On the constants in a Kato inequality for the Euler and Navier-Stokes equations

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Abstract

We continue an analysis, started in [10], of some issues related to the incompressible Euler or Navier-Stokes (NS) equations on a d-dimensional torus $\mathbf{T}^d$. More specifically, we consider the quadratic term in these equations; this arises from the bilinear map $(v, w) \mapsto v \bullet \partial w$, where $v, w : \mathbf{T}^d \to \mathbf{R}^d$ are two velocity fields. We derive upper and lower bounds for the constants in some inequalities related to the above bilinear map; these bounds hold, in particular, for the sharp constants $G_{nd} \equiv G_n$ in the Kato inequality $|\langle v \bullet \partial w | w \rangle_n| \leqslant G_n \|v\|_n \|w\|^2_n$, where $n \in (d/2+1, +\infty)$ and v, w are in the Sobolev spaces $\mathbb{H}^n_{\Sigma 0}, \mathbb{H}^{n+1}_{\Sigma 0}$ of zero mean, divergence free vector fields of orders n and $n+1$, respectively. As examples, the numerical values of our upper and lower bounds are reported for $d = 3$ and some values of n. When combined with the results of [10] on another inequality, the results of the present paper can be employed to set up fully quantitative error estimates for the approximate solutions of the Euler/NS equations, or to derive quantitative bounds on the time of existence of the exact solutions with specified initial data; a sketch of this program is given.

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1 Introduction

The present paper continues our previous work on some inequalities related to the Euler or Navier-Stokes (NS) equations. We work on a d-dimensional torus $\mathbf{T}^d$, and write these equations as

$$\frac{\partial u}{\partial t} = -\mathfrak{L}(u \bullet \partial u) + \nu \Delta u + f \ , \tag{1.1}$$

where: $u = u(x,t)$ is the divergence free velocity field; $x = (x_s)_{s=1,...,d} \in \mathbf{T}^d$ are the space variables (yielding the derivatives $\partial_s := \partial/\partial x_s$); $\Delta := \sum_{s=1}^d \partial_{ss}$ is the Laplacian; $(u \bullet \partial u)_r := \sum_{s=1}^d u_s \partial_s u_r$ $(r = 1, ..., d)$; $\mathfrak{L}$ is the Leray projection onto the space of divergence free vector fields; $\nu = 0$ for the Euler equations; $\nu \in (0, +\infty)$ (in fact $\nu = 1$, after rescaling) for the NS equations; $f = f(x,t)$ is the Leray projected density of external forces. As already noted [8], the analysis of the above equations can be reduced to the case where the (spatial) means $\langle u \rangle := (2\pi)^{-d} \int_{\mathbf{T}^d} u \, dx$ and $\langle f \rangle$ are zero at all times.

A precise functional setting for the above framework can be built using, for suitable (integer or noninteger) values of n, the Sobolev spaces

$$\mathbb{H}^n_0(\mathbf{T}^d) \equiv \mathbb{H}^n_0 := \{ v : \mathbf{T}^d \to \mathbf{R}^d \mid \sqrt{-\Delta}^{\,n} v \in \mathbb{L}^2(\mathbf{T}^d), \ \langle v \rangle = 0 \} \ , \tag{1.2}$$

$$\mathbb{H}^n_{\Sigma 0}(\mathbf{T}^d) \equiv \mathbb{H}^n_{\Sigma 0} := \{ v \in \mathbb{H}^n_0 \mid \operatorname{div} v = 0 \} \tag{1.3}$$

(the subscripts 0, Σ recall the vanishing of the mean and of the divergence, respectively). For each n, we equip $\mathbb{H}^n_0$ with the standard inner product and the norm

$$\langle v | w \rangle_n := \langle \sqrt{-\Delta}^{\,n} v | \sqrt{-\Delta}^{\,n} w \rangle_{L^2} \ , \qquad \| v \|_n := \sqrt{\langle v | v \rangle_n} \ , \tag{1.4}$$

which can be restricted to the (closed) subspace $\mathbb{H}^n_{\Sigma 0}$.

Our aim is to analyze quantitatively, in terms of the Sobolev inner products, the quadratic map appearing in (1.1). Some aspects of this map have been already examined in the companion paper [10]; here we have considered the bilinear maps sending two vector fields v, w on $\mathbf{T}^d$ into $v \bullet \partial w$ or $\mathfrak{L}(v \bullet \partial w)$, and we have discussed some inequalities about them, the basic one being

$$\| \mathfrak{L}(v \bullet \partial w) \|_n \leqslant K_n \| v \|_n \| w \|_{n+1} \qquad \text{for } n \in (\frac{d}{2}, +\infty), \ v \in \mathbb{H}^n_{\Sigma 0}, \ w \in \mathbb{H}^{n+1}_{\Sigma 0} \ . \tag{1.5}$$

Our attention has been focused on the sharp constants $K_n \equiv K_{nd}$ appearing therein, for which we have given fully quantitative upper and lower bounds.

In the present work we discuss other inequalities related to the quadratic Euler/NS nonlinearity, discovered by Kato in [6], and establish upper and lower bounds for the unknown sharp constants appearing therein. First of all we consider the inequality

$$| \langle v \bullet \partial w | w \rangle_n | \leqslant G'_n \| v \|_n \| w \|^2_n \quad \text{for } n \in (\frac{d}{2} + 1, +\infty), \ v \in \mathbb{H}^n_{\Sigma 0}, \ w \in \mathbb{H}^{n+1}_0 \ , \tag{1.6}$$

writing $G'_n \equiv G'_{nd}$ for the sharp constants therein. With the additional assumption that w be divergence free, we can write

$$|\langle v \bullet \partial w | w \rangle_n| \leqslant G_n \|v\|_n \|w\|_n^2 \quad \text{for } n \in (\frac{d}{2}+1, +\infty),\ v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0}\ , \tag{1.7}$$

with the sharp constant $G_n \equiv G_{nd}$ fulfilling the obvious relation $G_n \leqslant G'_n$. Let us observe that (1.7) can be rephrased in terms of the Leray projection $\mathfrak{L}$; indeed, with the assumptions therein we have $w = \mathfrak{L}w$ and this fact, combined with the symmetry of $\mathfrak{L}$ in the Sobolev inner product, gives

$$\langle v \bullet \partial w | w \rangle_n = \langle v \bullet \partial w | \mathfrak{L} w \rangle_n = \langle \mathfrak{L}(v \bullet \partial w) | w \rangle_n \qquad \text{for } v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0}\ . \tag{1.8}$$

Due to (1.8), Eq. (1.7) is more directly related to the incompressible Euler/NS equations (1.1); in the sequel, (1.7) is referred to as the Kato inequality, and we call (1.6) the auxiliary Kato inequality.
These inequalities (and similar ones) are well known, but little has been done previously to evaluate with some accuracy the constants which appear therein. On the other hand, quantitative bounds on such constants are useful to estimate the time of existence of the solution of (1.1) for a given initial datum, or its distance from any approximate solution.
In the present paper we derive fully computable upper and lower bounds $G^{\pm}_n \equiv G^{\pm}_{nd}$ such that

$$G^-_n \leqslant G_n \leqslant G'_n \leqslant G^+_n \tag{1.9}$$

for all $n > d/2+1$. As examples, the bounds $G^{\pm}_n$ are computed in dimension $d = 3$, for some values of n. In these cases the upper and lower bounds are not too far, at least for the purpose to apply them to the Euler/NS equations.
To be more precise about such applications, let us exemplify a framework already mentioned in [10]; the starting point of this setting is a result of Chernyshenko, Constantin, Robinson and Titi [4], that can be stated as follows. Consider the Euler/NS equation (1.1) with a specified initial condition $u(x,0) = u_0(x)$; let $u_{ap} : \mathbf{T}^d \times [0, T_{ap}] \to \mathbf{R}^d$ be an approximate solution of this Cauchy problem with errors $\epsilon : \mathbf{T}^d \times [0, T_{ap}] \to \mathbf{R}^d$ on the equation and $\epsilon_0 : \mathbf{T}^d \to \mathbf{R}$ on the initial condition, by which we mean that

$$\epsilon := \frac{\partial u_{ap}}{\partial t} + \mathfrak{L}(u_{ap} \bullet \partial u_{ap}) - \nu \Delta u_{ap} - f\ , \qquad \epsilon_0 := u_{ap}(\cdot, 0) - u_0\ . \tag{1.10}$$

Fix $n \in (d/2+1, +\infty)$; then, Eq. (1.1) with datum u_0 has a (strong) exact solution u in $\mathbb{H}^n_{\Sigma 0}$ on a time interval $[0,T] \subset [0, T_{ap}]$, if T and u_{ap} fulfill the inequality

$$\|\epsilon_0\|_n + \int_0^T \|\epsilon(t)\|_n dt < \frac{1}{G_n T}\, e^{-\int_0^T (G_n \|u_{ap}(t)\|_n + K_n \|u_{ap}(t)\|_{n+1}) dt} \tag{1.11}$$

$(u_{ap}(t) := u_{ap}(\cdot, t)$, $\epsilon(t) := \epsilon(\cdot, t))$. For a given datum u_0, one can try a practical implementation of the above criterion after choosing a suitable u_{ap} (say, a Galerkin approximate solution). Of course, T can be evaluated via (1.11) only in the presence of quantitative information on K_n and G_n, which are missing in [4]. In a forthcoming paper [11], our estimates on K_n and G_n will be employed together with the existence condition (1.11) (or with some refinement of it, suited as well to get bounds on $\|u(t) - u_{ap}(t)\|_n$).

For completeness we wish to mention that a program similar to the one described above, but based on technically different inequalities, has been developed in [8] [9] for the incompressible NS equations in Sobolev spaces of lower order. For example, in [9] we have considered the NS equations in $\mathbb{H}^1_{\Sigma 0}(\mathbf{T}^3)$; here we have derived a fully quantitative upper bound on the vorticity $\|\mathrm{curl}\, u_0\|_{L^2}$ of the initial datum, which ensures global existence of the solution.

Again for completeness, we remark that the fully quantitative attitude proposed here for the Euler/NS equations is more or less close to the viewpoints of other authors about these equations, or about different nonlinear evolutionary PDEs [1] [3] [7] [12] [13] [14].

Organization of the paper. Section 2 summarizes our standards about Sobolev spaces on $\mathbf{T}^d$ and the Euler/NS quadratic nonlinearity.

Section 3 states the main results of the paper; here we present our upper and lower bounds $G_n^{\pm}$ on the constants in the inequalities (1.6) (1.7), which are treated by Propositions 3.5 and 3.7. The upper bounds are determined by the sup of a positive function $\mathcal{G}_n$, defined on the space $\mathbf{Z}^d \setminus \{0\}$ of nonzero Fourier wave vectors; at each point $k \in \mathbf{Z}^d \setminus \{0\}$, $\mathcal{G}_n(k)$ is a sum (of convolutional type) over $\mathbf{Z}^d \setminus \{0, k\}$. The lower bounds are determined by suitable trial functions. As examples, in Eq. (3.21) we report the numerical values of $G_n^{\pm}$, for $d = 3$ and $n = 3, 4, 5, 10$.

Section 4 contains the proofs of the previously mentioned Propositions 3.5, 3.7.

Several appendices are devoted to the practical evaluation of the function $\mathcal{G}_n$ mentioned before, and of the bounds $G_n^{\pm}$. Appendix A presents some preliminary notations and results. Appendix B contains the main theorem (Proposition B.1) about the evaluation of $\mathcal{G}_n$ and of its sup. Appendix C gives details on the computation of $\mathcal{G}_n$, and on the corresponding upper bounds G_n^{+}, for the previously mentioned cases $d = 3$, $n = 3, 4, 5, 10$. Appendix D describes the computation of the bounds G_n^{-}, for the same values of d and n.

For all the numerical computations required by this paper, as well as for some lengthy symbolic manipulations, we have used systematically the software MATHEMATICA. Throughout the paper, an expression like $r = a.bcde...$ means the following: computation of the real number r via MATHEMATICA produces as an output $a.bcde$, followed by other digits not reported for brevity.

2 Some preliminaries

We use for Sobolev spaces and the Euler/NS bilinear map the same notations proposed in [10]; for the reader's convenience, these are summarized hereafter. Throughout the paper, we work in any space dimension

$$d \geqslant 2 \ ; \tag{2.1}$$

we use r, s as indices running from 1 to d. For $a, b \in \mathbf{C}^d$ we put

$$a \bullet b := \sum_{r=1}^{d} a_r \, b_r \ ; \qquad |a| := \sqrt{\overline{a} \bullet a} \tag{2.2}$$

where $\overline{a} := (\overline{a_r})$ is the complex conjugate of a. We often refer to the d-dimensional torus

$$\mathbf{T}^d := \underbrace{\mathbf{T} \times ... \times \mathbf{T}}_{d \text{ times}} \ , \qquad \mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z}) \ , \tag{2.3}$$

whose elements are typically written $x = (x_r)_{r=1,...d}$.

Distributions on $\mathbf{T}^d$, Fourier series and Sobolev spaces. The space of periodic distributions $D'(\mathbf{T}^d, \mathbf{C}) \equiv D'_{\mathbf{C}}$ is the (topological) dual of $C^\infty(\mathbf{T}^d, \mathbf{C}) \equiv C^\infty_{\mathbf{C}}$; $\langle v, f \rangle \in \mathbf{C}$ denotes the action of a distribution $v \in D'_{\mathbf{C}}$ on a test function $f \in C^\infty_{\mathbf{C}}$.
Each $v \in D'_{\mathbf{C}}$ has a unique (weakly convergent) Fourier series expansion

$$v = \sum_{k \in \mathbf{Z}^d} v_k e_k \ , \quad e_k(x) := \frac{1}{(2\pi)^{d/2}} \, e^{ik \bullet x} \text{ for } x \in \mathbf{T}^d \ , \quad v_k := \langle v, e_{-k} \rangle \ \in \mathbf{C} \ . \tag{2.4}$$

The complex conjugate of a distribution $v \in D'_{\mathbf{C}}$ is the unique distribution $\overline{v}$ such that $\overline{\langle v, f \rangle} = \langle \overline{v}, \overline{f} \rangle$ for each $f \in C^\infty_{\mathbf{C}}$; one has $\overline{v} = \sum_{k \in \mathbf{Z}^d} \overline{v_k} \, e_{-k}$.
The *mean* of $v \in D'_{\mathbf{C}}$ and the space of *zero mean* distributions are

$$\langle v \rangle := \frac{1}{(2\pi)^d} \langle v, 1 \rangle = \frac{1}{(2\pi)^{d/2}} v_0 \ , \qquad D'_{\mathbf{C}0} := \{ v \in D'_{\mathbf{C}} \mid \langle v \rangle = 0 \} \tag{2.5}$$

(of course, $\langle v, 1 \rangle = \int_{\mathbf{T}^d} v \, dx$ if $v \in L^1(\mathbf{T}^d, \mathbf{C}, dx)$). The relevant Fourier coefficients of zero mean distributions are labeled by the set

$$\mathbf{Z}^d_0 := \mathbf{Z}^d \setminus \{0\} \ . \tag{2.6}$$

The distributional derivatives $\partial/\partial x_s \equiv \partial_s$ and the Laplacian $\Delta := \sum_{s=1}^d \partial_{ss}$ send $D'_{\mathbf{C}}$ into $D'_{\mathbf{C}0}$ and, for each v, $\partial_s v = i \sum_{k \in \mathbf{Z}^d_0} k_s v_k e_k$, $\Delta v = -\sum_{k \in \mathbf{Z}^d_0} |k|^2 v_k e_k$. For any $n \in \mathbf{R}$, we further define

$$\sqrt{-\Delta}^{\,n} : D'_{\mathbf{C}} \to D'_{\mathbf{C}0} \ , \qquad v \mapsto \sqrt{-\Delta}^{\,n} v := \sum_{k \in \mathbf{Z}^d_0} |k|^n v_k e_k \ . \tag{2.7}$$

The space of *real* distributions is

$$D'(\mathbf{T}^d,\mathbf{R}) \equiv D' := \{ v \in D'_{\mathbf{C}} \mid \overline{v} = v \} = \{ v \in D'_{\mathbf{C}} \mid \overline{v_k} = v_{-k} \text{ for all } k \in \mathbf{Z}^d \} \ . \quad (2.8)$$

For $p \in [1,+\infty]$ we often consider the real space

$$L^p(\mathbf{T}^d,\mathbf{R},dx) \equiv L^p \ , \quad (2.9)$$

especially for $p = 2$. L^2 is a Hilbert space with the inner product $\langle v|w\rangle_{L^2} := \int_{\mathbf{T}^d} v(x)w(x)dx = \sum_{k\in\mathbf{Z}^d} \overline{v_k} w_k$ and the induced norm $\| \ \|_{L^2}$.
The zero mean parts of D' and L^p are

$$D'_{0} := \{ v \in D' \mid \ \langle v \rangle = 0 \} \ , \qquad L^p_0 := L^p \cap \mathbb{D}'_{0} \ ; \quad (2.10)$$

all the differential operators mentioned before send D' into D'_{0}.
For each $n \in \mathbf{R}$, the *zero mean Sobolev space* $H^n_{0}(\mathbf{T}^d,\mathbf{R}) \equiv H^n_{0}$ is defined by

$$H^n_{0} := \{ v \in D'_{0} \mid \sqrt{-\Delta}^{\,n} v \in L^2 \} = \{ v \in D'_{0} \mid \sum_{k\in\mathbf{Z}^d_0} |k|^{2n} |v_k|^2 < +\infty \ \} \ ; \quad (2.11)$$

this is a real Hilbert space with the inner product $\langle v|w\rangle_n := \langle \sqrt{-\Delta}^{\,n} v \,|\, \sqrt{-\Delta}^{\,n} w \rangle_{L^2}$ $= \sum_{k\in\mathbf{Z}^d_0} |k|^{2n} \, \overline{v_k} w_k$ and the induced norm $\| \ \|_n$. Of course, $H^0_0 = L^2_0$.

Spaces of vector valued functions on $\mathbf{T}^d$. If $V(\mathbf{T}^d,\mathbf{R}) \equiv V$ is any vector space of *real* functions or distributions on $\mathbf{T}^d$, we write

$$\mathbb{V}(\mathbf{T}^d) \equiv \mathbb{V} := \{ v = (v_1,...,v_d) \mid v_r \in V \quad \text{for all } r \} \ . \quad (2.12)$$

In this way we can define, e.g., the spaces $\mathbb{D}'(\mathbf{T}^d) \equiv \mathbb{D}'$, $\mathbb{L}^p(\mathbf{T}^d) \equiv \mathbb{L}^p$ ($p \in [1,+\infty]$), $\mathbb{H}^n_{0}(\mathbf{T}^d) \equiv \mathbb{H}^n_{0}$. Any $v = (v_r) \in \mathbb{D}'$ is referred to as a (distributional) *vector field* on $\mathbf{T}^d$. We note that v has a unique Fourier series expansion (2.4) with coefficients

$$v_k := (v_{rk})_{r=1,...,d} \in \mathbf{C}^d \ , \qquad v_{rk} := \langle v_r, e_{-k} \rangle \ ; \quad (2.13)$$

as in the scalar case, the reality of v ensures $\overline{v_k} = v_{-k}$.
$\mathbb{L}^2$ is a real Hilbert space, with the inner product and the norm

$$\langle v|w\rangle_{L^2} := \int_{\mathbf{T}^d} v(x) \bullet w(x) dx = \sum_{k\in\mathbf{Z}^d} \overline{v_k} \bullet w_k \ , \qquad \|v\|_{L^2} := \sqrt{\langle v|v\rangle_{L^2}} \ . \quad (2.14)$$

We define componentwise the mean $\langle v \rangle \in \mathbf{R}^d$ of any $v \in \mathbb{D}'$ (see Eq. (2.5)); $\mathbb{D}'_{0}$ is the space of zero mean vector fields, and $\mathbb{L}^p_0 = \mathbb{L}^p \cap \mathbb{D}'_{0}$.
We similarly define componentwise the operators $\partial_s, \Delta, \sqrt{-\Delta}^{\,n} : \mathbb{D}' \to \mathbb{D}'_{0}$.
For any real n, the n-th Sobolev space of zero mean vector fields $\mathbb{H}^n_{0}(\mathbf{T}^d) \equiv \mathbb{H}^n_{0}$ is made of all d-uples v with components $v_r \in H^n_{0}$; an equivalent definition can be

given via Eq.(2.11), replacing therein L^2 with $\mathbb{L}^2$. $\mathbb{H}^n_{0}$ is a real Hilbert space with the inner product and the induced norm

$$\langle v|w\rangle_n := \langle \sqrt{-\Delta}^{\,n} v\,|\sqrt{-\Delta}^{\,n} w\rangle_{L^2} = \sum_{k\in\mathbf{Z}^d_0} |k|^{2n}\, \overline{v_k}\bullet w_k \ , \qquad (2.15)$$

$$\|v\|_n = \|\sqrt{-\Delta}^{\,n} v\|_{L^2} = \sqrt{\sum_{k\in\mathbf{Z}^d_0} |k|^{2n}\, |v_k|^2} \ .$$

Divergence free vector fields. Let $\mathrm{div} : \mathbb{D}' \to D'_{0}$, $v \mapsto \mathrm{div}\, v := \sum_{r=1}^d \partial_r v_r = i \sum_{k\in\mathbf{Z}^d_0}(k\bullet v_k)e_k$. Hereafter we introduce the space $\mathbb{D}'_{\Sigma}$ of *divergence free (or solenoidal) vector fields* and some subspaces of it, putting

$$\mathbb{D}'_{\Sigma} := \{v \in \mathbb{D}' \mid \mathrm{div}\, v = 0\} = \{v \in \mathbb{D}' \mid k\bullet v_k = 0 \ \forall k \in \mathbf{Z}^d\ \} \ ; \qquad (2.16)$$

$$\mathbb{D}'_{\Sigma 0} := \mathbb{D}'_{\Sigma} \cap \mathbb{D}'_{0} \ , \quad \mathbb{L}^p_{\Sigma} := \mathbb{L}^p \cap \mathbb{D}'_{\Sigma} \ , \ \mathbb{L}^p_{\Sigma 0} := \mathbb{L}^p \cap \mathbb{D}'_{\Sigma 0} \quad (p \in [1, +\infty]) \ , \qquad (2.17)$$

$$\mathbb{H}^n_{\Sigma 0} := \mathbb{D}'_{\Sigma} \cap \mathbb{H}^n_{0} \quad (n \in \mathbf{R}). \qquad (2.18)$$

$\mathbb{H}^n_{\Sigma 0}$ is a closed subspace of the Hilbert space $\mathbb{H}^n_{0}$, that we equip with the restrictions of $\langle\ |\ \rangle_n$, $\|\ \|_n$. The *Leray projection* is the (surjective) map

$$\mathfrak{L} : \mathbb{D}' \to \mathbb{D}'_{\Sigma} \ , \qquad v \mapsto \mathfrak{L}v := \sum_{k\in\mathbf{Z}^d} (\mathfrak{L}_k v_k) e_k \ , \qquad (2.19)$$

where, for each k, $\mathfrak{L}_k$ is the orthogonal projection of $\mathbf{C}^d$ onto the orthogonal complement of k; more explicitly, if $c \in \mathbf{C}^d$,

$$\mathfrak{L}_0 c = c \ , \qquad \mathfrak{L}_k c = c - \frac{k\bullet c}{|k|^2}\, k \quad \text{for } k \in \mathbf{Z}^d_0 \ . \qquad (2.20)$$

From the Fourier representations of $\mathfrak{L}$, $\langle\ \rangle$, etc., one easily infers

$$\langle \mathfrak{L}v\rangle = \langle v\rangle \text{ for } v \in \mathbb{D}', \quad \mathfrak{L}\mathbb{D}'_{0} = \mathbb{D}'_{\Sigma 0}, \quad \mathfrak{L}\mathbb{L}^2 = \mathbb{L}^2_{\Sigma}, \quad \mathfrak{L}\mathbb{H}^n_{0} = \mathbb{H}^n_{\Sigma 0} \text{ for } n \in \mathbf{R} \ . \quad (2.21)$$

Furthermore, $\mathfrak{L}$ is an orthogonal projection in each one of the Hilbert spaces $\mathbb{L}^2$, $\mathbb{H}^n_{0}$; in particular,

$$\|\mathfrak{L}v\|_n \leqslant \|v\|_n \qquad \text{for } v \in \mathbb{H}^n_{0} \ . \qquad (2.22)$$

Making contact with the Euler/NS equations. The quadratic nonlinearity in the Euler/NS equations is related to the bilinear map sending two (sufficiently regular) vector fields v, w on $\mathbf{T}^d$ into $v\bullet\partial w$; we are now ready to discuss this map. Hereafter we often refer to the case

$$v \in \mathbb{L}^2 \ , \quad \partial_s w \in \mathbb{L}^2 \ \ (s = 1, ..., d) \ ; \qquad (2.23)$$

the above condition on the derivatives of w implies $w \in \mathbb{L}^2$.
The results mentioned in the sequel are known: the proofs of Lemmas 2.1, 2.2 are found, e.g., in [10], and the proof of Lemma 2.3 is reported only for completeness.

2.1 *Lemma.* *For v, w as in (2.23), consider the vector field $v{\scriptstyle\bullet}\partial w$ on $\mathbf{T}^d$, of components*

$$(v{\scriptstyle\bullet}\partial w)_r := \sum_{s=1}^{d} v_s \partial_s w_r \ ; \tag{2.24}$$

this is well defined and belongs to $\mathbb{L}^1$. With the additional assumption $\operatorname{div} v = 0$, one has $\langle v{\scriptstyle\bullet}\partial w\rangle = 0$ (which also implies $\langle \mathfrak{L}(v{\scriptstyle\bullet}\partial w)\rangle = 0$, see (2.21)).

2.2 *Lemma.* *Assuming (2.23), $v{\scriptstyle\bullet}\partial w$ has Fourier coefficients*

$$(v{\scriptstyle\bullet}\partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d} [v_h {\scriptstyle\bullet} (k-h)] w_{k-h} \qquad \textit{for all } k \in \mathbf{Z}^d \ . \tag{2.25}$$

2.3 *Lemma.* *Besides (2.23), assume $\operatorname{div} v = 0$ and $v{\scriptstyle\bullet}\partial w \in \mathbb{L}^2$. Then*

$$\langle v{\scriptstyle\bullet}\partial w | w \rangle_{L^2} = 0 \ . \tag{2.26}$$

Proof. Suppose for a moment that $v, w : \mathbf{T}^d \to \mathbf{R}^d$ are C^1, with no other condition; then (integrating by parts in one passage)

$$\langle v{\scriptstyle\bullet}\partial w | w \rangle_{L^2} = \sum_{r,s=1}^{d} \int_{\mathbf{T}^d} v_s (\partial_s w_r) w_r \, dx = \frac{1}{2} \sum_{r,s=1}^{d} \int_{\mathbf{T}^d} v_s \partial_s (w_r^2) \, dx$$

$$= -\frac{1}{2} \sum_{r,s=1}^{d} \int_{\mathbf{T}^d} (\partial_s v_s) w_r^2 \, dx = -\frac{1}{2} \int_{\mathbf{T}^d} (\operatorname{div} v) |w|^2 \, dx \ .$$

In particular, (2.26) holds if v, w are C^1 and $\operatorname{div} v = 0$. By a density argument, one extends (2.26) to all v, w as in the statement of the Lemma. □

The following result, essential for the sequel, is also well known (see, e.g., [10]).

2.4 *Proposition.* *Let $n \in (d/2, +\infty)$. If $v \in \mathbb{H}^n_{\Sigma 0}$ and $w \in \mathbb{H}^{n+1}_0$, one has $v{\scriptstyle\bullet}\partial w \in \mathbb{H}^n_0$. Furthermore, the map $(v, w) \mapsto v{\scriptstyle\bullet}\partial w$ is bilinear and continuous between the spaces mentioned before.*

3 The Kato inequality

Throughout this section we assume

$$n \in (\frac{d}{2} + 1, +\infty) \ . \tag{3.1}$$

The following Proposition 3.1 is known, dating back to [6] (see [5] for a more general formulation, similar to the one proposed hereafter). As a matter of fact, the quantitative analysis presented later in this paper also gives, as a byproduct, an alternative proof of this Proposition.

3.1 ***Proposition.*** *Let $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_0$ (so that $v {\scriptstyle\bullet} \partial w \in \mathbb{H}^n_0$). Then, there is $G' \in [0, +\infty)$, independent of v, w, such that*

$$|\langle v {\scriptstyle\bullet} \partial w | w \rangle_n| \leqslant G' \|v\|_n \|w\|^2_n \ . \tag{3.2}$$

3.2 ***Definition.*** *We put*

$$G'_{nd} \equiv G'_n \tag{3.3}$$

$$:= \min\{G' \in [0, +\infty) \mid |\langle v {\scriptstyle\bullet} \partial w | w \rangle_n| \leqslant G' \|v\|_n \|w\|^2_n \textit{ for all } v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_0\} \ ;$$

$$G_{nd} \equiv G_n \tag{3.4}$$

$$:= \min\{G \in [0, +\infty) \mid |\langle v {\scriptstyle\bullet} \partial w | w \rangle_n| \leqslant G \|v\|_n \|w\|^2_n \textit{ for all } v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0}\} \ .$$

(Note that all w's in (3.4) are divergence free, a property not required in (3.3).)

With the language of the Introduction, G'_n and G_n are, respectively, the sharp constants in the "auxiliary Kato inequality" (1.6) and in the Kato inequality (1.7); we recall that $\mathfrak{L}$ could be inserted into (3.4), due to the relation (1.8) $\langle v {\scriptstyle\bullet} \partial w | w \rangle_n = \langle \mathfrak{L}(v {\scriptstyle\bullet} \partial w) | w \rangle_n$. It is obvious that

$$G_n \leqslant G'_n \ ; \tag{3.5}$$

in the rest of the section (which is its original part) we present computable upper and lower bounds on G'_n and G_n, respectively.
The upper bound requires a more lengthy analysis; the final result relies on a function $\mathcal{G}_{nd} \equiv \mathcal{G}_n$, appearing in the forthcoming Definition 3.3. To build this function, as in [10] we refer to the exterior power $\bigwedge^2 \mathbf{R}^d$, identified with the space of real, skew-symmetric $d \times d$ matrices $A = (A_{rs})_{r,s,=1,\dots,d}$. We consider the (bilinear, skew-symmetric) operation $\wedge$ and the norm $|\ |$ defined by

$$\wedge : \mathbf{R}^d \times \mathbf{R}^d \to \bigwedge^2 \mathbf{R}^d, \quad (p, q) \mapsto p \wedge q \ \text{ s.t. } \ (p \wedge q)_{rs} := p_r q_s - q_r p_s \ ; \tag{3.6}$$

$$|\ | : \bigwedge^2 \mathbf{R}^d \to [0, +\infty), \qquad A = (A_{rs}) \mapsto |A| := \sqrt{\frac{1}{2} \sum_{r,s=1}^{d} |A_{rs}|^2} \ . \tag{3.7}$$

In the sequel, for $p, q \in \mathbf{R}^d$, we often use the relations

$$|p \wedge q| = \sqrt{|p|^2 |q|^2 - (p {\scriptstyle\bullet} q)^2} = |p||q| \sin \vartheta \ , \tag{3.8}$$

where $\vartheta \equiv \vartheta(p, q) \in [0, \pi]$ is the convex angle between p and q (defined arbitrarily, if $p = 0$ or $q = 0$); we use as well the inequality

$$|p \wedge q| \leqslant |p||q| \ . \tag{3.9}$$

Keeping in mind these facts, let us stipulate the following.

3.3 ***Definition.*** *We put*

$$\mathbf{Z}^d_{0k} := \mathbf{Z}^d \setminus \{0, k\} \qquad \text{for each } k \in \mathbf{Z}^d_0 \; ; \tag{3.10}$$

$$\mathcal{G}_{nd} \equiv \mathcal{G}_n : \mathbf{Z}^d_0 \to (0, +\infty), \quad k \mapsto \mathcal{G}_n(k) := \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|^2 (|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} \; . \tag{3.11}$$

3.4 Remarks. (i) For any $k \in \mathbf{Z}^d_0$ one has $\mathcal{G}_n(k) < +\infty$, as stated above, since

$$\frac{|h \wedge k|^2 (|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} = O(\frac{1}{|h|^{2n}}) \qquad \text{for } h \to \infty \; , \tag{3.12}$$

and $2n > d$.
(ii) Consider the reflection operators $R_r(k_1, .., k_r, ..., k_d) := (k_1, ..., -k_r, ..., k_d)$ ($r = 1, ..., d$) and the permutation operators $P_\sigma(k_1, ..., k_d) := (k_{\sigma(1)}, ..., k_{\sigma(d)})$ (σ a permutation of $\{1, ..., d\}$); then

$$\mathcal{G}_n(R_r k) = \mathcal{G}_n(k) \; , \quad \mathcal{G}_n(P_\sigma k) = \mathcal{G}_n(k) \qquad \text{for each } k \in \mathbf{Z}^d_0 \; . \tag{3.13}$$

The proof is very similar to the one employed for the analogous properties of the function $\mathcal{K}_n$ appearing in [10].
(iii) In Appendix B we will prove that

$$\sup_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k) < +\infty \; , \tag{3.14}$$

and give tools for the practical evaluation of $\mathcal{G}_{nd}$ and of its sup. □

The main result of the present section is the following.

3.5 ***Proposition.*** *The constant G'_n defined by (3.4) has the upper bound*

$$G'_n \leqslant G^+_n \; , \tag{3.15}$$

$$G_n := \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k)} \; \textit{(or any approximant for this).} \tag{3.16}$$

Proof. See Section 4. □

The practical calculation of the above upper bound is made possible by a general method, illustrated in Appendix B; the results of such calculations, for $d = 3$ and some illustrative choices of n, are reported at the end of this section.
Let us pass to the problem of finding a lower bound for the constant G_n; this can be obtained directly from the tautological inequality

$$G_n \geqslant \frac{|\langle v \bullet \partial w | w \rangle_n|}{\|v\|_n \|w\|^2_n} \qquad \text{for } v \in \mathbb{H}^n_{\Sigma 0} \setminus \{0\}, \, w \in \mathbb{H}^{n+1}_{\Sigma 0} \setminus \{0\} \; , \tag{3.17}$$

choosing for v and w two suitable non zero "trial functions"; hereafter we consider a choice where $v_k = 0$ for $k \in \mathbf{Z}_0^d \setminus V$ and $w_k = 0$ for $k \in \mathbf{Z}_0^d \setminus W$ with V, W two finite sets. For the sake of brevity in the exposition of the final result, let us stipulate the following.

3.6 *Definition.* *We put*

$$\mathcal{H}_d \equiv \mathcal{H} := \{(u_k)_{k\in U} \mid U \subset \mathbf{Z}_0^d \text{ finite}, -U = U; \tag{3.18}$$

$$u_k \in \mathbf{C}^d,\ \overline{u_k} = u_{-k},\ k \bullet u_k = 0 \text{ for all } k \in U\}$$

(the set U can depend on the family (u_k), and $-U := \{-k \mid k \in U\}$). □

3.7 *Proposition.* *Consider two nonzero families $(v_k)_{k\in V}$, $(w_k)_{k\in W} \in \mathcal{H}$; these give the lower bound*

$$G_n \geqslant G_n^- \ , \tag{3.19}$$

where

$$G_n^- := \frac{1}{(2\pi)^{d/2}} \frac{|P_n((v_k),(w_k))|}{N_n((v_k))N_n^2((w_k))} \text{ (or any lower approximant for this)} \ , \tag{3.20}$$

$$N_n((v_k)) := \Big(\sum_{k\in V} |k|^{2n}|v_k|^2\Big)^{1/2}, \qquad N_n((w_k)) := \Big(\sum_{k\in V} |k|^{2n}|w_k|^2\Big)^{1/2} \ ,$$

$$P_n((v_k),(w_k)) := -i \sum_{h\in V, \ell\in W, h+\ell\in W} |h+\ell|^{2n}(\overline{v_h}\bullet\ell)(\overline{w_\ell}\bullet w_{h+\ell}) \ .$$

Proof. See Section 4. Here, we anticipate the main idea: the vector fields $v := \sum_{k\in V} v_k e_k$, $w := \sum_{k\in W} w_k e_k$ belong to $\mathbb{H}^m_{\Sigma 0}$ for each real m, and $\|v\|_n = N_n((v_k))$, $\|w\|_n = N_n((w_k))$, $\langle v\bullet\partial w|w\rangle_n = (2\pi)^{-d/2} P_n((v_k),(w_k))$; so, (3.19) is just the relation (3.17) for this choice of v, w. □

Putting together Eqs. (3.5) (3.15) (3.19) we obtain a chain of inequalities, anticipated in the Introduction,

$$G_n^- \leqslant G_n \leqslant G_n' \leqslant G_n^+ \ ;$$

here, the bounds $G_n^\pm$ can be computed explicitly from their definitions (3.16) (3.20).

3.8 Examples. For $d = 3$ and $n = 3, 4, 5, 10$, Eq. (3.16) and Eq. (3.20) (with suitable choices of (v_k), (w_k)), give

$$G_3^- = 0.114\,,\ G_3^+ = 0.438\,; \qquad G_4^- = 0.181\,,\ G_4^+ = 0.484\,; \tag{3.21}$$

$$G_5^- = 0.280\,,\ G_5^+ = 0.749\,; \qquad G_{10}^- = 2.41\,,\ G_{10}^+ = 7.56$$

(see Appendices C and D for the upper and lower bounds, respectively). In the above, the ratios G_n^-/G_n^+ are 0.260..., 0.373..., 0.373..., 0.318... for $n = 3, 4, 5, 10$, respectively. To avoid misunderstandings related to these examples, we repeat that the approach of this paper applies as well to noninteger values of n.

4 Proof of Propositions (3.1 and) 3.5, 3.7

For the reader's convenience, we report a Lemma from [10].

4.1 ***Lemma.*** *Let*

$$p, q \in \mathbf{R}^d \setminus \{0\}\ , \quad z \in \mathbf{C}^d,\ p \bullet z = 0\ , \tag{4.1}$$

and $\vartheta(p,q) \equiv \vartheta \in [0,\pi]$ *be the convex angle between* q *and* p*. Then*

$$|q \bullet z| \leqslant \sin\vartheta\, |q||z| = \frac{|p \wedge q|}{|p|}\, |z|\ . \tag{4.2}$$

From now on, $n \in (\frac{d}{2}+1, +\infty)$. Hereafter we present an argument proving (Proposition 3.1 and, simultaneously) Proposition 3.5. This is divided in several steps; in particular, Step 1 relies on an idea of Constantin and Foias [5]. These authors use their idea to obtain a proof of the Kato inequalities, but are not interested in the quantitative evaluation of the sharp constants therein; our forthcoming argument can be regarded as a refined, fully quantitative version of their approach, developed for the specific purpose to estimate G'_n.

Proof of Propositions 3.1, 3.5. We choose $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_0$ and proceed in some steps.

Step 1. We have $v \in \mathbb{L}^\infty_{\Sigma 0}$, $\sqrt{-\Delta}^n w \in \mathbb{H}^1_0$, $v \bullet \partial(\sqrt{-\Delta}^n w) \in \mathbb{L}^2_0$, $v \bullet \partial w \in \mathbb{H}^n_0$ *and* $\sqrt{-\Delta}^n (v \bullet \partial w) \in \mathbb{L}^2_0$*; furthermore, the vector field*

$$z := \sqrt{-\Delta}^n (v \bullet \partial w) - v \bullet \partial(\sqrt{-\Delta}^n w) \in \mathbb{L}^2_0 \tag{4.3}$$

fulfills the equality

$$\langle v \bullet \partial w | w \rangle_n = \langle z | \sqrt{-\Delta}^n w \rangle_{L^2}\ , \tag{4.4}$$

which implies

$$|\langle v \bullet \partial w | w \rangle_n| \leqslant \|z\|_{L^2} \|w\|_n\ . \tag{4.5}$$

To prove all this, we first recall the Sobolev imbedding $H^n_0 \subset L^\infty$, holding because $n > d/2$ (see, e.g., [2]); this obviously implies $\mathbb{H}^n_{\Sigma 0} \subset \mathbb{L}^\infty_{\Sigma 0}$, so $v \in \mathbb{L}^\infty_{\Sigma 0}$. Of course, $\sqrt{-\Delta}^n$ sends $\mathbb{H}^{n+1}_0$ into $\mathbb{H}^1_0$, thus $\sqrt{-\Delta}^n w \equiv u \in \mathbb{H}^1_0$. This implies $\partial_s u_r \in L^2$

that, with $v_s \in L^\infty$, gives $(v\bullet\partial u)_r = \sum_{s=1}^d v_s \partial_s u_r \in L^2$. Summing up, $v\bullet\partial u \in \mathbb{L}^2$; furthermore, $v\bullet\partial u \in \mathbb{L}^2_0$ due to Lemma 2.1. The statement $v\bullet\partial w \in \mathbb{H}^n_0$ holds due to Proposition 2.4; since $\sqrt{-\Delta}^n$ sends $\mathbb{H}^n_0$ into $\mathbb{H}^0_0 = \mathbb{L}^2_0$, we finally obtain $\sqrt{-\Delta}^n(v\bullet\partial w) \in \mathbb{L}^2_0$.
To go on, we note that

$$\langle v\bullet\partial w | w\rangle_n = \langle \sqrt{-\Delta}^n(v\bullet\partial w) | \sqrt{-\Delta}^n w\rangle_{L^2}$$
$$= \langle \sqrt{-\Delta}^n(v\bullet\partial w) - v\bullet\partial(\sqrt{-\Delta}^n w) | \sqrt{-\Delta}^n w\rangle_{L^2} = \langle z | \sqrt{-\Delta}^n w\rangle_{L^2} \ .$$

In the above: the first equality corresponds to the definition of $\langle \ | \ \rangle_n$, the second one holds because $\langle v\bullet\partial(\sqrt{-\Delta}^n w) | \sqrt{-\Delta}^n w\rangle_{L^2} = 0$ by Lemma 2.3 (here applied to the vector fields $v, \sqrt{-\Delta}^n w$); the last equality corresponds to the definition of z, and proves Eq. (4.4). Now, the Schwartz inequality yields $|\langle v\bullet\partial w | w\rangle_n| \leqslant \|z\|_{L^2} \|\sqrt{-\Delta}^n w\|_{L^2} = \|z\|_{L^2}\|w\|_n$, as in (4.5).
Step 2. The vector field z in (4.3) has Fourier coefficients

$$z_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} [v_h \bullet (k-h)](|k|^n - |k-h|^n) w_{k-h} \quad \textit{for all } k \in \mathbf{Z}^d_0 \ . \tag{4.6}$$

To prove this, let us start from the Fourier coefficients of $v\bullet\partial w$; this has zero mean, so $(v\bullet\partial w)_0 = 0$. The other coefficients are

$$(v\bullet\partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} [v_h \bullet (k-h)] w_{k-h} \qquad \text{for all } k \in \mathbf{Z}^d_0 \ ; \tag{4.7}$$

this follows from (2.25) taking into account that, in the sum therein, the term with $h = 0$ vanishes due to $v_0 = 0$, and the term with $h = k$ is zero for evident reasons. Consider any $k \in \mathbf{Z}^d_0$; Eq. (4.7) implies

$$[\sqrt{-\Delta}^n(v\bullet\partial w)]_k = |k|^n (v\bullet\partial w)_k = \frac{i|k|^n}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} [v_h \bullet (k-h)] w_{k-h} \ . \tag{4.8}$$

The analogue of Eq. (4.7) for the pair $v, \sqrt{-\Delta}^n w$ reads $[v\bullet\partial(\sqrt{-\Delta}^n w)]_k = i(2\pi)^{-d/2} \sum_{h \in \mathbf{Z}^d_{0k}} [v_h \bullet (k-h)](\sqrt{-\Delta}^n w)_{k-h}$, i.e.,

$$[v\bullet\partial(\sqrt{-\Delta}^n w)]_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} [v_h \bullet (k-h)] |k-h|^n w_{k-h} \ . \tag{4.9}$$

Subtracting (4.9) from (4.8), we obtain the thesis (4.6).
Step 3. Estimating the Fourier coefficients of z. Let $k \in \mathbf{Z}^d_0$; Eq. (4.6) implies

$$|z_k| \leqslant \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} |v_h \bullet (k-h)| \, \Big| |k|^n - |k-h|^n \Big| \, |w_{k-h}| \ . \tag{4.10}$$

To go on, we note that $h{\scriptstyle\bullet}v_h = 0$ due to the assumption $\mathrm{div}\, v = 0$; so, we can apply Eq. (4.2) with $p = h$, $q = k - h$ and $z = v_h$, which gives

$$|v_h{\scriptstyle\bullet}(k-h)| \leqslant \frac{|h \wedge (k-h)|}{|h|}|v_h| = \frac{|h \wedge k|}{|h|}|v_h| \tag{4.11}$$

(recall that $h \wedge (k-h) = h \wedge k$). Inserting the inequality (4.11) into (4.10), we get

$$|z_k| \leqslant \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|}{|h|}|v_h| \,\Big| |k|^n - |k-h|^n \Big| \,|w_{k-h}| \tag{4.12}$$

$$= \frac{1}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k| \Big||k|^n - |k-h|^n\Big|}{|h|^{n+1}|k-h|^n} \Big(|h|^n|v_h||k-h|^n|w_{k-h}|\Big) \ .$$

Now, Hölder's inequality $|\sum_h \ a_h b_h|^2 \leqslant \Big(\sum_h |a_h|^2\Big)\Big(\sum_h \ |b_h|^2\Big)$ gives

$$|z_k|^2 \leqslant \frac{1}{(2\pi)^d}\, \mathcal{G}_n(k)\mathcal{Q}_n(k) \text{ for all } k \in \mathbf{Z}^d_0, \tag{4.13}$$

$$\mathcal{G}_n(k) := \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|^2(|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} \quad \text{as in (3.11)},$$

$$\mathcal{Q}_n(k) \equiv \mathcal{Q}_n(v,w)(k) := \sum_{h \in \mathbf{Z}^d_{0k}} |h|^{2n}|v_h|^2|k-h|^{2n}|w_{k-h}|^2$$

(in the definition of $\mathcal{Q}_n(k)$ one can write as well $\sum_{h \in \mathbf{Z}^d_0}$, since the general term of the sum vanishes for $h = k$).

Step 4. Estimates on $\|z\|_{L^2}$. Eq. (4.13) implies

$$\|z_n\|^2_{L^2} = \sum_{k \in \mathbf{Z}^d_0} |z_k|^2 \leqslant \frac{1}{(2\pi)^d} \sum_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k)\mathcal{Q}_n(k) \ \leqslant \frac{1}{(2\pi)^d}\Big(\sup_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k)\Big)\Big(\sum_{k \in \mathbf{Z}^d_0} \mathcal{Q}_n(k)\Big) \ .$$

The sup of $\mathcal{G}_n$ is finite, as we will show (by an independent argument) in Proposition B.1; making reference to the definition of G^+_n in terms of this sup (see Eq. (3.16)), we can write the last result as

$$\|z_n\|^2_{L^2} \leqslant (G^+_n)^2 \sum_{k \in \mathbf{Z}^d_0} \mathcal{Q}_n(k) \ . \tag{4.14}$$

On the other hand,

$$\sum_{k \in \mathbf{Z}^d_0} \mathcal{Q}_n(k) = \sum_{h \in \mathbf{Z}^d_0} |h|^{2n}|v_h|^2 \sum_{k \in \mathbf{Z}^d_0} |k-h|^{2n}|w_{k-h}|^2 = \sum_{h \in \mathbf{Z}^d_0} |h|^{2n}|v_h|^2 \sum_{\ell \in \mathbf{Z}^d_{0h}} |\ell|^{2n}|w_\ell|^2$$

$$\leqslant \sum_{h \in \mathbf{Z}_0^d} |h|^{2n} |v_h|^2 \sum_{\ell \in \mathbf{Z}_0^d} |\ell|^{2n} |w_\ell|^2 = \|v\|_n^2 \|w\|_n^2 . \tag{4.15}$$

Inserting this result into (4.14), we obtain

$$\|z\|_{L^2} \leqslant G_n^+ \|v\|_n \|w\|_n . \tag{4.16}$$

Step 5. Concluding the proofs of Propositions 3.1, 3.5. Eqs. (4.5) (4.16) imply

$$|\langle v \bullet \partial w | w \rangle_n| \leqslant G_n^+ \|v\|_n \|w\|_n^2 ; \tag{4.17}$$

so, Proposition 3.1 is proved. Eq. (4.17) also indicates that the sharp constant G'_n in (3.3) fulfills $G'_n \leqslant G_n^+$; this proves Eq. (3.15) and Proposition 3.5. □

We conclude this section proving the statements of Section 3 on the lower bounds G_n^-.

Proof of Proposition 3.7. Let us recall the definition (3.18) of $\mathcal{H}$; our argument is divided in some steps.
Step 1. Let $(u_k)_{k \in U} \in \mathcal{H}$. Then,

$$u := \sum_{k \in U} u_k e_k \tag{4.18}$$

belongs to $\mathbb{H}^m_{\Sigma 0}$ for each real m, and

$$\|u\|_m = \left(\sum_{k \in U} |k|^{2m} |u_k|^2 \right)^{1/2} \equiv N_m((u_k)) . \tag{4.19}$$

These statements are self-evident; of course, the conditions $\overline{u_k} = u_{-k}$ and $k \bullet u_k = 0$ in (3.18) ensure u to be real, and divergence free.
Step 2. Consider two families $(v_k)_{k \in V}$, $(w_k)_{k \in W} \in \mathcal{H}$, and define $v := \sum_{k \in V} v_k e_k$, $w := \sum_{k \in W} w_k e_k$. Then

$$\langle v \bullet \partial w | w \rangle_n = \frac{1}{(2\pi)^{d/2}} P_n((v_k), (w_k)) \tag{4.20}$$

where, as in (3.20), $P_n(v, w) := -i \sum_{h \in V, \ell \in W, h+\ell \in W} |h + \ell|^{2n} (\overline{v_h} \bullet \ell)(\overline{w_\ell} \bullet w_{h+\ell})$.
In fact, the Fourier coefficients of $v \bullet \partial w$ have the expression (2.25)

$$(v \bullet \partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d} [v_h \bullet (k - h)] w_{k-h} ;$$

this implies

$$\langle v{\bullet}\partial w | w\rangle_n = \sum_{k\in\mathbf{Z}^d} |k|^{2n} \overline{(v{\bullet}\partial w)_k} \bullet w_k \tag{4.21}$$

$$= -\frac{i}{(2\pi)^{d/2}} \sum_{h,k\in\mathbf{Z}^d} |k|^{2n} [\overline{v_h}\bullet (k-h)](\overline{w_{k-h}}\bullet w_k) \; = -\frac{i}{(2\pi)^{d/2}} \sum_{h,\ell\in\mathbf{Z}^d} |h+\ell|^{2n} (\overline{v_h}\bullet \ell)(\overline{w_\ell}\bullet w_{h+\ell})$$

$$= -\frac{i}{(2\pi)^{d/2}} \sum_{h\in V,\ell\in W,h+\ell\in W} |h+\ell|^{2n} (\overline{v_h}\bullet \ell)(\overline{w_\ell}\bullet w_{h+\ell}) = -\frac{i}{(2\pi)^{d/2}} P_n((v)_k,(w)_k) \ ,$$

which proves the thesis (4.20). In the above chain of equalities, the third passage relies on a change of variable $k = h + \ell$, and the fourth passage depends on the relations $v_h = 0$ for $h \in \mathbf{Z}^d \setminus V$, $w_\ell = 0$ for $\ell \in \mathbf{Z}^d \setminus W$.

Step 3. Conclusion of the proof. We consider two nonzero families $(v_k)_{k\in V}$, $(w_k)_{k\in W} \in \mathcal{H}$, and define $v := \sum_{k\in V} v_k e_k$, $w := \sum_{k\in W} w_k e_k$. According to Steps 1 and 2, we have $\|v\|_n = N_n((v_k))$, $\|w\|_n = N_n((w_k))$, $\langle v{\bullet}\partial w|w\rangle_n = (2\pi)^{-d/2} P_n((v_k),(w_k))$; so, the inequality $G_n \geqslant |\langle v{\bullet}\partial w|w\rangle_n|/\|v\|_n\|w\|^2_n$ takes the form (3.19-3.20). □

A Some tools preparing the analysis of the function $\mathcal{G}_n$

In the sequel $d \in \{2, 3, ...\}$. Let us fix some notations, to be used throughout the Appendices.

A.1 ***Definition.*** *(i) $\theta : \mathbf{R} \to \{0, 1\}$ is the Heaviside function such that $\theta(z) := 1$ if $z \in [0, +\infty)$ and $\theta(z) := 0$ if $z \in (-\infty, 0)$.*
(ii) Γ is the Euler Gamma function, $\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$ are the binomial coefficients.
(iii) We put $\mathbf{S}^{d-1} := \{u \in \mathbf{R}^d \mid |u| = 1\}$. For each $p \in \mathbf{R}^d \setminus \{0\}$, the versor of p is $\widehat{p} := \dfrac{p}{|p|} \in \mathbf{S}^{d-1}$.

A.2 ***Lemma.*** *For any function $f : \mathbf{Z}^d_0 \to \mathbf{R}$ and $k \in \mathbf{Z}^d_0$, $\rho \in (1, +\infty)$, one has*

$$\sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho \, or \, |k-h|<\rho} f(h) = \sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho} f(h) + \theta(|k-h| - \rho) f(k-h) \ . \tag{A.1}$$

Proof. See [10]. □

A.3 ***Lemma.*** *For any $n \in (1, +\infty)$, the following holds.*
(i) Consider the function

$$c_n : [0, 4] \times [0, 1] \to [0, +\infty) \tag{A.2}$$

$$c_n(z, u) := \begin{cases} \dfrac{z(4-z)[(1 - zu + zu^2)^{n/2} - (1-u)^n]^2}{2u^2[u^{2n-2} + (1-u)^{2n-2}]} & \text{if } u \in (0, 1] \ , \\ \dfrac{n^2 z(4-z)(2-z)^2}{8} & \text{if } u = 0 \ . \end{cases}$$

This is well defined and continuous, which implies existence of

$$C_n := \max_{z \in [0,4], u \in [0,1]} c_n(z, u) \in (0, +\infty) \ . \tag{A.3}$$

(ii) For all $p, q \in \mathbf{R}^d$, one has

$$|p \wedge q|^2 (|p + q|^n - |q|^n)^2 \leqslant \frac{C_n}{2} |p|^4 |q|^2 \Big[|p|^{2n-2} + |q|^{2n-2} \Big] \ . \tag{A.4}$$

Proof. (i) Well definedness and continuity of c_n are checked by elementary means, the main point being the computation of $\lim_{u\to 0^+} c_n(z,u)$.
(ii) Eq. (A.4) is obvious if $p = 0$ or $q = 0$, due to the vanishing of both sides; hereafter we prove (A.4) for $p, q \in \mathbf{R}^d \setminus \{0\}$. Let $\vartheta(p,q) \equiv \vartheta \in [0,\pi]$ denote the convex angle between p and q; we have the relations

$$|p \wedge q|^2 = |p|^2 |q|^2 \sin^2\vartheta \ , \qquad |p+q|^2 = |p|^2 + |q|^2 + 2|p||q|\cos\vartheta \ ,$$

which imply

$$\frac{2|p\wedge q|^2(|p+q|^n - |q|^n)^2}{|p|^4|q|^2\Big[|p|^{2n-2} + |q|^{2n-2}\Big]} = \frac{2\sin^2\vartheta[(|p|^2+|q|^2+2|p||q|\cos\vartheta)^{n/2} - |q|^n]^2}{|p|^2\Big[|p|^{2n-2}+|q|^{2n-2}\Big]} \ . \tag{A.5}$$

To go on, we define $z \in [0,4]$, $u \in (0,1)$ through the equations

$$\cos\vartheta = 1 - \frac{z}{2} \ , \quad |p| = \frac{u}{1-u}|q| \tag{A.6}$$

(note that $|p| = \xi|q|$ for a unique $\xi \in (0,+\infty)$; on the other hand, the map $u \mapsto u/(1-u)$ is one-to-one between $(0,1)$ and $(0,+\infty)$). Returning to (A.5), after some computations we get

$$\frac{2|p\wedge q|^2(|p+q|^n - |q|^n)^2}{|p|^4|q|^2\Big[|p|^{2n-2}+|q|^{2n-2}\Big]} = c_n(z,u) \ . \tag{A.7}$$

But $c_n(z,u) \leqslant C_n$, so we obtain the thesis (A.4). □

A.4 Examples. Let c_n, C_n be defined as in the previous Lemma. For $n = 3,4,5,10$ we have the following numerical results, to be employed later:

$$C_3 = c_3(0.69603..., 0.46453...) = 14.814... \ ; \tag{A.8}$$

$$C_4 = c_4(0.61987..., 0.47822...) = 58.460... \ ;$$

$$C_5 = c_5(0.55023..., 0.48569...) = 215.97... \ ;$$

$$C_{10} = c_{10}(0.33289..., 0.49672...) = 1.3467... \times 10^5 \ .$$

A.5 ***Lemma.*** *Let $\nu \in (d,+\infty)$. For any $\rho \in (2\sqrt{d},+\infty)$, one has*

$$\sum_{h\in\mathbf{Z}^d, |h|\geqslant\rho} \frac{1}{|h|^\nu} \leqslant \frac{2\pi^{d/2}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \binom{d-1}{i} \frac{d^{d/2-1/2-i/2}}{(\nu-i-1)(\rho-2\sqrt{d})^{\nu-i-1}} \ . \tag{A.9}$$

Proof. This is just Lemma C.2 of [9] (with the variable λ of the cited reference related to ρ by $\lambda = \rho - 2\sqrt{d}$). $\square$

A.6 ***Lemma.*** *Let $\rho \in (1, +\infty)$ and $\varphi : [1, \rho) \to \mathbf{R}$. Then, for each $k \in \mathbf{R}^d$,*

$$\sum_{h \in \mathbf{Z}^d_0, |h| < \rho} (h \bullet k)^2 \varphi(|h|) = \frac{|k|^2}{d} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} |h|^2 \varphi(|h|) \ . \tag{A.10}$$

Proof. See [10]. $\square$

A.7 ***Definition.*** *Let us introduce the domain*

$$\mathcal{E} := \{(c, \xi) \in \mathbf{R}^2 \mid c \in [-1, 1], \, \xi \in [0, +\infty), \, (c, \xi) \neq (1, 1)\} \ ; \tag{A.11}$$

furthermore, let $n \in \mathbf{R}$.
(i) We put

$$D_n : \mathcal{E} \to [0, +\infty) \ , \tag{A.12}$$

$$(c, \xi) \mapsto D_n(c, \xi) := \begin{cases} \dfrac{(1-c^2)[1-(1-2c\xi+\xi^2)^{n/2}]^2}{\xi^2(1-2c\xi+\xi^2)^n} & \textit{if } \xi \neq 0, \\ n^2(c^2 - c^4) & \textit{if } \xi = 0 \ ; \end{cases}$$

$$E_n : \mathcal{E} \to [0, +\infty) \ , \quad (c, \xi) \mapsto E_n(c, \xi) := \frac{1 - c^2}{(1 - 2c\xi + \xi^2)^{n+1}} \ . \tag{A.13}$$

(D_n is C^∞, as shown by an elementary analysis of the term $\xi^{-2}[1-(1-2c\xi+\xi^2)^{n/2}]^2$; E_n already appeared in [10], and is C^∞ as well.)
(ii) For $\ell = 0, 1, 2, ...$, we put

$$D_{n\ell}, E_{n\ell} : [-1, 1] \to \mathbf{R}, \ D_{n\ell}(c) := \frac{1}{\ell!} \frac{\partial^\ell D_n}{\partial \xi^\ell}(c, 0), \ E_{n\ell}(c) := \frac{1}{\ell!} \frac{\partial^\ell E_n}{\partial \xi^\ell}(c, 0) \ . \tag{A.14}$$

(iii) For $t = 1, 2, ...$,

$$Q_{nt}, R_{nt} : \mathcal{E} \to \mathbf{R} \tag{A.15}$$

are the unique C^∞ functions such that, for all $(c, \xi) \in \mathcal{E}$,

$$D_n(c, \xi) = \sum_{\ell=0}^{t-1} D_{n\ell}(c) \xi^\ell + Q_{nt}(c, \xi) \xi^t, \quad E_n(c, \xi) = \sum_{\ell=0}^{t-1} E_{n\ell}(c) \xi^\ell + R_{nt}(c, \xi) \xi^t. \tag{A.16}$$

(iv) For $t = 1, 2, ...$, we put

$$\lambda_{nt} := \min_{c \in [-1,1], \, \xi \in [0,1/2]} Q_{nt}(c, \xi) \ , \qquad \mu_{nt} := \min_{c \in [-1,1], \, \xi \in [0,1/2]} R_{nt}(c, \xi) \ , \tag{A.17}$$

$$\Lambda_{nt} := \max_{c \in [-1,1], \, \xi \in [0,1/2]} Q_{nt}(c, \xi) \ ; \qquad M_{nt} := \max_{c \in [-1,1], \, \xi \in [0,1/2]} R_{nt}(c, \xi) \ . \tag{A.18}$$

A.8 Remarks. (i) The first $D_{n\ell}$ functions are

$$D_{n0}(c) = n^2(c^2 - c^4), \qquad D_{n1}(c) = -n^2 c + (3n^2 + n^3)c^3 - (2n^2 + n^3)c^5, \quad \text{(A.19)}$$

$$D_{n2}(c) := \frac{n^2}{4} - (\frac{13}{4}n^2 + \frac{3}{2}n^3)c^2 + (\frac{20}{3}n^2 + \frac{9}{2}n^3 + \frac{7}{12}n^4)c^4 - (\frac{11}{3}n^2 + 3n^3 + \frac{7}{12}n^4)c^6.$$

The first $E_{n\ell}$ functions are reported in [10].
(ii) In general, $D_{n\ell}$ and $E_{n\ell}$ are polynomials in c of degrees $\ell+4$ and $\ell+2$, respectively; as functions of c, these have the same parity as ℓ.
(iii) Eq. (A.16) characterizes $Q_{nt}(c,\xi)\xi^t$ and $R_n(c,\xi)\xi^t$ as the reminders of two Taylor expansions. One can solve the equations in (A.16) with respect to $Q_{nt}(c,\xi)$, $R_{n,t}(c,\xi)$; the expressions obtained in this way can be used for the practical computation of these functions, and of their minima and maxima defined by (A.17) (A.18). Typically, the evaluation of the cited minima and maxima will be numerical.
(iv) For future use, we report here the minima and maxima, determined numerically from the definitions (A.17) (A.18) with $n = 3$, $t = 8$ and $n = 4, 5, 10$, $t = 6$:

$$\lambda_{38} = -72.563..., \ \Lambda_{38} = 202.91...; \qquad \mu_{38} = -159.61..., \ M_{38} = 930.73...; \quad \text{(A.20)}$$

$$\lambda_{46} = -112.95..., \ \Lambda_{46} = 904.92...; \quad \lambda_{56} = -432.09..., \ \Lambda_{56} = 4970.4...;$$

$$\lambda_{10,6} = -1.3678... \times 10^4..., \ \Lambda_{10,6} = 5.0076... \times 10^6 .$$

(Some of the subsequent computations require as well the values of m_{n6}, M_{n6} for $n = 4, 5, 10$; these are reported in [10].) □

In the sequel we present a lemma on a function of two vector variables h, k, to be used later (see Eq.(B.4)); as indicated below, this is related to the functions D_n, E_n in (A.13) and to their Taylor expansions.

A.9 ***Lemma.*** *Let $h, k \in \mathbf{R}^d \setminus \{0\}$, $h \neq k$, and let $\vartheta(h,k) \equiv \vartheta$ be the convex angle between them. Furthermore, let $n \in \mathbf{R}$; then the following holds.*
(i) One has

$$|h \wedge k|^2 \left[\frac{(|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} + \frac{(|k|^n - |h|^n)^2}{|h|^{2n}|k-h|^{2n+2}} \right] \quad \text{(A.21)}$$

$$= \frac{1}{|h|^{2n-2}} \left[D_n\Big(\cos\vartheta, \frac{|h|}{|k|}\Big) + \Big(1 - \frac{|h|^n}{|k|^n}\Big)^2 E_n\Big(\cos\vartheta, \frac{|h|}{|k|}\Big) \right] .$$

(ii) Let $|k| \geqslant 2|h|$. For any $t \in \{1, 2, ...,\}$, Eq. (A.21) implies

$$\frac{1}{|h|^{2n-2}}\left[\sum_{\ell=0}^{t-1} D_{n\ell}(\cos\vartheta)\frac{|h|^\ell}{|k|^\ell} + \lambda_{nt}\frac{|h|^t}{|k|^t} + \Big(1-\frac{|h|^n}{|k|^n}\Big)^2\Big(\sum_{\ell=0}^{t-1} E_{n\ell}(\cos\vartheta)\frac{|h|^\ell}{|k|^\ell} + \mu_{nt}\frac{|h|^t}{|k|^t}\Big)\right]$$

$$\leqslant |h\wedge k|^2\left[\frac{(|k|^n-|k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} + \frac{(|k|^n-|h|^n)^2}{|h|^{2n}|k-h|^{2n+2}}\right] \qquad \text{(A.22)}$$

$$\leqslant \frac{1}{|h|^{2n-2}}\left[\sum_{\ell=0}^{t-1} D_{n\ell}(\cos\vartheta)\frac{|h|^\ell}{|k|^\ell} + \Lambda_{nt}\frac{|h|^t}{|k|^t} + \Big(1-\frac{|h|^n}{|k|^n}\Big)^2\Big(\sum_{\ell=0}^{t-1} E_{n\ell}(\cos\vartheta)\frac{|h|^\ell}{|k|^\ell} + M_{nt}\frac{|h|^t}{|k|^t}\Big)\right]$$

(note that $\cos\vartheta = \widehat{h}\bullet\widehat{k}$*, with* $\widehat{\ }$ *denoting the versor).*

Proof. (i) We consider the function in the left hand side of (A.21), and reexpress it using the identities

$$|h\wedge k|^2 = |h|^2|k|^2(1-\cos^2\vartheta)\ , \qquad \text{(A.23)}$$

$$|k-h| = \sqrt{|k|^2 - 2|k||h|\cos\vartheta + |h|^2} = |k|\sqrt{1 - 2\,\cos\vartheta\frac{|h|}{|k|} + \frac{|h|^2}{|k|^2}}\ ,$$

$$|k|^n - |h|^n = |k|^n\Big(1-\frac{|h|^n}{|k|^n}\Big)\ ;$$

these readily yield the thesis (A.21).
(ii) Eqs. (A.16) (A.17) (A.18) imply

$$\sum_{\ell=0}^{t-1} D_{n\ell}(c)\xi^\ell + \lambda_{nt}\xi^t \leqslant D_n(c,\xi) \leqslant \sum_{\ell=0}^{t-1} D_{n\ell}(c)\xi^\ell + \Lambda_{nt}\xi^t\ , \qquad \text{(A.24)}$$

$$\sum_{\ell=0}^{t-1} E_{n\ell}(c)\xi^\ell + \mu_{nt}\xi^t \leqslant E_n(c,\xi) \leqslant \sum_{\ell=0}^{t-1} E_{n\ell}(c)\xi^\ell + M_{nt}\xi^t \quad \text{for } (c,\xi)\in[-1,1]\times[0,1/2].$$

Let us apply these inequalities with $c := \cos\vartheta$ and $\xi := |h|/|k|$ (noting that $0\leqslant\xi\leqslant 1/2$, by the assumption $|k|\geqslant 2|h|$). In this way, from Eqs. (A.21) and (A.24) we readily get the thesis (A.22). □

To conclude, let us introduce some variants $\hat{D}_{n\ell}$ and $\hat{E}_{n\ell}$ of the polynomials defined before ($\hat{E}_{n\ell}$ was already considered in [10]).

A.10 ***Definition.*** *For* $\ell = 0, 2, ...,$ $\hat{D}_{n\ell d} \equiv \hat{D}_{n\ell}$ *and* $\hat{E}_{n\ell d} \equiv \hat{E}_{n\ell}$ *are the polynomials obtained from* $D_{n\ell}$ *and* $E_{n\ell}$*, replacing the term* c^2 *with* $1/d$*.*

A.11 Example. The expressions of D_{n0}, D_{n2} in (A.19) imply

$$\hat{D}_{n0}(c) = \frac{n^2}{d} - n^2 c^4 \quad , \tag{A.25}$$

$$\hat{D}_{n2}(c) = \frac{n^2}{4} - (\frac{13}{4}n^2 + \frac{3}{2}n^3)\frac{1}{d} + (\frac{20}{3}n^2 + \frac{9}{2}n^3 + \frac{7}{12}n^4)c^4 - (\frac{11}{3}n^2 + 3n^3 + \frac{7}{12}n^4)c^6 \ .$$

B The function $\mathcal{G}_n$

Throughout the appendix $n \in (d/2+1, +\infty)$. For $k \in \mathbf{Z}^d_0$, we recall the definition (3.11)

$$\mathcal{G}_n(k) := \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|^2 (|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} \in (0, +\infty) \ , \qquad (\mathbf{Z}^d_{0k} := \mathbf{Z}^d \setminus \{0, k\}) \ .$$

B.1 *Proposition.* *Let us choose a "cutoff"*

$$\rho \in (2\sqrt{d}, +\infty) \ ; \tag{B.1}$$

then, the following holds (with the functions and quantities $\mathcal{g}_n$, $\delta\mathcal{g}_n$,... mentioned in the sequel depending parametrically on d and ρ: $\mathcal{g}_n(k) \equiv \mathcal{g}_{nd}(k,\rho)$, $\delta\mathcal{g}_n \equiv \delta\mathcal{g}_{nd}(\rho)$,...).
(i) The function $\mathcal{G}_n$ can be evaluated using the inequalities

$$\mathcal{g}_n(k) < \mathcal{G}_n(k) \leqslant \mathcal{g}_n(k) + \delta\mathcal{g}_n \quad \textit{for all } k \in \mathbf{Z}^d_0 \ . \tag{B.2}$$

Here

$$\mathcal{g}_n(k) := \sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho \; or \; |k-h|<\rho} \frac{|h \wedge k|^2 (|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} \ ; \tag{B.3}$$

this function can be reexpressed as

$$\mathcal{g}_n(k) = \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} |h \wedge k|^2 \left[\frac{(|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} + \theta(|k-h| - \rho) \frac{(|k|^n - |h|^n)^2}{|h|^{2n}|k-h|^{2n+2}} \right] \tag{B.4}$$

(with θ as in Definition A.1). If $|k| \geqslant 2\rho$, in Eq. (B.4) one can replace $\mathbf{Z}^d_{0k}$ with $\mathbf{Z}^d_0$ and $\theta(|k-h|-\rho)$ with 1. Furthermore

$$\delta\mathcal{g}_n := \frac{2\pi^{d/2} C_n}{\Gamma(d/2)} \sum_{i=0}^{d-1} \binom{d-1}{i} \frac{d^{d/2-1/2-i/2}}{(2n-3-i)(\rho - 2\sqrt{d})^{2n-3-i}} \ , \tag{B.5}$$

with C_n *as in (A.3).*
(ii) As in Remark 3.4, consider the reflection operators R_r $(r = 1, ..., d)$ *and the permutation operators* P_σ *(*σ *a permutation of* $\{1, ..., d\}$*). Then*

$$\mathcal{G}_n(R_r k) = \mathcal{G}_n(k) \ , \quad \mathcal{G}_n(P_\sigma k) = \mathcal{G}_n(k) \qquad \text{for each } k \in \mathbf{Z}^d_0 \tag{B.6}$$

(so, the computation of $\mathcal{G}_n(k)$ *can be reduced to the case* $k_1 \geqslant k_2 \geqslant ... \geqslant k_d \geqslant 0$*).*
(iii) Let $t \in \{2, 4, ...\}$*. One has*

$$\sum_{\ell=0,2,...,t-2} \frac{1}{|k|^\ell} \left(\mathcal{P}_{n\ell}(\widehat{k}) + \frac{\mathcal{P}'_{n\ell}(\widehat{k})}{|k|^n} + \frac{\mathcal{P}''_{n\ell}(\widehat{k})}{|k|^{2n}} \right) + \frac{1}{|k|^t} \left(w_{nt} + \frac{w'_{nt}}{|k|^n} + \frac{w''_{nt}}{|k|^{2n}} \right)$$

$$\leqslant \mathcal{G}_n(k) \tag{B.7}$$

$$\leqslant \sum_{\ell=0,2,...,t-2} \frac{1}{|k|^\ell} \left(\mathcal{P}_{n\ell}(\widehat{k}) + \frac{\mathcal{P}'_{n\ell}(\widehat{k})}{|k|^n} + \frac{\mathcal{P}''_{n\ell}(\widehat{k})}{|k|^{2n}} \right) + \frac{1}{|k|^t} \left(W_{nt} + \frac{W'_{nt}}{|k|^n} + \frac{W''_{nt}}{|k|^{2n}} \right) \text{ for } k \in \mathbf{Z}^d_0,\ |k| \geqslant 2\rho.$$

In the above, $\widehat{k} \in \mathbf{S}^{d-1}$ *is the versor of* k *(see Definition A.1). Furthermore,*

$$\mathcal{P}_{n\ell}, \mathcal{P}'_{n\ell}, \mathcal{P}''_{n\ell} : \mathbf{S}^{d-1} \to \mathbf{R}, \tag{B.8}$$

$$\mathcal{P}_{n\ell}(u) := \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \frac{\hat{D}_{n\ell}(\widehat{h} \bullet u) + \hat{E}_{n\ell}(\widehat{h} \bullet u)}{|h|^{2n-2-\ell}}, \quad \mathcal{P}'_{n\ell}(u) := -2 \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \frac{\hat{E}_{n\ell}(\widehat{h} \bullet u)}{|h|^{n-2-\ell}} \ ,$$

$$\mathcal{P}''_{n\ell}(u) := \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \hat{E}_{n\ell}(\widehat{h} \bullet u) |h|^{2+\ell} \qquad (\hat{D}_{n\ell},\ \hat{E}_{n\ell} \text{ as in Definition A.10}) \ ;$$

$$w_{nt} := (\lambda_{nt} + \mu_{nt}) \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \frac{1}{|h|^{2n-2-t}}, \quad w'_{nt} := -2\mu_{nt} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \frac{1}{|h|^{n-2-t}}, \tag{B.9}$$

$$w''_{nt} := \mu_{nt} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} |h|^{2+t} \qquad (\lambda_{nt},\ \mu_{nt} \text{ as in Eq. (A.17)}) \ ;$$

$$W_{nt} := (\Lambda_{nt} + M_{nt}) \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \frac{1}{|h|^{2n-2-t}}, \quad W'_{nt} := -2M_{nt} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \frac{1}{|h|^{n-2-t}}, \tag{B.10}$$

$$W''_{nt} := M_{nt} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} |h|^{2+t} \qquad (\Lambda_{nt},\ M_{nt} \text{ as in Eq. (A.18)}) \ .$$

For each ℓ*,* $\mathcal{P}_{n\ell}$*,* $\mathcal{P}'_{n\ell}$ *and* $\mathcal{P}''_{n\ell}$ *are polynomial functions on* $\mathbf{S}^{d-1}$*; setting*

$$p_{n\ell} := \min_{u \in \mathbf{S}^{d-1}} \mathcal{P}_{n\ell}(u), \quad p'_{n\ell} := \min_{u \in \mathbf{S}^{d-1}} \mathcal{P}'_{n\ell}(u), \quad p''_{n\ell} := \min_{u \in \mathbf{S}^{d-1}} \mathcal{P}''_{n\ell}(u), \tag{B.11}$$

$$P_{n\ell} := \max_{u \in \mathbf{S}^{d-1}} \mathcal{P}_{n\ell}(u), \quad P'_{n\ell} := \max_{u \in \mathbf{S}^{d-1}} \mathcal{P}'_{n\ell}(u), \quad P''_{n\ell} := \max_{u \in \mathbf{S}^{d-1}} \mathcal{P}''_{n\ell}(u),$$

one infers from (B.7) that

$$\sum_{\ell=2,4,\dots,t-2} \frac{1}{|k|^\ell} \left(p_{n\ell} + \frac{p'_{n\ell}}{|k|^n} + \frac{p''_{n\ell}}{|k|^{2n}} \right) + \frac{1}{|k|^t} \left(w_{nt} + \frac{w'_{nt}}{|k|^n} + \frac{w''_{nt}}{|k|^{2n}} \right) \leqslant \mathcal{G}_n(k) \tag{B.12}$$

$$\leqslant \sum_{\ell=2,4,\dots,t-2} \frac{1}{|k|^\ell} \left(P_{n\ell} + \frac{P'_{n\ell}}{|k|^n} + \frac{P''_{n\ell}}{|k|^{2n}} \right) + \frac{1}{|k|^t} \left(W_{nt} + \frac{W'_{nt}}{|k|^n} + \frac{W''_{nt}}{|k|^{2n}} \right) \ \textit{for } k \in \mathbf{Z}^d_0,\ |k| \geqslant 2\rho.$$

Consider a sequence $(k_i)_{i=0,1,2,\dots}$ *in* $\mathbf{Z}^d_0$*; then the inequalities (B.12), with* $t = 2$*, imply*

$$\mathcal{G}_n(k_i) \to \mathcal{P}_{n0}(u) \qquad \textit{for } i \to +\infty, \textit{ if } k_i \to \infty \textit{ and } \widehat{k_i} \to u \in \mathbf{S}^{d-1} \ . \tag{B.13}$$

Finally, we have

$$\liminf_{k \in \mathbf{Z}^d_0, k \to \infty} \mathcal{G}_n(k) = p_{n0} \ , \qquad \limsup_{k \in \mathbf{Z}^d_0, k \to \infty} \mathcal{G}_n(k) = P_{n0} \ . \tag{B.14}$$

(iv) Items (i) and (iii) imply

$$\sup_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k) \leqslant \sup_{k \in \mathbf{Z}^d_0} \mathscr{G}_n(k) \leqslant \Big(\sup_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k) \Big) + \delta \mathcal{G}_n < +\infty \ . \tag{B.15}$$

Proof. We fix a cutoff ρ as in (B.1). Our argument is divided in several steps; more precisely, Steps 1-5 give proofs of statements (i)(ii), while Steps 6-9 prove statements (iii)(iv). The assumption (B.1) $\rho > 2\sqrt{d}$ is essential in Step 3.
Step 1. One has

$$\mathscr{G}_n(k) = \mathcal{G}_n(k) + \Delta \mathcal{G}_n(k) \qquad \textit{for all } k \in \mathbf{Z}^d_0 \ , \tag{B.16}$$

where, as in (B.3), $\mathcal{G}_n(k) := \displaystyle\sum_{h \in \mathbf{Z}^d_{0k}, |h| < \rho \,\mathrm{or}\, |k-h| < \rho} \frac{|h \wedge k|^2 (|k|^n - |k-h|^n)^2}{|h|^{2n+2} |k-h|^{2n}}$*, while*

$$\Delta \mathcal{G}_n(k) := \sum_{h \in \mathbf{Z}^d_0, |h| \geqslant \rho, |k-h| \geqslant \rho} \frac{|h \wedge k|^2 (|k|^n - |k-h|^n)^2}{|h|^{2n+2} |k-h|^{2n}} \ \in (0, +\infty) \ . \tag{B.17}$$

The above decomposition follows noting that $\mathbf{Z}^d_{0k}$ is the disjoint union of the domains of the sums defining $\mathcal{G}_n(k)$ and $\Delta \mathcal{G}_n(k)$. $\mathcal{G}_n(k)$ is finite, involving finitely many summands; $\Delta \mathcal{G}_n(k)$ is finite as well, since we know that $\mathscr{G}_n(k) < +\infty$.

Step 2. For each $k \in \mathbf{Z}^d_0$, one has the representation (B.4)

$$\mathcal{G}_n(k) = \sum_{h \in \mathbf{Z}^d_{0k}, |h| < \rho} |h \wedge k|^2 \left[\frac{(|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}} + \theta(|k-h| - \rho) \frac{(|k|^n - |h|^n)^2}{|h|^{2n}|k-h|^{2n+2}} \right] .$$

If $|k| \geqslant 2\rho$, in the above one can replace $\mathbf{Z}^d_{0k}$ with $\mathbf{Z}^d_0$ and $\theta(|k-h| - \rho)$ with 1.
To prove (B.4) we reexpress the sum in Eq. (B.3), using Eq. (A.1) with $f(h) \equiv f_k(h) := \dfrac{|h \wedge k|(|k|^n - |k-h|^n)^2}{|h|^{2n+2}|k-h|^{2n}}$ (note that $f(k-h)$ contains a term $|(k-h) \wedge k| = |h \wedge k|$). To go on, assume $|k| \geqslant 2\rho$; then, for all $h \in \mathbf{Z}^d_0$ with $|h| < \rho$ one has $|k-h| \geqslant |k| - |h| > \rho$; this implies $h \neq k$ (i.e., $h \in \mathbf{Z}^d_{0k}$) and $\theta(|k-h| - \rho) = 1$, two facts which justify the replacements indicated above.
Step 3. For each $k \in \mathbf{Z}^d_0$ one has

$$0 < \Delta\mathcal{G}_n(k) \leqslant \delta\mathcal{G}_n \ , \tag{B.18}$$

with $\delta\mathcal{G}_n$ as in Eq. (B.5). The obvious relation $0 < \Delta\mathcal{G}_n(k)$ was already noted; in the sequel we prove that $\Delta\mathcal{G}_n(k) \leqslant \delta\mathcal{G}_n$. The definition (B.17) of $\Delta\mathcal{G}_n(k)$ contains the term $|h \wedge k|^2(|k|^n - |k-h|^n)^2$, for which we have:

$$\begin{aligned} |h \wedge k|^2(|k|^n - |k-h|^n)^2 &= |h \wedge (k-h)|^2(|k|^n - |k-h|^n)^2 \\ &\leqslant \frac{C_n}{2}|h|^4|k-h|^2\Big[|h|^{2n-2} + |k-h|^{2n-2}\Big] \end{aligned} \tag{B.19}$$

(the last inequality follows from (A.4), with $p = h$ and $q = k - h$). Inserting (B.19) into (B.17), we obtain

$$\Delta\mathcal{G}_n(k) \leqslant \frac{C_n}{2} \sum_{h \in \mathbf{Z}^d_0, |h| \geqslant \rho, |k-h| \geqslant \rho} \frac{|h|^{2n-2} + |k-h|^{2n-2}}{|h|^{2n-2}|k-h|^{2n-2}} \tag{B.20}$$

$$= \frac{C_n}{2} \Big(\sum_{h \in \mathbf{Z}^d_0, |h| \geqslant \rho, |k-h| \geqslant \rho} \frac{1}{|k-h|^{2n-2}} + \sum_{h \in \mathbf{Z}^d_0, |h| \geqslant \rho, |k-h| \geqslant \rho} \frac{1}{|h|^{2n-2}} \Big) .$$

The domain of the above two sums is contained in each one of the sets $\{h \in \mathbf{Z}^d \mid |h| \geqslant \rho\}$ and $\{h \in \mathbf{Z}^d \mid |k-h| \geqslant \rho\}$; so,

$$\Delta\mathcal{G}_n(k, \rho) \leqslant \frac{C_n}{2} \Big(\sum_{h \in \mathbf{Z}^d_0, |k-h| \geqslant \rho} \frac{1}{|k-h|^{2n-2}} + \sum_{h \in \mathbf{Z}^d_0, |h| \geqslant \rho} \frac{1}{|h|^{2n-2}} \Big) . \tag{B.21}$$

Now, the change of variable $h \mapsto k - h$ in the first sum shows that it is equal to the second one, so

$$\Delta\mathcal{G}_n(k) \leqslant C_n \sum_{h \in \mathbf{Z}^d_0, |h| \geqslant \rho} \frac{1}{|h|^{2n-2}} . \tag{B.22}$$

Finally, Eq. (B.22) and Eq. (A.9) with $\nu = 2n-2$ give

$$\Delta \mathcal{G}_n(k) \leqslant \frac{2\pi^{d/2} C_n}{\Gamma(d/2)} \sum_{i=0}^{d-1} \left(\begin{array}{c} d-1 \\ i \end{array} \right) \frac{d^{d/2-1/2-i/2}}{(2n-3-i)(\rho - 2\sqrt{d})^{2n-3-i}} \;\; = \delta\mathcal{G}_n \text{ as in (B.5)} \; .$$

Step 4. One has the inequalities (B.2) $\mathcal{G}_n(k) < \mathcal{G}_n(k) \leqslant \mathcal{G}_n(k) + \delta\mathcal{G}_n$. These relations follow immediately from the decomposition (B.16) $\mathcal{G}_n(k) = \mathcal{G}_n(k) + \Delta\mathcal{G}_n(k)$ and from the bounds (B.18) on $\Delta\mathcal{G}_n(k)$.
Step 5. One has the equalities (B.6) $\mathcal{G}_n(R_r k) = \mathcal{G}_n(k)$, $\mathcal{G}_n(P_\sigma k) = \mathcal{G}_n(k)$, *involving the reflection and permutation operators* R_r, P_σ. Again, we can invoke the argument employed for the analogous properties of the function $\mathcal{K}_n$ in [10].
Step 6. Let $t \in \{2, 4, ...\}$. *One has the inequalities (B.7) for* $\mathcal{G}_n$. As an example, for any $k \in \mathbf{Z}^d_0$ with $|k| \geqslant 2\rho$ we prove the upper bound (B.7)

$$\mathcal{G}_n(k) \leqslant \sum_{\ell=0,2,...,t-2} \frac{1}{|k|^\ell} \left(\mathcal{P}_{n\ell}(\widehat{k}) + \frac{\mathcal{P}'_{n\ell}(\widehat{k})}{|k|^n} + \frac{\mathcal{P}''_{n\ell}(\widehat{k})}{|k|^{2n}} \right) + \frac{1}{|k|^t} \left(W_{nt} + \frac{W'_{nt}}{|k|^n} + \frac{W''_{nt}}{|k|^{2n}} \right) .$$

Since $|k| \geqslant 2\rho$, we can express $\mathcal{G}_n(k)$ via Eq. (B.4), replacing therein $\mathbf{Z}^d_{0k}$ with $\mathbf{Z}^d_0$ and $\theta(|k-h| - \rho)$ with 1 (see the final statement in Step 2). So,

$$\mathcal{G}_n(k) = \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} |h \wedge k|^2 \left[\frac{(|k|^n - |k-h|^n)^2}{|h|^{2n+2} |k-h|^{2n}} + \frac{(|k|^n - |h|^n)^2}{|h|^{2n} |k-h|^{2n+2}} \right] . \tag{B.23}$$

In this expression we insert the upper bound of Eq. (A.22), writing therein $\cos\vartheta = \widehat{h} \bullet \widehat{k}$ (note that (A.22) can be used, since $|h|/|k| < \rho/(2\rho) < 1/2$ for each h in the sum). After some elementary manipulations (such as expanding the square $(1 - |h|^n/|k|^n)^2$, and reorganizing the terms that arise in this way), we conclude

$$\mathcal{G}_n(k) \leqslant \sum_{\ell=0,1,...,t-1} \frac{1}{|k|^\ell} \left(\mathcal{P}_{n\ell}(\widehat{k}) + \frac{\mathcal{P}'_{n\ell}(\widehat{k})}{|k|^n} + \frac{\mathcal{P}''_{n\ell}(\widehat{k})}{|k|^{2n}} \right) + \frac{1}{|k|^t} \left(W_{nt} + \frac{W'_{nt}}{|k|^n} + \frac{W''_{nt}}{|k|^{2n}} \right),$$

where $W_{nt}, W'_{nt}, W''_{nt}$ are as in (B.10) and, for each $\ell \in \{0, ..., t-1\}$, we have provisionally defined

$$\mathcal{P}_{n\ell}, \mathcal{P}'_{n\ell}, \mathcal{P}''_{n\ell} : \mathbf{S}^{d-1} \to \mathbf{R}, \tag{B.24}$$

$$\mathcal{P}_{n\ell}(u) := \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \frac{D_{n\ell}(\widehat{h} \bullet u) + E_{n\ell}(\widehat{h} \bullet u)}{|h|^{2n-2-\ell}}, \quad \mathcal{P}'_{n\ell}(u) := -2 \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \frac{E_{n\ell}(\widehat{h} \bullet u)}{|h|^{n-2-\ell}} \; ,$$

$$\mathcal{P}''_{n\ell}(u) := \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} E_{n\ell}(\widehat{h} \bullet u) |h|^{2+\ell} \qquad (E_{n\ell},\, D_{n\ell} \text{ as in Definition A.7}) \; .$$

Now, the thesis follows if we prove the following relations:

$$\mathcal{P}_{n\ell}(u) = 0, \quad \mathcal{P}'_{n\ell}(u) = 0, \quad \mathcal{P}''_{n\ell}(u) = 0 \qquad \text{for } \ell \in \{1, 3, ..., t-1\}, \; u \in \mathbf{S}^{d-1} \; ; \tag{B.25}$$

$$\mathcal{P}_{n\ell}(u), \mathcal{P}'_{n\ell}(u), \mathcal{P}''_{n\ell}(u) \text{ are as in (B.8), for } \ell \in \{0, 2, 4, ..., t-2\}, \; u \in \mathbf{S}^{d-1} \; . \tag{B.26}$$

The relations (B.25) are proved recalling that, for ℓ odd, the functions $c \mapsto E_{n\ell}(c)$, $D_{n\ell}(c)$ are odd as well; this implies that the general term of the sum (B.24) changes its sign under a transformation $h \mapsto -h$.

Now, let us prove (B.26) for any even ℓ. As an example, we consider the case of $\mathcal{P}_{n\ell}$; the sum defining it in (B.24) contains the even polynomials

$$D_{n\ell}(c) = \sum_{j=0,2,\dots,\ell+4} D_{n\ell j} c^j, \qquad E_{n\ell}(c) = \sum_{j=0,2,\dots,\ell+2} E_{n\ell j} c^j, \tag{B.27}$$

so (B.24) implies

$$\mathcal{P}_{n\ell}(u) = \sum_{j=0,2,\dots,\ell+4} D_{n\ell j} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(\widehat{h} \bullet u)^j}{|h|^{2n-2-\ell}} + \sum_{j=0,2,\dots,\ell+2} E_{n\ell j} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(\widehat{h} \bullet u)^j}{|h|^{2n-2-\ell}} \; ; \tag{B.28}$$

in particular, for the $j = 2$ terms in both sums above we have (writing $\widehat{h} = h/|h|$)

$$\sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(\widehat{h} \bullet u)^2}{|h|^{2n-2-\ell}} = \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(h \bullet u)^2}{|h|^{2n-\ell}} = \frac{1}{d} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{1}{|h|^{2n-2-\ell}} \; , \tag{B.29}$$

where the last passage follows from the identity (A.10) (with k replaced by u and $\varphi(|h|) = 1/|h|^{2n-\ell}$). Eqs. (B.28) (B.29) imply

$$\mathcal{P}_{n\ell}(u) = \sum_{j=0,4,6,\dots,\ell+4} D_{n\ell j} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(\widehat{h} \bullet u)^j}{|h|^{2n-2-\ell}} + \frac{D_{n\ell 2}}{d} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{1}{|h|^{2n-2-\ell}} \tag{B.30}$$

$$+ \sum_{j=0,4,6\dots,\ell+2} E_{n\ell j} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(\widehat{h} \bullet u)^j}{|h|^{2n-2-\ell}} + \frac{E_{n\ell 2}}{d} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{1}{|h|^{2n-2-\ell}} \; .$$

On the other hand, Definition A.10 of $\hat{D}_{n\ell}$, $\hat{E}_{n\ell}$ prescribes

$$\hat{D}_{n\ell}(c) = \sum_{j=0,4,6,\dots,\ell+4} D_{n\ell j} c^j + \frac{D_{n\ell 2}}{d} \; , \qquad \hat{E}_{n\ell}(c) = \sum_{j=0,4,6,\dots,\ell+2} E_{n\ell j} c^j + \frac{E_{n\ell 2}}{d} \; ; \tag{B.31}$$

comparing this with (B.30), we conclude

$$\mathcal{P}_{n\ell}(u) = \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{\hat{D}_{n\ell}(\widehat{h} \bullet u|) + \hat{E}_{n\ell}(\widehat{h} \bullet u)}{|h|^{2n-2-\ell}}, \quad \text{as in (B.8).}$$

So, statement (B.26) is proved for $\mathcal{P}_{n\ell}$; one proceeds similarly for $\mathcal{P}'_{n\ell}$ and $\mathcal{P}''_{n\ell}$.
Step 7. Let $t \in \{2,4,...\}$. For $\ell \in \{0,2,4,...,t-2\}$, $\mathcal{P}_{n\ell}$, $\mathcal{P}'_{n\ell}$ and $\mathcal{P}''_{n\ell}$ are polynomial function on $\mathbf{S}^{d-1}$; considering their minima and maxima $p_{n\ell}$, $P_{n\ell}$, etc., one infers from (B.7) the inequalities (B.12)

$$\sum_{\ell=0,2,4,...,t-2} \frac{1}{|k|^\ell}\left(p_{n\ell} + \frac{p'_{n\ell}}{|k|^n} + \frac{p''_{n\ell}}{|k|^{2n}}\right) + \frac{1}{|k|^t}\left(w_{nt} + \frac{w'_{nt}}{|k|^n} + \frac{w''_{nt}}{|k|^{2n}}\right) \leqslant \mathcal{G}_n(k)$$

$$\leqslant \sum_{\ell=0,2,4,...,t-2} \frac{1}{|k|^\ell}\left(P_{n\ell} + \frac{P'_{n\ell}}{|k|^n} + \frac{P''_{n\ell}}{|k|^{2n}}\right) + \frac{1}{|k|^t}\left(W_{nt} + \frac{W'_{nt}}{|k|^n} + \frac{W''_{nt}}{|k|^{2n}}\right) \quad \textit{for } |k| \geqslant 2\rho.$$

The polynomial nature of the functions $\mathcal{P}_{n\ell}$, $\mathcal{P}'_{n\ell}$ and $\mathcal{P}''_{n\ell}$ follows from their definition (B.8) in terms of the polynomials $\hat{E}_{n\ell}, \hat{D}_{n\ell}$. The inequalities (B.12) are obvious.
Step 8. Consider a sequence $(k_i)_{i=0,1,2,...}$ in $\mathbf{Z}^d_0$; then the inequalities (B.12), with $t=2$, imply statement (B.13)

$$\mathcal{G}_n(k_i) \to \mathcal{P}_{n0}(u) \qquad \textit{for } i \to +\infty, \textit{ if } k_i \to \infty \textit{ and } \widehat{k_i} \to u \in \mathbf{S}^{d-1} .$$

Finally, we have the results (B.14)

$$\liminf_{k\in\mathbf{Z}^d_0, k\to\infty} \mathcal{G}_n(k) = p_{n0} , \qquad \limsup_{k\in\mathbf{Z}^d_0, k\to\infty} \mathcal{G}_n(k) = P_{n0} .$$

To prove all this we start from any sequence $(k_i)_{i=0,1,...}$ in $\mathbf{Z}^d_0$ and note that (B.7), with $t=2$ and $k=k_i$, gives

$$\mathcal{P}_{n0}(\widehat{k_i}) + \frac{\mathcal{P}'_{n0}(\widehat{k_i})}{|k_i|^n} + \frac{\mathcal{P}''_{n0}(\widehat{k_i})}{|k_i|^{2n}} + \frac{1}{|k_i|^2}\left(w_{n2} + \frac{w'_{n2}}{|k_i|^n} + \frac{w''_{n2}}{|k_i|^{2n}}\right) \leqslant \mathcal{G}_n(k_i) \tag{B.32}$$

$$\leqslant \mathcal{P}_{n0}(\widehat{k_i}) + \frac{\mathcal{P}'_{n0}(\widehat{k_i})}{|k_i|^n} + \frac{\mathcal{P}''_{n0}(\widehat{k_i})}{|k_i|^{2n}} + \frac{1}{|k_i|^2}\left(W_{n2} + \frac{W'_{n2}}{|k_i|^n} + \frac{W''_{n2}}{|k_i|^{2n}}\right) \text{for } |k_i| \geqslant 2\rho .$$

Now, assume $k_i \to \infty$ and $\widehat{k_i} \to u \in \mathbf{S}^{d-1}$; then, both the lower and the upper bounds to $\mathcal{G}_n(k_i)$ in (B.32) tend to $\mathcal{P}_{n0}(u)$ and we obtain Eq. (B.13).
Let us pass to the proof of Eq. (B.14); as an example, we derive the statement about $\limsup_{k\to\infty} \mathcal{G}_n(k)$. By definition,

$$\limsup_{k\in\mathbf{Z}^d_0, k\to\infty} \mathcal{G}_n(k) = \sup_{(k_i)\in\mathfrak{C}} \lim_{i\to+\infty} \mathcal{G}_n(k_i) , \tag{B.33}$$

$$\mathfrak{C} := \{\text{sequences } (k_i)_{i=0,1,2,...} \text{ in } \mathbf{Z}^d_0 \text{ such that } k_i \to \infty, \lim_{i\to+\infty} \mathcal{G}_n(k_i) \text{ exists}\} .$$

Consider any sequence $(k_i) \in \mathfrak{C}$; applying the upper bound in Eq. (B.12), with $t=2$ and $k=k_i$, we get

$$\mathcal{G}_n(k_i) \leqslant P_{n0} + \frac{P'_{n0}}{|k_i|^n} + \frac{P''_{n0}}{|k_i|^{2n}} + \frac{1}{|k_i|^2}\left(W_{n2} + \frac{W'_{n2}}{|k_i|^n} + \frac{W''_{n2}}{|k_i|^{2n}}\right) \tag{B.34}$$

for all i such that $|k_i| \geqslant 2\rho$. Let $i \to +\infty$; then $k_i \to \infty$, and the previous inequality implies

$$\lim_{i \to +\infty} \mathcal{G}_n(k_i) \leqslant P_{n0} \ . \tag{B.35}$$

Now, let $\mathfrak{u} \in \mathbf{S}^{d-1}$ be such that

$$\mathcal{P}_{n\ell}(\mathfrak{u}) = P_{n0} \ , \tag{B.36}$$

and let us consider a sequence $(k_i)_{i=0,1,2,\dots}$ in $\mathbf{Z}^d_0$ such that

$$k_i \to \infty \ , \ \widehat{k_i} \to \mathfrak{u} \qquad \text{for } i \to +\infty \tag{B.37}$$

(e.g., $k_i := ([i\mathfrak{u}_1], ..., [i\mathfrak{u}_d])$, where $[\ ]$ is the integer part). Eqs. (B.37) (B.13) and (B.36) give

$$\lim_{i \to +\infty} \mathcal{G}_n(k_i) = P_{n0} \ . \tag{B.38}$$

The results (B.35) and (B.38) imply $\limsup_{k\to\infty} \mathcal{G}_n(k) \leqslant P_{n0}$ and $\limsup_{k\to\infty} \mathcal{G}_n(k) \geqslant P_{n0}$, respectively, yielding the desired relation

$$\limsup_{k \in \mathbf{Z}^d_0, k \to \infty} \mathcal{G}_n(k) = P_{n0} \ . \tag{B.39}$$

Step 9. Proof of the inequalities (B.15)

$$\sup_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k) \leqslant \sup_{k \in \mathbf{Z}^d_0} \mathscr{G}_n(k) \leqslant \Big(\sup_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k) \Big) + \delta\mathcal{G}_n < +\infty \ .$$

The first two inequalities are obvious consequences of the relations (B.2) $\mathcal{G}_n(k) < \mathscr{G}_n(k) \leqslant \mathcal{G}_n(k) + \delta\mathcal{G}_n$; the third inequality above holds if we show that

$$\sup_{k \in \mathbf{Z}^d_0} \mathcal{G}_n(k) < +\infty \ , \tag{B.40}$$

and this follows from the finiteness of $\limsup_{k\to\infty} \mathcal{G}_n(k)$ (see Step 8). $\square$

C Appendix. The upper bounds G_n^+, for $d = 3$ and $n = 3, 4, 5, 10$

Eq. (3.16) defines G_n^+ in terms of $\sup_{k \in \mathbf{Z}_0^3} \mathcal{G}_n(k)$, or of any upper approximant for this sup. In all the cases analyzed hereafter, we produce both an upper and a lower approximant; the lower one is given only to indicate the uncertainty in our evaluation of $\sup \mathcal{G}_n$.

Some details on the evaluation of $\mathcal{G}_3$ and of its sup. Among the examples presented here, the case of $\mathcal{G}_3$ is the one requiring more expensive computations.
To evaluate $\mathcal{G}_3$, we apply Proposition B.1 with a fairly large cutoff

$$\rho = 20 \; ; \tag{C.1}$$

thus, following Proposition B.1, we must often sum over the set $\{h \in \mathbf{Z}_0^3 \mid |h| < 20\}$.
Eq. (B.5) (with the value of C_3 in (A.8)) gives

$$\delta \mathcal{G}_3 = 12.478... \; , \tag{C.2}$$

and it remains to evaluate the function $\mathcal{G}_3$.
To compute $\mathcal{G}_3(k)$, we start from the $k's$ in $\mathbf{Z}_0^3$ with $|k| < 2\rho = 40$. Using directly the definition (B.4) for all such k's ([2]), we obtain

$$\max_{k \in \mathbf{Z}_0^3, |k| < 40} \mathcal{G}_3(k) = \mathcal{G}_3(9, 9, 9) = 34.901... \; . \tag{C.3}$$

Let us pass to the case $|k| \geqslant 40$. Here, our main tool is the upper bound in (B.12) with $t = 8$; after some computations, this gives

[2] In fact, due to the symmetry properties (B.6), computation of $\mathcal{G}_3(k)$ can be limited to points k such that $k_1 \geqslant k_2 \geqslant k_3 \geqslant 0$.

$$\mathcal{G}_3(k) \leqslant 33.725 + \frac{1070.6}{|k|^2} - \frac{3337.9}{|k|^3} + \frac{2.9764 \times 10^5}{|k|^4} - \frac{2.6596 \times 10^6}{|k|^5} \tag{C.4}$$

$$+\frac{1.3451 \times 10^8}{|k|^6} - \frac{1.7663 \times 10^9}{|k|^7} + \frac{2.5858 \times 10^{12}}{|k|^8} - \frac{1.0476 \times 10^{12}}{|k|^9} + \frac{4.7461 \times 10^{12}}{|k|^{10}}$$

$$-\frac{2.3621 \times 10^{16}}{|k|^{11}} + \frac{3.1212 \times 10^{15}}{|k|^{12}} + \frac{7.2378 \times 10^{19}}{|k|^{14}} \leqslant 34.792 \quad \text{for } k \in \mathbf{Z}_0^3,\ |k| \geqslant 40$$

([3]). (For completeness, we mention that the $t = 8$ lower bound in (B.12) and Eq. (B.14) imply $\inf_{k \in \mathbf{Z}_0^3, |k| \geqslant 40} \mathcal{G}_3(k) = \liminf_{k \in \mathbf{Z}_0^3, k \to \infty} \mathcal{G}_3(k) = 23.627...$, while $\limsup_{k \in \mathbf{Z}_0^3, k \to \infty} \mathcal{G}_3(k) = 33.724...$. ([4]).)

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[3] Let us give some supplementary information on the computations yielding (C.4). The $t = 8$ upper bound in Eq. (B.12) reads:

$$\mathcal{G}_3(k) \leqslant \sum_{\ell = 0,2,4,6} \frac{1}{|k|^\ell} \left(P_{3\ell} + \frac{P'_{3\ell}}{|k|^3} + \frac{P''_{3\ell}}{|k|^6} \right) + \frac{1}{|k|^8} \left(W_{38} + \frac{W'_{38}}{|k|^3} + \frac{W''_{38}}{|k|^6} \right) \ \text{for } k \in \mathbf{Z}_0^d,\ |k| \geqslant 40.$$

The constants W_{38}, W'_{38}, W''_{38} are computed directly from the definition (B.10) (this requires previous knowledge of $M_{38} = 930.73...$ and $\Lambda_{38} = 202.91...$, see Eq. (A.20)). For $\ell = 0, 2, 4, 6$, $P_{3\ell}$, $P'_{3\ell}$ and $P''_{3\ell}$ are the maxima of the polynomial functions $\mathcal{P}_{3\ell}$, $\mathcal{P}'_{3\ell}$ and $\mathcal{P}''_{3\ell}$ on $\mathbf{S}^2$; for example, Eq. (B.8) with $n = 3$, $\ell = 0$ gives

$$\mathcal{P}_{30}(u) = 58.311... - 39.076...(u_1^2 u_2^2 + u_1^2 u_3^2 + u_2^2 u_3^2) - 34.683...(u_1^4 + u_2^4 + u_3^4)$$

for all $u \in \mathbf{S}^2$, and one finds that $P_{30} = \mathcal{P}_{30}(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = 33.724....$ Computing the other polynomials mentioned above and their maxima, and rounding up from above the numerical outputs, we obtain the first inequality (C.4) $\mathcal{G}_3(k) \leqslant 33.725 + 1070.6\,|k|^{-2} +$ etc., holding for $|k| \geqslant 40$; on the other hand, $33.725 + 1070.6\,|k|^{-2} +$ etc. $\leqslant 34.792$ for all such k's, which explains the second inequality (C.4).

[4] Let us explain how to derive these statements. First of all, Eq. (B.14) gives

$$\liminf_{k \in \mathbf{Z}_0^d, k \to \infty} \mathcal{G}_3(k) = p_{30}\ , \qquad \limsup_{k \in \mathbf{Z}_0^d, k \to \infty} \mathcal{G}_3(k) = P_{30}\ ,$$

where p_{30} and P_{30} are the minimum and the maximum of the polynomial $\mathcal{P}_{30}$ over $\mathbf{S}^2$. The explicit expression of $\mathcal{P}_{30}$ is given in the previous footnote; it turns out that $p_{30} = \mathcal{P}_{30}(1, 0, 0) = 23.627...$ and (as stated before) $P_{30} = \mathcal{P}_{30}(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = 33.724...$.
Now, let us use the lower bound (B.12) with $n = 3$, $t = 8$; computing all the necessary constants, after some round up we get

$$\mathcal{G}_3(k) \geqslant p_{30} + \frac{1042.9}{|k|^2} - \frac{3338.0}{|k|^3} + \frac{2.9617 \times 10^5}{|k|^4} - \frac{2.6755 \times 10^6}{|k|^5} + \frac{1.3449 \times 10^8}{|k|^6} - \frac{1.7822 \times 10^9}{|k|^7}$$

$$-\frac{5.2231 \times 10^{11}}{|k|^8} - \frac{1.0729 \times 10^{12}}{|k|^9} + \frac{4.6822 \times 10^{12}}{|k|^{10}} + \frac{4.0510 \times 10^{15}}{|k|^{11}} + \frac{3.0213 \times 10^{15}}{|k|^{12}} - \frac{1.2413 \times 10^{19}}{|k|^{14}}$$

for $k \in \mathbf{Z}_0^3$, $|k| \geqslant 40$. On the other hand, one has $1042.9\,|k|^{-2} - 3338.0|k|^{-3} + ... \geqslant 0$ for $|k| \geqslant 40$; so, $\inf_{k \in \mathbf{Z}_0^3, |k| \geqslant 40} \mathcal{G}_3(k) \geqslant p_{30}$. It is obvious that $\inf_{k \in \mathbf{Z}_0^3, |k| \geqslant 40} \mathcal{G}_3(k) \leqslant \liminf_{k \in \mathbf{Z}_0^3, k \to \infty} \mathcal{G}_3(k)$; the latter equals p_{30}, thus $\inf \mathcal{G}_3 = \liminf \mathcal{G}_3 = p_{30}$.

The results (C.3) (C.4) yield

$$\sup_{k \in \mathbf{Z}_0^3} \mathcal{G}_3(k) = \mathcal{G}_3(9,9,9) = 34.901... \ . \tag{C.5}$$

We now pass to the function $\mathscr{G}_3$; according to (B.15) we have $\sup_{k\in\mathbf{Z}_0^3} \mathcal{G}_3(k) \leqslant \sup_{k\in\mathbf{Z}_0^3} \mathscr{G}_3(k) \leqslant \Big(\sup_{k\in\mathbf{Z}_0^3} \mathcal{G}_3(k)\Big) + \delta\mathcal{G}_3$, and the numerical results (C.2) (C.5) give

$$34.901 < \sup_{k \in \mathbf{Z}_0^3} \mathscr{G}_3(k) < 47.381 \ . \tag{C.6}$$

(The uncertainty on this sup is fairly large, due to the value of $\delta\mathcal{G}_3$ in (C.2); the error $\delta\mathcal{G}_3$ could be significantly reduced choosing a cutoff $\rho \gg 20$, but the related computations would be much more expensive.)

The upper bound G_3^+. According to the definition (3.16), we have

$$G_3^+ = \frac{1}{(2\pi)^{3/2}} \sqrt{\sup_{k \in \mathbf{Z}_0^3} \mathscr{G}_3(k)} \quad \text{(or any upper approximant for this)} \ . \tag{C.7}$$

Due to (C.6), we can take $G_3^+ = (2\pi)^{-3/2}\sqrt{47.381}$; rounding up to three digits we can write

$$G_3^+ = 0.438 \,, \tag{C.8}$$

as reported in (3.21).

Preparing the examples with $n = 4, 5, 10$. To evaluate $\mathscr{G}_n$ for the cited values of n, we apply Proposition B.1 with a cutoff

$$\rho = 10 \ ; \tag{C.9}$$

thus, all sums over h in Proposition B.1 are over the set $\{h \in \mathbf{Z}_0^3 \mid |h| < 10\}$.

Some details on the evaluation of $\mathscr{G}_4$ and of its sup. Eq. (B.5) (with the value of C_4 in (A.8)) gives

$$\delta\mathcal{G}_4 = 1.2626... \ , \tag{C.10}$$

and it remains to evaluate the function $\mathcal{G}_4$.

To compute $\mathcal{G}_4(k)$, we start from the $k's$ in $\mathbf{Z}_0^3$ with $|k| < 2\rho = 20$. Using directly the definition (B.4) for all such k's, we obtain

$$\max_{k \in \mathbf{Z}_0^3, |k| < 20} \mathcal{G}_4(k) = \mathcal{G}_4(2,1,0) = 56.628... \ . \tag{C.11}$$

Let us pass to the case $|k| \geqslant 20$. Here we use the upper bound in (B.12) with $t = 6$, giving

$$\mathcal{G}_4(k) \leqslant 31.379 + \frac{193.19}{|k|^2} + \frac{3740.3 \times 10^5}{|k|^4} + \frac{1.1291 \times 10^7}{|k|^6} - \frac{8.6865 \times 10^6}{|k|^8} \qquad \text{(C.12)}$$

$$-\frac{6.3946 \times 10^{10}}{|k|^{10}} + \frac{2.4366 \times 10^5}{|k|^{12}} + \frac{2.0079 \times 10^{14}}{|k|^{14}} \leqslant 32.056 \quad \text{for } k \in \mathbf{Z}^3_0,\ |k| \geqslant 20\ .$$

(For completeness we mention that the $t = 6$ lower bound in (B.12) and Eq. (B.14) imply $\inf_{k \in \mathbf{Z}^3_0, |k| \geqslant 20} \mathcal{G}_4(k) = \liminf_{k \in \mathbf{Z}^3_0, k \to \infty} \mathcal{G}_4(k) = 11.716...$, $\limsup_{k \in \mathbf{Z}^3_0, k \to \infty} \mathcal{G}_4(k) = 31.378...$).
The results (C.11) (C.12) yield

$$\sup_{k \in \mathbf{Z}^3_0} \mathcal{G}_4(k) = \mathcal{G}_4(2, 1, 0) = 56.628...\ . \qquad \text{(C.13)}$$

We now pass to the function $\mathcal{G}_4$; according to (B.15) we have $\sup_{k \in \mathbf{Z}^3_0} \mathcal{G}_4(k) \leqslant \sup_{k \in \mathbf{Z}^3_0} \mathcal{G}_4(k) \leqslant \Big(\sup_{k \in \mathbf{Z}^3_0} \mathcal{G}_4(k) \Big) + \delta \mathcal{G}_4$, and the numerical results (C.10) (C.13) give

$$56.628 < \sup_{k \in \mathbf{Z}^3_0} \mathcal{G}_4(k) < 57.892\ . \qquad \text{(C.14)}$$

The upper bound G_4^+. According to the definition (3.16), we have

$$G_4^+ = \frac{1}{(2\pi)^{3/2}} \sqrt{\sup_{k \in \mathbf{Z}^3_0} \mathcal{G}_4(k)} \quad \text{(or any upper approximant for this)}\ . \qquad \text{(C.15)}$$

Due to (C.14), we can take $G_4^+ = (2\pi)^{-3/2} \sqrt{57.892}$; rounding up to three digits we can write

$$G_4^+ = 0.484\,, \qquad \text{(C.16)}$$

as reported in (3.21).

Some details on the evaluation of $\mathcal{G}_5$ and of its sup. Eq. (B.5) (with the value of C_5 in (A.8)) gives

$$\delta \mathcal{G}_5 = 0.067895...\ , \qquad \text{(C.17)}$$

and it remains to evaluate the function $\mathcal{G}_5$.
To compute $\mathcal{G}_5(k)$, we start from the $k's$ in $\mathbf{Z}^3_0$ with $|k| < 2\rho = 20$. Using directly the definition (B.4) for all such k's, we obtain

$$\max_{k \in \mathbf{Z}^3_0, |k| < 20} \mathcal{G}_5(k) = \mathcal{G}_5(2, 1, 0) = 138.96...\ . \qquad \text{(C.18)}$$

Let us pass to the case $|k| \geqslant 20$. Here we use the upper bound in (B.12) with $t = 6$, giving

$$\mathcal{G}_5(k) \leqslant 40.612 + \frac{271.13}{|k|^2} + \frac{1970.7}{|k|^4} - \frac{43.608}{|k|^5} + \frac{1.4210\times 10^6}{|k|^6} - \frac{8949.1}{|k|^7} \tag{C.19}$$

$$-\frac{2.4425\times 10^6}{|k|^9} + \frac{1.6428\times 10^5}{|k|^{10}} - \frac{2.9673\times 10^{10}}{|k|^{11}} + \frac{1.2866\times 10^5}{|k|^{12}}$$

$$+\frac{5.3524\times 10^5}{|k|^{14}} + \frac{7.9455\times 10^{14}}{|k|^{16}} \leqslant 41.325 \qquad \text{for } k\in\mathbf{Z}_0^3,\ |k|\geqslant 20\ .$$

(For completeness we mention that the $t = 6$ lower bound in (B.12) and Eq.(B.14) imply $\inf_{k\in\mathbf{Z}_0^3,|k|\geqslant 20}\mathcal{G}_5(k) = \liminf_{k\in\mathbf{Z}_0^3,k\to\infty}\mathcal{G}_5(k) = 8.5405...$ and $\limsup_{k\in\mathbf{Z}_0^3,k\to\infty}\mathcal{G}_5(k) = 40.611...$.)
The results (C.18) (C.19) yield

$$\sup_{k\in\mathbf{Z}_0^3}\mathcal{G}_5(k) = \mathcal{G}_5(2,1,0) = 138.96...\ . \tag{C.20}$$

We now pass to the function $\mathcal{G}_5$; according to (B.15) we have $\sup_{k\in\mathbf{Z}_0^3}\mathcal{G}_5(k) \leqslant \sup_{k\in\mathbf{Z}_0^3}\mathcal{G}_5(k) \leqslant \left(\sup_{k\in\mathbf{Z}_0^3}\mathcal{G}_5(k)\right) + \delta\mathcal{G}_5$, and the numerical results (C.17) (C.20) give

$$138.96 < \sup_{k\in\mathbf{Z}_0^3}\mathcal{G}_5(k) < 139.04\ . \tag{C.21}$$

The upper bound G_5^+. According to the definition (3.16), we have

$$G_5^+ = \frac{1}{(2\pi)^{3/2}}\sqrt{\sup_{k\in\mathbf{Z}_0^3}\mathcal{G}_5(k)} \quad \text{(or any upper approximant for this)}\ . \tag{C.22}$$

Due to (C.14), we can take $G_5^+ = (2\pi)^{-3/2}\sqrt{139.04}$; rounding up to three digits we can write

$$G_5^+ = 0.749\,, \tag{C.23}$$

as reported in (3.21).

Some details on the evaluation of $\mathcal{G}_{10}$ and of its sup. Eq. (B.5) (with the value of C_{10} in (A.8)) gives

$$\delta\mathcal{G}_{10} = 1.0366...\times 10^{-7}\ , \tag{C.24}$$

and it remains to evaluate the function $\mathcal{G}_{10}$.
To compute $\mathcal{G}_{10}(k)$, we start from the $k's$ in $\mathbf{Z}_0^3$ with $|k| < 2\rho = 20$. Using directly the definition (B.4) for all such k's, we obtain

$$\max_{k\in\mathbf{Z}_0^3,|k|<20}\mathcal{G}_{10}(k) = \mathcal{G}_{10}(2,1,0) = 1.4143...\times 10^4\ . \tag{C.25}$$

Let us pass to the case $|k| \geqslant 20$. Here we use the upper bound in (B.12) with $t = 6$, giving

$$\mathcal{G}_{10}(k) \leqslant 137.62 + \frac{3125.7}{|k|^2} + \frac{3.2133 \times 10^4}{|k|^4} + \frac{5.9819 \times 10^7}{|k|^6} - \frac{9.2610}{|k|^{10}} \tag{C.26}$$

$$-\frac{78.735}{|k|^{12}} - \frac{1.1360 \times 10^4}{|k|^{14}} - \frac{1.0781 \times 10^9}{|k|^{16}} + \frac{1.6428 \times 10^5}{|k|^{20}} + \frac{4.9586 \times 10^8}{|k|^{22}}$$

$$+\frac{6.8396. \times 10^{11}}{|k|^{24}} + \frac{5.0800 \times 10^{17}}{|k|^{26}} \leqslant 146.57 \quad \text{for } k \in \mathbf{Z}_0^3,\ |k| \geqslant 20\ .$$

(For completeness we mention that the $t = 6$ lower bound in (B.12) and Eq. (B.14) imply $\inf_{k \in \mathbf{Z}_0^3, |k| \geqslant 20} \mathcal{G}_{10}(k) = \liminf_{k \in \mathbf{Z}_0^3, k \to \infty} \mathcal{G}_{10}(k) = 4.4157...$ and $\limsup_{k \in \mathbf{Z}_0^3, k \to \infty} \mathcal{G}_{10}(k) = 137.61...$.)
The results (C.25) (C.26) yield

$$\sup_{k \in \mathbf{Z}_0^3} \mathcal{G}_{10}(k) = \mathcal{G}_{10}(2,1,0) = 1.4143... \times 10^4\ . \tag{C.27}$$

We now pass to the function $\mathcal{G}_3$; according to (B.15) we have $\sup_{k \in \mathbf{Z}_0^3} \mathcal{G}_{10}(k) \leqslant \sup_{k \in \mathbf{Z}_0^3} \mathcal{G}_{10}(k) \leqslant \Big(\sup_{k \in \mathbf{Z}_0^3} \mathcal{G}_{10}(k) \Big) + \delta \mathcal{G}_{10}$, and the numerical results (C.24) (C.27) give ([5])

$$\sup_{k \in \mathbf{Z}_0^3} \mathcal{G}_{10}(k) = 1.4143... \times 10^4\ . \tag{C.28}$$

The upper bound G^+_{10}. According to the definition (3.16), we have

$$G^+_{10} = \frac{1}{(2\pi)^{3/2}} \sqrt{\sup_{k \in \mathbf{Z}_0^3} \mathcal{G}_{10}(k)} \quad \text{(or any upper approximant for this)}\ . \tag{C.29}$$

Using (C.28), and rounding up to three digits the final result, we can write

$$G^+_{10} = 7.56\,, \tag{C.30}$$

as reported in (3.21).

[5] In the MATHEMATICA output for $\mathcal{G}_{10}(2,1,0)$, 1.4143 is followed by a digit different from 9; so, the digits 1.4143 do not change when $\delta\mathcal{G}_{10}$ is added to this output.

D Appendix. The lower bounds G_n^-, for $d = 3$ and $n = 3, 4, 5, 10$

Let $n \in (5/2, +\infty)$; according to Proposition 3.7, for all nonzero families $(v_k)_{k \in V}$, $(w_k)_{k \in W}$ in the space $\mathcal{H}$ of (3.18), we have the lower bound (3.20)

$$G_n^- := \frac{1}{(2\pi)^{3/2}} \frac{|P_n((v_k),(w_k))|}{N_n((v_k)) N_n^2((w_k))} \text{ (or any lower approximant for this),}$$

$$N_n((v_k)) := \Big(\sum_{k \in V} |k|^{2n} |v_k|^2 \Big)^{1/2}, \qquad N_n((w_k)) := \Big(\sum_{k \in V} |k|^{2n} |w_k|^2 \Big)^{1/2},$$

$$P_n((v_k),(w_k)) := -i \sum_{h \in V, \ell \in W, h+\ell \in W} |h+\ell|^{2n} (\overline{v_h} \bullet \ell)(\overline{w_\ell} \bullet w_{h+\ell}) \ .$$

Let us consider the choices

$$V := \{\pm(1,0,0)\} \ , \qquad v_{\pm(1,0,0)} := (0, P \pm iQ, 0) \quad (P, Q \in \mathbf{R}) \ ; \tag{D.1}$$

$$W := \{\pm(0,1,0), \pm(1,1,0), \pm(1,-1,0), \pm(2,1,0), \pm(2,-1,0)\} \ ; \tag{D.2}$$

$$w_{\pm t} := (0, 0, X_t \pm iY_t) \quad (X_t, Y_t \in \mathbf{R})$$
$$\text{for } t = (0,1,0), (1,1,0), (1,-1,0), (2,1,0), (2,-1,0)$$

(with $(P,Q) \neq 0$ and $(X_t, Y_t)_{t=(0,1,0),\dots,(2,-1,0)} \neq 0$). For any n, the expressions of $N_n((v_k))$, $N_n((w_k))$ and $P_n((v_k),(w_k))$ can be computed from the above definitions. One gets

$$N_n^2((v_k)) = 2(P^2 + Q^2) \ , \tag{D.3}$$

$$N_n^2((w_k)) = 2(X_{(0,1,0)}^2 + Y_{(0,1,0)}^2) + 2^{n+1} \sum_{t=(1,\pm1,0)} (X_t^2 + Y_t^2) + 2 \times 5^n \sum_{t=(2,\pm1,0)} (X_t^2 + Y_t^2) \ ;$$

$$
\begin{aligned}
&P_n((v_k),(w_k)) = \qquad \text{(D.4)}\\
&2\Big(- QX_{(0,1,0)}X_{(1,-1,0)} + QX_{(0,1,0)}X_{(1,1,0)} + PX_{(1,-1,0)}Y_{(0,1,0)} + PX_{(1,1,0)}Y_{(0,1,0)}\\
&+PX_{(0,1,0)}Y_{(1,-1,0)} + QY_{(0,1,0)}Y_{(1,-1,0)} - PX_{(0,1,0)}Y_{(1,1,0)} + QY_{(0,1,0)}Y_{(1,1,0)}\Big)\\
&+2^{n+1}\Big(QX_{(0,1,0)}X_{(1,-1,0)} - QX_{(0,1,0)}X_{(1,1,0)} - QX_{(1,-1,0)}X_{(2,-1,0)} + QX_{(1,1,0)}X_{(2,1,0)}\\
&-PX_{(1,-1,0)}Y_{(0,1,0)} - PX_{(1,1,0)}Y_{(0,1,0)} - PX_{(0,1,0)}Y_{(1,-1,0)} - PX_{(2,-1,0)}Y_{(1,-1,0)}\\
&-QY_{(0,1,0)}Y_{(1,-1,0)} + PX_{(0,1,0)}Y_{(1,1,0)} + PX_{(2,1,0)}Y_{(1,1,0)} - QY_{(0,1,0)}Y_{(1,1,0)}\\
&+PX_{(1,-1,0)}Y_{(2,-1,0)} - QY_{(1,-1,0)}Y_{(2,-1,0)} - PX_{(1,1,0)}Y_{(2,1,0)} + QY_{(1,1,0)}Y_{(2,1,0)}\Big)\\
&+2\times 5^n\Big(QX_{(1,-1,0)}X_{(2,-1,0)} - QX_{(1,1,0)}X_{(2,1,0)} + PX_{(2,-1,0)}Y_{(1,-1,0)} - PX_{(2,1,0)}Y_{(1,1,0)}\\
&-PX_{(1,-1,0)}Y_{(2,-1,0)} + QY_{(1,-1,0)}Y_{(2,-1,0)} + PX_{(1,1,0)}Y_{(2,1,0)} - QY_{(1,1,0)}Y_{(2,1,0)}\Big)\ .
\end{aligned}
$$

For any n, inserting the expressions (D.3) (D.4) into Eq. (3.20) we get a lower bound G^-_n depending on the real variables P, Q, X_t, Y_t. Of course, to get the best lower bound of this type one should choose P, Q, X_t, Y_t so as to maximize the ratio $|P_n((v_k),(w_k))|/N_n((v_k))N_n^2((w_k))$ in the right hand side of (3.20).
A search of the maximum has been done for $n = 3, 4, 5, 10$, using the maximization algorithms of MATHEMATICA. The program suggests that the maxima should be attained close to the points (P, Q, X_t, Y_t) reported below. It is not granted that such values actually produce the wanted maxima; in any case, the numbers obtained from (3.20) with these choices of P, Q, X_t, Y_t are lower bounds on G_n, and are the best derivable by the above algorithms.
The values provided by MATHEMATICA are as follows:

$$
\begin{gathered}
n = 3: \qquad P = 1, Q = -7.0796..., \qquad \text{(D.5)}\\
X_{(0,1,0)} = 1, Y_{(0,1,0)} = -5.8246..., X_{(1,-1,0)} = -0.063853..., Y_{(1,-1,0)} = -2.1489...,\\
X_{(1,1,0)} = 0.65657..., Y_{(1,1,0)} = -2.0472..., X_{(2,-1,0)} = -0.043617..., Y_{(2,-1,0)} = 0.39270...,\\
X_{(2,1,0)} = 0.17210..., Y_{(2,1,0)} = -0.35566...\ ;
\end{gathered}
$$

$$
\begin{gathered}
n = 4: \qquad P = 1, Q = -7.0768..., \qquad \text{(D.6)}\\
X_{(0,1,0)} = 1, Y_{(0,1,0)} = -2.7437..., X_{(1,-1,0)} = -0.16319..., Y_{(1,-1,0)} = -0.76896...,\\
X_{(1,1,0)} = 0.36987..., Y_{(1,1,0)} = -0.69363..., X_{(2,-1,0)} = 0.0065160..., Y_{(2,-1,0)} = 0.094627...,\\
X_{(2,1,0)} = 0.055900..., Y_{(2,1,0)} = -0.076628...\ ;
\end{gathered}
$$

$$n = 5: \qquad P = 1, Q = -7.0768..., \tag{D.7}$$

$$X_{(0,1,0)} = 1, Y_{(0,1,0)} = -2.7618..., X_{(1,-1,0)} = -0.12151..., Y_{(1,-1,0)} = -0.57858...,$$

$$X_{(1,1,0)} = 0.27707..., Y_{(1,1,0)} = -0.52225..., X_{(2,-1,0)} = 0.0031227..., Y_{(2,-1,0)} = 0.046786...,$$

$$X_{(2,1,0)} = 0.027554..., Y_{(2,1,0)} = -0.037939... ;$$

$$n = 10: \qquad P = 1, Q = -7.0769..., \tag{D.8}$$

$$X_{(0,1,0)} = 1, Y_{(0,1,0)} = -2.8038..., X_{(1,-1,0)} = -0.031443..., Y_{(1,-1,0)} = -0.15337...,$$

$$X_{(1,1,0)} = 0.072707..., Y_{(1,1,0)} = -0.13865..., X_{(2,-1,0)} = 8.9903 \times 10^{-5}..., Y_{(2,-1,0)} = 0.0014520...,$$

$$X_{(2,1,0)} = 8.4924 \times 10^{-4}..., Y_{(2,1,0)} = -0.0011812... .$$

(Note that the ratio $|P_n((v_k), (w_k))|/N_n((v_k))N_n^2((w_k))$ is invariant under any rescaling $(v_k) \mapsto (\lambda v_k)$, $(w_k) \mapsto (\mu w_k)$, with $\lambda, \mu \in \mathbf{R} \setminus \{0\}$; the normalizations for P and $X_{(0,1,0)}$ adopted above arise from the possibility of such rescalings.)
With the above choices of P, Q, X_t, Y_t (i.e., of (v_k) and (w_k)), one has

$$G_n^- = \begin{cases} 0.11433... & \text{for } n = 3, \\ 0.18128... & \text{for } n = 4, \\ 0.28013... & \text{for } n = 5, \\ 2.4155... & \text{for } n = 10. \end{cases} \tag{D.9}$$

Rounding down to three digits the above numbers, we obtain the results in (3.21).

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