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Abstract In this paper, we suggest an alternative way to calculate the concentration index and a new matrix form approach, which can be applied both to the Gini and to the concentration index. For both indices, this approach yields expressions that are decomposable by groups and easily comparable. Our findings are illustrated by applying them to the Atkinson Plotnick Kakwani (APK) reranking measure.

Keywords Inequality Measure Decomposition, Reranking, Reranking Decomposition.

JEL D63, H23.

1 Introduction

Consider two attributes of population units, the concentration index for one of the two attributes with respect to the other can be calculated by lining up the values of the first characteristic by the ordering of the second one. If the values of an attribute are ranked according to their own ordering, the concentration index is then the Gini index (Lambert, 2001). The literature offers a number of alternative ways to express the Gini index. In particular, Pyatt (1976), Silber (1989) and Yao (1999) endorse matrix form approaches. Differently from the Gini index, the concentration index is usually evaluated in terms of Lorenz curve. In this paper, we suggest an alternative expression of the concentration index and a new matrix form approach that can be applied both to Gini and to concentration index. This approach yields easily comparable expressions for the two indices. This is our first result. Moreover, we show that the matrix form expressions we propose for the two indices are decomposable into the sum of three components. The overall indices and their components are expressed as functions of a same matrix. This is our second result. Representing the Gini index, the concentration index and their components as functions of a same basic matrix facilitates comparisons among indices and fit to several problems. Here we illustrate our results analysing the reranking induced by a tax system, then from the beginning of the paper we consider income units with two attributes: their before and after-tax values. When a population is not homogenous, the reranking measure can be decomposed following the suggestion of the conventional Gini index decomposition. We show that reranking within groups

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and the reranking related with the overlapping term cannot be considered as measures of the unfairness of the tax system. This is our third result.The paper is organized as follows. In Section 2, we present the matrix form for the Gini and the concentration index. In the same Section, the decomposition of the two indices is obtained. In Section 3, we present the matrix form of the reranking index R^{APK} and discuss its decompositions. Section 4 concludes.

2 Gini and concentration index in matrix form

Let be X and Y the before and the after-tax income distribution for a population of K individuals, $K \in \mathbb{N}$. We denote by x_i and by y_i the before and the after-tax income of the i^{th} individual (i=1, 2……, K). The weight p_i, $\sum_{i=1}^{K} p_i = N$, is associated to the pair (x_i , y_i). X-ordering denotes the ordering of the (x_i, y_i, p_i) sequence when all elements are lined up in the non-decreasing ordering of X. Y-ordering denotes the ordering of the (x_i, y_i, p_i) sequence when all elements are sorted in the non-decreasing ordering of Y.

Given the expression of the Gini index (G_Y) in terms of Mean Difference, we write (G_Y) and the concentration index for after-tax incomes w.r.t before-tax incomes ($C_{Y|X}$) [2] as in expressions (1) and (2).

$$2\mu_Y N^2 G_Y = \sum_{i=1}^{K} \sum_{j=1}^{K} \left(y_i - y_j\right) p_i p_j \cdot I\left\{r_Y\left(y_i\right) - r_Y\left(y_j\right)\right\}, \tag{1}$$

$$2\mu_Y N^2 C_{Y|X} = \sum_{i=1}^{K} \sum_{j=1}^{K} \left(y_i - y_j\right) p_i p_j \cdot I\left\{r_X\left(y_i\right) - r_X\left(y_j\right)\right\}, \tag{2}$$

where μ_Y denotes the after-tax weighed average income, $r_Y\left(y_i\right)$ [3] is the rank of y_i in the Y-ordering ordering, $r_X\left(y_i\right)$ is the rank of y_i in the X-ordering and $I\{z\}$ is an indicator function [4] such that $I\{z\}$ =1 if z>0, $I\{z\}$ =0 if z=0 and $I\{z\}$ =-1 if z<0.

Let us introduce the following notation:

$\mathbf{y}$ is the K×1 vector where after-tax incomes are stacked in Y-ordering;
$\mathbf{y}_X$ is the K×1 vector where after-tax incomes are stacked in X-ordering;
$\mathbf{p}_Y$ is the K×1 vector containing weights stacked as the elements of $\mathbf{y}$;
$\mathbf{p}_X$ is the K×1 vector containing weights stacked as the elements of $\mathbf{y}_X$;
$\mathbf{E}$ is a K×K permutation matrix, [5] such that $\mathbf{p}_X = \mathbf{E}\mathbf{p}_Y$, $\mathbf{p}_Y = \mathbf{E}'\mathbf{p}_X$, $\mathbf{y}_X = \mathbf{E}\mathbf{y}$, $\mathbf{y} = \mathbf{E}'\mathbf{y}_X$;
$\mathbf{S}$ denotes a K×K emi-symmetric matrix with diagonal elements equal to zero, super-diagonal elements equal to 1 and sub-diagonal elements equal to -1;
$\mathbf{j}$ is a K×1 vector with entries equal to 1;

[2] The concentration index for after-tax incomes is calculated lining up after-tax incomes according to the corresponding before-tax ordering.
[3] When two subjects have the same after-tax income value, their relative positions in the after-tax ranking remain as in the before-tax income parade.
[4] On the indicator function, see Faliva (2000).
[5] Observe that $\mathbf{E}^{-1} = \mathbf{E}'$ **;** for definitions concerning permutation matrices, see Faliva (1996).

$\mathbf{D}_Y$ and $\mathbf{D}_{Y|X}$ denote the $K\times K$ matrices $\mathbf{D}_Y = (\mathbf{jy'} - \mathbf{yj'})$ and $\mathbf{D}_{Y|X} = (\mathbf{jy}_X' - \mathbf{y}_X\mathbf{j'})$.

Using the Hadamard product $\odot$,[6] we rewrite expression (1) as

$$2\mu_Y N^2 G_Y = \mathbf{p}_Y'(\mathbf{S}\odot\mathbf{D}_Y)\mathbf{p}_Y . \tag{3}$$

Then, following the definition of the concentration index one has

$$2\mu_Y N^2 C_{Y|X} = \mathbf{p}_X'(\mathbf{S}\odot\mathbf{D}_{Y|X})\mathbf{p}_X . \tag{4}$$

Observing that $\mathbf{E'D}_{Y|X}\mathbf{E} = (\mathbf{jy}_X'\mathbf{E} - \mathbf{E'y}_X\mathbf{j'}) = \mathbf{D}_Y$ and applying Hadamard product properties, expression (4) rewrites as (see Vernizzi 2009)

$$2\mu_Y N^2 C_{Y|X} = \mathbf{p}_Y'(\mathbf{E'SE}\odot\mathbf{D}_Y)\mathbf{p}_Y . \tag{5}$$

Expression (5) is the matrix form for expression (2). Equations (3) and (5) pave the way for a representation of both indices as functions of the same difference matrix $\mathbf{D}_Y$.

In a non-homogeneous population, income earners can be partitioned into H groups, $H\in\mathbb{N}$. The Gini index decomposes by groups as a sum of three components: the within, the between and the overlapping component. The between component is represented as a function of the difference between the means of the groups and the overlapping is generally obtained as a residual. Dagum (1997), as defined in Monti (2008), rewrites the Gini coefficient as the sum of within-groups component, G_Y^W , and across-groups component, G_Y^{AG} ,

$$2\mu_Y N^2 G_Y^W = \sum_{h=1}^{H}\sum_{i=1}^{K_h}\sum_{j=1}^{K_h}\left|y_{h,i} - y_{h,j}\right| p_{h,i}p_{h,j} , \tag{6}$$

$$2\mu_Y N^2 G_Y^{AG} = \sum_{h=1}^{H}\sum_{\substack{g=1\\h\neq g}}^{H}\left(\sum_{i=1}^{K_h}\sum_{j=1}^{K_g}\left|y_{h,i} - y_{g,j}\right| p_{h,i}p_{g,j}\right). \tag{7}$$

Then, by sorting the H groups according to the ranking of their averages and lining up incomes in non-decreasing order within each group, the AG component decomposes into the sum of the between and overlapping components. The overlapping component, G_Y^T , can be expressed as the weighed sum of transvariations as in (8) (see Monti 2008, Monti and Santoro 2007)

$$\mu_Y N^2 G_Y^T = \sum_{h=2}^{H}\sum_{g=1}^{h-1}\left(\sum_{i=1}^{K_h}\sum_{j=1}^{K_g} 2\left|y_{h,i} - y_{g,j}\right| p_{h,i}p_{g,j}\right) \quad \forall\left(y_{h,i} - y_{g,j}\right) < 0 \tag{8}$$

and the between-group component, G_Y^B , can be written as

$$\mu_Y N^2 G_Y^B = \sum_{h=2}^{H}\sum_{g=1}^{h-1}\left(\sum_{i=1}^{K_h}\sum_{j=1}^{K_g}\left(y_{h,i} - y_{g,j}\right) p_{h,i}p_{g,j}\right). \tag{9}$$

Expression (9) defines G_Y^B as a function of the differences between incomes rather than as a function of the group mean differences. Expressions (7), (8) and (9) allow us to express the three terms of the Gini decomposition as a function of $\mathbf{D}_Y$. To show the latter, we introduce the matrices $\mathbf{W}_Y$ and $(\mathbf{J}\text{-}\mathbf{W}_Y)$. The matrix $\mathbf{W}_Y = \sum_{h=1}^{H}\mathbf{w}_{Y,h}\mathbf{w}_{Y,h}'$ is a $K\times K$ matrix where $\mathbf{w}_h$ denotes a $K\times 1$ column vector: the j-th entry of $\mathbf{w}_h$ can be equal to 1 or to 0. The j-th entry equals 1 if the income with rank j in the Y-ordering belongs

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[6] See Faliva (1996) for Hadamard product properties.

to group h, equals zero otherwise. The matrix, $\mathbf{W}_Y$, applied to $\mathbf{D}_Y$ by the Hadamard product, can be used to detect the $\sum_{h=1}^{H} K_h^2$ differences belonging to the same group from the whole set of K^2 income differences. Defining $\mathbf{J}$ as a $K \times K$ matrix with all elements equal to one, the matrix $(\mathbf{J}-\mathbf{W}_Y)$, when applied to $\mathbf{D}_Y$, identifies the set of the $\left(K^2 - \sum_{h=1}^{H} K_h^2\right)$ differences between incomes belonging to different groups. Using $\mathbf{W}_Y$ and $(\mathbf{J}-\mathbf{W}_Y)$ in (3), we obtain

$$2\mu_Y N^2 G_Y^W = \mathbf{p}_Y'\left(\mathbf{W}_Y \odot \mathbf{S} \odot \mathbf{D}_Y\right)\mathbf{p}_Y , \tag{10}$$

$$2\mu_Y N^2 G_Y^{AG} = \mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right) \odot \mathbf{S} \odot \mathbf{D}_Y\right]\mathbf{p}_Y . \tag{11}$$

Then, introducing the permutation matrix $\mathbf{A}_Y$, [7] the matrix form for G_Y^B becomes

$$2\mu_Y N^2 G_Y^B = \mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right) \odot \mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y \odot \mathbf{D}_Y\right]\mathbf{p}_Y . \tag{12}$$

Subtracting (12) from G_Y^{AG} as in (11), we obtain

$$2\mu_Y N^2 G_Y^T = \mathbf{p}_Y'\left\{\left(\mathbf{J}-\mathbf{W}_Y\right) \odot \left(\mathbf{S}-\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y\right) \odot \mathbf{D}_Y\right\}\mathbf{p}_Y . \tag{13}$$

Expression (13) represents the matrix form of the overlapping component expressed as weighed sum of transvariations, as in equation (8).

We decompose the concentration index following the same approach used for G_Y. By first, we decompose this index into two terms $C_{Y|X}^W$ and $C_{Y|X}^{AG}$, respectively analogous to G_Y^W and G_Y^{AG}. $C_{Y|X}^W$ and $C_{Y|X}^{AG}$ are calculated substituting in the expression of the before-tax Gini index the values of the after-tax incomes. In order to formalize the matrix form for $C_{Y|X}^{AG}$ and $C_{Y|X}^W$, we introduce the selection matrix $\mathbf{W}_X = \sum_{h=1}^{H} \mathbf{w}_{X,h}\mathbf{w}_{X,h}'$. Applying the matrix $\mathbf{W}_X$ to the matrix $\mathbf{D}_{Y|X}$ one detects the differences belonging to a same group from the whole set of K^2 income differences. Conversely, applying to $\mathbf{D}_{Y|X}$ the matrix $(\mathbf{J}\text{-}\mathbf{W}_X)$ one detects the $K^2 - \sum_{h=1}^{H} K_h^2$ differences between incomes belonging to different groups. Remembering that $\mathbf{D}_{Y|X} = \left(\mathbf{j}\mathbf{y}_X' - \mathbf{y}_X\mathbf{j}'\right)$, $\mathbf{y}_X = \mathbf{E}\mathbf{y}$ and $\mathbf{y} = \mathbf{E}'\mathbf{y}_X$, after some manipulations one obtains

$$2\mu_Y N^2 C_{Y|X}^W = \mathbf{p}_Y'\left[\mathbf{W}_Y \odot \mathbf{E}'\mathbf{S}\mathbf{E} \odot \mathbf{D}_Y\right]\mathbf{p}_Y , \tag{14}$$

$$2\mu_Y N^2 C_{Y|X}^{AG} = \mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right) \odot \mathbf{E}'\mathbf{S}\mathbf{E} \odot \mathbf{D}_Y\right]\mathbf{p}_Y . \tag{15}$$

Analogously to G_Y^{AG}, the across-group concentration index $C_{Y|X}^{AG}$ decomposes as the sum of $C_{Y|X}^B$ and $C_{Y|X}^T$. They are calculated substituting in the between component and in the overlapping component of the before-tax Gini index the values of the after-tax incomes.

Introducing the permutation matrix $\mathbf{A}_X$[8] we obtain

[7] In $\mathbf{A}_Y\mathbf{y}$, after-tax incomes are lined up in a non decreasing ordering within each group and groups follow the non decreasing ordering of their mean. Given $\mu_{Yh} \le \mu_{Yh+1}$, one has $(\mathbf{A}_Y\mathbf{y})' = \left[\left(y_{1,1}, y_{1,2}, \ldots y_{1,K_1}\right), \ldots, \left(y_{h,1}, y_{h,2}, \ldots y_{h,K_h}\right), \ldots, \left(y_{H,1}, y_{H,2}, \ldots y_{H,K_H}\right)\right]$, $y_{h,i} \le y_{h,i+1}$.

[8] $(\mathbf{A}_X\mathbf{x})' = \left[\left(x_{1,1}, x_{1,2}, \ldots x_{1,K_1}\right), \ldots, \left(x_{h,1}, x_{h,2}, \ldots x_{h,K_h}\right), \ldots, \left(x_{H,1}, x_{H,2}, \ldots x_{H,K_H}\right)\right]$, $x_{h,i} \le x_{h,i+1}$ and $\mu_{Xh} \le \mu_{Xh+1}$.

$$2\mu_Y N^2 C_{Y|X}^{B} = \mathbf{p}_Y'\left\{\left(\mathbf{J}-\mathbf{W}_Y\right)\odot \mathbf{E}'\mathbf{A}_X'\mathbf{S}\mathbf{A}_X\mathbf{E}\odot \mathbf{D}_Y\right\}\mathbf{p}_Y . \tag{16}$$

Then subtracting (16) from $C_{Y|X}^{AG}$ (expression (15)) one has

$$2\mu_Y N^2 C_{Y|X}^{T} = \mathbf{p}_Y'\left\{\left(\mathbf{J}-\mathbf{W}_Y\right)\odot\left(\mathbf{E}'\mathbf{S}\mathbf{E}-\mathbf{E}'\mathbf{A}_X'\mathbf{S}\mathbf{A}_X\mathbf{E}\right)\odot \mathbf{D}_Y\right\}\mathbf{p}_Y . \tag{17}$$

3 The reranking index $\mathbf{R}^{\mathbf{APK}}$ and its decomposition

A tax system induces reranking when the rank ordering of after-tax incomes is different from that of before-tax incomes. For a homogeneous population, Atkinson (1980), Plotnick (1981) and Kakwani (1984) proposed a reranking measure given by the difference between the after-tax (*a.t.*) Gini index and the concentration index for after-tax incomes w.r.t before-tax (*b.t.*) incomes. Using (3) and (5) we write the APK reranking index R^{APK} in matrix form as

$$2\mu_Y N^2 R^{APK} = \mathbf{p}_Y'\left[\left(\mathbf{S}-\mathbf{E}'\mathbf{S}\mathbf{E}\right)\odot \mathbf{D}_Y\right]\mathbf{p}_Y = \mathbf{p}_Y'\left[\mathbf{S}^{APK}\odot \mathbf{D}_Y\right]\mathbf{p}_Y . \tag{18}$$

When a population is not homogeneous, (18) can be decomposed by groups. The difference between expressions (10) and (14) yields the reranking within each group, R^W.

$$2\mu_Y N^2 R^{W} = \mathbf{p}_Y'\left[\mathbf{W}_Y\odot\left(\mathbf{S}-\mathbf{E}'\mathbf{S}\mathbf{E}\right)\odot \mathbf{D}_Y\right]\mathbf{p}_Y = \mathbf{p}_Y'\left[\mathbf{S}^{W}\odot \mathbf{D}_Y\right]\mathbf{p}_Y . \tag{19}$$

The difference between expressions (11) and (15) yields the reranking across the groups, R^{AG}. R^{AG} evaluates the whole reranking between incomes belonging to different groups

$$2\mu_Y N^2 R^{AG} = \mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right)\odot\left(\mathbf{S}-\mathbf{E}'\mathbf{S}\mathbf{E}\right)\odot \mathbf{D}_Y\right]\mathbf{p}_Y = \mathbf{p}_Y'\left[\mathbf{S}^{AG}\odot \mathbf{D}_Y\right]\mathbf{p}_Y . \tag{20}$$

To show how (18) yields the reranking index R^{APK}, we analyze the effects of the permutation matrices $\mathbf{E}$ on the matrix of signs $\mathbf{S}=\left[s_{ij}\right]$: the matrix $\mathbf{E}'\mathbf{S}\mathbf{E}=\left[s_{ij}^{e}\right]$ maps the permutation of each income from the X-ordering to the Y-ordering. If we consider only the super-diagonal elements of $\mathbf{D}_Y=\left[d_{ij}^{Y}\right]$, $d_{ij}^{Y}=y_j-y_i\geq 0$, s_{ij}^{e} is +1 if both $y_j\geq y_i$ and $x_j > x_i$, conversely s_{ij}^{e} is −1 if $y_j > y_i$ but $x_j < x_i$; as a consequence, in (18) the elements $\left(s_{ij}-s_{ij}^{e}\right)$ are +2 whenever a reranking occurs, otherwise they are zero. [9] So, as the super-diagonal elements, $d_{ij}^{Y}=y_j-y_i\geq 0$, can be associated either to 0 or to 2, expression (18) confirms the well-known result $0\leq R^{APK}\leq 2G_Y$. Moreover, as (18) splits into R^W and R^{AG}, we obtain $0\leq R^{W}\leq 2G_Y^{W}$ and $0\leq R^{AG}\leq 2G_Y^{AG}$.

The across-group reranking R^{AG} decomposes as sum of two terms: $R^{B}=G_Y^{B}-C_{Y|X}^{B}$ and $R^{T}=G_Y^{T}-C_{Y|X}^{T}$. We obtain R^B subtracting (16) from (12)

$$2\mu_Y N^2 R^{B} = \mathbf{p}_Y'\left[\left(\mathbf{J}-\mathbf{W}_Y\right)\odot\left(\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y-\mathbf{E}'\mathbf{A}_X'\mathbf{S}\mathbf{A}_X\mathbf{E}\right)\odot \mathbf{D}_Y\right]\mathbf{p}_Y = \mathbf{p}_Y'\left[\mathbf{S}^{B}\odot \mathbf{D}_Y\right]\mathbf{p}_Y . \tag{21}$$

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[9] We observe that, in case of reranking, in the sub-diagonal part of $\mathbf{D}_Y$ we have $d_{ji}^{Y}=y_i-y_j\leq 0$. Then, these terms are multiplied by $\left(s_{ji}-s_{ji}^{e}\right)=-2$.

We obtain R^T subtracting (17) from (13)

$$2\mu_Y N^2 R^T = \mathbf{p}_Y'\left\{\left(\mathbf{J}-\mathbf{W}_Y\right)\odot\left(\mathbf{S}-\mathbf{A}_Y'\mathbf{S}\mathbf{A}_Y-\mathbf{E}'\mathbf{S}\mathbf{E}+\mathbf{E}'\mathbf{A}_X'\mathbf{S}\mathbf{A}_X\mathbf{E}\right)\odot\mathbf{D}_Y\right\}\mathbf{p}_Y = \\ = \mathbf{p}_Y'\left\{\mathbf{S}^T\odot\mathbf{D}_Y\right\}\mathbf{p}_Y \ . \quad (22)$$

If after tax group averages do not change their ranking w.r.t. their before tax ranking, it can be verified that $\mathbf{A}_Y = \mathbf{A}_X\mathbf{E}$ and then it follows immediately that R^B is zero and that, consequently $R^{AG}\equiv R^T$. In order to evaluate the effect of the matrices $\mathbf{S}^{APK}$, $\mathbf{S}^{W}$, $\mathbf{S}^{AG}$ $\mathbf{S}^{B}$ on $\mathbf{D}_Y$, we give the following example.

Consider two groups (denoted by the subscripts *\$* and *£*) containing, respectively, three and four incomes

$$\mathbf{y}' = \begin{bmatrix}4_£ & 5_\$ & 8_\$ & 9_£ & 10_£ & 24_\$ & 32_£\end{bmatrix}; \quad \mathbf{y}_X' = \begin{bmatrix}5_\$ & 4_£ & 10_£ & 8_\$ & 9_£ & 24_\$ & 32_£\end{bmatrix}$$

If both $\mu_{Y\$}<\mu_{Y£}$ and $\mu_{X\$}<\mu_{X£}$, one has e.g.

$$\mathbf{y}'\mathbf{A}_Y' = \begin{bmatrix}5_\$ & 8_\$ & 24_\$ & 4_£ & 9_£ & 10_£ & 32_£\end{bmatrix}; \ \mathbf{y}_X'\mathbf{A}_X' = \begin{bmatrix}5_\$ & 8_\$ & 24_\$ & 4_£ & 10_£ & 9_£ & 32_£\end{bmatrix}$$

As the two group averages do not rerank, R^B is zero and $\mathbf{S}^B = \mathbf{0}$. Matrices $\mathbf{S}^{APK}$, $\mathbf{S}^{W}$ and $\mathbf{S}^{AG}$ are

$$\mathbf{S}^{APK} = \begin{bmatrix}0&2&0&0&0&0&0\\-2&0&0&0&0&0&0\\0&0&0&0&2&0&0\\0&0&0&0&2&0&0\\0&0&-2&-2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}, \ \mathbf{S}^{W} = \begin{bmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&2&0&0\\0&0&0&-2&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}, \ \mathbf{S}^{AG} = \begin{bmatrix}0&2&0&0&0&0&0\\-2&0&0&0&0&0&0\\0&0&0&0&2&0&0\\0&0&0&0&0&0&0\\0&0&-2&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}. \quad (23)$$

Let's now consider group means reranking.

In this case, we have $0 < R^B \le 2G_Y^B$, with $R^B = 2G_Y^B$ when all group averages permute their reciprocal positions.

In Table 1 we summarize the super-diagonal elements in $\mathbf{S}^{AG} = \left[s_{ij}^{AG}\right]$, $\mathbf{S}^{B} = \left[s_{ij}^{B}\right]$ and $\mathbf{S}^{T} = \left[s_{ij}^{T}\right]$, (associated to the non-negative differences in $\mathbf{D}_Y$ that belong to different groups). We observe what follows:

1) (row *a*, column *a*) s_{ij}^{AG}=0, s_{ij}^{B}=+2, s_{ij}^{T}=−2: *b.t.* the inequality between group means, expressed by $\mu_G(x_j)–\mu_G(x_i)$, has an opposite sign w.r.t. $x_j–x_i$. *a.t.* group means inequality assumes the same direction as $y_j–y_i$.

2) (row *b*, column *a*) s_{ij}^{AG}=+2, s_{ij}^{B}=+2, s_{ij}^{T}=0: *a.t.* and *b.t.* inequalities between incomes and between group means have the same sign, that is, both incomes and group means rerank towards the same direction;

3) (row *a*, column *b*) s_{ij}^{AG}=0, s_{ij}^{B}=−2, s_{ij}^{T}=+2: *b.t.* income inequalities and between group means have the same sign; conversely *a.t.* inequality between incomes $y_j–y_i$ maintains the same sign as $x_j–x_i$, while $\mu_G(y_j)–\mu_G(y_i)$ changes its sign w.r.t. that of $\mu_G(x_j)–\mu_G(x_i)$. In this case *a.t* incomes do not rerank w.r.t. the *b.t.* situation.

4) (row *b*, column *b*) s_{ij}^{AG} =+2, s_{ij}^{B} =−2, s_{ij}^{T} =+4: both *b.t.* and *a.t.* inequalities between incomes are opposite w.r.t. inequalities between their group means. Both incomes and their averages rerank, but towards opposite directions.

Table 1 *Group means re-rank.* s_{ij}^{AG} , s_{ij}^{B} , s_{ij}^{T} .

	$r_Y\left(y_j\right) > r_Y\left(y_i\right)$	*a* ${}_Y r_{\mu_G(y_j)} > {}_Y r_{\mu_G(y_i)}$ ${}_X r_{\mu_G(y_j)} < {}_X r_{\mu_G(y_i)}$			*b* ${}_Y r_{\mu_G(y_j)} < {}_Y r_{\mu_G(y_i)}$ ${}_X r_{\mu_G(y_j)} > {}_X r_{\mu_G(y_i)}$		
		s_{ij}^{AG}	s_{ij}^{B}	s_{ij}^{T}	s_{ij}^{AG}	s_{ij}^{B}	s_{ij}^{T}
a	$r_X\left(y_j\right) > r_X\left(y_i\right)$	0	+2	−2	0	−2	+2
b	$r_X\left(y_j\right) < r_X\left(y_i\right)$	+2	+2	0	+2	−2	+4

The income pairs that contribute positively to R^B are those that fall in cases 1 or in case 2. The income pairs that falls in cases 3 and 4 smooth R^B. If all income pairs falls in 1 and in 3, group means reranking occurs without any across-group income reranking, consequently R^{AG}=0 and $-R^B = R^T$. If income pair behaviour is as in case 2 for all income pairs, $R^{AG}=R^B$ and R^T=0. Cases 3 and 4, that oppose to average reranking effect, and case 1 cannot occur by themselves. If only cases 1 and 3 occur, one has average reranking without any income reranking. From this it derives that neither R^B nor R^T can be considered as unfairness indicators.

The term R^B measures the reranking (in average) of the entire groups. The term R^T captures how differences between incomes belonging to different groups react to inequality direction changes of their group means. An interpretation of R^T may be given by considering specific cases, however a global interpretation of R^T seems more difficult. In what follows, we give an example when group averages rerank.

In this example, we suppose two groups with $\mu_{Y\$}<\mu_{Y£}$ and $\mu_{X\$}>\mu_{X£}$:

$$\mathbf{y}'=\left[4_{£}\quad 5_{\$}\quad 8_{\$}\quad 9_{£}\quad 10_{£}\quad 24_{\$}\quad 32_{£}\right];\quad \mathbf{y}_X'=\left[5_{\$}\quad 4_{£}\quad 10_{£}\quad 8_{\$}\quad 9_{£}\quad 32_{£}\quad 24_{\$}\right]$$

$$\mathbf{y}'\mathbf{A}_Y'=\left[5_{\$}\quad 8_{\$}\quad 24_{\$}\quad 4_{£}\quad 9_{£}\quad 10_{£}\quad 32_{£}\right];\quad \mathbf{y}_X'\mathbf{A}_X'=\left[4_{£}\quad 10_{£}\quad 9_{£}\quad 32_{£}\quad 5_{\$}\quad 8_{\$}\quad 24_{\$}\right]$$

$$\mathbf{S}^{AG}=\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{bmatrix},\ \mathbf{S}^{B}=\begin{bmatrix} 0 & -2 & -2 & 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & 2 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 2 & 0 & 2 \\ 0 & -2 & -2 & 0 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & 2 & 2 & 0 & 2 \\ 0 & -2 & -2 & 0 & 0 & -2 & 0 \end{bmatrix},\ \mathbf{S}^{T}=\begin{bmatrix} 0 & 4 & 2 & 0 & 0 & 2 & 0 \\ -4 & 0 & 0 & -2 & -2 & 0 & -2 \\ -2 & 0 & 0 & -2 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\mathbf{S}^W$ is the same as in (23).

Case 1 is represented by pairs ($9_£$−$5_\$$), ($10_£$−$5_\$$), ($32_£$−$5_\$$), ($9_£$−$8_\$$) and ($32_£$−$8_\$$); case 2 is represented by pairs ($10_£$−$8_\$$) and ($32_£$−$24_\$$); case 3 is represented by pairs ($8_\$$−$4_£$), ($24_\$$−$4_£$), ($24_\$$−$9_£$) and ($24_\$$−$10_£$); case 4 is represented by the pair ($5_\$$−$4_£$).

4 Conclusions

This paper has accomplished two tasks. First, it has provided a matrix-based approach for the decomposition of the Gini and the concentration indices. These matrix form expressions are decomposable into the sum of three components, each of them expressed as a function of a same matrix. Second, this matrix approach enables to further analyse the reranking effect of taxation. It decomposes the overall reranking index, R^{APK}, as a sum of within and across-groups reranking. The across group reranking can be further decomposed yielding two components: a component capturing the entire group reranking, plus a residual which is both an adjusting term of R^B and a measure of the extent to which pairs of incomes belonging to different groups do not present a rank change analogous to the one of their group averages. In any cases, neither R^B nor R^T can be considered as unfairness indicators.

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