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Stationary entanglement in N-atom subradiant degenerate cascade systems

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We address ultracold N-atom degenerate cascade systems and show that stationary subradiant states, already observed in the semiclassical regime, also exist in a fully quantum regime and for a small number of atoms. We explicitly evaluate the amount of stationary entanglement for the two-atom configuration and show full inseparability for the three-atom case. We also show that a continuous variable description of the systems is not suitable to detect entanglement due to the non-Gaussianity of subradiant states.

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I. INTRODUCTION

Atomic subradiant states have recently gained wide attention because of their exceptionally slow decoherence [1,2]. This stability of quantum superpositions inside the subradiant subspaces originates from the low probability of photon emission, which means very weak interaction between the atoms and their environment. Hence, the subradiant states span a decoherence-free subspace [3–5] of the atomic Hilbert space, and consequently can become important from the viewpoint of quantum computation [6,7].

Subradiance in an extended pencil-shaped sample was observed in a unique experiment in 1985 by Pavolini *et al.* [8]. Among different schemes of multilevel systems, Crubellier *et al.* [9–12] investigated in detail the arising of subradiance in three-level atoms with a degenerate transition. More precisely, they considered the case of three-level atoms with two transitions sharing either the upper level (Λ configuration) or the lower level (V configuration) or the intermediate one ("cascade" configuration), when both the transitions have the same frequency and the same polarization. The first two configurations may be experimentally realized by rather specific atomic-level systems, having a small hyperfine structure, whereas the degenerate cascade configuration, which contains two cascading transitions of the same frequency, would in fact be encountered in atomic-level systems only in the presence of an external field which suitably modifies the atomic frequencies [10]. Recently, the same degenerate cascade among three equally spaced levels has been investigated for collective momentum states of an ultracold atomic gas in a high-finesse ring cavity driven by a two-frequency laser field [13]. This system is particularly attractive for the observation of subradiance, since the absence of Doppler broadening and collisions at subrecoil temperatures avoid other undesirable decoherence mechanisms. Furthermore, the experimental control of the external parameters, such as, for instance, the intensities of the pump fields, allows for a continuous tuning through different subradiant states of the system, which instead should turn out to be difficult or impossible for internal atomic transitions.

More specifically, the system described in [13] consists in N two-level ultracold bosonic atoms (at $T \approx 0$) placed in a ring cavity with linewidth κ and driven by two pump fields E_0 and E_1 of frequency ω and $\omega + \Delta$, respectively. The atoms scatter the photons of the two-frequency pump into a single counterpropagating cavity mode with frequency ω_s. The pump frequency ω is sufficiently detuned from the atomic resonance ω_0 to neglect absorption. The general atom-field dynamics is described by the following Hamiltonian written in the space of the atomic momentum eigenstates $|m\rangle$ [with $\vec{p}|m\rangle = m(\hbar\vec{q})|m\rangle$, where $\vec{q} = \vec{k} - \vec{k}_s$ is the transferred momentum]:

$$H = \sum_m \{\hbar\omega_m c_m^\dagger c_m - i\hbar g[\alpha(t) a c_m^\dagger c_{m+1} - \text{H.c.}]\}, \qquad (1)$$

where $\omega_m = m(m\omega_r - \delta)$, $\omega_r = q^2/2M$ is the recoil frequency, M is the atomic mass, $\delta = \omega - \omega_s$, $g = (\Omega_0/2\Delta_0)(\omega d^2/2\hbar\epsilon_0 V)^{1/2}$, $\Omega_0 = dE_0/\hbar$, d is the atomic dipole, $\Delta_0 = \omega - \omega_0$ is the detuning of the pump frequency ω from the atomic resonance ω_0, and $\alpha(t) = 1 + \epsilon \exp(i\Delta t)$, where $\epsilon = |E_1/E_0|$. The scattering process can be described by two steps. In the first step, the atom, initially at rest, scatters the ω pump photon into a photon with frequency $\omega_{s1} = \omega - \omega_r$, and recoils with a momentum $\vec{p} = \hbar\vec{q}$. In the second step, the atom scatters the $\omega + \Delta$ pump photon into a photon with frequency $\omega_{s2} = \omega + \Delta - 3\omega_r$, changing momentum from $\vec{p} = \hbar\vec{q}$ into $\vec{p}\,' = 2\hbar\vec{q}$. The recoil shift of $3\omega_r$ arises from the kinetic-energy conservation, that is, $\Delta E = (p'^2 - p^2)/2M = 3\hbar\omega_r$. If the frequency difference between the two pump fields is $\Delta = 2\omega_r$, then $\omega_{s1} = \omega_{s2} \equiv \omega_s$ and a degenerate cascade between three-momentum levels, $\vec{p} = m(\hbar\vec{q})$, with $m = 0,1,2$, is realized. Furthermore, if the cavity linewidth κ is smaller that $2\omega_r$, the cavity will support only a single frequency ω_s, avoiding further higher-order scattering with frequency $\omega_s \pm m\omega_r$. In this way, the transitions are restricted to only the first three-momentum states, forming a three-level degenerate cascade. Although the condition $\kappa \ll 2\omega_r$ is experimentally demanding, requiring a typical cavity finesse larger than 10^6, it is not very far from the state-of-the-art optical cavities [14,15]. An important feature of the momentum three-level cascade is that the ratio of the two transition rates is $\epsilon = |E_1/E_0|$, that is, it is an experimentally tunable parameter, contrary to the case of atomic transitions, where it is fixed by the branching ratios.

In this paper, we address N-atom degenerate cascade systems and show that stationary subradiant states also exist in a fully quantum regime. We solve explicitly the master equation governing the system for a small number of atoms and evaluate the amount of stationary entanglement [16]. The paper is structured as follows. In the next section, we give a full quantum description of the dynamics and we numerically solve a quantum master equation for $N = 2$ (3)-atom systems. In Sec. III, we study the entanglement properties of subradiant states looking at the system from both discrete-variable and continuous-variable (CV) points of view and show that CV criteria are not able to detect entanglement due to non-Gaussianity of subradiant states. Section IV closes the paper with some concluding remarks.

II. DISSIPATIVE DYNAMICS AND N-ATOM PURE SUBRADIANT STATES

Restricting the Hilbert space of the atoms to the one spanned by only the first three recoil momentum states, $m = 0,1,2$, the multimode Hamiltonian (1) reduces, for $\delta = \omega_r$, to

$$H = 2\hbar\omega_r c_2^\dagger c_2 - i\hbar g\{\alpha(t)a(c_0^\dagger c_1 + c_1^\dagger c_2) - \text{H.c.}\}. \quad (2)$$

The Hamiltonian (2) contains fast oscillating terms at the frequency $\Delta = 2\omega_r$. Redefining $c_2 \to c_2 \exp(2i\omega_r t)$ and neglecting the fast oscillating terms, it reduces to the following Hamiltonian describing a three-level degenerate cascade interaction:

$$H = -i\hbar g(ac_1c_0^\dagger + \epsilon ac_2c_1^\dagger - \text{H.c.}) \quad (3)$$

for the three atomic bosonic operators, c_m, with $m = 0,1,2$ and $[c_m,c_n^\dagger] = \delta_{m,n}$, and for the single-cavity radiation mode, a, with $[a,a^\dagger] = 1$; here, ϵ is the relative rate of the second transition, from $m = 1$ to 2. The state of the total system is described by a density operator ϱ which is defined on the overall Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_f$, $\mathcal{H}_i$ being the Hilbert-Fock space associated with the $m = i$ atomic mode and $\mathcal{H}_f$ the Hilbert-Fock space associated with the radiation mode a.

Introducing the Fock states $|n_0,n_1,n_2\rangle$, where n_i is the number of quanta in the $m = i$ mode, it has been demonstrated in [10,13] that such an N-atom system, with N even, admits $N/2$ subradiant pure states,

$$|\text{sr}\rangle_p = C_p \sum_{k=0}^{p} \left(\frac{-1}{2\epsilon}\right)^k \frac{1}{k!}\sqrt{\frac{(2k)!(N-p-k)!}{(p-k)!}} \times |p-k,2k,N-p-k\rangle, \quad (4)$$

where

$$C_p = \left(\frac{(N-p)!}{p!}\, {}_2F_1(-p,1/2;-N+p;\epsilon^{-2})\right)^{-1/2},$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function and $p = 1,\ldots,N/2$, to which we can add the ground state $|\text{sr}\rangle_0 = |0,0,N\rangle$ for $p = 0$. If $N \gg 1$, these states exist iff $0 \leqslant \epsilon \leqslant 1+\sqrt{2}$.

To study the time evolution of ϱ accounting for scattered photon dissipation too, in a pure quantum-mechanical framework, we introduce the following $T = 0$ Lindblad master equation:

$$\frac{d}{dt}\varrho = \frac{1}{i\hbar}[H_{\text{int}},\varrho] + 2\kappa\mathcal{L}(a)\varrho, \quad (5)$$

where $\mathcal{L}(a)\varrho = a\varrho a^\dagger - a^\dagger a\varrho/2 - \varrho a^\dagger a/2$ is the Lindblad operator describing the cavity damping of the mode a. In the following subsections, we solve numerically Eq. (5) for two and three atoms in terms of total Fock states $|n_0,n_1,n_2,n\rangle$, where n is the number of scattered photons. We will find that for every ϵ and for $\kappa \neq 0$, Eq. (5) always admits a stationary solution of the form

$$\varrho_s = [P_1|\text{sr}\rangle_{11}\langle\text{sr}| + (1-P_1)|\text{sr}\rangle_{00}\langle\text{sr}|] \otimes |0\rangle\langle 0|, \quad (6)$$

where $|\text{sr}\rangle_0, |\text{sr}\rangle_1$ are the only two populated states of two-atom and three-atom systems, $P_1 = \text{Tr}(\varrho|\text{sr}\rangle_{1\,1}\langle\text{sr}|0)$, and $|0\rangle$ is the vacuum photon state.

A. $N = 2$ system

From Eq. (4), it is straightforward to check that the two-atom system admits the subradiant state

$$|\text{sr}\rangle_1 = \frac{1}{\sqrt{1+2\varepsilon^2}}(|0,2,0\rangle - \sqrt{2}\varepsilon|1,0,1\rangle). \quad (7)$$

We choose as initial condition for Eq. (5), $\varrho_i = |2,0,0,0\rangle\langle 2,0,0,0|$. Figure 1(left) shows the time evolution of the populations $\langle N_i\rangle = \text{Tr}(c_i^\dagger c_i\varrho)$ (with $i = 0,1,2$) and $\langle N\rangle = \text{Tr}(a^\dagger a\varrho)$ for chosen values of κ and ϵ: we observe that $\langle N_0\rangle$ and $\langle N_1\rangle$ tend to nonzero values after the photons have escaped from the cavity. This result suggests that the atoms do not decay to the lower momentum state $m = 2$, but they arrange themselves in a superposition of the three-momentum states. Figure 2 (left) shows the time evolution of the probabilities $P_i = \text{Tr}(\varrho|\text{sr}\rangle_{ii}\langle\text{sr}|)$ for $i = 0,1$ for the same values of κ and ϵ. Asymptotically, $P_0 + P_1 = 1$ and ϱ approaches the steady-state density matrix ϱ_s of Eq. (6), where P_1 in general is a function of the parameters ϵ and κ. A similar behavior may be observed for any choice of the interaction parameters. As shown by Bonifacio *et al.* in [17] and Crubellier *et al.* in [10], if $\kappa \approx g\sqrt{N}$, the electromagnetic field will follow adiabatically the atomic motion. By means of the standard projection-operator technique [18], it is then possible to eliminate the field variables from the master equation (5) getting a closed equation for the atomic density operator ϱ_a

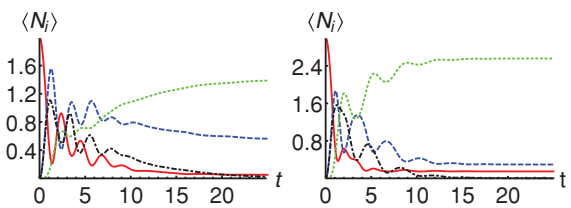

FIG. 1. (Color online) Time evolution of $\langle N_i\rangle$ for $m = 0$ (red), $m = 1$ (blue dashed), $m = 2$ (green dotted), and $\langle N_{\text{ph}}\rangle$ (black dot-dashed) with t in units of $1/g$. Left: $N = 2, \kappa = 0.2g, \epsilon = 0.3$; right: $N = 3, \kappa = 0.3g, \epsilon = 0.5$.

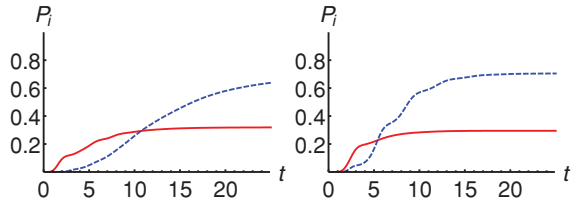

FIG. 2. (Color online) Time evolution of P_0 (blue dotted) and P_1 (red) with t in units of $1/g$. Left: $N = 2$, $\kappa = 0.2g$,$\epsilon = 0.3$; right: $N = 3$, $\kappa = 0.3g$,$\epsilon = 0.5$.

only. This is the well-known effective superradiance master equation

$$\frac{d\varrho_a}{dt} = \Gamma S^- \varrho_a S^+ - \frac{\Gamma}{2}(S^+ S^- \varrho_a + \varrho_a S^+ S^-), \tag{8}$$

where $S^- = c_1^\dagger c_0 + \varepsilon c_2^\dagger c_1$ and $\Gamma = g^2/\kappa$. Solving this equation analytically, we get an expression for the time evolution of P_1 and in the stationary limit we find a Γ-independent expression

$$P_1 = \frac{2(1-\epsilon^2)^2}{9\epsilon^2 + 2(1-\epsilon^2)^2}. \tag{9}$$

This expression is in excellent agreement with the solutions of Eq. (5) for any possible value of ϵ in the bad cavity regime. Figure 3 shows P_1 as a function of ϵ for several parameter regimes. The continuous black curve is the theoretical steady state given by Eq. (9) and the dashed curves are obtained as numerical solutions of Eq. (5). We see that, increasing κ/g and then approaching the bad cavity regime, the latter converges to the analytical one for each value of ϵ. In Fig. 4 (left), we focus our attention on the analytical form of P_1 only: it decreases from unit as ϵ increases and vanishes for $\epsilon = 1$, for which the degenerate three-level cascade has equal transition rates and the atoms decay superradiantly to the ground state. For $\epsilon > 1$ the subradiant probability increases and reaches asymptotically the unity. For large values of ϵ the steady-state is no longer a mixed state, $\varrho_s \approx |1,0,1\rangle\langle 1,0,1|$.

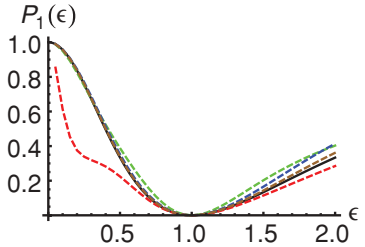

FIG. 3. (Color online) Steady-state probability P_1 as a function of ϵ for different values of κ (red dashed: $\kappa = 0.002g$; green dashed: $\kappa = 0.01g$; blue dashed: $\kappa = 0.04g$; brown dashed: $\kappa = 0.08g$) and as given by Eq. (9) (continuous black).

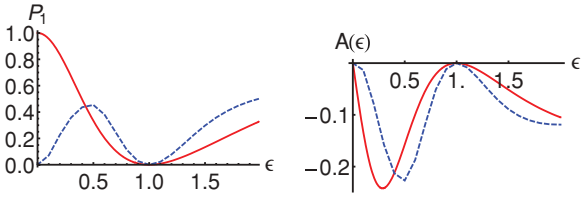

FIG. 4. (Color online) Left: steady-state probability as a function of ϵ for $N = 2$ (red) and $N = 3$ (blue dashed). Right: negative eigenvalue of the partially transposed state as a function of ϵ for $N = 2$ (red) and $N = 3$ (blue dashed). In the $N = 3$ case, we have set $\kappa = 0.8g$.

B. $N = 3$ system

In this case, we cannot directly use Eq. (4), which applies only for N even. The equation $S^-|\psi\rangle = 0$, where $|\psi\rangle$ is a generic three-atom state, is satisfied by the subradiant state

$$|\mathrm{sr}\rangle_1 = \frac{1}{\sqrt{1+4\epsilon^2}}(|0,2,1\rangle - 2\epsilon|1,0,2\rangle). \tag{10}$$

Figure 1 (right) shows the time evolution of the populations $\langle N_i\rangle$ with $i = 0,1,2$ and $\langle N\rangle$ for fixed values of ϵ and κ, obtained solving Eq. (5) with initial condition $\varrho_i = |3,0,0,0\rangle\langle 3,0,0,0|$. As in the $N = 2$ case, the mean occupation number of every atomic mode saturates to nonzero values and, asymptotically, the only atomic states surviving are the subradiant state $|\mathrm{sr}\rangle_1$ and the ground state $|\mathrm{sr}\rangle_0 = |0,0,3\rangle$. Hence also in this case we can write the reduced steady-state density operator in the form (6). Figure 2 (right) shows P_1 versus t and Fig. 4 (left) versus ϵ for a particular choice of κ. There are some important differences and analogies with the $N = 2$ case. First we note that this time P_1 vanishes for $\epsilon = 0$, whereas in the $N = 2$ case it has a maximum and is equal to unity. This is a consequence of the fact that the two-atom subradiant state (7) contains the state $|0,2,0\rangle$ which is the ground state for $\epsilon = 0$ (only the $m = 0 \to m = 1$ transition can take place). On the contrary, for $N = 3$ the steady-state density operator does not contain the state $|0,3,0\rangle$ which is the three-atom analog; moreover, it is a linear superposition of states with $n_2 \neq 0$, so, if the second transition is forbidden, it is impossible to populate these states observing subradiance. Secondly, whereas for $N = 2$, P_1 decreases with increasing ϵ and vanishes for $\epsilon = 1$, for $N = 3$, P_1 increases, reaches a maximum around $\epsilon \approx 0.5$, and decreases again falling off to zero for $\epsilon = 1$. For $\epsilon > 1$, the behavior of the two plots is quite similar.

C. Large N and p degeneracy

Let us consider now the $N \gg 1$ case. In [13], the following relation linking the p index and ϵ has been given:

$$p = \frac{N}{2}\left(\frac{1-2\epsilon^2}{1-\epsilon^2}\right), \quad 0 \leqslant \epsilon \leqslant 1/\sqrt{3}, \tag{11}$$

$$p = N\left(\frac{1-\epsilon^2}{1+\epsilon^2}\right)^2, \quad 1/\sqrt{3} \leqslant \epsilon \leqslant 1+\sqrt{2}. \tag{12}$$

Let us pay more attention to formulas (12). For $1/\sqrt{3} \leqslant \epsilon \leqslant \sqrt{3}$, it is easy to prove that two different values of ϵ exist

corresponding to the same value of p (except for $\epsilon = 1$ yielding $p = 0$, i.e., the ground state). This means that, once p is fixed, the two subradiant states $|\mathrm{sr}\rangle_{p(\epsilon_0)}, |\mathrm{sr}\rangle_{p(\epsilon_1)}$ are both acceptable, provided that they are actually different states. Now, let us assume that, for a fixed p, it is possible to build the following two states:

$$|\psi_p^{(\pm)}\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{sr}\rangle_{p(\epsilon_0)} \pm |\mathrm{sr}\rangle_{p(\epsilon_1)}\right). \tag{13}$$

It is straightforward to show that the following two states are orthonormal:

$$|\phi_p^{(\pm)}\rangle = \frac{1}{\sqrt{1 \pm \alpha}}|\psi_p^{(\pm)}\rangle, \tag{14}$$

where $\alpha = {}_{p(\epsilon_0)}\langle \mathrm{sr}|\mathrm{sr}\rangle_{p(\epsilon_1)} = {}_{p(\epsilon_1)}\langle \mathrm{sr}|\mathrm{sr}\rangle_{p(\epsilon_0)} \in \mathbb{R}$. We can compute the total kinetic energy of the N-atom system and also the energy difference between these two states $|\Delta_p| = |E_p^{(+)}(\epsilon) - E_p^{(-)}(\epsilon)|$, which turns out to be

$$|\Delta_p| = \left|\frac{4\{2N - 2p - 2[\bar{k}(\epsilon_0(p)) + \bar{k}(\epsilon_1(p))]\}}{\sqrt{2(1+\alpha)}} - \frac{2[\bar{k}(\epsilon_0(p)) + \bar{k}(\epsilon_1(p))]}{\sqrt{2(1-\alpha)}}\right|, \tag{15}$$

where

$$\bar{k}(\epsilon_j(p)) = \frac{p!(N-p-1)!}{2\epsilon_j^2(N-p)!(p-1)!} \times \frac{{}_2F_1\left(1-p, 3/2; 1+p-N; \epsilon_j^{-2}\right)}{{}_2F_1\left(-p, 1/2; p-N; \epsilon_j^{-2}\right)}.$$

Since this parameter depends on ϵ, in principle it should be possible to distinguish between the two states (14). More interesting, by applying an external potential coupling these two states, we could realize a fictitious two-level system where the many-body states $|\phi_p^{(\pm)}\rangle$ play the role of the logical-basis states $|0\rangle, |1\rangle$. A similar system has been proposed in [19] for a Λ configuration.

III. ENTANGLEMENT PROPERTIES

In this section, we address the entanglement properties of the state (6) either regarded as made of three bosonic modes or, upon considering the selection rule induced by the constraint on the number of atoms, as a finite-dimensional one. Generally speaking, there are several classes of entanglement in a tripartite quantum system, from genuine tripartite entanglement to full separability passing through bipartite separability, which are defined on the basis of all possible groupings of the three parties [20]. Our primary goal is to establish whether the stationary state is fully inseparable and if it is possible to quantify the amount of entanglement.

At first, we notice that our system consists of three atomic modes for which, being the atomic ensemble made of a finite and fixed number of atoms, only a finite number of quanta per mode is permitted from a minimum of $n_i = 0$ up to a maximum of $n_i = N$. This physical constraint corresponds to a geometrical constraint which identifies a subspace $\mathcal{S}$ of $\mathcal{H}_a = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$ defined as

$$\mathcal{S} = \left\{\varrho \in \mathcal{H} : \sum_{i=0}^{2}\langle N_i\rangle = N\right\}.$$

Hence, being that $\mathcal{S}$ is a finite-dimension subspace, we can make use of the positive partial transposition (PPT) criterion for discrete-variable quantum systems. In fact, the PPT condition is sufficient to establish entanglement of a state and, as we will see, it is enough to show that ϱ_s is genuinely tripartite entangled. Partially transposing with respect to a given party corresponds to a specific grouping of two parties with respect to the third: for example, when we partial-transpose with respect to the $m = 0$ mode, we are grouping together the $m = 1$ and 2 modes with respect to $m = 0$. Suppose we have partial-transposed with respect to $m = 0$, obtaining $\varrho_s^{T_0}$ and let the eigenvalues of $\varrho_s^{T_0}$ be $\{\lambda_k^{(0)}\}$. If at least one k exists such that $\lambda_k < 0$, we can conclude that $\varrho_s \neq \sum_j p_j \varrho_j^0 \otimes \varrho_j^{12}$. If the positivity condition is violated for $\varrho_s^{T_1}$ and $\varrho_s^{T_2}$ too, we can state that ϱ_s is genuinely tripartite entangled. We start with the $N = 2$ case and we take the reduced density operator $\mathrm{Tr}_{\mathrm{ph}}(\varrho_s)$, where ϱ_s is given by Eq. (6) with P_1 given by Eq. (9). The $\mathcal{S}$ subspace is spanned by six Fock states: $|2,0,0\rangle, |0,2,0\rangle, |0,0,2\rangle, |1,1,0\rangle, |1,0,1\rangle, |0,1,1\rangle$. If we compute the eigenvalues of every $\varrho_s^{T_j}$ with $j = 0,1,2$, we find that there always exists a negative eigenvalue $A_2(\epsilon)$,

$$A_2(\epsilon) = -\frac{\sqrt{2}}{2}\frac{\epsilon(\epsilon^2-1)^2}{\left(\frac{1}{2}+\epsilon^2\right)^2(2+\epsilon^2)} = -\frac{\epsilon^2(2\epsilon^2+3)^2+2}{4\sqrt{2}(\epsilon^2-1)^2}\langle N_0\rangle\langle N_1\rangle, \tag{16}$$

where the expression on the second line provides a link between the negative eigenvalue, which directly quantifies the amount of entanglement, and a couple of moments which are, at least in principle, amenable to direct measurements.

In the $N = 3$ case, $\mathcal{S}$ is spanned by the following ten states: $|3,0,0\rangle$, $|0,3,0\rangle$, $|0,0,3\rangle$, $|2,1,0\rangle$, $|2,0,1\rangle$, $|0,1,2\rangle$, $|0,2,1\rangle$, $|1,0,2\rangle$, $|1,2,0\rangle$, $|1,1,1\rangle$, and the PPT condition is still quite simple to employ to establish tripartite entanglement. On the other hand, for $N = 3$ we have not been able to give an analytical expression for P_1 and, therefore, we resort to numerical evaluation of the eigenvalues. As in the previous case, we find a negative eigenvalue $A_3(\epsilon)$ which turns out to be always the same for every possible partial transposition and for any value of ϵ. This means that also the three-atom ϱ_s is genuinely tripartite entangled. Figure 4 (right) shows the behavior of $A_N(\epsilon)$ both for $N = 2$ and 3. The behavior is qualitatively the same for both cases. The eigenvalues decrease from zero and reach a minimum after which they start to increase. As we should expect, they vanish for $\epsilon = 1$ and after this they start to decrease again.

A. Continuous-variable description

Let us now investigate the entanglement properties of our system as a continuous-variable one [21], that is, taking into account that we are dealing with three bosonic modes

describing the number of atoms within a given state (for instance, the momentum state). To this end, we start by reviewing a few facts about the description of a multimode continuous-variable system [22]. Upon introducing the canonical operators $q_j = (c_j + c_j^\dagger)/\sqrt{2}$ and $p_j = i(c_j^\dagger - c_j)/\sqrt{2}$ in terms of the single-mode operators c_j, $j = 0,1,2$, we may write the vector of operators $\boldsymbol{R} = (q_1, p_1, q_2, p_2, q_3, p_3)^T$. The so-called vector of mean values $\boldsymbol{X} = \boldsymbol{X}(\varrho)$ and covariance matrix (CM) $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\varrho)$ of a quantum state ϱ have components $X_j = \langle R_j \rangle$ and $R\sigma_{jk} = \frac{1}{2}\langle\{R_j, R_k\}\rangle - \langle R_j\rangle\langle R_j\rangle$, where $\langle O\rangle = \mathrm{Tr}(\varrho\, O)$, and $\{A,B\} = AB + BA$ denotes the anticommutator. The characteristic function $\chi(\varrho)(\boldsymbol{\lambda}) = \mathrm{Tr}[\varrho\, D(\boldsymbol{\lambda})]$ is defined in terms of the multimode displacement operator $D(\boldsymbol{\lambda}) = \otimes_{j=1}^n D(\lambda_j)$, with $\lambda = (\lambda_1, \ldots, \lambda_n)^T$, $\lambda_j \in \mathbb{C}$, and $D(\lambda_j) = \exp(\lambda_j c_j^\dagger - \lambda_j^* c_j)$. A quantum state ϱ_G is referred to as a Gaussian state if its characteristic function has the Gaussian form $\chi(\varrho_G)(\boldsymbol{\Lambda}) = \exp(-\frac{1}{2}\boldsymbol{\Lambda}^T\boldsymbol{\sigma}\boldsymbol{\Lambda} + \mathbf{X}^T\boldsymbol{\Omega}\boldsymbol{\Lambda})$, where $\boldsymbol{\Lambda}$ is the real vector $\boldsymbol{\Lambda} = (\mathrm{Re}\lambda_1, \mathrm{Im}\lambda_1, \ldots, \mathrm{Re}\lambda_n, \mathrm{Im}\lambda_n)$ and $\boldsymbol{\Omega}$ is the symplectic matrix $\boldsymbol{\Omega} = i \oplus_{j=1}^n \sigma_2$, σ_2 being the y Pauli matrix. Gaussian states are thus completely characterized by their mean values vector and their covariance matrix. For bipartite Gaussian states, the PPT condition is necessary and sufficient for separability [23], and it may be rewritten in terms of the CM, which uniquely determines the entanglement properties of the state under investigation. A mode of a Gaussian state is separable from the others iff $\Lambda_j \boldsymbol{\sigma} \Lambda_j + \frac{i}{2}\boldsymbol{\Omega} \geqslant 0$, where Λ_j is a diagonal matrix implementing the partial transposition at the level of CM, that is, inverting the sign of the jth momentum. In our case, $\Lambda_1 = \mathrm{diag}(1,-1,1,1,1,1)$, $\Lambda_2 = \mathrm{diag}(1,1,1,-1,1,1)$, and $\Lambda_3 = \mathrm{diag}(1,1,1,1,1,-1)$. When we have a non-Gaussian state, the violation of the previous condition is still a sufficient condition for entanglement, whereas the necessary part is lost.

Let us start by focusing attention on an N-atom pure subradiant state (4). The density operator is $\varrho_p = |\mathrm{sr}\rangle_{pp}\langle\mathrm{sr}|$ and the only nonvanishing terms of the CM $\boldsymbol{\sigma}(\varrho_p)$ are the diagonal ones, which are reported in Appendix A. This fact is a direct consequence of two selection rules affecting the geometrical form of subradiant states: the first one is the conservation of the number of atoms and the second one is the presence of the k index in every Fock component of $|\mathrm{sr}\rangle_p$. As a matter of fact, the CM $\boldsymbol{\sigma}(\varrho_p)$ satisfies the three separability conditions $\Lambda_j \boldsymbol{\sigma} \Lambda_j + \frac{i}{2}\boldsymbol{\Omega} \geqslant 0$, $j = 1,2,3$, and therefore we have no information about the entanglement properties of ϱ_p, that is, we cannot use the PPT condition translated to continuous-variable systems to analyze and quantify entanglement. On the other hand, we know from the previous analysis that ϱ_p is a genuinely tripartite entangled state, and this suggests that it should be far from being a Gaussian state. Indeed, the characteristic function is given by a Gaussian modulated by Laguerre polynomials and thus shows a distinctive non-Gaussian shape.

In the following, we will evaluate quantitatively the non-Gaussianity of the subradiant states upon the use of the non-Gaussianity measure introduced in [24], which is defined as

$$\delta(\varrho) = \frac{D_{\mathrm{HS}}^2(\varrho,\tau)}{\mu_\varrho} = \frac{\mu_\varrho + \mu_\tau - 2\kappa_{\varrho\tau}}{2\mu_\varrho}, \tag{17}$$

where $D_{\mathrm{HS}}(\varrho,\tau)$ denotes the Hilbert-Schmidt distance between the state ϱ under scrutiny and its reference Gaussian state τ, that is, a Gaussian state with the same mean value $\boldsymbol{X}(\tau) = \boldsymbol{X}(\varrho)$ and CM $\boldsymbol{\sigma}(\tau) = \boldsymbol{\sigma}(\varrho)$, $\mu_\varrho = \mathrm{Tr}(\varrho^2)$ is the purity of ϱ, and $\kappa_{\varrho\tau} = \mathrm{Tr}(\varrho\,\tau)$ is the overlap between ϱ and τ. The non-Gaussian measure $\delta(\varrho)$ possesses all the properties for a good measure of the non-Gaussian character and, in particular, $\delta(\varrho) = 0$ iff ϱ is a Gaussian state. For the pure subradiant states $\varrho_p = |\mathrm{sr}\rangle_{pp}\langle\mathrm{sr}|$, the reference Gaussian state has the form $\tau = \nu_0 \otimes \nu_1 \otimes \nu_2$, where $\nu_j = (1-y_j)\sum_s y_j^s |s\rangle\langle s|$, and $y_j = N_j/(1+N_j)$ is a (Gaussian) thermal state with N_j average thermal photons. For a three-mode thermal state, we have $\sigma_{11} = \sigma_{22} = \frac{1}{2}(N_0 + \frac{1}{2})$, $\sigma_{33} = \sigma_{44} = \frac{1}{2}(N_1 + \frac{1}{2})$, and $\sigma_{55} = \sigma_{66} = \frac{1}{2}(N_2 + \frac{1}{2})$. Since for a subradiant state the average occupation numbers are given by $N_0 = p - \langle k\rangle_p$, $N_1 = 2\langle k\rangle_p$, and $N_2 = N - p - \langle k\rangle_p$, we have, after some calculations, that the non-Gaussianity of ϱ_p is given by

$$\delta(\varrho_p) = \frac{1}{2} + \frac{1}{2}\prod_{j=0}^{2}\frac{1-y_j}{1+y_j} - |C_p|^2 \prod_{j=0}^{2}(1-y_j) \times \sum_{k=0}^{p} \beta_k^2\, y_0^{p-k}\, y_1^{2k}\, y_2^{N-p-k}. \tag{18}$$

For a large number of atoms $N \gg 1$, the non-Gaussianity of $\delta(\varrho_p)$ may be evaluated upon exploiting Eqs. (11) and (12), that is, the relation between p and ϵ. In practice, every value of ϵ corresponds to a single p and in turn to the pure subradiant state ϱ_p. In Fig. 5 (left), we show the behavior of the non-Gaussianity $\delta(\varrho_p)$ for $N = 50$ and $0 \leqslant \epsilon \leqslant 1 + \sqrt{2}$. As is apparent from the plot, the subradiant states are always non-Gaussian and the value of δ is almost constant. For $N = 2,3$, it is possible to evaluate explicitly the non-Gaussianity for the stationary state (6). Since the number of atoms is small, p and ϵ are independent of one another and we are not allowed to use relations (11) and (12). For $N = 2,3$, we have the ground state, ϱ_0, and a single subradiant state, ϱ_1, and the non-Gaussianity may be expressed as

$$\delta(\varrho_s) = P_1(P_1 - 1) + \frac{1+\mu_\tau}{2} - (1-P_1)\kappa_{\varrho_0\tau} - P_1\kappa_{\varrho_1\tau}.$$

The reference Gaussian state is again a three-mode thermal state, as it can be easily demonstrated by noticing that the CM of ϱ_s is the convex combination of those corresponding to ϱ_0 and ϱ_1, that is, $\boldsymbol{\sigma}(\varrho_s) = P_1\boldsymbol{\sigma}(\varrho_1) + (1-P_1)\boldsymbol{\sigma}(\varrho_0) = P_1$. P_1 is given by Eq. (9) for $N = 2$ and it may be evaluated numerically for $N = 3$. The overlaps are given in Appendix B. The behavior of the non-Gaussianity $\delta(\varrho_s)$ as a function of ϵ for

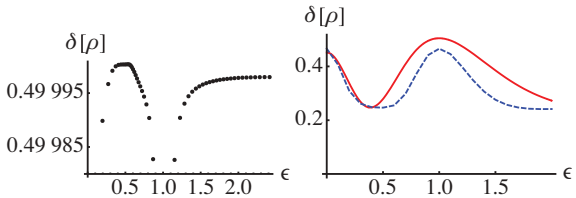

FIG. 5. (Color online) Left: $\delta(\varrho)$ for $N = 50$ vs ϵ accounting for relations (11) and (12). Right: $\delta(\varrho)$ for $N = 2$ (red) and $N = 3$ (blue dashed); in this second case we have set $\kappa = 0.8g$.

$N = 2,3$ is shown in Fig. 5 (right). Notice that the behavior of $\delta(\varrho_s)$ does not show a strict correlation to that of the negative eigenvalue $A(\epsilon)$ and thus we cannot use non-Gaussianity to assess quantitatively the failure of the CV PPT condition in detecting stationary entanglement in our system.

IV. CONCLUSIONS

In this paper, we have considered a systems made of N two-level ultracold bosonic atoms in a ring cavity where subradiance appears in the bad cavity limit. We provided a full quantum description of the dynamics and have shown the appearance of stationary entanglement among atoms. We have evaluated the amount of steady-state entanglement for $N = 2$ and 3 atoms as quantified by the negativity of the partially transposed density matrix. We have also investigated the entanglement properties of the systems as a continuous-variable one and have shown that it is not possible to detect entanglement due to the non-Gaussian character of subradiant states. It is important to notice that the lifetime of experimentally demonstrated entangled states is generally limited, due to their fragility under decoherence and dissipation. Therefore, to decrease dissipation and decoherence, strict isolation from the environment is usually considered. On the contrary, in our system, the action of the coupling to the environment drives it into a steady entangled state, which is robust with respect to the interaction parameters. Our results pave the way for entanglement characterization [25,26] of subradiant states and suggest further investigations for a large number of atoms.

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APPENDIX A: CM OF THE SUBRADIANT STATE

The covariance matrix of the subradiant state $\varrho_p = |\mathrm{sr}\rangle_{pp}\langle \mathrm{sr}|$ is diagonal. The nonzero matrix elements may be expressed as

$$\sigma_{11} = \sigma_{22} = \frac{|C_p|^2}{2}\sum_{k=0}^{p}\beta_k^2\left(p-k+\frac{1}{2}\right) = \frac{1}{2}\left(p-\langle k\rangle_p-\frac{1}{2}\right) \tag{A1}$$

$$\sigma_{33} = \sigma_{44} = \frac{|C_p|^2}{4}\sum_{k=0}^{p}\beta_k^2(4k+1) = \langle k\rangle_p+\frac{1}{4}, \tag{A2}$$

$$\sigma_{55} = \sigma_{66} = \frac{|C_p|^2}{2}\sum_{k=0}^{p}\beta_k^2\left(N-p-k+\frac{1}{2}\right) = \frac{1}{2}\left(N-p-\langle k\rangle_p+\frac{1}{2}\right), \tag{A3}$$

where we have defined

$$\beta_k = \left(-\frac{1}{2\epsilon}\right)^k\left(\frac{1}{k!}\right)\sqrt{\frac{(2k)!(N-p-k)!}{(p-k)!}}$$

and $\langle k\rangle_p = |C_p|^2\sum_{k=0}^{p}\beta_k^2 k$ is the mean value of k for the p-subradiant state.

APPENDIX B: OVERLAPS

The overlaps $\kappa^{(N)}_{\varrho_j\tau}$ between the states ϱ_j and their reference Gaussian states for N atoms, which appear in the expression of the non-Gaussianity $\delta(\varrho_s)$, are given by

$$\kappa^{(2)}_{\varrho_0,\tau} = \kappa^{(3)}_{\varrho_0,\tau} = y_2^2\prod_{j=0}^{2}\frac{1-y_j}{1+y_j},$$

$$\kappa^{(2)}_{\varrho_1,\tau} = \frac{y_1^2+2\epsilon^2 y_0\,y_2}{1+2\epsilon^2}\prod_{j=0}^{2}\frac{1-y_j}{1+y_j}, \tag{B1}$$

$$\kappa^{(3)}_{\varrho_1,\tau} = \frac{y_2\left(y_1^2+4\epsilon^2 y_0\,y_2\right)}{1+4\epsilon^2}\prod_{j=0}^{2}\frac{1-y_j}{1+y_j}.$$

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