Università degli Studi di Milano

Economics

Year 2009 *Paper* 28

Transitional Dynamics in a Growth Model with Government Spending, Technological Progress and Population Change

Alberto Bucci
University of Milan

Massimo Florio
University of Milan

Davide La Torre
University of Milan

Transitional Dynamics in a Growth Model with Government Spending, Technological Progress and Population Change

Abstract

Abstract This paper extends public spending-based growth theory along three directions: we assume that exogenous and constant technological progress does exist and that both population change and the ratio of government expenditure to income follow a logistic trajectory. By focusing on the choices of a benevolent social planner we find that, if the inverse of the intertemporal elasticity of substitution in consumption is sufficiently high, the ratio of consumption to private physical capital converges towards zero when time goes to infinity. Through two examples we see that, depending on the form of the underlying aggregate production function and on whether, for given production function, technological progress equals zero or a positive constant, our model may or may not yield an asymptotic balanced growth path (ABGP) equilibrium. When there is no exogenous technological progress, an equilibrium where population size, the ratio of government spending to total income and the ratio of consumption to private physical capital are all constant does exist and the equilibrium is a saddle point. In case of positive technological progress numerical simulations show that the model still exhibits an ABGP equilibrium.

Transitional Dynamics in a Growth Model with Government Spending, Technological Progress and Population Change

Alberto BUCCI
Massimo FLORIO*
Davide LA TORRE

University of Milan
Department of Economics, Business and Statistics

Abstract

This paper extends public spending-based growth theory along three directions: we assume that exogenous and constant technological progress does exist and that both population change and the ratio of government expenditure to income follow a logistic trajectory. By focusing on the choices of a benevolent social planner we find that, if the inverse of the intertemporal elasticity of substitution in consumption is sufficiently high, the ratio of consumption to private physical capital converges towards zero when time goes to infinity. Through two examples we see that, depending on the form of the underlying aggregate production function and on whether, for given production function, technological progress equals zero or a positive constant, our model may or may not yield an asymptotic balanced growth path (ABGP) equilibrium. When there is no exogenous technological progress, an equilibrium where population size, the ratio of government spending to total income and the ratio of consumption to private physical capital are all constant does exist and the equilibrium is a saddle point. In case of positive technological progress numerical simulations show that the model still exhibits an ABGP equilibrium.

KEY-WORDS: Economic Growth; Logistic Process; Dynamics; Population Change; Government Expenditure; Technological Progress

JEL CLASSIFICATION: O40; O41; H50

* ***Corresponding author***: Massimo Florio, Department of Economics, Business and Statistics, State University of Milan, Via Conservatorio 7, I-20122 Milan (Italy). Tel.: ++39-(0)2-50321.510; Fax: ++39-(0)2-50321.505; E-mail: massimo.florio@unimi.it. A previous version of this paper was presented at the *First Milan Workshop on Dynamics, Optimal Growth and Population Change: Theory and Applications* held at the Department of Economics, Business and Statistics of the University of Milan (September 18-19, 2008). We thank all the participants at this workshop for their insightful comments and remarks. Responsibility for any remaining errors and/or omissions lies exclusively with the authors.

1. Introduction

The neoclassical growth literature (Solow, 1956 and Swan, 1956) has long ago emphasized the importance of such factors as physical capital accumulation, changes in labor-force availability and exogenous technological progress in driving economic growth (both in the short and long-run). More recently, the new growth theory has made clear that other variables like purposeful R&D investment, human capital accumulation, population dynamics, institutions, and government spending also play a relevant role in increasing people's wealth and living standards.[1]

The contribution of our paper consists in modeling simultaneously, within the same theoretical framework, the dynamics of three growth-drivers: technological progress, population change, and the increase of productive public expenditure as a share of GDP. In other words, we combine in a consistent analytical structure three different hypothesis concerning, respectively, the cumulative nature of technical change, the self-limiting nature of population growth, and the logistic behavior of the public spending to GDP ratio over time.

Since the pioneering work of Barro (1990) and Barro and Sala-i-Martin (1992), it is widely recognized that government's purchases of goods and services can affect economic growth. Indeed, it is found[2] that in the long-run consumption, physical capital and output all grow at a common (constant) rate determined, among others, by the constant level of technology and of the labor force (strong scale effect)[3] and by the constant ratio of public spending to GDP. With respect to this strand of the literature our paper introduces three important novelties. First of all we assume that the level of technology might increase exogenously and exponentially over time. Secondly, we revive the Malthusian conjecture (Malthus, 1798) that there can exist an upper limit to population growth by modeling such a variable as a logistic process. Finally, we adopt a new view about the dynamics of the ratio of public expenditure to GDP and explicitly model the time-evolution of this ratio as a logistic process, as well.

While the use of a logistic-like function for population change is not new,[4] the logistic model assumption in describing the dynamics of the ratio of public expenditure to aggregate income is new in public economics and can be justified on strong theoretical and empirical grounds. Florio and Colautti (2005) were the first to introduce this hypothesis. In their work, they observe that over one hundred years the G/Y ratio not only does change significantly in countries like US, UK, France, Germany, and Italy, but also that it follows a *S*-shaped trajectory over time. This fact implies that the time derivative of G/Y is first increasing and then decreasing. According to them, this evolution can be explained by the simultaneous presence of an elasticity of the demand of public services to income (the Wagner's Law) greater than unity and of a brake represented by the excess burden of taxation (the Pigou's effect). Thus, they conclude that under some conditions, namely balanced budget and offsetting changes in income growth and in the elasticity of the demand for public services to income, the trajectory of the ratio of public spending to aggregate income may well be modeled as a logistic process. In their paper, however, Florio and Colautti (2005) did not include the dynamics of the G/Y ratio within a growth model, which represents instead the main novelty of the present work.

[1] Elegant surveys of the different approaches to the so-called Endogenous Growth Theory are offered, among others, by Acemoglu (2009), Aghion and Howitt (2009), Barro and Sala-i-Martin (2004), Helpman (2004) and Lucas (2002).
[2] See Barro and Sala-i-Martin (2004), p.221, equation (4.42).
[3] Empirical evidence (Jones, 2005) appears to reject the strong scale effect.
[4] The first analytical treatment is due to Verhulst (1838), who showed that the size of population is asymptotically constant under logistic growth for any initial positive value of population.

After assuming a standard Cobb-Douglas aggregate production function, augmented with public spending, we show through formal examples and numerical simulations that the solution to the intertemporal optimization problem faced by a social planner, being constrained by the accumulation of private physical capital and the exogenous dynamics of technology, population and public spending (as a share of GDP), may be represented by an asymptotic balanced growth path equilibrium.

In more detail, we organize the paper as follows. In section 2 we discuss the motivation behind our work and the conceptual framework we are going to use. Section 3 describes the model economy and presents the social planner's optimization problem under a generic, concave, individual, instantaneous utility function. In Section 4 we focus on the case of CIES (constant intertemporal elasticity of substitution) preferences and obtain the necessary first order conditions to the social planner's problem. In Section 5 we analyze the dynamics of the model under different assumptions on the inverse of the intertemporal elasticity of substitution in consumption and prove that, if this parameter is higher than a threshold value, then the ratio of aggregate consumption to aggregate private physical capital converges towards zero when time goes to infinity. By using two different examples we also show that, depending on the form of the underlying aggregate production function and on whether, for given aggregate production function, technological progress equals zero or a positive constant, our model may or may not yield an asymptotic balanced growth path equilibrium. In the same section we also see that, under a given set of parameters, the Barro (1990) model - in which the ratio G/Y is supposed to be always constant over time - can be interpreted as the *"limit"* of our approach when $t \to +\infty$. In Section 6 we focus on the special case where there is no exogenous technological progress and prove formally that an equilibrium in which population size, the ratio of government spending to total income and the one of consumption to private physical capital are all constant does exist and is stable. In Section 7 we turn to the more general case of positive, constant and exogenous technological progress and present numerical simulations of the dynamic behavior of some model's key endogenous variables under parameter values either dictated by existing empirical estimates or suggested by previous theoretical papers. The numerical simulations show that in its more general possible formulation the model exhibits, indeed, an asymptotic balanced growth path equilibrium. Section 8 concludes, summarizes and proposes possible paths for future theoretical research.

2. Motivation

In this section we justify in more detail how and why we model the dynamics of technology (A), the labor-force (L) and the ratio of government spending to aggregate output (G/Y) in the framework of our model. Indeed, A, L, and G/Y are all constant in the seminal papers by Barro (1990) and Barro and Sala-i-Martin (1992). This is clearly an unrealistic assumption and therefore we depart from it along three different directions.

Concerning the time-evolution of A, since in the rest of the paper we do not model the way new ideas are discovered and developed, we stick to neoclassical growth theory and assume that this variable changes over time at a constant and exogenous rate.[5]

The second difference with the two afore-mentioned works has to do with the way we model the population dynamics. Malthus (1798) was among the first to point out the existence of two distinct phases in the evolution of world population. The first phase is the one in which there is an explosion of the fertility rate (largely because of income growth) and, hence, of the population growth rate. The distinctive trait of the second phase, instead, is represented by a general tendency of the population growth rate to slow down (mainly because of the increase in the mortality rate, in turn induced by the competition across

[5] R&D-based endogenous growth models have already made clear that inventing new ideas becomes harder and harder over time and that one important factor that increases R&D difficulty is represented by the size of the market (Dinopoulos and Segerstrom, 1999). For a deeper discussion about the so-called *R&D difficulty-index* see, among others, the excellent surveys by Dinopoulos and Thompson (1999), Dinopoulos and Sener (2006) and Jones (2005).

individuals for the relatively scarce output of productive land). This idea was later on formalized as a logistic process by Verhulst (1838), Pearl and Reed (1920), Lotka (1925) and Volterra (1931). According to up-to-date demographic forecasts (United Nations, 2000), the world population annual growth rate is expected to fall gradually from 1.8% (1950-2000) to 0.9% (2000-2050), before reaching a value of 0.2% between the years 2050 and 2100. As stated by the same study, the world population will stabilize at a level of about eleven billion people by 2200. Thus, even from an empirical point of view, it seems reasonable to model population size as following a logistic process. If we assume that the work-intensity per person in the population equals one, then population and the aggregate labor-force (L) do coincide. Hence, in what follows we shall assume that L exhibits a logistic-type behavior over time.

The last, and probably the most important, difference with respect to Barro (1990) and Barro and Sala-i-Martin (1992) concern our assumption on the dynamics of G/Y. In our model G is understood as productive government expenditure, including public investment and the provision of public services. In this context we consider government expenditure (G) as net of depreciation. Unlike Barro (1990) and Barro and Sala-i-Martin (1992), we assume that this ratio follows a logistic process, as well. Considering the ratio of government spending to aggregate output (G/Y) constant is unattractive because over the last century we observe a sustained and significant dynamics of this ratio. For instance, in the United States (general government, including federal and states expenditures) G/Y was, according to Musgrave (1969) and Maddison (1995), in the range 0.068-0.079 around 1900 and it is well above 0.30 around 2000. This is nearly a fivefold increase over one hundred years. The corresponding increase was threefold in France and Italy, fourfold in Germany and UK. There is, however, clear evidence that the process of expansion of G/Y has slowed down since at least the 1980s and has almost stopped after 2000.

To explain why there could be a sustained demand for government services, one should think of public expenditures as complementary with other factor-inputs. As an example, public expenditure on education and health positively affects human capital investment, public infrastructures increase firms' productivity, law and order protect intellectual property rights and, hence, spur knowledge accumulation. One of the earliest contributions to the study of the long-term trend of public services expenditure is Wagner (1894), which prompted a huge flow of literature reviewed by Peacock and Scott (2000). Under the so-called *"Wagner's law"*, public services are considered as a bundle of goods with elasticity to aggregate income greater than one. This idea is consistent with several recent empirical analyses (see Tanzi and Schuknecht, 2000 for a review). Florio and Colautti (2005) prove that the combined action of the *"Wagner's law"* and the *"Pigou's effect"* (excess burden of taxation – Pigou, 1947)[6], together with other specific assumptions on the parameter values, lead to an equation describing the dynamics over time of the ratio of public expenditure to GDP of the following type:

$$\frac{d}{dt}\left(\frac{G_t}{Y_t}\right)=\mu\left(\frac{G_t}{Y_t}\right)-\gamma\left(\frac{G_t}{Y_t}\right)^2,$$

where μ and γ are given parameters[7]. Clearly, this is the equation of a logistic function. In the same paper, Florio and Colautti (2005), by looking at five countries (France, Germany, Italy, UK and US) for more than one century (1870-1990), also show that representing the G/Y ratio as a logistic process fits data better than an exponential process[8] (the only exception being Italy, a country with a notoriously high public debt).

Modeling the dynamics of G/Y as a logistic process has a number of attractive features. The most important one for empirical analysis is probably represented by the fact that the two parameters μ and γ have the property that their ratio (μ/γ) gives the upper limit of G/Y when $t\to+\infty$. Thus, if we could estimate these two parameters (together with a set of other country-specific variables) we might explicitly predict the exact behavior of this ratio both across countries and over time.

[6] This effect is quadratic in the tax rate (*T/Y*) and under balnced budget *G/Y = T/Y*.
[7] The parameters are respectively related to the elasticity to income of the demand for government services and to the elasticity to their tax-price.
[8] See also Sideris (2007) for Greece.

In this paper we explore the dynamics of a growth model in which there exist deterministic constraints represented by the demographic trends, the technological progress and the time-evolution of the ratio of public expenditure to GDP. By using both numerical examples and simulations, we see that the model may yield an asymptotic balanced growth path equilibrium and that the Barro's (1990) solution can be obtained (asymptotically) as a special case of our more general framework.

3. The Model

Consider a closed economy in which a homogeneous final good (*Y*) is produced by using private physical capital (hereafter simply physical capital, *K*), labor (*L*) and government's expenditure (*G*). As already mentioned, we assume that the work-intensity per person in the population equals one[9] and postulate an aggregate production function (the production function for a single representative firm) taking the Cobb-Douglas form:

$$Y_t = A_t K_t^{\alpha} L_t^{\theta} G_t^{1-\alpha-\theta}, \qquad \alpha \in (0,1), \qquad \theta \in [0,1), \tag{1}$$

where A_t is a variable representing the state of technology at time t.

Depending on the value of parameters α and θ, Eq. (1) suggests that, for given A and G, at the aggregate level there can be either decreasing ($0 < \alpha + \theta < 1$), or increasing ($\alpha + \theta > 1$), or else constant ($\alpha + \theta = 1$) returns to scale in the rival inputs (K and L) and that, for given L and A, instead, there can be either decreasing or at most constant returns to scale to physical capital (K) and public expenditure (G), jointly considered (*i.e.*, $0 < 1 - \theta \leq 1$). Finally, for given level of technology, the aggregate production function of Eq. (1) reveals constant returns to scale to labor, physical capital and public expenditure together. Therefore, Eq. (1) appears sufficiently general to be able to embed two different versions of the same type of endogenous growth model with public spending (Barro, 1990 and Barro and Sala-i-Martin, 1992). Both versions of the model take the index of the level of technology and the aggregate labor force constant.

In Barro (1990) the aggregate production function is:[10]

$$Y_t = A K_t^{\alpha} G_t^{1-\alpha}.$$

In Eq. (1), imposing $\theta = 0$ (for each $L_t > 0$) and $\alpha \in (0,1)$ yields the Barro's (1990) aggregate production function with a non-constant index of the technology (time-varying A), *i.e.*:

$$Y_t = A_t K_t^{\alpha} G_t^{1-\alpha}.$$

In a companion paper, Barro and Sala-i-Martin (1992) have analyzed the specific case in which the governmental services are subject to congestion (in this situation the public goods are rival but, to some extent, non-excludable). In its simplest formulation[11] the idea that some public services might be subject to congestion is formalized simply by re-writing the aggregate production function of Barro (1990) as:

$$Y_t = A K_t \left(\frac{G_t}{K_t} \right)^{1-\alpha}.$$

Hence, the production process is now visibly of the *AK-type* modified by the term that involves the *congestion effect* in public services (G_t / K_t). In particular, this formulation suggests that, as long as the

---

[9] Hence, population and aggregate labor-force are the same.

[10] Barro and Sala-i-Martin (2004, pp. 220-223) provide a simplified version of Barro (1990). If we normalize L to one in their Eq. (4.39), p. 221, we obtain exactly the aggregate production function written in the text.

[11] Barro and Sala-i-Martin (1992), Eq. (14), p.650.

government decides to maintain a given level of congestion (G/K), aggregate output displays constant returns to scale to private inputs (K). On the contrary, for given K, a decrease in G, by reducing the total amount of public services available at the aggregate level and, thus, increasing the congestion of these services, lowers Y. Clearly, we are able to obtain a production function displaying congestion effects in public goods and services by setting, again, $\theta = 0$ (for each $L_t > 0$) and $\alpha \in (0,1)$ in our model (Eq. 1). In this case the aggregate production function would read as:

$$Y_t = A_t K_t \left(\frac{G_t}{K_t}\right)^{1-\alpha} = A_t K_t f\left(\frac{G_t}{K_t}\right), \qquad f'(\cdot) > 0, \qquad f''(\cdot) < 0,$$

which is the aggregate production function of Barro and Sala-i-Martin (1992)[12] with a non-constant A.

Once produced, output (Y) can be either consumed (C), or used for expenditure in public goods and services (G) or invested in private capital (K). Thus, the law of motion of the aggregate stock of physical capital is given by:

$$\dot{K}_t = Y_t - \beta_K K_t - C_t - G_t, \tag{2}$$

where $\beta_K \in (0,1)$ is the exogenous, instantaneous depreciation rate of K.

As we said before, we depart from the two different versions of the same endogenous growth model with public spending mentioned above (*i.e.,* Barro, 1990 and Barro and Sala-i-Martin, 1992) for the following three assumptions. As for the level of technology, we postulate a simple exponential exogenous growth process:

$$\dot{A}_t = g_A A_t, \tag{3}$$

whereas we posit a logistic-type function for both population changes:

$$\dot{L}_t = nL_t - d\left(L_t\right)^2, \qquad d > 0, \qquad n - dL_0 > 0, \tag{4}$$

and the time-evolution of the ratio of aggregate public expenditure to total income (G_t / Y_t):

$$\frac{d}{dt}\left(\frac{G_t}{Y_t}\right) = \mu\left(\frac{G_t}{Y_t}\right) - \gamma\left(\frac{G_t}{Y_t}\right)^2, \qquad \gamma > 0, \qquad \mu - \gamma\left(\frac{G_0}{Y_0}\right) > 0. \tag{5}$$

The law of motion in Eq. (3) assumes that the technology can grow forever without bounds at a constant and exogenous rate, g_A. The law of motion of population size (Eq. 4) allows replicating the Malthusian conjecture that the latter variable becomes asymptotically constant - *i.e.*, population growth goes smoothly to zero when $t \to +\infty$. In the same equation n and d are two parameters and $L_0 > 0$ is the size of population at $t = 0$. Finally, while the ratio G/Y, although endogenously determined, is taken as constant in Barro (1990) and Barro and Sala-i-Martin (1992), in Eq. (5) we explicitly consider the case where the time-derivative of this ratio follows a logistic-type trajectory. As in the previous equation, in (5) μ and γ are two parameters and $(G_0/Y_0) > 0$ is the ratio of public expenditure to total income at $t = 0$.

Mathematically, modeling the dynamics of G/Y and population size as logistic processes has also the advantage, as we shall see in a moment, of contributing to make the dynamics of the underlying economic model much more interesting.

[12] See also Barro and Sala-i-Martin (2004), p. 223, Eq. (4.45). According to them: *"…The formulation assumes that G has to rise in relation to total output, Y… We could have assumed alternatively that G had to rise in relation to aggregate private capital, K, in order to raise the quantity of services. The results would be essentially the same under this specification"* (p. 223).

In what follows we analyze the choices of a benevolent social planner who seeks to maximize, under constraints, the intertemporal utility of a society including a population of L identical and infinitely-lived agents. The dynamic, intertemporal optimization problem that the benevolent social planner faces is:

$$\underset{\{c_t\}_{t=0}^{\infty}}{\text{Max}}\ U \equiv \int_0^{\infty} u(c_t) L_t e^{-\rho t} dt\,, \qquad u'(c) > 0\,, \qquad u''(c) < 0 \tag{6}$$

subject to:

$$\begin{cases} \dot{K}_t = Y_t - \beta_K K_t - C_t - G_t \\ \dot{A}_t = g_A A_t \\ \dot{L}_t = nL_t - d(L_t)^2 \\ \dfrac{d}{dt}\left(\dfrac{G_t}{Y_t}\right) = \mu\left(\dfrac{G_t}{Y_t}\right) - \gamma\left(\dfrac{G_t}{Y_t}\right)^2 \\ K_0 > 0;\quad A_0 > 0;\quad L_0 > 0;\quad (G_0/Y_0) > 0 \end{cases} \tag{7}$$

The constraints in Eq. (7) describe, respectively, the evolution over time of the private capital stock (K), the index of the state of technology (A), the population size (L) and the ratio of public expenditure to GDP (G/Y). In eq. (6), U is the intertemporal social welfare function, $u(c)$ is the instantaneous utility of each single member of the society, ρ is the constant subjective discount rate, L_t denotes population size at time *t* and, finally, K_0, A_0, L_0 and (G_0/Y_0) are the given initial conditions of the state variables. Note that the instantaneous utility of any single agent depends solely on the stream of real per capita consumption ($c_t \equiv C_t / L_t$).

The system of dynamic constraints (7) can be reduced to only one differential equation in K_t. In order to do this, first of all we recognize that the second differential equation is linear with solution given by:

$$A_t = A_0 e^{g_A t}\,. \tag{8}$$

The third and fourth differential equations are of the *Bernoulli-type* and can be solved in closed-form, as well. In particular, the solution to the third differential equation is:

$$L_t = \frac{n}{d + \left(\dfrac{n}{L_0} - d\right)e^{-nt}}\,, \tag{9}$$

while the solution to the forth differential equation is represented by:

$$\frac{G_t}{Y_t} = \frac{\mu}{\gamma + \left(\dfrac{\mu Y_0}{G_0} - \gamma\right)e^{-\mu t}}\,. \tag{10}$$

Combining Eqs. (10) and (1) implies:

$$G_t = \frac{\mu}{\gamma + \left(\dfrac{\mu Y_0}{G_0} - \gamma\right)e^{-\mu t}} A_t K_t^{\alpha} L_t^{\theta} G_t^{1-\alpha-\theta}\,. \tag{11}$$

According to Eq. (11) G_t is a function of A_t, L_t and K_t since:

$$G_t = \left[\frac{\mu}{\gamma + \left(\frac{\mu Y_0}{G_0} - \gamma \right) e^{-\mu t}} A_t K_t^{\alpha} L_t^{\theta} \right]^{\frac{1}{\alpha+\theta}} \tag{12}$$

After plugging Eq. (12) into the production function Y, the differential equation in K in (7) becomes:

$$\dot{K}_t = \left[1 - \frac{\mu}{\gamma + \left(\frac{\mu Y_0}{G_0} - \gamma \right) e^{-\mu t}} \right] \left[\frac{\mu}{\gamma + \left(\frac{\mu Y_0}{G_0} - \gamma \right) e^{-\mu t}} \right]^{\frac{1-\alpha-\theta}{\alpha+\theta}} A_t^{\frac{1}{\alpha+\theta}} K_t^{\frac{\alpha}{\alpha+\theta}} L_t^{\frac{\theta}{\alpha+\theta}} - \beta_K K_t - C_t \, . \tag{13}$$

Finally, defining a new variable ξ_t,

$$\xi_t \equiv \left(1 - \frac{G_t}{Y_t} \right) \left(\frac{G_t}{Y_t} \right)^{\frac{1-\alpha-\theta}{\alpha+\theta}} = \left[1 - \frac{\mu}{\gamma + \left(\frac{\mu Y_0}{G_0} - \gamma \right) e^{-\mu t}} \right] \left[\frac{\mu}{\gamma + \left(\frac{\mu Y_0}{G_0} - \gamma \right) e^{-\mu t}} \right]^{\frac{1-\alpha-\theta}{\alpha+\theta}} , \tag{14}$$

and inserting Eqs. (8) and (9) into (13) yields:

$$\dot{K}_t = \xi_t \left(A_0 e^{g_A t} \right)^{\frac{1}{\alpha+\theta}} K_t^{\frac{\alpha}{\alpha+\theta}} \left[\frac{n}{d + \left(\frac{n}{L_0} - d \right) e^{-nt}} \right]^{\frac{\theta}{\alpha+\theta}} - \beta_K K_t - C_t \, . \tag{15}$$

Before proceeding with the assumptions on the preferences, we now introduce the definitions of *Balanced Growth Path* (BGP) and *Asymptotic Balanced Growth Path* (ABGP) equilibrium.

DEFINITION: *Balanced Growth Path* (BGP) and *Asymptotic Balanced Growth Path* (ABGP) *Equilibrium*

A BGP equilibrium is a long-run equilibrium in which all variables depending on time grow at constant exponential rates and in which this constant growth could continue forever. A BGP equilibrium will be said non-degenerate if all growth rates are strictly positive.

An ABGP equilibrium is a long-run equilibrium in which the growth rates of all variables depending on time approach constant values when $t \to +\infty$. An ABGP equilibrium will be said non-degenerate if all these constant values are strictly positive.

4. Preferences

In this section we develop the analysis under the assumption that the instantaneous utility function of each member of this economy takes the CIES (constant intertemporal elasticity of substitution) form:

$$u(c_t) = \frac{c_t^{1-\varphi} - 1}{1-\varphi}, \qquad \varphi > 0, \tag{16}$$

where φ is the inverse of the intertemporal elasticity of substitution in consumption. Thus, the problem (6) can be recast as:

$$\underset{\{c_t\}_{t=0}^{\infty}}{\text{Max}} \ \text{U} \equiv \int_0^{\infty} \frac{c_t^{1-\varphi} - 1}{1-\varphi} L_t e^{-\rho t} dt \tag{17}$$

Replacing in the above-equation per-capita consumption $(c_t \equiv C_t / L_t)$, in turn, implies:

$$\underset{\{C_t\}_{t=0}^{\infty}}{\text{Max}} \ \text{U} \equiv \int_0^{\infty} \frac{C_t^{1-\varphi} - L_t^{1-\varphi}}{(1-\varphi)} L_t^{\varphi} e^{-\rho t} dt \tag{17'}$$

where C_t denotes aggregate consumption at time *t*.[13]

Hence, the original dynamic optimization problem that a benevolent social planner faces (Eqs. 6 and 7) in the end can be written as:

$$\underset{\{C_t\}_{t=0}^{\infty}}{\text{Max}} \ \text{U} \equiv \int_0^{\infty} \frac{C_t^{1-\varphi}}{(1-\varphi)} L_t^{\varphi} e^{-\rho t} dt \tag{18}$$

subject to: $$\dot{K}_t = \xi_t \left(A_0 e^{g_A t}\right)^{\frac{1}{\alpha+\theta}} K_t^{\frac{\alpha}{\alpha+\theta}} \left[\frac{n}{d + \left(\frac{n}{L_0} - d\right) e^{-nt}}\right]^{\frac{\theta}{\alpha+\theta}} - \beta_K K_t - C_t .$$

The Hamilton function (H) associated with the latest problem is:

$$\text{H}_t = \frac{C_t^{1-\varphi}}{(1-\varphi)} L_t^{\varphi} e^{-\rho t} + \lambda_{K_t} \dot{K}_t ,$$

where λ_{K_t} is the co-state variable related to K_t and $\dot{K}_t$ is given by Eq. (15). The necessary first order conditions (henceforth FOCs) are:

$$\begin{cases} \dfrac{\partial \text{H}_t}{\partial C_t} = 0 \quad \Rightarrow \quad \left(\dfrac{L_t}{C_t}\right)^{\varphi} e^{-\rho t} - \lambda_{Kt} = 0 \\[2ex] \dfrac{\partial \text{H}_t}{\partial K_t} = -\dot{\lambda}_{Kt} \quad \Rightarrow \quad \lambda_{Kt} \left[\dfrac{\alpha}{\alpha+\theta} \xi_t \left(A_0 e^{g_A t}\right)^{\frac{1}{\alpha+\theta}} K_t^{\frac{\alpha}{\alpha+\theta}-1} \left(\dfrac{n}{d + \left(\frac{n}{L_0} - d\right) e^{-nt}}\right)^{\frac{\theta}{\alpha+\theta}} - \beta_K\right] = -\dot{\lambda}_{Kt} \end{cases} \tag{19}$$

[13] Since a constant has no influence on the maximization, the problem (17') is equivalent to:
$$\underset{\{C_t\}_{t=0}^{\infty}}{\text{Max}} \ \text{U} \equiv \int_0^{\infty} \frac{C_t^{1-\varphi}}{(1-\varphi)} L_t^{\varphi} e^{-\rho t} dt$$

together with the dynamic constraint (Eq. 15), the boundary conditions ($A_0, K_0, L_0, G_0 / Y_0$) and the transversality condition: $\lim_{t\to\infty} \lambda_{K_t} K_t = 0$.

5. Dynamics

We have already showed (paragraph 3) that the system of four dynamic constraints (respectively in K_t, A_t, L_t and G_t / Y_t) can be reduced to only one constraint (the differential equation in K_t, see Eqs. 7 and 15). Hence, in problem (18) aggregate consumption (C_t) represents the control variable and aggregate physical capital (K_t) is now the sole state variable. In this section we characterize the dynamics of the ratio C_t / K_t[14] under different values of the inverse of the intertemporal elasticity of substitution in consumption (φ). At this aim, note that the first equation in (19) yields:

$$\varphi \frac{\dot{L}_t}{L_t} - \varphi \frac{\dot{C}_t}{C_t} - \rho = \frac{\dot{\lambda}_{K_t}}{\lambda_{K_t}} \tag{20}$$

Using (20), (15) and the second equation in (19) leads to:

$$\frac{\dot{C}_t}{C_t} = -\frac{\beta_K}{\varphi} + \frac{\alpha}{\varphi(\alpha+\theta)}\left(\frac{\dot{K}_t}{K_t} + \beta_K + \frac{C_t}{K_t}\right) + \frac{\dot{L}_t}{L_t} - \frac{\rho}{\varphi}. \tag{21}$$

We can now state the following theorem.

THEOREM 1:

Suppose $\alpha \in (0,1)$, $\theta \in (0,1)$ *and* $\varphi = \dfrac{\alpha}{\alpha+\theta} < 1$. *In this case the differential equation (21) has a solution given by:*

$$\frac{C_t}{K_t} = \frac{L_t}{L_0} \frac{e^{-\left[\frac{\rho(\alpha+\theta)+\beta_K\theta}{\alpha}\right]t}}{\left[\frac{K_0}{C_0} - \int_0^t \frac{L_s}{L_0} e^{-\left[\frac{\rho(\alpha+\theta)+\beta_K\theta}{\alpha}\right]s} ds\right]} \tag{22}$$

Proof: See Appendix B.

[14] Suppose that $\theta = 0$ and $\alpha = 1$ (for each $L_t > 0$ and $G_t > 0$). If these two parameter values were simultaneously possible, then the aggregate production function (1) would become: $Y_t = A_t K_t$. Thus, under this parameterization, our model would be also able to obtain an AK model with a non-constant index of the technology (time-varying A). As in the AK model, we can study the transitional dynamics by analyzing the behaviour of the ratio C_t / K_t over time (see Barro and Sala-i-Martin, 2004, p. 208).

Eq. (22) gives the optimal time-path of the ratio of aggregate consumption to the total stock of private capital (C_t / K_t) under the assumption that φ equals $\frac{\alpha}{\alpha+\theta}$. From Theorem 1 the following corollary follows:

COROLLARY 1:

Assume $\alpha, \theta \in (0,1)$, $\varphi = \frac{\alpha}{\alpha+\theta} < 1$ *and* $\frac{K_0}{C_0} > \int_0^{+\infty} \frac{L_s}{L_0} e^{-\left[\frac{\rho(\alpha+\theta)+\beta_K \theta}{\alpha}\right]s} ds$. *Then* $\lim_{t\to+\infty} \frac{C_t}{K_t} = 0$.

Proof: The proof immediately follows from Eq. (22). ■

We are now interested in characterizing the dynamics of C_t / K_t in the case in which $\varphi \geq \frac{\alpha}{(\alpha+\theta)}$. Indeed, using econometric and/or calibration methods, Hall (1988), Patterson and Pesaran (1992), Guvenen (2005), Favero (2005) and Harashima (2005), among others, have made clear that the intertemporal elasticity of substitution in consumption ($1/\varphi$) is lower that one, implying $\varphi > 1$. Therefore, analyzing what happens when $\varphi \geq \frac{\alpha}{(\alpha+\theta)}$ seems to be both more realistic and in line with economic theory.

When $\varphi \geq \frac{\alpha}{(\alpha+\theta)}$, since A_t, C_t, K_t and L_t are positive variables, if the pair (C_t, K_t) is a solution of the FOCs then it has to satisfy the following differential inequality:

$$\frac{\dot{C}_t}{C_t} \leq -\frac{\beta_K}{\varphi} + \frac{\dot{K}_t}{K_t} + \beta_K + \frac{C_t}{K_t} + \frac{\dot{L}_t}{L_t} - \frac{\rho}{\varphi},$$

that is:

$$\left(\frac{C_t}{K_t}\right)' \leq \frac{C_t}{K_t}\left(-\frac{\beta_K}{\varphi} + \beta_K + \frac{\dot{L}_t}{L_t} - \frac{\rho}{\varphi}\right) + \left(\frac{C_t}{K_t}\right)^2,$$

where $\left(\frac{C_t}{K_t}\right)' \equiv \frac{d}{dt}\left(\frac{C_t}{K_t}\right)$.

By applying a classical comparison theorem (see Szarski, 1967, Ch. III, p. 44), we may conclude that when $\varphi \geq \frac{\alpha}{(\alpha+\theta)}$ the following inequality must be satisfied:

$$\frac{C_t}{K_t} \leq \frac{L_t}{L_0} \frac{e^{\left[-\frac{\beta_K}{\varphi}+\beta_K-\frac{\rho}{\varphi}\right]t}}{\left[\frac{K_0}{C_0} - \int_0^t \frac{L_s}{L_0} e^{\left[-\frac{\beta_K}{\varphi}+\beta_K-\frac{\rho}{\varphi}\right]s} ds\right]} \tag{23}$$

Eq. (23) implies that $\frac{C_t}{K_t}$ is bounded from above when $\varphi \geq \frac{\alpha}{(\alpha+\theta)}$. We can now enunciate the following two corollaries.

COROLLARY 2:

Assume $\alpha, \theta \in (0,1)$, $\frac{\alpha}{\alpha+\theta} \leq \varphi < 1 + \frac{\rho}{\beta_K}$ *and* $\frac{K_0}{C_0} > \int_0^{+\infty} \frac{L_s}{L_0} e^{\left[-\frac{\beta_K}{\varphi}+\beta_K-\frac{\rho}{\varphi}\right]s} ds$. *Then* $\lim_{t\to+\infty} \frac{C_t}{K_t} = 0$.

Proof: The proof immediately follows from equation 23. ■

COROLLARY 3*:*

Assume $\alpha, \theta \in (0,1)$, $\frac{\alpha}{\alpha+\theta} \leq \varphi < 1 + \frac{\rho}{\beta_K}$ *and* $\frac{K_0}{C_0} > \int_0^{+\infty} \frac{L_s}{L_0} e^{\left[-\frac{\beta_K}{\varphi}+\beta_K-\frac{\rho}{\varphi}\right]s} ds$ *and let* (C_t, K_t) *be a solution of the FOCs. Then* $g_K \equiv \lim_{t\to+\infty} \frac{\dot{K}_t}{K_t}$ *is finite if and only if* $g_C \equiv \lim_{t\to+\infty} \frac{\dot{C}_t}{C_t}$ *is finite and the following relation does hold:*

$$g_C = \frac{-\beta_K\theta + \alpha g_K - \rho(\alpha+\theta)}{\varphi(\alpha+\theta)}.$$

Proof: From Corollary 2 we get that $\lim_{t\to+\infty} \frac{C_t}{K_t} = 0$. Now the thesis follows from Eq. (21). ■

Corollary 1 and 2 together imply that, if (as we would expect from empirical evidence) the inverse of the intertemporal elasticity of substitution satisfies the inequality $\frac{\alpha}{\alpha+\theta} \leq \varphi < 1 + \frac{\rho}{\beta_K}$, then the condition

$$\lim_{t\to+\infty} \frac{C_t}{K_t} = 0$$

is checked. We now present two examples showing, however, that compliance with this condition does not necessarily imply the existence of an asymptotic balanced growth path equilibrium. For the sake of simplicity, both examples will assume logarithmic preferences ($\varphi = 1$) and differ solely in the shape of the underlying aggregate production function, Y, that is whether technological progress is missing or not.

EXAMPLE 1

Consider the case in which the instantaneous utility is logarithmic ($\varphi = 1$), and the parameters of the aggregate production function are such that $\theta = 0$ and $\alpha \in (0,1)$. Eq. (1) can be written as:

$$Y_t = A_t K_t^{\alpha} G_t^{1-\alpha} = A_t K_t \left(\frac{G_t}{K_t}\right)^{1-\alpha}.$$

The equation above resembles the aggregate production function that Barro (1990) and Barro and Sala-i-Martin (1992) use, respectively, in their own versions of the growth model with productive government spending. The only difference with respect to their approach is that we postulate a non-constant index of the technology (A is time-varying). Under these two parameter values, the maximization problem becomes:

$$\underset{\{C_t\}_{t=0}^{\infty}}{\text{Max}}\ \text{U} \equiv \int_0^{\infty} \log(C_t) L_t e^{-\rho t} dt$$

$$\text{subject to:} \quad \dot{K}_t = \xi_t \left(A_0 e^{g_A t}\right)^{\frac{1}{\alpha}} K_t - \beta_K K_t - C_t .$$

and the FOCs are:

$$\begin{cases} \dfrac{\dot{C}_t}{C_t} = \dfrac{\dot{K}_t}{K_t} + \dfrac{C_t}{K_t} + \dfrac{\dot{L}_t}{L_t} - \rho \\ \dfrac{\dot{K}_t}{K_t} = \xi_t A_0 e^{\frac{g_A}{\alpha} t} - \beta_K - \dfrac{C_t}{K_t} \end{cases}$$

By using the results provided earlier, if (C_t, K_t) is a solution of the FOCs, then:

$$\frac{C_t}{K_t} = \frac{L_t}{L_0} \frac{e^{-\rho t}}{\left[\dfrac{K_0}{C_0} - \displaystyle\int_0^t \frac{L_s}{L_0} e^{-\rho s} ds\right]} .$$

Suppose that $\dfrac{K_0}{C_0} > \displaystyle\int_0^{+\infty} \frac{L_s}{L_0} e^{-\rho s} ds$. We can see that $\lim\limits_{t\to+\infty} \dfrac{C_t}{K_t} = 0$ even if $\lim\limits_{t\to+\infty} \dfrac{\dot{K}_t}{K_t} = \lim\limits_{t\to+\infty} \dfrac{\dot{C}_t}{C_t} = +\infty$. In this case the model would display no asymptotic balanced growth path equilibrium.

EXAMPLE 2

Consider now the same case as before, that is $\varphi = 1$, $\theta = 0$ and $\alpha \in (0;1)$, but with the further assumption that there is no technological progress ($g_A = 0$). If we normalize to one the initial level of technology ($A_0 = 1$) the aggregate production function becomes:

$$Y_t = K_t^{\alpha} G_t^{1-\alpha} = K_t \left(\frac{G_t}{K_t}\right)^{1-\alpha} .$$

Again, this production function is similar to those used by Barro (1990) and Barro and Sala-i-Martin (1992) in that it displays a constant index of the technology, set equal to one. It is possible to show that in this situation the solution is represented by an asymptotic balanced growth equilibrium. Indeed, we now have:

$$\lim_{t\to+\infty} \frac{\dot{C}_t}{C_t} = \xi^* - \beta_K - \rho \qquad \text{and} \qquad \lim_{t\to+\infty} \frac{\dot{K}_t}{K_t} = \xi^* - \beta_K ,$$

where

$$\xi^* \equiv \lim_{t\to+\infty} \xi_t = \left(1 - \frac{\mu}{\gamma}\right)\left(\frac{\mu}{\gamma}\right)^{\frac{1-\alpha}{\alpha}}$$

and

$$\frac{\mu}{\gamma} = \lim_{t \to +\infty} \frac{G_t}{Y_t}.$$

Therefore, the ratio G/Y is only asymptotically constant. In turn, this implies that:

$$\lim_{t \to +\infty} \frac{\dot{C}_t}{C_t} = \left(1 - \frac{G}{Y}\right)\left(\frac{G}{Y}\right)^{\frac{1-\alpha}{\alpha}} - \beta_K - \rho. \tag{24}$$

If we compare Eq. (24) with the corresponding result in Barro (1990) it is immediate to verify that the latter model, in which the ratio G/Y is supposed to be always constant over time, is the *"limit"* of our model when $t \to +\infty$ (see Appendix A for further details).

In brief, with $\varphi = 1$, $\theta = 0$, $\alpha \in (0;1)$ and $g_A = 0$, our model yields an asymptotic balanced growth path equilibrium. When $t \to +\infty$, our growth-rate solution approximates that of the basic Barro's (1990) model. Therefore, our framework can be seen as a generalization of the latter to the case of non-null (exponential) technological progress and logistic behavior over time of both population size and the ratio of public expenditure to GDP.

As a final comment, by comparing the two examples presented above, one can also emphasize the importance of technological progress in this class of growth models with public spending. In particular, we realize that with logarithmic preferences and with labor not entering the aggregate production function as an input, the absence of any disembodied technological progress (example 2) may yield an asymptotic balanced growth path equilibrium. In the next section we further develop this point.

6. Characterization of the *"No-Technological-Progress"* Equilibrium

In the previous section we demonstrated that the ratio $\frac{C_t}{K_t}$ converges towards zero when $t \to \infty$ and $\frac{\alpha}{\alpha+\theta} \le \varphi < 1 + \frac{\rho}{\beta_K}$. Moreover, the two previous examples have also showed that, depending on the shape of the aggregate production function and the fact that the constant and exogenous technological progress equals zero or not, the model can admit an asymptotic balanced growth path equilibrium.

The aim of this paragraph is to prove that, when the index of the technology is constant ($g_A = 0$), there exists an equilibrium in which, for each value of φ, population size (L), the ratio of government spending to total income (G/Y) and the ratio of consumption to physical capital (C/K) are constant as well, and that such constants are all positive.

At this aim we introduce the following new variables: $\Psi_t \equiv \frac{C_t}{K_t}$ and $\Sigma_t \equiv \xi_t A_t^{\frac{1}{\alpha+\theta}} K_t^{\frac{\alpha}{\alpha+\theta}-1} L_t^{\frac{\theta}{\alpha+\theta}}$.

By employing the two definitions of Ψ_t and Σ_t, Eqs. (21), (15), (8) and (9) and our assumptions on $\frac{d}{dt}\left(\frac{G_t}{Y_t}\right)$, $\dot{A}_t/A_t$ and $\dot{L}_t/L_t$ in (7), we end up with the following system of differential equations:

$$
\begin{cases}
\dfrac{\dot{\Psi}_t}{\Psi_t} = \left(\dfrac{\varphi-1}{\varphi}\right)\beta_K + \left[\dfrac{\alpha-\varphi(\alpha+\theta)}{\varphi(\alpha+\theta)}\right]\Sigma_t + \Psi_t + \dfrac{\dot{L}_t}{L_t} - \dfrac{\rho}{\varphi} \\
\dfrac{\dot{\Sigma}_t}{\Sigma_t} = \dfrac{\dot{\xi}_t}{\xi_t} + \dfrac{g_A}{(\alpha+\theta)} - \left(\dfrac{\theta}{\alpha+\theta}\right)\left[\Sigma_t - \beta_K - \Psi_t - \dfrac{\dot{L}_t}{L_t}\right] \\
\dfrac{\dot{A}_t}{A_t} = g_A \\
\dfrac{\dot{L}_t}{L_t} = n - dL_t \\
\dfrac{d}{dt}\left(\dfrac{G_t}{Y_t}\right) = \mu\left(\dfrac{G_t}{Y_t}\right) - \gamma\left(\dfrac{G_t}{Y_t}\right)^2
\end{cases}
$$

Using the definition of ξ_t (Eq. 14), one can show that:

$$
\frac{\dot{\xi}_t}{\xi_t} = -\frac{\left[\mu\dfrac{G_t}{Y_t} - \gamma\left(\dfrac{G_t}{Y_t}\right)^2\right]}{\left(1-\dfrac{G_t}{Y_t}\right)} + \left(\frac{1}{\alpha+\theta}-1\right)\left[\mu - \gamma\left(\frac{G_t}{Y_t}\right)\right].
$$

Hence, the previous system of differential equations can be recast as:

$$
\begin{cases}
\dfrac{\dot{\Psi}_t}{\Psi_t} = \left(\dfrac{\varphi-1}{\varphi}\right)\beta_K + \left[\dfrac{\alpha-\varphi(\alpha+\theta)}{\varphi(\alpha+\theta)}\right]\Sigma_t + \Psi_t + \dfrac{\dot{L}_t}{L_t} - \dfrac{\rho}{\varphi} \\
\dfrac{\dot{\Sigma}_t}{\Sigma_t} = -\dfrac{\left[\mu\dfrac{G_t}{Y_t} - \gamma\left(\dfrac{G_t}{Y_t}\right)^2\right]}{\left(1-\dfrac{G_t}{Y_t}\right)} + \left(\dfrac{1}{\alpha+\theta}-1\right)\left[\mu - \gamma\left(\dfrac{G_t}{Y_t}\right)\right] + \dfrac{g_A}{(\alpha+\theta)} - \left(\dfrac{\theta}{\alpha+\theta}\right)\left[\Sigma_t - \beta_K - \Psi_t - \dfrac{\dot{L}_t}{L_t}\right] \\
\dfrac{\dot{A}_t}{A_t} = g_A \\
\dfrac{\dot{L}_t}{L_t} = n - dL_t \\
\dfrac{d}{dt}\left(\dfrac{G_t}{Y_t}\right) = \mu\left(\dfrac{G_t}{Y_t}\right) - \gamma\left(\dfrac{G_t}{Y_t}\right)^2
\end{cases}
\quad (25)
$$

The following theorem states the existence and stability of an equilibrium for the system of differential equations (25).

THEOREM 2:

Assume $\alpha \in (0,1)$, $\theta \in (0,1)$ and $\varphi > 0$. Suppose also that there is no technological progress, $g_A = 0$. There exists an equilibrium in which, for each value of φ, population size (L), the ratio of government spending to total income (G/Y), the ratio of consumption to physical capital ($\Psi \equiv C/K$) and variable Σ equal, respectively, some positive constants. Moreover, such equilibrium is a saddle point.

Proof:

Solving the last system of differential equations with $g_A = 0$, $\dot{L}_t = 0$, $d(G_t/Y_t)/dt = 0$, $\dot{\Psi}_t = 0$ and $\dot{\Sigma}_t = 0$, after some algebra yields the following result:

$$\begin{cases} \Psi^* = \left(\dfrac{\alpha+\theta}{\alpha}\right)(\beta_K+\rho)-\beta_K \\ \Sigma^* = \left(\dfrac{\alpha+\theta}{\alpha}\right)(\beta_K+\rho) \\ L^* = \dfrac{n}{d} \\ \left(\dfrac{G}{Y}\right)^* = \dfrac{\mu}{\gamma} \end{cases}$$

It is easy to see that, as long as α, θ, β_K, μ, γ, d, n and ρ are positive, Ψ^*, Σ^*, L^* and $(G/Y)^*$ will also be positive and independent of φ.

To assess the stability of the equilibrium, we need to find the *Jacobian matrix* associated with the log-linearized system of differential equations in (25) and evaluate it at the equilibrium values Ψ^*, Σ^*, L^* and $(G/Y)^*$ written above. This produces:

$$\begin{bmatrix} \left(\dfrac{\alpha+\theta}{\alpha}\right)(\beta_K+\rho)-\beta_K & \left(\rho+\dfrac{\theta}{\alpha}(\beta_K+\rho)\right)\left(\dfrac{\alpha-\varphi(\alpha+\theta)}{\varphi(\alpha+\theta)}\right) & \cdots & \cdots \\ \dfrac{\theta}{\alpha}(\beta_K+\rho) & -\dfrac{\theta}{\alpha}(\beta_K+\rho) & \cdots & \cdots \\ 0 & 0 & -n & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix}$$

The eigenvalues of the Jacobian matrix are $\lambda_1 = -n$, $\lambda_2 = -\mu$, λ_3 and λ_4 with $\lambda_3 + \lambda_4 = \rho$ and

$$\lambda_1\lambda_2 = -\frac{\theta}{\alpha}(\beta_K+\rho)\left[\frac{\rho\alpha+\theta(\beta_K+\rho)}{\varphi(\alpha+\theta)}\right] < 0$$

It follows that one eigenvalue is positive and the others are negative and this implies the equilibrium is a saddle point (Simon and Blume (1994, Theorem 23.9)). ■

7. A numerical simulation

We now turn back to the most general possible version of the model (including positive, constant and exogenous technological progress) and present a numerical simulation of the dynamic behavior of some of its endogenous variables under a given set of parameter values. In choosing such values we draw either on existing empirical estimates or on baseline specifications coming from previous works. We use the following parameter-values:

- $\alpha = 1/3$. This is the conventional (see Denison, 1962; Maddison, 1982; Jorgenson *et al.*, 1987; Mankiw *et al.*, 1992) share of gross income accruing to a narrow concept of physical capital (structures and equipment);
- $(1-\alpha-\theta) = 0.46$. If we interpret $(1-\alpha-\theta)$ as the share of GDP allocated to government expenditures, then empirical evidence (Tanzi and Schuknecht, 2000, Table 1.1) for a sample of eleven OECD countries[15] over the time-period 1990-1999 suggests that the average value of such share is approximately equal to 46% ;
- $\theta = 0.2$. With $\alpha = 1/3$ and $(1-\alpha-\theta) = 0.46$ it follows that the share of gross income accruing to labor, θ, should be set to about 20%;
- $g_A = 2\%$. This parameter is obtained by comparing two different studies in the tradition of the so-called *growth-accounting*, respectively Christensen *et al.* (1980) and Jorgensen and Yip (2001).[16] The first paper covers Canada, France, Germany, Italy, Japan, Netherlands, UK and US for the time-period 1947-1973 and finds that the annual growth rate of TFP has been on average (both across countries and time) of about 2.7%. The second paper, instead, reports data for the same set of countries as before (except Netherlands), but covers a more recent period, 1960-1995. In this case the authors find than on average (across countries and time) TFP has grown solely by 1.3% per year. The value we give to g_A (*i.e.*, 2%) derives from a combination of the results obtained by the two above-mentioned studies;[17]
- $n = 0.0144$. The parameter n in the logistic function giving the change of population size over time represents the so-called *"growth rate coefficient"*. We attribute to this parameter the value of 0.0144, which is the average growth rate of the labor-force in the U.S. private business sector over the period 1948-1997 (Jones and Williams, 2000, Table 1, p.73). Such a value is not distant from the estimate of 0.019 obtained for the population of Great Britain from 1801 to 1971 (see Oliver, 1982, p. 360);
- $d = 0.0144/63$. In the same paper, Oliver (1982) also estimates the saturation level of the population of Great Britain (assumed to follow a logistic process) and finds that it is close to 63 million people. Accordingly, with this value and $n = 0.0144$, the last parameter of interest in the logistic curve for population (d) will equal 0.0144/63;

[15] The sample includes: United States, Australia, Canada, Denmark, France, Germany, Italy, Japan, Spain, Sweden, United Kingdom.
[16] See the discussion in Barro and Sala-i-Martin (2004, p. 439, Table 10.1).
[17] Aghion and Howitt (2009, Table 5.1, p. 109) show that for OECD countries (1960-2000) the average annual growth rate of TFP has been of 1.61%. It is worth mentioning here, however, that none of the works within the *growth accounting*-tradition mentioned thus far includes explicitly public expenditures as an input in the aggregate production function.

- $\varphi = 2$; $\rho = 0.04$; $\beta_K = 0.05$. Concerning the inverse of the intertemporal elasticity in consumption (φ), the subjective discount rate (ρ) and the instantaneous depreciation rate of physical capital (β_K) we follow the baseline specification of Mulligan and Sala-i-Martin (1993, p. 761);

- $\mu = 0.05$; $\gamma = 0.10$. These values are based on estimates in Florio and Colautti (2005). They report $\mu = 0.05$ for US, UK and Germany, 0.07 for France and 0.03 for Italy, while the limit of G/Y is on average slightly above 0.50. This implies $\gamma = 0.10$, because μ/γ is the upper limit of G/Y.

- Finally, we use the following values for the initial conditions: $L(0) = 1$, $A(0) = 1$, $K(0) = 1$ and $G_0 / Y_0 = 8.5\%$. In particular, the latter value is close to the average level of G/Y around 1900 in countries like United States, France, Italy, Germany and United Kingdom (as estimated by Florio and Colautti, 2005, based on a set of different data sources drawn from quantitative economic historians).

Inspection of the following figures clearly suggests that, under the parameterization discussed above, our more general model leads to an asymptotic balanced growth path equilibrium.

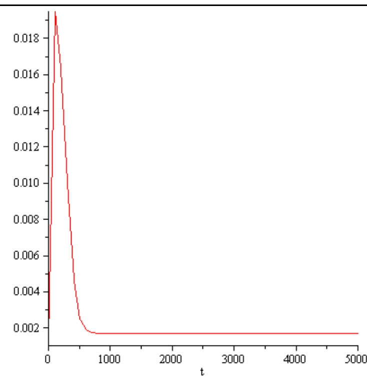

Figure 1: $\dfrac{\dot{C}_t}{C_t}$

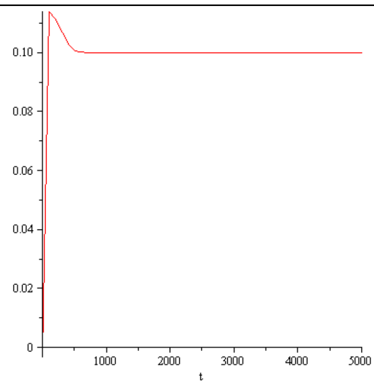

Figure 2: $\dfrac{\dot{K}_t}{K_t}$

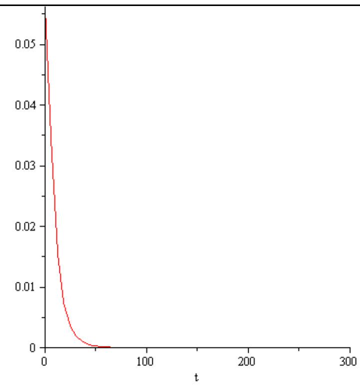

Figure 3: $\dfrac{C_t}{K_t}$

8. Concluding Remarks

Since the pioneering contributions of Barro (1990) and Barro and Sala-i-Martin (1992), it is widely recognized that public expenditure plays an important role in raising the economic potential of an economy. This paper contributes to this branch of growth literature by building an aggregative growth model that combines within the same framework three different assumptions that are absent in the first generation of public spending-based growth theories. First of all, we assume that the model economy never runs short of new ideas and that the rate at which new ideas are discovered is exogenous and remains constant over time (in other words, the variable reflecting the state of technology follows an exponential process). Moreover, we postulate a logistic-type evolution over time for both population size and the ratio of public expenditure to GDP. While the view that population size might follow a logistic behavior with respect to time is not new (as it goes back to Malthus (1798) and Verhulst (1838)), recent forecasts (United Nations, 2000) having the world as unit of analysis, confirm that the annual growth rate of population is expected to fall gradually until 2100 and that world population will stabilize at a level of about eleven billion people by 2200. Thus, not only theoretically but also empirically, it seems reasonable to model population size as following a logistic process. On the other hand, the use of such a process in modeling the behavior over time of public expenditure as a fraction of total GDP is wholly new in the context of growth theory, and is due to Florio and Colautti (2005). When this idea, that has solid theoretical and empirical grounds, is introduced in a deterministic growth framework where a benevolent social planner is assumed to maximize the intertemporal utility of a society including L identical and infinitely-lived agents, we find that the ratio of consumption to physical capital converges towards zero when time goes to infinity, provided that the inverse of the intertemporal elasticity of substitution in consumption is larger than a given threshold level. This requirement is consistent with the empirical evidence showing that the intertemporal elasticity of substitution in consumption is, indeed, lower than one. By using two different examples we also show that, depending on the form of the underlying aggregate production function and on whether, for given aggregate production function, technological progress equals zero or a positive constant, our model may or may not yield an asymptotic balanced growth path equilibrium. In particular, we see that with logarithmic preferences and in the absence of any technical change, our growth-rate solution approximates, when $t \to +\infty$, that of Barro (1990), in which the ratio G/Y is supposed to be always constant over time. When there is no exogenous technological progress at all, we formally prove that for each value of the intertemporal elasticity of substitution in consumption an equilibrium in which population size, the ratio of government spending to GDP and the ratio of consumption to physical capital are all constant. On the other hand, if technological progress is assumed to be positive, numerical simulations (using parameter values drawn either on existing empirical estimates or on previous works) show that the model still exhibits an asymptotic balanced growth path equilibrium.

While the model we have proposed in this paper is extremely simplified in many respects, it has definitely the advantage of being more realistic than Barro (1990) and Barro and Sala-i-Martin (1992), as it removes from these approaches the assumption of a constant population and a time-invariant G/Y ratio.

As for future research, we believe that our framework can be extended along different possible paths. We mention here just two of them. First of all, the stylized fact of increasing government expenditures over the last two centuries and the more recent fiscal restraint in many countries since the 1980s are both expression of a unique dynamic pattern, resulting from the combined effect of preferences for public services and welfare losses due to distortionary taxation. A more comprehensive approach would explicitly derive the key parameters of the public expenditure logistic function from the underlying optimization problem of the consumer-tax-payer.

A second extension would consist in introducing in the model endogenous, rather than exogenous, technological progress, that is a separate R&D sector that produces ideas for the whole economy. This would certainly go in the direction of having a model closer to real world in which nothing comes in the form of "*manna from heaven*".

APPENDIX A

In this appendix we compare our model (when $\varphi = 1$, $\theta = 0$, $\alpha \in (0,1)$ and $g_A = 0$) with Barro (1990). When $\theta = 0$ and the initial index of the technology is normalized to one ($A_0 = 1$), equation (1) in the main text becomes:

$$Y_t = K_t^{\alpha} G_t^{1-\alpha}. \tag{A1}$$

Note that this production function coincides exactly with that of Barro, 1990 (see Barro and Sala-i-Martin, 2004, p. 221, equation 4.39) once we set $A_t = L_t = 1$ for each t in that model.[18] Equation (A1) states that the only two inputs entering the production of the homogeneous final good (Y) are private physical capital (K) and public expenditure (G). Therefore, labor (L) is not an input into the production process.[19] With $L = 1$, per worker ($c \equiv C/L$) and aggregate (C) consumption are clearly the same. Moreover, with A also equal to one and logarithmic preferences ($\varphi = 1$), equation 4.42, p. 221, in Barro and Sala-i-Martin (2004) delivers:

$$\frac{\dot{C}}{C} = \alpha \left(\frac{G}{Y}\right)^{\frac{1-\alpha}{\alpha}} - \delta - \rho, \tag{A2}$$

where G/Y is the constant ratio of government purchases to GDP, δ is the instantaneous depreciation rate of private physical capital and ρ is the subjective discount rate. If G/Y is taken to be equal to $(1-\alpha)$, then it is showed (see Barro and Sala-i-Martin, 2004, p.222) that the decentralized growth-rate solution coincides with that of a social planner. With $G/Y = (1-\alpha)$, equation (A2) above becomes:

$$\frac{\dot{C}}{C} = \left[\alpha (1-\alpha)^{\frac{1-\alpha}{\alpha}} - \delta - \rho\right]. \tag{A3}$$

If we set $G/Y = (1-\alpha)$ in Eq. (24) in the body-text, we would get:

$$\lim_{t \to +\infty} \frac{\dot{C}_t}{C_t} = \left[\alpha (1-\alpha)^{\frac{1-\alpha}{\alpha}} - \beta_K - \rho\right], \tag{A4}$$

where $\beta_K = \delta$ represents the instantaneous depreciation rate of private physical capital. Equation (A4) states that the solution for the equilibrium growth rate of consumption that our model suggests in the specific case in which $\varphi = 1$, $\alpha \in (0,1)$, $\theta = 0$ and $g_A = 0$ coincides, when $t \to +\infty$, with that of the basic Barro's (1990) model when the ratio G/Y equals $1-\alpha$ (*i.e.*, when the decentralized and the social planner's growth-rate solutions do coincide).

It is worth emphasizing that, unlike Barro (1990) - where G/Y is postulated to be always constant - in line with empirical evidence we explicitly assume that such a ratio follows a logistic trajectory. It is exactly because of this assumption we have that only at the limit, *i.e.* when $t \to +\infty$, G/Y equals a constant given by the ratio μ/γ, where μ and γ are the parameters of the logistic process postulated for G/Y.

---

[18] In the simplified version of the Barro's model presented in Barro and Sala-i-Martin (2004, pp. 220-223) there is no disembodied technological progress and no growth in the aggregate labor force (*A* and *L* are given constants). If we normalize *A* and *L* to one, eq. 4.39 in Barro and Sala-i-Martin (2004, p.221) collapses to the one written in (A1).

[19] We can think of labor as included implicitly in the form of human capital and lumped together with private physical capital, K (see Rebelo, 1991 as an example).

APPENDIX B: Proof of *Theorem 1*

Since we assume $\varphi = \dfrac{\alpha}{\alpha+\theta}$ equation (21) in the main text can be rewritten as:

$$\frac{\dot{C}_t}{C_t} - \frac{\dot{K}_t}{K_t} = -\frac{\beta_K(\alpha+\theta)}{\alpha} + \beta_K + \frac{C_t}{K_t} + \frac{\dot{L}_t}{L_t} - \frac{\rho(\alpha+\theta)}{\alpha}. \qquad \text{(B1)}$$

By defining:

$$\left(\frac{C_t}{K_t}\right)' \equiv \frac{d}{dt}\left(\frac{C_t}{K_t}\right),$$

Eq. (B1), can be written as:

$$\left(\frac{C_t}{K_t}\right)' \cdot \frac{K_t}{C_t} = -\frac{\beta_K(\alpha+\theta)}{\alpha} + \beta_K + \frac{C_t}{K_t} + \frac{\dot{L}_t}{L_t} - \frac{\rho(\alpha+\theta)}{\alpha}. \qquad \text{(B2)}$$

As a final step, write (B2) as:

$$\left(\frac{C_t}{K_t}\right)' = \frac{C_t}{K_t}\left[\frac{C_t}{K_t} + \frac{\dot{L}_t}{L_t} - \frac{\rho(\alpha+\theta)+\theta\beta_K}{\alpha}\right]$$

which is a *Bernoulli-type* equation whose solution is:

$$\frac{C_t}{K_t} = \frac{L_t}{L_0}\frac{e^{-\left[\frac{\rho(\alpha+\theta)+\beta_K\theta}{\alpha}\right]t}}{\left[\frac{K_0}{C_0} - \int_0^t \frac{L_s}{L_0} e^{-\left[\frac{\rho(\alpha+\theta)+\beta_K\theta}{\alpha}\right]s} ds\right]} \blacksquare$$

REFERENCES

Acemoglu D. (2009). *Introduction to Modern Economic Growth*. Princeton, NJ: Princeton University Press.

Aghion P., Howitt P. (2009). *The Economics of Growth*. Cambridge, MA: MIT Press.

Barro R.J. (1990), "Government Spending in a Simple Model of Endogenous Growth", *Journal of Political Economy*, 98(5), Part 2, pp. S103-26.

Barro R.J., Sala-i-Martin X. (1992), "Public Finance in Models of Economic Growth", *Review of Economic Studies*, 59(4), pp. 645-61.

Barro R.J., Sala-i-Martin X. (2004). *Economic Growth* (Second Edition). Cambridge, MA: MIT Press.

Christensen L.R., Cummings D., Jorgenson, D.W. (1980), "Economic Growth, 1947-1973: An International Comparison". In J.W. Kendrick and B. Vaccara (eds.), *New Developments in Productivity Measurement and Analysis*, NBER Conference Report. Chicago: Chicago University Press.

Denison E.F. (1962), "Sources of Growth in the United States and the Alternatives Before Us", *Supplement Paper 13*. New York: Committee for Economic Development.

Dinopoulos E., Segerstrom P. (1999), "A Schumpeterian Model of Protection and Relative Wages", *American Economic Review*, 89(3), pp. 450-72.

Dinopoulos E., Sener F. (2007), "New Directions in Schumpeterian Growth Theory". In H. Hanusch and A. Pyka (eds.), *Edgar Companion to Neo-Schumpeterian Economics*, Cheltenham, UK: Edward Elgar.

Dinopoulos E., Thompson P. (1999), "Scale Effects in Schumpeterian Models of Economic Growth", *Journal of Evolutionary Economics*, 9(2), pp. 157-85.

Florio M., Colautti S. (2005), "A Logistic Growth Theory of Public Expenditures: A Study of Five Countries over 100 Years", *Public Choice*, 122(3-4), pp. 355-93.

Guvenen F. (2006), "Reconciling Conflicting Evidence on the Elasticity of Intertemporal Substitution: A Macroeconomic Perspective", *Journal of Monetary Economics*, 53(7), pp 1451-72.

Hall R.E. (1988), "Intertemporal Substitution in Consumption", *Journal of Political Economy*, 96(2), pp. 339-57.

Harashima T. (2005), "An Estimate of the Elasticity of Intertemporal Substitution in a Production Economy", University of Tsukuba (August), *mimeo*.

Helpman E. (2004). *The Mystery of Economic Growth*. Cambridge, Cambridge, MA: The Belknap Press of Harvard University Press.

Jones C.I. (2005), "Growth and Ideas". In P. Aghion and S.N. Durlauf (eds.), *Handbook of Economic Growth*, Volume 1B, Amsterdam: Elsevier (North-Holland), pp. 1063-1111.

Jones C.I., Williams J.C. (2000), “Too Much of a Good Thing? The Economics of Investment in R&D”, *Journal of Economic Growth*, 5(1), pp 65-85.

Jorgenson D.W., Gollop F.M., Fraumeni B.M. (1987). *Productivity and U.S. Economic Growth*. Cambridge, MA: Harvard University Press.

Jorgenson D.W., Yip E. (2001), “Whatever Happened to Productivity Growth?”. In E.R. Dean, M.J. Harper and C. Hulten (eds.), *New Developments in Productivity Analysis*. Chicago: Chicago University Press, pp. 205-46.

Lotka A.J. (1925). *Elements of Physical Biology*. Baltimore: Williams and Wilkins Company.

Lucas R.E. (2002). *Lectures on Economic Growth*. Cambridge, MA: Harvard University Press.

Maddison A. (1982). *Phases of Capitalist Development*. Oxford: Oxford University Press.

Maddison (1995). *Monitoring the World Economy, 1820–1992*. Paris: Development Centre Studies, OECD.

Malthus T.R. (1798). A*n Essay on the Principle of Population, as It Affects the Future Improvement of Society with Remarks on the Speculations of Mr. Godwin, M. Condorcet, and Other Writers*. London: J. Johnson (First Edition).

Mankiw N.G., Romer D., Weil D.N. (1992), “A Contribution to the Empirics of Economic Growth”, *Quarterly Journal of Economics*, 107(2), pp. 407-37.

Mulligan C.B., Sala-i-Martin X. (1993), “Transitional Dynamics in Two-Sector Models of Endogenous Growth”, *Quarterly Journal of Economics*, 108(3), pp 739-73.

Musgrave R.A. (1969). *Fiscal Systems*. New Haven, CT: Yale University Press.

Oliver F.R. (1982), “Notes on the Logistic Curve for Human Populations”, *Journal of the Royal Statistical Society*, Series A, 145(0), pp. 359-63, Part 3.

Patterson K.D., Pesaran B. (1992), “The Intertemporal Elasticity of Substitution in Consumption in the United States and the United Kingdom”, *Review of Economics and Statistics*, 74(4), pp 573-84.

Peacock A., Scott A. (2000), “The Curious Attraction of Wagner’s Law”, *Public Choice*, 102(1-2), pp 1-17.

Pearl R., Reed L.J. (1920), “On the Rate of Growth of the Population of the United States since 1790 and its Mathematical Representation”, *Proceedings of the National Academy of Sciences of the USA*, 6, pp. 275-88.

Pigou A.C. (1947). *A Study in Public Finance*. London: Macmillan (3rd Edition).

Rebelo S. (1991), “Long Run Policy Analysis and Long Run Growth”, *Journal of Political Economy*, 99(3), 500-21.

Sideris D. (2007), "Wagner's Law in 19th Century Greece: A Cointegration and Causality Analysis", *Bank of Greece, Working Paper No. 64* (December).

Simon C.P., Blume L. (1994). *Mathematics for Economists*. New York: W.W. Norton & Co.

Solow R.M. (1956), "A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics*, 70(1), pp. 65-94.

Swan T.W. (1956), "Economic Growth and Capital Accumulation", *Economic Record*, 32(63), pp. 334-61.

Szarski J. (1967). *Differential Inequalities*. Warszawa: PWN.

Tanzi V., Schuknecht L. (2000). *Public Spending in the 20th Century: A Global Perspective*. Cambridge, UK: Cambridge University Press.

United Nations (2000). *World Population Prospects: The 2000 Revision*. New York: United Nations Population Division.

Verhulst P.F. (1838), "Notice sur la Loi que la Population suit dans son Accroissement", *Correspondance Mathématique et Physique*, 10, pp. 113-21.

Volterra V. (1931), "Variations and Fluctuations of the Number of Individuals in Animal Species living Together". In R.N. Chapman (ed.), *Animal Ecology*, New York: McGraw-Hill, pp. 409-48.

Wagner A. (1894). *Grundlegung der Politischen Oekonomie*. Leipzig: C.F. Winter.