PROJECTABLE VERONESE VARIETIES

ALBERTO ALZATI AND EDOARDO BALLICO

Abstract. Let X be a non degenerate, reduced, reducible algebraic variety embedded in $\mathbb{P}^N$, of pure dimension $m \geq 3$. X is said to be an x-projectable Veronese variety if, assuming $N \geq m + x + 1$, X is of minimal degree, connected in codimension 1 and isomorphically projectable into a linear space of dimension $m + x$.

In this paper we classify 2 and 3-projectable Veronese varieties and x-projectable Veronese varieties having only linear components.

1. Introduction

Let $\mathbb{P}^N$ be the N-dimensional projective space over $\mathbb{C}$. In this paper a variety will always be an algebraic, reduced, of pure dimension, projective scheme embedded in some projective space $\mathbb{P}^N$. For any integer $k \geq 0$, a variety $V \subset \mathbb{P}^N$ is said to be connected in codimension k if for any subvariety $W \subset V$, such that $cod_V(W) > k$, the algebraic set $V \backslash W$ is connected. For any variety $V \subset \mathbb{P}^N$ and for any λ-dimensional linear subspace $\Lambda \subset \mathbb{P}^N$ we say that V projects isomorphically to Λ if there exists a linear projection $\pi_{\mathcal{L}} : \mathbb{P}^N - -- > \Lambda$, from a suitable $(N - \lambda - 1)$-dimensional linear space $\mathcal{L}$, disjoint from V, such that $\pi_{\mathcal{L}}(V)$ is isomorphic to V via the projection $\pi_{\mathcal{L}}$.

In [A-B] we have classified all reducible Veronese surfaces, according to the following definition.

Definition 1. *For any positive integer $n \geq 1$, we will call reducible Veronese surface any algebraic surface $X \subset \mathbb{P}^{n+4}$ such that:*

i) X is a non degenerate, reducible variety of pure dimension 2;

ii) $\deg(X) = n + 3$, $cod(X) = n + 2$, so that X is a minimal degree surface;

iii) $\dim[Sec(X)] \leq 4$, so that it is possible to choose a generic linear space $\mathcal{L}$ of dimension $n - 1$ in $\mathbb{P}^{n+4}$ such $\pi_{\mathcal{L}}(X)$ is isomorphic to X via $\pi_{\mathcal{L}}$, where $\pi_{\mathcal{L}}$ is the the rational projection $\pi_{\mathcal{L}} : \mathbb{P}^{n+4} - -- > \Lambda$, from $\mathcal{L}$ to a generic target $\Lambda \simeq \mathbb{P}^4$;

iv) X is connected in codimension 1, i.e. if we drop any finite number (eventually 0) of points $Q_1, ..., Q_r$ from X we have $X \backslash \{Q_1, ..., Q_r\}$ is connected;

v) X is a locally Cohen-Macaulay surface.

Examples of reducible Veronese surfaces are the surfaces Σ_n introduced by the following:

Date: March, 30th 2010.

1991 *Mathematics Subject Classification.* Primary 14J25; Secondary 14N20.

Key words and phrases. reducible varieties, projectability.

This work is within the framework of the national research projects: "Geometry on Algebraic Varieties" Cofin 2008 of MIUR and "Geometric Properties of Real and Complex Varieties" Cofin 2007 of MIUR..

Definition 2. *For any positive integer* $n \geq 1$, *let us choose a plane* Π_0 *and* $n+2$ *distinct points* $P_1, ..., P_{n+2}$ *in general position in* $\mathbb{P}^{n+4}$, *so that* $\langle \Pi_0 \cup P_1 \cup ... \cup P_{n+2} \rangle = \mathbb{P}^{n+4}$. *Let us choose* $n+2$ *planes* Π_i, $i = 1, ..., n+2$, $P_i \in \Pi_i$, *such that* $\Pi_i \cap \Pi_0$ *is a line* L_i *and the* $n+2$ *lines* L_i *are in general position on* Π_0. *(i.e. the curve given by their union has no triple points). Let us call* Σ_n *any surface in* $\mathbb{P}^{n+4}$ *which is the union* $\Pi_0 \cup \Pi_1 ... \cup \Pi_{n+2}$.

In [A-B] we proved the following:

Theorem 1. *(see Theorems 2 and 3 of* [A-B]*) Let* X *be a reducible Veronese surface. Then we have only three possibilities:*

i) X *is a surface* Σ_n *for some* $n \geq 1$;

ii) $X = Q \cup X_1 \cup X_2$, *where* Q *is a smooth quadric,* X_1 *and* X_2 *are planes, and* Q, X_1, X_2 *intersect transversally along a unique line* $L := Q \cap X_1 \cap X_2$;

iii) $X = Q \cup X_1 \cup X_2$ *as above and* X_1 *and* X_2 *cut* Q *along two lines intersecting at a point* $P := X_1 \cap X_2$.

Here we will consider x-projectable Veronese varieties $X \subset \mathbb{P}^N$ of dimension $m \geq 3$, according to the following definition. Note that the x-projectability is a natural geometric property which a minimal degree algebraic subset may have.

Definition 3. *For any positive integers* $m \geq x \geq 2$, *we will call* x*-projectable Veronese variety any algebraic variety* $X \subset \mathbb{P}^N$, $N \geq m + x + 1$, *such that:*

i) X *is a non degenerate, reducible variety of dimension* $m \geq 3$;

ii) $\deg(X) = cod(X) + 1$, *so that* X *is a minimal degree variety*;

iii) it is possible to choose a generic linear space $\mathcal{L}$ *of dimension* $N - m - 1 - x$ *in* $\mathbb{P}^N$ *such* $\pi_{\mathcal{L}}(X)$ *is isomorphic to* X *via* $\pi_{\mathcal{L}}$, *where* $\pi_{\mathcal{L}}$ *is the the rational projection* $\pi_{\mathcal{L}} : \mathbb{P}^N - -- > \Lambda$, *from* $\mathcal{L}$ *to a generic target* $\Lambda \simeq \mathbb{P}^{m+x}$;

iv) X *is connected in codimension* 1, *i. e. it is possible to arrange the components of* X *in such a way that* $X = X_1 \cup X_2 \cup ... \cup X_r$ *and* $cod_{X_j}[(X_1 \cup ... \cup X_{j-1}) \cap X_j] = 1$ *for any* $j \geq 2$;

v) X *is a locally Cohen-Macaulay variety.*

Remark 1. *Actually v) implies iv) by Corollary* 2.4 *of* [Ha], *however it is more useful to give the above definitions 1 and 3 because condition iv) is crucial to get the classifications. The assumption* $m \geq x$ *implies that* $m + x \leq 2m$.

Remark 2. *Note that if condition iii) of Definition 3 is satisfied then* $\dim[Sec(X)] \leq m + x$. *We will often use this fact to prove that iii) is not satisfied.*

In this paper we give a precise description of all x-projectable Veronese varieties X having only linear components (see Theorem 5), we prove that there are no 2-projectable varieties with $m \geq 3$ (see Theorem 4) and we give a complete classification of 3-projectable varieties by the following result.

Theorem 2. *Let* $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^N$ *be a 3-projectable Veronese variety, according to Definition 3*, $\dim(X) = m \geq 3$. *Then* $\dim(X) = 3$ *and* X *is one of the following:*

1) $X \subset \mathbb{P}^{r+2}$ *has only linear components,* $r \geq 5$ *and there exists a fixed component* X_i *such that all other components intersect* X_i *along planes of* X_i *in general position;*

2) $X \subset \mathbb{P}^{r+2}$ *has only linear components,* $r \geq 5$ *and there are two fixed components* X_i *and* X_j *such that* $X_i \cap X_j := \Pi \simeq \mathbb{P}^2$ *and all other components intersect* X_i

along planes in general position, or intersect X_j along planes in general position, and intersect each other along lines in general position in Π;

3) $X = X_1 \cup X_2 \subset \mathbb{P}^7$, *$X_1$ is a cone of degree* 3 *having a line E_1 as vertex and a twisted cubic as base, X_2 is a cone of degree* 2 *having a point E_2 as vertex and $X_1 \cap X_2$ is a plane containing E_1 and E_2 with* $E_1 \cap E_2 = \emptyset$;

4) $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{P}^7$; *$X_1$ is a quadric cone having a line E_1 as vertex, X_2 is a quadric cone having a point E_2 as vertex, $X_3 \simeq \mathbb{P}^3$; $X_1 \cap X_2 \cap X_3$ is a plane F, the vertexes are disjoint (the role of X_2 and X_3 can be exchanged);*

5) $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{P}^7$; *$X_1$ is a quadric cone having a line E_1 as vertex, X_2 is a quadric cone having a point E_2 as vertex, $X_3 \simeq \mathbb{P}^3$; $X_1 \cap X_2$ is a plane F, $X_3 \cap X_1$ is another plane F', $X_3 \cap X_2 = E_1 \subset F$, the vertexes are disjoint (the role of X_2 and X_3 can be exchanged);*

6) $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^{r+3}$, $r \geq 4$; *X_1 is a quadric cone having a line E_1 as vertex, $X_i \simeq \mathbb{P}^3$ for $i \geq 2$; $X_1 \cap X_2$ is a plane F; $X_i \cap X_2$ are planes in generic position in X_2 intersecting lines $L_i \subset F$ in generic position for* $i \geq 3$;

7) $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^{r+3}$, $r \geq 4$; *X_1 is a quadric cone having a point E_1 as vertex, $X_i \simeq \mathbb{P}^3$ for $i \geq 2$; $X_1 \cap X_2$ is a plane F; $X_i \cap X_2$ are planes in generic position in X_2 intersecting lines $L_i \subset F$ in generic position for $i \geq 3$; possibly one of the components X_p, $3 \leq p \leq r$, is exceptional: it intersects X_1 along another plane F', cutting F along a line l in generic position with respect to the set* $\{L_i,\, i \neq p\}$;

8) $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^{r+3}$, $r \geq 4$; *X_1 is a quadric cone having a point E_1 as vertex, $X_i \simeq \mathbb{P}^3$ for $i \geq 2$; $X_1 \cap X_2$ is a plane F; $X_i \cap X_2$ are planes in generic position in X_2 intersecting lines $L_i \subset F$ in generic position for $i = 3, ..., r-1$; there exists a fixed j, $3 \leq j \leq r-1$, such that $X_r \cap X_j$ is a plane and $X_r \cap X_1 = L_j$ ($E_1 \notin L_j$);*

9) $X = X_1 \cup X_2 \cup X_3 \cup X_4 \subset \mathbb{P}^7$; *$X_1$ is a quadric cone having a point E_1 as vertex, $X_i \simeq \mathbb{P}^3$ for $i \geq 2$; $X_1 \cap X_2 \cap X_3$ is a plane F; $X_4 \cap X_2$ is a plane Π, $X_4 \cap X_3 = X_4 \cap X_1$ is a line $L = \Pi \cap F$, not passing through E_1, or $X_4 \cap X_3$ is a plane Π, $X_4 \cap X_2 = X_4 \cap X_1$ is a line $L = \Pi \cap F$, not passing through E_1.*

Our strategy will be: firstly to consider the case in which all components of X are linear spaces, secondly to use Remark 2 to get a short list of possibilities, thirdly to prove (or disprove) that X projects isomorphically by checking that the union of all Zariski tangent spaces at points $P \in X$ is disjoint with a general linear subspace of dimension $N - m - 1 - x$ of $\mathbb{P}^N$ (see Theorem 6.5 of [Ho], page 168, and Propositions 4.2 and 5.3 for the definiton of $Zarsec(X, i)$, see also Corollary 2 of [J]).

We will use the following definitions:

$\langle V_1 \cup ... \cup V_r \rangle$: linear span in $\mathbb{P}^N$ of the varieties $V_i \subset \mathbb{P}^N$, $i = 1, ..., r$; when $V_1, ..., V_r$ are linear spaces we often write $V_1 \cup ... \cup V_r$ instead of $\langle V_1 \cup ... \cup V_r \rangle$;

$Sing(V)$: singular locus of the subscheme $V \subset \mathbb{P}^N$;

$T_P(V)$: Zariski tangent space at a point P of V; it is a projective subspace of $\mathbb{P}^N$ whose dimension is the embedding dimension of the local ring $\mathcal{O}_{V,P}$;

$\mathbb{T}_P$: union of $T_P(V_i)$ for all irreducible components $V_i \subset V$ containing P; note that the linear span of $\mathbb{T}_P$ is always contained in $T_P(V)$ for any point $P \in V$;

$[V; W] : \overline{\{ \bigcup_{v \in V, w \in W, v \neq w} \langle v \cup w \rangle \}} \subset \mathbb{P}^N$, join of V and W, for any pair of distinct irreducible varieties $V, W \subset \mathbb{P}^N$. In case $V = W$, $[V; V] = Sec(V)$

$Sec(V) : \overline{\{\bigcup_{v_1 \neq v_2 \in V} \langle v_1 \cup v_2 \rangle\}} \subset \mathbb{P}^N$ for any variety $V \subset \mathbb{P}^N$,
if $V = V_1 \cup ... \cup V_r$ is reducible then $Sec(V) = \bigcup_{i,j=1,...,r} [V_i; V_j]$.

We will often use the following facts: if V is an irreducible variety, not a linear space, for which there exists a linear space L, such that for any generic point $P \in V$, $T_P(V) \supseteq L$, then V is a cone whose vertex contains L (see [A2], page 17, but recall that, in our paper, a cone is never a linear space); if $V = V_1 \cup ... \cup V_r$ is reducible and x-projectable then $\dim([V_i; V_j]) \leq m + x$ for any $i, j = 1, ..., r$.

Let $a_1 \geq a_2 \geq ... \geq a_k > 0$ be a set of integers, $k \geq 1$. Let us consider the rank $m \geq k$ vector bundle $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) .... \oplus \mathcal{O}_{\mathbb{P}^1}(a_k) \oplus \mathcal{O}_{\mathbb{P}^1} .... \oplus \mathcal{O}_{\mathbb{P}^1}$ over $\mathbb{P}^1$ and let V be $\mathbb{P}(\mathcal{E})$. If $m = k \geq 2$, we say that V is a *smooth rational scroll* of degree $\sum_{i=1}^{k} a_i$, embedded in a projective space of dimension $\sum_{i=1}^{k} a_i + k - 1$ as a linearly normal variety. If $m > k \geq 2$, we say that V is a *cone over a smooth rational scroll*, having a vertex E of dimension $e := m - k - 1$; in this case $\deg(V) = \sum_{i=1}^{k} a_i$, but V is embedded in a projective space of dimension $\sum_{i=1}^{k} a_i + m - 1$ as a linearly normal variety. If $m > k = 1$, $a_1 \geq 2$, we say that V is a *cone over a rational normal curve* of degree a_1, having a vertex E of dimension $e = m - 2$; in this case $\deg(V) = a_1$ and V is embedded in a projective space of dimension $a_1 + m - 1$ as a linearly normal variety. In all these cases V is a variety of minimal degree, i.e. $\deg(V) = cod(V) + 1$, in its span which has always dimension $\sum_{i=1}^{k} a_i + m - 1$.

Acknowledgements: we wish to thank the referee for his very careful revision of the manuscript and for having pointed out a missing case in the first version.

2. Xambó's result and first remarks

In Theorem 1 of [X] Xambó proves the following result:

Theorem 3. *Let $V = V_1 \cup .... \cup V_r \subset \mathbb{P}^N$ be a non degenerate, reducible, reduced, variety of pure dimension $m \geq 2$, whose irreducible components are $V_1, ..., V_r$. Assume that V is connected in codimension 1 and that it has minimal degree, then:*

- any irreducible component V_i of V is an irreducible variety of dimension m and minimal degree in its span $\langle V_i \rangle$;

- there is at least an ordering $V_1, V_2, ..., V_r$ such that, for any $j = 2, ..., r$, $V_j \cap (V_1 \cup ... \cup V_{j-1}) = \langle V_j \rangle \cap \langle V_1 \cup ... \cup V_{j-1} \rangle$ and this intersection is always a linear space of dimension exactly $m - 1$.

From now on an ordering given by Theorem 3 will be called a *good ordering.*

Corollary 1. *Let V be any variety as in Theorem 3. Let d_i be the degree of V_i. Then:*

i) for any pair of irreducible components V_j, $V_k \subset V$ we have only three possibilities:

- $V_j \cap V_k = \langle V_j \rangle \cap \langle V_k \rangle = \emptyset$

- $V_j \cap V_k = \langle V_j \rangle \cap \langle V_k \rangle$ is a point

- $V_j \cap V_k = \langle V_j \rangle \cap \langle V_k \rangle$ is a linear space of dimension δ with $1 \leq \delta \leq m - 1$;

ii) for any $j \geq 2$, there is at least a component V_k with $k < j$ such that $V_j \cap V_k \simeq \mathbb{P}^{m-1}$ and the other intersections $V_j \cap V_i$, with $i < j$, are linear spaces contained in $V_j \cap V_k$, possibly coincident with it;

iii) $N = \dim(\langle V \rangle) = \sum_{i=1}^{r} \dim(\langle V_i \rangle) - (r-1)(m-1) = \sum_{i=1}^{r} d_i + m - 1$;

iv) for any point $P \in V$, $T_P(V) = \langle \mathbb{T}_P \rangle$.

Proof. *i)* Let us assume that $V_j \cap V_k \neq \emptyset$ and that $k > j$ in a good ordering for the components of V. Then $V_j \cap V_k \subseteq V_k \cap (V_1 \cup ... V_j \cup ... \cup V_{k-1})$ which is a linear space of dimension $m-1$, as a scheme, because it is the intersection of two linear spaces in $\mathbb{P}^N$. By Theorem 0.4 of [E-G-H-P] V is small according to the definition of [E-G-H-P], p.1364, hence $V_j \cap V_k = \langle V_i \rangle \cap \langle V_j \rangle$ is a linear space by Proposition 2.4 of [E-G-H-P]. As $V_j \cap V_k$ is contained in a linear space of dimension $m-1$, Corollary 1 *i)* follows.

ii) By *i)* we know that $V_j \cap (V_1 \cup ... \cup V_{j-1}) = (V_j \cap V_1) \cup ... \cup (V_j \cap V_{j-1})$ is the union of linear spaces of dimension $m-1$ at most. On the other hand we know that $V_j \cap (V_1 \cup ... \cup V_{j-1})$ is in fact a unique linear space of dimension exactly $m-1$ by Theorem 3 and *ii)* follows.

iii) The first equality follows from the fact that, for any $j = 2, ..., r$, $\dim(\langle V_j \rangle \cap \langle V_1 \cup ... \cup V_{j-1} \rangle) = m-1$; the second equality follows from the fact that $\dim(\langle V_j \rangle) = m + d_j - 1$ for any j.

iv) Recall that $\mathbb{T}_P$ is the union of the Zariski tangent spaces $T_P(V_i)$ for all irreducible components $V_i \subset V$ containing P. In our case $T_P(V)$ is the linear span of $\mathbb{T}_P$ thanks to property *i)*. ∎

By Theorem 3 it follows that any irreducible component of an x-projectable Veronese variety X is an irreducible, m-dimensional variety of minimal degree in its span containing a linear space of dimension $m-1$. From the well known classification of such varieties (see for instance Theorem 0.1 of [E-G-H-P]) we have the following:

Corollary 2. *Let X_i be any irreducible component of an x-projectable Veronese variety $X \subset \mathbb{P}^N$, of dimension $m \geq 3$. Then, a priori, we have only the following possibilities:*

i) X_i is a linear space in $\mathbb{P}^N$ of dimension m;

ii) X_i is a cone, having a vertex of dimension $e_i \geq 0$ over a smooth scroll having fibres of dimension $m - e_i - 2 \geq 1$;

iii) X_i is a cone, having a vertex of dimension $e_i \geq 0$ over a rational normal curve;

iv) X_i is a smooth rational scroll and $m \leq x + 1$.

Proof. By looking at Theorem 0.1 of [E-G-H-P], where irreducible, m-dimensional varieties of minimal degree in their spans are listed, we get X_i may be a linear space or a hyperquadric, or as in *ii), iii)*, or a smooth rational scroll, or a cone over a Veronese surface in $\mathbb{P}^5$. As X_i must contain a linear space of dimension $m-1$ we can exclude cones over Veronese surfaces because such a surface does not contain lines. For the same reason we can exclude hyperquadrics of rank ≥ 5; a hyperquadric of rank 4 is a cone over a smooth quadric of $\mathbb{P}^3$ which is a smooth rational scroll; a hyperquadric of rank 3 is a cone over a rational normal curve: a smooth conic. If X_i is a smooth rational scroll its fibres of dimension $m-1$ are disjoint, so they have to remain disjoint after projecting X in $\mathbb{P}^{m+x}$, but this is not possible if $m \geq x + 2$. ∎

Proposition 1. *Let V be a variety as in Theorem 3.*

i) Let V_i be any component of V. Then there exists at least a good ordering for the r components of V such that $V_i = V_1$;

ii) Let $W = W_1 \cup ... \cup W_k \subset V$ be the union of some components of V, $k < r$, such that the assumptions of Theorem 3 are true for W. Then there exists at least a good ordering for the r components of V such that $V_i = W_i$ for $i = 1, ..., k$.

Proof. *i)* Let $W \subset V$ be a proper subvariety of V such that $W = V_1 \cup .... \cup V_\rho$ with $1 \leq \rho < r$. Let us assume that W is connected in codimension 1. Then we claim the existence of at least a component $V_i \subset V$ such that $\dim(W \cap V_i) = m - 1$ and $W \cup V_i$ is connected in codimension 1.

In fact, if $\dim(W \cap V_i) \leq m - 2$ for any irreducible component $V_i \subset V$ with $\rho < i \leq r$, then $\dim[W \cap (V_{\rho+1} \cup ... \cup V_r)] \leq m-2$, but this is not possible, otherwise $V \backslash [W \cap (V_{\rho+1} \cup ... \cup V_r)]$ would be not connected while we are assuming that V is connected in codimension 1. Hence, by changing the ordering of $V_{\rho+1}, ..., V_r$ if necessary, we can assume that $\dim(W \cap V_{\rho+1}) \geq m - 1$. It is not possible that $\dim(W \cap V_{\rho+1}) \geq m$, otherwise the irreducible surface $V_{\rho+1}$ would be a component of W, so that $\dim(W \cap V_{\rho+1}) = m - 1$. Let us consider $W \cup V_{\rho+1}$. W is connected in codimension 1 by assumptions, $V_{\rho+1}$ is connected in codimension 1 because it is an irreducible variety of dimension m; as $\dim(W \cap V_{\rho+1}) = m - 1$ we have $W \cup V_{\rho+1}$ is connected in codimension 1 too.

Now, let us choose a component $V' \subset V$ and let us consider a good ordering. If $V' = V_1$ we have nothing to prove, otherwise, in any case, there exists at least another component V_j such that $V' \cap V_j \simeq \mathbb{P}^{m-1}$. Let us prove that there exists another good ordering having V' at the first position. Set $V_1 = V'$ and $V_2 = V_j$. If $r = 2$ we are done, if not we can apply the above remark with $W = V_1 \cup V_2$ and we get another component V_3 such that $\dim(W \cap V_3) = m-1$ and $W \cup V_3$ is connected in codimension 1. By applying the above remark a suitable number of times we get an ordering $V_1, ..., V_r$ such that $V_1 = V'$, $\dim[V_j \cap (V_1 \cup ... \cup V_{j-1})] = m - 1$ for any $j \geq 2$, and $V_1 \cup ... \cup V_j$ is connected in codimension 1. As $V_j \cap (V_1 \cup ... \cup V_{j-1}) \subseteq (\langle V_j \rangle \cap \langle V_1 \cup ... \cup V_{j-1} \rangle)$ for any $j \geq 2$, we have only to prove that $\dim(\langle V_j \rangle \cap \langle V_1 \cup ... \cup V_{j-1} \rangle) = m - 1$ to get $V_j \cap (V_1 \cup ... \cup V_{j-1}) = \langle V_j \rangle \cap \langle V_1 \cup ... \cup V_{j-1} \rangle \simeq \mathbb{P}^{m-1}$ for any $j \geq 2$, hence to prove the Proposition.

Let us put $\deg(V_i) = d_i$; obviously $\dim(\langle V_i \rangle) = m - 1 + d_i$.

Let us put $a_j := \dim(\langle V_j \rangle \cap \langle V_1 \cup ... \cup V_{j-1} \rangle)$ for any $j \geq 2$, so that:

$\dim(\langle V_1 \cup V_2 \rangle) = m - 1 + d_1 + m - 1 + d_2 - a_2$

$\dim(\langle V_1 \cup V_2 \cup V_3 \rangle) = \dim(\langle \langle V_1 \cup V_2 \rangle \cup \langle V_3 \rangle \rangle) =$

$= m - 1 + d_1 + m - 1 + d_2 - a_2 + m - 1 + d_3 - a_3$

....

$\dim(\langle V_1 \cup V_2 \cup ... \cup V_r \rangle) = r(m - 1) + \sum\limits_{j=1}^{r} d_j - \sum\limits_{j=2}^{r} a_j = N.$

On the other hand: $\dim(\langle V \rangle) = N = m - 1 + \deg(V) = m - 1 + \sum\limits_{j=1}^{r} d_j$ so that we have: $r(m - 1) - \sum\limits_{j=2}^{r} a_j = m - 1$, i.e. $(r - 1)(m - 1) = \sum\limits_{j=2}^{r} a_j$.

As $a_j \geq m - 1$ for any $j \geq 2$ we get $a_j = m - 1$ for any $j \geq 2$ and we are done.

ii) As in the proof of *i)* we can do induction, starting from W instead of $V' = V_1$, knowing that $a_j = m - 1$ for $2 \leq j \leq k$. ∎

Let us recall the Terracini's lemma:

Lemma 1. *Let us consider a pair of irreducible varieties $V, W \subset \mathbb{P}^N$ and a generic point $R \in [V;W]$ such that $R \in \langle P \cup Q \rangle$, with $P \in V$ and $Q \in W$, (hence P and Q are generic points of V and W, respectively), then $T_R([V;W]) = \langle T_P(V) \cup T_Q(W) \rangle$ and $\dim([V;W]) = \dim(\langle T_P(V) \cup T_Q(W) \rangle)$.*

Proof. See Corollary 1.11 of [A1]. ∎

Corollary 3. *Let $V, W \subset \mathbb{P}^N$ be two irreducible varieties such that $V \cap W = \emptyset$, then $\dim([V;W]) = \dim(V) + \dim(W) + 1$.*

Proof. Obviously $\dim([V;W]) \leq \dim(V) + \dim(W) + 1$; if $V \cap W = \emptyset$ we have $\dim([V;W]) \geq \dim(V) + \dim(W) + 1$, see for instance Corollary 2.5 of [A1]. ∎

Lemma 2. *Let $X \subset \mathbb{P}^N$ be an x-projectable Veronese variety. Then:*
i) any irreducible component $X_i \subset X$ can be isomorphically projected in $\mathbb{P}^{m+x}$;
ii) for any pair of irreducible components X_j and X_k of X we have $\dim(X_j \cap X_k) \geq m - x$, and, unless X_j and X_k are both linear spaces, $\dim(X_j \cap X_k) \geq m - x + 1$;
iii) for any irreducible, not linear, component $X_i \subset X$ let Y_i be any linear space of maximal dimension $m-1$ contained in X_i, then $Y_j \cap Y_k \neq \emptyset$ if $m \geq x+2$ and $X_j \cap Y_k \neq \emptyset$ if $m \geq x+1$.

Proof. i) Obvious.
ii) Since X is an x-projectable Veronese variety, there exists a projection $\pi_{\mathcal{L}} : \mathbb{P}^N - - - > \Lambda$ from a suitable linear space $\mathcal{L}$ to a suitable linear space $\Lambda \subset \mathbb{P}^N$, $\Lambda \simeq \mathbb{P}^{m+x}$, such that $\pi_{\mathcal{L}}(X) \simeq X$. This implies that, for any $i = 1, ..., r$, $\pi_{\mathcal{L}}(X_i) \simeq X_i$, and, for any pair $X_j, X_k \subset X$, $\pi_{\mathcal{L}}(X_j \cap X_k) \simeq \pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k) \simeq X_j \cap X_k = \langle X_j \rangle \cap \langle X_k \rangle$ is a linear space by Corollary 1. As $\pi_{\mathcal{L}}(X_j)$ and $\pi_{\mathcal{L}}(X_k)$ are two irreducible varieties of dimension m in $\mathbb{P}^{m+x}$ we have $\dim[\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)] \geq m - x$. If $\dim[\pi_{\mathcal{L}}(X_j) \cap \pi_{\mathcal{L}}(X_k)] = m - x$ we have two irreducible varieties of dimension m in $\mathbb{P}^{m+x}$ whose scheme-theoretic intersection is a linear space of dimension $m - x$. By Bezout's theorem this is possibly only if they are both linear spaces.
iii) Let us consider any pair of linear spaces Y_j and Y_k. By contradiction let us suppose that $Y_j \cap Y_k = \emptyset$, then, by Corollary 3, their join has dimension $m - 1 + m - 1 + 1 = 2m - 1 \leq m + x$, because $\dim[Sec(X)] \leq m + x$, and this is not possible if $m \geq x + 2$. Let us consider any pair X_j and Y_k. By contradiction let us suppose that $X_j \cap Y_k = \emptyset$, then, by Corollary 3, their join has dimension $m + m - 1 + 1 = 2m \leq m + x$, because $\dim[Sec(X)] \leq m + x$, and this is not possible if $m \geq x + 1$. ∎

Lemma 3. *Let $X \subset \mathbb{P}^N$ be an x-projectable Veronese variety. Let P be a singular point of X and let $X_1^P, \ldots, X_s^P$ be the irreducible components of X containing P with $s \geq 2$. For any $i = 1, ..., s$ let T_i be the tangent space of X_i^P at P in $\langle X_i^P \rangle$ and let us assume that the natural ordering of $X_1^P, \ldots, X_s^P$ is coherent with a good ordering. Then, for any $j = 2, ..., s$, $T_j \nsubseteq \langle T_1 \cup ... \cup T_{j-1} \rangle$ and $\dim[T_j \cap \langle T_1 \cup ... \cup T_{j-1} \rangle] \leq m - 1$.*

Proof. By contradiction, let us assume that $T_j \subseteq \langle T_1 \cup ... \cup T_{j-1} \rangle$, hence $T_j \subseteq T_j \cap \langle T_1 \cup ... \cup T_{j-1} \rangle \subseteq \langle X_j^P \rangle \cap \langle X_1^P \cup ... \cup X_{j-1}^P \rangle$. As we are assuming that the natural ordering of $X_1^P, ..., X_s^P$ is coherent with a good ordering, we have $\dim[\langle X_j^P \rangle \cap \langle X_1^P \cup ... \cup X_{j-1}^P \rangle] \leq m - 1$. Moreover $\dim(T_j) = m$ if P is a smooth point of X_j^P

and $\dim(T_j) \geq m+1$ if P is a singular point of X_j^P. So that in any case we get a contradiction. By the way we have also proved that $\dim[T_j \cap \langle T_1 \cup ... \cup T_{j-1}\rangle] \leq m-1$. ∎

Lemma 4. *Let $X \subset \mathbb{P}^N$ be an x-projectable Veronese variety. Let P be any point of X and let $X_1^P, \ldots, X_s^P$ be the irreducible components of X containing P, $s \geq 1$. For any $i = 1, ..., s$ let T_i be the Zariski tangent space of X_i^P at P in $\langle X_i^P \rangle$ and let us define $\mathbb{T}_P := \bigcup_{i=1}^{s} T_i$. Then $\dim(\langle \mathbb{T}_P \rangle) \leq m+x$. Moreover, if $P \in Sing(X_i)$ $T_i = \langle X_i \rangle$ and, for any $P \in X$, we have $T_P(X) = \langle \mathbb{T}_P \rangle$.*

Proof. If $s = 1$ we have $\langle \mathbb{T}_P \rangle = T_1$ and $\dim(T_1) \leq m+x$ by Lemma 2 *i*). If $s \geq 2$, $\mathbb{T}_P$ is the union of s linear spaces, of dimensions $\geq m$, passing through P according a certain configuration $\mathcal{C}_P \subset \mathbb{P}^N$. By contradiction, let us assume that $\dim(\langle \mathbb{T}_P \rangle) \geq m+x+1$. Let $\pi_{\mathcal{L}} : \mathbb{P}^N - -- > \Lambda$ be any linear projection, from a suitable $(N-m-1-x)$-dimensional linear space $\mathcal{L}$ to a suitable $\Lambda \subset \mathbb{P}^N$, $\Lambda \simeq \mathbb{P}^{m+x}$, such that $\pi_{\mathcal{L}}(X)$ is isomorphic to X, hence $\pi_{\mathcal{L}}(\mathcal{C}_P)$ is isomorphic to $\mathcal{C}_P$. As $\dim(\langle \mathbb{T}_P \rangle) \geq m+x+1$ there is a non empty linear space $\mathcal{L}' := \mathcal{L} \cap \langle \mathbb{T}_P \rangle$ such that $\pi_{\mathcal{L}}(\mathcal{C}_P) = \pi_{\mathcal{L}'}(\mathcal{C}_P)$ where $\pi_{\mathcal{L}'} : \langle \mathbb{T}_P \rangle - -- > \Lambda$. But, as $\dim(\Lambda) < \dim(\langle \mathbb{T}_P \rangle)$, it is not possible that $\pi_{\mathcal{L}'}(\mathcal{C}_P) \simeq \mathcal{C}_P$, otherwise isomorphic configurations of linear spaces would have linear spans of different dimensions, so that we get a contradiction.

Now, by Corollary 2, we have $T_i = \langle X_i \rangle$ if $P \in Sing(X_i)$ and, by Corollary 1 *iv*) the Zariski tangent space $T_P(X)$ is exactly $\langle \mathbb{T}_P \rangle$ for any point $P \in X$. ∎

Corollary 4. *Let $X \subset \mathbb{P}^N$ be an x-projectable Veronese variety,* $\dim(X) = m \geq 3$. *Let P be any point of X. Then there are at most $x+1$ irreducible components of X passing through P.*

Proof. Let us assume that there are at least two irreducible components of X passing through P, so that $P \in Sing(X)$. We know that $\dim(\langle \mathbb{T}_P \rangle) \leq m+x$ by Lemma 4, on the other hand we can apply Lemma 3 to the set $\{X_1^P, ..., X_s^P\}$ of irreducible components containing P and we have $\dim(T_1) \geq m$ and $\dim(\langle T_1 \cup ... \cup T_j \rangle)$ increases of a unity at least for $j = 2, ..., s$. If $s \geq x+2$ we would have $\dim(\langle \mathbb{T}_P \rangle) = \dim(\langle T_1 \cup ... \cup T_s \rangle) \geq m+s-1 \geq m+x+1$, contradiction. ∎

Proposition 2. *Let $X \subset \mathbb{P}^N$ be an x-projectable Veronese variety,* $\dim(X) = m \geq 3$. *Let X_i be an irreducible component of X, of degree d_i, which is neither a linear space nor a smooth scroll, then $X_i = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle over $\mathbb{P}^1$ of the following type:*

$\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_{k_i}) \oplus \mathcal{O}_{\mathbb{P}^1} .... \oplus \mathcal{O}_{\mathbb{P}^1}$, $X_i \subset \mathbb{P}^{m+d_i-1}$, $d_i = \sum_{j=1}^{k_i} a_j \leq x$.

If $x = 3$ we have only the following possibilities:

a) $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} ... \oplus \mathcal{O}_{\mathbb{P}^1}$, $X_i \subset \mathbb{P}^{m+2}$, $d_i = 3$

b) $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} ... \oplus \mathcal{O}_{\mathbb{P}^1}$, $X_i \subset \mathbb{P}^{m+2}$, $d_i = 3$

c) $\mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1} ... \oplus \mathcal{O}_{\mathbb{P}^1}$, $X_i \subset \mathbb{P}^{m+2}$, $d_i = 3$

d) $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} ... \oplus \mathcal{O}_{\mathbb{P}^1}$, $X_i \subset \mathbb{P}^{m+1}$, $d_i = 2$

e) $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} ... \oplus \mathcal{O}_{\mathbb{P}^1}$, $X_i \subset \mathbb{P}^{m+1}$, $d_i = 2$.

The above cones have a vertex E_i of dimension $e_i = m - k_i - 1 \geq 0$ and none of them can be isomorphically projected into a linear space unless $\dim(\langle X_i \rangle) = m + d_i - 1 \leq m + x - 1$ *and the linear span of the cone is isomorphically projected.*

Proof. Let us fix a good ordering among the irreducible components of X; we know that X_i is a variety of minimal degree in its span. Let us assume that X_i is a cone as in *ii*) or *iii*) of Corollary 2 and let P be a point in $Sing(X_i) = E_i$. By Lemma 4 we know that $\dim[T_P(X_i)] \leq m + x$, hence $\dim \langle X_i \rangle = \dim[T_P(X_i)] \leq m + x$.

By contradiction, let us assume that $\dim \langle X_i \rangle = \dim[T_P(X_i)] = m + x$. If $i \neq 1$ we have $\langle X_1 \cup ... \cup X_{i-1} \rangle \cap \langle X_i \rangle = (X_1 \cup ... \cup X_{i-1}) \cap X_i \simeq \mathbb{P}^{m-1}$ and we can choose a component X_j, with $j < i$, such that $X_j \cap X_i = \langle X_j \rangle \cap \langle X_i \rangle \simeq \mathbb{P}^{m-1}$. If $i = 1$ let us consider X_2 and we have $X_1 \cap X_2 = \langle X_1 \rangle \cap \langle X_2 \rangle \simeq \mathbb{P}^{m-1}$. In any case we can find another component X_j of X such that $X_j \cap X_i = \langle X_j \rangle \cap \langle X_i \rangle \simeq \mathbb{P}^{m-1}$. Note that $P \in X_j$ because E_i is contained in all $(m-1)$-dimensional linear spaces contained in X_i and one of them is containd also in X_j. Let us consider $T_P(X_j) \cup T_P(X_i)$, by Lemma 4 we know that $\dim[T_P(X_j) \cup T_P(X_i)] \leq m + x$, hence $T_P(X_j) \subseteq T_P(X_i) = \langle X_i \rangle$, so that $T_P(X_j) \subseteq \langle X_j \rangle \cap \langle X_i \rangle$. As $\dim[T_P(X_j)] \geq m$ this is a contradiction.

The conclusion is that $\dim \langle X_i \rangle = \dim[T_P(X_i)] \leq m + x - 1$. Now let us recall that for a cone as *ii*) or *iii*) of Corollary 2 the linear span has dimension: $\sum_{j=1}^{k} a_j + m - 1 = d_i + m - 1$. Hence we get $d_i + m - 1 \leq m + x - 1$ and $d_i \leq x$. It is easy to see that if $x = 3$ the only possibilities are the above ones.

Let X_i be a cone as above, over a smooth base B_i and having a vertex E_i. Note that $E_i \cap \langle B_i \rangle = \emptyset$; $\dim(X_i) = \dim(B_i) + \dim(E_i) + 1$; $\dim(\langle X_i \rangle) = \dim \langle B_i \rangle + \dim(E_i) + 1$. Hence $Sec(X_i)$ contains the cone over $Sec(B_i)$ having vertex E_i and $\dim[Sec(X_i)] \geq \dim[Sec(B_i)] + \dim(E_i) + 1$. Therefore X_i can be isomorphically projected into a linear space of dimension smaller than the dimension of its linear span (i.e. $\dim[Sec(X_i) < \dim(\langle X_i \rangle))$ only if this is true also for B_i (i.e. $\dim[Sec(B_i)] < \dim(\langle B_i \rangle)$). By the well known classification of smooth irreducible varieties of small degree, which are all projectively normal, it follows that this is not possible, so that this is not possible for every X_i too. ∎

Proposition 3. *Let $X \subset \mathbb{P}^N$ be an x-projectable Veronese variety,* $\dim(X) = m \geq 3$. *Let X_i be an irreducible component of X which is a smooth scroll of degree d_i, then $X_i = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle over $\mathbb{P}^1$ of the following type:*

$\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_m)$, $X_i \subset \mathbb{P}^{m+d_i-1}$, $\sum_{j=1}^{m} a_j - 1 = d_i - 1 \leq x$, $m \leq x + 1$.

If $x = 3$ we have only the following possibilities:

$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $X_i \subset \mathbb{P}^6$, $m = 3$, $\deg(X_i) = 4$

$\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $X_i \subset \mathbb{P}^5$, $m = 3$, $\deg(X_i) = 3$.

None of the above scrolls can be isomorphically projected into a linear space unless the linear span of the scroll is isomorphically projected.

Proof. By Corollary 2 *iv*), $m \leq x + 1$. By Lemma 2 *i*) we know that X_i must be isomorphically projected into $\mathbb{P}^{m+x}$, but $\dim[Sec(X_i)] = \min\{\dim(\langle X_i \rangle), 2m + 1\}$ (see [C]), hence it must be $\min\{\dim(\langle X_i \rangle), 2m + 1\} \leq m + x$. As $2m + 1 > m + x$, it must be $\dim(\langle X_i \rangle) \leq m + x$. As in the proof of Proposition 2 $\dim(\langle X_i \rangle) = \sum_{j=1}^{m} a_j + m - 1 = d_i + m - 1$, hence $d_i + m - 1 \leq m + x$.

If $x = 3$ it is easy to see that the only possibilities are the above ones and the following one: $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $X_i \subset \mathbb{P}^7$, $m = 4$, $\deg(X_i) = 4$.

However this last possibility cannot occur. By contradiction, let us assume that X has a component as above. By Proposition 1 we can assume that the component is X_1, note that $X_1 \simeq \mathbb{P}^1 \times \mathbb{P}^3$. Let Z_{1j} be the linear space which is the intersection $X_1 \cap X_j = \langle X_1 \rangle \cap \langle X_j \rangle$ for $j \geq 2$. By Lemma 2 *ii*), $\dim(Z_{1j}) \in \{3, 2\}$ and, by Lemma 2 *iii*), Z_{1j} must intersect any linear space of dimension 3 contained in X_1. But it is easy to see that in $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ all linear spaces of dimension 2 and 3 are contained in a unique linear space of dimension 3. ∎

Corollary 5. *Let $X \subset \mathbb{P}^N$ be an x-projectable Veronese variety,* $\dim(X) = m \geq 3$. *Let us assume that X contains an irreducible component X_i which is a cone of degree d_i, having a vertex E_i of dimension $e_i \geq 0$. Let Z_{ij} be the linear space which is the intersection $X_i \cap X_j = \langle X_i \rangle \cap \langle X_j \rangle$ for $j \neq i$, then* $\dim(E_i \cap Z_{ij}) \geq e_i - x + 2$. *When $x = 3$ this fact implies:*

a) if X_i is a cone of degree 3, $e_i = m - 4$ (necessarily $m \geq 4$) $Z_{ij} \cap E_i \neq \emptyset$;
b) if X_i is a cone of degree 3, $e_i = m - 3$, $Z_{ij} \cap E_i \neq \emptyset$ when $m \geq 4$;
c) if X_i is a cone of degree 3, $e_i = m - 2$, $Z_{ij} \cap E_i \neq \emptyset$;
d) if X_i is a cone of degree 2, $e_i = m - 3$, $Z_{ij} \cap E_i \neq \emptyset$ when $m \geq 4$;
e) if X_i is a cone of degree 2, $e_i = m - 2$, $Z_{ij} \cap E_i \neq \emptyset$.

Proof. By Corollary 1 and Lemma 2 *ii*) we know that $m-1 \geq \dim(Z_{ij}) \geq m-x+1$. If $\dim(Z_{ij}) = m - 1$ then Z_{ij} contains the vertex E_i of X_i and we have nothing to prove. Let us assume that $m-2 \geq \dim(Z_{ij})$. Let B_i be the base of the cone X_i. Note that $E_i \cap \langle B_i \rangle = \emptyset$ and that $X_i = [E_i; B_i]$, hence $\langle X_i \rangle = \langle [E_i; B_i] \rangle = \langle E_i \cup \langle B_i \rangle \rangle$, moreover if we project Z_{ij} from E_i onto $\langle B_i \rangle$ we get in fact a linear space of B_i having dimension $\dim[(E_i \cup Z_{ij}) \cap \langle B_i \rangle]$. Recall that the linear spaces contained in B_i have dimension $\dim(B_i) - 1$ at most, hence $\dim[(E_i \cup Z_{ij}) \cap \langle B_i \rangle] \leq \dim(B_i) - 1$. Obviously:

$\dim(E_i \cup Z_{ij}) = e_i + \dim(Z_{ij}) - \dim(E_i \cap Z_{ij}) \geq e_i + m - x + 1 - \dim(E_i \cap Z_{ij})$
$\dim(\langle X_i \rangle) = \dim(\langle E_i \cup \langle B_i \rangle \rangle) = \dim(\langle E_i \cup Z_{ij} \cup \langle B_i \rangle \rangle) = \dim(E_i \cup Z_{ij} \cup \langle B_i \rangle)$
$\dim(E_i \cup Z_{ij} \cup \langle B_i \rangle) = \dim(E_i \cup Z_{ij}) + \dim(\langle B_i \rangle) - \dim[(E_i \cup Z_{ij}) \cap \langle B_i \rangle]$,

hence:

$\dim[(E_i \cup Z_{ij}) \cap \langle B_i \rangle] \geq e_i + m - x + 1 - \dim(E_i \cap Z_{ij}) + \dim(\langle B_i \rangle) - \dim(\langle X_i \rangle)$
$e_i + m - x + 1 - \dim(E_i \cap Z_{ij}) + \dim(\langle B_i \rangle) - \dim(\langle X_i \rangle) \leq \dim(B_i) - 1$
$e_i + m - x + 1 - \dim(E_i \cap Z_{ij}) + (d_i + \dim(B_i) - 1) - (m + d_i - 1) \leq \dim(B_i) - 1$
$e_i - x + 2 \leq \dim(E_i \cap Z_{ij})$.

When $x = 3$, $\dim(E_i \cap Z_{ij}) \geq e_i - 1$ and we have only the five possibilities listed by Proposition 2.

a) In this case $e_i = m - 4 \geq 0$, hence $\dim(E_i \cap Z_{ij}) \geq m - 5 \geq 0$ if $m \geq 5$. When $m = 4$, $\dim(E_i) = 0$ and $\dim(Z_{ij}) = 2$; if $E_i \cap Z_{ij} = \emptyset$ the plane Z_{ij} projects isomorphically from E_i onto a plane contained in $B_i \simeq \mathbb{P}^1 \times \mathbb{P}^2$, hence Z_{ij} is contained only in a unique 3-dimensional linear space of X_i, but this is not possible by Lemma 2 *iii*). Therefore $E_i \cap Z_{ij} \neq \emptyset$ when $m = 4$ too.

b) In this case $e_i = m - 3$, hence $\dim(E_i \cap Z_{ij}) \geq m - 4 \geq 0$ if $m \geq 4$.
c) In this case $e_i = m - 2$, hence $\dim(E_i \cap Z_{ij}) \geq m - 3 \geq 0$.
d) In this case $e_i = m - 3$, hence $\dim(E_i \cap Z_{ij}) \geq m - 4 \geq 0$ if $m \geq 4$.
e) In this case $e_i = m - 2$, hence $\dim(E_i \cap Z_{ij}) \geq m - 3 \geq 0$. ∎

Corollary 6. *Let $X \subset \mathbb{P}^N$ be an x-projectable Veronese variety,* $\dim(X) = m \geq 3$. *Let X_i and X_j be two irreducible components of X which are cones of degree d_i and d_j, having vertexes E_i and E_j respectively. Then:*

i) if $E_i \cap E_j \neq \emptyset$, $d_i + d_j \leq x + 1$;

ii) if $x = 3$ and $d_i = 3$, X contains only two components, unless $m = 3$ and the vertex of X_i (and X_j) is a point.

Proof. *i*) Let P be a point in $E_i \cap E_j$. By Corollary 1 $\dim(\langle X_j \rangle \cap \langle X_i \rangle) \leq m-1$. By Lemma 4 $\dim[T_P(X_i) \cup T_P(X_j)] \leq m+x$. Therefore, if we consider the tangent spaces at P, we get $\dim[T_P(X_i) \cap T_P(X_j)] = m+d_i-1+m+d_j-1-\dim[T_P(X_i) \cup T_P(X_j)] \leq m-1$. Hence $2m+d_i+d_j-2 \leq \dim[T_P(X_i) \cup T_P(X_j)]+m-1 \leq 2m+x-1$.

ii) By Proposition 1 we can assume that the cone of the statement is X_1; let E_1 be its vertex. Then $\langle X_1 \rangle \cap \langle X_2 \rangle = X_1 \cap X_2 \simeq \mathbb{P}^{m-1}$ and $X_2 \supset E_1$. If X has another component X_3 we have $Z_{13} := X_3 \cap X_1$ is a linear space of dimension at least $m-2$ by Corollary 1 and Lemma 2 *ii*), moreover $Z_{13} \cap E_1 \neq \emptyset$ by Corollary 5, unless $m = 3$ and the vertex of X_1 is a point.

Let P be a point in $Z_{13} \cap E_1$; as we have seen $P \in X_2$ too. By Lemma 4 we have: $m+3 \geq \dim[T_P(X_1) \cup T_P(X_2)] = m+2+\dim[T_P(X_2)] - \dim[T_P(X_1) \cap T_P(X_2)]$ hence: $m \leq \dim[T_P(X_2)] \leq 1+\dim[T_P(X_1) \cap T_P(X_2)] \leq 1+m-1 = m$ and therefore: $\dim[T_P(X_2)] = m$, $\dim[T_P(X_1) \cap T_P(X_2)] = m-1$, $\dim[T_P(X_1) \cup T_P(X_2)] = m+3$. This is not possible: as $P \in X_3$, by Lemma 3 $T_P(X_3) \nsubseteq [T_P(X_1) \cup T_P(X_2)]$ and we get a contradiction with Lemma 4. Hence X_3 cannot exist unless $m = 3$ and the vertex of X_1 is a point. ∎

Corollary 7. *Let $X \subset \mathbb{P}^N$ be a 3-projectable Veronese variety, $\dim(X) = m \geq 3$. Let X_i be an irreducible component of X such that X_i is a cone of vertex E_i and degree $d_i = 2$. Then:*

i) for any other irreducible component X_j such that X_j is a cone of vertex E_j and degree d_j, if $E_i \cap E_j \neq \emptyset$, we have $d_j = 2$;

ii) for any other irreducible component X_j such that X_j is a cone of vertex E_j, if $X_j \cap X_i = \langle X_j \rangle \cap \langle X_i \rangle \simeq \mathbb{P}^{m-2}$, we have $E_i \cap E_j = \emptyset$;

iii) if $m \geq 4$, X has three components at most unless $m = 4$, X_1 is a cone of degree 2 having a line E_1 as vertex, $X_j \simeq \mathbb{P}^4$ for any $j \geq 3$.

Proof. *i*) By Corollary 6 *i*) we have: $2+d_j \leq 4$.

ii) By contradiction, let us assume that $E_i \cap E_j \neq \emptyset$ and let P be a point in $E_i \cap E_j$. We have: $\dim[T_P(X_i)] = m+1$ and $\dim[T_P(X_j)] \geq m+1$. Let us consider the tangent spaces at P: $\dim[T_P(X_i) \cap T_P(X_j)] = m+1+\dim[T_P(X_j)] - \dim[T_P(X_i) \cup T_P(X_j)] \leq m-2$. It follows: $2m+2 \leq m+1+\dim[T_P(X_j)] \leq \dim[T_P(X_i) \cup T_P(X_j)]+m-2$ and $m+2 \leq 1+\dim[T_P(X_j) \leq \dim[T_P(X_i) \cup T_P(X_j)]-2$. Hence $\dim[T_P(X_i) \cup T_P(X_j)] \geq m+4$ and this is not possible by Lemma 4.

iii) By Proposition 1 we can assume that the cone of the statement is X_1. Then $\langle X_1 \rangle \cap \langle X_2 \rangle = X_1 \cap X_2 \simeq \mathbb{P}^{m-1}$ and $X_2 \supset E_1$. By contradiction let us assume that X contains two other components X_3 and X_4 at least. By Corollary 5 we know that $Z_{13} \cap E_1 \neq \emptyset$ and $Z_{14} \cap E_1 \neq \emptyset$.

If $\dim(E_1) = m-2$ we have $Z_{13} \cap Z_{14} \cap E_1 \neq \emptyset$ (as $\dim(Z_{13} \cap E_1) \geq m-3$ and $\dim(Z_{14} \cap E_1) \geq m-3$) so that there exists at least a point $P \in X_1 \cap X_2 \cap X_3 \cap X_4$, but this is not possible because $\dim[T_P(X_1)] = m+1$ and by applying Lemma 3 we would get a contradiction with Lemma 4.

If $\dim(E_1) = m-3$ we can argue in the same way unless $m = 4$. If $m = 4$ we can argue in the same way unless, for any $j \geq 3$, Z_{1j} are planes intersecting the line E_1 at different points P_{1j}. Let us show that this is possible only if $X_j \simeq \mathbb{P}^4$ for any $j \geq 3$. Every plane Z_{1j} belongs to a unique 3-dimensional linear space contained in

X_1 because Z_{1j} projects from E_1 onto a line of the quadric which is the base of the cone X_1. Let F be a generic 3-dimensional linear space contained in X_1. We have $F \cap Z_{1j} = P_{1j}$, hence $F \cap X_j = F \cap \langle X_j \rangle = P_{1j}$. Let us consider $[F; X_j]$. By Lemma 1 we have $\dim([F; X_j]) = 8$, unless all tangent spaces at smooth points of X_j pass through P_{1j}. If $\dim([F; X_3]) = 8$ we would have $\dim[Sec(X)] \geq 8$, contradiction; in the other case X_j can be a cone, having a vertex intersecting E_1, or a linear space, but if X_j is a cone we would get a contradiction with *ii)* as $\dim(Z_{1j}) = 2$, hence X_j is a linear space. ∎

3. 2-projectable Veronese varieties

The direct generalization of Definition 1 would be the definition of 2-projectable Veronese varieties. In fact the target space for surfaces has dimension $4 = \dim(X) + 2$. However the previous definition is not very interesting by the following theorem.

Theorem 4. *There are no 2-projectable Veronese varieties of dimension $m \geq 3$.*

Proof. Let $X \subset \mathbb{P}^N$ be a 2-projectable Veronese variety of dimension $m \geq 3$. First of all let us remark that we can argue as in Lemmas 2, 3, 4 and we can prove that for any point $P \in X$ there pass at most 3 irreducible components of X.

Let us choose a generic linear space L in $\mathbb{P}^N$ of codimension $m-2$. Let S be $X \cap L$, then S is a reducible Veronese surface according to Definition 1. In fact L cuts Λ along a 4 -dimensional linear space and the projection of S into it from $L \cap \mathcal{L}$ is an isomorphism because $\pi_{\mathcal{L}}(X)$ is isomorphic to X, hence $\dim[Sec(S)] \leq 4$. Obviously *i)* and *ii)* of Definition 1 are satisfied for some n by conditions *i)* and *ii)* of Definition 3. By Proposition 2.1 of [E-G-H-P] S is small, hence we can give an ordering to the irreducible components $\{S_j\}$ of S such that $(S_1 \cup ... \cup S_j) \cap S_{j+1} = \langle S_1 \cup ... \cup S_j \rangle \cap \langle S_{j+1} \rangle$ for any $j \geq 1$, (Theorem 0.4 of [E-G-H-P]). The dimension of these linear spaces must be 1, otherwise X could not be connected in codimension 1, and this implies that S is connected in codimension 1 too. Moreover the coordinate ring of S is Cohen Macaulay by Theorem 1.4 of [E-G-H-P], hence S is locally Cohen Macaulay. Therefore S is one of the surfaces classified by Theorem 1.

If S is a surface Σ_n for some $n \geq 1$, then X is the union of $n+3$ linear spaces of dimension m. One of them, say X_0, is cut from L along Π_0 and the other ones, say X_i, are cut from L along Π_i. It follows that every X_i cut $X_0 \simeq \mathbb{P}^m$ along a hyperplane, and as for any point $P \in X_0$ at most other two components of X can pass, in $X_0 \simeq \mathbb{P}^m$ three or more hyperplanes can not intersect. This is possible only if $n = 2$ as $m \geq 3$ and, in this case, $X = X_0 \cup X_1 \cup X_2$, $\dim(X_0 \cap X_1) = \dim(X_0 \cap X_2) = m-1$, $\dim(X_1 \cap X_2) = m-2$. However, in this case, $\dim(\langle X \rangle) = m+2$, while N must be $m+3$ at least.

If S is the union of a smooth quadric $Q \subset \mathbb{P}^3$ and two planes as in Theorem 1, then X must be the union of a quadric Q' and two linear spaces of dimension m and they must cut Q' along a linear space of dimension $m-1$ by Theorem 3. As in the proof of Corollary 2 the quadric Q' must be a cone over Q having a vertex of dimension $m-3$, so that X is a cone over S having a vertex of the same dimension. For any point $P \in Sing(X)$, $\dim[T_P(X)] = \dim(\langle X \rangle) = m - 3 + \dim(\langle S \rangle) + 1 = m + 3 = N$, so that, in this case too, X is not 2-projectable because the Zariski tangent space at P concides with $\mathbb{P}^N$. ∎

4. x-projectable Veronese varieties having only linear components

We need the following Lemma.

Lemma 5. *Let $X = X_1 \cup .... \cup X_r \subset \mathbb{P}^N$ be an x-projectable Veronese variety, such that all components of X are linear spaces, $\dim(X) = m \geq 3$. Then, for any good ordering $X_1, ..., X_r$ of the components of X:*

i) $r \geq x + 2$;

ii) for any integer q, $r \geq q \geq 2, \dim(X_1 \cap X_2 \cap ... \cap X_q) \geq m + 1 - q$;

iii) X cannot contain three components X_i, X_j, X_k, such that $X_i \cap X_j \cap X_k = \mathbb{P}^{m-1}$.

Proof. *i)* We have $N = \dim(\langle X \rangle) = m + r - 1 \geq m + x + 1$. Thus implies $r \geq x + 2$.

ii) Let us prove the following claim: for any integer p, $r \geq p \geq 2$, $\dim(X_1 \cap ... \cap X_{p-1}) \geq \dim(X_1 \cap ... \cap X_p) \geq \dim(X_1 \cap ... \cap X_{p-1}) - 1$.

In fact, by Corollary 1 *ii)* there exists at least a component X_s, with $s < p$, such that $X_p \cap X_s \simeq \mathbb{P}^{m-1}$. Therefore $X_1 \cap ... \cap X_p$ is the intersection of the linear space $X_1 \cap ... \cap X_{p-1} \subset X_j \simeq \mathbb{P}^m$ with a hyperplane of X_s, hence the claim follows.

From the claim it follows that $\dim(X_1 \cap ... \cap X_q) \geq \dim(X_1) - (q - 1) = m + 1 - q$.

iii) By contradiction, let us assume that X contains three components X_i, X_j, X_k, such that $X_i \cap X_j \cap X_k := \Pi \simeq \mathbb{P}^{m-1}$. By Proposition 1 we can choose a good ordering among the components of X, such that $X_1 = X_i$. We can also assume that $2 \leq j < k$.

The claim proved in *ii)* implies that, for any integer q with $r \geq q \geq 2$, if there exists an integer q' with $2 \leq q' \leq q$ such that $X_1 \cap ... \cap X_{q'-1} = X_1 \cap ... \cap X_{q'}$ then $\dim(X_1 \cap ... \cap X_q) \geq m + 2 - q$.

Let us consider three cases, by recalling that $r \geq x + 2$ by *i)* and that $m \geq x$.

a) $2 \leq j < k \leq x + 2$. In this case $X_1 \cap ... \cap X_{k-1} = X_1 \cap ... \cap X_k$, hence $\dim(X_1 \cap ... \cap X_{x+2}) \geq m + 2 - (x + 2) \geq 0$ by *ii)*. Then there is a point, at least, contained in $x + 2$ components of X and this is a contradiction with Corollary 4.

b) $2 \leq j \leq x + 1 < k$. In this case $\dim(X_1 \cap ... \cap X_{x+1}) \geq m + 1 - (x + 1) \geq 0$ by *ii)*. Let P be a point in $X_1 \cap ... \cap X_{x+1}$, $P \in X_k$ too by assumptions, then there is a point, at least, contained in $x + 2$ components of X and this is a contradiction with Corollary 4.

c) $2 < x < j < k$. In this case $\dim(X_1 \cap ... \cap X_x) \geq m + 1 - x \geq 1$, by *ii)*. Therefore in X_1 there is at least a point $P \in X_1 \cap ... \cap X_x \cap \Pi$, hence there is a point, at least, contained in $x + 2$ components of X and this is a contradiction with Corollary 4. ∎

Lemma 6. *Let $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^N$ be an x-projectable Veronese variety, $\dim(X) = m \geq 3$. If all components of X are linear spaces then the components of X are the vertexes of a connected graph $\mathcal{A}$ which is a tree, having X_1 as root; the edges of $\mathcal{A}$ correspond to pairs X_i, X_j such that $X_i \cap X_j \simeq \mathbb{P}^{m-1}$, moreover $m = x$ and no simple path is longer than x. If $x = m = 3$ we have only the following possibilities:*

i) there is a distinguished component X_i and all other components intersect X_i along planes of X_i in general position, $r \geq 5$;

ii) there are two distinguished components X_i and X_j such that $X_i \cap X_j := \Pi \simeq \mathbb{P}^2$ and all other components intersect X_i along planes in general position, or intersect X_j along planes in general position, and intersect each other along lines in general position in Π; $r \geq 5$.

Proof. Let us cut X with a linear space of codimension $m-1$ in general position. Let C be the linear section. $C = C_1 \cup C_2 \cup ... \cup C_r$ is a curve having exactly r irreducible components which are lines. Note that $C_i \cap C_j \neq \emptyset$ if and only if $\dim(X_i \cap X_j) = m-1$. As X is 1-connected C is connected and by Lemma 5 there are no points on C for which there pass more than two lines, moreover C is small according to the definition of [E-G-H-P]. By Proposition 2.1 of that paper there are no 3-secant lines for C, no 4-secant planes for C, ... , no k-secant linear spaces of dimension $k-2$ (see Theorem 2.2 of [E-G-H-P]). It follows that if we consider a graph $\mathcal{A}$ such that the vertexes of $\mathcal{A}$ are the components of C and the edges of $\mathcal{A}$ correspond to the intersections among the components of C, then $\mathcal{A}$ is a (connected) tree, because there are no circuits in $\mathcal{A}$.

Now, if we consider the weighted graph $\mathcal{G}_X$ associated to X as in [E-G-H-P], page 1381, in our case $\mathcal{G}_X$ is a complete graph by Lemma 2 *ii*), the graph $\mathcal{A}$ is a spanning tree for $\mathcal{G}_X$ and X_1 gives the root of $\mathcal{A}$. Moreover the order among the components of X is compatible with the natural order of vertexes in $\mathcal{A}$ (see Theorem 5.1 (b) of [E-G-H-P]).

By contradiction, let us assume that $m \geq x+1$. By Lemma 5 *i*), we have $r \geq x+2$. By taking $q = x+2$, Lemma 5 *ii*) implies that $\dim(X_1 \cap X_2 \cap ... \cap X_{x+2}) \geq m-(x+2)+1 \geq 0$ and this is a contradiction with Corollary 4; hence $m \leq x$. As $m \geq x$ in any case, we get $m = x$.

Let us consider a fixed component X_k in X, $k \neq 1$. There is a unique path γ in $\mathcal{A}$ joining X_k with the root X_1. We want to prove that γ has no more than x edges. Let us assume that $X_{1=p_1}$, $X_{p_2}, ..., X_{p_t} = X_k$ give rise to the unique path in $\mathcal{A}$ joining X_k with X_1 and let us prove that $X_{p_1} \cap X_{p_t} = X_{p_1} \cap X_{p_2} \cap ... \cap X_{p_t}$. Obviously we have only to prove that $X_{p_1} \cap X_{p_t} \subseteq X_{p_1} \cap X_{p_2} \cap ... \cap X_{p_t}$. Let us do induction on t: if $t = 2$ it is true. If it is true for $2, ..., t-1$ we have $X_{p_1} \cap X_{p_{t-1}} \subseteq X_{p_1} \cap X_{p_2} \cap ... \cap X_{p_{t-1}}$, on the other hand $X_{p_1} \cap X_{p_t} \subseteq X_{p_{t-1}}$ by Corollary 1 *ii*) because $X_{p_t} \cap X_{p_{t-1}} \simeq \mathbb{P}^{m-1}$. Hence $X_{p_1} \cap X_{p_t} \subseteq X_{p_1} \cap X_{p_{t-1}} \subseteq X_{p_1} \cap X_{p_2} \cap ... \cap X_{p_{t-1}} \subseteq X_{p_1} \cap X_{p_2} \cap ... \cap X_{p_t}$.

By contradiction, let us assume that $t \geq x+2$. As $\dim(X_1 \cap X_k) \geq 0$ by Lemma 2 *ii*) we have $\dim(X_{p_1} \cap X_{p_2} \cap ... \cap X_{p_t}) \geq 0$ by the previous claim, proved by induction, but this is a contradiction with Corollary 4.

Now let us consider any two components of X, say X_j and X_k, and, by contradiction, let us assume that there exists a path γ in $\mathcal{A}$, joining X_k with X_j, having more than x edges. Let us recall that there exists a good ordering such that $X_j = X'_1$. With respect to this new order we get another tree $\mathcal{A}'$ for which the root is X_j, however $\mathcal{A}'$ has the same edges of $\mathcal{A}$, so that we would have a path in $\mathcal{A}'$, having more than x edges, joining X_k and the root of $\mathcal{A}'$: contradiction.

If $x = m = 3$, in $\mathcal{A}$ every path passing through the root X_1 has 3 edges at most. It is easy to see that the only possibilities for X are *i*) or *ii*), obviously with $r \geq 5$ since $N \geq 7$. ∎

Now we can give the complete classification of x-projectble varieties X when all the components of X are linear spaces.

Theorem 5. *Let* $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^N$ *be a variety satisfying the assumptions of Theorem 3,* $\dim(X) = m \geq 3$, *such that all components of* X *are linear spaces. Then* X *is* x*-projectable if and only if:*

i) $r \geq x+2$;

ii) $\dim(X_i \cap X_j) \geq 0$ *for any pair of components of* X;

iii) the components of X *are the vertexes of a connected graph* $\mathcal{A}$ *which is a tree, having* X_1 *as root; the edges of* $\mathcal{A}$ *correspond to pairs* X_i, X_j *such that* $X_i \cap X_j \simeq \mathbb{P}^{m-1}$; *the natural ordering of the vertexes of* A *is compatible with a good ordering;*

iv) every point $P \in X$ *is contained in at most* $x+1$ *components of* X.

Proof. By Lemma 2 *ii)*, Corollary 4, Lemma 5 *i)* and Lemma 6 we know that the above assumptions are necessary.

As X satisfies the assumptions of Theorem 3, we know that $i), ii), iv)$ and $v)$ of Definition 3 are satisfied (for $v)$ recall the proof of Theorem 4). By looking at the proofs of Lemma 2 and Lemma 6 we see that the above assumptions imply that $m = x$ and that in $\mathcal{A}$ there are no paths whose lenght is longer than x.

Moreover assumption $i)$ implies that $N = m + r - 1 \geq m + x + 1$ and $ii)$ implies that $\dim([X_i; X_j]) = \dim(X_i \cup X_j) \leq m + x$ for any pair of components of X, hence $\dim[Sec(X)] \leq m + x$. To get $iii)$ of Definition 3 we have only to prove that the union of all Zariski tangent spaces at any point $P \in X$ has dimension at most $m + x = 2m$. As X is an union of linear spaces it suffices to prove that, for any point $P \in X$, the linear span of all linear spaces of X containing P has dimension at most $2m$.

Let P be a point of X, by Proposition 1 we can assume that $P \in X_1$, the root of $\mathcal{A}$. Let us assume that P is contained in s components of X and let X_k be one of these components. There is a unique path γ_k, $X_{1=p_1} \leftarrow X_{p_2} \leftarrow ... \leftarrow X_{p_t} = X_k$, joining X_k and X_1 in $\mathcal{A}$, with $1 = p_1 < p_2 < ... < p_t = k$ and, by the proof of Lemma 6, we know that $P \in X_{p_1} \cap X_{p_2} \cap ... \cap X_{p_t}$.

We consider the components X_k containing P such that the corresponding path γ_k is of maximal lenght; let us call them m_P-components. Let Γ_k be the set of vertexes of $\mathcal{A}$ involved by γ_k. We can give a complete, not unique, ordering among the distinct m_P-components $X_{k_1} \geq X_{k_2} \geq X_{k_3} ... \geq X_{k_i} \geq ...$ induced by a corresponding complete, not unique, ordering on the sets $\{\Gamma_k\}$ defined in the following way:

$\Gamma_{k_1} \geq \Gamma_{k_2} \geq \Gamma_{k_3} ... \geq \Gamma_{k_i} \geq ...$ if and only if

$\Gamma_{k_1} \supseteq \Gamma_{k_1} \cap \Gamma_{k_2} \supseteq (\Gamma_{k_1} \cup \Gamma_{k_2}) \cap \Gamma_{k_3} ... \supseteq (\Gamma_{k_1} \cup \Gamma_{k_2} \cup ... \cup \Gamma_{k_{i-1}}) \cap \Gamma_{k_i} \supseteq ...$

For instance, in the following tree:

$$\begin{array}{ccccccccc}
 & & & & \bullet\, X_a & & & & \\
 & & & \swarrow & & & & & \\
X_1 \bullet & \leftarrow & \bullet & & & & \bullet & \leftarrow & \bullet\, X_b \\
 & & & \nwarrow & & \swarrow & & & \\
 & & & & \bullet & & & & \\
 & & & & & \nwarrow & & & \\
 & & & & & & \bullet & \leftarrow & \bullet\, X_c
\end{array}$$

$X_c \geq X_b \geq X_a$ or $X_b \geq X_c \geq X_a$ are allowable orderings, $X_a \geq X_b \geq X_c$ is not allowable.

Now let us consider the first m_P-component X_{k_1} and the corresponding path γ_{k_1} as above. We have $\dim(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_t}) = m + t - 1$. Let us prove it by induction on $t \geq 2$: if $t = 2$ it is true; let us assume that it is true for $2, ..., t-1$, then:

$\dim(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_t}) = \dim[(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_{t-1}}) \cup X_{p_t}] =$

$= \dim(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_{t-1}}) + \dim(X_{p_t}) - \dim[(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_{t-1}}) \cap X_{p_t}] =$

$= m + t - 2 + m - (m-1) = m + t - 1$,
because $(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_{t-1}}) \cap X_{p_t} = X_{p_{t-1}} \cap X_{p_t}$ by the properties of $\mathcal{A}$ given by assumption *iv*). In fact $\{X_{p_1}, X_{p_2}, ..., X_{p_{t-1}}\}$ are among $\{X_1, X_2, ..., X_{p_t-1}\}$, as the two orderings are compatible, and $(X_1 \cup X_2 \cup ... \cup X_{p_t-1}) \cap X_{p_t} = X_{p_{t-1}} \cap X_{p_t}$ by Corollary 1 *ii*).

Let us consider the second m_P-component X_{k_2} (if any). We can argue as in the previous case: there is a unique path γ_{k_2}, $X_{1=q_1} \leftarrow X_{q_2} \leftarrow ... \leftarrow X_{q_\tau} = X_{k_2}$, joining X_{k_2} and X_1 in $\mathcal{A}$, with $1 = q_1 < q_2 < ... < q_\tau = k_2$, $P \in X_{q_1} \cap X_{q_2} \cap ... \cap X_{q_\tau}$, and $\dim(X_{q_1} \cup X_{q_2} \cup ... \cup X_{q_\tau}) = m + \tau - 1$. Of course there are some components common to γ_{k_1} and γ_{k_2} (at least X_1): if $p_1 = q_1$, $p_2 = q_2, ..., p_z = q_z$ for some integer $z \geq 1$, then $\dim(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_t} \cup X_{q_1} \cup X_{q_2} \cup ... \cup X_{q_\tau}) =$
$= \dim(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_t}) + \dim(X_{q_1} \cup X_{q_2} \cup ... \cup X_{q_\tau})$
$- \dim[(X_{p_1} \cup X_{p_2} \cup ... \cup X_{p_t}) \cap (X_{q_1} \cup X_{q_2} \cup ... \cup X_{q_\tau})] \leq$
$\leq (m + t - 1) + (m + \tau - 1) - \dim(X_{q_1} \cup X_{q_2} \cup ... \cup X_{q_z}) =$
$= 2m + t + \tau - 2 - (m + z - 1) = m + t + \tau - z - 1$.

Note that $t + \tau - z$ is exactly the number of components of X containing P and involved by γ_{k_1} and γ_{k_2}.

Now let X_{k_3} be the third m_P-component (if any). As above, there is a unique path γ_{k_3}, $X_{1=v_1} \leftarrow X_{v_2} \leftarrow ... \leftarrow X_{v_\theta} = X_{k_3}$, joining X_{k_3} and X_1 in $\mathcal{A}$, with $1 = v_1 < v_2 < ... < v_\theta = k_3$, $P \in X_{v_1} \cap X_{v_2} \cap ... \cap X_{v_\theta}$, and $\dim(X_{v_1} \cup X_{v_2} \cup ... \cup X_{v_\theta}) = m + \theta - 1$. Of course there are some components common to γ_{k_1}, γ_{k_2} and γ_{k_3} (at least X_1). Thanks to the given ordering among the m_P-components, we have to consider only the case in which $p_1 = q_1 = v_1$, $p_2 = q_2 = v_2, ..., p_w = q_w = v_w$ for some integer $w \geq 1$. Then $\dim(X_{p_1} \cup ... \cup X_{p_t} \cup X_{q_1} \cup ... \cup X_{q_\tau} \cup X_{v_1} \cup ... \cup X_{v_\theta}) =$
$= \dim(X_{p_1} \cup ... \cup X_{p_t} \cup X_{q_1} \cup ... \cup X_{q_\tau}) + \dim(X_{v_1} \cup ... \cup X_{v_\theta})$
$- \dim[(X_{p_1} \cup ... \cup X_{p_t} \cup X_{q_1} \cup ... \cup X_{q_\tau}) \cap (X_{v_1} \cup ... \cup X_{v_\theta})] \leq$
$\leq (m + t + \tau - z - 1) + (m + \theta - 1) - \dim(X_{v_1} \cup X_{v_2} \cup ... \cup X_{v_w}) =$
$= 2m + t + \tau + \theta - z - 2 - (m + w - 1) = m + t + \tau + \theta - z - w - 1$.

Note that $t + \tau + \theta - z - w$ is exactly the number of components of X containing P and involved by γ_{k_1}, γ_{k_2} and γ_{k_3}.

By iterating the same calculation for all m_P-components of X, we have that the dimension of the linear span of all components containing P is at most $m + s - 1$. As $s \leq x + 1$ by assumption *iv*), it is at most $m + x = 2m$. ∎

5. 3-projectable Veronese varieties with $m \geq 4$

In this section we will prove the following theorems:

Theorem 6. *Let $X \subset \mathbb{P}^N$ be a 3-projectable Veronese variety,* $\dim(X) = m \geq 4$. *Then X cannot contain a cone of degree* 3.

Proof. By contradiction, let us assume that X contains a cone X_i of degree 3 as a component. By Corollary 5 there exists a good ordering among the components of X, such that $i = 1$. Let E_1 be the vertex of X_1. By Corollary 6 we know that there exists only another component X_2 such that $X_2 \cap X_1 = \langle X_2 \rangle \cap \langle X_1 \rangle \simeq \mathbb{P}^{m-1}$. If $X_2 \simeq \mathbb{P}^{m-1}$ then $\dim(\langle X_1 \cup X_2 \rangle) = (m+2) + m - (m-1) = m + 3$, but we are assuming $N > m + 3$, so that X_2 cannot be a linear space. By Propositions 2, 3 we know that X_2 is a cone; let E_2 be its vertex. By Corollary 6 *i*) we have: $E_1 \cap E_2 = \emptyset$.

Let F_1 be a generic $(m-1)$-dimensional linear space contained in X_1, different from $X_2 \cap X_1$. Let F_2 be a generic $(m-1)$-dimensional linear space contained in

X_2, different from $X_2 \cap X_1$. If $m \geq 5$, by Lemma 2 *iii*) we know that $F_1 \cap F_2 \neq \emptyset$, but this implies $E_1 \cap E_2 \neq \emptyset$, as $F_1 \cap \langle X_2 \rangle = E_1$ and $F_2 \cap \langle X_1 \rangle = E_2$, and this is not possible. The only possibility is $m = 4$.

If $m = 4$, $X_2 \cap X_1 = \langle X_2 \rangle \cap \langle X_1 \rangle \simeq \mathbb{P}^3$. Let us consider $[F_1; X_2]$. As X is a 3-projectable Veronese variety we know that $\dim([F_1; X_2]) \leq 7$. On the other hand, by Lemma 1, $\dim([F_1; X_2]) = 3 + 4 - \dim[F_1 \cap T_P(X_2)]$ where P is a generic point of X_2, hence $\dim[F_1 \cap T_P(X_2)] \geq 0$. As $F_1 \cap \langle X_2 \rangle = E_1$ we get: $E_1 \cap T_P(X_2) \neq \emptyset$ for generic points $P \in X_2$. By arguing in the same way we get: $E_2 \cap T_Q(X_1) \neq \emptyset$ for generic points $Q \in X_1$. It follows that E_1 and E_2 cannot be points, otherwise they would belong to the vertex of X_2 or of X_1, respectively (see [A2], page 17), and this is not possible as $E_1 \cap E_2 = \emptyset$. E_1 and E_2 cannot be planes as $E_1 \cap E_2 = \emptyset$. So that the only possibility is that $\dim(E_1) = \dim(E_2) = 1$.

Let us consider the line E_2 in X_1. As E_2 is contained in $X_2 \cap X_1$ and it is disjoint with E_1, it is contained in a unique 3-dimensional linear space of X_1, which is a cone over a cubic scroll $B_1 \subset \mathbb{P}^4$ because E_2 projects from E_1 onto a line l of B_1. Let us call π the projection. As $E_2 \cap T_Q(X_1) \neq \emptyset$ for generic points $Q \in X_1$ we get $l \cap T_{\pi(Q)}(B_1) \neq \emptyset$ for generic points $Q \in X_1$, hence $l \cap T_b(B_1) \neq \emptyset$ for generic points $b \in B_1$ and this is not possible. ∎

Theorem 7. *Let $X \subset \mathbb{P}^N$ be a 3-projectable Veronese variety,* $\dim(X) = m \geq 4$. *If X contains a cone X_i of degree 2 then $m = 4$ and:*

1) *if X has at most 3 components, $X = X_1 \cup X_2 \cup X_3$, $X_1 \cap X_2 \cap X_3 \simeq \mathbb{P}^3$, X_1 and X_2 are cones with a line as vertex, X_3 is a cone of the same type and the three vertexes are disjoint, or $X_3 \simeq \mathbb{P}^4$; in the first case $X \subset \mathbb{P}^9$ in the second case $X \subset \mathbb{P}^8$;*

2) *if X has more than 3 components, $X = X_1 \cup X_2 \cup X_3 \cup X_4$, X_1 is a cone with a line as vertex, X_i are linear spaces for $i \geq 2$, $X_1 \cap X_2 \simeq \mathbb{P}^3$, $X_1 \cap X_2 \cap X_3 \simeq \mathbb{P}^2$, $X_1 \cap X_2 \cap X_3 \cap X_4 \simeq \mathbb{P}^1$, (disjoint with the vertex of X_1), $X \subset \mathbb{P}^8$.*

Proof. By Proposition 1 there exists a good ordering among the components of X, such that $i = 1$. Let E_1 be the vertex of X_1. If X has only two components X_1 and X_2 we know that $X_2 \cap X_1 = \langle X_2 \rangle \cap \langle X_1 \rangle \simeq \mathbb{P}^{m-1}$, and, in any case, $\dim(\langle X_2 \cup X_1 \rangle) \leq m + 3$, hence this possibility does not occur as $N > m + 3$ and X has 3 components at least. By Proposition 3 and Theorem 6, X_i can be only a quadric cone or a linear space for any $i \geq 2$. By Corollary 7 *iii*) we have only two possibilities.

1) Let us assume that X has exactly 3 components. We have to consider the following four cases:

a) X_2 and X_3 are quadric cones;

b) X_2 is a quadric cone and X_3 is a linear space;

c) X_2 and X_3 are linear spaces;

d) X_2 is a linear space and X_3 is a quadric cone.

a) Let E_2 be the vertex of X_2. Let F_1 be a generic $(m-1)$-dimensional linear space contained in X_1, different from $X_2 \cap X_1$. Let F_2 be a generic $(m-1)$-dimensional linear space contained in X_2, different from $X_2 \cap X_1$.

If $m \geq 5$, by Lemma 2 *iii*) we know that $F_1 \cap F_2 \neq \emptyset$, but this implies $E_1 \cap E_2 \neq \emptyset$, as $F_1 \cap \langle X_2 \rangle = E_1$ and $F_2 \cap \langle X_1 \rangle = E_2$. Let us consider X_3. We know that $X_3 \cap (X_1 \cup X_2) \simeq \mathbb{P}^{m-1}$, hence $X_3 \cap X_1 \simeq \mathbb{P}^{m-1}$ or $X_3 \cap X_2 \simeq \mathbb{P}^{m-1}$. In the first case $E_1 \subset X_3$ and $E_1 \cap E_2 \cap X_3 \neq \emptyset$. In the second case $E_2 \subset X_3$ and $E_1 \cap E_2 \cap X_3 \neq \emptyset$. So that there exists at least a point $P \in E_1 \cap E_2 \cap X_3$, but this is a contradiction with Lemmas 3 and 4 because $\dim[T_P(X_1) \cup T_P(X_2)] = (m+1)+(m+1)-(m-1) = m+3$.

If $m = 4$ the above argument shows that $E_1 \cap E_2 = \emptyset$, hence $\dim(E_1) = \dim(E_2) = 1$ as $X_2 \cap X_1 \simeq \mathbb{P}^3$ (recall Proposition 2). By Lemma 2 *ii)* we have to consider 3 subcases:

- $X_3 \cap X_1 \simeq \mathbb{P}^3$, different from $X_2 \cap X_1$, and $X_3 \cap X_2 \simeq \mathbb{P}^2$ with $(X_3 \cap X_2) \subset (X_3 \cap X_1)$. Hence $(X_3 \cap X_2) \subset (X_2 \cap X_1)$ and $(X_3 \cap X_2) \subset [(X_3 \cap X_1) \cap (X_2 \cap X_1)] = E_1$ but this is not possible as $X_3 \cap X_2 \simeq \mathbb{P}^2$.

- $X_3 \cap X_2 \simeq \mathbb{P}^3$, different from $X_2 \cap X_1$, and $X_3 \cap X_1 \simeq \mathbb{P}^2$ with $(X_3 \cap X_1) \subset (X_3 \cap X_2)$. We can argue as before and this subcase is not possible.

- $X_3 \cap X_2 = X_2 \cap X_1 \simeq \mathbb{P}^3$, hence $X_1 \cap X_2 \cap X_3 \simeq \mathbb{P}^3$. X_3 must be a cone having a line E_3 as vertex, because $E_1 \cap E_3 = \emptyset$ and $E_2 \cap E_3 = \emptyset$ by the above argument, and this subcase is possible.

b) We can argue as in case a), the only difference is that here $X_3 \simeq \mathbb{P}^4$.

c) In this case $\dim(\langle X_2 \cup X_1 \rangle) = m+2$ and $\dim(\langle X_3 \cup X_2 \cup X_1 \rangle) = \dim(\langle X_3 \rangle) + (m+2) - \dim(\langle X_3 \rangle \cap \langle X_2 \cup X_1 \rangle) = (m+1) + (m+2) - (m-1) = m+3$, hence this possibility does not occur as $N > m+3$.

d) If $X_3 \cap X_1 \simeq \mathbb{P}^{m-1}$ and $X_3 \cap X_2 \simeq \mathbb{P}^{m-2}$ or $X_3 \cap X_2 = X_3 \cap X_1$ we can consider a different good ordering among the components of X, according to Theorem 3: X_1, X_3, X_2 and we can get case b). Hence, by Lemma 2 *ii)*, we can assume: $X_3 \cap X_2 \simeq \mathbb{P}^{m-1}$ and $X_3 \cap X_1 \simeq \mathbb{P}^{m-2}$ with $(X_3 \cap X_1) \subset (X_3 \cap X_2)$. Let E_3 be the vertex of X_3, by Corollary 7 *ii)* we have: $E_1 \cap E_3 = \emptyset$. Now, $(E_1 \cap X_3 \cap X_1) \subset (X_3 \cap X_1) \simeq \mathbb{P}^{m-2}$ and $E_1 \cap X_3 \cap X_1$ has dimension $\dim(E_1) - 1 = m - \varepsilon_1 - 1$ because $E_1 \subset X_2 \cap X_1$ and $X_3 \cap X_1$ is a divisor in $X_2 \cap X_1$; moreover $(E_3 \cap X_3 \cap X_1) \subset (X_3 \cap X_1) \simeq \mathbb{P}^{m-2}$ and $E_3 \cap X_3 \cap X_1$ has dimension $\dim(E_3) - 1 = m - \varepsilon_3 - 1$ because $E_3 \subset X_2 \cap X_3$ and $X_3 \cap X_1$ is a divisor in $X_2 \cap X_3$. As $E_1 \cap E_3 = \emptyset$ it must be: $m - \varepsilon_1 - 1 + m - \varepsilon_3 - 1 < m - 2$, i.e. $m < \varepsilon_1 + \varepsilon_3 \leq 6$, as $\varepsilon_i \in \{2, 3\}$. We have to consider 3 subcases:

- $m = 5$, $\dim(E_1) = \dim(E_3) = 2$. It must be $\dim([X_1; X_3]) \leq m + 3 = 8$, hence, by Lemma 1, it must be $\dim(\langle T_P(X_1) \cup T_Q(X_3) \rangle) \leq 8$ for generic points $P \in X_1$ and $Q \in X_3$. We claim that this implies that there is a fixed plane $H \subset X_3 \cap X_1 = \langle X_1 \rangle \cap \langle X_3 \rangle \simeq \mathbb{P}^3$ such that $H \subset T_P(X_1)$ and $H \subset T_Q(X_3)$. In fact $\dim[T_P(X_1) \cap T_Q(X_3)] \geq 2$ and $T_P(X_1) \cap T_Q(X_3) \subseteq K := \langle X_1 \rangle \cap \langle X_3 \rangle$. If $T_P(X_1) = K$ for any generic pont $P \in X_1$, X_1 would be a cone with a vertex containing K and this is not possible, hence $\dim[T_P(X_1) \cap K] = 2$, analogously $\dim[T_Q(X_3) \cap K] = 2$ for any generic pont $Q \in X_3$; hence $\dim[T_P(X_1) \cap T_Q(X_3)] = 2$ for generic points $P \in X_1$ and $Q \in X_3$. Let us fix a generic point $\overline{P} \in X_1$ and let $\overline{H}$ be the plane $T_{\overline{P}}(X_1) \cap K$. As $\dim[T_{\overline{P}}(X_1) \cap T_Q(X_3)] = 2$ for any generic point $Q \in X_3$ and $T_{\overline{P}}(X_1) \cap T_Q(X_3) \subseteq K$, we have $T_{\overline{P}}(X_1) \cap T_Q(X_3) = T_{\overline{P}}(X_1) \cap T_Q(X_3) \cap K = \overline{H} \cap T_Q(X_3)$, hence $\overline{H} \subseteq T_Q(X_3)$ and $\overline{H} \subseteq T_Q(X_3) \cap K$. As $\dim[T_Q(X_3) \cap K] = 2$, it follows that $\overline{H} = T_Q(X_3) \cap K$ for any generic point $Q \in X_3$. By changing the role of X_1 and X_3 we get the claim.

However this is not possible: let $\pi : X_1 \to B_1$ be the natural projection of X_1 onto the base B_1 of the cone, B_1 is a smooth quadric in $\mathbb{P}^3$; $\pi(H)$ is a point or a line in B_1 and the tangent space $T_{\pi(P)}(B_1)$ would contain $\pi(H)$ for any generic point $P \in X_1$: contradiction.

- $m = 4$, $\dim(E_1) = \dim(E_3) = 1$. It must be $\dim([X_1; X_3]) \leq m + 3 = 7$, hence, by Lemma 1, it must be $\dim(\langle T_P(X_1) \cup T_Q(X_3) \rangle) \leq 7$ for generic points $P \in X_1$

and $Q \in X_3$. As above, this implies that there is a fixed line $L \subset X_3 \cap X_1 = \langle X_1 \rangle \cap \langle X_3 \rangle \simeq \mathbb{P}^2$ such that $L \subset T_P(X_1)$ and $L \subset T_Q(X_3)$. But this is not possible: let $\pi : X_1 \to B_1$ be the projection of X_1 onto the base B_1 of the cone, B_1 is a smooth quadric in $\mathbb{P}^3; \pi(L)$ is a point or a line in B_1 and the tangent space $T_{\pi(P)}(B_1)$ would contain $\pi(L)$ for any generic point $P \in X_1$: contradiction.

- $m = 4$, $\dim(E_1) = 1, \dim(E_3) = 2$ (the case $\dim(E_1) = 2, \dim(E_3) = 1$ is analogous). It must be $\dim([X_1; X_3]) \leq m + 3 = 7$, hence, by Lemma 1, it must be $\dim(\langle T_P(X_1) \cup T_Q(X_3) \rangle) \leq 7$ for generic points $P \in X_1$ and $Q \in X_3$. As above, this implies that there is a fixed line $L \subset X_3 \cap X_1 = \langle X_1 \rangle \cap \langle X_3 \rangle \simeq \mathbb{P}^2$ such that $L \subset T_P(X_1)$ and $L \subset T_Q(X_3)$. But this is not possible: let $\pi : X_1 \to B_1$ be the projection of X_1 onto the base B_1 of the cone, B_1 is a smooth quadric in $\mathbb{P}^3; \pi(L)$ is a point or a line in B_1 and the tangent space $T_{\pi(P)}(B_1)$ would contain $\pi(L)$ for any generic point $P \in X_1$: contradiction.

2) Now let us assume that X has more than 3 components. By Corollary 7 we know that X_j is a linear space for any $j \geq 3$ and by the above proof we know that X_2 can be only a quadric cone, having a line as vertex, or a linear space. In the first case, by the above proof, $X_1 \cap X_2 \cap X_3$ is a 3-dimensional linear space containing the two disjoint vertex of X_1 and X_2, so that X_4 would cut another 3-dimensional linear space on X_3, or on X_2, or on X_1, by Corollary 1 *ii*). Therefore X_4 would cut and at least a point on the two vertexes, but this is not possible by Lemma 4. Therefore X_j is a linear space for any $j \geq 2$.

Let F be $X_1 \cap X_2 \simeq \mathbb{P}^3$, obviously $F \supset E_1$, the vertex of X_1. X_3 cuts a 3-dimensional linear space on X_1, or on X_2. If $X_3 \cap X_1 \simeq \mathbb{P}^3$ then $E_1 \subset X_3 \cap X_1$, hence $E_1 \subset X_3 \cap X_2 \cap X_1$ and X_4 would cut at least a point on E_1: this is not possible by Lemma 4. Therefore $X_3 \cap X_2 \simeq \mathbb{P}^3$, $X_3 \cap X_1 := H \simeq \mathbb{P}^2$ (recall that $\dim(X_3 \cap X_1) \geq 2$ by Lemma 2 *ii*)), $H \subset F$ and $H = X_3 \cap X_2 \cap X_1$. Let us consider $E_1 \cap H$; if $E_1 \subset H$ then X_4 would cut a point on E_1, at least, and this is not possible by Lemma 4, hence $E_1 \cap H$ is a point P. By Lemma 4 no other component of X passes through P, hence $X_4 \cap X_1$ cannot be a 3-dimensional linear space, hence $X_4 \cap X_2 \simeq \mathbb{P}^3$ or $X_4 \cap X_3 \simeq \mathbb{P}^3$, in any case $X_4 \cap H = X_4 \cap X_3 \cap X_2 \cap X_1$ is a line L, not passing through P, and, by Lemma 4, no other component of X intersects L. Now it is easy to see that X cannot contain other components: X_5 would cut a point on L, at least, or it would pass through P. Both cases are not possible by Lemma 4. ∎

6. 3-projectable Veronese varieties with $m = 3$

A direct computation gives the following lemma.

Lemma 7. *Let $V = \mathbb{P}(\mathcal{E})$ be a smooth rational scroll over $\mathbb{P}^1$ such that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ or $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ or $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Let F be a fixed fibre of anyone of the above scrolls and let P be a generic point of V. Then:*

- in the first case and in the third case $T_p(V) \cap F = \emptyset$;

- in the second case $T_p(V) \cap F$ is a point and if P belongs to another fibre F', the map $F' \to F$, given by $P \to T_p(V) \cap F$, is an isomorphism.

Now we consider 3-projectable Veronese varieties containing scrolls and cones.

Proposition 4. *Let $X \subset \mathbb{P}^N$ be a 3-projectable Veronese variety,* $\dim(X) = 3$. *Let us assume that X contains a smooth scroll X_i. Then: $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{P}^7$,* *or* $X = X_1 \cup X_2 \cup X_3 \cup X_4 \subset \mathbb{P}^8$*,* $X_1 \simeq \mathbb{P}^1 \times \mathbb{P}^2$*,* $X_2 \simeq X_3 \simeq X_4 \simeq \mathbb{P}^3$*, and all components intersect along a fixed* 2*-dimensional fibre of the scroll* X_1*.*

Proof. We can always assume that X_1 is a scroll by Proposition 1. Then $X_1 \cap X_2 = \langle X_1 \rangle \cap \langle X_2 \rangle \simeq \mathbb{P}^2$ is a fibre F of the scroll. It must be $\dim([X_1; X_2]) \leq 6$, hence $T_p(X_1) \cap T_Q(X_2) \neq \emptyset$ by Lemma 1 for generic points $P \in X_1$ and $Q \in X_2$ and $T_p(X_1) \cap T_Q(X_2) \subseteq F$. If $X_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, by Lemma 7, $T_p(X_1) \cap F = \emptyset$ hence X cannot contain such a scroll. If $X_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \simeq \mathbb{P}^1 \times \mathbb{P}^2$ and $T_Q(X_2) \cap F \neq F$ then for generic points $P \in X_1$ we have $T_p(X_1) \cap T_Q(X_2) = \emptyset$ by Lemma 7. Hence $T_Q(X_2) \cap F = F$, $X_2 \simeq \mathbb{P}^3$ and $\dim(\langle X_1 \cup X_2 \rangle) = 6$. As $N \geq 7$ there exist other components in X. Let us consider X_3. By arguing as before we have $X_3 \simeq \mathbb{P}^3$ too and $X_3 \cap X_1$ is a fibre of the scroll. Since: $X_3 \cap X_2 \neq \emptyset$, $X_3 \cap X_2 \subseteq X_3 \cap X_1$ by Corollary 1 *ii*) and the fibres of X_1 are disjoint, we have $X_3 \cap X_1 = F$. The same is true for X_4 if there is another component. X cannot contain other components by Corollary 4.

We have no other cases to consider thanks to Proposition 3. ∎

From now on we can assume that X does not contain scrolls as components.

Proposition 5. *Let* $X \subset \mathbb{P}^N$ *be a* 3*-projectable Veronese variety,* $\dim(X) = 3$*. Let us assume that* X *contains a cone* X_i *of degree* 3*. Then* $X_i = X_1$*,* $X = X_1 \cup X_2 \subset \mathbb{P}^7$*,* X_1 *has a line* E_1 *as vertex,* X_2 *is a cone of degree* 2 *having a point* E_2 *as vertex and* $X_1 \cap X_2$ *is a plane containing* E_1 *and* E_2 *with* $E_1 \cap E_2 = \emptyset$*.*

Proof. We can always assume that X_1 a cone of degree 3 by Proposition 1. Then $X_1 \cap X_2 = \langle X_1 \rangle \cap \langle X_2 \rangle \simeq \mathbb{P}^2$ is a linear subspace F of maximal dimension for the cone. It must be $\dim([X_1; X_2]) \leq 6$, hence $T_p(X_1) \cap T_Q(X_2) \neq \emptyset$ by Lemma 1 for generic points $P \in X_1$ and $Q \in X_2$ and $T_p(X_1) \cap T_Q(X_2) \subseteq F$.

If X_1 has a point E_1 as vertex and a base $B_1 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ then $T_p(X_1) \cap F = E_1$, for generic points $P \in X_1$, otherwise, by looking at the projection from the vertex of the cone onto B_1, we would get a contradiction with Lemma 7. Hence $E_1 \subset T_Q(X_2)$ and $X_2 \simeq \mathbb{P}^3$ or X_2 is a cone with vertex containing E_1, but this is not possible by Corollary 6 *i*). Therefore $X_2 \simeq \mathbb{P}^3$. By arguing in this way we can conclude that all components X_i of X with $i \geq 2$ are isomorphic to $\mathbb{P}^3$ and they pass through E_1 (also in case $\dim(X_i \cap X_1) = 1$), but this is not possible by considering $\mathbb{T}_{E_1}$ and Lemma 4. The only possibility is $X = X_1 \cup X_2$, but in this case $\dim(\langle X_1 \cup X_2 \rangle) = 6$. As $N \geq 7$, X cannot contain such a cone.

If X_1 has a line E_1 as vertex and a base B_1 which is a rational space cubic then $T_p(X_1) \cap F = E_1$, for generic points $P \in X_1$. Hence $E_1 \cap T_Q(X_2) \neq \emptyset$ for generic points $Q \in X_2$. Let us examine the possibilities for X_2.

- If X_2 is a cone of degree 3, or of degree 2, having a line (on F) as vertex we would get $E_1 \cap E_2 \neq \emptyset$, but this is not possible by Corollary 6 *i*).

- If X_2 is a cone of degree 3 having a point as vertex we can reorder the components of X changing the role of X_1 and X_2; we can argue as above and we can exclude this case.

- If $X_2 \simeq \mathbb{P}^3$, $\dim(\langle X_1 \cup X_2 \rangle) = 6$, then it must exist X_3; however X_3 would cut a plane on X_1 or on X_2 and it would intersect E_1 at least at a point P, but this is not possible by considering $\mathbb{T}_P$ and Lemma 4.

- If X_2 is a cone of degree 2 having a point as vertex, for any generic point $Q \in X_2$ we have $T_Q(X_2) \cap F$ is a line passing through E_2 so that the condition $E_1 \cap T_Q(X_2) \neq \emptyset$ is fullfilled. Of course it must be $E_1 \cap E_2 = \emptyset$ otherwise we would have a contradiction with Lemma 4. X cannot contain another component X_3: X_3 would cut a plane or a line on X_1, hence there would be a point $P \in X_3 \cap E_1$ and this is not possible by considering $\mathbb{T}_P$ and Lemma 4. ∎

For the sequel we need the following Lemma.

Lemma 8. *Let $X \subset \mathbb{P}^N$ be a 3-projectable Veronese variety,* $\dim(X) = 3$. *Let us assume that X contains two cones X_i and X_j of degree 2, then $X_i \cap X_j$ is a plane.*

Proof. By Lemma 2 *ii*) we know that $X_i \cap X_j = \langle X_i \rangle \cap \langle X_j \rangle$ is a plane or a line. By contradiction, let us assume that $X_i \cap X_j$ is a line L and let π be the isomorphic projection of X onto $\mathbb{P}^6$. In this case $\pi(\langle X_i \rangle) \simeq \langle X_i \rangle \simeq \langle X_j \rangle \simeq \pi(\langle X_j \rangle) \simeq \mathbb{P}^4$ and $\pi(\langle X_i \rangle) \cap \pi(\langle X_j \rangle)$ is a plane Π, moreover there are two planes $\Pi_i \subset \langle X_i \rangle$ and $\Pi_j \subset \langle X_j \rangle$ such that $L = \Pi_i \cap \Pi_j$ and $\Pi = \pi(\Pi_i) = \pi(\Pi_j)$. Let $C'_i \supset L$ and $C'_j \supset L$ be the two plane conics $\Pi_i \cap X_i$ and $\Pi_j \cap X_j$. As π is an isomorphism, $C_i = \pi(C'_i)$ and $C_j = \pi(C'_j)$ are two plane conics in Π and $L = X_i \cap X_j \simeq \pi(X_i \cap X_j) \simeq \pi(X_i) \cap \pi(X_j) = C_i \cap C_j$: contradiction, as the intersection of two plane conics cannot be isomorphic to a (reduced) line. ∎

Proposition 6. *Let $X \subset \mathbb{P}^N$ be a 3-projectable Veronese variety,* $\dim(X) = 3$. *Let us assume that X does not contain cones of degree 3 and that X contains a cone X_i of degree 2 having a line E_i as vertex. Then:*

i) $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{P}^8$; *$X_1$ is a quadric cone having a line E_1 as vertex, X_2 and X_3 are quadric cones having points E_2 and E_3 as vertexes; $X_1 \cap X_2 \cap X_3$ is a plane F, the vertexes are disjoint;*

ii) $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{P}^7$; *$X_1$ is a quadric cone having a line E_1 as vertex, X_2 is a quadric cone having a point E_2 as vertex, $X_3 \simeq \mathbb{P}^3$; $X_1 \cap X_2 \cap X_3$ is a plane F, the vertexes are disjoint (the role of X_2 and X_3 can be exchanged);*

iii) $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{P}^7$; *$X_1$ is a quadric cone having a line E_1 as vertex, X_2 is a quadric cone having a point E_2 as vertex, $X_3 \simeq \mathbb{P}^3$; $X_1 \cap X_2$ is a plane F, $X_3 \cap X_1$ is another plane F', $X_3 \cap X_2 = E_1 \subset F$, the vertexes are disjoint (the role of X_2 and X_3 can be exchanged);*

iv) $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^{r+3}$, $r \geq 4$; *X_1 is a quadric cone having a line E_1 as vertex, $X_i \simeq \mathbb{P}^3$ for $i \geq 2$; $X_1 \cap X_2$ is a plane F; $X_i \cap X_2$ are planes in generic position in X_2 intersecting lines $L_i \subset F$ in generic position for $i \geq 3$.*

Proof. As usual we can assume that X_1 is a quadric cone having a line E_1 as vertex by Proposition 1. We have to consider three cases.

Case 1: X_2 is a quadric cone having a line E_2 as vertex, intersecting X_1 along a plane F. Let P be a point in $E_1 \cap E_2 \neq \emptyset$. If X contains a component X_3 then X_3 cuts a plane on X_2 or on X_1 (or on both), in any case X_3 passes through P, but this is not possible by Lemma 4. Moreover X cannot be $X_1 \cup X_2$ because $N \geq 7$.

Case 2: X_2 is a quadric cone having a point E_2 as vertex, intersecting X_1 along a plane F. If $E_2 \in E_1 \subset F$ we can argue as in Case 1 and we have no possibilities, so we can assume $E_2 \notin E_1$. X cannot be $X_1 \cup X_2$ because $N \geq 7$. Let us consider X_3.

2*a*) X_3 is a quadric cone. Hence it cuts a plane on X_1 and X_2 by Lemma 8 and the two planes coincide with F by Corollary 1 *ii*). By Lemma 4 the vertex E_3 of X_3 does not intersect E_1 or E_2, hence E_3 is a point in F in generic position with respect E_1 and E_2. X cannot contain X_4 because X_4 would cut a plane on some

X_i, $i \leq 3$, hence it would pass through some vertex E_i but this is not possible by Lemma 4. We have i).

$2b$) $X_3 \simeq \mathbb{P}^3$, $X_3 \cap X_2$ is a plane $F' \neq F$, $X_3 \cap X_1$ is a line $L \subset F'$. As $F' \cap X_1 = E_2$, the assumptions imply that $F' = F$, so that this subcase is not possible.

$2c$) $X_3 \simeq \mathbb{P}^3$, $X_3 \cap X_2 = F = X_3 \cap X_1$. If X contains X_4 then X_4 cuts a plane on some X_i, $i \leq 3$, hence X_4 intersects E_1 or passes through E_2, but this is not possible by Lemma 4. We have ii).

$2d$) $X_3 \simeq \mathbb{P}^3$, $X_3 \cap X_1$ is a plane F', $X_3 \cap X_2$ is a line $L \subset F'$. Unless $L = E_1$ we get $F' = F$, the previous considered case. If X contains X_4 then X_4 cuts a plane on some X_i, $i \leq 3$. If X_4 cuts a plane on X_1, or on X_3, it intersects E_1 at least at a point P, but this is not possible by Lemma 4. If X_4 cuts a plane F'' on X_2, then it must also cut a line $L' \subset F''$ on X_1 hence $F'' = F$ and X_4 intersects E_1 in this case too. We have iii).

Case 3: $X_2 \simeq \mathbb{P}^3$, intersecting X_1 along a plane F. X cannot be $X_1 \cup X_2$ because $N \geq 7$. Let us consider X_3.

$3a$) X_3 is a quadric cone. By Lemma 8, $X_3 \cap X_1$ is a plane F'. If $F' = F$ we can argue as in Case $2c$) changing the role of X_2 and X_3. If $F' \neq F$, $X_3 \cap X_2 = E_1$ because $F' \cap X_2 = E_1$ (recall Corollary 1 ii)), moreover the vertex E_3 of X_3 cannot be a line, otherwise $E_3 \cap E_1$ would be non empty and this is not possible by Lemma 4. Hence E_3 is a point on F', $E_3 \notin E_1$. We can argue as in Case $2d$) changing the role of X_2 and X_3.

$3b$) $X_3 \simeq \mathbb{P}^3$, $X_3 \cap X_1$ is a plane $F' \neq F$, $X_3 \cap X_2$ is a line $L \subset F'$. Necessarily $L = E_1$ (otherwise $F' = F$). If X contains X_4, then X_4 cuts a plane on some X_i, $i \leq 3$. In any case X_4 intersects E_1 at least at a point P, but this is not possible by Lemma 4. Moreover X cannot be $X_1 \cup X_2 \cup X_3$ as $N \geq 7$, so we have no possibilities.

$3c$) $X_3 \simeq \mathbb{P}^3$, $X_3 \cap X_1 = F = X_3 \cap X_2$. We can argue as in the previous case and we have no possibilities.

$3d$) $X_3 \simeq \mathbb{P}^3$, $X_3 \cap X_2$ is a plane $\Pi \neq F$, $X_3 \cap X_1$ is a line $L \subset \Pi$. If $L = E_1$ X cannot contain X_4 because X_4 would intersect E_1 in any case and this is not possible by Lemma 4; on the other hand X cannot be $X_1 \cup X_2 \cup X_3$ as $N \geq 7$, so we can assume that $L \neq E_1$. Let H be $L \cap E_1$, note that no other component can pass through H by Lemma 4. Let us consider X_4. If X_4 is a quadric cone then it must cut a plane on X_1 by Lemma 8, hence it would pass through H: contradiction. Therefore $X_4 \simeq \mathbb{P}^3$. X_4 cuts a plane on some X_i with $i \leq 3$, but $X_4 \cap X_1$ cannot be a plane, otherwise X_4 would contain H. If $X_4 \cap X_3$ is a plane then $X_4 \cap X_1$ would be a line, necessarily contained in this plane (recall Corollary 1 ii)), but, as $X_3 \cap X_1 = L$, it would imply: $X_4 \cap X_1 = L$, hence $H \in X_4$: contradiction. Hence there is only one possibility: $X_4 \cap X_2$ is a plane Π_4 (not passing through H), $X_4 \cap X_1$ is a line $L_4 \subset F$, $X_4 \cap X_3$ is the line $\Pi_4 \cap \Pi$.

Now let us proceed by induction on the number of components of X. Let us put $L_3 = L$ and $\Pi_3 = \Pi$. In this subcase Proposition 6 is true when X has $r = 4$ components. Let us prove Proposition 6 assuming that it is true when X has $4, 5, ..., r-1$ components and considering the case in which X has $r \geq 5$ components. Let us consider X_r. If X_r is a quadric cone then it must cut a plane on X_1 by Lemma 8, hence it would pass through H: contradiction. Therefore $X_r \simeq \mathbb{P}^3$. X_r cuts a plane on some X_i with $i \leq r-1$, but $X_r \cap X_1$ cannot be a plane, otherwise X_r would contain H. If $X_r \cap X_i$ is a plane for some $i = 3, ..., r-1$, then $X_r \cap X_1$

would be a line, necessarily contained in this plane (recall Corollary 1 *ii*)), but, as $X_i \cap X_1 = L_i \subset F = X_1 \cap X_2$ by induction, this would imply: $X_r \cap X_1 = L_i$, hence there would be at least a point in $X_1 \cap X_2 \cap X_i \cap X_r$ and this is not possible by Lemma 4. Hence there is only one possibility: $X_r \cap X_2$ is a plane Π_r (in generic position in X_2), $X_r \cap X_1$ is a line $L_r \subset F$, $X_r \cap X_i$ is the line $\Pi_r \cap \Pi_i$ for $i = 3, ..., r-1$, and we have *iv*). ∎

Proposition 7. *Let $X \subset \mathbb{P}^N$ be a* 3*-projectable Veronese variety,* $\dim(X) = 3$. *Let us assume that X does not contain cones of degree* 3*, or of degree* 2 *having a line as vertex, and that X contains a cone X_i of degree* 2 *having a point E_i as vertex. Then:*

i) $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^{r+3}$, $r \geq 4$*; X_1 is a quadric cone having a point E_1 as vertex, $X_i \simeq \mathbb{P}^3$ for $i \geq 2$; $X_1 \cap X_2$ is a plane F; $X_i \cap X_2$ are planes in generic position in X_2 intersecting lines $L_i \subset F$ in generic position for $i \geq 3$; possibly one of the components X_p, $3 \leq p \leq r$, is exceptional: it intersects X_1 along another plane F', cutting F along a line l in generic position with respect to the set* $\{L_i,\ i \neq p\}$;

ii) $X = X_1 \cup X_2 \cup ... \cup X_r \subset \mathbb{P}^{r+3}$, $r \geq 4$; *X_1 is a quadric cone having a point E_1 as vertex, $X_i \simeq \mathbb{P}^3$ for $i \geq 2$; $X_1 \cap X_2$ is a plane F; $X_i \cap X_2$ are planes in generic position in X_2 intersecting lines $L_i \subset F$ in generic position for $i = 3, ..., r-1$; there exists a fixed j, $3 \leq j \leq r-1$, such that $X_r \cap X_j$ is a plane and $X_r \cap X_1 = L_j$ ($E_1 \notin L_j$);*

iii) $X = X_1 \cup X_2 \cup X_3 \cup X_4 \subset \mathbb{P}^7$; *$X_1$ is a quadric cone having a point E_1 as vertex, $X_i \simeq \mathbb{P}^3$ for $i \geq 2$; $X_1 \cap X_2 \cap X_3$ is a plane F; $X_4 \cap X_2$ is a plane Π, $X_4 \cap X_3 = X_4 \cap X_1$ is a line $L = \Pi \cap F$, not passing through E_1, or $X_4 \cap X_3$ is a plane Π, $X_4 \cap X_2 = X_4 \cap X_1$ is a line $L = \Pi \cap F$, not passing through E_1.*

Proof. As usual we can assume that X_1 is a quadric cone having a point E_1 as vertex by Proposition 1. By the previous Propositions we know that all other components of X are linear spaces. $X_2 \cap X_1$ is a plane F. We consider two cases.

Case 1: for any $i \geq 3$, $X_i \cap X_1$ is a line L_i. Let us consider X_3. X_3 must cut a plane Π_3 on X_2 and $L_3 = \Pi_3 \cap F$ on X_1. X must contain another component as $N \geq 7$.

Let us consider X_4. X_4 has to cut a plane on X_3 or on X_2 (or on both). If X_4 cuts a plane on X_3 containing the line $X_4 \cap X_1 = L_4$ then $L_4 = L_3$ as $X_3 \cap X_1 = L_3$. This case is possible only if $E_1 \notin L_3$, otherwise we would get a contradiction with Lemma 4, and in this case no other component of X can intersect $L_4 = L_3 \subset F$ by Corollary 4, on the other hand another component X_5 would cut a plane on some X_i with $2 \leq i \leq 4$, hence X_5 would intersect L_3: contradiction. Then we get *ii*) with $r = 4$ and $j = 3$.

If X_4 cuts a plane Π_4 on X_2, then $X_4 \cap X_1 = \Pi_4 \cap F$ is a line L_4, while $X_4 \cap X_3$ is the line $\Pi_4 \cap \Pi_3$ (the position of E_1 with respect to L_4 is not important). Here we get *i*) with $r = 4$.

Now we can proceed by induction on the number of components of X. Let us assume that if X has $r-1$ components, $r \geq 5$, such that for any $i \geq 3$, $X_i \cap X_1$ is a line L_i and let us assume that *i*) or *ii*) holds for X. Let us prove that, if X has r components, $r \geq 5$, such that for any $i \geq 3$, $X_i \cap X_1$ is a line L_i, then *i*) or *ii*) holds for X. Let us consider $Y := X_1 \cup ... \cup X_{r-1}$; by induction we know that *i*) or *ii*) holds for Y. In fact *ii*) cannot hold: $X_{r-1} \cap X_1 = L_{r-1} = L_j \subset F$ for some j with $3 \leq j \leq r-2$; X_r must cut a plane on some X_i with $2 \leq i \leq r-1$ and a line

L_r on X_1 such that L_r is contained in that plane, hence $L_r \subset F$ and L_r intersects $L_{r-1} = L_j$, then there would be at least a point on five components of X: X_1, X_2, X_j, X_{r-1}, X_r: contradiction with Corollary 4. Therefore *i*) holds for Y. Now X_r must cut a plane on some X_i with $2 \leq i \leq r-1$ and a line L_r on X_1 such that L_r is contained in that plane, hence $L_r \subset F$. If L_r is distinct from L_j with $3 \leq j \leq r-1$, and necessarily in general position on F to avoid contradiction with Corollary 4, then we have *i*) for X; if L_r concides with a line L_j with $3 \leq j \leq r-1$, necessarily not passing through E_1, then we have *ii*) for X.

Case 2: there exists at least a component X_i, $i \geq 3$, such that $X_i \cap X_1$ is a plane.

2*a*) $X_3 \cap X_1$ is a plane. Firstly let us assume that $X_3 \cap X_1 := \Pi \neq F$ and $\Pi \cap F = \emptyset$ (then $X_3 \cap X_2 = E_1$, recall Corollary 1 *ii*)). X must contain another component as $N \geq 7$. Let us consider X_4. X_4 has to cut a plane on X_i, $i \leq 3$. If $X_4 \cap X_1$ is a plane X_4 contains E_1; if $X_4 \cap X_2$ is a plane then X_4 cuts a line or a point on X_3, hence $E_1 \in X_4$; if $X_4 \cap X_3$ is a plane then X_4 cuts a line or a point on X_2, hence $E_1 \in X_4$; therefore X_4 contains E_1 in any case, but this is not possible by Lemma 4.

Secondly let us assume that $X_3 \cap X_1 = F$. X must contain another component X_4, as $N \geq 7$, and, by arguing as above, we get $X_4 \cap X_2$ is a plane Π intersecting F (and X_1 and X_3) along a line L not passing through E_1, or $X_4 \cap X_3$ is a plane Π intersecting F (and X_1 and X_2) along a line L not passing through E_1. In both cases X cannot contain another component X_5: X_5 would cut a plane on X_i, $i \leq 4$, hence it would cut L: contradiction with Corollary 4, or it would contain E_1: contradiction with Lemma 4. We get *iii*).

Thirdly let us assume that $X_3 \cap X_1 := \Pi \neq F$ and $\Pi \cap F := l$ is a line. X must contain another component X_4, as $N \geq 7$, and X_4 cuts a plane on X_i, $i \leq 3$. As $X_4 \cap X_1$ cannot be a plane, by looking at $\mathbb{T}_{E_1}$, we have that $X_4 \cap X_3$ or $X_4 \cap X_2$ is a plane. We can assume that $X_4 \cap X_2 := \Pi_4$ is a plane intersecting F along a line L_4, cutting l at a point $P_4 \neq E_1$. Note that $X_3 \cap X_4 = P_4$. X can contain another component X_5. It cannot cut neither a plane on X_1 nor on X_3 or X_4: if $X_5 \cap X_4$ would be a plane it would pass through P_4 (recall Corollary 1 *ii*)) and this is not possible by Corollary 4. Hence $X_5 \cap X_2$ is a plane Π_5 cutting F along a line L_5 in generic position with respect to l and L_4 and *i*) holds for $X_1 \cup X_2 \cup X_4 \cup X_5$. Now we can proceed as in Case 1 because, by arguing as above, the other components X_i must cut X_2 along a plane Π_i, cutting F along a line L_i in generic position with respect to the previous ones. Note that X_i cannot cut F along a line L_j, $i < r$, (as in *ii*)) otherwise at least five components of X would pass through $l \cap L_j$. We get *i*) with $p = 3$.

2*b*) $X_3 \cap X_1$ is a line. By arguing as in Case 1 we get $X_3 \cap X_2$ is a plane Π_2, intersecting F and X_1 along a line L_3. X must contain another component as $N \geq 7$. Let us consider X_4. X_4 has to cut a plane on some X_i, $i \leq 3$.

Let us assume that X_4 cuts a plane F' on X_1. If $F' \neq F$ and $F' \cap F = E_1$, then $X_4 \cap X_3 = \emptyset$ and this is not possible by Lemma 2 *ii*). If $F' = F$ we have $X_4 \cap X_3 = L_3$. X cannot have another component X_5: X_5 would cut a plane on X_i, $i \leq 4$, hence it would cut L_3: contradiction with Corollary 4 or it would contain E_1: contradiction with Lemma 4. We get *iii*). If $F' \neq F$ and $F' \cap F := l$ is a line, we are in Case 2*a*) with X_4 in the role of X_3. We can proceed as in 2*a*) and we get *i*), with $p = 4$.

Let us assume that X_4 cuts a line L_4 on X_1, then $Y := X_1 \cup X_2 \cup X_3 \cup X_4$ is as in Case 1 and we get *ii*) or *i*) for Y. Only in this second case ($X_4 \cap X_2$ is a plane Π_2 cutting a line L_4 on F in generic position with respect L_3 and E_1), there can be another component X_5 for X. X_5 has to cut a plane on some X_i, $i \leq 4$. By arguing as above we can exclude that $X_5 \cap X_1$ is a plane different from F. On the other hand $X_5 \cap X_1$ cannot be F, otherwise there would be at least a point on five components of X (i.e. $L_4 \cap L_3 \subset F$) and this would be a contradiction with Corollary 4. Therefore $X_5 \cap X_1$ must be a plane intersecting F along a line l and we get *i*) with $p = 5$, or $X_5 \cap X_1$ is a line and so on. By repeating the above arguments it is easy to see that we get *i*) with an exceptional component X_p. ∎

7. Proof of Theorem 2

In this section we give the proof of Theorem 2 by showing that only the nine varieties listed in Theorem 2 are 3-projectable among all remaining varieties, after having imposed necessary conditions in the previous sections. In any case it suffices to prove or disprove that $\dim(\bigcup_{P \in X} T_P(X)) \leq m + 3$.

Firstly let us consider the case in which all components of X are linear spaces. By Lemma 6 we know that the only possibilities for X are *i*) and *ii*) described in that Lemma, with $r \geq 5$ by Lemma 5 *i*). Theorem 5 says that such an X is in fact 3-projectable. We get 1) and 2).

Let us assume $m \geq 4$. By Section 5 we know that the possibilities for X are the three cases described in Theorem 7.

In the first case $X \subset \mathbb{P}^9$. If we consider a point $P \in X_1 \cap X_2 \cap X_3$ we have, for any $i = 1, 2, 3$, $T_P(X_i) \simeq \mathbb{P}^4$ is generated by $\langle E_i \cup P \rangle$ and the tangent plane at $\pi_i(P)$ to the smooth quadric Q_i, base of the cone X_i, where π_i is the natural projection from E_i onto Q_i. Let L_i be the line $X_1 \cap X_2 \cap X_3 \cap Q_i$, then $T_P(X) \simeq \mathbb{P}^6$ is generated by $X_1 \cap X_2 \cap X_3$ and the 3 lines $L'_{P,i} \subset Q_i$ intersecting L_i at $\pi_i(P)$. Viceversa, if we take 3 generic points $R_i \in Q_i$, if we consider the 3 points R'_i intersected on L_i by the unique line L'_{R_i} of Q_i passing through R_i, not belonging to the ruling of L_i, if we consider the point $P := \langle E_1 \cup R'_1 \rangle \cap \langle E_2 \cup R'_2 \rangle \cap \langle E_3 \cup R'_3 \rangle \in X_1 \cap X_2 \cap X_3 \simeq \mathbb{P}^3$, then $T_P(X) = \left\langle (X_1 \cap X_2 \cap X_3) \cup L'_{R_1} \cup L'_{R_2} \cup L'_{R_3} \right\rangle$. It follows that the union of all the 6-dimensional linear spaces $T_P(X)$, for generic points $P \in X_1 \cap X_2 \cap X_3$, has dimension 9, and X is not 3-projectable into $\mathbb{P}^7$.

In the second case $X \subset \mathbb{P}^8$. We can argue as in the previous case and we can prove that the union of all the 6-dimensional linear spaces $T_P(X)$, for generic points $P \in X_1 \cap X_2 \cap X_3$, has dimension 8, and X is not 3-projectable into $\mathbb{P}^7$.

In the third case $X \subset \mathbb{P}^8$. Let us consider the line $L := X_1 \cap X_2 \cap X_3 \cap X_4$ and recall that L is disjoint from the line E_1, vertex of X_1 and that $X_1 \cap (X_2 \cup X_3 \cup X_4) = F := X_1 \cap X_2 \simeq \mathbb{P}^3$. For any point $Q \in L$, $\dim[T_Q(X)] = 7$ and $T_Q(X) = \langle T_Q(X_1) \cup X_2 \cup X_3 \cup X_4 \rangle$. As the projection of L from E_1 is a line on the 2-dimensional smooth quadric B_1 which is the base of X_1, when Q moves in L, $T_Q(X_1)$ varies in a pencil of 4-dimensional linear spaces, containing F, spanned by E_1 and the tangent plane to B_1 at $\pi_1(Q)$, hence $\dim(\bigcup_{Q \in L} T_Q(X)) = 8$ and X is not 3-projectable into $\mathbb{P}^7$.

From now on let us assume that $m = 3$.

By Proposition 4 we know that, if X contains a scroll, the possibilities are those described in that Proposition.

In the first case, let F be the fixed fibre of the scroll X_1 where all components of X intersect. For any point $P \in F$, $T_p(X) = \langle T_P(X_1) \cup X_2 \cup X_3 \rangle \simeq \mathbb{P}^5$, moreover for any other point $P' \in F$, $T_{P'}(X) = \langle T_{P'}(X_1) \cup X_2 \cup X_3 \rangle \neq T_p(X)$, because $T_P(X_1) \neq T_{P'}(X_1)$ by Lemma 7 and $\langle X_2 \cup X_3 \rangle \cap \langle X_1 \rangle = F$. It follows that the union of all the 5-dimensional linear spaces $T_P(X)$, for generic points $P \in F$, has dimension 7, and X is not 3-projectable into $\mathbb{P}^6$.

In the second case $X \subset \mathbb{P}^8$. We can argue as in the previous case and we can prove that the union of all the 6-dimensional linear spaces $T_P(X)$, for generic points $P \in F$, has dimension 8, and X is not 3-projectable into $\mathbb{P}^6$.

Proposition 5 describes the unique possibility for X when it contains a cubic cone. $X \subset \mathbb{P}^7$, to prove that X is 3-projectable into $\mathbb{P}^6$ it suffices to show that $\dim(\bigcup_{P \in X} T_P(X)) \leq 6$. Obviously it suffices to control points belonging to $X_1 \cap X_2 \simeq \mathbb{P}^2$, because if P belongs to a unique X_i, $T_P(X) \subset \langle X_i \rangle \subset \mathbb{P}^7$.

If $P \in (X_1 \cap X_2) \backslash (E_1 \cup E_2)$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \rangle \simeq \mathbb{P}^4$ so that $\dim(\bigcup_{P \in (X_1 \cap X_2) \backslash (E_1 \cup E_2)} T_P(X)) \leq 6$.

If $P \in E_1$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \rangle \simeq \mathbb{P}^6$ and this linear space is fixed as P varies in E_1. If $P = E_2$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \rangle \simeq \mathbb{P}^5$. Hence X is 3-projectable into $\mathbb{P}^6$ and we get 3).

By Proposition 6 we know that, if X does not contain cubic cones but it contains a quadric cone having a line as vertex, the possibilities are those described in that Proposition. Let us examine them.

i) $X \subset \mathbb{P}^8$. For any generic $P \in F$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \cup T_P(X_3) \rangle \simeq \mathbb{P}^5$. As P varies in F, $T_P(X_1)$ is fixed, but $T_P(X_2)$ and $T_P(X_3)$ are generated, respectively, by F and the two lines, of the smooth base quadrics Q_2 and Q_3, different from $L_2 := F \cap Q_2$, $L_3 := F \cap Q_3$ and passing through the points $\pi_2(P)$ and $\pi_3(P)$, where π_i is the natural projection from the vertex E_i of the cone X_i onto its base Q_i. Viceversa, if we take two generic points $R_2 \in Q_2$ and $R_3 \in Q_3$, if we consider the two points R_i' intersected on L_i by the unique line L'_{R_i} of Q_i passing through R_i, not belonging to the ruling of L_i $(i = 1, 2)$, if we consider the point $P := \langle E_2 \cup R_2' \rangle \cap \langle E_3 \cup R_3' \rangle \in F \simeq \mathbb{P}^2$, then $T_P(X) = \left\langle T_P(X_1) \cup L'_{R_2} \cup L'_{R_3} \right\rangle$. We claim that $\dim[\bigcup_{generic\ P \in F} T_P(X)] = 7$, excluding the x-projectability of X. Indeed, since the intersection of two general elements of this family is not $\mathbb{P}^5$, $\bigcup_{generic\ P \in F} T_P(X)$ cannot be contained in a linear space of dimension 6. Moreover, no non-linear 6-dimensional variety contains a 2-dimensional family of linear spaces of dimension 5 (cut with a general linear subspace and use that a plane is the unique integral surface of a projective space containing a 2-dimensional family of lines). It follows that the union of all the 5-dimensional linear spaces $T_P(X)$, for generic points $P \in F$, has dimension 7. Hence X is not 3-projectable into $\mathbb{P}^6$.

ii) $X \subset \mathbb{P}^7$. To prove that X is 3-projectable into $\mathbb{P}^6$ it suffices to show that $\dim(\bigcup_{P \in X} T_P(X)) \leq 6$. Obviously it suffices to control points belonging to two or more components of X, because if P belongs to a unique X_i, $T_P(X) \subset \langle X_i \rangle \subset \mathbb{P}^7$.

If $P \in F \backslash (E_1 \cup E_2)$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \cup T_P(X_3) \rangle \simeq \mathbb{P}^5$. As P varies in $F \backslash (E_1 \cup E_2)$, $T_P(X_1)$ and $T_P(X_3)$ are fixed, while $T_P(X_2)$ is generated by

F and the lines, of the smooth base quadrics Q_2, different from $F \cap Q_2$, and passing through the points $\pi_2(P)$. Hence the union of all the 5 dimensional linear spaces $T_P(X)$, for points $P \in F \backslash (E_1 \cup E_2)$, has dimension 6. If $P \in E_1$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \cup T_P(X_3) \rangle \simeq \mathbb{P}^6$ and this space is fixed as P varies in E_1. If $P = E_2$, $\dim[T_p(X)] = 6$. Hence X is 3-projectable into $\mathbb{P}^6$ and we get 4).

iii) $X \subset \mathbb{P}^7$. To prove that X is 3-projectable into $\mathbb{P}^6$ it suffices to show that $\dim(\bigcup_{P \in X} T_P(X)) \leq 6$. Obviously it suffices to control points belonging to two or more components of X.

If $P \in F \backslash (E_1 \cup E_2)$, $T_P(X) = \langle T_P(X_1) \cup T_P(X_2) \rangle \simeq \mathbb{P}^4$, so that $\dim(\bigcup_{P \in F \backslash (E_1 \cup E_2)} T_P(X)) \leq 6$.

If $P \in F' \backslash (E_1)$, $T_P(X) = \langle T_P(X_1) \cup T_P(X_3) \rangle \simeq \mathbb{P}^4$, so that $\dim(\bigcup_{P \in F' \backslash (E_1)} T_P(X)) \leq 6$.

If $P \in E_1$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \cup T_P(X_3) \rangle \simeq \mathbb{P}^6$ and this space is fixed as P varies in E_1. If $P = E_2$, $\dim[T_p(X)] = 5$. Hence X is 3-projectable into $\mathbb{P}^6$ and we get 5).

iv) $X \subset \mathbb{P}^{r+3}$. To prove that X is 3-projectable into $\mathbb{P}^6$ it suffices to show that $\dim(\bigcup_{P \in X} T_P(X)) \leq 6$. Obviously it suffices to control points belonging to two or more components of X.

If $P \in F \backslash (E_1 \cup L_2 \cup ... \cup L_r)$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \rangle \simeq \mathbb{P}^4$, so that $\dim(\bigcup_{P \in F \backslash (E_1 \cup L_2 \cup ... \cup L_r)} T_P(X)) \leq 6$.

If $P \in E_1 \backslash [(E_1 \cap L_2) \cup ... \cup (E_1 \cap L_r)]$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \rangle \simeq \mathbb{P}^5$ and this space is fixed as P varies in $E_1 \backslash [(E_1 \cap L_2) \cup ... \cup (E_1 \cap L_r)]$.

If $P \in L_i \backslash [(E_1 \cap L_i) \bigcup_{j \geq 3,\ j \neq i} (L_i \cap L_j)]$, $i \geq 3$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \cup T_P(X_i) \rangle \simeq \mathbb{P}^5$ and this space is fixed as P varies in $L_i \backslash [(E_1 \cap L_i) \bigcup_{j \geq 3,\ j \neq i} (L_i \cap L_j)]$.

If $P \in (X_i \cap X_j) \backslash (F \bigcup_{k \geq 3,\ k \neq i, k \neq j} X_k)$, $i, j \geq 3$, $T_p(X) = \langle T_P(X_i) \cup T_P(X_j) \cup T_P(X_2) \rangle \simeq \mathbb{P}^5$ and this space is fixed as P varies in $(X_i \cap X_j) \backslash (F \bigcup_{k \geq 3,\ k \neq i, k \neq j} X_k)$.

If P is in the remaining discrete sets of a finite number of points we know that $\dim[T_p(X)] \leq 6$ because this necessary condition, given by Corollary 4, was checked in the proof of Proposition 6. Hence X is 3-projectable into $\mathbb{P}^6$ and we get 6).

By Proposition 7 we know that the last three possibilities for X are those described in that Proposition. Let us examine them.

i) We can argue as in the previous case *iv*) of Proposition 6 and conclude that X is 3-projectable into $\mathbb{P}^6$, it is 7). Note that the existence of an exceptional component X_p is not important, since for any generic point $P \in l := X_p \cap F$ we have $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \cup T_P(X_p) \rangle \simeq \mathbb{P}^5$ and these linear spaces are fixed for generic $P \in l$. For the remaining discrete sets of points on l we know that $\dim[T_p(X)] \leq 6$.

ii) We can argue as in the previous case *iv*) of Proposition 6 unless $P \in X_r$.

If $P \in (X_r \cap X_j) \backslash L_j$, $T_p(X) = \langle T_P(X_j) \cup T_P(X_r) \rangle \simeq \mathbb{P}^4$

and this space is fixed as P varies in $(X_r \cap X_j)\backslash L_j$.
If $P \in L_j\backslash(\bigcup_{k\geq 3,\ k\neq r, k\neq j} X_k)$, $T_p(X) = \langle T_P(X_1) \cup T_P(X_2) \cup T_P(X_j) \cup T_P(X_r)\rangle \simeq \mathbb{P}^6$ and this space is fixed as P varies in $L_j\backslash(\bigcup_{k\geq 3,\ k\neq r, k\neq j} X_k)$.

If P is in the remaining discrete sets of a finite number of points we know that $\dim[T_p(X)] \leq 6$ because this necessary condition, given by Corollary 4, was checked in the proof of Proposition 7. Hence X is 3-projectable into $\mathbb{P}^6$ and we get 8).

iii) We can argue as in previous cases: for $P \in F\backslash L$, $P \in \Pi\backslash L$, $P \in L$ the tangent space is fixed as P varies in these sets and $\dim[T_P(X)] \leq 6$. If $P = E_1$ $\dim[T_p(X)] = 5$. Hence X is 3-projectable into $\mathbb{P}^6$ and we get 9).

As there are no other possibilities for X we have proved Theorem 2.

References

[A1] B.Adlandsvik: "Joins and higher secant varieties" Math. Scand. **62** (1987), p. 213-222.

[A2] B.Adlandsvik: "Varieties with an extremal numbers of degenerate higher secant varieties" J. Reine Angew. Math. **392** (1988) p. 16-26.

[A-B] A.Alzati-E.Ballico: "Reducible Veronese surfaces". To appear on Advances in Geometry.

[B-S] M.C.Beltrametti-A.J.Sommese: "The adjunction theory of complex projective varieties" Walter de Gruyter, Berlin, 1995.

[B-H] W.Bruns-J.Herzog: "Cohen-Macaulay rings" Cambridge studies in advanced math. 39, Cambridge Univ. Press 1993.

[C] M.L.Catalano-Johnson: "The Possible Dimensions of the Higher Secant Varieties" Amer. J. Math. **118** (2) (1996), p. 355-361.

[E] D.Eisenbud: "Commutative Algebra with a View Towards Algebraic Geometry" G.T.M. 150, Springer, New York, 1995.

[F] G.Fløystad: "Monads on projective spaces" Comm. Algebra **28** (12) (2000), p. 5503-5516.

[E-G-H-P] D.Eisenbud-M.Green-K.Hulek-S.Popescu: "Small schemes and varieties of minimal degree" Amer. J. Math. **128** (6) (2006), p. 1363-1389.

[J] K.W.Johnson: "Immersion and embeddings of projective varieties" Acta Math. **140** (1978), p. 49-74.

[Ha] R.Hartshorne: "Complete intersections and connectedness" Amer. J. Math. **84** (1962), p. 497-508.

[Ho] A.Holme: "Embedding obstructions for singular algebraic varieties in $\mathbb{P}^N$" Acta Math. **135** (1975), p. 155-185.

[X] S.Xambó: "On projective varieties of minimal degree" Collectanea Math. **32** (2) (1981), p. 149-163.

Dipartimento di Matematica Univ. di Milano, via C. Saldini 50 20133-Milano (Italy)
E-mail address: `alberto.alzati@unimi.it`

Dipartimento di Matematica Univ. di Trento, via Sommarive 14 38050-Povo (TN) (Italy)
E-mail address: `ballico@science.unitn.it`