathbf{W}_i$ is a subalgebra of $\mathbf{L}_n$. The following are equivalent:*

(a) *Every finite BL_n-chain embeds into $\bigoplus_{i\in I}\mathbf{W}_i$.*

(b) *$\mathbf{W}_{i_0}$ is isomorphic to $\mathbf{L}_n$ and for infinitely many i, $\mathbf{W}_i$ is isomorphic to $\mathbf{L}_n$.*

Proof. The proof is easy and it is left to the reader. □

Theorem 8.3.4. *Let $\mathbf{A} = \bigoplus_{i\in I}\mathbf{W}_i$ be a BL-chain, where the $\mathbf{W}_i$ are totally ordered Wajsberg hoops, I is a totally ordered set with minimum i_0 and $\mathbf{W}_{i_0}$ is bounded; assume also that every $\mathbf{W}_i$ is a subalgebra of $\mathbf{L}_n$. Then $\mathbf{A}$ is strongly generic for $\mathcal{BL}_n$ iff the following conditions hold.*

(a) *$\mathbf{W}_{i_0}$ is isomorphic to $\mathbf{L}_n$.*

(b) *For infinitely many i, $\mathbf{W}_i$ is isomorphic to $\mathbf{L}_n$.*

Proof. ⇒ Suppose that $\mathbf{A}$ generates the variety $\mathcal{BL}_n$, and assume, by way of contradiction, that condition (a) does not hold. Then by Lemma 8.3.1, the equation ($\neg\neg\mathrm{div}_{n,n}$) holds in $\mathbf{A}$, but it fails e.g. in $\mathbf{L}_n$, and hence it is not valid in $\mathcal{BL}_n$. Hence, $\mathbf{A}$ is does not generate $\mathcal{BL}_n$. Now suppose that $\mathbf{A}$ does not satisfy condition (b), i.e., suppose that $\mathbf{A}$ has only a limited number, k say, of components isomorphic to $\mathbf{L}_n$. Then the equation

$$(\varepsilon_k) \qquad \bigwedge_{i=1}^{k}(x_{i+1}\uparrow x_i) \leq \bigvee_{i=1}^{k+1} x_i \vee \bigvee_{i=1}^{k+1}(d_{n,n}(x_i)\to x_i^n)$$

holds in $\mathbf{A}$. Indeed, let $a_1,...,a_{k+1}\in \mathbf{A}\setminus\{1\}$ be given. If for all $i\leq k$, $a_i \ll a_{i+1}$, then $a_1,\ldots,a_{k+1}$ belong to different components, and at least one of them is a proper subalgebra of $\mathbf{L}_n$. If a_i belongs to that component, then $d_{n,n}(a_i)\to a_i^n = 1$ and the equation (ε_k) holds.

On the other hand, if for some i the condition $a_i \ll a_{i+1}$ fails, then $\bigwedge_{i=1}^{k}(a_{i+1}\uparrow a_i) \leq \bigvee_{i=1}^{k+1} a_i$ and again the equation (ε_k) holds.

Now the equation (ε_k) is not a valid equation of $\mathcal{BL}_n$, because it may be invalidated in the algebra $\mathbf{L}_n^1\oplus...\oplus\mathbf{L}_n^{k+1}$, where for $i=1,...,k+1$, $\mathbf{L}_n^i$ is an isomorphic copy of $\mathbf{L}_n$. Indeed, let for $i=1,...,k+1$, a_i be the isomorphic copy, $(\frac{n-1}{n})_i$, of $\frac{n-1}{n}$ in $\mathbf{L}_n^i$. Then, $\bigwedge_{i=1}^{k}(a_{i+1}\uparrow a_i) = 1$, $\bigvee_{i=1}^{k+1} a_i = (\frac{n-1}{n})_{k+1} < 1$, $d_{n,n}(a_i) = 1$ and $\bigvee_{i=1}^{k+1}(d_{n,n}(a_i)\to a_i^n)$ is the minimum element of the last component. Thus, $\bigvee_{i=1}^{k+1} a_i \vee \bigvee_{i=1}^{k+1}(d_{n,n}(a_i)\to a_i^n) = (\frac{n-1}{n})_{k+1} < 1$.

Once again, we have that $\mathbf{A}$ satisfies an equation which is not valid in $\mathcal{BL}_n$, and hence it does not generate $\mathcal{BL}_n$. We have obtained a contradiction from the assumption that $\mathbf{A}$ generates $\mathcal{BL}_n$ and does not satisfy either condition (a) or condition (b).

Suppose now that $\mathbf{A}$ is strongly generic for $\mathcal{BL}_n$: it follows that $\mathbf{A}$ is generic and hence it must satisfy (a) and (b).

$\Leftarrow$ If $\mathbf{A}$ satisfies conditions (a) and (b) of the theorem then, by corollary 8.3.2, every finite BL_n-chain embeds into $\mathbf{A}$. Since $\mathcal{BL}_n$ is generated as a quasivariety by the class of its finite chains, it is generated as a quasivariety by $\mathbf{A}$.

$\square$

Example 8.3.1. *Let I be any infinite totally ordered set with minimum, and let, for $i \in I$, $\mathbf{L}_n^i$ be an isomorphic copy of $\mathbf{L}_n$. Then, $\bigoplus_{i\in I} \mathbf{L}_n^i$ is strongly generic for $\mathcal{BL}_n$.*

For $\mathcal{SBL}^n$ we also have a strongly generic chain. Let for $m \in \omega$, $r(m)$ be the remainder of the division of m by n, let $\mathbf{W}_m$ be (an isomorphic copy of) $\mathbf{L}_{r(m)+1}$ (hence, in particular, $\mathbf{W}_0$ is isomorphic to $\mathbf{L}_1$), and let $\mathbf{C} = \bigoplus_{m\in\omega} \mathbf{W}_m$.

Theorem 8.3.5. $\mathbf{C}$ *is an SBL^n-chain and generates the variety $\mathcal{SBL}^n$ as a quasivariety, that is, it is a strongly generic SBL^n-chain.*

Proof. That $\mathbf{C}$ is an SBL^n-chain follows from the fact that its first component is isomorphic to $\mathbf{L}_1$ and the remaining components are isomorphic to $\mathbf{L}_h$ for some $h \leq n$. In order to prove that $\mathbf{C}$ generates $\mathcal{SBL}^n$ as a quasivariety, it suffices to prove that every finite SBL^n-chain $\mathbf{D}$ embeds into $\mathbf{C}$. To this purpose, note that $\mathbf{D}$ has the form $\mathbf{D} = \bigoplus_{i=0}^r \mathbf{L}_{k_i}$ with $k_0 = 1$ and $1 \leq k_i \leq n$. Moreover, for every $m \in \omega$ and for every h with $1 \leq h \leq n$, there is a $k > m$ such that $\mathbf{W}_k$ is isomorphic to $\mathbf{L}_h$. Hence, we can find natural numbers $0 = m_0 < m_1 < ... < m_r$ such that for $i = 0, ..., r$, $\mathbf{W}_{m_i}$ is isomorphic to $\mathbf{L}_{k_i}$. It follows that $\mathbf{D} = \bigoplus_{i=0}^r \mathbf{L}_{k_i}$ embeds into $\mathbf{C} = \bigoplus_{m\in\omega} \mathbf{W}_m$ and the claim is proved. $\square$

Let $\mathbf{E} = \bigoplus_{m\in\omega} \mathbf{U}_m$ where $\mathbf{U}_0$ is isomorphic to $\mathbf{L}_1$ and for $m > 0$, $\mathbf{U}_m$ is isomorphic to $\mathbf{L}_n$. Then:

Theorem 8.3.6. $\mathbf{E}$ *is strongly generic for $\mathcal{SBL}_n$. More generally, an SBL$_n$-chain is strongly generic for $\mathcal{SBL}_n$ iff it has infinitely many components isomorphic to $\mathbf{L}_n$.*

Proof. The proof is an easy adaptation of the proof of Theorem 8.3.4, and hence it is left to the reader. $\square$

We now investigate the problem of finding, for every natural number k, a finite k-generic algebra for $\mathcal{BL}^n$ and for $\mathcal{BL}_n$. We start from the easy case of $\mathcal{BL}_n$.

Lemma 8.3.2. *(a) Every k-generated BL-chain $\mathbf{A}$ is the ordinal sum of $k+1$ components at most.*

(b) *Let $\mathcal{V}$ be a variety of BL-algebras. Then, every quasiequation in k variables which is not valid in $\mathcal{V}$ can be invalidated in a chain in $\mathcal{V}$ which is the ordinal sum of $k+1$ Wajsberg components.*

Proof. (a) Let $a_1, ..., a_k$ be the generators of $\mathbf{A}$, let $\mathbf{W}_1, ..., \mathbf{W}_k$ be the components they belong to (possibly, we may have $\mathbf{W}_i = \mathbf{W}_j$ for some $i \neq j$, that is, a_i and a_j may belong to the same component), and let $\mathbf{W}_0$ be the first component. Then by induction on $t(x_1, ..., x_k)$ we can see that for every term $t(x_1, ..., x_k)$, $t^{\mathbf{A}}(a_1, ..., a_k)$ belongs to one of $\mathbf{W}_0, \mathbf{W}_1, ..., \mathbf{W}_k$.

(b) Any invalid quasiequation ε in k variables $x_1, ..., x_k$ may be invalidated in a BL-chain $\mathbf{A}$ and by some valuation v. Let $v(x_1) = a_1, ..., v(x_k) = a_k$. Then ε may be invalidated in the subchain of A generated by $a_1, ..., a_k$, and the claim follows from (a).

□

Theorem 8.3.7. *Let $\mathbf{L}_n^1, ..., \mathbf{L}_n^{k+1}$ be isomorphic copies of $\mathbf{L}_n$. Then, $\mathbf{L}_n^1 \oplus ... \oplus \mathbf{L}_n^{k+1}$ is strongly k-generic for $\mathcal{BL}_n$, i.e., every quasiequation in k variables which is not valid in $\mathcal{BL}_n$ can be invalidated in $\mathbf{L}_n^1 \oplus ... \oplus \mathbf{L}_n^{k+1}$.*

Proof. Every invalid quasiequation ε in k variables can be invalidated in a k-generated BL_n chain $\mathbf{A}$. By Lemma 8.3.2, such a chain is the ordinal sum of $k+1$ components at most. Moreover, each component of $\mathbf{A}$ is a subalgebra of $\mathbf{L}_n$ and hence $\mathbf{A}$ is a subalgebra of $\mathbf{L}_n^1 \oplus ... \oplus \mathbf{L}_n^{k+1}$. It follows that every invalid quasiequation can be invalidated in $\mathbf{L}_n^1 \oplus ... \oplus \mathbf{L}_n^{k+1}$. □

We have just seen that for every k there is a strongly k-generic model for $\mathcal{BL}_n$ with cardinality $n+1+kn$. Thus for fixed n, there is a strongly k-generic model with cardinality linear in k. The situation of $\mathcal{BL}^n$ is slightly more problematic. For instance, $\mathbf{L}_n^1 \oplus ... \oplus \mathbf{L}_n^{k+1}$ is not generic for $\mathcal{BL}^n$ because if $1 < m < n$ and m does not divide n, then $(\mathrm{div}_{n,m})$ holds in $\mathbf{L}_n^1 \oplus ... \oplus \mathbf{L}_n^{k+1}$ but is not a valid equation for $\mathcal{BL}^n$. In order to construct a generic algebra define, for $h = 1, ..., n$ and for $m = 0, ..., kn$, $\mathbf{W}_m^h$ as follows: $\mathbf{W}_0^h = \mathbf{L}_h$ and for $m > 0$, $\mathbf{W}_m^h = \mathbf{L}_{r(m)+1}$. Finally, let for $h = 1, ..., n$, $\mathbf{B}_h^{n,k} = \bigoplus_{m=0}^{nk} \mathbf{W}_m^h$.

Theorem 8.3.8. *For every quasiequation $\varepsilon(x_1, ..., x_k)$ in k variables which is not valid in $\mathcal{BL}^n$ there is an h with $1 \leq h \leq n$ such that $\varepsilon(x_1, ..., x_k)$ is not valid in $\mathbf{B}_h^{n,k}$.*

Proof. Every invalid quasiequation in k variables may be invalidated in a k-generated BL chain $\mathbf{A}$ which is the ordinal sum of an ordered family of $k+1$ MV-chains with cardinality at most $n+1$. Now if the first component of $\mathbf{A}$ is $\mathbf{L}_i$, then by induction on k we see that $\mathbf{A}$ embeds into $\mathbf{B}_i^{n,k}$, and the claim follows. □

Corollary 8.3.3. *There is a k-generic BL^n-algebra with cardinality* $\leq ((k+1)n(n+1))^n$.

Proof. The desired algebra is $\prod_{i=1}^{n} \mathbf{B}_i^{n,k}$. Since $\mathbf{B}_i^{n,k}$ has cardinality $i+1+k\frac{n(n+1)}{2} \leq (k+1)n(n+1)$, the claim follows. $\square$

For $\mathcal{SBL}^n$ the above upper bound can be improved, because the algebra $\mathbf{B}_1^{n,k}$ is strongly k-generic for $\mathcal{SBL}^n$. Moreover, the algebra $\mathbf{L}_1 \oplus \mathbf{L}_n^1 \oplus ... \oplus \mathbf{L}_n^k$ (where $\mathbf{L}_n^i$ is an isomorphic copy of $\mathbf{L}_n$) is a strongly k-generic chain for $\mathcal{SBL}_n$. Therefore:

Theorem 8.3.9. *(1) There is a strongly k-generic SBL^n-chain with cardinality* $2+k\frac{n(n+1)}{2}$.

(2) There is a strongly k-generic SBL$_n$-chain with cardinality $2+kn$.

8.4 n-contractive BL-logics, completeness and complexity

It follows from [CEG+09] that an algebraizable fuzzy logic L with corresponding algebraic semantic $\mathcal{L}$ is complete with respect to a class $\mathcal{K} \subseteq \mathcal{L}$ if $\mathcal{K}$ generates $\mathcal{L}$, and is finitely strongly complete with respect to $\mathcal{K}$ if $\mathcal{K}$ generates $\mathcal{L}$ as a quasivariety. Finally, if in addition $\mathcal{K}$ is a class of chains in $\mathcal{L}$, then L is strongly complete with respect to $\mathcal{K}$ if every countable chain in $\mathcal{L}$ embeds into some algebra in $\mathcal{K}$. Hence, theorems 8.3.3, 8.3.4, 8.3.5 and 8.3.6 give us:

Theorem 8.4.1.

1. *BL^n is finitely strongly complete with respect to the set* $\{\mathbf{B}_\infty^1, \mathbf{B}_\infty^2, ..., \mathbf{B}_\infty^n\}$.
2. *SBL^n is finitely strongly complete with respect to* $\mathbf{B}_\infty^1$.
3. *BL$_n$ is finitely strongly complete with respect to* $\bigoplus_{i\in\omega} \mathbf{L}_n^i$, *where* $\mathbf{L}_n^i$ *is an isomorphic copy of* $\mathbf{L}_n$.
4. *SBL$_n$ is finitely strongly complete with respect to* $\bigoplus_{i\in\omega} \mathbf{W}_i$ *where* $\mathbf{W}_0$ *is an isomorphic copy of* $\mathbf{L}_1$ *and for* $i>0$, $\mathbf{W}_i$ *is an isomorphic copy of* $\mathbf{L}_n$.

We have seen that there is no BL^n-chain $\mathbf{A}$ such that BL^n is complete with respect to $\mathbf{A}$, and a fortiori there is no BL^n-chain $\mathbf{A}$ such that BL^n is strongly complete with respect to $\mathbf{A}$. We wonder if there is a BL$_n$-chain (SBL^n-chain, SBL$_n$-chain respectively) $\mathbf{A}$ such that BL$_n$ (SBL^n, SBL$_n$ respectively) is strongly complete with respect to $\mathbf{A}$. The answer to this question is affirmative. Let Q^+ be the set of all non-negative rationals, and let a positive natural number n be given. We partition $Q^+ \setminus \{0\}$ into n dense and mutually disjoint subsets $Q_1, ..., Q_n$. For instance, let $p_1, ..., p_{n-1}$ be the first $n-1$ prime numbers, let for $i = 1, ..., n-1$, Q_i be the set of rational numbers whose denominator is a power of p_i, and let $Q_n = Q^+ \setminus \left(\bigcup_{i=1}^{n-1} Q_i\right)$. Let for all $q \in Q^+$, $\mathbf{W}_q = \mathbf{L}_n$. Moreover, let $\mathbf{U}_0 = \mathbf{L}_1$ and let, for $q > 0$, $\mathbf{U}_q = \mathbf{L}_i$ if $q \in Q_i$, $i = 1, ..., n$. Next, let $\mathbf{V}_0 = \mathbf{L}_1$ and let for $q > 0$, $\mathbf{V}_q = \mathbf{L}_n$. Finally, let $\mathbf{W} = \bigoplus_{q \in Q^+} \mathbf{W}_q$, $\mathbf{U} = \bigoplus_{q \in Q^+} \mathbf{U}_q$ and $\mathbf{V} = \bigoplus_{q \in Q^+} \mathbf{V}_q$.

Theorem 8.4.2. *(1) BL_n is strongly complete with respect to a BL_n-chain* $\mathbf{A}$ *iff* $\mathbf{W}$ *is a subalgebra of* $\mathbf{A}$*.*

(2) SBL^n is strongly complete with respect to an SBL^n-chain $\mathbf{B}$ *iff* $\mathbf{U}$ *is a subalgebra of* $\mathbf{B}$*.*

(3) SBL_n is strongly complete with respect to an SBL_n-chain $\mathbf{C}$ *iff* $\mathbf{V}$ *is a subalgebra of* $\mathbf{C}$*.*

Proof. (1) BL_n is strongly complete with respect to $\mathbf{A}$ iff every countable BL_n-chain embeds into $\mathbf{A}$. Hence if BL_n is strongly complete with respect to $\mathbf{A}$, then $\mathbf{W}$ is a subalgebra of $\mathbf{A}$. It remains to prove that every countable BL_n-chain $\mathbf{D}$ embeds into $\mathbf{W}$. Now $\mathbf{D}$ can be represented as $\mathbf{D} = \bigoplus_{i \in I} \mathbf{H}_i$, where I is a countable ordered set with minimum and each $\mathbf{H}_i$ is a subalgebra of $\mathbf{L}_n$. Now I can be order-embedded into Q^+ by an embedding h preserving the minimum. This fact is well-known and it is a special case of a more general fact that will be proved later.

Moreover, for every $i \in I$ there is a unique embedding f_i of $\mathbf{H}_i$ into $\mathbf{W}_{h(i)}$. Now define, for $a \in D$, $g(a)$ as follows:

- if $a = 1$, then $g(a) = 1$.
- otherwise, let i be the unique index such that $a \in H_i \setminus \{1\}$. Then, let $g(a) = f_i(a)$.

It is easy to verify that g is an embedding of $\mathbf{D}$ into $\mathbf{W}$.

(2) Let $\mathbf{D} = \bigoplus_{i \in I} \mathbf{H}_i$ be any countable SBL^n-chain, and let i_0 be the minimum of I. Let for $h = 1, ..., n$, I_h be the set of all $i \in I$ such that $\mathbf{H}_i$ is isomorphic to $\mathbf{L}_h$. We claim that there is an embedding g of I into Q^+ such that $g(i_0) = 0$ and for $h = 1, ..., n$, if $i \in I_h$, then $g(i) \in Q_h$. Let $I = \{a_0, ..., a_n, ...\}$. Without loss of generality, we may assume that $a_0 = i_0$.

Step 0. We define $g(a_0) = 0$.

Step $m + 1$. Assume that at step m we have defined $g(a_0), ..., g(a_m)$ in such a way that: (a) $g(i_0) = 0$; (b) for $i, j = 0, ..., m$, $a_i < a_j$ iff $g(a_i) < g(a_j)$; (c) for $i = 0, ..., m$, and for $h = 1, ..., n$, $a_i \in I_h$ iff $g(a_i) \in Q_h$. Let h be such that $a_m \in I_h$. Distinguish the following cases: (1) if a_{m+1} is greater than $a_0, ..., a_m$, then since Q_h is dense in Q^+, there is $q_{m+1} \in Q_h$ such that q_{m+1} is greater than $g(a_0), ..., g(a_m)$.

Choose such a q_{m+1} (to make the procedure deterministic, take all q greater than $g(a_0), ..., g(a_m)$ with smallest denominator and among them choose q_{m+1} with smallest numerator) and put $g(a_{m+1}) = q_{m+1}$.
(2) If a_{m+1} is not greater than all $a_0, ..., a_m$, then a_{m+1} cannot be smaller than all $a_0, ..., a_m$ since a_0 is the minimum. Hence there is a greatest lower bound a_i and a least upper bound a_j of a_{m+1} in $\{a_0, ..., a_m\}$. Then, using again the density of Q_h, choose $q_{m+1} \in Q_h$ such that $g(a_i) < q_{m+1} < g(a_j)$ (it is possible to make the procedure deterministic by a trick as in case (1)), and let $g(a_{m+1}) = q_{m+1}$.

Eventually, in this way we define g on the whole of I in such a way that g is order-preserving, $g(i_0) = 0$ and for $h = 1, ..., n$, if $i \in I_h$, then $g(i) \in Q_h$.

Finally, we obtain an embedding f of $\mathbf{D}$ into $\mathbf{U}$, letting $f(1) = 1$, and for all $x \in \mathbf{H}_i \setminus \{1\}$, $f(x) = t_i(x)$, where t_i is the unique isomorphism from $\mathbf{H}_i$ onto $\mathbf{U}_{g(i)}$.

(3) The proof of (3) is obtained from the proof of (1) with obvious changes.

□

Theorems 8.3.8, 8.3.7, 8.3.9 allow us to derive some complexity theoretic results. In order to introduce them, we need some definitions.

Definition 8.4.1. *Let $\mathcal{K}$ be a class of BL-algebras. A BL-formula ϕ is said to be:*
(a) a $\mathcal{K}$-1-tautology (abbreviated as $\phi \in \mathcal{K}$-1-TAUT) if for every $\mathbf{A} \in \mathcal{K}$ and for every valuation v in $\mathbf{A}$, $v(\phi) = 1$.
(b) a $\mathcal{K}$-positive-tautology (abbreviated as $\phi \in \mathcal{K}$-pos-TAUT) if for every $\mathbf{A} \in \mathcal{K}$ and for every valuation v in $\mathbf{A}$, $v(\phi) > 0$.
(c) $\mathcal{K}$-1-satisfiable (abbreviated as $\phi \in \mathcal{K}$-1-SAT) if there is $\mathbf{A} \in \mathcal{K}$ and a valuation v in $\mathbf{A}$ such that $v(\phi) = 1$.
(d) $\mathcal{K}$-positively-satisfiable (abbreviated as $\phi \in \mathcal{K}$-pos-SAT) if there is $\mathbf{A} \in \mathcal{K}$ and a valuation v in $\mathbf{A}$ such that $v(\phi) > 0$.

Theorem 8.4.3. *(1) $\mathcal{BL}^n$-1-TAUT, $\mathcal{SBL}^n$-1-TAUT, $\mathcal{BL}_n$-1-TAUT, $\mathcal{SBL}_n$-1-TAUT, $\mathcal{BL}^n$-pos-TAUT, $\mathcal{SBL}^n$-pos-TAUT, $\mathcal{BL}_n$-pos-TAUT and $\mathcal{SBL}_n$-pos-TAUT are Co-NP complete.*

(2) $\mathcal{BL}^n$-1-SAT, $\mathcal{SBL}^n$-1-SAT, $\mathcal{BL}_n$-1-SAT, $\mathcal{SBL}_n$-1-SAT, $\mathcal{BL}^n$-pos-SAT, $\mathcal{SBL}^n$-pos-SAT, $\mathcal{BL}_n$-pos-SAT and $\mathcal{SBL}_n$-pos-SAT are NP complete.

(3) There is a deterministic algorithm for checking if $\phi \in \mathcal{BL}^n$-1-TAUT ($\phi \in \mathcal{BL}^n$-pos-TAUT, $\phi \in \mathcal{BL}^n$-1-SAT, $\phi \in \mathcal{BL}^n$-pos-SAT respectively) which works in time bounded by $Chn((k+1)n(n+1))^k$, where k is the number of variables in ϕ, h is the complexity of ϕ, and C is a suitable constant.

(4) *There is a deterministic algorithm for checking if* $\phi \in \mathcal{SBL}^n$-1-*TAUT* ($\phi \in \mathcal{SBL}^n$-*pos-TAUT*, $\phi \in \mathcal{SBL}^n$-1-*SAT*, $\phi \in \mathcal{SBL}^n$-*pos-SAT respectively) which works in time bounded by* $Ch((k+1)n(n+1))^k$, *where* k *is the number of variables in* ϕ, h *is the complexity of* ϕ, *and* C *is a suitable constant.*

(5) *There is a deterministic algorithm for checking if* $\phi \in \mathcal{BL}_n$-1-*TAUT* ($\phi \in \mathcal{BL}_n$-*pos-TAUT*, $\phi \in \mathcal{BL}_n$-1-*SAT*, $\phi \in \mathcal{BL}_n$-*pos-SAT respectively) which works in time bounded by* $Ch((k+1)n+1)^k$, *where* k *is the number of variables in* ϕ, h *is the complexity of* ϕ, *and* C *is a suitable constant.*

(6) *There is a deterministic algorithm for checking if* $\phi \in \mathcal{SBL}_n$-1-*TAUT* ($\phi \in \mathcal{SBL}_n$-*pos-TAUT*, $\phi \in \mathcal{SBL}_n$-1-*SAT*, $\phi \in \mathcal{SBL}_n$-*pos-SAT respectively) which works in time bounded by* $Ch((k+1)n+1)^k$, *where* k *is the number of variables in* ϕ, h *is the complexity of* ϕ, *and* C *is a suitable constant.*

Proof. (1) and (2). We exhibit a non-deterministic polynomial time algorithm for the complement of $\mathcal{BL}^n$-1-TAUT. Guess non-deterministically a natural number h with $1 \leq h \leq n$. Guess non-deterministically k elements of $\mathbf{B}_h^{n,k}$ (cf Theorem 8.3.8) call them $a_1, ..., a_k$. Using binary representations, these guesses can be done in time proportional to $\ln(n)$ and to $k\ln(k)\ln(n(n+1))$. Then, compute $\phi(a_1, ..., a_k)$, i.e., $v(\phi)$ where for $i = 1, ..., k$, $v(p_i) = a_i$, in time proportional to h. If $\phi(a_1, ..., a_k) < 1$, we have verified that $\phi \notin \mathcal{BL}^n$-1-TAUT. The algorithm for checking $\phi \notin \mathcal{BL}^n$-pos-TAUT ($\phi \in \mathcal{BL}^n$-1-SAT, $\phi \in \mathcal{BL}^n$-pos-SAT respectively) is similar, the only difference being that in order to verify that $\phi \notin \mathcal{BL}^n$-pos-TAUT ($\phi \in \mathcal{BL}^n$-1-SAT, $\phi \in \mathcal{BL}^n$-pos-SAT respectively) we need to obtain $\phi(a_1, ..., a_k) = 0$ ($\phi(a_1, ..., a_k) = 1$, $\phi(a_1, ..., a_k) > 0$ respectively). For $\mathcal{BL}_n$, the proof is similar, the only differences being that we need not guess a natural number h with $1 \leq h \leq n$ and instead of guessing k elements $\mathbf{B}_h^{n,k}$, we have to guess k elements of $\mathbf{L}_n^1 \oplus ... \oplus \mathbf{L}_n^{k+1}$, $\mathbf{L}_n^1, ...\mathbf{L}_n^{k+1}$ being isomorphic copies of $\mathbf{L}_n$. For $\mathcal{SBL}^n$, the proof is similar, the only differences being that we need not guess a natural number h with $1 \leq h \leq n$ and instead of guessing k elements $\mathbf{B}_h^{n,k}$, we have to guess k elements of $\mathbf{B}_1^{n,k}$. For $\mathcal{SBL}_n$, the proof is similar, the only differences being that we need not guess a natural number h with $1 \leq h \leq n$ and instead of guessing k elements $\mathbf{B}_h^{n,k}$, we have to guess k elements of $\mathbf{L}_1 \oplus \mathbf{L}_n^1 ... \oplus \mathbf{L}_n^k$, $\mathbf{L}_n^1, ..., \mathbf{L}_n^k$ being isomorphic copies of $\mathbf{L}_n$.

As regards to Co-NP hardness (NP-hardness respectively), just note that classical logic can be reduced to any of BL^n, SBL^n, BL_n or SBL_n. Indeed, let $f(x) = \neg\neg(x^n)$. Then, $f(x)$ is either 0 or 1. Moreover, $f(0) = 0$ and $f(1) = 1$. It follows that for every formula $\phi(p_1, ..., p_n)$ we have that ϕ is a 1-tautology in classical logic iff $\phi(f(p_1), ..., f(p_n))$ is a 1-tautology (or a positive tautology) in any of BL^n or SBL^n or BL_n or SBL_n. The same relation holds between 1-satisfiability in classical logic and 1-satisfiability (or positive satisfiability) over BL^n or SBL^n or BL_n or SBL_n.

(3) We exhibit a deterministic algorithm for checking whether $\phi \in \mathcal{BL}^n$-1-TAUT, the algorithms for $\mathcal{BL}^n$-pos-TAUT, $\mathcal{BL}^n$-1-SAT and $\mathcal{BL}^n$-pos-SAT being similar.

For $h = 1, ..., n$, do the following: list all k-tuples $(a_1, ..., a_k)$ of elements of $\mathbf{B}_h^{n,k}$ (where as usual k is the number of variables of ϕ) and compute $\phi(a_1, ..., a_k)$.

Then $\phi \in \mathcal{BL}^n$-1-TAUT iff for $h = 1, ..., n$ and for all k-tuples $(a_1, ..., a_k)$ of elements of $\mathbf{B}_h^{n,k}$, $\phi(a_1, ..., a_k) = 1$. Checking whether or not $\phi(a_1, ..., a_k) = 1$ requires a time proportional to the complexity of ϕ, and the total number of k-tuples of elements of $\mathbf{B}_h^{n,k}$ is $\left(h + 1 + k\frac{n(n+1)}{2}\right)^k \leq ((k+1)n(n+1))^k$. We have to repeat this operation for $h = 1, ..., n$, hence we need a number of computation bounded by $Cn\,((k+1)n(n+1))^k$, C being a suitable constant.

The proofs of (4) and (5) and (6) are similar, the only difference being that we have to check only one algebra instead of n algebras, and that the algebra in question has cardinality $\leq (k+1)n(n+1)$ in the case of $\mathcal{SBL}^n$, $\leq n + 1 + kn$ in the case of $\mathcal{BL}_n$, and in the case of $\mathcal{SBL}_n$. $\square$

8.5 Amalgamation and interpolation in varieties of n-contractive BL-algebras

A relevant difference between the varieties $\mathcal{BL}^n$ and $\mathcal{SBL}^n$ on one side and $\mathcal{BL}_n$ and $\mathcal{SBL}_n$ on the other side, is that $\mathcal{BL}_n$ and $\mathcal{SBL}_n$ have the amalgamation property, while $\mathcal{BL}^n$ and $\mathcal{SBL}^n$ do not. In any variety of (bounded) commutative residuated lattices, amalgamation is equivalent to the deductive interpolation property of the corresponding logic, cf [GJKO07]. It follows that BL_n and SBL_n have the deductive interpolation property, while BL^n and SBL^n do not. We recall the definitions of amalgamation property and of interpolation.

Definition 8.5.1. *Let $\mathcal{K}$ be a class of algebras of the same type. A* V-formation *in $\mathcal{K}$ is a system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ such that $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and i and j are embeddings of $\mathbf{A}$ into $\mathbf{B}$ and into $\mathbf{C}$ respectively.*
Given a V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$, an amalgam of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ in $\mathcal{K}$ is a system $(\mathbf{D}, h, k)$ such that $\mathbf{D} \in \mathcal{K}$, h and k are embeddings of $\mathbf{B}$ and of $\mathbf{C}$ respectively into $\mathbf{D}$, and for all $a \in A$, $h(i(a)) = k(j(a))$.
A class $\mathcal{K}$ is said to have the amalgamation property (AP for short) if every V-formation in $\mathcal{K}$ has an amalgam in $\mathcal{K}$.

Definition 8.5.2. *A logic L has the* deductive interpolation property *if for any theory Γ and for any formula ψ of L, if $\Gamma \vdash_L \psi$, then there is a formula*

γ such that $\Gamma \vdash_L \gamma$, $\gamma \vdash_L \psi$ and every propositional variable occurring in γ occurs both in Γ and in ψ.
A logic L having an implication connective $\to$ has the Craig interpolation property *iff for any two formulas ϕ and ψ of L, if $\vdash_L \phi \to \psi$, then there is a formula γ such that $\vdash_L \phi \to \gamma$, $\vdash_L \gamma \to \psi$ and every propositional variable occurring in γ occurs both in ϕ and in ψ.*

We start from the positive result:

Theorem 8.5.1. *$\mathcal{BL}_n$ and $\mathcal{SBL}_n$ have the amalgamation property.*

Proof. It follows from [Mon06] that a variety $\mathcal{V}$ of BL-algebras has the AP iff every V-formation in $\mathcal{V}$ consisting of totally ordered algebras has an amalgam in $\mathcal{V}$. Hence, it is sufficient to prove that any V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ consisting of totally ordered BL$_n$-algebras (SBL$_n$-algebras respectively) has an amalgam. Thus let $\mathbf{A} = \bigoplus_{m\in M} \mathbf{U}_m$, $\mathbf{B} = \bigoplus_{s\in S} \mathbf{V}_s$ and $\mathbf{C} = \bigoplus_{t\in T} \mathbf{W}_t$ where M, S and T are totally ordered sets with minimum m_0, s_0 and t_0 respectively, for $m \in M$, $s \in S$ and $t \in T$, $\mathbf{U}_m$, $\mathbf{V}_s$ and $\mathbf{W}_t$ are totally ordered Wajsberg hoops and $\mathbf{U}_{m_0}$, $\mathbf{V}_{s_0}$ and $\mathbf{W}_{t_0}$ are bounded.

Before prosecuting with the proof, we need the following

Lemma 8.5.1. *Let i be an embedding of a BL-chain $\mathbf{A}$ into a BL-chain $\mathbf{B}$ and let $x, y \in A$. Then:*

(1) $x = 1$ iff $i(x) = 1$.

(2) If x, y are in the same component of $\mathbf{A}$, then $i(x)$ and $i(y)$ belong to the same component of $\mathbf{B}$.

(3) If $x, y \neq 1$ and $x \ll y$ (i.e., $x < y$ and x, y do not belong to the same component of $\mathbf{A}$), then $i(x) \ll i(y) \neq 1$, i.e., $i(x) < i(y)$, and $i(x), i(y)$ are not in the same component of $\mathbf{B}$.

(4) x belongs to the first component of $\mathbf{A}$ iff $i(x)$ belongs to the first component of $\mathbf{B}$.

Proof.

(1) is trivial. (2) x, y are in the same component of $\mathbf{A}$ iff $(x \to y) \to y = (y \to x) \to x$ iff $(i(x) \to i(y)) \to i(y) = (i(y) \to i(x)) \to i(x)$ iff $i(x)$ and $i(y)$ are in the same component of $\mathbf{B}$.

(3) If $x, y \neq 1$ and $x \ll y$, then $i(x), i(y) \neq 1$, $(i(y) \to i(x)) \to i(x) = i((y \to x) \to x) = i(1) = 1$ and hence $i(x) \ll i(y)$.

(4) x belongs to the first component of $\mathbf{A}$ iff x and 0 belong to the same component of $\mathbf{A}$, and the claim follows from (2) and (3). $\square$

We continue the proof of Theorem 8.5.1. By Lemma 8.5.1, for each $m \in M$, there is a unique $s = i^*(m) \in S$ and a unique $t = j^*(m) \in T$ such that for all $x \in U_m \backslash \{1\}$, $i(x) \in V_s \backslash \{1\}$ and $j(x) \in W_t \backslash \{1\}$. Hence, we obtain two maps i^* and j^* from M into S and into T respectively. By Lemma 8.5.1, i^* and j^* are one-one, order preserving, and $i^*(m_0) = s_0$ and $j^*(m_0) = t_0$. Now, one moment's reflection shows that the V-formation (M, S, T, i^*, j^*) has an amalgam (Y, h^*, k^*), that is, there are a totally ordered set Y with minimum y_0 and two order preserving and one-one maps h^* from S into Y and k^* from T into Y such that $h^*(s_0) = k^*(t_0) = y_0$ and for every $m \in M$, $h^*(i^*(m)) = k^*(j^*(m))$.

Indeed, modulo isomorphism we may assume that for all $m \in M$, $i^*(m) = j^*(m) \in S \cap T$, and that $S \setminus i^*(M)$ and $T \setminus j^*(M)$ are disjoint. Then we can take $Y = S \cup T$. Moreover, denoting the orders on S and on T by $\leq_S$ and by $\leq_T$ respectively, we define the order $\leq_Y$ on Y as follows:

(1) If, $y_1, y_2 \in S$, then $y_1 \leq_Y y_2$ iff $y_1 \leq_S y_2$.

(2) If, $y_1, y_2 \in T$, then $y_1 \leq_Y y_2$ iff $y_1 \leq_T y_2$. (Note that if $y_1, y_2 \in S \cap T$, then the clauses (1) and (2) do not conflict, because $y_1 \leq_S y_2$ iff there are $m_1, m_2 \in M$ such that $i^*(m_1) = y_1$, $i^*(m_2) = y_2$ and $m_1 \leq_M m_2$ iff there are $m_1, m_2 \in M$ such that $j^*(m_1) = y_1$, $j^*(m_2) = y_2$ and $m_1 \leq_M m_2$ iff $y_1 \leq_T y_2$).

(3) If $y_1 \in S \setminus T$ and $y_2 \in T \setminus S$, then $y_1 \leq_Y y_2$ iff for all $m \in M$, if $i^*(m) \leq_S y_1$, then $j^*(m) \leq_T y_2$. Otherwise, $y_2 <_Y y_1$.

We are now ready to construct an amalgam $(\mathbf{D}, h, k)$ of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$. Consider the amalgamation in $\mathcal{BL}_n$ first. Then, $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{BL}_n$ and for all $s \in S$ and $t \in T$, $\mathbf{V}_s$ and $\mathbf{W}_t$ are subalgebras of $\mathbf{L}_n$. Let for all $y \in Y$, $\mathbf{L}_n^y$ be an isomorphic copy of $\mathbf{L}_n$, and let $\mathbf{D} = \bigoplus_{y \in Y} \mathbf{L}_n^y$. Let for all $s \in S$ and for all $t \in T$, h_s and k_t be the unique embeddings of $\mathbf{V}_s$ and of $\mathbf{W}_t$ respectively into $\mathbf{L}_n^y$. Moreover, let for all $v \in B$ and $w \in C$, $h(v)$ and $k(w)$ be defined as follows:

-if $v = 1$, then $h(v) = 1$ and if $w = 1$, then $k(w) = 1$.

-if $v < 1$, let $s(v)$ be the unique element of S such that $v \in V_{s(v)}$, and let $h(v) = h_{s(v)}(v)$.

-if $w < 1$, let $t(w)$ be the unique element of T such that $w \in W_{t(w)}$, and let $k(w) = k_{t(w)}(w)$.

It is rather straightforward to check that $(\mathbf{D}, h, k)$ is an amalgam in $\mathcal{BL}_n$ of the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$.

The proof for $\mathcal{SBL}_n$ is similar, the only differences being that: (1) $\mathbf{U}_{m_0}, \mathbf{V}_{s_0}, \mathbf{W}_{t_0}$ are isomorphic to $\mathbf{L}_1$; (2) in the definition of $\mathbf{D}$, the first component must be replaced by an isomorphic copy, $\mathbf{L}_1^{y_0}$, of $\mathbf{L}_1$ (the other components remain unchanged); (3) if $v = 0$ (if $w = 0$ respectively), then $h_{s(v)}$ ($k_{t(w)}$ respectively) has to be the unique isomorphism of $\mathbf{V}_{s_0}$ ($\mathbf{W}_{t_0}$ respectively) into $\mathbf{L}_1^{y_0}$. $\square$

Corollary 8.5.1. *(1) BL_n and SBL_n have the deductive interpolation prop-*

erty.
(2) Although, for $n > 1$, BL_n and SBL_n do not have Craig's interpolation property, they enjoy the following weak form of Craig interpolation. If L is any of BL_n or SBL_n, and if $\vdash_L \phi^n \to \psi$, then there is a formula γ such that $\vdash_L \phi^n \to \gamma$, $\vdash_L \gamma^n \to \psi$ and every propositional variable occurring in γ occurs both in ϕ and in ψ.

Proof. (1) It follows from [GJKO07] that for commutative substructural logics, the deductive interpolation property is equivalent to the amalgamation property for its corresponding variety.

(2) That BL_n and SBL_n do not have Craig interpolation follows from a general result of [Mon06], where it is shown that there are only four schematic extensions of BL with Craig's interpolation property, namely, Gödel logic, the three-valued Gödel logic, classical logic and the inconsistent logic. The weak form of Craig interpolation follows from the fact that if L is any n-contractive extension of BL, then for any two formulas ϕ and ψ, one has $\phi \vdash_L \psi$ iff $\vdash_L \phi^n \to \psi$ (see [HNP07, theorem 3.3]). □

Theorem 8.5.2. *If $n > 2$, then none of BL^n or SBL^n has the AP.*

Proof. We start from the following remark. If $n > 2$, then $\mathbf{L}_{n-1}$ is not a subalgebra of $\mathbf{L}_n$. Moreover, it follows from a result of Di Nola and Lettieri [dL00] that if $\mathbf{L}_m$ and $\mathbf{L}_n$ embed into an MV-algebra $\mathbf{A}$, then also $\mathbf{L}_{\mathrm{lcm}(n,m)}$ embeds into $\mathbf{A}$. Thus, if $\mathbf{L}_{n-1}$ and $\mathbf{L}_n$ embed into an MV-algebra $\mathbf{A}$, then $\mathbf{L}_{\mathrm{lcm}(n,n-1)}$ embeds into $\mathbf{A}$, and $\mathbf{A}$ is not n-contractive.

Now consider $\mathcal{BL}^n$. Let i and j be the embeddings of $\mathbf{L}_1$ into $\mathbf{L}_n$ and into $\mathbf{L}_{n-1}$ respectively, and suppose, by way of contradiction, that $(\mathbf{D}, h, k)$ is an amalgam of $(\mathbf{L}_1, \mathbf{L}_n, \mathbf{L}_{n-1}, i, j)$, with $\mathbf{D} \in \mathcal{BL}^n$. Let $\mathbf{H}$ be the subalgebra of $\mathbf{D}$ generated by $h(L_n) \cup k(L_{n-1})$. Note that $\mathbf{H}$ is finitely generated, and hence it is finite. Decompose $\mathbf{H}$ into totally ordered factors, $\mathbf{H}_1, ..., \mathbf{H}_k$. Then, for each generator x of $\mathbf{H}$ different from 1, we have $x^n = 0$. It follows that for $i = 1, ..., k$, we have $x_i^n = 0$, and x_i belongs to the first component of $\mathbf{H}_i$. But then each generator of $\mathbf{H}_i$ is in the first component, and hence $\mathbf{H}_i$ has only one component, that is, $\mathbf{H}_i$ is an MV-chain. It follows that $\mathbf{H}$ is an MV-algebra, and since $\mathbf{L}_n$ and $\mathbf{L}_{n-1}$ embed into $\mathbf{H}$, by the remark made at the beginning of this proof, $\mathbf{H}$ is not n-contractive. Thus, $\mathbf{H} \notin \mathcal{BL}^n$ and $\mathbf{D} \notin \mathcal{BL}^n$, a contradiction. Hence, $\mathcal{BL}^n$ does not have the AP.

Next, consider $\mathcal{SBL}^n$. Let i and j be the embeddings of $\mathbf{L}_1 \oplus \mathbf{L}_1$ into $\mathbf{L}_1 \oplus \mathbf{L}_n$ and into $\mathbf{L}_1 \oplus \mathbf{L}_{n-1}$ respectively, and suppose, by way of contradiction, that $(\mathbf{D}, h, k)$ is an amalgam of $(\mathbf{L}_1 \oplus \mathbf{L}_1, \mathbf{L}_1 \oplus \mathbf{L}_n, \mathbf{L}_1 \oplus \mathbf{L}_{n-1}, i, j)$, with $\mathbf{D} \in \mathcal{SBL}^n$. Let $\mathbf{H}$ be the subalgebra of $\mathbf{D}$ generated by $h(L_1 \oplus L_n) \cup k(L_1 \oplus L_{n-1})$. Note that $\mathbf{H}$ is finitely generated, and hence it is finite. Moreover, let $a = h(\min(L_n)) = k(\min(L_{n-1}))$. Then, $a > 0$ is an idempotent element, and every generator of $\mathbf{H}$ except 0 is $\geq a$. By induction on the generation of $\mathbf{H}$, we have that every element of $\mathbf{H}$ is either 0 or $\geq a$. Indeed, every generator has this property, and if $x, y \geq a$, then $x \& y \geq a^2 = a$ and $x \to y \geq y \geq a$. Moreover $x \& 0 = 0 \& x = 0$, $0 \to x = 1 \geq a$, and if $x \geq a$, then $x \to 0 = 0$. Thus, a is the unique atom of $\mathbf{H}$, and $\mathbf{H}$ can be decomposed as $\mathbf{L}_1 \oplus \mathbf{K}$. Moreover, $\mathbf{K}$ is generated by $h(L_n) \cup k(L_{n-1})$, and reasoning as in the case of $\mathcal{BL}^n$, we see that $\mathbf{K}$ is an MV-algebra with minimum a. Since $\mathbf{L}_n$ and $\mathbf{L}_{n-1}$ embed into $\mathbf{K}$, $\mathbf{K}$ is not n-contractive. Thus, $\mathbf{K} \notin \mathcal{BL}^n$, $\mathbf{H} \notin \mathcal{BL}^n$ and $\mathbf{D} \notin \mathcal{BL}^n$, a contradiction. $\square$

Corollary 8.5.2. *BL^n and SBL^n do not have deductive interpolation.*

8.6 First-order n-contractive BL-logics

For the concepts of first-order many-valued logics (syntax and semantics), we refer to chapter 5.

If ϕ is a sentence, then $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e}$ does not depend on e, and hence we will write $\|\phi\|^{\mathbf{A}}_{\mathbf{M}}$ instead of $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e}$. Moreover, if the only free variables in ϕ are $x_1, ..., x_n$, then $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e}$ only depends on $(\mathbf{A}, \mathbf{M})$ and $e(x_1), ..., e(x_n)$. Thus if for $i = 1, ..., n$, $e(x_i) = d_i$, then sometimes we will write $\|\phi(d_1, ..., d_n)\|^{\mathbf{A}}_{\mathbf{M}}$ instead of $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e}$.

Definition 8.6.1. *Let $\mathcal{K}$ be a class of BL-chains, let Γ be a set of formulas of $L\forall$, and let ϕ be a formula of $L\forall$. We say that ϕ is* a semantic consequence of Γ in $\mathcal{K}$ *(and we write $\Gamma \models_{\mathcal{K}} \phi$) iff for every first-order safe interpretation $(\mathbf{A}, \mathbf{M}, e)$ with $\mathbf{A} \in \mathcal{K}$, if $\|\psi\|^{\mathbf{A}}_{\mathbf{M},e} = 1$ for all $\psi \in \Gamma$, then $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e} = 1$.*

Definition 8.6.2. *Let $\mathcal{K}$ be a class of BL-chains. A formula ϕ is said to be: a* $\mathcal{K}$-1-tautology *(abbreviated as $\phi \in \mathcal{K}$-1-TAUT) if for every first-order safe interpretation $(\mathbf{A}, \mathbf{M}, e)$ with $\mathbf{A} \in \mathcal{K}$ we have $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e} = 1$; a* $\mathcal{K}$-positive tautology *(abbreviated as $\phi \in \mathcal{K}$-pos-TAUT) if for every first-order safe interpretation $(\mathbf{A}, \mathbf{M}, e)$ with $\mathbf{A} \in \mathcal{K}$, $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e} > 0$;* $\mathcal{K}$-1-satisfiable *(abbreviated as $\phi \in \mathcal{K}$-1-SAT) if there is a first-order safe interpretation $(\mathbf{A}, \mathbf{M}, e)$ with $\mathbf{A} \in \mathcal{K}$, such that $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e} = 1$;* $\mathcal{K}$-positively-satisfiable *(abbreviated as $\phi \in \mathcal{K}$-pos-SAT) if there is a first-order safe interpretation $(\mathbf{A}, \mathbf{M}, e)$ with $\mathbf{A} \in \mathcal{K}$, such that $\|\phi\|^{\mathbf{A}}_{\mathbf{M},e} > 0$.*
We say that $L\forall$ is strongly complete *with respect to a class $\mathcal{K}$ of BL-chains if for every set Γ of formulas, the set of formulas derivable from Γ in $L\forall$ coincides with the set of semantic consequences of Γ in $\mathcal{K}$.*
We say that $L\forall$ is finitely strongly complete *with respect to $\mathcal{K}$ if the above condition holds for all* finite *sets Γ of formulas, and that $L\forall$ is* complete *with respect to $\mathcal{K}$ if the above condition holds for $\Gamma = \emptyset$.*

In [Háj98b], the following is shown:

Theorem 8.6.1. *Let L be a schematic extension of BL and let $\mathcal{L}$ be its corresponding variety. Then, $L\forall$ is strongly complete with respect to the class of all chains in $\mathcal{L}$.*

For reader's convenience we recall the definition of σ-embedding (previously introduced in chapter 7).

Definition 8.6.3. *Let L and $\mathcal{L}$ be as in Theorem 8.6.1, and let $\mathbf{A}$, $\mathbf{B}$ be chains in $\mathcal{L}$. An embedding h from $\mathbf{A}$ into $\mathbf{B}$ is said to be a σ* embedding *if it preserves all existing suprema and infima, i.e., for every non-empty set $X \subseteq A$, if $\sup(X)$ ($\inf(X)$ respectively) exists in $\mathbf{A}$, then $\sup(h(X))$ ($\inf(h(X))$ respectively) exists in $\mathbf{B}$, and $h(\sup(X)) = \sup(h(X))$ ($h(\inf(X)) = \inf(h(X))$ respectively).*

In [CEG$^+$09], the following is shown:

Theorem 8.6.2. *Let L and $\mathcal{L}$ be as in Theorem 8.6.1, and let $\mathcal{K}$ be a class of chains in $\mathcal{L}$ such that for every countable chain $\mathbf{A}$ in $\mathcal{L}$ there is a σ embedding of $\mathbf{A}$ into some $\mathbf{B} \in \mathcal{K}$. Then, $L\forall$ is strongly complete with respect to $\mathcal{K}$.*

As a consequence of Theorem 8.6.1, we get:

Theorem 8.6.3. *Let $\mathcal{K}$ be the set of all BL^n-chains (SBL^n-chains, BL_n-chains, SBL_n-chains respectively). Then the sets $\mathcal{K}$-1-TAUT and $\mathcal{K}$-pos-TAUT are Σ_1-complete and the sets $\mathcal{K}$-1-SAT and $\mathcal{K}$-pos-SAT are Π_1-complete.*

Proof. Thanks to [MN10, corollary 3.17], we immediately obtain the Σ_1-completeness (of $\mathcal{K}$-1-TAUT and $\mathcal{K}$-pos-TAUT) as well as the fact that $\mathcal{K}$-1-SAT is Π_1-complete and $\mathcal{K}$-pos-SAT is Π_1.

To conclude the proof, let for every formula ϕ, ϕ^* be the formula obtained by replacing every atomic subformula ψ of ϕ by $\neg\neg(\psi^n)$. The set of classical satisfiable formulas reduces to $\mathcal{K}$-pos-SAT via the map $\phi \mapsto \phi^*$, and hence $\mathcal{K}$-pos-SAT is Π_1-complete. $\square$

Since the class of BL^n-chains (SBL$_n$-chains, BL$_n$-chains, SBL$_n$-chains respectively) is closed under MacNeille completions (this result is showed in [BM09]. For more general information about MacNeille completions see [Mac37], [DP02], [GJKO07]. Other studies concerning the MacNeille completions of MTL-chains have been done in [Lv08],[van10]), and since any residuated lattice embeds into its MacNeille completion by a σ embedding, we have:

Theorem 8.6.4. *Let $\mathcal{K}$ be as in Theorem 8.6.3 and let $\mathcal{H}$ be the class of all elements of $\mathcal{K}$ which are complete with respect to the order. Then $\mathcal{K}$-1-TAUT= $\mathcal{H}$-1-TAUT, $\mathcal{K}$-pos-TAUT= $\mathcal{H}$-pos-TAUT, $\mathcal{K}$-1-SAT= $\mathcal{H}$-1-SAT and $\mathcal{K}$-pos-SAT= $\mathcal{H}$-pos-SAT. Hence, $\mathcal{H}$-1-TAUT and $\mathcal{H}$-pos-TAUT are Σ_1-complete and $\mathcal{H}$-1-SAT and $\mathcal{H}$-pos-SAT are Π_1-complete.*

We cannot hope to have any of standard completeness (i.e., completeness with respect to the class of BL^n-chains on $[0, 1]$) or rational completeness

(completeness with respect to the class of BL^n-chains on the rational interval $[0, 1]$) or even hyperreal completeness (completeness with respect to the class of BL^n-chains over non-standard extensions of $[0, 1]$) for $BL^n\forall$, because all these chains are densely ordered, and even for $n > 1$, the unique standard BL^n-chain is the standard Gödel algebra, which, being a BL^1-algebra, does not generate the whole variety of BL^n-algebras. For the same reason, we cannot have any of the above kind of completeness for $SBL^n\forall$ or for $BL_n\forall$ or for $SBL_n\forall$.

Unlike the propositional case, we cannot even have completeness of any of the above logics with respect to the class of the corresponding finite chains. Let $\mathcal{K}_{fin}$ be any of the classes of finite BL^n-chains (SBL^n-chains, BL_n-chains, SBL_n-chains respectively) and let $\mathcal{K}$ be as in Theorem 8.6.3. Then, $\exists x(P(x) \to \forall y P(y))$ and $\exists x(\exists y P(y) \to P(x))$ are two examples of formulas in $\mathcal{K}_{fin}$-1-TAUT, but not in $\mathcal{K}$-1-TAUT.

Moreover, it follows from [MN10] that $\mathcal{K}_{fin}$-1-TAUT is Π_2-complete and hence it is not in Σ_1.

To conclude this section, we investigate the following problem: Let L be any of BL^n, SBL^n, BL_n or SBL_n. Is it true that there is an L-chain $\mathbf{K}$ such that L$\forall$ is (strongly) complete with respect to $\mathbf{K}$?

We already know that if $n > 2$, then the answer is negative for BL^n. We will prove that for BL_n with n not a prime number the answer is still negative. Then we will prove that the answer for SBL^n and for SBL_n is positive. As a warm-up, we prove that $BL_n\forall$ ($SBL_n\forall$ respectively) are not complete with respect to the algebras $\mathbf{W}$ ($\mathbf{V}$ respectively) introduced just before Theorem 8.4.2, although their propositional versions are. Indeed, we have:

Theorem 8.6.5. *Let $n > 3$ be a non-prime natural number, and let ϕ be the formula*

$$\exists x(P(x) \to \forall y P(y)) \vee ((\forall y P(y))^{n-1} \leftrightarrow (\forall y P(y) \to (\forall y P(y))^n).$$

Then, ϕ is valid in $\mathbf{W}$, but it is not a theorem of $BL_n\forall$.
Moreover, $\neg(\forall x P(x)) \vee \phi$ is valid in $\mathbf{V}$, but is not provable in $SBL_n\forall$.

Proof. Let $(\mathbf{W}, \mathbf{M}, e)$ be any safe interpretation, and let D be the domain of $\mathbf{M}$. Let $Z = \left\{\|P(d)\|^{\mathbf{A}}_{\mathbf{M}} : d \in D\right\}$. If Z has a minimum, or equivalently if there is a $d \in D$ such that $\|P(d)\|^{\mathbf{A}}_{\mathbf{M}} = \|\forall x P(x)\|^{\mathbf{W}}_{\mathbf{M}}$, then $\|\exists x(P(x) \to \forall y P(y))\|^{\mathbf{W}}_{\mathbf{M}} = 1$. Otherwise, the set S of all $q \in Q^+$ such that for some $d \in D$, $\|P(d)\|^{\mathbf{A}}_{\mathbf{M}} \in \mathbf{W}_q \setminus \{1\}$ has no minimum (because each component is finite, and if S had a minimum, then Z would have in turn a minimum). However, S must have an infimum s, otherwise $\|\forall x P(x)\|^{\mathbf{W}}_{\mathbf{M}}$ would not be defined and the interpretation would not be safe. It follows that $\|\forall x P(x)\|^{\mathbf{W}}_{\mathbf{M}}$ is the coatom, a, of $\mathbf{W}_s$. Since $\mathbf{W}_s$ is isomorphic to $\mathbf{L}_n$, $a^{n-1} = (a \to a^n)$, and $\|((\forall yP(y))^{n-1} \leftrightarrow (\forall yP(y) \to (\forall yP(y))^n)\|_{\mathbf{M}}^{\mathbf{W}} = 1$. However, if $1 < m < n$ and m divides n, then ϕ is not valid in the ordinal sum $\mathbf{H} = \bigoplus_{q\in Q^+} \mathbf{L}_m^q$, where for all $q \in Q^+$, $\mathbf{L}_m^q$ is an isomorphic copy of $\mathbf{L}_m$. Indeed, if we take the domain D of $\mathbf{M}$ to be the set of all rationals in the half-open interval $(\frac{1}{2}, 1]$, and we define, for every $d \in D$, $P^{\mathbf{H}}(d)$ as being the coatom of $\mathbf{L}_m^d$, then $\|\forall xP(x)\|_{\mathbf{M}}^{\mathbf{H}}$ is the coatom, c, of $\mathbf{L}_m^{\frac{1}{2}}$ and since $c^{n-1} = c^n = \min(\mathbf{L}_m^{\frac{1}{2}})$ and $c \to c^n > \min(\mathbf{L}_m^{\frac{1}{2}})$, we have $\|((\forall yP(y))^{n-1} \leftrightarrow (\forall yP(y) \to (\forall yP(y))^n)\|_{\mathbf{M}}^{\mathbf{H}} < 1$ and $\|\exists x(P(x) \to \forall yP(y))\|_{\mathbf{M}}^{\mathbf{H}} < 1$. Hence, $\|\phi\|_{\mathbf{M}}^{\mathbf{H}} < 1$.

The proof for $\mathrm{SBL}_n\forall$ is similar, and hence we only discuss the parts where the two proofs diverge. To check the validity of $\phi \vee \neg(\forall yP(y))$ in $\mathbf{V}$, note that if $\|\forall yP(y)\|_{\mathbf{M}}^{\mathbf{V}} = 0$, then $\|\neg\forall yP(y)\|_{\mathbf{M}}^{\mathbf{V}} = 1$. Otherwise, $\|\forall yP(y)\|_{\mathbf{M}}^{\mathbf{V}}$, $\|\exists x(P(x) \to \forall yP(y))\|_{\mathbf{M}}^{\mathbf{V}}$ and $\|\phi\|_{\mathbf{M},e}^{\mathbf{V}}$ belong to a component different from the first component, and since $\mathbf{V}$ and $\mathbf{W}$ only differ on the first component, we have $\|\phi\|_{\mathbf{M}}^{\mathbf{V}} = 1$. Moreover $\phi \vee \neg\forall yP(y)$ can be invalidated in the interpretation $(\mathbf{H}', \mathbf{M})$ (the choice of the evaluation e is irrelevant), where $\mathbf{H}'$ is obtained from the BL_n-chain $\mathbf{H}$ defined in the first part of the present proof, by replacing the first component by $\mathbf{L}_1$ and $\mathbf{M}$ is defined as in the first part of this proof, the only difference being that in this case $\mathbf{M}$ is thought of as an $\mathbf{H}'$ structure and not as an $\mathbf{H}$ structure. Then, $\|\phi\|_{\mathbf{M}}^{\mathbf{H}'} < 1$ and $\|\forall yP(y)\|_{\mathbf{M}}^{\mathbf{H}'} > 0$. It follows that $\|\neg\forall yP(y)\|_{\mathbf{M}}^{\mathbf{H}'} = 0$, as $\mathbf{H}'$ is an SBL-chain, and finally $\|\phi \vee \neg\forall yP(y)\|_{\mathbf{M}}^{\mathbf{H}'} = \|\phi\|_{\mathbf{M}}^{\mathbf{H}'} < 1$. □

Theorem 8.6.6. *Let $n > 3$ be a non-prime natural number. For every BL_n-chain $\mathbf{K}$, $BL_n\forall$ is not complete with respect to $\mathbf{K}$.*

Proof. Let $\mathbf{K} = \bigoplus_{i\in I} \mathbf{K}_i$, where I is a totally ordered set with minimum i_0 and each $\mathbf{K}_i$ is a subalgebra of $\mathbf{L}_n$. Distinguish the following cases:

(1) If i_0 is not a limit point in I, then the formula $(\forall x\neg\neg P(x)) \to (\neg\neg\forall xP(x))$ is valid in $\mathbf{K}$, because, for every $\mathbf{K}$ structure $\mathbf{M}$ with domain D, if for all $d \in D$, $\|P(d)\|_{\mathbf{M}}^{\mathbf{K}} \notin \mathbf{K}_{i_0}\setminus\{1\}$, then $\|\forall xP(x)\|_{\mathbf{M}}^{\mathbf{K}} \notin \mathbf{K}_{i_0}\setminus\{1\}$, as i_0 is not a limit point, and hence $\|\neg\neg\forall xP(x)\|_{\mathbf{M}}^{\mathbf{K}} = 1$. If for some $d \in D$, $\|P(d)\|_{\mathbf{M}}^{\mathbf{K}} \in \mathbf{K}_{i_0}\setminus\{1\}$, then $\|\neg\neg\forall xP(x)\|_{\mathbf{M}}^{\mathbf{K}} = \|\forall xP(x)\|_{\mathbf{M}}^{\mathbf{K}} = \|\forall x\neg\neg P(x)\|_{\mathbf{M}}^{\mathbf{K}}$, and in any case $\|(\forall x\neg\neg P(x)) \to (\neg\neg\forall xP(x))\|_{\mathbf{M}}^{\mathbf{K}} = 1$. But $(\forall x\neg\neg P(x)) \to (\neg\neg\forall xP(x))$ is not valid e.g. in the Gödel algebra on $[0, 1]$, and hence it is not a theorem of BL_n.

(2) If i_0 is a limit point in I and $\mathbf{K}_{i_0}$ is isomorphic to $\mathbf{L}_m$ for some $m < n$, then the formula $(\neg\neg Q)^m \to (\neg\neg Q)^{m+1}$, where Q is a zero-ary predicate, is valid in $\mathbf{K}$, but it is not a theorem of BL_n, because it is not valid, e.g., in $\mathbf{L}_n$.

(3) If i_0 is a limit point of I and $\mathbf{K}_{i_0}$ is isomorphic to $\mathbf{L}_n$, then the formula $\neg\neg\forall xP(x) \vee \phi$, where ϕ is as in Theorem 8.6.5, is valid in $\mathbf{K}$. Indeed, the claim is trivial if for all $d \in D$, $\|P(d)\|_{\mathbf{M}}^{\mathbf{K}} = 1$. If for some $d \in D$,

$\|P(d)\|^{\mathbf{K}}_{\mathbf{M}} < 1$, then let $S = \Big\{i \in I : \exists d \in D\ (\|P(d)\|^{\mathbf{K}}_{\mathbf{M}} \in K_i \setminus \{1\})\Big\}$. Note that $S \neq \emptyset$, and if $(\mathbf{K}, \mathbf{M}, e)$ is safe, then $\inf(S)$ must exist, otherwise $\|\forall x P(x)\|^{\mathbf{K}}_{\mathbf{M}}$ would be undefined. Now if $\inf(S) > i_0$, then $\|\neg\neg\forall x P(x)\|^{\mathbf{K}}_{\mathbf{M}} = 1$. Moreover if $\inf(S) = \min(S) = i_0$, then $\|\exists x(P(x) \to \forall y P(y))\|^{\mathbf{K}}_{\mathbf{M}} = 1$ and $\|\phi\|^{\mathbf{K}}_{\mathbf{M}} = 1$. If S has no minimum and $\inf(S) = i_0$, then $\|\forall x P(x)\|^{\mathbf{K}}_{\mathbf{M}}$ is the coatom of $\mathbf{K}_{i_0}$ and, by the argument used in the proof of Theorem 8.6.5, $\|\phi\|^{\mathbf{K}}_{\mathbf{M}} = 1$.

Now $\neg\neg\forall x P(x) \vee \phi$ is not provable in $\mathrm{BL}_n\forall$. Indeed, let Q^+ be the set of non-negative rational numbers, and let $\mathbf{T} = \bigoplus_{q \in Q^+} \mathbf{T}_q$ where $\mathbf{T}_q$ is isomorphic to $\mathbf{L}_n$ if $q > 0$ and is isomorphic to $\mathbf{L}_m$ where $1 < m < n$ and m divides n if $q = 0$. Moreover, let $\mathbf{M}$ be the $\mathbf{T}$ structure with domain D equal to the set of strictly positive rational numbers and such that $P^{\mathbf{M}}(q)$ is the coatom of $\mathbf{T}_q$, for every positive rational number q. Then, $\|\neg\neg\forall x P(x)\|^{\mathbf{K}}_{\mathbf{M}}$ is the coatom of $\mathbf{T}_0$, call it c. Hence, $c^{n-1} = c^n = 0$, and $c \to c^n = \neg c > 0$. It follows that $\|\phi\|^{\mathbf{K}}_{\mathbf{M}} < 1$ and $\|\phi \vee \neg\neg\forall x P(x)\|^{\mathbf{K}}_{\mathbf{M}} < 1$.

Thus, in any case there is a formula which is not a theorem of BL_n but is valid in $\mathbf{K}$, and hence $\mathrm{BL}_n\forall$ is not complete with respect to any BL_n-chain. □

We are going to prove that both $\mathrm{SBL}^n\forall$ and $\mathrm{SBL}_n\forall$ are strongly complete with respect to a single SBL^n chain $\mathbf{A}^n$ (SBL_n chain $\mathbf{A}_n$ respectively).

We start from a partition of the set Q^+ of all non-negative rationals into n dense subsets, $Q_1, ..., Q_n$ such that $0 \in Q_1$. For $q \in Q^+$, let i_q be the unique natural number such that $q \in Q_{i_q}$, and define for $q \in Q^+$, $\mathbf{W}_q = \mathbf{L}_{i_q}$. Finally, let $\mathbf{A}^n = \bigoplus_{q \in Q^+} \mathbf{W}_q$.

As regards to $\mathbf{A}_n$, let $i_1, ..., i_h$ be the natural numbers which divide n, with $i_1 = 1$, let $Q_1, ..., Q_h$ be a partition of Q^+ into h dense subsets such that $0 \in Q_1$, and let for $q \in Q^+$, h_q be the unique natural number such that $q \in Q_{h_q}$. Define, for $q \in Q^+$, $\mathbf{U}_q = \mathbf{L}_{i_{h_q}}$. Finally, let $\mathbf{A}_n = \bigoplus_{q \in Q^+} \mathbf{U}_q$. Clearly, $\mathbf{A}^n$ is an SBL^n chain and $\mathbf{A}_n$ is an SBL_n chain.

Theorem 8.6.7. *(1) Every countable SBL^n chain can be σ-embedded into $\mathbf{A}^n$ and hence $SBL^n\forall$ is strongly complete with respect to $\mathbf{A}^n$.*

(2) Every countable SBL_n chain can be σ-embedded into $\mathbf{A}_n$, and hence $SBL_n\forall$ is strongly complete with respect to $\mathbf{A}_n$.

Proof. We prove (1), the proof of (2) being quite similar. Let $\mathbf{B} = \bigoplus_{i \in I} \mathbf{H}_i$ be any countable SBL^n-chain, where I is a (finite or) countable totally ordered set with minimum i_0, $\mathbf{H}_{i_0}$ is isomorphic to $\mathbf{L}_1$ and for $i > i_0$, $\mathbf{H}_i$ is isomorphic to one of $\mathbf{L}_1, ..., \mathbf{L}_n$. Let for $h = 1, ..., n$, I_h be the set of all $i \in I$ such that $\mathbf{H}_i$ is isomorphic to $\mathbf{L}_h$.

We prosecute the proof with the following

Lemma 8.6.1. *Suppose that f is an order preserving one-one map from I into Q^+ such that: (1) $f(i_0) = 0$; (2) for $i \in I$ and for $h = 1, ..., n$, if $i \in I_h$, then $f(i) \in Q_h$; (3) f preserves all suprema and infima existing in I. Let for $i \in I$, g_i be an isomorphism between $\mathbf{H}_i$ and $\mathbf{W}_{f(i)}$, and define for $a \in B$,*

$$g(a) = \begin{cases} 1 & \textit{if } a = 1 \\ g_i(a) & \textit{if } a \in \mathbf{H}_i \setminus \{1\} \end{cases}$$

Then, g is a σ-embedding of $\mathbf{B}$ into $\mathbf{A}^n$.

Proof. That g is an embedding from $\mathbf{B}$ into $\mathbf{A}^n$ can be proved as in Theorem 8.4.2, and it is left to prove that suprema and infima existing in $\mathbf{B}$ are preserved. Thus let X be a non-empty subset of B and assume first that $\sup(X)$ exists in $\mathbf{B}$. That $g(\sup(X)) = \sup(g(X))$ is clear if $\sup(X) = \max(X)$. Otherwise, note that $1 \notin X$. Now let I_X be the set of $i \in I$ for which there is an $x \in X$ such that $x \in H_i \setminus \{1\}$. Then, I_X has a supremum, i_X say, but not a maximum (otherwise, X would have a maximum, because all components are finite). It follows that $\sup(X)$ is the bottom of $\mathbf{H}_{i_X}$, and $g(\sup(X))$ is the bottom of $\mathbf{W}_{f(i_X)}$. Since f preserves all suprema and infima existing in I, $f(\sup(I_X)) = f(i_X) = \sup(f(I_X))$. We claim that $\sup(g(X))$ is the minimum of $\mathbf{W}_{f(i_X)}$. Indeed, it is clear that $\min(\mathbf{W}_{f(i_X)})$ is an upper bound of $g(X)$. Now let $z < \min(\mathbf{W}_{f(i_X)})$, and let q_z be such that $z \in \mathbf{W}_{q_z}$. Then, $q_z < \sup(f(I_X))$. Hence, there is $i \in I_X$ such that $q_z < f(i)$. Let $x \in X \cap (H_i \setminus \{1\})$. Then, $z < g(x)$, and z is not an upper bound of $g(X)$. This shows that $\min(\mathbf{W}_{f(i_X)}) = g(\sup(X)) = \sup(g(X))$, as desired.

Now assume that $\inf(X)$ exists in $\mathbf{B}$. Without loss of generality, we may assume that $1 \notin X$. As in the previous part, we may assume without loss of generality that $\inf(X)$ is not the minimum of X. Thus, the set I_X of all $i \in I$ for which there is an $x \in X$ such that $x \in H_i \setminus \{1\}$ has an infimum, j_X say, which is not a minimum. It follows that $\inf(X)$ is the coatom of $\mathbf{H}_{j_X}$, and by condition (2), $\mathbf{H}_{j_X}$ and $\mathbf{W}_{f(j_X)}$ are isomorphic. Hence, $g(\inf(X))$ is the coatom of $\mathbf{W}_{f(j_X)}$. Since f preserves all suprema and infima existing in I, $f(\inf(I_X)) = f(j_X) = \inf(f(I_X))$. Moreover, it is easy to check that $\inf(g(X))$ is the coatom of $\mathbf{W}_{f(i_X)}$, that is, $g(\inf(X)) = \inf(g(X))$, as desired. □

Continuing with the proof of Theorem 8.6.7, it suffices to find an order preserving one-one map from I into Q^+ satisfying conditions (1), (2) and (3) in Lemma 8.6.1. Let $I = \{a_0, a_1, ..., a_n, ...\}$. Without loss of generality, we may assume $a_0 = i_0 = \min(I)$. We define the desired f by steps:

Step 0: define $f(a_0) = 0$.

Step $n+1$. Assume that after step n we have defined $f(a_0), ..., f(a_n)$ so that, besides $f(a_0) = 0$, the following conditions are satisfied: (a) for $i = 0, ..., n$, $0 \leq f(a_i) < 1$; (b) for $i, j = 1, ..., n$, $a_i < a_j$ iff $f(a_i) < f(a_j)$; (c) for $i = 2, ..., n$, suppose that there are $j, h < i$ such that $a_j < a_i < a_h$ and for all $k < i$, if $a_k < a_i$, then $a_k \leq a_j$ and if $a_k > a_i$, then $a_k \geq a_h$; then, $f(a_j) + \frac{1}{3}(f(a_h) - f(a_j)) < f(a_i) < f(a_j) + \frac{2}{3}(f(a_h) - f(a_j))$. Note that condition (c) implies that $|f(a_i) - f(a_j)| < \frac{2}{3}|f(a_h) - f(a_j)|$ and $|f(a_i) - f(a_h)| < \frac{2}{3}|f(a_h) - f(a_j)|$. Distinguish the following cases:

(i) For $i = 0, ..., n$, $a_{n+1} > a_i$. Then, let r be such that $a_{n+1} \in I_r$. By the density of Q_r, we can choose a $q_{n+1} \in Q_r$ such that for $i = 1, ...n$, $1 > q_{n+1} > f(a_i)$. Then, we define $f(a_{n+1}) = q_{n+1}$.

(ii) There are $j, h \leq n$ such that $a_j < a_{n+1} < a_h$ and for all $k \leq n$, if $a_k < a_{n+1}$, then $a_k \leq a_j$ and if $a_k > a_{n+1}$, then $a_k \geq a_h$. Let k be such that $a_{n+1} \in I_k$. Since Q_k is dense in Q^+, there is a $q_{n+1} \in Q_k$ such that $f(a_j) + \frac{1}{3}(f(a_h) - f(a_j)) < q_{n+1} < f(a_j) + \frac{2}{3}(f(a_h) - f(a_j))$. Then let $f(a_{n+1}) = q_{n+1}$. It is readily seen that conditions (a), (b) and (c) are preserved.

We claim that the above defined function f is increasing and satisfies conditions (1), (2) and (3) of Lemma 8.6.1. The only non-trivial property is condition (3). We only prove that f preserves suprema, the proof that f preserves infima being similar. Thus suppose $\emptyset \neq X \subseteq I$ and $i = \sup(X)$. If $i = \max(X)$, then trivially $f(\sup(X)) = \sup(f(X))$. Otherwise, since f is increasing, we have $f(i) = f(\sup(X)) \geq \sup(f(X))$ and it remains to prove that for all $\varepsilon > 0$ there is $x \in X$ such that $|f(i) - f(x)| < \varepsilon$. Let h be such that $i = a_h$, and let $k < h$ be such that $a_k < a_h$ and for all $j < h$, if $a_j < a_h$, then $a_j \leq a_k$. Define recursively: $n_0 = k$: $n_{i+1} = \min\{j : a_{n_i} < a_j < a_h\}$. Then, by our construction, $|f(a_{n_{i+1}}) - f(i)| < \frac{2}{3}|f(a_{n_i}) - f(i)|$. It follows that $|f(a_{n_k}) - f(i)| < \left(\frac{2}{3}\right)^k |f(a_{n_0}) - f(i)| < \left(\frac{2}{3}\right)^k$.

Now let $\varepsilon > 0$ be arbitrary, let k be such that $\left(\frac{2}{3}\right)^k < \varepsilon$ and let $x \in X$ be such that $a_{n_k} < x < i$. (such an x exists because $\sup(X) = i$ and $a_{n_k} < i$). Then, $|f(x) - f(i)| < |f(a_{n_k}) - f(i)| < \left(\frac{2}{3}\right)^k < \varepsilon$ and the claim is proved.

The proof that f preserves infima is similar.

(2) The proof of claim (2) is quite similar to the proof of claim (1).

□

Chapter 9

First-order Nilpotent Minimum Logics: first steps

First-order Nilpotent Minimum Logic was introduced in [EG01]; in [Gis03] it is showed that every infinite NM-chain with negation fixpoint is complete w.r.t. the logic NM. In this thesis we have shown that this last result, in the first-order case, does not hold. We have studied the sets of first-order tautologies of some subalgebras of $[0,1]_{NM}$: in particular finite NM-chains and other four infinite NM-chains (with and without negation fixpoint). Moreover we have found a connection between the validity, in an NM-chain, of certain first-order formulas and its order type. Finally, we have analyzed axiomatization, undecidability and the monadic fragments.

This study has been inspired by the work done, for first-order Gödel logic, in [BPZ07] and [BCF07]: when possible, we have pointed out the analogies and the differences with the Gödel logic case. All these results have been submitted for publications in [Bia10].

9.1 First-order Nilpotent Minimum Logics

In this chapter we will assume that the reader is acquainted with first-order many-valued logics (introduced in chapter 5). Moreover, we will largely use the NM-chains listed in section 4.3.6.

9.1.1 $\mathbf{NM}_\infty$, $\mathbf{NM'}_\infty$, $\mathbf{NM}_\infty^-$, $\mathbf{NM'}_\infty^-$ and finite NM-chains

Remark 9.1.1. *In the following we will assume that the first-order language is fixed.*

Let $\mathcal{A}$ be an NM-chain: with the notation $TAUT_{\mathcal{A}}\forall$ we will denote the first-order tautologies of $\mathcal{A}$.

Theorem 9.1.1. *For every NM-chain $\mathcal{A}$ it holds that $TAUT_{[0,1]_{NM}}\forall \subseteq TAUT_{\mathcal{A}}\forall$.*

Proof. Immediate from theorem 5.3.2 and chain completeness theorem for NM∀ (see [EG01, theorem 7]). □

Now we analyze the differences between the (first-order) tautologies of $[0,1]_{NM}$ and those of the other four infinite chains that we have introduced.

Theorem 9.1.2.

1. $TAUT_{NM_\infty}\forall \subset TAUT_{NM_\infty^-}\forall$, $TAUT_{NM'_\infty}\forall \subset TAUT_{NM'^-_\infty}\forall$.
2. $TAUT_{[0,1]_{NM}}\forall \subset TAUT_{NM_\infty}\forall$, $TAUT_{[0,1]_{NM}}\forall \subset TAUT_{NM'^-_\infty}\forall$ *and* $TAUT_{NM_\infty}\forall \neq TAUT_{NM'_\infty}\forall$. *In fact the formula*

   $$(*) \qquad (\forall x)(\varphi(x)\&\nu) \leftrightarrow ((\forall x)\varphi(x)\&\nu)$$

   where x does not occur freely in ν, is a tautology for NM_∞ and NM'^-_∞, but it fails in NM'_∞ (and hence, from theorem 9.1.1, it fails in $[0,1]_{NM}$).

Proof. 1. Immediate from theorem 4.3.5 and the fact that $NM_\infty^- \hookrightarrow NM_\infty, NM'^-_\infty \hookrightarrow NM'_\infty$ preserving all inf and sup.

2. First of all we show that (*) fails in NM'_∞. Consider the formula $(\forall x)(P(x)\&p) \leftrightarrow ((\forall x)P(x)\&p)$, where p is a predicate of arity zero. Construct a model $\mathbf{M}$ (that is necessarily safe, since NM'_∞ is complete) such that $M = (\frac{1}{2},1] \cap NM'_\infty$, p is interpreted as $\frac{1}{2}$ and $r_P(m) = m$, for each $m \in M$. An easy check shows that $\|(\forall x)(P(x)\&p) \leftrightarrow ((\forall x)P(x)\&p)\|_{\mathbf{M}}^{NM'_\infty} = \frac{1}{2}$ and hence $NM'_\infty \not\models (*)$.

   Now we show that $NM_\infty \models (*)$. We have to check that, for each $W \subseteq NM_\infty$ (observe that NM_∞ is a complete lattice) and $y \in NM_\infty$, it holds that $\inf_{w\in W}(w*y) = \inf(W)*y$. Note that, if W has minimum m, then $\inf(W)*y = m*y = \inf_{w\in W}(w*y)$. Suppose then that W has infimum but not minimum: an easy check shows that $\inf(W) = 0$. In this last case we have that $\inf(W)*y = 0 = \inf_{w\in W}(w*1) \geq \inf_{w\in W}(w*y)$.

   Finally we analyze NM'^-_∞. We have to show that $\inf_{w\in W}\{w*x\} = \inf(W)*x$, for each $W \subseteq NM'^-_\infty$ and $x \in NM'^-_\infty$, when these inf exist. If W has a minimum, say m, then $\inf_{w\in W}\{w*x\} = m*x = \inf(W)*x$; if W does not have minimum, then it does not have inf and we are not interested to this case.

□

Remark 9.1.2. *In [BPZ07] are studied the Gödel-chains $G_\uparrow$, whose universe is $\{1-\frac{1}{n} : n \in \mathbb{N}^+\}\cup\{1\}$ and $G_\downarrow$, whose universe is $\{\frac{1}{n} : n \in \mathbb{N}^+\}\cup\{0\}$. In our case, since the negation is involutive, if we construct the NM-chain generated by (the universe of) $G_\uparrow$ or $G_\downarrow$ and $n(x) = 1-x$, then we obtain NM_∞.*

Lemma 9.1.1. *Let $\mathcal{A}$ be an NM-chain: an element a has precedessor (successor) if and only if $n(a)$ has successor (predecessor).*

Proof. Immediate from the properties of the negation. □

Theorem 9.1.3. *Consider the following formulas:*

$C_\uparrow$	$(\exists x)(\varphi(x) \to \forall y\varphi(y))$
$C_\downarrow$	$(\exists x)(\exists y\varphi(y) \to \varphi(x))$.

The formulas $C_\uparrow$ and $C_\downarrow$ hold in every NM-chain $\mathcal{A}$ in which every element of $\mathcal{A} \setminus \{0,1\}$ has a predecessor in $\mathcal{A}$. They both fail in any other NM-chain.

Proof. Let $\mathcal{B}$ be an NM-chain that has an element $x \in \mathcal{B} \setminus \{0,1\}$ without predecessor in $\mathcal{B}$.

Consider the set $W = \{w \in \mathcal{B} : \ w < x\}$: direct inspection shows that $\sup_{w\in W}\{\sup(W) \Rightarrow w\} = \sup_{w\in W}\{x \Rightarrow w\} = \sup_{w\in W}\{\max(n(x), w)\} < 1$. This shows that $\mathcal{B} \not\models C_\downarrow$.

From lemma 9.1.1 we know that $n(x)$ does not have successor. Construct the set $W = \{w \in \mathcal{B} : \ w > n(x)\}$: direct inspection shows that $\sup_{w\in W}\{w \Rightarrow \inf(W)\} = \sup_{w\in W}\{w \Rightarrow n(x)\} = \sup_{w\in W}\{\max(n(w), n(x))\} < 1$. This shows that $\mathcal{B} \not\models C_\uparrow$.

Consider now the NM-chain $\mathcal{A}$ of the theorem: note that every element of $\mathcal{A} \setminus \{0,1\}$ has predecessor and successor in $\mathcal{A}$. We have to check that $\sup_{w\in W}\{w \Rightarrow \inf(W)\} = 1$ and $\sup_{w\in W}\{\sup(W) \Rightarrow w\} = 1$, for every W in which these inf and sup exist. If W has minimum m, then $\sup_{w\in W}\{w \Rightarrow \inf(W)\} = m \Rightarrow m = 1$; if W has maximum n, then $\sup_{w\in W}\{\sup(W) \Rightarrow w\} = n \Rightarrow n = 1$. If W has infimum, but not minimum, then $\inf(W) = 0$ and $\sup_{w\in W}\{w \Rightarrow \inf(W)\} = \sup_{w\in W}\{n(w)\} = 1$. Finally, if W has supremum, but not maximum, then $\sup(W) = 1$ and $\sup_{w\in W}\{\sup(W) \Rightarrow w\} = \sup_{w\in W}\{1 \Rightarrow w\} = 1$. □

Corollary 9.1.1.

- *$C_\downarrow$ and $C_\uparrow$ belong to $TAUT_{NM_\infty}\forall$, $TAUT_{NM_\infty^-}\forall$, $TAUT_{NM'^-_\infty}\forall$ and $TAUT_{NM_n}\forall$, for every $1 < n < \omega$.*
- *$C_\downarrow$ and $C_\uparrow$ fail in $[0,1]_{NM}$ and NM'_∞.*

Remark 9.1.3. *Continuing with the analogies with Gödel logic, it can be showed (see [BPZ07] and [BLZ96]) that $C_\downarrow$ and $C_\uparrow$ are tautologies in $G_\uparrow$ and in every finite Gödel chain, whilst $G_\downarrow \not\models C_\uparrow$ and $G_\downarrow \models C_\downarrow$. Both the formulas fail in $G\forall$ (see [BLZ96]).*

We prosecute our analysis of first-order tautologies with the following

Theorem 9.1.4. *Let φ be an NM$\forall$ formula. For every integer $n > 1$ and every* even *integer $m > 1$ it holds that*

- *if* $NM_n \not\models \varphi$, *then* $NM_\infty \not\models \varphi$ *and* $NM'_\infty \not\models \varphi$.
- *if* $NM_m \not\models \varphi$, *then* $NM^-_\infty \not\models \varphi$ *and* $NM'^-_\infty \not\models \varphi$.

Proof. It is enough to show that $NM_n \hookrightarrow NM_\infty$, $NM_n \hookrightarrow NM'_\infty$, $NM_m \hookrightarrow NM^-_\infty$, $NM_m \hookrightarrow NM'^-_\infty$ preserving all inf and sup.

We begin with the case of NM_∞.

Let $0 = c_1 < c_2 < \cdots < c_n = 1$ be the elements of NM_n: consider a map ϕ such that

- $\phi(c_1) = 0$ and $\phi(c_n) = 1$.
- If NM_n has a fixpoint f, then $\phi(f) = \frac{1}{2}$.
- Let c_k be the least positive element: we set $\phi(c_j) = 1 - \frac{1}{3+(j-k)}$ for every $c_n > c_j \geq c_k$.
- Let c_h be the greatest negative element: we set $\phi(c_i) = \frac{1}{3+(h-i)}$ for every $c_1 < c_i \leq c_h$.

Direct inspection shows that ϕ is an embedding from the two chains. Moreover, since NM_n is finite, then for each $W \subseteq NM_n$, $\phi(\inf(W)) = \phi(\min(W)) = \min(\phi(W))$; analogously for sup.

Concerning the case of NM'_∞ we have only to modify the map ϕ and the proof proceeds analogously to the previous case.

Let $0 = c_1 < c_2 < \cdots < c_n = 1$ be the elements of NM_n: consider a map ϕ such that

- $\phi(c_1) = 0$ and $\phi(c_n) = 1$.
- If NM_n has a fixpoint f, then $\phi(f) = \frac{1}{2}$.
- Let c_k be the greatest positive element of $NM'_\infty \setminus \{1\}$: we set $\phi(c_j) = \frac{1}{2} + \frac{1}{2(2+k-j)}$ for every $c_n > c_k \geq c_j$.
- Let c_h be the least negative element of $NM'_\infty \setminus \{0\}$: we set $\phi(c_i) = \frac{1}{2} - \frac{1}{2(2+i-h)}$ for every $c_1 < c_h \leq c_i$.

Finally the proofs for NM^-_∞ and NM'^-_∞ are identical to the previous ones, except for the absence of the negation fixpoint. □

Corollary 9.1.2. *For every integer* $n > 1$ *we have* $TAUT_{[0,1]_{NM}}\forall \subset TAUT_{NM_n}\forall$, $TAUT_{NM_\infty}\forall \subset TAUT_{NM_n}\forall$, $TAUT_{NM'_\infty}\forall \subset TAUT_{NM_n}\forall$. *Moreover, if* n *is even, then* $TAUT_{NM^-_\infty}\forall \subset TAUT_{NM_n}\forall$, $TAUT_{NM'^-_\infty}\forall \subset TAUT_{NM_n}\forall$.

Proof. From theorems 9.1.1, 9.1.4 we have the non-strict inclusions. To prove the strictness, direct inspection shows that the formula $\bigvee_{0<i<n}(p_i \to p_{i+1})$ (where each p_i is a predicate of arity zero) is a first-order tautology of NM_n, but it fails in every infinite NM-chain. □

By contrast with the results of [BPZ07] for G_n, it cannot be showed that $TAUT_{NM_{n+1}}\forall \subset TAUT_{NM_n}\forall$. In fact, if NM_n has negation fixpoint, then (see theorem 4.3.5) $NM_n \not\models \neg(\neg p^2)^2 \leftrightarrow (\neg(\neg p)^2)^2$, where p is a predicate of arity zero. However $NM_{n+1} \models \neg(\neg p^2)^2 \leftrightarrow (\neg(\neg p)^2)^2$.

However, we have the following

Theorem 9.1.5. *For each pair of integers m, n such that $1 < m < n$, if m, n are both even (odd), then $TAUT_{NM_n}\forall \subset TAUT_{NM_m}\forall$.*

Proof. First of all, note that NM_n has negation fixpoint if and only if n is odd. This solves the problem previously cited, about the formula $\neg(\neg p^2)^2 \leftrightarrow (\neg(\neg p)^2)^2$.

From these facts, using an easy adaptation of the proof of theorem 9.1.4 it can be proved that $NM_m \hookrightarrow NM_n$, preserving all inf and sup. This shows that $TAUT_{NM_n}\forall \subseteq TAUT_{NM_m}\forall$.

To conclude, note that $NM_m \models \bigvee_{0<i<m}(p_i \to p_{i+1})$, but $NM_n \not\models \bigvee_{0<i<m}(p_i \to p_{i+1})$. $\square$

Moreover, by inspecting the previous proof, we obtain

Corollary 9.1.3. *For every* even *integer $n > 1$, it holds that $TAUT_{NM_{n+1}}\forall \subset TAUT_{NM_n}\forall$.*

Now we study the relation between $TAUT_{NM_\infty}\forall$ and $\bigcap_{n\geq 2} TAUT_{NM_n}\forall$. First of all, we need the following lemma.

Lemma 9.1.2. *Let $\mathbf{M} = \langle M, \{r_p\}_{p\in\mathbf{P}}, \{m_c\}_{c\in\mathbf{C}}\rangle$ be an NM_∞-model. For $\alpha \in NM_\infty$, consider the NM_∞-model $\mathbf{M}_\alpha = \left\langle M, \{r'_p\}_{p\in\mathbf{P}}, \{m_c\}_{c\in\mathbf{C}}\right\rangle$ such that, for each atomic formula ψ and every evaluation v*

$$\text{(m)} \qquad \|\psi\|_{\mathbf{M}_\alpha,v} = \begin{cases} 1 & \text{if } \|\psi\|_{\mathbf{M},v} > |\alpha| \\ 0 & \text{if } \|\psi\|_{\mathbf{M},v} < n(|\alpha|) \\ \|\psi\|_{\mathbf{M},v} & \text{otherwise} \end{cases}$$

Where $|\alpha| = \max(\alpha, n(\alpha))$.

Then (m) holds for every first-order formula φ.

Proof. By structural induction. Since $\mathbf{M}_\alpha$ and $\mathbf{M}_{n(\alpha)}$ define the same model we will assume, without loss of generality, that $\alpha \geq \frac{1}{2}$ (otherwise we set $\alpha = n(\alpha)$).

- If φ is atomic or $\bot$, then there is nothing to prove.
- $\varphi := \psi \wedge \chi$ and the claim holds for ψ and χ. First of all note that $\|\psi \wedge \chi\|_{\mathbf{M},v} = \min(\|\psi\|_{\mathbf{M},v}, \|\chi\|_{\mathbf{M},v})$ and $\|\psi\wedge\chi\|_{\mathbf{M}_\alpha,v} = \min(\|\psi\|_{\mathbf{M}_\alpha,v}, \|\chi\|_{\mathbf{M}_\alpha,v})$:

from the induction hypothesis, if $\|\psi\|_{\mathbf{M},v} = \|\chi\|_{\mathbf{M},v}$, then the lemma holds.

For the other cases, note that if $\|\psi\|_{\mathbf{M},v} < \|\chi\|_{\mathbf{M},v}$ ($>$), then $\|\psi\|_{\mathbf{M}_\alpha,v} \leq \|\chi\|_{\mathbf{M}_\alpha,v}$ ($\geq$). Suppose that $\|\psi\|_{\mathbf{M},v} < \|\chi\|_{\mathbf{M},v}$. If $\|\psi\|_{\mathbf{M}_\alpha,v} < \|\chi\|_{\mathbf{M}_\alpha,v}$ then, applying the induction hypothesis, we have the result. The other case is $\|\psi\|_{\mathbf{M}_\alpha,v} = \|\chi\|_{\mathbf{M}_\alpha,v} \in \{0,1\}$: clearly either $\|\chi\|_{\mathbf{M},v} < n(\alpha)$ or $\|\psi\|_{\mathbf{M},v} > \alpha$. Again, applying the induction hypothesis, the claim follows.

- $\varphi := \psi \& \chi$ and the claim holds for ψ and χ. We have two cases:
  - $\|\varphi\|_{\mathbf{M},v} = 0$: this happens if and only if $\|\psi\|_{\mathbf{M},v} \leq n(\|\chi\|_{\mathbf{M},v})$. Direct inspection shows that this implies $\|\psi\|_{\mathbf{M}_\alpha,v} \leq n(\|\chi\|_{\mathbf{M}_\alpha,v})$ and hence $\|\varphi\|_{\mathbf{M}_\alpha,v} = 0$.
  - $\|\varphi\|_{\mathbf{M},v} = \min(\|\psi\|_{\mathbf{M},v}, \|\chi\|_{\mathbf{M},v}) > 0$: this happens if and only if $\|\psi\|_{\mathbf{M},v} > n(\|\chi\|_{\mathbf{M},v})$.
    If $\|\psi\|_{\mathbf{M},v} < n(\alpha)$ then $\|\varphi\|_{\mathbf{M},v} < n(\alpha)$ and $\|\psi\|_{\mathbf{M}_\alpha,v} = 0 = \|\varphi\|_{\mathbf{M}_\alpha,v}$.
    If $n(\alpha) \leq \|\psi\|_{\mathbf{M},v} \leq \alpha$, then $\|\psi\|_{\mathbf{M},v} = \|\psi\|_{\mathbf{M}_\alpha,v}$ and $n(\|\chi\|_{\mathbf{M},v}) < \alpha$: if $n(\alpha) \leq n(\|\chi\|_{\mathbf{M}_\alpha,v})$, then $\|\varphi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M}_\alpha,v}$, otherwise $n(\|\chi\|_{\mathbf{M},v}) < n(\alpha)$, $\|\chi\|_{\mathbf{M},v} > \alpha$ and hence $\|\varphi\|_{\mathbf{M},v} = \|\psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M}_\alpha,v}$, since $\|\chi\|_{\mathbf{M}_\alpha,v} = 1$, thanks to the induction hypothesis.
    Finally, suppose that $\|\psi\|_{\mathbf{M},v} > \alpha$. We have that $\|\psi\|_{\mathbf{M}_\alpha,v} = 1$: if $n(\|\chi\|_{\mathbf{M},v}) > \alpha$, then $\|\chi\|_{\mathbf{M},v} < n(\alpha)$ and hence $\|\varphi\|_{\mathbf{M},v} = \|\chi\|_{\mathbf{M},v}$, from which we have $\|\chi\|_{\mathbf{M}_\alpha,v} = 0 = \|\varphi\|_{\mathbf{M}_\alpha,v}$. If $n(\alpha) \leq n(\|\chi\|_{\mathbf{M},v}) \leq \alpha$, then the same holds for $\|\chi\|_{\mathbf{M},v}$ and we have $\|\varphi\|_{\mathbf{M},v} = \|\chi\|_{\mathbf{M},v} = \|\chi\|_{\mathbf{M}_\alpha,v} = \|\varphi\|_{\mathbf{M}_\alpha,v}$. If $n(\|\chi\|_{\mathbf{M},v}) < n(\alpha)$, then $\|\chi\|_{\mathbf{M},v} > \alpha$ and hence $\|\chi\|_{\mathbf{M}_\alpha,v} = \|\psi\|_{\mathbf{M}_\alpha,v} = 1 = \|\varphi\|_{\mathbf{M}_\alpha,v}$.
- $\varphi := \psi \to \chi$ and the claim holds for ψ and χ. We have two cases.
  - $\|\psi\|_{\mathbf{M},v} \leq \|\chi\|_{\mathbf{M},v}$: as we have already noticed, this implies $\|\psi\|_{\mathbf{M}_\alpha,v} \leq \|\chi\|_{\mathbf{M}_\alpha,v}$ and we have that $\|\varphi\|_{\mathbf{M},v} = 1 = \|\varphi\|_{\mathbf{M}_\alpha,v}$.
  - $\|\psi\|_{\mathbf{M},v} > \|\chi\|_{\mathbf{M},v}$: it is not difficult to check that $\|\psi\|_{\mathbf{M}_\alpha,v} \geq \|\chi\|_{\mathbf{M}_\alpha,v}$.
    If the equality holds, then $\|\psi\|_{\mathbf{M}_\alpha,v} = \|\chi\|_{\mathbf{M}_\alpha,v} \in \{0,1\}$ and either $\|\chi\|_{\mathbf{M},v} > \alpha$ or $\|\psi\|_{\mathbf{M},v} < n(\alpha)$: in both the cases $\|\varphi\|_{\mathbf{M},v} = \max(n(\|\psi\|_{\mathbf{M},v}), \|\chi\|_{\mathbf{M},v})$. If $\|\chi\|_{\mathbf{M},v} > \alpha$, then $n(\|\psi\|_{\mathbf{M},v}) < n(\alpha)$ and $\|\varphi\|_{\mathbf{M},v} = \|\chi\|_{\mathbf{M},v} > \alpha$: from these facts we have $\|\psi\|_{\mathbf{M}_\alpha,v} = \|\chi\|_{\mathbf{M}_\alpha,v} = 1 = \|\varphi\|_{\mathbf{M}_\alpha,v}$. If $\|\psi\|_{\mathbf{M},v} < n(\alpha)$, then $n(\|\psi\|_{\mathbf{M},v}), \|\varphi\|_{\mathbf{M},v} > \alpha$ and from the induction hypothesis we have $\|\psi\|_{\mathbf{M}_\alpha,v} = 0 = \|\chi\|_{\mathbf{M}_\alpha,v}$ and $\|\varphi\|_{\mathbf{M}_\alpha,v} = 1$.
    The last case is $\|\psi\|_{\mathbf{M}_\alpha,v} > \|\chi\|_{\mathbf{M}_\alpha,v}$: we have that $\|\varphi\|_{\mathbf{M},v} = \max(n(\|\psi\|_{\mathbf{M},v}), \|\chi\|_{\mathbf{M},v})$ and $\|\varphi\|_{\mathbf{M}_\alpha,v} = \max(n(\|\psi\|_{\mathbf{M}_\alpha,v}), \|\chi\|_{\mathbf{M}_\alpha,v})$.

There are two subcases.

$n(\|\psi\|_{\mathbf{M},v}) > \|\chi\|_{\mathbf{M},v}$: clearly $\|\varphi\|_{\mathbf{M},v} = n(\|\psi\|_{\mathbf{M},v})$. If $n(\alpha) \leq \|\psi\|_{\mathbf{M},v} \leq \alpha$, then we have that $\|\psi\|_{\mathbf{M},v} = \|\psi\|_{\mathbf{M}_\alpha,v}$, $n(\|\psi\|_{\mathbf{M},v}) = n(\|\psi\|_{\mathbf{M}_\alpha,v})$ and $\|\varphi\|_{\mathbf{M}_\alpha,v} = \|\varphi\|_{\mathbf{M},v} = n(\|\psi\|_{\mathbf{M},v})$ (noting that $\|\chi\|_{\mathbf{M}_\alpha,v} \leq \|\chi\|_{\mathbf{M},v}$, since $\|\psi\|_{\mathbf{M}_\alpha,v} > \|\chi\|_{\mathbf{M}_\alpha,v}$). If $\|\psi\|_{\mathbf{M},v} > \alpha$, then $\|\psi\|_{\mathbf{M}_\alpha,v} = 1$, $n(\|\psi\|_{\mathbf{M},v}) < n(\alpha)$ and $n(\|\psi\|_{\mathbf{M}_\alpha,v}) = 0$: from these facts and the hypothesis we obtain $n(\alpha) > \|\varphi\|_{\mathbf{M},v} = n(\|\psi\|_{\mathbf{M},v}) > \|\chi\|_{\mathbf{M},v}$ and hence $\|\varphi\|_{\mathbf{M}_\alpha,v} = 0 = n(\|\psi\|_{\mathbf{M}_\alpha,v}) = \|\chi\|_{\mathbf{M}_\alpha,v}$. The last case is $\|\psi\|_{\mathbf{M},v} < n(\alpha)$: we have that $\|\varphi\|_{\mathbf{M},v} = n(\|\psi\|_{\mathbf{M},v}) > \alpha$ and hence $1 = n(\|\psi\|_{\mathbf{M}_\alpha,v}) = \|\varphi\|_{\mathbf{M}_\alpha,v}$.

$\|\chi\|_{\mathbf{M},v} > n(\|\psi\|_{\mathbf{M},v})$: we proceed analogously with the previous case.

- $\varphi := (\forall x)\psi(x)$ and the claim holds for $\psi(x)$: this means that, from the induction hypothesis, for every $v' \equiv_x v$ we have that (m) holds for $\|\psi(x)\|_{\mathbf{M},v'}$ and $\|\psi(x)\|_{\mathbf{M}_\alpha,v'}$.

  We have three cases.

  - $\|(\forall x)\psi(x)\|_{\mathbf{M},v} < n(\alpha)$. Clearly there exists a $v' \equiv_x v$ such that $\|\psi(x)\|_{\mathbf{M},v'} < n(\alpha)$ and hence, applying the induction hypothesis, we have $\|\psi(x)\|_{\mathbf{M}_\alpha,v'} = 0 = \|(\forall x)\psi(x)\|_{\mathbf{M}_\alpha,v}$.
  - $\|(\forall x)\psi(x)\|_{\mathbf{M},v} > \alpha$. Clearly for each $v' \equiv_x v$ it holds that $\|\psi(x)\|_{\mathbf{M},v'} > \alpha$, $\|\psi(x)\|_{\mathbf{M}_\alpha,v'} = 1$ (thanks to the induction hypothesis) and hence we have $\|(\forall x)\psi(x)\|_{\mathbf{M}_\alpha,v} = 1$.
  - $n(\alpha) \leq \|(\forall x)\psi(x)\|_{\mathbf{M},v} \leq \alpha$. We have that $\|\psi(x)\|_{\mathbf{M},v'} \geq n(\alpha)$ for every $v' \equiv_x v$. Moreover there is at least a $v'' \equiv_x v$ such that $\|\psi(x)\|_{\mathbf{M},v''} \leq \alpha$: thanks to the induction hypothesis for every such v'' we have that $\|\psi(x)\|_{\mathbf{M}_\alpha,v''} = \|\psi(x)\|_{\mathbf{M},v''}$. Applying again the induction hypothesis we have that $\|(\forall x)\psi(x)\|_{\mathbf{M}_\alpha,v} = \|(\forall x)\psi(x)\|_{\mathbf{M},v}$.

We do not analyze the case $\varphi := (\exists x)\psi(x)$, since the two quantifiers are inter-definable, in NM$\forall$, as in classical logic (see [CH10, theorem 2.31]). $\square$

Remark 9.1.4. *It is not difficult to see that the previous lemma holds even for $[0,1]_{NM}$, using the same proof. This remark will be useful for the subsequent results.*

Theorem 9.1.6. $TAUT_{NM_\infty}\forall = \bigcap_{n\geq 2} TAUT_{NM_n}\forall$.

Proof. The fact that $TAUT_{NM_\infty}\forall \subseteq \bigcap_{n\geq 2} TAUT_{NM_n}\forall$ follows from corollary 9.1.2.

Concerning the reverse inclusion, suppose that $\|\varphi\|^{NM_\infty}_{\mathbf{M},v} = \alpha < 1$. Take $\alpha < \beta < 1$: thanks to lemma 9.1.2 it is easy to check that $\|\varphi\|^{NM_\infty}_{\mathbf{M}_\beta,v} \leq \alpha$. Since $\mathbf{M}_\beta$ uses only a finite number of truth values, it is easy to construct a model $\mathbf{M}'_\beta$ (starting from $\mathbf{M}_\beta$ and modifying the range of the various r'_P's) over an appropriate NM_k such that $\|\varphi\|^{NM_k}_{\mathbf{M}'_\beta,v} = \|\varphi\|^{NM_\infty}_{\mathbf{M}_\beta,v}$. □

Before going on, we need to introduce the following family of NM-chains.

For $\alpha \in (0,1)$, let $\mathcal{A}_\alpha$ be the NM-chain defined over the universe $[1-|\alpha|, |\alpha|] \cup \{0,1\}$ and $n(x) = 1-x$ (recall that $|\alpha| = \max(|\alpha|, n(|\alpha|))$): observe that $\mathcal{A}_\alpha$ and $\mathcal{A}_{n(\alpha)}$ are isomorphic and every chain of this type forms a complete lattice.

Thanks to remark 9.1.4 and theorem 9.1.1, with a proof very similar to the one of theorem 9.1.6, we obtain the following result.

Theorem 9.1.7. $TAUT_{[0,1]_{NM}}\forall = \bigcap_{\alpha\in(0,1)} TAUT_{\mathcal{A}_\alpha}\forall$.

Now we analyze the "quantifiers shifting" laws.

Theorem 9.1.8. *Consider the following formulas:*

(1) $(\forall x)(\varphi(x) \wedge \nu) \leftrightarrow ((\forall x)\varphi(x) \wedge \nu)$

(2) $(\exists x)(\varphi(x) \wedge \nu) \leftrightarrow ((\exists x)\varphi(x) \wedge \nu)$

(3) $(\forall x)(\varphi(x) \vee \nu) \leftrightarrow ((\forall x)\varphi(x) \vee \nu)$

(4) $(\exists x)(\varphi(x) \vee \nu) \leftrightarrow ((\exists x)\varphi(x) \vee \nu)$

(5) $(\forall x)(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \wedge (\forall x)\psi(x))$

(6) $(\exists x)(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\exists x)\varphi(x) \wedge (\exists x)\psi(x))$

(7) $(\forall x)(\varphi(x) \vee \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \vee (\forall x)\psi(x))$

(8) $(\exists x)(\varphi(x) \vee \psi(x)) \leftrightarrow ((\exists x)\varphi(x) \vee (\exists x)\psi(x))$

(9) $(\exists x)(\varphi(x)\&\nu) \leftrightarrow ((\exists x)\varphi(x)\&\nu)$

(10) $(\exists x)(\varphi(x)\&\psi(x)) \leftrightarrow ((\exists x)\varphi(x)\&(\exists x)\psi(x))$

(11) $(\forall x)(\varphi(x) \to \nu) \leftrightarrow ((\exists x)\varphi(x) \to \nu)$

(12) $(\forall x)(\nu \to \varphi(x)) \leftrightarrow (\nu \to (\forall x)\varphi(x))$

(13) $\neg(\exists x)\varphi(x) \leftrightarrow (\forall x)\neg\varphi(x)$

(14) $\neg(\forall x)\varphi(x) \leftrightarrow (\exists x)\neg\varphi(x)$

(15) $(\forall x)(\varphi(x)\&\nu) \leftrightarrow ((\forall x)\varphi(x)\&\nu)$

(16) $(\forall x)(\varphi(x)\&\psi(x)) \leftrightarrow ((\forall x)\varphi(x)\&(\forall x)\psi(x))$

(17) $(\exists x)(\varphi(x) \to \nu) \leftrightarrow ((\forall x)\varphi(x) \to \nu)$

(18) $(\exists x)(\nu \to \varphi(x)) \leftrightarrow (\nu \to (\exists x)\varphi(x))$

where x does not occurs freely in ν. We have that

- *The formulas (1)-(14) hold in every NM-chain.*

- *The formulas (15)-(18) hold in every NM-chain $\mathcal{A}$ in which every element of $\mathcal{A} \setminus \{0, 1\}$ has a predecessor in $\mathcal{A}$.*
- *The formulas (15)-(18) fail in any NM-chain distinct from $\mathcal{A}$.*

Proof. • We prove that (1)-(14) are theorems of MTL∀. In [CH10, theorem 2.26] it is showed that (1)-(3),(5),(8),(9),(11)-(13) are theorems of MTL∀. Consider now the formulas (4),(6),(7),(10): direct inspection shows that in every standard MTL-algebra $\mathcal{A}$ it holds that $\sup_{w\in W}\{\max(w, x)\} = \max(\sup(W), x)$, $\sup_{\langle v,w\rangle\in V\times W}\{\min(v, w)\} = \min(\sup(V), \sup(W))$, $\inf_{\langle v,w\rangle\in V\times W}\{\max(v, w)\} = \max(\inf(V), \inf(W))$, $\sup_{\langle v,w\rangle\in V\times W}\{v * w\} = \sup(V) * \sup(W)$, for every $V, W \subseteq \mathcal{A}$ and $x \in \mathcal{A}$ (take the lexicographic order over $V\times W$). As regards to (14), thanks [CH10, theorem 2.31] we have $NM\forall \vdash (\exists x)\varphi(x) \leftrightarrow \neg(\forall x)\neg\varphi(x)$: since NM∀ satisfies the double negation law, then $NM\forall \vdash (14)$.

- Consider the NM-chain $\mathcal{A}$ of the theorem: from the properties of the negation we have that every element of $\mathcal{A} \setminus \{0, 1\}$ has predecessor and successor in $\mathcal{A}$.

  We have to check that

  - $\inf(W) * x = \inf_{w\in W}\{w * x\}$, formula (15).
  - $\inf_{\langle v,w\rangle\in V\times W}\{v * w\} = \inf(V) * \inf(W)$, formula (16).
  - $\sup_{w\in W}\{w \Rightarrow x\} = \inf(W) \Rightarrow x$, formula (17)
  - $\sup_{w\in W}\{x \Rightarrow w\} = x \Rightarrow \sup(W)$, formula (18).

  For every $W, V \subseteq \mathcal{A}$, $x \in \mathcal{A}$ and by taking the lexicographic order over $V \times W$.

  Take a set $W \subseteq \mathcal{A}$ and an element $x \in \mathcal{A}$: we have four cases.

  W has minimum m. In this case it is easy to check that $\inf(W) * x = m * x = \inf_{w\in W}\{w * x\}$ and $\sup_{w\in W}\{w \Rightarrow x\} = m \Rightarrow x = \inf(W) \Rightarrow x$. This solves the first case for (15),(17). Consider now a set $V \subseteq \mathcal{A}$: if V has minimum m', then $\inf_{\langle v,w\rangle\in V\times W}\{v * w\} = m' * m = \inf(V) * \inf(W)$. If V has infimum, but not minimum, then necessarily $\inf(V) = 0$ and $\inf(V) * \inf(W) = 0 = \inf_{v\in V}\{v * 1\} \geq \inf_{\langle v,w\rangle\in V\times W}\{v * w\}$. This solves the general case for (16), thanks to the commutativity of $*$.

  W has maximum n: we immediately see that $\sup_{w\in W}\{x \Rightarrow w\} = x \Rightarrow n = x \Rightarrow \sup(W)$.

  W has infimum m, but not minimum. The only possibility is that $m = 0$: from this fact we have that $\inf(W)*x = 0$ and $\inf_{w\in W}\{w*x\} \leq \inf_{w\in W}\{w*1\} = 0$. Moreover $\inf(W) \Rightarrow x = 1$ and $\sup_{w\in W}\{w \Rightarrow x\} =$

1: this last equality is immediate if there exists $y \in W$ such that $y \leq x$, otherwise $\sup_{w\in W}\{w \Rightarrow x\} = \sup_{w\in W}\{\max(n(w), x)\} = 1$.

W has supremum n, but not maximum: we have that $n = 1$. In this case we have that $x \Rightarrow \sup(W) = 1 = \sup_{w\in W}\{x \Rightarrow w\}$: the second equality is obvious if there exists $y \in W$ such that $x \leq y$, otherwise $\sup_{w\in W}\{x \Rightarrow w\} = \sup_{w\in W}\{\max(n(x), w)\} = 1$.

- Let $\mathcal{B}$ be an NM-chain that has an element $x \in \mathcal{B} \setminus \{0, 1\}$ without predecessor in $\mathcal{B}$.

  Consider the set $W = \{w \in \mathcal{B} : \ w < x\}$: direct inspection shows that $x \Rightarrow \sup(W) = x \Rightarrow x = 1$, but $\sup_{w\in W}\{x \Rightarrow w\} = \sup_{w\in W}\{\max(n(x), w)\} = \max(n(x), x) < 1$. Hence we have that that $\mathcal{B} \not\models (18)$.

  From lemma 9.1.1 we know that $n(x)$ does not have successor. Construct the set $W = \{w \in \mathcal{B} : \ w > n(x)\}$: direct inspection shows that $\inf(W) \Rightarrow n(x) = n(x) \Rightarrow n(x) = 1$, but $\sup_{w\in W}\{w \Rightarrow n(x)\} = \sup_{w\in W}\{\max(n(w), n(x))\} = \max(x, n(x)) < 1$. It follows that $\mathcal{B} \not\models (17)$. Moreover we have that $\inf(W) * x = 0$, whilst $\inf_{w\in W}\{w * x\} = \inf_{w\in W}\{\min(w, x)\} = \min(n(x), x) > 0$. This proves that $\mathcal{B} \not\models (15)$ and $\mathcal{B} \not\models (16)$.

$\square$

Corollary 9.1.4.

- *The formulas (1)-(18) belong to* $TAUT_{NM_\infty}\forall$, $TAUT_{NM_\infty^-}\forall$, $TAUT_{NM'^-_\infty}\forall$ *and* $TAUT_{NM_n}\forall$, *for every* $1 < n < \omega$.
- *The formulas (1)-(14) belong to* $TAUT_{[0,1]_{NM}}\forall$ *and* $TAUT_{NM'_\infty}\forall$.
- *The formulas (15)-(18) fail in* $[0, 1]_{NM}$ *and* NM'_∞.

Finally, we summarize relationship between the sets of tautologies of the NM-chains studied.

Theorem 9.1.9. *For every integer* $n > 1$ *and every* even *integer* $m > 1$

1. $TAUT_{[0,1]_{NM}}\forall = \bigcap_{\alpha\in(0,1)} TAUT_{\mathcal{A}_\alpha}\forall$.
2. $TAUT_{[0,1]_{NM}}\forall \subset TAUT_{NM_\infty}\forall \subset TAUT_{NM_n}\forall$.
3. $TAUT_{NM_\infty}\forall \subset TAUT_{NM_\infty^-}\forall \subset TAUT_{NM_m}\forall$, $TAUT_{NM'_\infty}\forall \subset TAUT_{NM'^-_\infty}\forall \subset TAUT_{NM_m}\forall$.
4. $TAUT_{[0,1]_{NM}}\forall \subseteq TAUT_{NM'_\infty}\forall \subset TAUT_{NM_n}\forall$.
5. $TAUT_{NM'_\infty}\forall \neq TAUT_{NM_\infty}\forall = \bigcap_{n\geq 2} TAUT_{NM_n}\forall$ *and hence* $TAUT_{NM'_\infty}\forall \subset TAUT_{NM_\infty}\forall$.

This theorem can be improved: in fact in the next section we will show that $TAUT_{NM'_\infty}\forall$ is not recursively enumerable. As a consequence, we have that $TAUT_{[0,1]_{NM}}\forall \subset TAUT_{NM'_\infty}\forall$.

9.1.2 Axiomatization and undecidability

Now we characterize the first-order logics associated to the finite NM-chains: for the other chains, previously discussed, we have undecidability results and an open problem.

From [Gis03, theorem 3] we can state

Theorem 9.1.10. *For every integer $n \geq 1$*

- *NM_{2n} is complete with respect to the logic LNM_{2n}: it is obtained from NM with the axioms $S_n(x_0, \ldots, x_n)$ and $BP(x)$.*
- *NM_{2n+1} is complete with respect to the logic LNM_{2n+1}: it is obtained from NM with the axiom $S_n(x_0, \ldots, x_n)$.*

As regards to the first-order version of these logics, we have

Theorem 9.1.11. *For each integer $n > 1$ and each $NM\forall$ formula φ,*

$$LNM_n\forall \vdash \varphi \qquad \textit{iff} \qquad NM_n \models \varphi$$

Proof. The soundness follows from the chain-completeness for axiomatic extensions of MTL∀ (see [EG01]).

For the completeness, note that each LNM$_n$-chain has at most n elements (this follows from the axiomatization of LNM$_n$ and theorem 4.3.5). Moreover, it easy to see that every LNM$_n$-chain embeds into NM$_n$ preserving all inf and sup. To conclude, from chain completeness theorems and the previous results we have that if $LNM_n\forall \nvdash \varphi$, then $NM_n \not\models \varphi$. □

For the other cases, we need some other machinery.

Let φ be a formula. Define φ^*, inductively, as follows:

- If φ is atomic, then $\varphi^* := \varphi^2$.
- If $\varphi := \bot$, then $\varphi^* := \bot$.
- If $\varphi := \psi \wedge \chi$, then $\varphi^* := \psi^* \wedge \chi^*$.
- If $\varphi := \psi \& \chi$, then $\varphi^* := \psi^* \& \chi^*$.
- If $\varphi := \psi \to \chi$, then $\varphi^* := (\psi^* \to \chi^*)^2$.
- If $\varphi := (\forall x)\chi$, then $\varphi^* := ((\forall x)\chi^*)^2$.

Lemma 9.1.3. *Let $\varphi, \mathcal{A}, \mathbf{M} = \langle M, \langle m_c \rangle_{c \in \mathbf{C}}, \langle r_P \rangle_{P \in \mathbf{P}} \rangle$ be a formula, an NM-chain (call $\mathcal{A}^+$ the set of its positive elements) and a safe $\mathcal{A}$-model. Construct an $\mathcal{A}$-model $\mathbf{M}^+ = \left\langle M, \langle m_c \rangle_{c \in \mathbf{C}}, \langle r'_P \rangle_{P \in \mathbf{P}} \right\rangle$ such that, for every evaluation v and atomic formula ψ*

$$\|\psi\|^{\mathcal{A}}_{\mathbf{M}^+,v} = \begin{cases} \|\psi\|^{\mathcal{A}}_{\mathbf{M},v} & \textit{if } \|\psi\|^{\mathcal{A}}_{\mathbf{M},v} \in \mathcal{A}^+ \\ 0 & \textit{otherwise.} \end{cases}$$

Then $\|\varphi^*\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi^*\|^{\mathcal{A}}_{\mathbf{M}^+,v}$, for every v.

Proof. By structural induction over φ: if $\varphi := \bot$ the claim is immediate. If φ is atomic, then $\varphi^* := \varphi^2$ and the claim easily follows from the definition of $\mathbf{M}^+$.

If $\varphi := \psi \circ \chi$, with $\circ \in \{\wedge, \&, \rightarrow\}$, then the claim follows from the induction hypothesis over ψ and χ.

Finally, if $\varphi := (\forall x)\chi$, then from the induction hypothesis it holds that $\|\chi^*\|^{\mathcal{A}}_{\mathbf{M},w} = \|\chi^*\|^{\mathcal{A}}_{\mathbf{M}^+,w}$, for every $w \equiv_x v$ and hence $\|\varphi^*\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi^*\|^{\mathcal{A}}_{\mathbf{M}^+,v}$. □

Theorem 9.1.12. *Let φ be a formula and $\mathcal{A}$ be an NM-chain.*

1. *$\mathcal{A} \models \varphi^*$ iff $\|\varphi^*\|^{\mathcal{A}}_{\mathbf{M}^+,v}$, for every safe $\mathcal{A}$-model $\mathbf{M}$ and evaluation v.*
2. *Let $\mathcal{B}$ be a* complete *NM-chain without negation fixpoint: call $\mathcal{B}^f$ its version with negation fixpoint f. It holds that*

$$\mathcal{B} \models \varphi^* \quad \textit{iff} \quad \mathcal{B}^f \models \varphi^*.$$

Proof. 1. Immediate from lemma 9.1.3.

2. Thanks to 1 it is enough to check that $\|\psi\|^{\mathcal{B}^f}_{\mathbf{M}^+,v} \neq f$, for every formula ψ and every $\mathcal{A}$-model $\mathbf{M}$ and evaluation v. This can be done by induction over ψ.
   - If ψ is atomic or $\bot$ the claim is immediate.
   - If $\psi := \theta \circ \chi$, with $\circ \in \{\wedge, \&, \rightarrow\}$, then the claim follows from the induction hypothesis over θ and χ.
   - Finally, if $\psi := (\forall x)\chi$, then from the induction hypothesis it holds that $\|\chi\|^{\mathcal{B}^f}_{\mathbf{M},w} \neq f$, for every $w \equiv_x v$: if $\|\chi\|^{\mathcal{B}^f}_{\mathbf{M},w} < f$, for some $w \equiv_x v$, then $\|(\forall x)\chi\|^{\mathcal{B}^f}_{\mathbf{M},v} < f$.

     Suppose that $\|\chi\|^{\mathcal{B}^f}_{\mathbf{M},w} > f$, for every $w \equiv_x v$: moreover, by contradiction, assume that $\|(\forall x)\chi\|^{\mathcal{B}^f}_{\mathbf{M},v} = \inf_{w \equiv_x v}\{\|\chi\|^{\mathcal{B}^f}_{\mathbf{M},w}\} = f$. This means that the set of positive elements of $\mathcal{B}$ does not have infimum, in contrast with the hypothesis that $\mathcal{B}$ is complete.

□

Consider now a Gödel chain (i.e. an MTL-chain satisfying the equation $x^2 = x$) $\mathcal{A}$: construct an NM-chain $\mathcal{A}_{NM}$ such that

- $A_{NM} = B \cup \{f\} \cup B'$, where $\langle B, \leq_{\mathcal{A}_{NM}}\rangle = \langle A \setminus \{0\}, \leq_{\mathcal{A}}\rangle$ and $\langle B' = \{b' : b \in B\}, \leq_{\mathcal{A}_{NM}}\rangle \simeq \langle B, \geq_{\mathcal{A}_{NM}}\rangle$.
- For every $x \in B, y \in B'$ set $x >_{\mathcal{A}_{NM}} f >_{\mathcal{A}_{NM}} y$.
- Define a strong negation function $n : A_{NM} \rightarrow A_{NM}$ such that $n(f) = f$, $n(a) = a'$ and $n(b') = b$, for every $a \in B$ and $b' \in B'$.

It is easy to see that $\mathcal{A}_{NM}$ has negation fixpoint f, B is the set of positive elements and B' the set of negative elements: note that $1 = \max(B)$ and $1' = \min(B')$ are the maximum and minimum of $\mathcal{A}_{NM}$. The element $1'$ will be called 0.

Remark 9.1.5. *An immediate consequence of this construction is that $\langle A, \leq_{\mathcal{A}}\rangle$ is order isomorphic to $\langle B \cup \{f\}, \leq_{\mathcal{A}_{NM}}\rangle$. From this fact it is easy to check that $\mathcal{A}$ is complete if and only if $\mathcal{A}_{NM}$ it is.*

Theorem 9.1.13. *Let φ be a formula, $\mathcal{A}$ be a Gödel chain. Consider a safe $\mathcal{A}$-model $\mathbf{M} = \langle M, \langle m_c\rangle_{c\in\mathbf{C}}, \langle r_P\rangle_{P\in\mathbf{P}}\rangle$: construct an $\mathcal{A}_{NM}$-model $\mathbf{M'} = \left\langle M, \langle m_c\rangle_{c\in\mathbf{C}}, \langle r'_P\rangle_{P\in\mathbf{P}}\right\rangle$ such that, for every evaluation v and* atomic *formula ψ*

$$\|\psi\|^{\mathcal{A}}_{\mathbf{M},v} = \|\psi\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v}.$$

Then it holds that

$$\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v},$$

for every evaluation v.

Proof. By structural induction over φ.

- If φ is $\bot$ or atomic, then the claim is immediate.

- $\varphi := \psi \circ \chi$, with $\circ \in \{\wedge, \&\}$ and the claim holds for ψ and χ. It follows that $\|\theta\|^{\mathcal{A}}_{\mathbf{M},v} = \|\theta^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v}$, for every v and with $\theta \in \{\psi, \chi\}$: noting that these values are 0 or idempotent elements the claim follows.

- $\varphi := \psi \to \chi$ and the claim holds for ψ and χ: it follows that $\|\theta\|^{\mathcal{A}}_{\mathbf{M},v} = \|\theta^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v}$, for every v and with $\theta \in \{\psi, \chi\}$. As previously noted, these values are idempotent elements or 0. Since $\varphi^* := (\psi^* \to \chi^*)^2$, an easy check shows that $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v}$, for every v.

- $\varphi := (\forall x)\psi$ and the claim holds for ψ. We have that $\|\psi\|^{\mathcal{A}}_{\mathbf{M},w} = \|\psi^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},w}$, for every w: if there is $w \equiv_x v$ such that $\|\psi\|^{\mathcal{A}}_{\mathbf{M},w} = 0$, then the claim is immediate.

  Suppose that $\|\psi\|^{\mathcal{A}}_{\mathbf{M},w} > 0$, for every $w \equiv_x v$.

  If $\|(\forall x)\psi\|^{\mathcal{A}}_{\mathbf{M},v} > 0$, then $\|(\forall x)\psi\|^{\mathcal{A}}_{\mathbf{M},v} = \|(\forall x)\psi^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v} = \|((\forall x)\psi^*)^2\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v} = \|\varphi^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v}$.

  If $\|(\forall x)\psi\|^{\mathcal{A}}_{\mathbf{M},v} = 0$, then $\|(\forall x)\psi^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v} = f$ and $\|((\forall x)\psi^*)^2\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v} = \|\varphi^*\|^{\mathcal{A}_{NM}}_{\mathbf{M'},v} = 0$.

□

Corollary 9.1.5. *Let φ be a formula, $\mathcal{A}$ be a Gödel chain. We have that*

$$\mathcal{A} \models \varphi \quad \textit{iff} \quad \mathcal{A}_{NM} \models \varphi^*.$$

Proof. An easy consequence of theorems 9.1.12, 9.1.13. □

Recall that a subset of $[0,1]$ is complete if and only if it is compact with respect to the order topology (see for example [SS95]). Now, in [BPZ07] it is showed that

Theorem 9.1.14 ([BPZ07])**.** *Let $\mathcal{A}$ be a countable topologically closed subalgebra of $[0,1]_G$ (i.e. a countable complete subalgebra of $[0,1]_G$). Then $TAUT_{\mathcal{A}}\forall$ is not recursively enumerable.*

In our case, we have

Theorem 9.1.15. *Let $\mathcal{A}$ be a countable topologically closed subalgebra of $[0,1]_{NM}$ (i.e. a countable complete subalgebra of $[0,1]_{NM}$). Then $TAUT_{\mathcal{A}}\forall$ is not recursively enumerable.*

Proof. Let $\mathcal{A}$ be a countable complete NM-chain.

If $\mathcal{A}$ has negation fixpoint then, thanks to the observations of remark 9.1.5, we can easily find a countable complete Gödel chain $\mathcal{B}$ such that $\mathcal{B}_{NM} \simeq \mathcal{A}$. From theorem 9.1.13 we have that $\varphi \in TAUT_{\mathcal{B}}\forall$ if and only if $\varphi^* \in TAUT_{\mathcal{A}}\forall$: since $TAUT_{\mathcal{B}}\forall$ is not recursively enumerable (theorem 9.1.14), then the same holds for $TAUT_{\mathcal{A}}\forall$.

If $\mathcal{A}$ does not have negation fixpoint, from theorem 9.1.12 we have that $\varphi^* \in TAUT_{\mathcal{A}}\forall$ if and only if $\varphi^* \in TAUT_{\mathcal{A}^f}\forall$, for every φ. Applying the argument of the previous case to $\mathcal{A}^f$, we have the theorem. □

Corollary 9.1.6. *For $\mathcal{A} \in \{NM_\infty, NM_\infty^-, NM'_\infty\}$, $TAUT_{\mathcal{A}}\forall$ is not recursively enumerable.*

Problem 9.1.1. *Which is the arithmetical complexity of $TAUT_{NM'^-_\infty}\forall$? Is it recursively axiomatizable ?*

9.1.3 Monadic fragments

In this final section, we will analyze the decidability of monadic fragments associated to some subalgebras of $[0,1]_{NM}$. Recall that in monadic first-order logic the language contains *only* unary predicates (and hence we have neither constants nor predicates of arity different from one).

Let $\mathcal{A}$ be a subalgebra of $[0,1]_{NM}$: with $monTAUT_{\mathcal{A}}\forall$ we indicate the first-order monadic tautologies associated to $\mathcal{A}$.

In [BCF07, theorem 1] it is showed that the monadic fragment of finite Gödel chains is decidable, but as noted in the subsequent remark, the proof applies to the monadic fragments of arbitrary finite-valued logics. As a consequence we have

Theorem 9.1.16. *Let $\mathcal{A}$ be a finite NM-chain: we have that $monTAUT_{\mathcal{A}}\forall$ is decidable.*

However, for the infinite case the situation is worst; in fact

Theorem 9.1.17 ([BCF07]). *Let $\mathcal{A}$ be an infinite complete subalgebra of $[0,1]_G$: with the possible exception of $\mathcal{A} = G_\uparrow$, $monTAUT_\mathcal{A}\forall$ is undecidable.*

Moving to the NM case, we obtain

Theorem 9.1.18. *Let $\mathcal{A}$ be an infinite complete subalgebra of $[0,1]_{NM}$: with the possible exception of $\mathcal{A} \in \{NM_\infty, NM_\infty^-\}$, $monTAUT_\mathcal{A}\forall$ is undecidable.*

Proof. The proof is an easy adaptation of the one of theorem 9.1.15. □

Corollary 9.1.7. *$monTAUT_{NM'_\infty}\forall$ is undecidable.*

Problem 9.1.2. *For $\mathcal{A} \in \{NM_\infty, NM_\infty^-, NM'^-_\infty\}$, is $monTAUT_\mathcal{A}\forall$ decidable ?*

9.2 Conclusions

In this chapter we have obtained some results about the first-order tautologies associated with particular NM-chains: moreover, we have showed some decidability and undecidability results, even in monadic case. Furthermore, when possible, we have compared our results with the ones presented in [BPZ07] and [BCF07], for (first-order) Gödel logics.

We have left two problems open.

Problem 9.1.1 is particularly interesting: if $TAUT_{NM'^-_\infty}\forall$ will result recursively axiomatizable, then the next step will be the search for a first-order logic complete with respect to NM'^-_∞. This logic could be a relevant infinite-valued logic, because NM'^-_∞ satisfies the quantifiers shifting rules and hence we could work with formulas in prenex normal form.

Consider now problem 9.1.2: for what concerns NM'^-_∞, this is a particular case of problem 9.1.1. As regards to $\mathcal{A} \in \{NM_\infty, NM_\infty^-\}$, instead, the solution is strictly connected with the analogous problem for Gödel logic with the chain $G_\uparrow$.

Finally, the full classification of the (existence of) first-order logics associated to the various subalgebras of $[0,1]_{NM}$ remains an interesting problem that should be faced in future papers.

Chapter 10

On some logical and algebraic properties of Nilpotent Minimum logic

In this chapter we will analyze various logical and algebraic properties of Nilpotent Minimum logic. Some of the cited properties are: disjunction property, Halldén completeness, deductive Maksimova's variable separation property, pseudo-relevance property, amalgamation property, deductive interpolation property and a weak version of Craig interpolation theorem (see the next section for the definitions of these properties). Moreover it has been presented an alternative definition of relation of semantic consequence, over standard Nilpotent minimum algebra, that is equivalent to the usual one.

10.1 Algebraic and logical properties of NM

In this section we will analyze some relevant (in the context of substructural logics: see for example [GJKO07, chapter 5]) algebraic and logical properties, for Nilpotent Minimum logic: some of them will be equivalent, for this logic. We will conclude by showing an alternative (and equivalent to the usual one) semantic consequence relation, over $[0,1]_{NM}$.

Remark 10.1.1. *Given an NM-chain defined over a set A, the set of its positive (negative) elements will be denoted with A^+ (A^-). See chapter 4 for the definition of positive and negative elements.*

Definition 10.1.1. *We say that a logic L has the* disjunctive property *(DP) if $\vdash_L \varphi \vee \psi$ implies that $\vdash_L \varphi$ or $\vdash_L \psi$.*

For example the intuitionistic logic enjoys this property: however it fails for many superintuitionistic logics (see [CZ91] for a survey) and for classical logic (for this last one $x \vee \neg x$ is a counterexample).

For the case of axiomatic extensions of MTL, we obtain a negative result

Theorem 10.1.1. *Let L be an axiomatic extension of MTL: then DP fails for L.*

Proof. The formula $(x \to y) \vee (y \to x)$ is a theorem of L. Consider now the direct product $\mathbf{2} \times \mathbf{2}$ of two copies of two elements boolean algebra: clearly this algebra belongs to the variety of L-algebras. By taking a $\mathbf{2} \times \mathbf{2}$-evaluation v such that $v(x) = \langle 0, 1\rangle$ and $v(y) = \langle 1, 0\rangle$, we obtain $v((x \to y) \vee (y \to x)) = 1$, whilst $v(x \to y) < 1$ and $v(y \to x) < 1$. From chain completeness theorem ([EG01]) we have $\nvdash_L x \to y$, $\nvdash_L y \to x$. □

Corollary 10.1.1. *Nilpotent Minimum logic does not have the DP.*

There is a property weaker than DP: the Halldén completeness.

Definition 10.1.2. *A logic L has the* Halldén completeness *(HC) if for every formulas φ, ψ with no variables in common, $\vdash_L \varphi \vee \psi$ implies that $\vdash_L \varphi$ or $\vdash_L \psi$.*

There is an interesting algebraic characterization of HC, for substructural logics

Theorem 10.1.2 ([GJKO07, corollary 5.30])**.** *Let L be a substructural logic over FL_{ew} that is n-contractive (i.e. $\vdash_L \varphi^n \to \varphi^{n+1}$, for some $n \geq 1$). The following are equivalent*

1. *L has the Halldén completeness.*
2. *There is a subdirectly irreducible FL_{ew}-algebra $\mathcal{A}$ such that*

$$\models_{\mathcal{A}} \varphi \quad \textit{iff} \quad \vdash_L \varphi.$$

Now,

Proposition 10.1.1 ([Nog06, proposition 8.11])**.** *Let $\mathcal{A}$ be an n-contractive MTL-chain. Then $\mathcal{A}$ is subdirectly irreducible if and only if the set of idempotent elements has a coatom.*

Noting that, in every NM-chain $\mathcal{A}$, the set of idempotent elements coincides with $A^+ \cup \{0\}$, and that $\vdash_{NM} \varphi^2 \to \varphi^3$, we obtain

Proposition 10.1.2. *An NM-chain is subdirectly irreducible if and only if it has a coatom.*

Recall now that

Theorem 10.1.3 ([Gis03, corollary 2])**.** *The variety of NM-algebras is generated by any infinite NM-chain with negation fixpoint.*

We arrive at the following

Theorem 10.1.4. *Nilpotent Minimum logic enjoys the HC.*

Proof. In [GJKO07, section 3.5.2] it is pointed out that MTL (and hence NM) is an axiomatic extension of FL_{ew}: from these fact and theorem 10.1.2 it is enough to construct a subdirectly irreducible NM-algebra that generates the variety associated to NM. Consider the chain NM'_∞ (see section 4.3.6): thanks to proposition 10.1.2 and theorem 10.1.3 we have that NM'_∞ is a subdirectly irreducible generic NM-chain. It follows that NM enjoys the HC. □

Now,

Theorem 10.1.5 ([GJKO07])**.** *The following conditions are equivalent for every substructural logic L over FL_{ew}:*

- *L is Halldén complete.*
- *L is meet irreducible in the lattice of all substructural logics over FL, i.e. L it is not the intersection (from the axiomatic point of view) of all the logics strictly larger than L, see [GJKO07, chapter 5].*

We immediately obtain

Corollary 10.1.2. *Nilpotent Minimum logic is meet irreducible in the lattice of axiomatic extensions of MTL (i.e. if $NM = x \wedge y$ then $NM = x$ or $NM = y$).*

Another property, similar to the HC, is the following

Definition 10.1.3. *A substructural logic L has the* deductive Maksimova's variable separation property *(DMVP), if for all sets of formulas $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ that have no variables in common, $\Gamma, \Sigma \vdash_L \varphi \vee \psi$ implies $\Gamma \vdash_L \varphi$ or $\Sigma \vdash_L \psi$.*

We have that

Theorem 10.1.6 ([GJKO07, theorem 5.35])**.** *The following conditions are equivalent for every substructural logic L over FL_{ew}:*

- *L has the DMVP.*
- *All pairs of subdirectly irreducible L-algebras are jointly embeddable into a subdirectly irreducible L-algebra.*

Theorem 10.1.7. *Nilpotent Minimum logics enjoys the DMVP.*

Proof. Thanks to theorem 10.1.6 and proposition 10.1.2 it is enough to check that, for every pair of NM-chains with coatom, there is an NM-chain with coatom in which both of them embed to.

Let $\mathcal{A}, \mathcal{B}$ be two NM-chains with coatoms $c_{\mathcal{A}}, c_{\mathcal{B}}$ and negations $n_{\mathcal{A}}, n_{\mathcal{B}}$: assume that $A \cap B = \{0, 1\}$ (otherwise take two isomorphic copies of $\mathcal{A}$ and $\mathcal{B}$ that satisfy that condition, by renaming their elements). Define $A' = A^+ \cup A^-$ and $B' = B^+ \cup B^-$ and set $C = A' \cup \{f\} \cup B'$.

Define $\leq$ over C such that $0 < x < y < f < 1$, for every $x \in A^- \setminus \{0\}$ and $y \in B^- \setminus \{0\}$.

Define a negation $n : C \to C$ such that, for every $a \in C$

$$n(a) = \begin{cases} n_{\mathcal{A}}(a) & \text{if } a \in A' \\ n_{\mathcal{B}}(a) & \text{if } a \in B' \\ f & \text{if } a = f \end{cases}$$

Finally, for every $x, y \in A^+ \cup B^+$, set $f < x$, $f < y$: define $x < y$ if and only if $n(x) > n(y)$. Note that $\leq$ is a total order over D.

Clearly n is involutive: the fact that n is order reversing follows from the definitions of $\leq$ and n. Call $\mathcal{C}$ the NM-chain obtained from $\langle C, \leq \rangle$ and n: note that $\mathcal{C}$ is subdirectly irreducible, since it has $c_{\mathcal{A}}$, as coatom.

Construct the maps $h : A \to C$ and $k : B \to C$ in the way that $h(a) = a$ and $k(b) = b$, for every $a \in A', b \in B'$ and $h(f_A) = f$, $h(f_B) = f$ (if the fixpoints f_B, f_C exist in B and C): direct inspection shows that they extend to a pair of homorphisms h, k such that $\mathcal{A} \overset{h}{\hookrightarrow} \mathcal{C}$, $\mathcal{B} \overset{k}{\hookrightarrow} \mathcal{C}$. □

Consider now

Definition 10.1.4. *A substructural logic L has the* pseudo-relevance property *(PRP), if for all pairs of formulas φ, ψ with no variables in common, $\vdash_L \varphi \to \psi$ implies either $\vdash_L \neg\varphi$ or $\vdash_L \psi$.*

Since it holds that

Theorem 10.1.8. *Let L be a substructural logic. If every pair of subdirectly irreducible L-algebras are jointly embeddable into an L-algebra, then L enjoys the PRP.*

Then, from theorems 10.1.6, 10.1.7, we immediately obtain

Theorem 10.1.9. *Nilpotent Minimum logic enjoys the PRP.*

Remark 10.1.2. *If we drop, from the PRP, the requirement that φ, ψ have no variables in common, then the property that we obtain fails in every axiomatic extension of MTL: the formula $x \to x$ is a counterexample.*

An interesting algebraic property (that, as we will see, is strictly connected with a logical one), already studied in chapter 8 for other logics, is the following

Definition 10.1.5. *We say that a variety K of MTL-algebras has the* amalgamation property *(AP) if for every tuple $\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, i, j\rangle$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K$ and $\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{B}$, $\mathcal{A} \overset{j}{\hookrightarrow} \mathcal{C}$, there is a tuple $\langle \mathcal{D}, h, k\rangle$, with $\mathcal{D} \in K$, $\mathcal{B} \overset{h}{\hookrightarrow} \mathcal{D}$, $\mathcal{C} \overset{k}{\hookrightarrow} \mathcal{D}$, such that $h \circ i = k \circ j$.*

Theorem 10.1.10 ([Mon06][lemmas 3.3, 3.4])**.** *Let K be a variety of BL-algebras and K_{lin} be the set of its chains. If K_{lin} has the AP then the same holds for K.*

Now, inspecting the proof of these lemmas it is easy to see that the same holds for MTL. Hence we have

Theorem 10.1.11. *Let K be a variety of MTL-algebras and K_{lin} be the set of its chains. If K_{lin} has the AP then the same holds for K.*

Now, concerning NM, in [ABM09b] it is showed that finite NM-algebras enjoy the AP. We now prove the general case[1]

Theorem 10.1.12. *The variety of NM-algebras enjoys the AP.*

Proof. Thanks to theorem 10.1.11 it is enough to show the claim for the NM-chains. The proof is an adaptation of the one given in [GM05, proposition 6.20] for Gödel's case.

Consider the tuple $\langle \mathcal{A}, \mathcal{B}, \mathcal{C}, i, j\rangle$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are NM-chains and $\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{B}$, $\mathcal{A} \overset{j}{\hookrightarrow} \mathcal{C}$.

Distinguish the following cases:

1. $\mathcal{A}, \mathcal{B}, \mathcal{C}$ have negation fixpoint or none of them have it.

   We can assume that $A = B \cap C$, otherwise take two isomorphic copies of $\mathcal{B}$ and $\mathcal{C}$ that satisfy that condition, by renaming their elements.

   Set $D = B \cup C$ and define $\leq$ over $B^+ \cup C^+$ such that for every $x, y \in B^+ \cup C^+$, $x < y$ iff

   (a) $x, y \in B^+$ and $x <_B y$ or
   (b) $x, y \in C^+$ and $x <_C y$ or
   (c) $x \in B^+, y \in C^+$ and there exists $z \in A$ such that $x <_B z$ and $z <_C y$ or
   (d) $x \in C^+, y \in B^+$ and there exists $z \in A$ such that $x <_C z$ and $z <_B y$

   It is easy to see that $\leq$ is a partial order over $B^+ \cup C^+$: extend it to a total order $\leq$ over $B^+ \cup C^+$.

[1] This result has been independently proved in [Mar10].

For every $x \in B^+ \cup C^+$ and $y \in B^- \cup C^-$ set $y < x$: if A has negation fixpoint f, then set $y < f < x$.

Define a negation $n : D \to D$ such that, for every $a \in D$

$$n(a) = \begin{cases} n_{\mathcal{A}}(a) & \text{if } a \in A \\ n_{\mathcal{B}}(a) & \text{if } a \in B \setminus A \\ n_{\mathcal{C}}(a) & \text{if } a \in C \setminus A \end{cases}$$

Finally, for every $x, y \in B^- \cup C^-$, set $x < y$ if and only if $n(x) > n(y)$. Note that $\leq$ is a total order over D.

Clearly n is involutive: the fact that n is order reversing follows from the definitions of $\leq$ and n. Call $\mathcal{D}$ the NM-chain obtained from $\langle D, \leq \rangle$ and n.

Construct the maps $h : B \to D$ and $k : C \to D$ in the way that $h(b) = b$ and $k(c) = c$, for every $b \in B, c \in C$: direct inspection shows that they extend to a pair of homorphisms h, k such that $\mathcal{B} \overset{h}{\hookrightarrow} \mathcal{D}$, $\mathcal{C} \overset{k}{\hookrightarrow} \mathcal{D}$ and $h \circ i = k \circ j$.

2. $\mathcal{A}$ does not have negation fixpoint, but $\mathcal{B}$ or $\mathcal{C}$ does. Call $B' = B^- \cup B^+$ and $C' = C^- \cup C^+$.

   We can assume that $A = B' \cap C'$, otherwise take two isomorphic copies of $\mathcal{B}$ and $\mathcal{C}$ that satisfy that condition, by renaming their elements.

   Set $D = B' \cup C' \cup \{f\}$ and define $\leq$ over $B^+ \cup C^+$ such that for every $x, y \in B^+ \cup C^+$, $x < y$ iff

   (a) $x, y \in B^+$ and $x <_B y$ or
   (b) $x, y \in C^+$ and $x <_C y$ or
   (c) $x \in B^+, y \in C^+$ and there exists $z \in A$ such that $x <_B z$ and $z <_C y$ or
   (d) $x \in C^+, y \in B^+$ and there exists $z \in A$ such that $x <_C z$ and $z <_B y$

   It is easy to see that $\leq$ is a partial order over $B^+ \cup C^+$: extend it to a total order $\leq$ over $B^+ \cup C^+$.

   Set $y < f < x$, for every $x \in B^+ \cup C^+$ and $y \in B^- \cup C^-$.

   Define a negation $n : D \to D$ such that, for every $a \in D$

$$n(a) = \begin{cases} n_{\mathcal{A}}(a) & \text{if } a \in A \\ n_{\mathcal{B}}(a) & \text{if } a \in B' \setminus A \\ n_{\mathcal{C}}(a) & \text{if } a \in C' \setminus A \\ f & \text{if } a = f \end{cases}$$

Finally, for every $x, y \in B^- \cup C^-$, set $x < y$ if and only if $n(x) > n(y)$. Note that $\leq$ is a total order over D.

Clearly n is involutive: the fact that n is order reversing follows from the definitions of $\leq$ and n. Call $\mathcal{D}$ the NM-chain obtained from $\langle D, \leq \rangle$ and n.

Construct the maps $h: B \to D$ and $k: C \to D$ in the way that $h(b) = b$ and $k(c) = c$, for every $b \in B', c \in C'$ and $h(f_B) = f$, $h(f_C) = f$ (if the fixpoints f_B, f_C exist in B and C): direct inspection shows that they extend to a pair of homorphisms h, k such that $\mathcal{B} \overset{h}{\hookrightarrow} \mathcal{D}$, $\mathcal{C} \overset{k}{\hookrightarrow} \mathcal{D}$ and $h \circ i = k \circ j$.

□

We recall a property, that we have already studied for n-contractive BL-logics in chapter 8:

Definition 10.1.6. *A logic L has the* deductive interpolation property *(DIP) if for any theory Γ and for any formula ψ of L, if $\Gamma \vdash_L \psi$, then there is a formula γ such that $\Gamma \vdash_L \gamma$, $\gamma \vdash_L \psi$ and every propositional variable occurring in γ occurs both in Γ and in ψ.*

Now, from the results of [GJKO07] (see even [GO06, theorem 5.8]) we have that in every axiomatic extension of MTL the DIP and AP (for the corresponding variety) are equivalent. It follows that

Theorem 10.1.13. *The logic NM enjoys the DIP.*

As showed in [ABM09b] the Craig interpolation theorem (see definition 8.5.2) does not hold, for NM. However, thanks to theorems 3.3.1, 10.1.13, we obtain

Theorem 10.1.14 (Weak Craig interpolation theorem)**.** *For every pair of formulas φ, ψ, if $\vdash_{NM} \varphi^2 \to \psi$, then there is a formula γ such that $\vdash_{NM} \varphi^2 \to \gamma$, $\vdash_{NM} \gamma^2 \to \psi$ and every propositional variable occurring in γ occurs both in φ and in ψ.*

We conclude by presenting an alternative definition of semantic consequence relation, inspired to the one introduced in [BPZ07] for Gödel logic.

Definition 10.1.7. *Let Γ, φ be a theory and a formula. We define*

$$\Gamma \Vdash_{[0,1]_{NM}} \varphi \quad \textit{iff} \quad v(\Gamma^2) \leq v(\varphi),$$

for every $[0,1]_{NM}$-evaluation v and with $v(\Gamma^2) = (\inf\{v(\psi^2) : \ \psi \in \Gamma\})^2$.

As can be noted, whilst the usual notion of semantic consequence is substantially analogous to the one of classical logic, since it refers only to the assignments that maps the formulas in 1, the previous notion is different. In fact it fixes the behavior of the assignments even for the truth values between 0 and 1.

Lemma 10.1.1. *Let v be a $[0,1]_{NM}$-evaluation. For $\alpha \in [0,1]$ construct a $[0,1]_{NM}$-evaluation v_α such that, for every propositional variable x*

$$v_\alpha(x) = \begin{cases} 1 & \text{if } v(x) > |\alpha| \\ 0 & \text{if } v(x) < 1 - |\alpha| \\ v(x) & \text{otherwise.} \end{cases} \tag{10.1}$$

With $|\alpha| = \max(\alpha, 1-\alpha)$. Then the equation (10.1) holds for every formula φ.

Proof. An easy structural induction over φ. □

We are now ready to show the equivalence between the two notions of semantic consequence

Theorem 10.1.15. *Let Γ, φ be a theory and a formula. We have that*

$$\Gamma \Vdash_{[0,1]_{NM}} \varphi \quad \textit{iff} \quad \Gamma \models_{[0,1]_{NM}} \varphi.$$

Proof.

- $\Rightarrow$ An easy check.
- $\Leftarrow$ Suppose that $\Gamma \nVdash_{[0,1]_{NM}} \varphi$, i.e. there exists a $[0,1]_{NM}$-evaluation v such that $v(\Gamma^2) > v(\varphi)$.

  Note that necessarily $v(\Gamma^2) = (\inf\{v(\psi^2) : \psi \in \Gamma\})^2 > \frac{1}{2}$, otherwise $v(\Gamma^2) = 0$, a contradiction.

  Take $\alpha \in (\frac{1}{2}, 1]$ such that $v(\varphi) < \alpha < v(\Gamma^2)$ and let v_α be as in lemma 10.1.1: we have that $v_\alpha(\varphi) < 1$, whilst, for every $\psi \in \Gamma$, $v_\alpha(\psi) \geq v_\alpha(\Gamma^2) = 1$. It follows that $\Gamma \not\models_{[0,1]_{NM}} \varphi$ and this concludes the proof.

□

Remark 10.1.3. *It is easy to see that lemma 10.1.1 can be extended to every NM-chain. From this fact, by inspecting the proof of theorem 10.1.15, we have that theorem 10.1.15 holds for every NM-chain that is complete and dense.*

From the previous results and standard completeness theorem we easily obtain

Corollary 10.1.3. *Let $\{\varphi_1, \dots, \varphi_n\}$ be a finite theory and ψ be a formula. Then*

$$\{\varphi_1, \dots, \varphi_n\} \vdash_{NM} \psi \quad \textit{iff} \quad \vdash_{NM} \varphi_1^2 \wedge \dots \wedge \varphi_n^2 \to \psi.$$

Finally, comparing our alternative notion of semantic consequence with the one introduced in [BPZ07] for Gödel logic, we have

Remark 10.1.4. *In [BPZ07] $\Gamma \Vdash_{[0,1]_G} \varphi$ is defined as $v(\Gamma) = \inf\{v(\psi) : \psi \in \Gamma\} \leq v(\varphi)$, for every $[0,1]_G$-evalutation v. This notion (reformulated over $[0,1]_{NM}$), however, is not equivalent to $\models_{[0,1]_{NM}}$, in NM: in fact $\{x, x \to y\} \models_{[0,1]_{NM}} y$, whilst by taking a $[0,1]_{NM}$-evaluation v such that $v(x) = \frac{1}{4}$ and $v(y) = \frac{1}{5}$ we have $v(\{x, x \to y\}) = \frac{1}{4} > \frac{1}{5} = v(y)$.*

However, it is possible to find syntactical conditions that are equivalent to the semantic consequence defined as "$\leq$", like the case of Gödel logic. Before doing this, we need some results.

Definition 10.1.8. *Let φ, ψ be two formulas. Then*

$$\varphi \models_{\leq} \psi \quad \textit{iff} \quad v(\varphi) \leq v(\psi),$$

for every $[0,1]_{NM}$-evaluation v.

An immediate consequence of this definition is

Lemma 10.1.2. *Let φ, ψ be two formulas. Then*

$$\varphi \models_{\leq} \psi \quad \textit{iff} \quad \neg\psi \models_{\leq} \neg\varphi.$$

Proof. Immediate from the previous definition and the properties of the negation on $[0,1]_{NM}$. □

Finally, we get the announced result

Theorem 10.1.16. *Let φ, ψ be two formulas. Then*

$$\varphi \vdash \psi, \neg\psi \vdash \neg\varphi \quad \textit{iff} \quad \varphi \models_{\leq} \psi.$$

Proof. Thanks to standard completeness theorem and theorem 10.1.15, it holds that (with $\models_*$ we mean $\models_{[0,1]_{NM}}$)

$$\varphi \vdash \psi, \neg\psi \vdash \neg\varphi \quad \text{iff} \quad \varphi \models_* \psi, \neg\psi \models_* \neg\varphi \quad \text{iff} \quad \varphi \Vdash \psi, \neg\psi \Vdash \neg\varphi. \tag{10.2}$$

Concerning the right-to-left direction of the theorem, thanks to lemma 10.1.2 and an easy check, we have that $\varphi \models_{\leq} \psi$ implies $\varphi \models_{[0,1]_{NM}} \psi$, $\neg\psi \models_{[0,1]_{NM}} \neg\varphi$.

It remains to prove the left-to-right direction: suppose that $\varphi \models_{[0,1]_{NM}} \psi$, $\neg\psi \models_{[0,1]_{NM}} \neg\varphi$, but $\varphi \not\models_{\leq} \psi$, i.e. there is a $[0,1]_{NM}$-assignment v such that $v(\varphi) > v(\psi)$. Thanks to the equivalence (10.2) we have that $\varphi \Vdash \psi$ and $\neg\psi \Vdash \neg\varphi$. If $v(\varphi)$ is an idempotent, then $v(\varphi) = v(\varphi^2) > v(\psi)$ and then $\varphi \nVdash \psi$. Suppose that $v(\varphi)$ is not an idempotent: it follows that $\frac{1}{2} \leq v(\neg\varphi) < v(\neg\psi) = v((\neg\psi)^2)$. We immediately see that $\neg\psi \nVdash \neg\varphi$. □

Chapter 11

Conclusions

In this thesis we have faced many topics concerning some axiomatic extensions of MTL as well as their semantics (algebraic and, for BL, also a temporal one); even the first-order case have been investigated.

We now summarize the problems left open.

Concerning supersound logics (7) there is an unanswered question.

Problem 11.0.1. *In theorem 7.2.1 we have showed that if L enjoys the CEP, then L∀ is supersound. Is the converse true ?*

Moreover, starting from the results of chapter 7, we can get a new result.

First of all, from theorems 7.2.1, 7.3.2, 7.3.3 we easily obtain

Corollary 11.0.4. *A variety of $\mathbb{L}$ of BL-algebras satisfy the equation $x^n = x^{n+1}$ if and only if the correspondent logic L enjoys the CEP.*

Now,

Theorem 11.0.17. *Let $\mathbb{L}$ be a subvariety of BL-algebras (call L the correspondent logic). The following are equivalent*

- *$\mathbb{L}$ satisfy the n-contraction law, for some n ($x^n = x^{n-1}$)*
- *Every L-chain can be embedded into a complete L-chain (in this case we say that $\mathbb{L}$ admits completions).*

Proof. Take a variety $\mathbb{L}$ of BL-algebras that is n-potent, for some n, and call L the correspondent logic: thanks to corollary 11.0.4, L enjoys the CEP. Take an L-algebra $\mathcal{A}$: it is isomorphic to a subdirect product of L-chains $\mathcal{A}_i, i \in I$ (for some index set I). Since L enjoys the CEP, then we have a family of complete L-chains $\mathcal{B}_i, i \in I$ such that $\mathcal{A}_i$ embeds into $\mathcal{B}_i$, for every $i \in I$. By taking the direct product of $\mathcal{B}_i$, we obtain a complete L-algebra in which $\mathcal{A}$ embeds to. Hence every L-chain can be embedded into a complete L-chain.

Conversely, suppose that there is no n such that L is n-contractive. Thanks to Lemma 7.3.5, we have that

> Either $\mathbb{L}$ contains the variety of product algebras or it contains the variety generated by Chang's algebra.

Now, in [KL08] it is shown that the varieties of product algebras and the variety generated by Chang's algebra do not admit completions (in the variety of BL-algebras). It follows that L does not admit completions. $\square$

Note that the equivalence stated in theorem 11.0.17 has been showed, in another way, in [BC10]: moreover, in this paper, it is showed that the two conditions of our theorem are equivalent to a third one, called dual canonicity.

The others open problems are relative to chapter 9, about first-order Nilpotent Minimum logics.

Problem 11.0.2. *Which is the arithmetical complexity of $TAUT_{NM'^{-}_{\infty}}\forall$? Is it recursively axiomatizable ?*

Problem 11.0.3. *For $\mathcal{A} \in \{NM_\infty, NM^-_\infty, NM'^{-}_\infty\}$, is $monTAUT_{\mathcal{A}}\forall$ decidable ?*

Appendices

Appendix A

Universal Algebra

A.1 Algebra and Universal algebra

In this section we will furnish the necessary background about universal algebra and some basic algebraic structures. Some reference textbooks are [BS81, MMT87, Grä08].

A.1.1 Partially ordered sets, lattices and algebraic structures

We briefly recall some preliminary algebraic notions: for further details the reader could consult [DP02, BS81].

Definition A.1.1. *Let A be a non-empty set: an n-ary operation over A is a map $f : A^n \to A$. n is said to be the arity of the operation.*

Definition A.1.2. *A* language *(or type) of an algebra is a pair $\langle \mathcal{F}, \nu \rangle$, where $\mathcal{F}$ is a set of function symbols and $\nu : \mathcal{F} \to \mathbb{N}$ indicates the arity of each of them.*

Definition A.1.3. *An* algebra *(algebraic structure) of type $\mathcal{F}$ is a pair $\mathcal{A} = \langle A, F \rangle$, where A is a non-empty set, said* universe *or support of the algebra and F is a set of operations over A such that, for every $f \in \mathcal{F}$, $\nu(f) = n$ there is an $f^{\mathcal{A}} : A^n \to A$.*[1] *$\mathcal{A}$ is finite if A it is, $\mathcal{A}$ is trivial if $|A| = 1$.*

Definition A.1.4. *A partially ordered set (poset) is a pair $\langle A, \leq \rangle$, where A is a set and $\leq$ is a binary relation over A such that*

(reflexivity)	$x \leq x$
(antisimmetry)	$x \leq y$ *and* $y \leq x$ *implies* $x = y$
(transitivity)	$x \leq y$ *and* $y \leq z$ *implies* $x \leq z$.

[1] In the case in which $\nu(f) = 0$, then $f^{\mathcal{A}} : \{\emptyset\} \to A$ is a constant.

For every $x, y, z \in A$.

A totally ordered set (or chain) is a poset $\langle A, \leq \rangle$ such that the following holds

(linearity) $$x \leq y \quad \textit{or} \quad y \leq x$$

For every $x, y \in A$.

We now introduce the concept of lattice: we will present two equivalent definitions.

Definition A.1.5. *A* lattice *is a poset $\langle A, \leq \rangle$ such that exist* $\inf \{x, y\}$ *and* $\sup \{x, y\}$*, for every $x, y \in A$.*

A lattice can be also seen as an algebraic structure:

Definition A.1.6. *A lattice is a structure $\langle A, \wedge, \vee \rangle$ such that*

$$\begin{array}{ll} \text{(associativity laws)} & (x \wedge (y \wedge z)) = ((x \wedge y) \wedge z) \\ & (x \vee (y \vee z)) = ((x \vee y) \vee z) \\ \text{(commutativity laws)} & x \wedge y = y \wedge x \\ & x \vee y = y \vee x \\ \text{(idempotency laws)} & x \wedge x = x \\ & x \vee x = x \\ \text{(absorption laws)} & x \wedge (x \vee y) = x \\ & x \vee (x \wedge y) = x. \end{array}$$

As previously pointed out these definitions are equivalent: the proof of the following proposition can be found in [DP02, BS81].

Proposition A.1.1. *Definitions A.1.5 and A.1.6 are equivalent, by setting*

$$x \wedge y = x \quad \textit{if and only if} \quad x \leq y.$$

Remark A.1.1. *Usually, the type of an algebraic structure is denoted also with the sequence of the arieties of its operations (assuming that there is a finite number of them).*

For example, a lattice $\langle A, \wedge, \vee \rangle$ has type $\langle 2, 2 \rangle$.

Since all the algebraic structures, considered in this thesis, have a finite number of operations, then we will follow the previous notation.

Other interesting types of lattices are the following

Definition A.1.7. *A lattice $\langle A, \leq \rangle$ is*

- *complete if there exist* $\inf X$ *and* $\sup X$*, for every $X \subseteq A$.*

- *bounded if there exist two elements x, y such that $x \leq z \leq y$, for every $z \in A$: x and y are called, respectively,* bottom *and* top *element.*
- *distributive if, by considering the corresponding algebraic structure $\langle A, \wedge, \vee \rangle$, the following hold, for every $a, b, c \in A$*

  distributivity $\quad (a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$
  $\quad (a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$.

Some well known algebraic structures are particular types of lattices:

Example A.1.1. *A Boolean algebra is an algebra $\mathcal{B} = \langle B, \wedge, \vee, ', 0, 1 \rangle$ such that, for every $x \in B$*

(B1) $\langle B, \wedge, \vee, 0, 1 \rangle$ *is a bounded distributive lattice*

(B2) $x \wedge x' = 0;\ x \vee x' = 1$

Other examples of lattices are presented in chapter 4: these structures will represent the semantics for the logics studied in this thesis.

A.1.2 Semigroups, monoids, groups

It is useful to recall some algebraic structures:

Definition A.1.8. *A* semigroup *is a system $\langle A, * \rangle$ such that*

$$x * (y * z) = (x * y) * z.$$

For every $x, y, z \in A$.

Definition A.1.9. *A* monoid *is a system $\langle A, *, 1 \rangle$ such that $\langle A, * \rangle$ is a semigroup and*

$$x * 1 = x = 1 * x.$$

For every $x \in A$.

Definition A.1.10. *A* linearly ordered monoid *is a structure $\mathcal{M} = \langle M, *, \leq, 1 \rangle$ such that $\langle M, *, 1 \rangle$ is a monoid and $\langle M \leq \rangle$ is a chain, where $\leq$ is compatible with $*$, that is, for every $x, y, z \in M$*

$$\text{if } x \leq y \text{ then } x * z \leq y * z,$$

if, moreover, 1 is the maximum, then the monoid is called integral.

Definition A.1.11. *A* group *is a system $\langle A, *, ^{-1}, 1 \rangle$ such that $\langle A, *, 1 \rangle$ is a monoid and*

$$x * x^{-1} = 1$$

For every $x \in A$.

Definition A.1.12. *A* commutative monoid *is a monoid* $\langle A, *, 1\rangle$ *such that*

$$x * y = y * x.$$

For every $x, y \in A$.

A commutative group is said to be an *abelian group*.

A.1.3 Isomorphic structures and subalgebras

The term isomorphism it is used to denote the fact that two (algebraic) structures are essentially the same, except (eventually) for the name of their elements or of their operations.

Definition A.1.13. *Given two poset* $\langle S, \leq\rangle$ *and* $\langle P, \leq'\rangle$, *a map* $\Phi : S \to T$ *is said to be* order-preserving *if, given* $x, y \in S$ *such that* $x \leq y$, *then* $\Phi(x) \leq' \Phi(y)$. *Moreover, if it holds that* $x \leq y$ *iff* $\Phi(x) \leq' \Phi(y)$, *then* Φ *is called* order-embedding. *An order-preserving map that is bijective (as a function) is called* order isomorphism.

Theorem A.1.1. *If* Φ *is a surjective order-embedding map, then* Φ *is an order isomorphism.*

We now introduce others general constructions, that preserve the structure of the algebras.

Definition A.1.14. *Let* $\langle A, F\rangle$, $\langle B, F'\rangle$ *be two algebras of the same type* $\mathcal{F}$: *a map* $\Phi : A \to B$ *is an* homomorphism *if for every* $f \in \mathcal{F}$, $\nu(f) = n$ *and for every* $a_1, \ldots, a_n \in A$ *it holds that*

$$\Phi(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\Phi(a_1), \ldots, \Phi(a_n))$$

- *an* injective homomorphism *is called* monomorphism
- *a* surjective homomorphism *is called* epimorphism
- *a* bijective homomorphism *is called* isomorphism

In the case in which Φ *is an epimorphism, then* $\langle B, F'\rangle$ *is said to be an homomorphic image of* $\langle A, F\rangle$.
An homomorphism $\Phi : A \to A$ *is said* endomorphism , *an isomorphism* $\psi : A \to A$ *is called* automorphism.

A monomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ is called embedding of $\mathcal{A}$ into $\mathcal{B}$: sometimes we will denote it with $\mathcal{A} \hookrightarrow \mathcal{B}$.

Definition A.1.15. *Let* $\mathcal{A} = \langle A, F\rangle$ *be an algebra of type* $\mathcal{F}$: *a subset* $X \subseteq A$ *is said to be a* subuniverse *of* $\mathcal{A}$ *if it is closed under all the operations in* F, *i.e., for every* $f \in \mathcal{F}$, $\nu(f) = n$, *and every* $x_1, \ldots, x_n \in X$ *we have* $f^{\mathcal{A}}(x_1, \ldots, x_n) \in X$.

We now introduce the notion of subalgebra: as can be seen it is strictly connected to the one of subuniverse.

Definition A.1.16. *Let $\mathcal{A} = \langle A, F\rangle$, $\mathcal{B} = \langle B, F'\rangle$ be two algebras of the same type $\mathcal{F}$: $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ when*

- $A \subseteq B$
- *for every $f \in \mathcal{F}$, $\nu(f) = n$ we have that $f^{\mathcal{A}} = f^{\mathcal{B}} \restriction A$, i.e. for every $(a_1, \ldots, a_n) \in A$ it holds that $f^{\mathcal{A}}(a_1, \ldots, a_n) = f^{\mathcal{B}}(a_1, \ldots, a_n)$*

In particular, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}$, then A is a subuniverse of B; conversely, if A is a subuniverse of B, then by endowing A with all the operations of $\mathcal{B}$ restricted to A we obtain a(n algebraic structure that is a) subalgebra of $\mathcal{B}$.

Definition A.1.17. *An algebra $\langle A, F\rangle$ of type $\mathcal{F}$ is a reduct of an algebra $\langle A, F'\rangle$ of type $\mathcal{F}'$ to $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{F}'$ and F if the restriction of F' to $\mathcal{F}$. A subalgebra $\langle B, F\rangle$ of $\langle A, F\rangle$ is said a subreduct of $\langle A, F'\rangle$ to $\mathcal{F}$.*

Definition A.1.18. *Given an algebra $\mathcal{A}$ we define, for every $X \subseteq A$*

$$Sg(X) = \bigcap\{B : X \subseteq B \text{ and } B \text{ is a subuniverse of } A\}.$$

We will "read" $Sg(X)$ as "the subuniverse generated by X": thanks to the relation between subuniverses and subalgebras we will "talk about" the subalgebra generated from X, and this fact will be denoted by $\mathcal{A} = Sg(X)$. In the case in which X is finite, $\mathcal{A} = Sg(X)$ will be called finitely generated.

It is possible to show that, by defining

$$E(X) = X \cup \{f^{\mathcal{A}}(a_1, \ldots, a_n), f \in \mathcal{F}, \nu(f) = n, a_1, \ldots, a_n \in X\}$$

and, by setting $E^0(X) = X$, $E^1(X) = E(X)$, $E^{n+1}(X) = E(E^n(X))$, we have

$$Sg(X) = E^*(X) = \bigcup_{i \in \mathbb{N}} E^i(X).$$

We conclude with the following theorem

Theorem A.1.2. *If Φ is an homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, then $\Phi(A) = \{b \in B : b = \Phi(a) \text{ for some } a \in A\} = \{\Phi(a) : a \in A\}$ is a subuniverse of B.*

A.1.4 Congruences, quotient, kernel

Definition A.1.19. *A binary relation θ over a set A is called* equivalence relation *when, for every $x, y, z \in A$, the following hold*

reflexivity	$x\theta x$
simmetry	*if* $x\theta y$ *then* $y\theta x$
transitivity	*if* $x\theta y$ *and* $y\theta z$, *then* $x\theta z$

Moreover, given $a \in A$, then the set

$$[a]_\theta = \{b \in A : b\theta a\}$$

is called equivalence class *of A.*

Two particular equivalence relations are $\Delta_{\mathcal{A}} = \{\langle a, a\rangle : a \in A\}$ and $\nabla_{\mathcal{A}} = \{\langle a, b\rangle : a, b \in A\}$: the first one is called diagonal relation, whilst the second one is known as total relation.

Note that these two are, respectively, the smallest and the largest equivalence relations that can be constructed, over a set.

Definition A.1.20. *Let $\mathcal{A} = \langle A, F\rangle$ be an algebra of type $\mathcal{F}$ and let θ be an equivalence relation over A: θ is a* congruence *over $\mathcal{A}$ if it is compatible with the operations of $\mathcal{A}$. This means that: for every $f \in \mathcal{F}$, $\nu(f) = n$ and every pair of n-tuples $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in A$, if $a_i\theta b_i$ (where $i \in \{1, \ldots, n\}$), then $f^{\mathcal{A}}(a_1, \ldots, a_n)\theta f^{\mathcal{A}}(b_1, \ldots, b_n)$.*

With the notation $Con(\mathcal{A})$ we will denote the set of congruences of the algebra $\mathcal{A}$.

Definition A.1.21. *Given a congruence θ over $\mathcal{A}$ we can create the quotient algebra $\mathcal{A}/\theta$ of $\mathcal{A}$ by θ. Denoted by $\mathcal{A}/\theta$, this is an algebra of type $\mathcal{F}$ such that $A/\theta = \{[a]_\theta : a \in A\}$ and, for every $f \in \mathcal{F}$, $\nu(f) = n$ and $a_1, \ldots, a_n \in A$*

$$f^{\mathcal{A}/\theta}([a_1]_\theta, \ldots, [a_n]_\theta) = [f^{\mathcal{A}}(a_1, \ldots, a_n)]_\theta.$$

Definition A.1.22. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an homomorphism and let R be the relation (over A)*

$$xRy \textit{ iff } \Phi(x) = \Phi(y).$$

This last one is called kernel *of Φ and denoted by $ker\Phi$.*

Theorem A.1.3. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an homomorphism: then $ker\Phi$ is a congruence over $\mathcal{A}$.*

Theorem A.1.4. *Let $R \in Con(\mathcal{A})$ and*

$$\pi_R : x \in A \mapsto [x]_R \in A/R$$

then π_R is an epimorphism said natural map*.*

We conclude the section with a result of notable importance (known as "first homomorphism theorem")

Theorem A.1.5. *Let $\alpha : \mathcal{A} \to \mathcal{B}$ be an epimorphism, then there exists an isomorphism $\beta : \mathcal{A}/ker\alpha \to B$ such that $\alpha = \beta \circ \nu$, where $\nu = \pi_{ker\alpha}$.*

Figure A.1 represents graphically what is pointed out in the theorem.

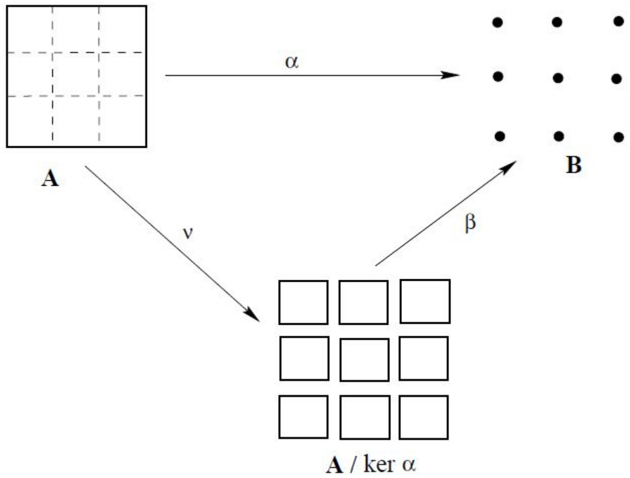


Figure A.1: First homomorphism theorem.

A.1.5 Direct products, subdirect products, reduced products

We now present some constructions that permit to generate new algebras, by starting from a class of them. We begin with direct products.

Definition A.1.23. *Let $\mathcal{A}, \mathcal{B}$ be two algebras of the same type $\mathcal{F}$: the direct product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times \mathcal{B}$, is the algebra of type $\mathcal{F}$ whose universe is $A \times B$; moreover, for every symbol $f \in \mathcal{F}$, $\nu(f) = n$ and $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$ it holds that*

$$f^{\mathcal{A}\times\mathcal{B}}(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle) := \left\langle f^{\mathcal{A}}(a_1, \ldots, a_n), f^{\mathcal{B}}(b_1, \ldots, b_n)\right\rangle.$$

In general we can note that two algebras of the same type $\mathcal{A}_1, \mathcal{A}_2$ are homomorphic images of $\mathcal{A}_1 \times \mathcal{A}_2$ by using the projections, i.e.

$$\begin{array}{lll} \pi_1 : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_1 & \text{such that} & \pi_1(\langle a_1, a_2\rangle) = a_1 \\ \pi_2 : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_2 & \text{such that} & \pi_2(\langle a_1, a_2\rangle) = a_2. \end{array}$$

It is possible to show that π_1, π_2 are epimorphisms; moreover, thanks to theorem A.1.5, $\mathcal{A}_1 \times \mathcal{A}_2 / ker\pi_i$ is isomorphic to $\mathcal{A}_i$.

Definition A.1.24. *Given $R_1, R_2 \in Con\mathcal{A}$ such that*

- $R_1 \cap R_2 = \Delta_{\mathcal{A}}$
- $R_1 \vee R_2 = \nabla_{\mathcal{A}}$
- *For every $a, b \in A$ there exist $c, d \in A$ such that aR_1cR_2b and aR_1dR_2b; that is R_1 and R_2 permute,*

where the operation $\vee$ is relative to the order relation $\subseteq$. Then the pair R_1, R_2 is called pair of factor congruences on $\mathcal{A}$.

Moreover, it can be shown that $\langle Con\mathcal{A}, \cap, \vee \rangle$ is a complete lattice.

Theorem A.1.6. *If R_1 and R_2 are a pair of factor congruences on $\mathcal{A}$, then $\mathcal{A}$ is isomorphic to $\mathcal{A}/R_1 \times \mathcal{A}/R_2$ under the map $\alpha(a) = \langle [a]_{R_1}, [a]_{R_2} \rangle$.*

Definition A.1.25. *An algebra $\mathcal{A}$ is* directly indecomposable *if $\mathcal{A}$ is not isomorphic to a direct product of two non trivial algebras.*

Corollary A.1.1. *An algebra $\mathcal{A}$ is directly indecomposable iff the only pair of factor congruences on $\mathcal{A}$ is $\Delta_{\mathcal{A}}$, $\nabla_{\mathcal{A}}$, in fact*

$$\mathcal{A}/\Delta_{\mathcal{A}} = \mathcal{A} \qquad |\mathcal{A}/\nabla_{\mathcal{A}}| = 1.$$

We now introduce a generalized version of direct product, extended to an arbitrary number of algebras.

Definition A.1.26. *Let $\{\mathcal{A}_i : i \in I\}$ be a family of algebras of type $\mathcal{F}$. The direct product of the family is the algebra $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$ of type $\mathcal{F}$, whose universe is $\prod_{i\in I} A_i$ and whose elements are functions $a : I \to \bigcup_{i\in I} A_i$ (observe figure A.2) such that $a(i) \in A_i$; moreover, for every symbol $f \in \mathcal{F}$, $\nu(f) = n$ and $a_1, \ldots, a_n \in \prod_{i\in I} A_i$ we have*

$$f^{\mathcal{A}}(a_1, \ldots, a_n)(i) = f^{\mathcal{A}_i}(a_1(i), \ldots, a_n(i)).$$

We can, as previously done, define the projection maps $\pi_j : \prod_{i\in I} A_i \to A_j$ with $j \in I$, such that $\pi_j(a) = a(j)$, under the ones we obtain the epimorphism

$$\pi_j : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_j.$$

If $I = \{1, \ldots, n\}$, we will write $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$; if I is arbitrary, but $\mathcal{A}_i = \mathcal{A}$ for every $i \in I$, then we will denote the direct product with $\mathcal{A}^I$ and we will call it direct power *of $\mathcal{A}$. $\mathcal{A}^{\emptyset}$ is the trivial algebra.*

We have the following result.

Theorem A.1.7. *Every finite algebra is (isomorphic to) the direct product of directly indecomposable algebras.*

The previous theorem, however, does not hold for infinite algebras, in general. For this reason it is necessary to introduce another construction

Definition A.1.27. *An algebra $\mathcal{A}$ is the* subdirect product *of a family $\{\mathcal{A}_i : i \in I\}$ when*

- *$\mathcal{A}$ is a subalgebra of $\prod_{i\in I} \mathcal{A}_i$.*

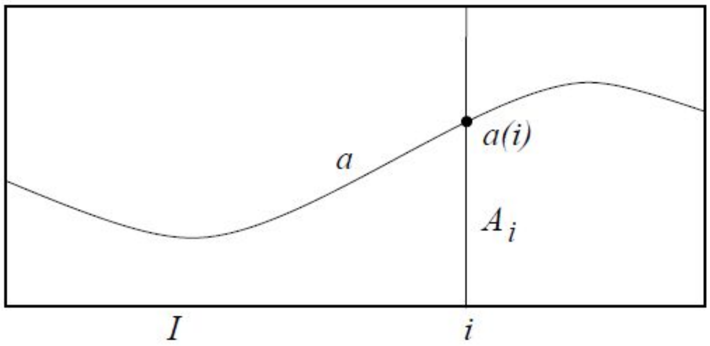


Figure A.2: Generalized direct product.

- $\pi_i(\mathcal{A}) = \mathcal{A}_i$.

Definition A.1.28. *A* subdirect embedding *is a monomorphism* $\alpha : \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i$ *such that* $\alpha(A)$ *is a subdirect product of* $\{A_i : i \in I\}$.

Definition A.1.29. *An algebra* $\mathcal{A}$ *is* subdirectly irreducible *if, for every subdirect embedding* $\alpha : \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i$ *there exists an index* $i \in I$ *such that* $\pi_i \circ \alpha$ *is an isomorphism between* $\mathcal{A}$ *and* $\mathcal{A}_i$.

Finally,

Theorem A.1.8. *Every algebra* $\mathcal{A}$ *is isomorphic to a subdirect product of subdirectly irreducible algebras (that are homomorphic images of* $\mathcal{A}$*).*

Theorem A.1.9. *An algebra* $\mathcal{A}$ *is subdirectly irreducible iff* $\mathcal{A}$ *is trivial or* $Con\mathcal{A} \backslash \{\Delta_{\mathcal{A}}\}$ *has a minimum. In this last case the minimum is* $\bigcap (Con\mathcal{A} \setminus \{\Delta_{\mathcal{A}}\})$ *and the lattice of congruences of* $\mathcal{A}$ *looks like as in figure A.3.*

Definition A.1.30. *An algebra* $\mathcal{A}$ *is* simple *if* $Con\mathcal{A} = \{\Delta_{\mathcal{A}}, \nabla_{\mathcal{A}}\}$. *A congruence* θ *on* $\mathcal{A}$ *is maximal if the interval* $[\theta, \nabla_{\mathcal{A}}]$ *of* $Con\mathcal{A}$ *contains exactly two elements.*

We conclude with reduced products: they will be fundamental for the definition of the concept of quasivariety. To begin with, we need to introduce the notion of filter:

Definition A.1.31. *Let* X *be a set, a* filter *on* X *is a set* F *of subsets of* X *such that*

1. $X \in F$
2. *If* $A, B \subseteq X$, $A \subseteq B$, $A \in F$, *then* $B \in F$
3. *If* $A, B \in F$ *then* $A \cap B \in F$.

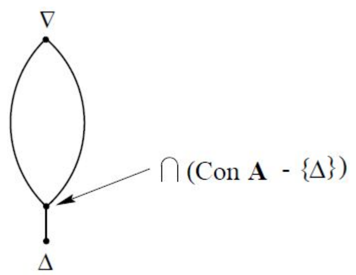


Figure A.3: The lattice of congruences of a non trivial subdirectly irreducible algebra.

A filter F (on a set X) is proper if $\emptyset \notin F$. An ultrafilter *F (on a set X) is a maximal proper filter: for every other proper filter F' (on X), if $F \subseteq F'$, then $F = F'$.*

Definition A.1.32. *Let $\{\mathcal{A}_i \in I\}$ be a family of algebras of the same type and F be a filter on I. We define a binary relation θ_F on $\prod_{i \in I} \mathcal{A}_i$ as follows*

$$\langle a, b\rangle \in \theta_F \quad \textit{iff} \quad \{i \in I : a(i) = b(i)\} \in F.$$

Lemma A.1.1. *θ_F is a congruence on $\prod_{i \in I} \mathcal{A}_i$.*

Definition A.1.33. *Given a family of algebras $\mathcal{A}_{i \in I}$ of type $\mathcal{F}$ and F be a proper filter ($\emptyset \notin F$) on I, we define the* reduced product *$\prod_{i \in I} \mathcal{A}_i/F$ as follows. Its universe, $\prod_{i \in I} A_i/F$, is $\prod_{i \in I} A_i/\theta_F$ and a/F indicates a/θ_F: moreover, for every $f \in \mathcal{F}$, $\nu(f) = n$ and $a_1, \ldots, a_n \in \prod_{i \in I} A_i/F$, it holds that*

$$f(a_1/F, \ldots, a_n/F) = f(a_1, \ldots, a_n)/F.$$

When F is an ultrafilter, then $\mathcal{A}_{i \in I}$ is called ultraproduct.

A.1.6 Varieties, equational classes, term algebra

Let K be a class of algebras of type $\mathcal{F}$. Consider the following algebraic operators:

- $\mathbb{I}(K)$ = the class of all algebras isomorphic to members of K
- $\mathbb{H}(K)$ = the class all homomorphic images of members of K
- $\mathbb{S}(K)$ = the class of all subalgebras of members of K
- $\mathbb{P}(K)$ = the class of all direct products of members of K
- $\mathbb{P}_R(K)$ = the class of all reduced products of members of K

Definition A.1.34. *A class K of algebras of type $\mathcal{F}$ is a* variety *if it is closed under the operators $\mathbb{I}, \mathbb{S}, \mathbb{H}, \mathbb{P}$, that is*

$$K = \mathbb{I}(K) = \mathbb{S}(K) = \mathbb{H}(K) = \mathbb{P}(K)$$

Now, as the intersection of a class of varieties of type $\mathcal{F}$ is again a variety and as all the algebras of type $\mathcal{F}$ form a variety, then we can conclude that for every class K of algebras of the same type there exists a smallest variety that contains K.

Definition A.1.35. *If K is a class of algebras of the same type, let $\mathbb{V}(K)$ be the smallest variety containing K. We will say that $\mathbb{V}(K)$ is the* variety generated *by K; in the case in which K has only an element $\mathcal{A}$, then we will write simply $V(\mathcal{A})$. A variety is* finitely generated *if $V = \mathbb{V}(K)$ for some finite set K of algebras. In the case in which $V = \mathbb{V}(K) = \mathbb{V}(\{\mathcal{A}\})$ we will say that $\mathcal{A}$ is generic for V .*

The following theorem gives a method to construct the variety generated by a class of algebras.

Theorem A.1.10. $\mathbb{V}(K) = \mathbb{HSP}(K)$

We can reformulate theorem A.1.8 as follows

Theorem A.1.11. *If K is a variety, then each of its members is (isomorphic to) a subdirect product of subdirectly irreducible algebras of K.*

Corollary A.1.2. *A variety is determined by its subdirectly irreducible members.*

Definition A.1.36. *A* subvariety *is a subclass of a variety that is again a variety.*

Finally, we introduce the following concept

Definition A.1.37. *An algebra (of a certain type) $\mathcal{A}$ is* locally finite *if every finitely generated subalgebra (i.e. generated by a finite subset of $\mathcal{A}$) is finite. A class K of algebras is locally finite if every member of K is locally finite.*

Terms, term algebra and free algebras

Definition A.1.38. *Let X be a set of symbols, called variables: the set $T(X)$ of terms of type $\mathcal{F}$ over X is the smallest set such that*

1. $X \cup \{f \in F : \nu(f) = 0\} \subseteq T(X)$
2. *if* $p_1, \ldots, p_n \in T(X)$ *and* $f \in \mathcal{F}$, $\nu(f) = n$, *then* $f(p_1, \ldots, p_n) \in T(X)$.

Note that the terms are purely syntactical objects: to move to the semantics, we need to interpret them in some algebra

Definition A.1.39. *Given a term $p(x_1, \ldots, x_n)$ of type $\mathcal{F}$ over some set X and given an algebra $\mathcal{A}$ of type $\mathcal{F}$ we define the mapping $p^{\mathcal{A}} : A^n \to A$ as follows:*

1. *if p is a variable x_i, then*

$$p^{\mathcal{A}}(a_1, \ldots, a_n) = a_i,$$

with $a_1, \ldots, a_n \in A$; that is $p^{\mathcal{A}}$ is the ith projection map

2. *if $p = f(p_1(x_1, \ldots, x_n), \ldots, p_k(x_1, \ldots, x_n))$ with $f \in \mathcal{F}$, $\nu(f) = k$, then*

$$p^{\mathcal{A}}(a_1, \ldots, a_n) = f^{\mathcal{A}}(p_1^{\mathcal{A}}(a_1, \ldots a_n), \ldots, p_k^{\mathcal{A}}(a_1, \ldots a_n)).$$

It is possible to transform, in a natural way, the set $T(X)$ in an algebra

Definition A.1.40. *Given $\mathcal{F}$ and X, if $T(X) \neq \emptyset$ then the* term algebra *of type $\mathcal{F}$ over X, written $\mathcal{T}(X)$, is the algebra whose universe is $T(X)$, and whose operations are of the form*

$$f^{\mathcal{T}(X)} : \langle p_1, \ldots, p_n \rangle \mapsto f(p_1, \ldots, p_n)$$

with $f \in \mathcal{F}$, $\nu(f) = n$ and $p_1, \ldots, p_n \in T(X)$.

Note that $\mathcal{T}(X)$ is generated by X.

Definition A.1.41. *Let K be a class of algebras of type $\mathcal{F}$ and let $\mathcal{U}(X)$ be the algebra of type $\mathcal{F}$ generated by X. If, for every $\mathcal{A} \in K$ and every map*

$$\alpha : X \to A$$

there is an homomorphism

$$\beta : \mathcal{U}(X) \to \mathcal{A}$$

that extends α ($\beta(x) = \alpha(x)$ for every $x \in X$), then we will say that $\mathcal{U}(X)$ has the universal mapping property *for K over X. X is called set of free generators of $\mathcal{U}(X)$ and $\mathcal{U}(X)$ is freely generated by X.*

It is easy to show that a such homomorphism α, if it exists, then is unique.

The term algebra represents also the most elementary example of algebra that enjoys the universal mapping property

Theorem A.1.12. *For any type $\mathcal{F}$ and set $X \neq \emptyset$ of variables, the term algebra $\mathcal{T}(X)$ has the universal mapping property for the class of all algebras of type $\mathcal{F}$ over X.*

Definition A.1.42. *Let K be a family of algebras of type $\mathcal{F}$. Given a set X of variables we define the congruence $\theta_K(X)$ on $\mathcal{T}(X)$ as*

$$\theta_K(X) = \bigcap \Phi_K(X)$$

where

$$\Phi_K(X) = \{\phi \in Con\mathcal{T}(X) : \mathcal{T}(X)/\phi \in \mathbb{IS}(K)\}.$$

*The (K-)*free algebra *over X is defined as*

$$F_K(X) = \mathcal{T}(X)/\theta_K(X).$$

Theorem A.1.13. *$F_K(X)$ has the universal mapping property for K over X.*

Finally, we present a result that connects the concepts of free algebras and varieties.

Theorem A.1.14. *Given a class of algebras of the same type $K \neq \emptyset$, we have $F_K(X) \in \mathbb{ISP}(K)$. Hence, if K is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, in particular if K is a variety, then $F_K(X) \in K$.*

Identities and equational classes

In this section we will show the relations between free algebras, varieties and equational classes.

Definition A.1.43. *Let $p = p(x_1, \dots, x_n)$, $q = q(x_1, \dots, x_n)$ be terms of type $\mathcal{F}$ over X. An* identity *(equation) is an expression of the form*

$$p \approx q.$$

As can be noted, they are purely syntactical objects: the next step is the one of define their semantics

Definition A.1.44. *We say that an algebra $\mathcal{A}$ of type $\mathcal{F}$ satisfy $p \approx q$ if, for every $a_1, \dots, a_n \in A$ we have*

$$p^{\mathcal{A}}(a_1, \dots, a_n) = q^{\mathcal{A}}(a_1, \dots, a_n),$$

this is denoted with the notation $\mathcal{A} \models p \approx q$.

The previous definition can be extended to classes of algebras in an easy way:

Definition A.1.45. *A class K of algebras of type $\mathcal{F}$ satisfy $p \approx q$ (this is denoted by $K \models p \approx q$) if $\mathcal{A} \models p \approx q$ for every $\mathcal{A} \in K$.*

In the case in which we have more than one identity, instead

Definition A.1.46. *Let Σ be a set of identities of type $\mathcal{F}$ over X. K satisfy Σ if $K \models p \approx q$ for every $p \approx q \in \Sigma$; in symbols $K \models \Sigma$.*

Moreover, it is possible to define

Definition A.1.47.

$$Id_K(X) = \{p \approx q : p, q \in Id(X), K \models p \approx q\}$$

where $Id(X)$ indicates the set of identities of type $\mathcal{F}$ over X.

The following theorems connect some of the objects that we have introduced:

Theorem A.1.15. *Let $p, q \in T(X)$ and K be a class of algebras of type $\mathcal{F}$. We have that*

$$\begin{aligned} & K \models p \approx q \quad \textit{iff} \\ & F_K(X) \models p \approx q \quad \textit{iff} \\ & \overline{p} = \overline{q} \textit{ in } F_K(X) \quad \textit{iff} \\ & \langle p, q \rangle \in \theta_K(X). \end{aligned}$$

Definition A.1.48. *Let Σ be a set of identities of type $\mathcal{F}$ and define $M(\Sigma)$ as the class of algebras that satisfy Σ. A class K of algebras is an* equational class *if there exists a set of identities Σ such that $K = M(\Sigma)$. In this case we will say that K is defined, or axiomatized, by Σ.*

Theorem A.1.16 (Birkhoff)**.** *Let K be a class of algebras of type $\mathcal{F}$. K is an equational class iff K is a variety.*

A.1.7 Quasivarieties and quasiequational classes

Definition A.1.49. *A* quasivariety *is a class of algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}_R$.*

Moreover, analogously to the definition of equation:

Definition A.1.50. *A* quasiequation *(or quasidentity) is an equation of the form $(p_1 \approx q_1 \& \ldots \& p_{n-1} \approx q_{n-1}) \to p_n \approx q_n$.*

Definition A.1.51. *We say that an algebra $\mathcal{A}$ of type $\mathcal{F}$ satisfy $(p_1 \approx q_1 \& \ldots \& p_{n-1} \approx q_{n-1}) \to p_n \approx q_n$ iff for every $a_1, \ldots, a_m \in A$*

$$\begin{aligned} & p_n^{\mathcal{A}}(a_1, \ldots, a_m) = p_n^{\mathcal{A}}(a_1, \ldots, a_m) \textit{ when} \\ & p_i^{\mathcal{A}}(a_1, \ldots, a_m) = p_i^{\mathcal{A}}(a_1, \ldots, a_m), \textit{ for every } i < n. \end{aligned}$$

This is characterized with the notation $\mathcal{A} \models p_1 \approx q_1 \& \ldots \& p_{n-1} \approx q_{n-1} \to p_n \approx q_n$.

The extension of this notion to classes of algebras, as well as the definitions of set of quasiequations and quasiequational classes, is analogous to the case of equations.

Analogously to what happens to varieties and equations, we have:

Theorem A.1.17. *Let K be a class of algebras of type $\mathcal{F}$. K is a quasivariety iff it is a quasiequational class.*

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