ON HUMAN CAPITAL AND ECONOMIC GROWTH

WITH RANDOM TECHNOLOGY SHOCKS

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ON HUMAN CAPITAL AND ECONOMIC GROWTH WITH RANDOM TECHNOLOGY SHOCKS[†]

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Abstract

We embed the Uzawa-Lucas human capital accumulation technology into the Mankiw-Romer-Weil exogenous growth model. The paper is divided into two parts. In the first part we assume that the rate of technological progress is exogenous and deterministic and study the local dynamics of the model around its steady-state equilibrium. The first order conditions lead to a system of four nonlinear differential equations. By reducing the dimension of the system to three, we find that the equilibrium is a saddle point. If the equations system is attacked in its original dimension, and by making use of an arbitrage condition, we prove that the equilibrium is unstable. In the second part of the paper technology is assumed to be subject to random shocks driven by a geometric Brownian motion. Using the Hamilton-Jacobi-Bellman equation, and through numerical simulations, we discuss the effects of technology shocks on the optimal policies of consumption and the allocation of human capital across sectors.

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1. INTRODUCTION

Since Uzawa (1965), and more recently Lucas (1988) and Mankiw *et al.* (1992), the importance of human capital accumulation in explaining economic growth both at the theoretical and empirical level is widely recognized.

Mankiw *et al.* (1992) were the first to suggest that international differences in income per-capita are best understood using a Solow-type growth model augmented with the inclusion of human capital investment. On the other hand, Uzawa (1965) and later on Lucas (1988) –henceforth simply Uzawa-Lucas– have showed that purposeful accumulation of human capital by rational, forward-looking agents represents an important engine of growth in real per-capita incomes. The reason why the Uzawa-Lucas model continues to be one of the most studied growth models is twofold (see Boucekkine and Ruiz-Tamarit, 2008 for a deeper discussion). First of all it is a two-sector, rather than one-sector, growth model and, as such, it differs from the *AK-type* endogenous growth models. In more detail, it is postulated that agents have to allocate their human capital across two production activities: a final output sector that produces, with constant returns to scale, a homogeneous good (that can be, in turn, either consumed or invested in physical capital) and an education sector (being relatively intensive in human capital), where individuals can augment their own level of skills. Secondly, it gives rise to a sophisticated dynamical system with two control variables (consumption and the share of human capital to be allocated across sectors) and two state variables (human and physical capital).

In this paper we embed the Uzawa-Lucas human capital accumulation technology into the Mankiw *et al.* exogenous growth model. In so doing, we extend both approaches along different directions. The first departure from Uzawa-Lucas consists in adding (exogenous) technological progress to that model. In this respect we postulate that aggregate output is obtained by combining not only human and physical capital (as in the original version of Uzawa-Lucas), but also *ideas*, a proxy of an economy's level of technology, whose evolution over time is taken as exogenous in our paper. Secondly, we also assume that technology might be subject to random shocks. In its original formulation the Uzawa-

Lucas model was set in a purely deterministic framework.[1] The differences between Mankiw *et al.* (1992) and our contribution are equally clear. The first is that, unlike Mankiw *et al.*, the objective of this paper is not empirical. In other words, we are not interested in a better understanding of cross-country international differences in income per-capita. On the contrary, our analysis here is rather motivated by theoretical aims, one of which is, as just mentioned, to extend the human capital-based growth theory of Uzawa-Lucas by including exogenous technological progress (both deterministic and stochastic). Secondly, we explicitly consider the case of different depreciation rates for human and physical capital. Furthermore, the choice of how much income to save and invest in physical capital accumulation is endogenous in our model. Finally, and this is probably the most salient departure from Mankiw *et al.* (1992), we assume that the production of human capital is an economic activity being relatively intensive in human capital (in their original formulation they consider the case where human capital is built from final output[2]). However, like in Mankiw *et al.* (1992, p. 417, equation 11), the steady-state growth rate of our model economy turns out to be equal to the (exogenous) rate of technological progress.

Our paper is divided into two parts. In the first part we assume that the rate of technical change is not only exogenous, but also deterministic. Under a particular parameterization, our model allows recasting the original deterministic Uzawa-Lucas approach. In this sense our contribution represents a generalization of that model. The main objective of this part of our work is to study the local dynamics of the model around its steady-state equilibrium. Indeed, the first order conditions of the intertemporal optimization problem we analyze lead to a system of four nonlinear differential equations. By reducing the dimension of the system to three, we find that the equilibrium is a saddle point. If the equations system is attacked in its original dimension, by making use of an arbitrage condition, we prove that the equilibrium is unstable. In the second part

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[1] The seminal paper in stochastic growth is Brock and Mirman (1972). Olson and Roy (2006) provide an excellent recent survey of this literature.

[2] *"…We are assuming that the same production function applies to human capital, physical capital, and consumption. In other words, one unit of consumption can be transformed costlessly into either one unit of physical capital or one unit of human capital. …Lucas (1988) models the production function for human capital as fundamentally different from that for other goods. We believe that, at least for an initial examination, it is natural to assume that the two types of production functions are similar"* (Mankiw *et al.*, 1992, p. 416).

of the article we formally introduce the hypothesis that technology might be subject to random shocks driven by a geometric Brownian motion. Using the Hamilton-Jacobi-Bellman equation, and through numerical simulations, our aim is to discuss the effects of technology shocks on the optimal policies of consumption and the sectoral allocation of human capital, respectively.

As far as we know there are only a few attempts at characterizing the solution to a stochastic growth model. Two among the most recent are represented, respectively, by Bethmann (2007) and Smith (2007). Bethmann (2007) extends the basic stochastic one-sector growth model with logarithmic preferences and full depreciation of physical capital (that is, the textbook McCallum (1989) real business cycle model) to the case with two capital goods (physical and human capital). The theoretical model we analyze in the present paper differs from Bethmann (2007) in three main respects. First of all, it is set in continuous time. Secondly, the level of technology evolves according to a geometric Brownian motion (in Bethmann, 2007 the logarithm of total factor productivity follows a first-order autoregressive process). Finally, we do not assume full depreciation of human and physical capital. With respect to Smith (2007), instead, the most significant difference is that in our paper we add a human capital accumulation sector, completely missing in the Smith's work.

The article is structured as follows. In section 2 we specify the general model. In section 3 we analyze and characterize the local dynamics of it with exogenous and deterministic technological progress. In section 4 we introduce the assumption that technical change might follow a stochastic Brownian motion. In this section we study the effects of technology shocks on the optimal policies of consumption (C^*) and the sectoral allocation of human capital (u^*). As usual, the last section summarizes, concludes and proposes possible paths for future research.

2. THE GENERAL MODEL

The economy is closed. Output is the numeraire good (its price is normalized to one) and is produced competitively by combining physical capital, human capital and labor in

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efficiency units. Aggregate income is allocated either to consumption or to (gross) physical capital investment. The aggregate production function is similar to the one used by Mankiw *et al.* (1992, p.416, Eq. 8):

$$Y(t) = \left(A(t)L\right)^{\gamma}\left(H_Y(t)\right)^{\alpha}K(t)^{1-\alpha-\gamma}, \qquad \alpha \in (0;1), \qquad \gamma \in (0;1), \qquad \alpha + \gamma \in (0;1),$$

with $K(t)$, $H_Y(t)$ and $\left(A(t)L\right)$ being, respectively, the stock of physical and human capital and the number of effective units of labor employed at time t in the production of the homogeneous consumption good (Y). The aggregate production function written above displays constant returns to scale to the three factor-inputs, jointly considered. Moreover, since output is produced under perfect competition conditions, each input is remunerated according to its own marginal productivity. Therefore, α, $(1-\alpha-\gamma)$ and γ are the shares of income accruing to human capital $H_Y(t)$, physical capital $K(t)$ and labor in efficiency units $A(t)L$, respectively. Note that we are treating the size of population (or raw labor, L) as a constant. This is done because, unlike Mankiw *et al.* (1992), we are not interested in the effects of population (labor-force) growth. Instead, it is one of the objective of (the second part of) this paper to analyze the impact of technology shocks on the optimal policies of consumption and the sectoral allocation of human capital. Hence, for the sake of simplicity and without any loss of generality, throughout the entire paper we set $L=1$. Under this hypothesis, the technology for the production of output becomes:

$$Y(t) = A(t)^{\gamma}\left(u(t)H(t)\right)^{\alpha}K(t)^{1-\alpha-\gamma}, \qquad (1)$$

$$\alpha \in (0;1), \qquad \gamma \in (0;1), \qquad \alpha + \gamma \in (0;1), \qquad u(t) \in [0,1], \qquad u(t)H(t) \equiv H_Y(t)$$

In (1) we depart from Mankiw *et al.* (1992) and follow Uzawa-Lucas in postulating that the total stock of human capital, $H(t)$, be allocated partially (*i.e.*, in the proportion $u(t)$) to the production of goods and partially (in the proportion $1-u(t)$) to the

acquisition of new human capital. This sectoral allocation of human capital is endogenous to our model.[3]

Even though we explicitly consider the more general case where $\gamma \in (0;1)$, it is still interesting to see what the aggregate production function (1) would be like to in two extremely specific situations (respectively, $\gamma = 0$ and $\gamma = 1$). When $\gamma = 0$ our model gives rise to the Uzawa-Lucas one. In this case there would be no disembodied technological progress ($A(t) = A = 1$ for each t) and the aggregate production function would display constant returns to scale to human and physical capital, only. On the other hand, when $\gamma = 1$ the aggregate technology would read as:

$$Y_t = A_t \left(\frac{u_t H_t}{K_t} \right)^\alpha, \qquad \alpha \in (0;1) \qquad (2)$$

Following Howitt (1999), the term $\left(u_t H_t / K_t \right)^\alpha$ could be interpreted as capturing the fact that production tends to become more human capital intensive through time as physical capital accumulates. In other words, for given A, the economy would need to accumulate human capital ($H_Y(t)$) when physical capital $K(t)$ increases in order for economic growth to be sustainable in the long run.

The laws of motion of physical and human capital are the following:

$$\dot{K}(t) = A(t)^\gamma \left(u(t) H(t) \right)^\alpha K(t)^{1-\alpha-\gamma} - \beta_K K(t) - C(t) \qquad (3)$$

$$\dot{H}(t) = \eta \left(1 - u(t) \right) H(t) - \beta_H H(t), \qquad \eta > 0 \qquad (4)$$

with the initial conditions $K(0) = K_0$ and $H(0) = H_0$ given. It is evident from (4) that the growth of human capital does not depend on the physical capital stock, but depends solely on the effort devoted to the accumulation of human capital, $1 - u(t)$, as well as on the already attained human capital stock, $H(t)$. In the same equation η represents the productivity of human capital in the (gross) production of new human capital. Instead, in Eq. (3) $C(t)$ is the stream of real consumption of the single good. Both physical and

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[3] In Mankiw *et al.* (1992, p. 416) a constant and exogenous fraction of income (s_h) is invested in human capital and another constant and exogenous fraction of it (s_k) is invested in physical capital.

human capital depreciate at constant rates ($\beta_K \in [0;1]$ and $\beta_H \in [0;1]$, respectively). As already mentioned, we also assume that the two depreciation rates differ for both types of capital ($\beta_H \neq \beta_K$).

As for the level of technology, we maintain that $A(t)$ might be a stochastic variable evolving over time according to the following geometric Brownian motion:

$$dA(t) = \mu A(t)dt + \sigma A(t)dW(t), \qquad A(0) = A_0, \qquad \mu > 0, \qquad \sigma \geq 0 \qquad (5)$$

In (5) $\mu A(t)$ is the expected instantaneous drift rate and $dW(t)$ is the increment of a Wiener process such that $E[dW(t)] = 0$ and $var[dW(t)] = dt$.

The optimal decision problem can be formulated as:

$$\max_{C(t),u(t)} E \int_0^\infty \ln(C(t)) e^{-\rho t} dt, \qquad \rho > 0 \qquad (6)$$

subject to (3), (4) and (5) and the initial conditions. Individuals get utility from the consumption of the homogeneous final good. We also assume that the instantaneous utility function is logarithmic. Finally, the parameter ρ measures the rate of time-preference, or discount rate.

3. THE DETERMINISTIC MODEL

In this section it is our objective to study the local dynamics of the model under the assumption that technological progress (*i.e.*, $\dot{A}(t)/A(t)$) is both exogenous and deterministic. This is the case when we set $\sigma = 0$ in Eq. (5). Accordingly, in the deterministic model we have:

$$\dot{A}(t) = \mu A(t), \qquad (7)$$

The Hamiltonian function $H(C, u, K, H, A, \lambda_K, \lambda_H, \lambda_A)$ associated with the intertemporal optimization problem formulated in the previous section is:

$$H(\cdot) = \ln(C) e^{-\rho t} + \lambda_K \left[A^\gamma K^{1-\alpha-\gamma} (uH)^\alpha - \beta_K K - C \right] + \lambda_H \left[\eta(1-u) - \beta_H \right] H + \lambda_A \mu A \qquad (8)$$

where λ_K, λ_H and λ_A are the co-state variables for K, H and A, respectively.

The necessary first order conditions read as:

$$\frac{e^{-\rho t}}{C} - \lambda_K = 0 \tag{9}$$

$$\frac{\alpha \lambda_K A^\gamma K^{1-\alpha-\gamma} \left(uH \right)^\alpha}{u} - \eta \lambda_H H = 0 \tag{10}$$

$$\dot{\lambda}_K = -\lambda_K \left[\frac{\left(1-\alpha-\gamma\right) A^\gamma K^{1-\alpha-\gamma} \left(uH \right)^\alpha}{K} - \beta_K \right] \tag{11}$$

$$\dot{\lambda}_H = -\frac{\alpha \lambda_K A^\gamma K^{1-\alpha-\gamma} \left(uH \right)^\alpha}{H} - \lambda_H \left[\eta \left(1-u\right) - \beta_H \right] \tag{12}$$

$$\dot{\lambda}_A = -\frac{\gamma \lambda_K A^\gamma K^{1-\alpha-\gamma} \left(uH \right)^\alpha}{A} - \lambda_A \mu \tag{13}$$

together with the initial conditions K_0, H_0 and A_0, the dynamic constraints:

$$\dot{K} = A^\gamma \left(uH \right)^\alpha K^{1-\alpha-\gamma} - \beta_K K - C$$

$$\dot{H} = \eta \left(1-u\right) H - \beta_H H$$

$$\dot{A} = \mu A$$

and the transversality conditions:

$$\lim_{t \to \infty} \lambda_K K = 0$$

$$\lim_{t \to \infty} \lambda_H H = 0$$

$$\lim_{t \to \infty} \lambda_A A = 0.$$

If we combine (9) and (11) we obtain:

$$\frac{\dot{C}}{C} = \left(1-\alpha-\gamma\right) A^\gamma K^{-\alpha-\gamma} \left(uH \right)^\alpha - \rho - \beta_K.$$

By introducing the intensive variables $x(t) \equiv H(t)/K(t)$, $y(t) \equiv C(t)/K(t)$ and $z(t) \equiv A(t)/K(t)$, it is possible to write the following system of equations of motion:[4]

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[4] In (14), the growth rate of u is obtained by combining the dynamic constraints and equations (10), (11) and (12).

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$$\frac{\dot{x}(t)}{x(t)} = \eta\big(1-u(t)\big)-\beta_H - z(t)^\gamma u(t)^\alpha x(t)^\alpha + \beta_K + y(t)$$

$$\frac{\dot{y}(t)}{y(t)} = -(\alpha+\gamma)z(t)^\gamma u(t)^\alpha x(t)^\alpha - \rho + y(t) \qquad (14)$$

$$\frac{\dot{z}(t)}{z(t)} = \mu - z(t)^\gamma u(t)^\alpha x(t)^\alpha + \beta_K + y(t)$$

$$\frac{\dot{u}(t)}{u(t)} = \left(\frac{1}{\alpha-1}\right)\big[-\gamma\mu - (\alpha+\gamma)\beta_K + (1-\alpha-\gamma)y(t) - \eta\alpha + \eta(\alpha-1)u(t) + \alpha\beta_H\big].$$

With $\phi(t) \equiv z(t)^\gamma u(t)^\alpha x(t)^\alpha$, the previous equations-system simplifies to:

$$\frac{\dot{\phi}(t)}{\phi(t)} = \gamma\big[\mu - \phi(t) + \beta_K + y(t)\big] +$$

$$\frac{\alpha}{\alpha-1}\big[-\gamma\mu - (\alpha+\gamma)\beta_K + (1-\alpha-\gamma)y(t) - \eta\alpha + \eta(\alpha-1)u(t) + \alpha\beta_H\big] +$$

$$\alpha\big[\eta\big(1-u(t)\big) - \beta_H - \phi(t) + \beta_K + y(t)\big]$$

$$\frac{\dot{y}(t)}{y(t)} = -(\alpha+\gamma)\phi(t) - \rho + y(t) \qquad (15)$$

$$\frac{\dot{u}(t)}{u(t)} = \left(\frac{1}{\alpha-1}\right)\big[-\gamma\mu - (\alpha+\gamma)\beta_K + (1-\alpha-\gamma)y(t) - \eta\alpha + \eta(\alpha-1)u(t) + \alpha\beta_H\big].$$

In what follows we characterize the steady-state equilibrium (to be defined in a moment) of the deterministic model and develop the analysis of local dynamics separately for both equation-systems (14) and (15). The main difference between the two systems is that the first one (14) will turn out to be not fully determined (since we need an extra-condition in order to obtain a solution for the steady-state equilibrium). The second system (15), instead, is fully determined since variable $\phi(t)$ wholly describes the behavior of $z(t)^\gamma u(t)^\alpha x(t)^\alpha$. Economically, $\phi(t)$ represents the gross average product of physical capital in the production of goods (Y_t / K_t).

3.1. STEADY-STATE ANALYSIS AND LOCAL DYNAMICS OF EQUATIONS- SYSTEM (15)

We start our analysis by studying the local dynamics of equations-system (15). However, before proceeding we introduce a formal definition of steady-state equilibrium.

DEFINITION 3.1.: $\left(\phi^*, y^*, u^*\right)$ is said to be a steady-state equilibrium of equations-system (15) if it solves $\dot{\phi}_t = \dot{y}_t = \dot{u}_t = 0$.

On applying the definition given above, in the steady-state equilibrium we have:[5]

$$0 = \gamma\left[\mu - \phi^* + \beta_K + y^*\right] +$$
$$\frac{\alpha}{\alpha-1}\left[-\gamma\mu - (\alpha+\gamma)\beta_K + (1-\alpha-\gamma)y^* - \eta\alpha + \eta(\alpha-1)u^* + \alpha\beta_H\right] +$$
$$\alpha\left[\eta(1-u^*) - \beta_H - \phi^* + \beta_K + y^*\right]$$
$$0 = -(\alpha+\gamma)\phi^* - \rho + y^* \tag{16}$$
$$0 = \frac{1}{\alpha-1}\left[-\gamma\mu - (\alpha+\gamma)\beta_K + (1-\alpha-\gamma)y^* - \eta\alpha + \eta(\alpha-1)u^* + \alpha\beta_H\right].$$

Solving this three-equations system leads to the following steady-state values for ϕ, y and u:

$$\phi^* = \frac{\gamma(\rho+\mu) + (\alpha+\gamma)\beta_K + \alpha(\eta-\beta_H)}{(\alpha+\gamma)(1-\alpha-\gamma)}$$

$$y^* = \frac{\rho + \gamma\mu + (\alpha+\gamma)\beta_K + \alpha(\eta-\rho) - \alpha\beta_H}{(1-\alpha-\gamma)} \tag{17}$$

$$u^* = \frac{\rho}{\eta}.$$

Notice that, with our parameter values, a (sufficient) condition for y^* and ϕ^* to be strictly positive is:

$$\eta \geq \rho + \beta_H.$$

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[5] As it is economically plausible, we assume $y_t \equiv \dfrac{C_t}{K_t} > 0$, $\phi_t \equiv \dfrac{Y_t}{K_t} > 0$ and $u_t \in (0;1)$ for each t in the steady-state.

This condition, requiring that the productivity of education technology (η) is relatively large compared with the sum of the discount rate for future utility (ρ) and the depreciation rate of human capital (β_H), is standard in models with human capital supply functions *à la* Uzawa-Lucas.[6] Moreover, in the presence of a positive depreciation rate for human capital ($\beta_H > 0$) the condition $\eta \geq \rho + \beta_H > \rho$ is surely satisfied and, hence, an interior solution for u^* ($0 < u^* < 1$) does exist in the steady-state.

The Jacobian matrix associated with the log-linearized system of differential equations in (15) is the following:

$$\begin{pmatrix} -\gamma\phi - \phi\alpha & y\gamma + \dfrac{y\alpha(1-\alpha-\gamma)}{\alpha-1} + y\alpha & 0 \\ -(\alpha+\gamma)\phi & y & 0 \\ 0 & \dfrac{y(1-\alpha-\gamma)}{\alpha-1} & \eta u \end{pmatrix} \tag{18}$$

Evaluating the determinant of the Jacobian at the steady state values of variables $\phi(t)$, $y(t)$ and $u(t)$ given in (17) yields:

$$-\frac{\rho BD}{(\alpha-1)(\alpha-1+\gamma)} \tag{19}$$

where:

$$B \equiv (\alpha+\gamma)\beta_K + \mu\gamma + \rho(1-\alpha) + \alpha(\eta - \beta_H) > 0$$

$$D \equiv (\alpha+\gamma)\beta_K + \gamma(\mu+\rho) + \alpha(\eta - \beta_H) > 0.$$

Given our parameter values, it follows from (19) that the determinant of the Jacobian is negative. Since one eigenvalue equals ρ, which is strictly positive, and the determinant is strictly negative, by Theorem 23.9 of Simon and Blume (1994), there exists one negative eigenvalue and hence the equilibrium is a saddle point (Figure 1 shows the saddle-path in the $(\phi(t), y(t))$ space).

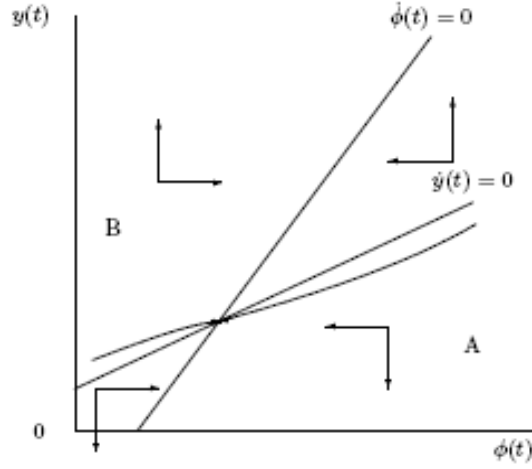[6] See, among others, Arnold, 1998, p. 85 (equation 1) and Strulik, 2005, p. 135 (equation 24).

Figure 1: Phase diagram in $(\phi(t), y(t))$ space

3.2. NUMERICAL SIMULATION

In this section we perform a numerical simulation concerning equations-system (15). We use the following parameter values: $\rho = 4/100$, $\eta = 12/100$, $\beta_K = 5/100$, $\mu = 2/100$, $\alpha = \gamma = 1/3$, $\beta_H = 6/100$. These parameter values can be explained as follows. The first three ($\rho = 4/100$, $\eta = 12/100$ and $\beta_K = 5/100$) are taken from Mulligan and Sala-i-Martin (1993, p. 761). The value of μ (*i.e.*, $\mu = 2/100$) represents the long-term growth rate of real GDP for the US economy (Barro and Sala-i-Martin, 2004, p. 58). We shall show in a moment that the restriction $\mu = \eta - \rho - \beta_H$ must be checked (see next section). Given $\mu = 2/100$, $\rho = 4/100$ and $\eta = 12/100$, it follows that β_H should be set equal to 6/100.[7] Finally, our assumption of $\alpha = \gamma = 1/3$ is consistent with the empirical evidence of a share of physical and human capital in income equal, respectively, to 1/3 (Mankiw *et al.*, 1992, p. 432). Under these parameterization, we have the following steady-state values:

$y^* = 13/50$, $\phi^* = 33/100$ and $u^* = 1/3$. Moreover, the Jacobian matrix becomes:

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[7] Note that, with $\mu > 0$ and $\eta = \mu + \rho + \beta_H$, the requirement $\eta \geq \rho + \beta_H$ is clearly satisfied. Moreover, note that we are using parameter values such that $\beta_H \neq \beta_K$, as mentioned earlier.

$$\begin{pmatrix} -\dfrac{11}{50} & \dfrac{13}{100} & 0 \\[2mm] -\dfrac{11}{50} & \dfrac{13}{50} & 0 \\[2mm] 0 & -\dfrac{13}{100} & \dfrac{1}{25} \end{pmatrix} \tag{20}$$

The eigenvalues are $1/25$ and $1/50 \pm 1/100\sqrt{290}$ and the determinant is -0.001. Figure 2 shows a three-dimensional plot of the direction field of the system, rendered in the (ϕ, y) space in Figure 3.
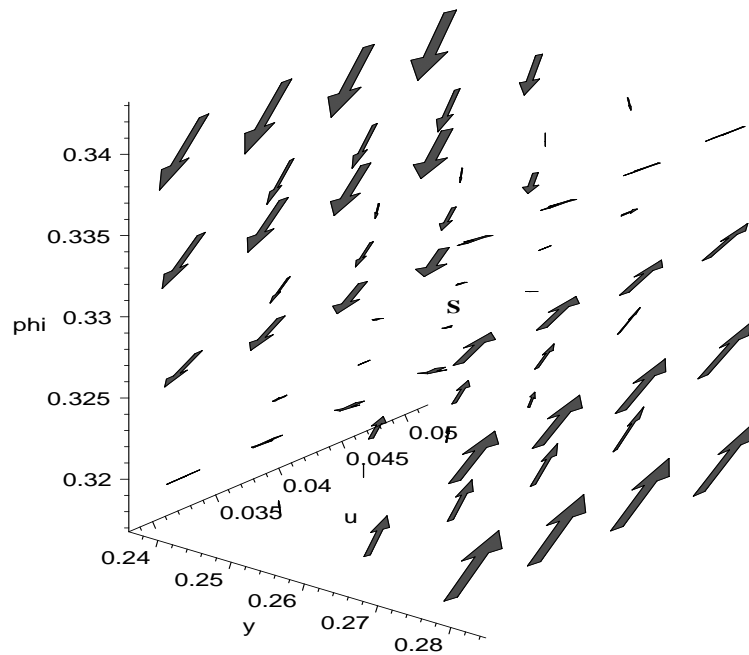

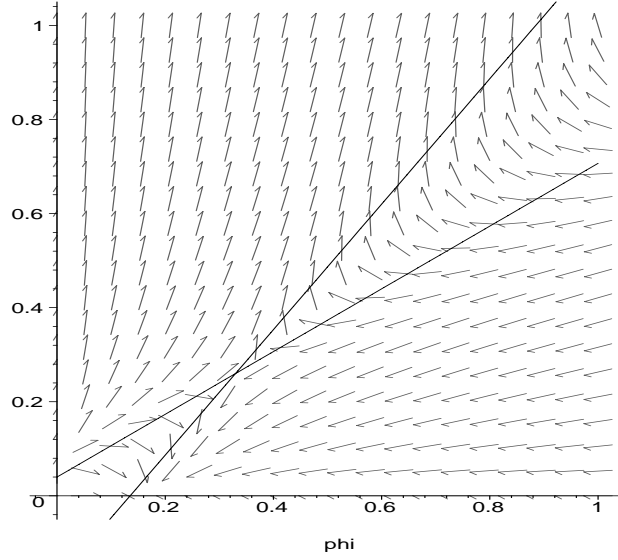Figure 2: Phase Diagram in the $(\phi(t), y(t), u(t))$ space

Figure 3: Phase Diagram in the $\left(\phi(t), y(t)\right)$ space

3.3. STEADY-STATE ANALYSIS AND LOCAL DYNAMICS OF EQUATIONS- SYSTEM (14)

We now turn to the analysis of local dynamics of equations-system (14). Unlike (15), this is a system of four (rather than three) differential equations in intensive variables x, y, z and in u. Again, and before proceeding, we introduce a formal definition of steady-state equilibrium.

DEFINITION 3.2.: $\left(\overline{x}, \overline{y}, \overline{z}, \overline{u}\right)$ is said to be a steady-state equilibrium of equations-system (14) if it solves $\overset{\bullet}{x}_t = \overset{\bullet}{y}_t = \overset{\bullet}{z}_t = \overset{\bullet}{u}_t = 0$.

On applying the definition given above to (14), we have:[8]

$$0 = \eta\left(1-\overline{u}\right) - \beta_H - \left(\overline{z}\right)^\gamma \left(\overline{u}\right)^\alpha \left(\overline{x}\right)^\alpha + \beta_K + \overline{y} \tag{21}$$

$$0 = -\left(\alpha+\gamma\right)\left(\overline{z}\right)^\gamma \left(\overline{u}\right)^\alpha \left(\overline{x}\right)^\alpha - \rho + \overline{y} \tag{22}$$

$$0 = \mu - \left(\overline{z}\right)^\gamma \left(\overline{u}\right)^\alpha \left(\overline{x}\right)^\alpha + \beta_K + \overline{y} \tag{23}$$

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[8] We continue to assume $y_t \equiv \dfrac{C_t}{K_t} > 0$, $z_t \equiv \dfrac{A_t}{K_t} > 0$, $u_t \in (0;1)$ and $x_t \equiv \dfrac{H_t}{K_t} > 0$ for each t in the steady-state.

$$0 = \left(\frac{1}{\alpha-1}\right)\left[-\gamma\mu - (\alpha+\gamma)\beta_K + (1-\alpha-\gamma)\overline{y} - \eta\alpha + \eta(\alpha-1)\overline{u} + \alpha\beta_H\right] \qquad (24)$$

It is possible to prove that the restriction $\mu = \eta - \rho - \beta_H$ must be checked in the steady-state equilibrium. Under this constraint, the system of nonlinear equations written above has a steady-state equilibrium given by (see Appendix A):

$$\overline{u} = \frac{\rho}{\eta}$$

$$\overline{y} = \frac{(\alpha+\gamma)(\mu+\beta_K)+\rho}{(1-\alpha-\gamma)} \qquad (25)$$

$$\left(\overline{z}\right)^\gamma \left(\overline{x}\right)^\alpha = \frac{\mu+\rho+\beta_K}{\left(\dfrac{\rho}{\eta}\right)^\alpha (1-\alpha-\gamma)}$$

It is clear from (25) that we need one more condition in order to get a solution for intensive variables z and x, separately. For this reason, we look for a steady-state equilibrium that satisfies some *arbitrage condition*. At this aim we know that, unlike physical capital (that is used exclusively in the goods sector), human capital may be employed either in the production of goods or in the production of new human capital. Therefore, we first of all need computing the (shadow) price of human capital in units of goods. This price (p) is obtained (see Barro and Sala-i-Martin, 2004, pp. 249-250) by taking the ratio of the marginal product of H in the production sector[9] (*i.e.* the wage rate, $\alpha A^\gamma H_Y^{\alpha-1} K^{1-\alpha-\gamma}$) to its marginal product in the education sector, η :[10]

$$p = \frac{\alpha A^\gamma \left(uH\right)^{\alpha-1} K^{1-\alpha-\gamma}}{\eta}, \qquad (26)$$

In words, equation (26) says that at the margin it should be indifferent for a decision-maker to invest one more unit of the available human capital stock in goods production or in the production of new (gross) human capital. This solves the first allocation-problem (whether to put one more unit of H into the production or education sector). As for the second allocation-problem (whether to invest in physical or human capital), we notice

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[9] Recall that in this economy final output, Y, acts as the numeraire good (the price of one unit of output equals one).

[10] From (4), gross investment in human capital is: $\dot{H} + \beta_H H = \eta(1-u)H$.

that if the markets for the two forms of capital are competitive (as we assume) in the very long-run the rates of return to K and H in terms of goods (respectively, the net marginal product of physical capital in the goods sector and the net shadow price of human capital in units of final output) be equalized, that is:[11]

$$(1-\alpha-\gamma)A^{\gamma}H_Y^{\alpha}K^{-\alpha-\gamma} - \beta_K = \frac{\alpha A^{\gamma}H_Y^{\alpha-1}K^{1-\alpha-\gamma}}{\eta} - \beta_H \tag{27}$$

Recalling the definitions of intensive variables $x(t)$ and $z(t)$, in the steady-state equilibrium Eq. (27) leads to:

$$\left(\overline{z}\right)^{\gamma} = \frac{\eta\left(\beta_K - \beta_H\right)}{\left[\eta(1-\alpha-\gamma)\left(\overline{u}\right)^{\alpha}\left(\overline{x}\right)^{\alpha} - \alpha\left(\overline{u}\right)^{\alpha-1}\left(\overline{x}\right)^{\alpha-1}\right]} \tag{28}$$

Therefore, in the steady-state, we have:

$$\overline{u} = \frac{\rho}{\eta}$$

$$\overline{y} = \frac{(\alpha+\gamma)(\mu+\beta_K)+\rho}{(1-\alpha-\gamma)}$$

$$\left(\overline{z}\right)^{\gamma}\left(\overline{x}\right)^{\alpha} = \frac{\mu+\rho+\beta_K}{\left(\frac{\rho}{\eta}\right)^{\alpha}(1-\alpha-\gamma)} \tag{29}$$

$$\left(\overline{z}\right)^{\gamma} = \frac{\eta\left(\beta_K - \beta_H\right)}{\left[\eta(1-\alpha-\gamma)\left(\overline{u}\right)^{\alpha}\left(\overline{x}\right)^{\alpha} - \alpha\left(\overline{u}\right)^{\alpha-1}\left(\overline{x}\right)^{\alpha-1}\right]}$$

That is:

---

[11] If this condition were not checked, it would always be preferable to invest in only one type of capital (physical or human). We look at a steady-state equilibrium where both forms of capital are *essential* in goods production (*Y*). See Barro and Sala-i-Martin (2004, p. 28) for a formal definition of *essentiality* within a neoclassical production function.

$$\bar{u} = \frac{\rho}{\eta}$$

$$\bar{y} = \frac{(\alpha + \gamma)(\mu + \beta_K) + \rho}{(1 - \alpha - \gamma)}$$

$$\bar{x} = \frac{\alpha(\mu + \rho + \beta_K)}{\rho(1 - \alpha - \gamma)(\mu + \rho + \beta_H)} \tag{30}$$

$$\bar{z} = \left[\frac{\eta^\alpha (\mu + \rho + \beta_H)}{\alpha \rho^{\alpha-1} (\bar{x})^{\alpha-1}} \right]^{\frac{1}{\gamma}}$$

The Jacobian matrix associated with the 4 x 4 system of differential equations in (14) is the following:

$$\begin{pmatrix} -z^\gamma u^\alpha x^\alpha \alpha & x & -\dfrac{xz^\gamma \gamma u^\alpha x^\alpha}{z} & x\left(-\eta - \dfrac{z^\gamma u^\alpha x^\alpha \alpha}{u}\right) \\[2ex] -\dfrac{y(\alpha+\gamma)z^\gamma u^\alpha x^\alpha \alpha}{x} & y & -\dfrac{y(\alpha+\gamma)z^\gamma \gamma u^\alpha x^\alpha}{z} & -\dfrac{y(\alpha+\gamma)z^\gamma u^\alpha x^\alpha \alpha}{u} \\[2ex] -\dfrac{z^{\gamma+1}u^\alpha x^\alpha \alpha}{x} & z & -z^\gamma \gamma u^\alpha x^\alpha & -\dfrac{z^{\gamma+1}u^\alpha x^\alpha \alpha}{u} \\[2ex] 0 & \dfrac{u(1-\alpha-\gamma)}{\alpha-1} & 0 & \eta u \end{pmatrix}$$

Evaluating the *Jacobian* at the steady-state values of intensive variables $x(t)$, $z(t)$ and $y(t)$ and control $u(t)$ given in (30) implies that the determinant is equal to zero. Moreover, out of the four eigenvalues, one equals zero and another equals $\rho > 0$. Since at least one real eigenvalue is positive, by Theorem 25.5 of Simon and Blume (1994) we can conclude that the steady-state equilibrium is unstable.

4. THE STOCHASTIC MODEL

We now move to the analysis of the following stochastic model:

$$\max_{C(t),u(t)} \mathrm{E} \int_0^{+\infty} \ln(C(t)) e^{-\rho t} dt \tag{31}$$

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subject to:

$$\overset{\bullet}{K}(t) = A(t)^{\gamma}\left(u(t)H(t)\right)^{\alpha}K(t)^{1-\alpha-\gamma} - \beta_K K(t) - C(t) \tag{32}$$

$$\overset{\bullet}{H}(t) = \eta\left(1-u(t)\right)H(t) - \beta_H H(t)$$

(33)

$$dA(t) = \mu A(t)\,dt + \sigma A(t)\,dW(t), \qquad \sigma > 0 \tag{34}$$

and the initial conditions $K(0) = K_0$, $H(0) = H_0$ and $A(0) = A_0$.

Let J be the value function associated to this stochastic optimization problem. The Hamilton-Jacobi-Bellman (HJB) equation can be written as:

$$
\begin{aligned}
0 &= \max_{C,u}\left[\ln(C) - \rho J + J_K \overset{\bullet}{K} + J_H \overset{\bullet}{H} + J_A\mu + \frac{J_{AA}\sigma^2 A^2}{2}\right] \\
&= \max_{C,u}\left[\ln(C) - \rho J + J_K\left(A^{\gamma}(uH)^{\alpha}K^{1-\alpha-\gamma} - \beta_K K - C\right) + J_H\left(\eta(1-u)H - \beta_H H\right) + J_A\mu + \frac{J_{AA}\sigma^2 A^2}{2}\right]
\end{aligned}
\tag{35}
$$

where J_K, J_H, J_A and J_{AA} are the partial derivatives of J with respect to the relevant variables. The first order conditions associated with the problem (35) lead to:

$$C^* = \frac{1}{J_K} \tag{36}$$

and

$$u^* = \frac{K^{\frac{1-\alpha-\gamma}{1-\alpha}}}{H}\left[\frac{\alpha A^{\gamma}J_K}{\eta J_H}\right]^{\frac{1}{1-\alpha}} \tag{37}$$

By substituting these expressions into (35) we get:

$$
0 = -\ln(J_K) - \rho J + J_K\left(A^{\gamma}K^{\frac{1-\alpha-\gamma}{1-\alpha}}\left(\frac{\alpha A^{\gamma}J_K}{\eta J_H}\right)^{\frac{1}{1-\alpha}-1} - \beta_K K\right) - 1 + J_H\left(\eta H - \eta K^{\frac{1-\alpha-\gamma}{1-\alpha}}\left(\frac{\alpha A^{\gamma}J_K}{\eta J_H}\right)^{\frac{1}{1-\alpha}} - \beta_H H\right) +
$$
$$
J_A\mu + \frac{J_{AA}\sigma^2 A^2}{2}
$$

(38)

We now look for a solution of this form:

$$J(H,K,A) = f\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\alpha+\gamma}{\alpha}}}{H}\right) + \frac{\alpha}{\rho(\alpha+\gamma)}\ln(H) + g(A) \tag{39}$$

where f and g are two unknown functions to be determined. After some algebraic computations (see Appendix B), the equation (38) can be split into the following ordinary differential equations:

$$-\ln\left(\left(\frac{\gamma+\alpha}{\alpha}\right)f'(x)\right) - \rho f(x) - \frac{\gamma}{\gamma+\alpha}\ln(x) + (\gamma+\alpha)\left(\frac{1}{\alpha}-1\right)f'(x)x\left[\frac{\eta^{-1}(\gamma+\alpha)f'(x)}{-f'(x)x+\frac{\alpha}{\rho(\alpha+\gamma)}}\right]^{\frac{1}{1-\alpha}-1} -$$

$$\beta_K\left(\frac{\alpha+\gamma}{\alpha}\right)f'(x)x - 1 + \left(-f'(x)x+\frac{\alpha}{\rho(\alpha+\gamma)}\right)(\eta-\beta_H) - \frac{\gamma\mu}{\alpha}f'(x)x + \left(\frac{\sigma\gamma}{\sqrt{2\alpha}}\right)^2 f''(x)x^2 -$$

$$\frac{\sigma^2\gamma}{2\alpha}\left(-\frac{\gamma}{\alpha}-1\right)f'(x)x = 0 \tag{40}$$

and

$$\left(\frac{\gamma}{\gamma+\alpha}\right)\ln(y) - \rho g(y) + g'(y)\mu y + \frac{g''(y)\sigma^2 y^2}{2} = 0 \tag{41}$$

where $x \equiv \dfrac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}$ and $y \equiv A$.

For a numerical simulation, we continue to use the following parameter values: $\rho = 4/100$, $\alpha = \gamma = 1/3$, $\eta = 12/100$, $\beta_H = 6/100$, $\beta_K = 5/100$, $\mu = 2/100$. Moreover, we set $\sigma = 0.0148$. This value has been recently suggested by Francis *et al.* (2008). The behavior of $f(x)$ against x is shown in Figure 4,
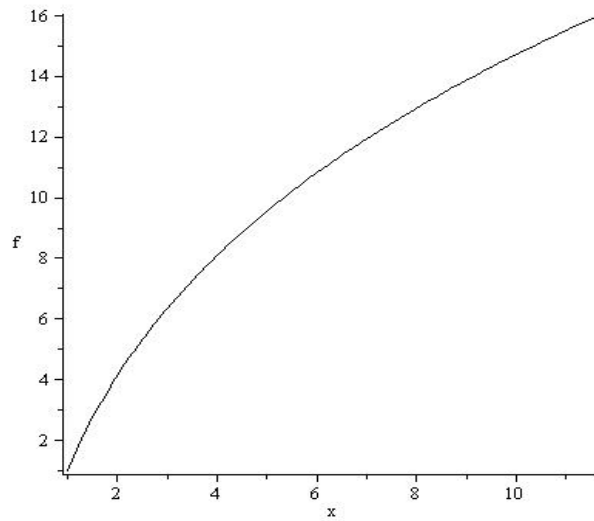
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Figure 4: $f(x)$ against x

while the plot $xf'(x)$ against x is shown in Figure 5.
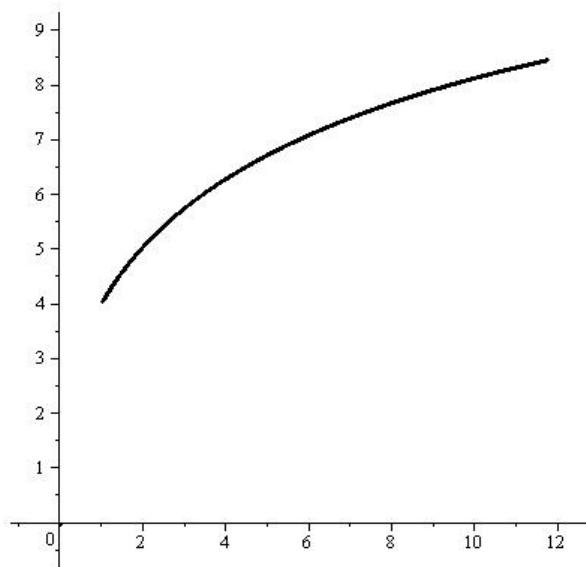

Figure 5: $xf'(x)$ against x

The values of C^* and u^* can be rewritten in terms of f and x as follows:

$$C^* = \frac{\alpha KH}{(\gamma + \alpha) xf'(x)} \qquad (42)$$

and

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$$u^* = \left[\frac{\eta^{-1}(\alpha+\gamma)x^{1-\alpha}f'(x)}{\dfrac{\alpha}{\rho(\alpha+\gamma)} - xf'(x)} \right]^{\frac{1}{1-\alpha}}$$

(43)

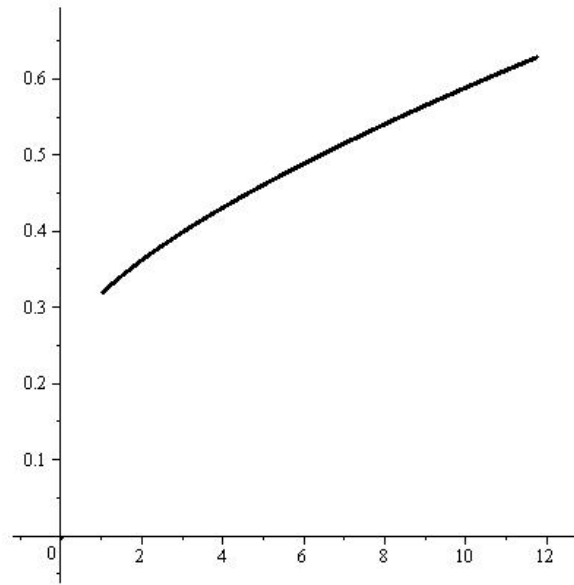The behavior of u^* as a function of x is shown in Figure 6

Figure 6: $u^*(x)$ against x

The following table summarizes the behavior of C^* and u^* following an increase of A:

$A\uparrow$	$x\downarrow$	$xf'(x)\downarrow$	$u^*(x)\downarrow$	$C^*(x)\uparrow$

As one can easily see from the table when a positive perturbation of A occurs, u^* decreases and C^* increases The intuition behind this result is the following: if the increase in A is positive then, *ceteris paribus*, the variation in the marginal productivities of both forms of capital in goods production (K and H_Y), respectively

$$\frac{\partial Y}{\partial K} = (1-\alpha-\gamma)\left(\frac{A}{K}\right)^{\gamma}\left(\frac{uH}{K}\right)^{\alpha}$$

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and

$$\frac{\partial Y}{\partial H_Y} = \alpha \left(\frac{A}{K} \right)^{\gamma} \left(\frac{K}{uH} \right)^{1-\alpha}$$

will be positive. Therefore, there will be an incentive to accumulate human capital faster (u^* decreases). At the same time, given the increase in factor-inputs productivities and in the level of technology as well, there will be a likely increase in total output and therefore in C^*.

5. SUMMARY AND CONCLUDING REMARKS

This paper has embedded an *Uzawa-Lucas*-type supply function of human capital into the framework traced by the model of Mankiw *et al.* (1992). Thus, we extended the latter model both by considering the investment in education as an economic activity relatively intensive in human capital and by endogenizing the rate at which agents save and invest in physical capital accumulation. With respect to the first model (Uzawa-Lucas) we considered the possibility that the technology level might grow over time, though at an exogenous rate.

In the first part of the article we postulated that technology grows deterministically and studied the local dynamics of the model around its steady-state equilibrium. The first order conditions of the intertemporal problem we have analyzed led to a system of four nonlinear differential equations. We split the analysis into two separate steps. In the first, by aggregating some of the intensive key-variables involved in the model, we considered a system of three differential equations. We found that the equilibrium is a saddle-point. In the second one, by making use of an arbitrage condition, we solved the system of equations in its original dimension and proved that in this case the equilibrium is unstable.

In the second part of the paper, instead, we allowed for random technology shocks driven by a geometric Brownian motion. We developed the analysis through the *Hamilton-Jacobi-Bellman* (HJB) equation. First of all we reduced the HJB partial differential equation to a system of nonlinear separated differential equations. Next, by using Maple 12 we showed a numerical simulation of the solutions. We used it to discuss

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the effects of technology shocks on the optimal policies of consumption (C^*) and the sectoral distribution of human capital (u^*). We found that, following a positive variation in technology, C^* increases and u^* decreases. The reason is that a positive technology noise increases marginal productivities of inputs used into production and, therefore, fosters both consumption and human capital investment.

For future research it would be interesting to analyze how the results we obtained in this paper (both in its deterministic and stochastic sections) might ultimately change in the presence of endogenous, rather than exogenous, technological progress driven by human capital. In this case skills would be employed not only to produce goods and to accumulate new human capital (as we have been arguing in the present work) but also to advance further the level of technology, an activity definitely human capital intensive.

Appendix A

In this appendix we prove formally the set of results written in equation (25) in the body text. For convenience, we re-write below the system (14) in the steady-state equilibrium (*i.e.*, with $\dot{x}_t = \dot{y}_t = \dot{z}_t = \dot{u}_t = 0$):

$$0 = \eta\left(1-\overline{u}\right) - \beta_H - \left(\overline{z}\right)^{\gamma}\left(\overline{u}\right)^{\alpha}\left(\overline{x}\right)^{\alpha} + \beta_K + \overline{y} \tag{A1}$$

$$0 = -\left(\alpha+\gamma\right)\left(\overline{z}\right)^{\gamma}\left(\overline{u}\right)^{\alpha}\left(\overline{x}\right)^{\alpha} - \rho + \overline{y} \tag{A2}$$

$$0 = \mu - \left(\overline{z}\right)^{\gamma}\left(\overline{u}\right)^{\alpha}\left(\overline{x}\right)^{\alpha} + \beta_K + \overline{y} \tag{A3}$$

$$0 = \left(\frac{1}{\alpha-1}\right)\left[-\gamma\mu - \left(\alpha+\gamma\right)\beta_K + \left(1-\alpha-\gamma\right)\overline{y} - \eta\alpha + \eta\left(\alpha-1\right)\overline{u} + \alpha\beta_H\right] \tag{A4}$$

Above we continue to assume that in the steady-state x_t, y_t and z_t are positive for each t and that $u_t \in \left(0;1\right)$. We denoted by $\overline{x}$, $\overline{y}$, $\overline{z}$ and $\overline{u}$ the steady-state values of x_t, y_t, z_t and u_t, respectively.

From (A1):

$$\left(\overline{z}\right)^{\gamma}\left(\overline{u}\right)^{\alpha}\left(\overline{x}\right)^{\alpha} - \beta_K - \overline{y} = \eta\left(1-\overline{u}\right) - \beta_H. \tag{A1'}$$

Instead, by using (A3):

$$\left(\overline{z}\right)^{\gamma}\left(\overline{u}\right)^{\alpha}\left(\overline{x}\right)^{\alpha}-\beta_{K}-\overline{y}=\mu. \tag{A3'}$$

If we combine (A1') and (A3') we get:

$$\eta\left(1-\overline{u}\right)-\beta_{H}=\mu. \tag{A5}$$

Plugging (A5) into (A4), after easy computations, yields:

$$\overline{y}=\frac{\left(\alpha+\gamma\right)\left(\eta+\beta_{K}-\beta_{H}\right)}{\left(1-\alpha-\gamma\right)}+\eta\overline{u}. \tag{A6}$$

Use now (A2):

$$\overline{y}=\left(\alpha+\gamma\right)\left(\overline{z}\right)^{\gamma}\left(\overline{u}\right)^{\alpha}\left(\overline{x}\right)^{\alpha}+\rho. \tag{A2'}$$

Plugging (A3') into (A2') in the end leads to:

$$\overline{y}=\frac{\left(\alpha+\gamma\right)\left(\mu+\beta_{K}\right)+\rho}{\left(1-\alpha-\gamma\right)}. \tag{A7}$$

Equalization of (A6) and (A7), and using (A5), implies:

$$\overline{u}=\frac{\rho}{\eta}. \tag{A8}$$

Given $\overline{u}$, (A5) delivers:

$$\mu=\eta-\rho-\beta_{H} \tag{A9}$$

This is the restriction on some of the key-parameters of the model that must be met in order for the equations-system (14) to have a steady-state equilibrium where $\dot{x}_{t}=\dot{y}_{t}=\dot{z}_{t}=\dot{u}_{t}=0$.

From (A1'):

$$\left(\overline{z}\right)^{\gamma}\left(\overline{x}\right)^{\alpha}=\frac{\left[\eta\left(1-\overline{u}\right)-\beta_{H}+\beta_{K}+\overline{y}\right]}{\left(\overline{u}\right)^{\alpha}}. \tag{A10}$$

Using (A7), (A8) and (A9) into (A10) leads to:

$$\left(\overline{z}\right)^{\gamma}\left(\overline{x}\right)^{\alpha}=\frac{\left(\eta-\beta_{H}\right)+\beta_{K}}{\left(\dfrac{\rho}{\eta}\right)^{\alpha}\left(1-\alpha-\gamma\right)}=\frac{\mu+\rho+\beta_{K}}{\left(\dfrac{\rho}{\eta}\right)^{\alpha}\left(1-\alpha-\gamma\right)}. \;\blacksquare \tag{A10'}$$

Appendix B

In this appendix we derive equations (40) and (41) in the main text.

By easy computations, we have:

$$J_{H}=-f^{'}\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H^{2}(t)}+\frac{\alpha}{\rho(\gamma+\alpha)H(t)} \tag{B1}$$

$$J_K = \left(\frac{\gamma+\alpha}{\alpha}\right) f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma}{\alpha}}(t)}{H(t)} \tag{B2}$$

$$J_A = -\frac{\gamma}{\alpha} f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)\frac{A^{-\frac{\gamma}{\alpha}-1}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)} + g'(A(t)) \tag{B3}$$

$$J_{AA} = \left(\frac{\gamma}{\alpha}\right)^2 f''\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)\left(\frac{A^{-\frac{\gamma}{\alpha}-1}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)^2$$

$$-\frac{\gamma}{\alpha}\left(-\frac{\gamma}{\alpha}-1\right) f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)\frac{A^{-\frac{\gamma}{\alpha}-2}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)} + g''(A(t)) \tag{B4}$$

By substituting these derivatives into 38 in the text leads to:

$$0 = -\ln\left(\left(\frac{\gamma+\alpha}{\gamma+\alpha-\gamma}\right) f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K(t)^{\frac{\gamma+\alpha}{\alpha}}}{H(t)}\right)\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma}{\alpha}}(t)}{H(t)} + \right.$$
$$\left. - \right)$$

$$-\rho\left(f\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right) + \frac{\alpha}{\rho(\gamma+\alpha)}\ln(H(t)) + g(A(t))\right) +$$

$$J_k K\left(\left[\frac{\eta^{-1}(\gamma+\alpha) f'\left(\frac{A^{-\frac{\gamma}{\alpha}}(t)K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)}{-f'\left(\frac{A^{-\frac{\gamma}{\alpha}}(t)K(t)^{\frac{\gamma+\alpha}{\alpha}}}{H(t)}\right)\frac{A^{-\frac{\gamma}{\alpha}}(t)K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)} + \frac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}-1} - \beta_K - 1 +\right)$$

$$J_H H\left(\eta - \eta\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}\left[\frac{\eta^{-1}(\gamma+\alpha) f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)}{-f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right)\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)} + \frac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}} - \beta_H\right)$$

$$+J_A\mu A + \frac{J_{AA}\sigma^2 A^2}{2}.$$

Now let $x \equiv \dfrac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}$ and $y \equiv A(t)$. We get

$$0 = -\ln\left(\left(\frac{\gamma+\alpha}{\alpha}\right)f^{'}(x)\right) - \rho f(x) - \frac{\gamma}{(\gamma+\alpha)}\ln(x) +$$

$$\left(\frac{\gamma+\alpha}{\alpha}\right)f'(x)x\left(\left[\frac{\eta^{-1}(\gamma+\alpha)f^{'}(x)}{-f^{'}(x)x + \dfrac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}-1} - \beta_K\right) - 1 +$$

$$\left(-f'(x)x + \frac{\alpha}{\rho(\gamma+\alpha)}\right)\left(\eta - \eta x\left[\frac{\eta^{-1}(\gamma+\alpha)f^{'}(x)}{-f^{'}(x)x + \dfrac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}} - \beta_H\right)$$

$$-\frac{\gamma\mu}{\alpha}f'(x)x + \left(\frac{\sigma\gamma}{\sqrt{2}\alpha}\right)^2 f''(x)x^2 - \frac{\sigma^2\gamma}{2\alpha}\left(-\frac{\gamma}{\alpha}-1\right)f'(x)x$$

$$+\left(\frac{\gamma}{\gamma+\alpha}\right)\ln(y) - \rho g(y) + g'(y)\mu y + \frac{g''(y)\sigma^2 y^2}{2}.$$

We can then split this equation into two differential equations to be solved separately in the variables x and y. We get

$$0 = -\ln\left(\left(\frac{\gamma+\alpha}{\alpha}\right)f^{'}(x)\right) - \rho f(x) - \frac{\gamma}{(\gamma+\alpha)}\ln(x) +$$

$$\left(\frac{\gamma+\alpha}{\alpha}\right)f'(x)x\left(\left[\frac{\eta^{-1}(\gamma+\alpha)f^{'}(x)}{-f^{'}(x)x + \dfrac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}-1} - \beta_K\right) - 1 +$$

$$\left(-f'(x)x + \frac{\alpha}{\rho(\gamma+\alpha)}\right)\left(\eta - \eta x\left[\frac{\eta^{-1}(\gamma+\alpha)f^{'}(x)}{-f^{'}(x)x + \dfrac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}} - \beta_H\right)$$

$$-\frac{\gamma\mu}{\alpha}f'(x)x + \left(\frac{\sigma\gamma}{\sqrt{2}\alpha}\right)^2 f''(x)x^2 - \frac{\sigma^2\gamma}{2\alpha}\left(-\frac{\gamma}{\alpha}-1\right)f'(x)x.$$

and

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$$0 = \left(\frac{\gamma}{\gamma + \alpha} \right) \ln(y) - \rho g(y) + g'(y)\mu y + \frac{g''(y)\sigma^2 y^2}{2} . \tag{B5}$$

It is easy to prove that the solution to equation (B5) is

$$g(y) = \frac{1}{\rho} \left(\frac{\gamma}{1-\alpha} \ln(y) + \frac{\mu y}{\rho(1-\alpha)} - \frac{\gamma\sigma^2}{2\rho(1-\alpha)} . \right)$$

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