J-EMBEDDABLE REDUCIBLE SURFACES

ALBERTO ALZATI AND EDOARDO BALLICO

Abstract. Here we classify J-embeddable surfaces, i.e. surfaces whose secant varieties have dimension at most 4, when the surfaces have two components at most.

1. Introduction

Let $\mathbb{P}^n$ be the n-dimensional complex projective space. In this paper a variety will be always a non degenerate, reduced subvariety of $\mathbb{P}^n$, of pure dimension. Surfaces and curves will be subvarieties of dimension 2 or 1, respectively.

In [J] the author introduces the definition of J-embedding: for any subvariety $V \subset \mathbb{P}^n$ and for any λ-dimensional linear subspace $\Lambda \subset \mathbb{P}^n$ we say that V projects isomorphically to Λ if there exists a linear projection $\pi_{\mathcal{L}} : \mathbb{P}^n - -- > \Lambda$, from a suitable $(n - \lambda - 1)$-dimensional linear space $\mathcal{L}$, disjoint from V, such that $\pi_{\mathcal{L}}(V)$ is isomorphic to V. We say that $\pi_{\mathcal{L}|V}$ is a J-embedding of V if $\pi_{\mathcal{L}|V}$ is injective and the differential of $\pi_{\mathcal{L}|V}$ is finite-to one (see [J], 1.2).

In this paper we want to give a complete classification of J-embeddable surfaces having at most two irreducible components. More precisely we prove (see Lemma 9 and Proposition 3) the following:

Theorem 1. *Let V be a non degenerate, surface in $\mathbb{P}^n$, $n \geq 5$. Assume that for a generic 4-dimensional linear subspace $\Lambda \subset \mathbb{P}^n$ the linear projection $\pi_{\mathcal{L}} : \mathbb{P}^n - -- >$ Λ is such that $\pi_{\mathcal{L}|V}$ is a J-embedding of V, and that V has at most two irreducible components. Then V is in the following list:*

1) *V is the Veronese surface in $\mathbb{P}^5$;*
2) *V is an irreducible cone;*
3) *V is the union of a Veronese surface in $\mathbb{P}^5$ and a tangent plane to it;*
4) *V is the union of two cones having the same vertex;*
5) *V is the union of a cone with vertex a point P and a plane passing though P;*
6) *V is the union of :*

- an irreducible surface S, such that the dimension of its linear span $\langle S \rangle$ is 4 and S is contained in a 3-dimensional cone having a line l as vertex,

- a plane cutting $\langle S \rangle$ along l.

Note that 6) is a particular case of Example 2.

By using our results it is possible to get a reasonable classification also for J-embeddable surfaces having at least three irreducible components. However the classification is very involved, consisting in a long list of cases and subcases, so that

Date: December, 18 2009.
1991 *Mathematics Subject Classification.* Primary 14J25; Secondary 14N20.
Key words and phrases. reducible surfaces, projectability.

we have only given some information about them in section 6. A longer version of this paper will be sent to ArXiv e-prints.

2. Notation-Definitions

If $M \subset \mathbb{P}^n$ is any scheme, $M \simeq \mathbb{P}^k$ means that M is a k-dimensional linear subspace of $\mathbb{P}^n$.

V_{reg} := subset of V consisting of smooth points.

$\langle V_1 \cup ... \cup V_r \rangle$:= linear span in $\mathbb{P}^n$ of the subvarieties $V_i \subset \mathbb{P}^n$, $i = 1, ..., r$.

$Sec(V) := \overline{\{ \bigcup_{v_1 \neq v_2 \in V} \langle v_1 \cup v_2 \rangle \}} \subset \mathbb{P}^n$ for any irreducible subvariety $V \subset \mathbb{P}^n$.

$[V; W] := \overline{\{ \bigcup_{v \in V, w \in W, v \neq w} \langle v \cup w \rangle \}} \subset \mathbb{P}^N$ for any pair of distinct irreducible subvarieties $V, W \subset \mathbb{P}^n$.

In case $V = W$, $[V; V] = Sec(V)$. In case $V = W$ is a unique point P we put $[V; W] = P$.

In case V is reducible, $V = V_1 \cup ... \cup V_r$, $Sec(V) := \{ \bigcup_{i=1}^{r} \bigcup_{j=1}^{r} [V_i; V_j] \}$.

In case V and W are reducible, without common components, $V = V_1 \cup ... \cup V_r$, $W = W_1 \cup ... \cup W_s$, we put $[V; W] := \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} [V_i; W_j]$ (with the reduced scheme structure).

$T_P(V)$:= embedded tangent space at a smooth point P of V.

$\mathcal{T}_v(V)$:= tangent star to V at v : it is the union of all lines l in $\mathbb{P}^n$ passing through v such that there exists afamily of lines $\langle v' \cup v'' \rangle \to l$ when $v', v'' \to v$ with $v', v'' \in V$. (see [J] page. 54).

$Vert(V) := \{ P \in V | \ [P; V] = V \}$.

Let us recall that $Vert(V)$ is always a linear space, moreover $Vert(V) = \bigcap_{P \in V} (T_P(V))$, (see [A2], page. 17).

We say that V is a cone of vertex $Vert(V)$ if and only if V is not a linear space and $Vert(V) \neq \emptyset$. If V is a cone the codimension in V of $Vert(V)$ is at least two.

Remark 1. *If V is an irreducible surface, not a plane, for which there exists a linear space L, such that for any generic point $P \in V$, $T_P(V) \supseteq L$, then L is a point and V is a cone over an irreducible curve with vertex L (see* [A2]*, page. 17).*

Caution: in this paper we distinguish among two dimensional cones and planes, so that a two dimensional cone will have a well determined point as vertex.

For any subvariety $V \subset \mathbb{P}^n$ let us denote by

$$V^* := \overline{\{ H \in \mathbb{P}^{n*} | \ H \supseteq T_P(V) \text{ for some point } P \in V_{reg} \}}$$

the dual variety of V, where $\mathbb{P}^{n*}$ is the dual projective space of $\mathbb{P}^n$ and H is a generic hyperplane of $\mathbb{P}^n$. Let us recall that $(V^*)^* = V$.

3. Background material

In this section we collect a few easy remarks about the previous definitions and some known results which will be useful in the sequel.

Proposition 1. *Let V be any subvariety of $\mathbb{P}^n$ and let P be a generic point of $\mathbb{P}^n$. If $P \notin [V; V]$ then $\pi_{P|V}$ is a J-embedding of V.*

Proof. See Proposition 1.5 c) of [Z], chapter II, page 37. ∎

Corollary 1. *Let V be any surface of $\mathbb{P}^n$, $n \geq 5$, and let Λ be a generic 4-dimensional linear space of $\mathbb{P}^n$. There exists a J-embedding $\pi_{P|V}$ for V, from a suitable $(n-5)$-dimensional linear space of $\mathbb{P}^n$ into $\Lambda \simeq \mathbb{P}^4$, if and only if* $\dim[Sec(V)] \leq 4$.

Proof. Apply Proposition 1. See also Theorem 1.13 c) of [Z], chapter II, page 40. ∎

Corollary 2. *Let $V = V_1 \cup ... \cup V_r$ be a reducible surface in $\mathbb{P}^n$, $n \geq 5$, and let Λ be a generic 4-dimensional linear space of $\mathbb{P}^n$. There exists a J-embedding $\pi_{P|V}$ for V, from a suitable $(n-5)$-dimensional linear space of $\mathbb{P}^n$ into $\Lambda \simeq \mathbb{P}^4$, if and only if* $\dim([V_i; V_j]) \leq 4$ *for all $i, j = 1, ..., r$, including cases $i = j$.*

Proof. Look at the definition of $Sec(V)$ and apply Corollary 1. ∎

Lemma 1. *For any pair of distinct irreducible subvarieties $V, W \subset \mathbb{P}^n$:*
1) *if V and W are linear spaces* $[V; W] = \langle V, W \rangle$;
2) *if V is a linear space, $[V; W]$ is a cone, having V as vertex;*
3) $\langle [V; W] \rangle = \langle \langle V \rangle \cup \langle W \rangle \rangle$;
4) $\langle V \rangle = \left\langle \bigcup_{P \in V} T_P(V) \right\rangle$, *$P$ generic point of V;*
5) $[V; [W; U]] = [[V; W]; U] = \overline{\{\bigcup_{v \in V, w \in W, u \in U, v \neq w, v \neq u, u \neq w} \langle v \cup w \cup u \rangle\}}$, *for any other irreducible subvariety U distinct from V and W.*

Proof. Immediate consequences of the definitions of $[V; W]$ and $\langle V \rangle$. ∎

Let us recall the Terracini's lemma:

Lemma 2. *Let us consider a pair of irreducible subvarieties $V, W \subset \mathbb{P}^n$ and a generic point $R \in [V; W]$ such that $R \in \langle P \cup Q \rangle$, with $P \in V$ and $Q \in W$. Then* $T_R([V; W]) = \langle T_P(V) \cup T_Q(W) \rangle$ *and* $\dim([V; W]) = \dim(\langle T_P(V) \cup T_Q(W) \rangle)$.

Proof. See Corollary 1.11 of [A1]. ∎

The following lemmas consider the join of two irreducible varieties of low dimensions.

Lemma 3. *Let C, C' be irreducible distinct curves in $\mathbb{P}^n$, $n \geq 2$, then* $\dim([C; C']) = 3$ *unless C and C' are plane curves, lying on the same plane, in this case* $\dim([C; C']) = 2$.

Proof. The claim follows from Corollary 1.5 of [A1] with $r = 2$. ∎

Lemma 4. *Let C be an irreducible curve, not a line, and let B be an irreducible surface in $\mathbb{P}^n$, $n \geq 2$. Then:*
i) $\dim([C; B]) \leq 4$;
ii) $\dim([C; B]) = 3$ *if and only if* $\langle C \cup B \rangle \simeq \mathbb{P}^3$;
iii) $\dim([C; B]) = 2$ *if and only if B is a plane and $C \subset B$.*

Proof. i) Obvious.
ii) If $\dim([C; B]) = 3 = 1 + \dim(B)$, by Proposition 1.3 of [A1], we have $C \subseteq Vert([C; B])$. If $[C; B] \simeq \mathbb{P}^3$ then $\langle C \cup B \rangle \simeq \mathbb{P}^3$ and we are done. If not the codimension of $Vert([C; B])$ in $[C; B]$ is at least 2 (see [A1] page. 214), hence $\dim\{Vert([C; B])\} \leq 1$, hence $Vert([C; B]) = C$, but this is a contradiction as C is not a line and $Vert([C; B])$ is a linear space.

iii) If $\dim([C;B]) = 2 = 1 + \dim(C)$, then Proposition 1.3 of [A1] implies $B \subseteq Vert([C;B])$. In this case $Vert([C;B]) = [C;B] = B$. Hence B is a plane and necessarily $C \subset B$ by Lemma 3. ∎

Lemma 5. *Let B be an irreducible surface and l any line in $\mathbb{P}^n$, $n \geq 2$. Then:*
i) $\dim([l;B]) \leq 4$;
ii) $\dim([l;B]) = 3$ *if and only if* $\langle l \cup B \rangle \simeq \mathbb{P}^3$ *or B is contained in a cone Ξ having l as vertex and an irreducible curve C as a basis.*
iii) $\dim([l;B]) = 2$ *if and only if B is a plane and $l \subset B$.*

Proof. *i*) Obvious.
ii) If $\dim([l;B]) = 3 = 1 + \dim(B)$, by Proposition 1.3 of [A1], we have $l \subset Vert([l;B])$. If $[l;B] \simeq \mathbb{P}^3$ we have $\langle l \cup B \rangle \simeq \mathbb{P}^3$, if not the codimension of $Vert([l;B])$ in $[l;B]$ is at least 2 (see [A1] page. 214). Hence $\dim\{Vert([l;B])\} \leq 1$, hence $Vert([l;B]) = l$ and Ξ is exactly $[l;B]$. Note that $\dim([l;B]) = 3$ if and only if $l \cap T_P(B) \neq \emptyset$ for any generic point $P \in B$.
iii) If $\dim([l;B]) = 2 = 1 + \dim(l)$, by Proposition 1.3 of [A1], we have $B \subset Vert([l;B])$. We can argue as in the proof of Lemma 4, *iii*). ∎

The following Lemmas consider the possible dimensions for the join of two surfaces according to the dimension of the intersection of their linear spans. Firstly we consider the case in which one of the two surface is a plane.

Lemma 6. *Let A be an irreducible, non degenerate surface in $\mathbb{P}^n$, $n \geq 3$, and let B be any fixed plane in $\mathbb{P}^n$. Let A' be the tangent plane at a generic point of A_{reg}. Then:*
i) $\dim([A;B]) = 5$ *if and only if* $A' \cap B = \emptyset$;
ii) $\dim([A;B]) = 4$ *if and only if* $\dim(A' \cap B) = 0$;
iii) $\dim([A;B]) = 3$ *if and only if* $\dim(A' \cap B) = 1$;
iv) $\dim([A;B]) = 3$ *if and only if* $\langle A, B \rangle \simeq \mathbb{P}^3$.

Proof. As $n \geq 3$, $\dim([A;B]) \geq 3$ and *i*), *ii*) and *iii*) are consequences of lemma 2. If $\langle A, B \rangle \simeq \mathbb{P}^3$ obviously $\dim(A' \cap B) = 1$. On the other hand, let us assume that $\dim(A' \cap B) = 1$ and let us consider two different generic points $P, Q \in A \backslash B$; we have $[A;B] \supseteq [P;B] \cup [Q;B]$ and $[P;B] \simeq [Q;B] \simeq \mathbb{P}^3$. If $P \notin [Q;B]$ we have $\dim([A;B]) \geq 4$, because $[A;B]$ is irreducible and it cannot contain the union of two distinct copies of $\mathbb{P}^3$, intersecting along a plane, unless $\dim([A;B]) \geq 4$, but this is a contradiction with $\dim(A' \cap B) = 1$ by *ii*). Hence $P \in [Q;B] \simeq \mathbb{P}^3$ and $A \subseteq [Q;B] \simeq \mathbb{P}^3$ as P is a generic point of A. ∎

Lemma 7. *Let A, B be two irreducible, surfaces in $\mathbb{P}^n$, $n \geq 5$. Let us assume that neither A nor B is a plane. Set $L := \langle A \rangle \cap \langle B \rangle$, $M := \langle A \cup B \rangle$, $m := \dim(M)$. Then:*
i) *if $L = \emptyset$, then* $\dim([A;B]) = 5$;
ii) *if L is a point P,* $\dim([A;B]) \leq 4$ *if and only if A and B are cones with vertex P*;
iii) *if* $\dim(L) = 1$, $\dim([A;B]) \leq 4$ *if and only if:*
- *there exists a point $P \in L$ such that A and B are cones with vertex P, or*
- $m \leq 4$;

iv) *if* $\dim(L) = 2$, $\dim([A;B]) \leq 4$ *if and only if:*
- *there exists a point $P \in L$ such that A and B are cones with vertex P, or*
- $\dim(\langle A \rangle) = \dim(\langle B \rangle) = 3$ *and* $m = 4$.

Proof. $i)$ let A' be the tangent plane at a generic point of A_{reg}. Let B' be the tangent plane at a generic point of B_{reg}. We have $A' \cap B' = \emptyset$ so that $i)$ follows from Lemma 2.

$ii)$ Obviously, in any case, if A and B are cones with a common vertex P, A' and B' contain P so that $\dim([A;B]) \leq 4$ by Lemma 2. On the other hand, if $L = P$, $A' \cap B' \neq \emptyset$ only if $A' \cap B' = P$ and this implies that the tangent planes at the generic points of A and B contain P. Hence A and B are cones with common vertex P.

$iii)$ If $m \leq 4$ obviously $\dim([A;B]) \leq 4$. Let us assume that $m \geq 5$ and $\dim([A;B]) \leq 4$. Lemma 2 implies $A' \cap B' \neq \emptyset$, while, obviously, $A' \cap B' \subseteq L$. Neither A' nor B' can contain L because A and B are not planes. Hence $A' \cap B'$ is a point $P \in L$ and we can argue as in $ii)$.

$iv)$ Let us assume that $\dim([A;B]) \leq 4$ and that A and B are not cones with a common vertex P. By Lemma 2 we have $A' \cap B' \neq \emptyset$, and, obviously, $A' \cap B' \subseteq L$. As A and B are not cones with a common vertex it is not possible that $A' \cap B'$ is a fixed point and it is not possible that $A' \cap B'$ is a fixed line because A and B are not planes. Hence $\dim(A' \cap L) = \dim(B' \cap L) = 1$ and in this case $\dim([A;L]) = \dim([B;L] = 3$ by Lemma 6 $iii)$. It follows that $\dim(\langle A \rangle) = \dim(\langle B \rangle) = 3$ by Lemma 6 $iv)$, hence $m = 4$. ∎

Lemma 8. *Let A, B be two irreducible surfaces in $\mathbb{P}^n$, $n \geq 5$. Set $L := \langle A \rangle \cap \langle B \rangle$, $M := \langle A \cup B \rangle$, $m := \dim(M)$. Let us assume that* $\dim(\langle A \rangle) = \dim(\langle B \rangle) = 4$, $\dim(L) = 3$, $m = 5$, $\dim([A;B]) \leq 4$. *Then A and B are cones with the same vertex.*

Proof. By Lemma 2 we know that for any pair of points $(P,Q) \in A_{reg} \times B_{reg}$, $\emptyset \neq T_P(A) \cap T_Q(B) \subseteq L$. As (P,Q) are generic, we can assume that $P \in A \backslash (A \cap L)$ and $Q \in B \backslash (B \cap L)$, so that $l_P := T_P(A) \cap L$ and $l_Q := T_Q(B) \cap L$ are lines, intersecting somewhere in L.

(a) Let us assume that $l_P \cap l_{P'} = \emptyset$ for any generic pair of points $(P, P') \in A \backslash (A \cap L)$. Then the lines $\{l_P | P \in A \backslash (A \cap L), P \in A_{reg}\}$ give rise to a smooth quadric $\mathcal{Q}$ in $L \simeq \mathbb{P}^3$ in such a way that the lines $\{l_P\}$ all belong to one of the two rulings of $\mathcal{Q}$. Note that $\mathcal{Q} \neq A$, because they have different spans. Now, for any smooth point $P \in A \backslash (A \cap L)$, let us consider a generic tangent hyperplane $H_P \subset M$ at P. Obviously $H_P \supset T_P(A)$ and, as H_P is generic, it cuts L only along a plane and this plane contains l_P. Hence it is a tangent plane for $\mathcal{Q}$. It follows that H_P is also a tangent hyperplane for $\mathcal{Q}$ in M. Therefore $A^* \subseteq \mathcal{Q}^*$ in M^*. If A is not a developable, ruled surface we have $A^* = \mathcal{Q}^*$ by looking at the dimension. Hence A $= (A^*)^* = (\mathcal{Q}^*)^* = \mathcal{Q}$: contradiction.

Now let us assume that A is a developable, ruled surface and let us consider the curve $C := A \cap L$, which is a hyperplane section of A. We claim that the support of C is not a line. In fact C must contain a directrix for A because C is a hyperplane section of A. So that if the support of C is a line l this line must be a directrix for A. Hence a direct local calculation shows that l is contained in every tangent plane at points of A_{reg}. It follows that $l_P = l$ for any point $P \in A_{reg}$: contradiction. Thus the claim is proved. On the other hand, for a fixed line $\overline{l_Q}$ we can consider $[\overline{l_Q}; C]$. Since the support of C is not a line $[\overline{l_Q}; C] = L$, moreover $[\overline{l_Q}; C] \subsetneq [\overline{l_Q}; A]$. Hence $\dim([\overline{l_Q}; A]) \geq 4$. This inequality contradicts Lemma 2 because $\overline{l_Q} \cap T_P(A) \neq \emptyset$, for any point $P \in A_{reg}$.

(b) From (a) it follows that $l_P \cap l_{P'} \neq \emptyset$ for any generic pair of points $(P, P') \in A \backslash (A \cap L)$. It is known (and a very easy exercise) that this is possible only if all lines $\{l_P\}$ pass through a fixed point $V_A \in L$ or all lines $\{l_P\}$ lie on a fixed plane $U_A \subset L$. In the same way we get $l_Q \cap l_{Q'} \neq \emptyset$ for any generic pair of points $(Q, Q') \in B \backslash (B \cap L)$ and that all lines $\{l_Q\}$ pass through a fixed point $V_B \in L$ or all lines $\{l_Q\}$ lie on a fixed plane $U_B \subset L$.

As for any pairs of points $(P, Q) \in A_{reg} \times B_{reg}$, $\emptyset \neq T_P(A) \cap T_Q(B) \subseteq L$, we have only four possibilities:

1) $V_A = V_B$, hence A and B are cones having the same vertex (recall that $T_P(A) \supset l_P \supset V_A$ and $T_Q(B) \supset l_Q \supset V_B$) and we are done;

2) $V_A \in U_B$, and all lines $\{l_Q\} \subset U_B$ pass necessarily through V_A, so that A and B are cones having the same vertex in this case too;

3) $V_B \in U_A$ and we can argue as in case 2);

4) there exist two planes U_A and U_B.

If $U_A \cap U_B$ is a line l, then the generic tangent planes $T_P(A)$ and $T_Q(B)$ would contain l and both A and B would be planes: contradiction. If $U_A = U_B$, by Lemma 2 we get $\dim([U_A; A]) = \dim([U_B; B]) = 3$ and they are (irreducible) cones as U_A and U_B are linear spaces. Hence they are 3-dimensional linear spaces containing A and B, respectively: contradiction. ∎

4. Examples of J-embeddable surfaces

In Section 4 we give some examples of J-embeddable surfaces and we prove a result concerning the Veronese surface which will be useful for the classification.

Example 1. *Let W be a fixed 2-dimensional linear subspace in $\mathbb{P}^n$, $n \geq 5$. Let m be a positive integer such that $1 \leq m \leq n-2$. Let us consider m distinct 3-dimensional linear subspaces $M_i \subset \mathbb{P}^n$, $1 \leq i \leq m$, such that $W \subset M_i$ for $i = 1, ..., m$ and $\langle M_1 \cup ... \cup M_m \rangle = \mathbb{P}^n$. For each $i = 1, \dots, m$ fix a reduced surface D_i of M_i in such a way that $X := \cup_{i=1}^m D_i$ spans $\mathbb{P}^n$. We claim that X can be J-projected into a suitable $\mathbb{P}^4$. By Corollary 1 it suffices to show that $\dim[Sec(X)] \leq 4$. Indeed, $\dim[Sec(D_i)] \leq 3$ for all i, while $\dim([D_i; D_j]) \leq 4$ for all $i \neq j$, because every $D_i \cup D_j$ is contained in the 4-dimensional linear space $\langle M_i \cup M_j \rangle$.*

Example 2. *Let N be a fixed 4-dimensional linear subspace in $\mathbb{P}^n$, $n \geq 5$. Let $A_i \subset N$ be irreducible surfaces, $i = 1, ..., s$. Assume that every A_i is contained in the intersection of some 3-dimensional cones $E_j \subset N$ having a line l_j as vertex and let $\{B_{jk_j}\}$ be a set of pairwise intersecting planes in $\mathbb{P}^n$ such that $B_{jk_j} \cap N = l_j$, with $j, k_j \geq 1$. Set $X := \{A_i \cup B_{jk_j}\}$. We claim that X can be J-projected into a suitable $\mathbb{P}^4$.*

By Corollary 1, it suffices to show that $\dim[Sec(X)] \leq 4$ and the only non trivial check is that $\dim([A_i; B_{jk_j}]) \leq 4$ for any A_i and for any plane B_{jk_j}. But this follows from Lemma 2 because for any j and for any point $P \in (A_i)_{reg} \cap (E_j)_{reg}$ the tangent plane $T_P(A_i)$ is contained in $T_P(E_j) \simeq \mathbb{P}^3$, hence $T_P(A_i) \cap l_j \neq \emptyset$.

Example 3. *Let $Y \subset \mathbb{P}^5$ be a Veronese surface. Fix a point $P \in Y$ and set $X := Y \cup T_P(Y)$. Let us recall that $\dim[Sec(Y)] = 4$. Hence, by Terracini's lemma, we know that $T_P(Y) \cap T_Q(Y) \neq \emptyset$ for any pair of points $P, Q \in Y$. Therefore $\dim[Y, T_p(Y)] = 4$ and $\dim[Sec(X)] = 4$ too. Then we can apply Corollary 1.*

The following proposition shows that the above example is in fact the only possibility for a surface $X = Y \cup B$ to have $\dim[Sec(X)] = 4$, where B is any irreducible surface.

Proposition 2. *Let $Y \subset \mathbb{P}^n$ be a Veronese surface embedded in $\langle Y \rangle \simeq \mathbb{P}^5$, $n \geq 5$, and let $B \subset \mathbb{P}^n$ be any irreducible surface. Set $X := Y \cup B$. Thus* $\dim[Sec(X)] = 4$ *if and only if B is a plane in $\langle Y \rangle$, tangent to Y at some point P.*

Proof. For the proof it is useful to choose a plane Π such that $\langle Y \rangle \simeq \mathbb{P}^5$ is the linear space parametrizing conics of Π, i.e. $\langle Y \rangle \simeq \mathbb{P}[H^0(\Pi, \mathcal{O}_\Pi(2))]$. Then Y can be considered as the subvariety of $\langle Y \rangle$ parametrizing double lines of Π, moreover Y can be also considered as the 2-Veronese embedding of Π^* via a map we call ν.

Firstly, let us consider the case in which B is a plane in $\langle Y \rangle$. Obviously $\dim[Sec(X)] = 4$ if and only if $\dim[Y; B] = 4$. Note that $\dim[Y; B] = 5$ if $B \cap Y = \emptyset$, because every point $P \in \mathbb{P}^5$ is contained in at least a line intersecting both B and Y. Then we have to consider all other possibilities for $B \cap Y$.

Let us remark that $\dim[Y; B] = 4$, if and only if the linear projection $\pi_B : \mathbb{P}^5 - -- > \Lambda$ is such that $\dim[\overline{\pi_B(Y \backslash B)}] = 1$, where $\Lambda \simeq \mathbb{P}^2$ is a generic plane, disjoint from B. In fact $\dim[Sec(X)] = 4$, if and only if $\dim([B; Y]) = 4$, if and only if $\dim(\overline{\bigcup_{y \in Y \backslash B} \langle B \cup y \rangle}) = 4$, if and only if $\dim[(\overline{\bigcup_{y \in Y \backslash B} \langle B \cup y \rangle}) \cap \Lambda] = 1$. But $(\overline{\bigcup_{y \in Y \backslash B} \langle B \cup y \rangle}) \cap \Lambda = \overline{\pi_B(Y \backslash B)}$.

Let us assume that $\dim(B \cap Y) = 1$. It is well known that Y does not contain lines or other plane curves different from smooth conics. If the scheme $B \cap Y$ contains a smooth conic γ, it is easy to see that the generic fibres of any linear projection as π_B are 0 -dimensional. Indeed, by considering the identification $\langle Y \rangle \simeq \mathbb{P}[H^0(\Pi, \mathcal{O}_\Pi(2))]$, for any point $P \in Y$, $T_P(Y)$ parametrizes the reducible conics of Π whose components are: a fixed line r of Π (such that $P \leftrightarrow r^2$) and any line of Π. While B parametrizes the reducible conics of Π having a singular point $Q \in \Pi$ such that the dual line $l \in \Pi^*$ corresponding to Q is such that $\nu(l) = \gamma$. Therefore, for generic $P \in Y$, $T_P(Y) \cap B = \emptyset$. It follows that $\dim[\overline{\pi_B(Y \backslash B)}] = 2$ and $\dim[Y, B] = 5$. This fact can also be checked by a direct computation with a computer algebra system, for instance Macaulay, taking into account that Y is a homogeneous variety, so that the computation can be made by using a particular smooth conic of Y.

Let us assume that $\dim(B \cap Y) = 0$ and that $B \cap Y$ is supported at a point $P \in Y$. We have to consider three cases:

i) B does not contain any line $l \in T_P(Y)$; in this case the intersection is transversal at P and the projection of Y from P into a generic $\mathbb{P}^4$ gives rise to a smooth cubic surface Y_P, (recall that Y has no trisecant lines). The projection of Y_P from a line to a generic plane has generic 0-dimensional fibres. Hence $\dim[\overline{\pi_B(Y \backslash B)}] = 2$ for any generic projection π_B as above and $\dim[Y; B] = 5$.

ii) B contains only a line $l \in T_P(Y)$; in this case the generic fibres of any linear projection as π_B are 0-dimensional. This fact can be proved by a direct computation with a computer algebra, for instance Macaulay; as above the computation can be made by using a particular line of Y. Hence dim $[\overline{\pi_B(Y \backslash B)}] = 2$ and $\dim[Y; B] = 5$.

iii) B contains all lines $l \in T_P(Y)$, i.e. $B = T_P(Y)$. In this case example 3 shows that $\dim[Sec(X)] = 4$.

Let us assume that $\dim(B \cap Y) = 0$ and that $B \cap Y$ is supported at two distinct points $P, Q \in Y$, at least. By the above analysis we have only to consider the case in which the intersection is transversal at P and at Q. In this case the projection of Y from the line $\langle P, Q \rangle$ into a generic $\mathbb{P}^3$ gives rise to a smooth quadric, (recall that Y has no trisecant lines), and any linear projection of a smooth quadric from a point of $\mathbb{P}^3$ has $\mathbb{P}^2$ as its image. Hence $\dim[\,\overline{\pi_B(Y \backslash B)}] = 2$ and $\dim[Y, B] = 5$.

Now let us consider the case in which B is a plane, but $B \not\subseteq \langle Y \rangle$. Note that $\dim[Sec(X)] = 4$ implies that $\dim[Y; B] \leq 4$. Hence $T_P(Y) \cap B \neq \emptyset$ for any generic point $P \in Y$ by Lemma 2. Let us consider $B \cap \langle Y \rangle$. If $B \cap \langle Y \rangle$ is a point R, we would have: $R \in T_P(Y)$ for any generic $P \in Y$ and this is not possible as Y is not a cone (recall Remark 1). If $B \cap \langle Y \rangle$ is a line L, it is not possible that $L \subseteq T_P(Y)$ for any generic $P \in Y$ as Y is not a cone (recall Remark 1). Then we would have: $\dim[T_P(Y) \cap L] = 0$ for any generic $P \in Y$ and for a fixed line $L \subset \langle Y \rangle$. This is not possible: $\langle Y \rangle$ can be considered as the space of conics lying on some $\mathbb{P}^2$, L is a fixed pencil of conics, $T_P(Y)$ is the web of conics reducible as a fixed line l_P and another line. For generic, fixed, l_P, the web does not contain any conic of the pencil L.

Now let us consider the case in which B is not a plane. As above, $\dim[Sec(X)] = 4$ implies that $\dim([Y; B]) \leq 4$. Let us consider $M := \langle Y \cup B \rangle$ and let us consider the dual varieties Y^* and B^* in M^*. As $Y \neq B$ we get $Y^* \neq B^*$ (otherwise $Y^* = B^*$ would imply $Y = B$). Hence the tangent plane B' at a generic point of B is not tangent to Y. By the above arguments we get $\dim([Y; B']) = 5$. It follows that $T_P(Y) \cap B' = \emptyset$ for the generic point $P \in Y$ by Lemma 2. Therefore $T_P(Y) \cap T_Q(B) = \emptyset$ for generic points $P \in Y$ and $Q \in B$ and $\dim([Y; B]) = 5$ by Lemma 2. ∎

Remark 2. *A priori, if* $\dim[\,\overline{\pi_B(Y \backslash B)}] = 1$ *for a generic* π_B *as above,* $\overline{\pi_B(Y \backslash B)}$ *is a smooth conic. In fact* $\overline{\pi_B(Y \backslash B)}$ *is an integral plane curve* Γ. *Let* $f : \mathbb{P}^1 \to \Gamma$ *be the normalization map given by a line bundle* $\mathcal{O}_{\mathbb{P}^1}(e)$, $e \geq 1$, *and let* $u : Y' \to Y$ *be the birational map such that* $\pi_B \circ u$ *is a morphism; we can assume that* Y' *is normal. The morphism* u *induces a morphism* $v : Y' \to \mathbb{P}^1$, *set* $D := v^*[\mathcal{O}_{\mathbb{P}^1}(1)]$. *We have* $h^0(Y', D) = 6$ *because* Y *is linearly normal and the restriction of* D *to the fibres of* u *is trivial. On the other hand, the map* f *induces an injection from* $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e))$ *into a* 3*-codimensional linear subspace of* $H^0(Y', D)$. *Hence* $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e)) = 3$, *hence* $e = 2$ *and* Γ *is a conic, necessarily smooth.*

5. Surfaces having at most two irreducible components

In this section we study the cases in which $\dim([A; B]) \leq 4$, where A and B are irreducible surfaces, eventually $A = B$. The following lemma, proved by Dale in [D], is the first step, concerning the case $A = B$.

Lemma 9. *Let* A *be an irreducible surface in* $\mathbb{P}^n$, *then* $\dim[Sec(A)] \leq 4$ *if and only if one of the following cases occurs:*
i) $\dim(\langle A \rangle) \leq 4$;
ii) A *is the Veronese surface in* $\langle A \rangle \simeq \mathbb{P}^5$;
iii) A *is a cone.*

Proof. Firstly let us prove that in all cases *i)*, *ii)*, *iii)* we have $\dim[Sec(A)] \leq 4$. For *i)* and *ii)* it is obvious. In case *iii)* A is a cone over a curve C and vertex P, then $[A; A]$ is a cone over $[C; C]$ and vertex P having dimension $1 + \dim([C; C])$ and $\dim([C; C]) \leq 3$. Note that, in case *iii)*, $\dim(\langle A \rangle)$ could be very big.

Now let us assume that $\dim[Sec(A)] \leq 4$ and that $\dim(\langle A \rangle) \geq 5$. If $Sec(A)$ is a linear space then $\dim(\langle A \rangle) \leq 4$. Hence we can assume that $Sec(A)$ is not a linear space. By [A2], page. 17, we have $\dim[Sec(A)] - \dim(A) \geq 2$, on the other hand $\dim[Sec(A)] - \dim(A) \leq 2$ in any case, so that $\dim[Sec(A)] - \dim(A) = 2$. By Proposition 2.6 of [A2] we have $Vert[Sec(A)] = Vert(A)$. Hence A is a cone if and only if $Sec(A)$ is a cone.

Let us assume that A is not a cone, by the previous argument we know that $Sec(A)$ is not a cone. Hence A is an $E_{2,1}$ variety according to Definition 2.4 of [A2]. Now Lemma 9 follows from Definition 2.7 and Theorem 3.10 of [A2]. ∎

Lemma 10. *Let A, B be two distinct, irreducible surfaces in $\mathbb{P}^n$, $n \geq 3$, such that A is a cone over an irreducible curve C and vertex P. Then* $\dim([A;B]) = 1 + \dim([C;B])$ *unless:*

i) $\dim(\langle A \cup B \rangle) \leq 4$;

ii) B is a cone over an irreducible curve C' and vertex P or a plane passing through P.

Proof. Note that C is not a line as A is not a plane. By Lemma 1, 5) we have $[A;B] = [[P;C];B] = [P;[C;B]]$ which is a cone over $[C;B]$ having vertex P. If $\dim([P;[C;B]]) = 1 + \dim([C;B])$ we are done. If not, we have $\dim([P;[C;B]]) = \dim([C;B])$. Hence $[P;[C;B]] = [C;B]$ because $[P;[C;B]] \supseteq [B;C]$ and they are irreducible with the same dimension. In this case we have $P \in Vert([C;B])$ by Proposition 1.3 of [A1].

If $\dim([C;B]) = 2$, by Lemma 4 we know that $Vert([C;B]) = [C;B] = B$ is a plane, but this is a contradiction as $P \in Vert([C;B])$ and A is not a plane. Assume $\dim([C;B]) = 3$. Lemma 4 gives that $Vert([C;B]) = [C;B] \simeq \mathbb{P}^3$. Hence $A = [P;C] \subset [C;B] \simeq \mathbb{P}^3$ and we are in case i).

We can assume that $\dim([C;B]) = 4$. Hence $\dim([A;B]) = \dim([P;[C;B]]) = \dim([C;B]) = 4$. If $\dim(\langle A \cup B \rangle) = 4$ we are in case i), otherwise $\dim(\langle A \cup B \rangle) \geq 5$.

Now let us consider generic pairs of points $c \in C$ and $b \in B$. As $[P;[C;B]] = [C;B]$ we have, for generic $(c,b) \in C \times B$, the union $\bigcup_{c \in C, b \in B} (\langle P \cup c \cup b \rangle)$ is contained in $[C;B]$ and has dimension 4, i.e. $[C;B] = \overline{\bigcup_{c \in C, b \in B, generic} (\langle P \cup c \cup b \rangle)}$. If, for generic $(c,b) \in C \times B$, $\dim(\langle P \cup c \cup b \rangle) = 1$, then the lines $\langle P \cup b \rangle$ are contained in $A = [P;C]$ for any generic $b \in B$, it would imply $B \subseteq A$: contradiction. Hence $\dim(\langle P \cup c \cup b \rangle) = 2$ for generic $(c,b) \in C \times B$. As $\dim([C;B]) = 4$ to have $\bigcup_{c \in C, b \in B, generic} (\langle P \cup c \cup b \rangle)$ of dimension 4, necessarily $\langle P \cup c \cup b \rangle = \langle P \cup c' \cup b' \rangle$ for infinitely many $(c', b') \in C \times B$. Let us fix a generic pair $(\overline{c}, \overline{b})$, it is not possible that infinitely many points $c' \in C$ belong to $\left\langle P \cup \overline{c} \cup \overline{b} \right\rangle$, otherwise C would be a plane curve and A would be a plane, so there is only a finite number of points $\overline{c'} \in C \cap \left\langle P \cup \overline{c} \cup \overline{b} \right\rangle$. Let us choose one of them; there exist infinitely many points $b' \in B$ such that $\left\langle P \cup \overline{c} \cup \overline{b} \right\rangle = \left\langle P \cup \overline{c'} \cup b' \right\rangle$. Hence there exists at least one plane curve $B_{\overline{c}} \subset B$, corresponding to $\overline{c}$, such that $\left\langle P \cup \overline{c} \cup \overline{b} \right\rangle = \left\langle P \cup \overline{c'} \cup B_{\overline{c}} \right\rangle = \langle P \cup \overline{c} \cup B_{\overline{c}} \rangle$. As $\overline{c} \in C$ was a generic point, we can say that, for any generic point $c \in C$, there exists a plane curve $B_c \subset B$ such that, for generic $(c,b) \in C \times B$, $\langle P \cup c \cup b \rangle = \langle P \cup c \cup B_c \rangle$. If, for generic $c \in C$, B_c is not a line we have $[C;B] = \overline{\bigcup_{c \in C, b \in B, generic} (\langle P \cup c \cup b \rangle)} = \overline{\bigcup_{c \in C, generic} (\langle P \cup c \cup B_c \rangle)}$

and $\dim\{\overline{\bigcup_{c\in C, generic}(\langle P\cup c\cup B_c\rangle)}\}\leq 3$, because $\langle P\cup c\cup B_c\rangle=\langle B_c\rangle$ and the set of plane curves $\{B_c|c$ generic, $c\in C\}$ would determine a family of planes of dimension at most 1. But this is not possible as $\dim([B;C])=4$, then B_c must be a line for generic $c\in C$ and $B=\overline{\bigcup_{c\in C, generic}(B_c)}$.

Note that $[C;B]$ must contain $\bigcup_{\overline{c}\in C, fixed, c\in C, generic}(\langle P\cup\overline{c}\cup B_c\rangle$ for any generic point $\overline{c}\in C$: if $[C;B]$ would contain only $\bigcup_{c\in C, generic}(\langle P\cup c\cup B_c\rangle$ it would have dimension at most 3. Moreover it is not possible that the lines $\{B_c|c$ generic, $c\in C\}$ cut the generic line $\langle P\cup\overline{c}\rangle\subset A$ at different points, otherwise $A\subset B$. Hence they cut $\langle P\cup\overline{c}\rangle$ at one point $P(\overline{c})$ and all lines $\{B_c|c$ generic, $c\in C\}$ pass through $P(\overline{c})$. By letting $\overline{c}$ vary in C we get a contradiction unless $P(\overline{c})=P$ (or B is a plane cutting a curve on A, but we are assuming $\dim(\langle A\cup B\rangle)\geq 5$). Hence B is covered by lines passing through P and we are in case *ii*). ∎

Proposition 3. *Let $V=A\cup B$ be the union of two irreducible surfaces in $\mathbb{P}^n$ such that* $\dim[Sec(V)]\leq 4$ *and* $\dim(\langle V\rangle)\geq 5$. *Then:*

i) B is the tangent plane at a point $P\in A_{reg}$ and A is a Veronese surface in $\langle A\rangle\simeq\mathbb{P}^5$ (or viceversa), in this case $\dim[Sec(A\cup B)]=4$;

ii) A and B are cones having the same vertex;

iii) A is a cone of vertex P and B is a plane passing through P;

iv) A is a surface, not a cone, such that $\langle A\rangle\simeq\mathbb{P}^4$ and such that A is contained in a 3-dimensional cone having a line l as vertex, B is a plane such that $B\cap\langle A\rangle=l$.

Proof. Obviously if $\dim[Sec(A\cup B)]\leq 4$ we have $\dim[Sec(A)]\leq 4$ and $\dim[Sec(B)]\leq 4$, so that, for both A and B, one of the conditions *i*), *ii*), *iii*) of Lemma 9 holds.

If A (or B) is a Veronese surface, Proposition 2 tells us that we are in case *i*). From now on we can assume that neither A nor B is a Veronese surface.

Let us assume that A is a cone of vertex P, over an irreducible curve C. If B is a cone of vertex P we are in case *ii*). Let us assume that B is a cone of vertex $P'\neq P$, over an irreducible curve C', we can assume that $P'\notin C$ by changing C if necessary. By Lemma 10 and Lemma 1, 5), we have: $\dim([A;B])=1+\dim([C;[C';P']])=1+\dim([[C;C'];P']=2+\dim([C;C']\geq 5$ unless C and C' are plane curves lying on the same plane (see Lemma 3), but in this case $\dim(\langle A=[P;C]\rangle)\leq 3,\dim(\langle B=[P';C']\rangle)\leq 3$ and $\dim(\langle A\cup B\rangle)\leq 4$.

Hence we can assume that B is not a cone and therefore $\dim(\langle B\rangle)\leq 4$ by Lemma 9. If B is a plane passing through P we are in case *iii*), in all other cases we have $\dim([A;B])=1+\dim([C;B])\leq 4$ by Lemma 10, hence $\dim([C;B])\leq 3$. By Lemma 4 we know that, in this case, $\dim(\langle C\cup B\rangle)\leq 3$ and this is not possible, otherwise $\dim(\langle A\cup B\rangle)\leq 4$.

By the above arguments we can assume that A is not a cone. For the same reason we can also assume that B is not a cone. Hence, by Lemma 9 we have $\dim(\langle A\rangle)\leq 4$ and $\dim(\langle B\rangle)\leq 4$ and $-1\leq\dim(\langle A\rangle\cap\langle B\rangle)\leq 3$. If neither A nor B is a plane, by Lemma 7, we have $\dim(\langle A\rangle\cap\langle B\rangle)=3$. This implies that $\dim(\langle A\rangle)=\dim(\langle B\rangle)=4$, otherwise we would have $\langle A\rangle\subseteq\langle B\rangle$ (or $\langle A\rangle\supseteq\langle B\rangle$) and this is not possible as $\dim(\langle A\rangle\cup\langle B\rangle)=\dim(\langle A\cup B\rangle)\geq 5$. Then we can apply Lemma 8 and we are done.

Hence we can assume that B, for instance, is a plane, $\dim(\langle B\rangle) = \dim(B) = 2$ and $\dim(\langle A\rangle) \leq 4$. If $\dim(\langle A\rangle) = 2$, A is a plane and it is not possible that $\dim(\langle A \cup B\rangle) \geq 5$ and $\dim([A; B]) \leq 4$. If $\dim(\langle A\rangle) = 3$ we have $\langle A\rangle \cap B$ is a point R as $\dim(\langle A \cup B\rangle) = \dim(\langle\langle A\rangle \cup B\rangle) \geq 5$, then for any point $P \in A_{reg}$, $T_P(A)$ passes through R, because $T_P(A) \cap B \neq \emptyset$ by Lemma 6 *ii*). Hence A would be a cone with vertex R and this is not possible. If $\dim(\langle A\rangle) = 4$ we have $\langle A\rangle \cap B$ is a line l, as $\dim(\langle A \cup B\rangle) = \dim(\langle\langle A\rangle \cup B\rangle) \geq 5$, and for any generic point $P \in A_{reg}$, $T_P(A) \cap l \neq \emptyset$ by arguing as above. Let us choose a generic plane $\Pi \subset \langle A\rangle$ and let us consider the rational map $\varphi : A - -- > \Pi$ given by the projection from l. φ cannot be constant, because A is not a plane, on the other hand the rank of the differential of φ is at most one by the assumption on $T_P(A)$, $P \in A_{reg}$. Hence $\overline{\mathrm{Im}(\varphi)}$ is a plane curve Γ and A is contained in the 3-dimensional cone generated by the planes $\langle l \cup Q\rangle$, where Q is any point of Γ. We get case *iv*). ∎

Remark 3. *Lemma 9 and Proposition 3 give the proof of Theorem 1.*

6. Surfaces having at least three irreducible components

In this section we want to give some information about the classification of J-embeddable surfaces $V = V_1 \cup ... \cup V_r$, $r \geq 3$. By Corollary 1 this property is equivalent to assume that $\dim[Sec(V)] \leq 4$. As any surface V is J-embeddable if $\dim(\langle V\rangle) \leq 4$ we will assume that $\dim(\langle V\rangle) \geq 5$. Note that V is J-embeddable if and only if $\dim([V_i; V_j]) \leq 4$ for any $i, j = 1, ..., r$, by Corollary 2.

Let us prove the following.

Lemma 11. *Let $V = V_1 \cup ... \cup V_r$, $r \geq 3$, be a reducible surface in $\mathbb{P}^n$ such that $\dim[Sec(V)] \leq 4$. Assume that there exists an irreducible component, say V_1, for which $\dim(\langle V_1 \cup V_j\rangle) \geq 5$ for any $j = 2, ..., r$. Then we have only one of the following cases:*

i) V_1 is a Veronese surface and the other components are tangent planes to V_1 at different points;

ii) V_1 is a cone, with vertex a point P, and every V_j, $j \geq 2$, is a plane passing through P or a cone having vertex at P;

iii) V_1 is a surface, not a cone, such that $\dim(\langle V_1\rangle) = 4$ and $V_2, ..., V_r$ are planes as in case $s = 1$ of example 2.

Proof. Let us consider V_1 and V_2. By assumption $\dim[Sec(V_1 \cup V_2)] \leq 4$ and $\dim(\langle V_1 \cup V_2\rangle) \geq 5$. By Proposition 3 we know that one possibility is that V_1 is a Veronese surface and V_2 is a tangent plane to V_1. In this case let us look at the pairs V_1, V_j, $j \geq 3$; we can argue analogously and we have *i*).

In the other two possibilities *ii*) and *iii*) of Proposition 3 for V_1 and V_2 we can assume that V_1 is a cone of vertex P. Now, by looking at the pairs V_1, V_j, $j \geq 3$ and by applying Proposition 3 to any pair, we have *ii*).

In the last case of Proposition 3 we can assume that V_1 is a surface, not a cone, such that $\dim(\langle V_1\rangle) = 4$. By looking at the pairs V_1, V_j, $j \geq 2$ and by applying Proposition 3 to any pair, we have any V_j, $j \geq 2$, is a plane cutting $\langle V_1\rangle$ along a line l_j which is the vertex of some 3-dimensional cone $E_j \subset \langle V_1\rangle$, $E_j \supset V_1$. Hence V is a surface as X in case $s = 1$ of Example 2. ∎

Thanks to Lemma 11 it is easy to give the classification of V when there exists an irreducible component V_i for which $\dim(\langle V_i\rangle) \geq 5$

Corollary 3. *Let $V = V_1 \cup ... \cup V_r$, $r \geq 3$, be a reducible surface in $\mathbb{P}^n$ such that $\dim[Sec(V)] \leq 4$. Assume that there exists an irreducible component, say V_1, for which $\dim(\langle V_1 \rangle) \geq 5$. Then we have case $i)$ or case $ii)$ of Lemma 11.*

Proof. As $\dim(\langle V_1 \rangle) \geq 5$ we have $\dim(\langle V_1 \cup V_j \rangle) \geq 5$ for any $j = 2, ..., r$, so we can apply Lemma 11, obviously case $iii)$ cannot occur. ∎

To complete the classification we would have to consider:

- the case in which all components V_i of V are such that $\dim(\langle V_i \rangle) \leq 4$ and there exists at least an irreducible component $V_{\bar{i}}$ such that $\dim(\langle V_{\bar{i}} \rangle) = 4$;

- the case in which all components V_i of V are such that $\dim(\langle V_i \rangle) \leq 3$ and there exist at least two components $V_{\bar{i}}$ and $V_{\bar{j}}$ such that $\dim(\left\langle V_{\bar{i}} \cup V_{\bar{j}} \right\rangle) \geq 5$;

- the case in which all components V_i of V are such that $\dim(\langle V_i \rangle) \leq 3$ and for any pair V_i, V_j, $\dim(\left\langle V_{\bar{i}} \cup V_{\bar{j}} \right\rangle) \leq 4$.

The complete analysys of the first two cases is very long and intricated and we think that it is not suitable to give it here. However we plan to present it in a separated enlarged version of this paper.

On the contrary, the last case can be studied very quickly and we give the following result in order to recover Example 1.

Theorem 2. *Let $V = V_1 \cup ... \cup V_r$, $r \geq 3$, be a reducible surface in $\mathbb{P}^n$ such that $\dim[Sec(V)] \leq 4$ and $\dim(\langle V \rangle) \geq 5$. Assume that $\dim(\langle V_i \rangle) \leq 3$ for $i = 1, ..., r$ and $\dim(\langle V_i \cup V_j \rangle) \leq 4$ for any $i, j = 1, ..., r$. Then either V is an union of planes pairwise intersecting at least at a point or the following conditions hold: $V_1 \cup ... \cup V_t \cup ... \cup V_r$ with $1 \leq t \leq r$ such that*

$i)$ $\dim(\langle V_i \rangle) = 3$ for any $1 \leq i \leq t$ and V_i is a plane for $t + 1 \leq i \leq r$ (if any);

$ii)$ $2 \leq \dim(\langle V_i \rangle \cap \langle V_j \rangle)$ for any $i, j = 1, ..., t$; $1 \leq \dim(\langle V_i \rangle \cap V_j)$ for any $i = 1, ..., t$ and $j = t + 1, ..., r$; $0 \leq \dim(V_i \cap V_j)$ for any $i, j = t + 1, ..., r$.

Let $V = V_1 \cup ... \cup V_r$, $r \geq 3$, be a reducible surface in $\mathbb{P}^n$ such that $\dim(\langle V \rangle) \geq 5$. Assume that $\dim(\langle V_i \rangle) \leq 3$ for $i = 1, ..., r$ and that V is either an union of planes, pairwise intersecting at least at a point, or $V_1 \cup ... \cup V_t \cup ... \cup V_r$, with $1 \leq t \leq r$, satisfying conditions $i), ii)$ above. Then $\dim[Sec(V)] \leq 4$.

Proof. Firstly let us assume that V is an union of planes. In this case, obviously, $\dim[Sec(V)] \leq 4$ if and only if every pair of planes intersects. From now on we can assume that V is not an union of planes.

Under our assumprtions V is as in $i)$. $ii)$ follows from the fact that, for any pair $V_i, V_j \in V$, $\dim(\langle V_i \cup V_j \rangle) = \dim(\langle V_i \rangle \cup \langle V_j \rangle) \leq 4$.

Conversely: if V is as in $i)$, condition $ii)$ implies that $\dim(\langle V_i \rangle \cup \langle V_j \rangle) = \dim(\langle V_i \cup V_j \rangle) \leq 4$ for any $i, j = 1, ..., r$. Hence $\dim([V_i; V_j]) \leq 4$ by Lemma 7; in any case $\dim[Sec(V)] \leq 4$. ∎

Remark 4. *Example 1 is a J-embeddable surface V considered by Theorem 2.*

To end the paper we give the following particular result in order to recover Example 2.

Theorem 3. *Let $V = V_1 \cup ... \cup V_r$, $r \geq 3$, be a reducible surface in $\mathbb{P}^n$ such that $\dim[Sec(V)] \leq 4$ and $\dim(\langle V \rangle) \geq 5$. Assume that $\dim(\langle V_i \rangle) \leq 4$ for $i = 1, ..., r$ and that there exists a component, say V_1, such that $\dim(\langle V_1 \rangle) = 4$ and V_1 is a surface, not a cone, contained in a 3-dimensional cone $E_2 \subset \langle V_1 \rangle$ having a line l_2 as vertex. Then:*

i) if E_2 is the unique 3-dimensional cone having a line as vertex and containing V_1, then V is the union of V_1, planes of $\mathbb{P}^n$ cutting $\langle V_1\rangle$ along l_2, cones in $\langle V_1\rangle$ whose vertex belongs to l_2, planes in $\langle V_1\rangle$ intersecting l_2, surfaces in $\langle V_1\rangle$ contained in 3-dimensional cones having l_2 as vertex;

ii) if there exist other cones as E_2, say $E_3, ..., E_k$, with lines $l_3, ..., l_k$ as vertices, then V is the union of V_1, other surfaces contained in $E_2 \cap ... \cap E_k$ (if any), planes pairwise intersecting and cutting $\langle V_1\rangle$ along at least some line l_j, cones in $\langle V_1\rangle$ having vertex belonging to $l_2 \cap ... \cap l_k$ (if not empty), planes in $\langle V_1\rangle$ intersecting $l_2 \cap ... \cap l_k$ (if not empty).

Proof. Note that it is not possible that $\dim(\langle V_1 \cup V_j\rangle) \leq 4$ for all $j = 2, ..., r$, otherwise $\dim(\langle V\rangle) = 4$, then there exists at least a component, say V_2, such that $\dim(\langle V_1 \cup V_2\rangle) \geq 5$. By applying Proposition 3 to V_1 and V_2 we have V_2 is a plane cutting $\langle V_1\rangle$ along l_2. Let us consider V_j, $j \geq 3$.

If $\dim(\langle V_1 \cup V_j\rangle) \geq 5$ then, by Proposition 3, V_j is a plane cutting $\langle V_1\rangle$ along a line l_j which is the vertex of some 3 -dimensional cone $E_j \subset \langle V_1\rangle$, $E_j \supset V_1$.

If $\dim(\langle V_1 \cup V_j\rangle) \leq 4$ then $V_j \subset \langle V_1\rangle$; in this case, to get $\dim([V_j; V_2]) \leq 4$, it must be $T_P(V_j) \cap l_2 \neq \emptyset$ for any point $P \in (V_j)_{reg}$ (recall that V_2 is a plane). Hence, either V_j is a cone whose vertex belong to l_2, or V_j is a plane intersecting l_2 or V_j is a surface contained in some 3-dimensional cone having l_2 as vertex.

Now, if E_2 is the unique cone of its type containing V_1, then V is as in case i), otherwise we are in case ii). ∎

Remark 5. *Example 2 is a J-embeddable surface V considered by Theorem 3.*

References

[A1] B. Adlandsvik: "Joins and higher secant varieties" Math. Scand. **62** (1987), 213-222.

[A2] B. Adlandsvik: "Varieties with an extremal numbers of degenerate higher secant varieties" J. Reine Angew. Math. **392** (1988) 16-26.

[D] M. Dale: "Severi's theorem on the Veronese-surface" J. London Math. Soc. **32** (3) (1985) 419-425.

[J] K. W. Johnson: "Immersion and embedding of projective varieties" Acta Math. **140** (1981) 49-74.

[Z] F. L. Zak: "Tangents and secants of algebraic varieties" Translations of Mathematical Monographs of A.M.S., **127**, Providence R.I., 1993.

Dipartimento di Matematica Univ. di Milano, via C. Saldini 50 20133-Milano (Italy)
E-mail address: alberto.alzati@unimi.it

Dipartimento di Matematica univ. di Trento, via Sommarive 14 38123-Povo (TN) (Italy)
E-mail address: ballico@science.unitn.it