Decompositions of $\mathcal{B}_n$ and Π_n using Symmetric Chains

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Abstract

We review the Green/Kleitman/Leeb interpretation of de Bruijn's symmetric chain decomposition of $\mathcal{B}_n$, and explain how it can be used to find a maximal collection of disjoint symmetric chains in the nonsymmetric lattice of partitions of a set.

1 Introduction

De Bruijn *et al* [4] and Griggs [6] have given symmetric chain decompositions of various lattices including in particular the Boolean lattice of subsets of a given set. Simpler characterizations of this decomposition were later found by Greene and Kleitman [5] and independently by K. Leeb [unpublished]. These results are of great utility in combinatorics, permitting, for example, various extensions of Sperner's theorem [12].

After a review of previous work for the Boolean lattice, we investigate whether a similar decomposition can be found for the lattice of partitions of a set. This lattice is not symmetric. For example, there are $2^{n-1} - 1$ partitions of an n-set into two blocks, but only $n(n-1)/2$ partitions of an n-set into $n-1$ blocks. Thus, there is no complete decomposition of the lattice into symmetric chains (for $n \geq 3$). However, using de Bruijn's decomposition together with a certain method to encode sets, we find a maximal collection of disjoint symmetric chains of partitions.

We conclude with other applications of this set encoding: it gives the relationship between the elementary and complete symmetric functions as well as a new formula for the Bell numbers.

2   de Bruijn's Method

A *symmetric chain* in a ranked lattice L is a sequence of elements $(x_i)_{i=1}^n$ such that x_{i+1} covers x_i and $r(x_1) + r(x_n) = r(L)$. A *partition* ϕ of a set S is an unordered collection of disjoint subsets called *blocks* whose union is S. (We write $\phi \vdash S$.) A *decomposition* of a lattice L into symmetric chains is a partition of L whose blocks are symmetric chains.

Proposition 1 (de Bruijn [4]) *Let L_1 and L_2 be lattices with symmetric chain decompositions. Then $L_1 \times L_2$ has a symmetric chain decomposition.*□

Proof: Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be decompositions of L_1 and L_2 respectively. It suffices to decompose $C \times D$ into symmetric chains for every $C \in \mathcal{D}_1$ and $D \in \mathcal{D}_2$. Suppose that $C = \{c_i\}_{i=1}^k$ and $D = \{d_j\}_{j=1}^l$. Then the sets $\{E_j\}_{j=0}^l$ are such a symmetric chain decomposition, where the individual chains E_j are given by

$$E_j = \{(c_1, d_j) < \ldots < (c_{k-j}, d_j) < (c_{k-j}, d_{j+1}) < \ldots < (c_{k-j}, d_l)\}.□$$

Corollary 2 (de Bruijn [4]) *Any finite chain product is symmetric (e.g.: $\mathcal{B}_n$).*□

Let $\mathcal{D}_n$ be the decomposition of $\mathcal{B}_n$ given by de Bruijn. Then $\mathcal{D}_{n+1}$ can be constructed from $\mathcal{D}_n$ as follows:

$$\mathcal{D}_{n+1} = \alpha(\mathcal{D}_n) \cup \beta(\mathcal{D}_n) - \{\emptyset\}$$

where $\emptyset$ denotes the empty chain and α and β are the maps on chains:

$$\begin{aligned} \alpha(c_1, \ldots, c_k) &= c_1 \cdots c_k (c_k \cup \{n+1\}) \\ \beta(c_1, \ldots, c_k) &= (c_1 \cup \{n+1\}) \cdots (c_k \cup \{n+1\}). \end{aligned}$$

Greene, Kleitman, Leeb give [5] a direct description of these symmetric chains. This description is equivalent to de Bruijn's; however, it has the advantage that a direct comparison of two sets can indicate whether they are in the same chain, and a complete chain can easily be generated given one of its elements.

We write $S \subseteq \{1, 2, \ldots, n\}$ as a word $w(S)$ of length n with a right or left parenthesis in position i according to whether $i \in S$,

$$w(S)_i = \begin{cases}) & \text{if } i \in S \\ (& \text{if } i \notin S. \end{cases}$$

For example, for $n = 10$ and $S = \{1, 3, 4, 8, 9\}$, $w(S)$ is)())((()(. We can then indicate matching parentheses in bold: $w(S) = $ **)()**)((**())**(. Those sets with the same matching parentheses belong to the same chain. Moreover, it is easy to see how to generate the rest of the chain. The parentheses in roman face must not match. Thus, they must consist of a certain number of right parentheses followed by a certain number of left parentheses. Thus,

$$\begin{aligned} &(()(((())((, \\ &)()(((())((, \\ &)())(((())((, \\ &)()))(())((, \quad \text{and} \\ &)()))(()) \end{aligned}$$

correspond respectively to the sets

$$\begin{aligned} &\{3, 8, 9\}, \\ &\{1, 3, 8, 9\}, \\ &\{1, 3, 4, 8, 9\} = S, \\ &\{1, 3, 4, 5, 8, 9\}, \quad \text{and} \\ &\{1, 3, 4, 5, 8, 9, 10\}. \end{aligned}$$

We note several easily verified facts.

Proposition 3   *1. In each chain, elements are added in increasing order.*

2. n is an element of the last set in each chain.

3. If $S \to S \cup \{i\}$ is a link in a chain, then $i + 1 \notin S$, and either $i = 1$ or $i - 1 \in S$.□

One should note, however, that not all symmetric chain decompositions are equivalent, even up to automorphism. For example, consider the De Bruijn decomposition of $\mathcal{B}_4$. The pair of chains:

$$(\{4\}, \{1, 4\}, \{1, 2, 4\}) \text{ and } (\{2, 4\})$$

can be replaced by the pair of chains

$$(\{4\}, \{2, 4\}, \{1, 2, 4\}) \text{ and } (\{1, 4\}).$$

The resulting decomposition is not isomorphic to the original.

See [8, 9] for other systems of "parentheses" used to find a Sperner decomposition of a poset into chains.

3   Partition Lattice

We will now describe how de Bruijn's decomposition of $\mathcal{B}_n$ into symmetric chains can be used to generate a similar decomposition of the lattice Π_{n+1} of partitions of the set $\{1, 2, \ldots, n + 1\}$.

A partition $\phi \vdash S$ is a *refinement* of another partition $\kappa \vdash S$ if each block of κ is a union of blocks of ϕ. This ordering results in a ranked lattice $\Pi_{|S|}$. Partitions $\pi \in \Pi_n$ of rank i consist of $n - i$ blocks.

Since there are $n(n - 1)/2$ partitions with $n - 1$ blocks and $2^{n-1} - 1$ partitions with two blocks, there is no total decomposition of Π_n into symmetric chains. However, amazingly enough we can find a set of disjoint symmetric chains which includes all partitions of rank $\leq \lfloor (n - 1)/2 \rfloor$. Such a collection is clearly maximal.

At first glance, such a collection may be surprising. However, it is not a Sperner decomposition, so it does not contradict the fact that Π_n is neither LYM [11] nor Sperner [2, 10, 7].

Definition 4 (Coding of Sets) *For each $S \subseteq \{1, 2, \ldots, n\}$, we associate a code $c(S) \in \mathbf{N}_0^{n+1}$ of length $n + 1$ as follows:*

$$c(S)_i = \begin{cases} 0 & \text{if } i \in S \\ i - \sum_{j=1}^{i-1} c(S)_j & \text{if } i \notin S. \end{cases}$$

For example, the code of $S = \{1, 2, 3, 7, 11, 12, 16, 18, 19\} \subseteq \{1, 2, \ldots, 20\}$ is

$$c(S) = (0, 0, 0, 4, 1, 1, 0, 2, 1, 1, 0, 0, 3, 1, 1, 0, 2, 0, 0, 3, 1).$$

Lemma 5   *1. If $\alpha \in \mathbf{N}_0^{n+1}$ is a vector such that $\alpha_{n+1} \neq 0$ and for all $1 \leq i \leq n + 1$ either $\alpha_i = 0$ or $\sum_{j=0}^{i} \alpha_j = i$, then $\alpha = c(S)$ for only and only one subset $S \subseteq \{1, \ldots, n\}$.*

2. Any α as above is determined by the placement of its zeros.

3. Any α as above is determined by the value and order of its nonzero elements.

Proof: (**1 and 2**) Let $S = \{i : \alpha_i = 0\}$.

(**3**) Each nonzero element a must be preceded by $a - 1$ zeros. Thus, the nonzero elements determine the zeros.□

We will write partitions with their parts arranged in lexicographical order. (This is the usual way one write partitions, and is of particular use, for example, in studying limited growth functions.) The *type* $t(\phi)$ of a partition ϕ is the ordered sequence giving the length of each of its blocks when listed in lexicographical order. For example, a partition ϕ of the integers from 1 to 20 according to their number of prime factors would be written like this:

$$\phi = \{\{1\}, \{2, 3, 5, 7, 11, 13, 17, 19\}, \{4, 6, 9, 10, 14, 15\}, \{8, 12, 18, 20\}, \{16\}\},$$

and would thus be of type $t(\phi) = (1, 8, 6, 4, 1)$.

For every set $S \subseteq \{1, \ldots, n\}$, we will associate a collection of partitions $\Pi_S \subseteq \Pi_{n+1}$ as follows: Π_S consists of the set of all partitions $\phi \vdash \{1, \ldots, n+1\}$ whose type when written backwards is the same as the code of S when written without its zeros.

Lemma 6 *The set of classes $\{\Pi_S : S \in \mathcal{B}_n\}$ forms a partition of Π_{n+1}.*

Proof: Lemma 5.□

Proposition 7 *Let $S \to S' = S \cup \{i\}$ be a link in a de Bruijn's decomposition of $\mathcal{B}_n$. Then there is a simply described order-preserving injection of Π_S into $\Pi_{S'}$.*

Proof: We will explicitly construct the injection $f : \Pi_S \to \Pi_{S'}$. Part 3 of Proposition 3 guarantees that difference between the codes $c(S)$ and $c(S')$ is that some substring of the form $k1$ is replaced by a string of the form $0(k + 1)$. Let $\phi \in \Pi_S$. Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b\}$ respectively be the blocks of ϕ which correspond to the k and the 1 in the substring $k1$ which is being replaced. Let $f(\phi)$ be the replacement of A and B with their union: $f(\phi) = \phi \cup \{A \cup B\} - \{A, B\}$. Clearly, $f(\phi)$ is a refinement of ϕ, and $f(\phi) \in \Pi_{S'}$. Moreover, f is injective, since we can recover ϕ from $f(\phi)$. That is to say, we can recover A and B from $A \cup B$, since $b = \min(A \cup B)$.□

By applying Proposition 7 to the entire decomposition, we obtain the following theorem.

Theorem 8 *For each link $S \to S'$ in de Bruijn's decomposition of $\mathcal{B}_n$ into symmetric chains, there are corresponding links between all of Π_S and part of $\Pi_{S'}$. Each resulting chain beginning at rank i ends at rank $n - i$ or higher. Thus, by pruning the tops of these chains we achieve a partial decomposition of Π_{n+1} into symmetric chains. This decomposition is maximal since it includes every partition with more than $\lfloor (n+1)/2 \rfloor$ parts.□*

The Stirling number S_{nk} gives the number of partitions of an n-set into k blocks.

Corollary 9 *The Stirling numbers of the second kind S_{nk} obey the following identities for all $n \geq 0$:*

$$S_{nn} \leq S_{n(n-1)} \leq \cdots \leq S_{n\lfloor (n+1)/2 \rfloor}$$

and for $k \leq n/2$

$$S_{nk} \geq S_{n(n-k)}. \square$$

Example 1 *The de Bruijn decomposition of $\mathcal{B}_3$ consists of the chains $(\emptyset, \{1\}, \{1,2\}, \{1,2,3\}), (\{2\}, \{2,3\}),$ and $(\{3\}, \{1,3\})$. Encoding the sets above yields the following partition types: $(1111, 112, 13, 4), (121, 31),$ and $(211, 22)$. We then compute the partitions of each type:*

$S \in \mathcal{B}_n$	$c(S)$	Π_S		
$\emptyset$	$1111 \rightarrow 1111$	$1/2/3/4$		
$\{1\}$	$0211 \rightarrow 112$	$1/2/34$		
$\{1,2\}$	$0031 \rightarrow 13$	$1/234$		
$\{1,2,3\}$	$0004 \rightarrow 4$	1234		
$\{2\}$	$1021 \rightarrow 121$	$1/23/4,$	$1/24/3$	
$\{2,3\}$	$1003 \rightarrow 31$	$123/4,$	$124/3,$	$134/2$
$\{3\}$	$1102 \rightarrow 211$	$12/3/4,$	$13/2/4,$	$14/2/3$
$\{1,3\}$	$0202 \rightarrow 22$	$12/34,$	$13/24,$	$14/23$

Reading down the columns marked Π_S gives the chains in the decomposition of Π_4.

4 Other Applications of Set Coding

The set coding defined above (Definition 4) has been found to have several applications. Indeed, in [3], this coding was used to give a simpler expression of the relationship between elementary and complete homogeneous symmetric functions.

Let us write e_i for the ith letter of $c(S)$ where $S \subseteq \{1, 2, \ldots, n-1\}$. That is to say,

$$e_i = c(S)_i = \begin{cases} 0 & \text{if } i \in S \\ i - \sum_{j=1}^{i-1} c(S)_j & \text{if } i \notin S \end{cases}$$

for $1 \leq i \leq n$. Then we have [3]

$$h_n = \sum_{\substack{c(S) \\ S \subseteq \{1,2,\ldots,n-1\}}} (-1)^{|S|} a_{e_1} a_{e_2} \cdots a_{e_n}$$

where we denote by h_i (resp. a_i) the ith complete (resp. elementary) homogeneous symmetric function.

Our coding also allows a straightforward derivation of the Bell numbers' generating function. Let g be a real function and let $F(x) = e^{g(x)}$. Now, let $g^{(i)}(x)$ and $F^{(i)}(x)$ denote the ith derivatives of $F(x)$ and $g(x)$ respectively $(i \geq 0)$. Then one can prove that

$$F^{(n)}(x) = F(x) \left(\sum_{\substack{c(S) \\ S \subseteq \{1,2,\ldots,n-1\}}} \prod_{\substack{i=1 \\ e_i \neq 0}}^{n} \binom{i-1}{e_i - 1} g^{(i)}(x) \right).$$

(See [3].)

Here we are interested in the case $g(x) = e^x - 1$. In such an event,

$$\exp(e^x - 1) = \sum_{k=0}^{\infty} \sum_{\substack{c(S) \\ S \subseteq \{1,2,\ldots,n-1\}}} \prod_{\substack{i=1 \\ e_i \neq 0}}^{n} \binom{i-1}{e_i - 1} \frac{x^k}{k!}.$$

Using lemmata 5 and 6, together with the concept of the type $t(\phi)$ of a partition, it can be shown that

$$B_n = \sum_{\substack{c(S) \\ S \subseteq \{1,2,\ldots,n-1\}}} \prod_{\substack{i=1 \\ e_i \neq 0}} \binom{i-1}{e_i - 1}.$$

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Note to compositor: Please be sure that the bold-face parentheses **()** are easily distinguished from regular parenthesis ().