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PROOF ANALYSIS IN TEMPORAL LOGIC

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Abstract

The logic of time is one of the most interesting modal logics, and its importance is widely acknowledged both for philosophical and formal reasons. In this thesis, we apply the method of internalisation of Kripke-style semantics into the syntax of sequent calculus to the proof-theoretical analysis of temporal logics.

Sequent systems for different flows of time are obtained as modular extensions of a basic temporal calculus, through the addition of appropriate mathematical rules that correspond to the properties of temporal frames: a general and uniform treatment is thus achieved for a wide range of temporal logics. All the calculi enjoy remarkable structural properties, in particular are contraction and cut free.

Linear discrete time is analysed by means of two infinitary calculi. The first is obtained by means of a rule with infinitely many premises, and the second through a new definition of provability which admits, under certain conditions, derivation trees with infinite branches.

The first calculus enjoys the desired structural properties, but the presence of an infinitary rule is harmful for proof analysis. Two finitary systems are identified by replacing the infinitary rule with a weaker finitary rule, and by bounding the number of its premises, respectively. Corresponding, somehow complementary, conservativity results are proved with respect to adequate fragments of the original calculus.

The second calculus stems from a closure algorithm which exploits the fixed-point equations for temporal operators and gives saturated sets of closure formulas from a given formula. Finitisation is obtained in the form of an upper bound to the proof-search procedure, and decidability follows as a major consequence.

Acknowledgements

First and most importantly I want to thank Dr. Sara Negri, who has been a present and patient guide for my research. In 2005, Sara Negri supervised my Master Degree Thesis on proof analysis in the intuitionistic theory of apartness; the contents of that work were subsequently improved and published in Boretti and Negri (2006). In the same year, Sara Negri formulated sequent calculi for modal and non-classical logics through the internalisation of the relational semantics described in Section 1.4, and proposed me to dedicate my Ph.D. Thesis to the development of this methodology for the study of temporal logic. We started thus a constant collaboration, which culminated in a Semester of study and research at the Department of Philosophy of the University of Helsinki (January-June 2007), and produced several papers and manuscripts. In particular, a different version of the results of Chapter 3 have been previously presented in Boretti and Negri (2007), and Boretti and Negri (2008a). Portions of the work of Chapter 4 have been presented in Boretti and Negri (2008b).

I also wish to thank my supervisor, Prof. Miriam Franchella, who has trusted me and has guided my steps since the beginning of my studies in Logic at the Department of Philosophy of the University of Milan. In all this time, she has orientated my Ph.D. research, while bearing with my moments of discouragement: I will treasure her precious advises for my future work and choices.

Last, but not least, I wish to thank my parents and Roberto for their long-lasting support, and for having been close to me when I was far away. To them this thesis is dedicated.

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Introduction

According to Augustine of Hippo we know what time is, but we are not able to communicate our knowledge through an explicit explanation: "Quid est tempus? Si nemo ex me quaerat, scio; si quarenti explicare velim, nescio"[1]. Nonetheless, time has been deeply analysed in its different issues both for its own interest, and because many disciplines have to some extent to cope with it:

> Time is ubiquitous. Look to such diverse fields as literature and computers, ethics and physics, logic and rhetoric, philosophy and natural science. If you are studying any of these subjects, professionally or *con amore*, you are very likely to come across temporality as a crucial factor to your studies.
>
> For this reason, people are led into the study of time from a variety of highly different disciplines. For the same reason, the study of time is useful and enlightening, both for its own sake and for a large number of specific purposes. [Øhrstrøm and Hasle (1995), p. vii. Authors' italics]

The birth of symbolic temporal logic in the late 1950's is intimately connected with the name of Arthur Prior and his interest in classical philosophical problems, such as the conflict between fatalism and free will: even if the logic of time cannot settle the quarrel, it can elucidate any hidden preconception and remote consequence of the choice for either rival conception. The study of the answers given to this question by ancient philosophers, including Aristotle and

[1] Augustine, *Confessiones* XI, xiv, 17.

Diodorus Cronus, and medieval ones such as Ockham and Peter de Rivo suggested him to consider propositions as endowed with temporal characterisations, and the work of Findlay (1941) gave him the idea to develop a logic of time on the model of the then nascent modal logic.

Temporal operators for future and for past were then to be formulated in analogy to the modalities $\Box$ and $\Diamond$ of necessity and possibility (see also Prior 1957, which bears the significant title "Time and Modality"). Expressions of the form $\mathbf{F}A$ and $\mathbf{P}A$ were introduced with the intended meanings 'it will be the case that A' and 'it has been the case that A', respectively, whereas the dual operators $\mathbf{G} \equiv \neg\mathbf{F}\neg$ and $\mathbf{H} \equiv \neg\mathbf{P}\neg$ have the intuitive interpretations 'it will always be the case that A' and 'it has always been the case that A'. Further operators were later introduced to denote the next and the previous moment (von Wright 1965, Scott 1965), and the introduction of the 'Until' and 'Since' operators into linear-time logic in Kamp (1968) allowed the formulation of a more expressive temporal logic. Several versions of temporal logic have been considered, each reflecting the properties of the intended flow of time (linearity, unboundness, discreteness, ... and so on).

In recent years temporal logic has been the focus of a great number of studies because of its applications in computer science, expecially in the specification and verification of reactive systems[2]. However, because of the specific purposes of such studies, the analysis of the future flow of time has almost always been privileged, to the detriment of a wholesome approach heeding past events too.

[2]It is worth noting that the Turing Award 2007 has been won by E. M. Clarke, E. A. Emerson and J. Sifakis for their contributions in developing Model Checking, a verification technology in which temporal logic plays a substantial rôle. See also `http://awards.acm.org/homepage.cfm`.

As a consequence of the increasing interest the logic of time has been gaining, semantical analysis of temporal logic has been well investigated both in its philosophical grounds (Schindler 1970, van Benthem 1984, Goldblatt 1992) and in its applicative potential (Gabbay *et al.* 1980, Manna and Pnueli 1981, Lichtenstein and Pnueli 2000, Huth and Ryan 2004).

On the other hand, a sufficiently developed proof-theoretical analysis for temporal logic is still lacking, and when syntactic systems are given (as for example in Nishimura 1980, Schmitt and Goubault-Larrecq 1997, Schwendimann 1998, Bolotov *et al.* 2006), different flows of time are dealt with separately, and not as modular extensions of a basic temporal calculus. A significant exception is represented by *display logic* (Belnap 1982, Wansing 1998), that covers a variety of temporal logics by means of a generalised, but rather complicated, syntax.

Therefore, we decided to dedicate this thesis to the formulation of appropriate sequent calculi for temporal logic, by means of a general methodology which allows to deal uniformly with several systems. For the sake of clarity, we recall here the main issues of sequent calculus.

Sequent calculus was formulated at the beginning of the 1930's by Gerhard Gentzen: it exploits a particular notation, which formalises multisuccedent arguments, while keeping track of the open assumptions at any step of the derivations. A sequent is a syntactic object of the form

$$A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$$

with the same informal meaning as the formula

$$(A_1 \,\&\, \ldots \,\&\, A_m) \supset (B_1 \vee \cdots \vee B_n)$$

The antecedent $A_1, \dots, A_n$ and the succedent $B_1, \dots, B_m$ (also indicated with the Greek capital letters Γ and Δ, respectively) are in general considered as multisets of formulas, where the number of occurrences of a given formula is relevant, but not the order in which the formulas appear.

Sequent calculus consists of logical rules, which introduce logical constant into the left- and right-hand side of sequents, and are formulated on the base of a natural correspondence with the intended interpretation. It also contains the structural rules of *Weakening*, *Contraction* and *Cut*, which, as their name suggests, do not add logical constants, but modify the structure of the sequents.

The rule of cut is the most important structural rule, and it is generally explained as follows: the derivation of a theorem is breaken down into two easier lemmas, which are then chained together according to the scheme of the rule

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut$$

The main achievement of Gentzen calculus, also called *Hauptsatz*, states that it is possible to do without cut rule: as a consequence, the *subformula property* is obtained, which says that every formula in a derivation is a subformula of the formulas in the endsequent.

Sequent calculus affords a fully satisfactory method of structural proof analysis for systems of pure logic (classical and intuitionistic predicate logic), but until the end of the 1990's there was a common belief that "the *Hauptsatz* fails for systems with proper axioms"[3]. However, in Negri and von Plato (1998), and in Negri (2003) a general method was proposed, that permits to transform certain axiomatic systems into systems of nonlogical (mathematical) rules of inference,

[3]Girard (1987), p. 125.

while preserving the *Hauptsatz* and the other remarkable structural properties of sequent calculus.

In Negri (2005) the method was generalised to treat normal modal logic as labelled sequent calculi extended with mathematical rules, in which the usual Kripke-style semantics becomes part of the formalism. Every formula in a sequent is either a labelled formula or a relational atom, which are the syntactic counterparts of the forcing relation and the accessibility relation of Kripke frames[4], respectively: the logical rules for modal operators are formulated on the base of their semantical explanation, and the conditions characterizing the accessibility relation are added in the form of mathematical rules.

In this thesis we propose to apply the methodology of the internalisation of the relational semantics into the syntax of sequent calculus to the proof-theoretical analysis of the most important systems of temporal logic.

Synopsis

Chapter 1 is dedicated to the illustration of some preliminary issues. First, we introduce tense logic with reference to its birth at the hands of A. N. Prior, and discuss the main problems encountered in formulating sequent calculi for temporal and modal logics. Then, we describe the general methodology of the internalisation of Kripke semantics into the syntax of sequent calculi, originally proposed in Negri (2005) for normal modal logics. After a short introduction of the main notions of relational semantics, we recall the results proved for the basic modal logic G3K and its extensions with mathematical rules for accessibility relation R expressible as universal axioms or geometric implications. Finally,

[4]See below, Section 1.3.

we discuss philosophical and methodological issues of labelled calculi with internalised semantics in response to some objections recently raised against them.

In Chapter 2, a labelled calculus is formulated for the basic temporal logic K_t by justifying the logical rules for the temporal operators **G**, **F**, **H** and **P** on the base of their meaning explanations in terms of the inteded relational semantics. Next, we show how several temporal systems can be obtained as modular extensions of the basic calculus $\mathrm{G3K}_t$, through the addition of mathematical rules for the frame properties of the accessibility relation. We prove that the calculus $\mathrm{G3K}_t$ and its extensions enjoy remarkable structural properties, as height-preserving admissibility of the substitution of labels, height-preserving invertibility of all the rules, and height-preserving admissibility of the structural rules of weakening and contraction. The syntactic proof of cut elimination allows to prove the weak subformula property: every formula in a derivation is a subformula of the endsequent or a relational atom. We also prove subterm property, stating that every label in a derivation is a label in the endsequent or an eigenvariable.

In Chapter 3, we consider, as a case study, the sequent calculus G3LT for Priorean linear discrete time logic. Rules for the next-time and the previous-time operators, **T** and **Y**, are formulated in analogy to those for **G** and **H**, and an infinitary mathematical rule is required corresponding to the definition of the accessibility relation $<$ for **G** and **H** as the transitive closure of the accessibility relation $\prec$ for **T** and **Y**. Two partial finitisation are then considered, and conservativity results are proved with respect to suitable fragments of the infinitary system G3LT. The labelled approach allows also to formulate temporal rules for Kamp's operators Until and Since. Structural properties are proved for G3LT and for its extension with the rules for Until and Since.

In Chapter 4, an alternative formulation of the temporal rules, based on their fixed-point definition, gives the system G3LT$_{cl}$. All the rules of the latter are finitary, but proof are generally constituted by derivation trees with (at least) an infinite branch. Decidability is then proved by showing that G3LT$_{cl}$ admits terminating proof search, with an exponential bound calculated on the lenght of the formula corresponding to the endsequent. The extension with fixed point rules for Until and Since is also considered. The system is finally compared with the infinitary calculus of Chapter 3.

The last chapter is dedicated to the conclusions, and to the description of possible directions for future work.

Chapter 1

Preliminaries

1.1 Prior's tense logic

Modern logic generally considers the truth value of a proposition as unchangeable with time: a statement of the form 'Socrates is sitting' has to be considered incomplete and inappropriate for a logical treatment unless its temporal reference is specified by an expression of the form 'at moment x'. This opinion should be correlated with the tight connection between modern logic and mathematics, where any proposition is unalterably true or unalterably false: a mathematical thesis, once proved, is true forever.

Tenses were then banned from logical statements, and replaced with the atemporal copula[1]. Propositions like 'Socrates was sitting' were thus rendered by complicated paraphrases, such as 'There exists an instant t, such that t is earlier than now and Socates is sitting at t' (where the verbs 'exists', 'is' and 'is sitting' should be interpreted atemporally): tensed propositions were

[1]Italian readers can find an interesting discussion of this question in the introduction of Pizzi (1974).

transformed into atemporal predicates of instants in a first-order logic with some kind of earlier-later relation.

This opinion was initially shared by A. N. Prior, who considered "not only correct but also traditional to think of propositions as incomplete, and not ready for accurate logical treatment, until all time-references had been so filled in"[2]. However, Prior's deep knowledge of the history of philosophy and his studies in ancient and medieval logic allowed him to rediscover that according to the scholastics and to Aristotle

> 'statements and opinions' vary in their truth and falsehood with the times at which they are made or held, just as concrete things have different qualities at different times; though the cases are different, because the changes in truth value of statements and opinions are not properly speaking changes in these statements and opinions themselves, but reflexions of changes in the objects to which they refer [Prior (1967), p. 16]

Moreover, a short hint by Findlay (1941) suggested him that "the calculus of tenses should have been included in the modern development of modal logics"[3]:

> Mary Prior has described the first occurrence of this idea: "I remember his waking me one night, coming and sitting on my bed, and reading a footnote from John Findlay's article on Time, and saying he thought one could make a formalised tense logic". [Øhrstrøm and Hasle (1995), p. 170]

Temporal operators **F** and **P** were to be introduced with the intuitive meaning of 'it will be the case that' and 'it has been the case that' respectively, and with dual operators $\mathbf{G} \equiv \neg\mathbf{F}\neg$ ('it will always be the case that') and $\mathbf{H} \equiv \neg\mathbf{P}\neg$ ('it has always been the case that'). In analogy with the modal operators 'it is possible that' and 'it is necessary that', tense-forming operators for past

[2] Prior (1967), pp. 15–16.
[3] Findlay (1941), p. 233.

and future should be considered as "expressions that form sentences from sentences"[4], whereas the present should be considered as a "zero tense-inflexion"[5] corresponding to the empty modality.

Several systems of temporal logic were then formulated according to the intended properties of the structure of time. The most general tense-logical system is usually known as K_t, the Hilbert-style axiom schemes of which are the following:

A1. p is a thesis for every propositional tautology p

A2. $\mathbf{G}(p \supset q) \supset (\mathbf{G}p \supset \mathbf{G}q)$

A3. $\mathbf{H}(p \supset q) \supset (\mathbf{H}p \supset \mathbf{H}q)$

A4. $p \supset \mathbf{HF}p$

A5. $p \supset \mathbf{GP}p$

Observe that axioms A2 and A3 say that the temporal operators $\mathbf{G}$ and $\mathbf{H}$ distribute over the implication, whereas axioms A4 and A5 state that a present event has been future at a past moment, and will be past at a future one, respectively. All the axioms are immediately provable for arbitrary well-formed formulas $p, q, \ldots$, whereas other theses are derivable by means of the following rules of inference:

R1. If $\vdash p$ and $\vdash p \supset q$, then $\vdash q$

R2. If $\vdash p$, then $\vdash \mathbf{G}p$

R3. If $\vdash p$, then $\vdash \mathbf{H}p$

R1 is the usual *modus ponens* rule. The rules R2 and R3 are usually called 'temporal generalisation rules', and are the tense-logical analogouses of 'necessitation rule' in modal logics.

[4]Prior (1967), p. 15.
[5]Prior (1967), p. 14.

The calculus K_t is the basic temporal system in the sense that it does not make any assumption on the flow of time. Richer systems can be obtained if special conditions are imposed on the structure of time by adding to the calculus some supplementary axioms. For example, the axiom

$\mathbf{FF}p \supset \mathbf{F}p$ (equivalently $\mathbf{PP}p \supset \mathbf{P}p$)

corresponds to transitivity ('if it will be the case that it will be the case that p, then it will be the case that p'); whereas the axiom

$\mathbf{F}p \supset \mathbf{FF}p$ (equivalently $\mathbf{P}p \supset \mathbf{PP}p$)

corresponds to density ('if it will be the case that p, then it will be the case that it will be the case that p'). We can also consider some axioms which produce a somehow asymmetric behaviour of past and future. For instance, if we add the thesis

$\mathbf{FP}p \supset (\mathbf{P}p \vee p \vee \mathbf{F}p)$

corresponding to backwards linearity ('if it will be the case that it has been the case that p, then it has been or it is or it will be the case that p'), we obtain a time structure with a tree shape which is linear towards the past, and possibly branching towards the future. A linear time structure is obtained by adding also the axiom of linearity towards the future ('if it has been the case that it will the case that p, then it has been or it is or it will be the case that p')

$\mathbf{PF}p \supset (\mathbf{P}p \vee p \vee \mathbf{F}p)$

Finally, the axioms for left seriality ('if it has always been the case that p, then it has been the case that p') and right seriality ('if it will always be the case that p, then it will be the case that p')

$\mathbf{H}p \supset \mathbf{P}p$

$\mathbf{G}p \supset \mathbf{F}p$

correspond to a time structure without a first and a last instant, respectively.

Prior did not formulate systematically an explicit semantics for tense logic; however, in a paper dating back to 1958, he suggested that tense logic (also called PF-calculus after the temporal operators **P** and **F**) could be interpreted within a first-order monadic logic for the earlier-later relation. In the latter calculus the variables $x, y, z, \ldots$ are supposed to range over dates or instants of time, whereas the propositional variables $p, q, \ldots$ are considered as functions of dates, with the expression px being read as 'p at time x'; a binary relation l is also added, with lxy being read as 'x is later than y'[6]. Tense-forming operators are then interpreted as follows, with z representing "the date at which the proposition under consideration is uttered" [7]:

$$\mathbf{F}p \equiv \exists x\,(lxz \,\&\, px) \qquad\qquad \mathbf{G}p \equiv \forall x\,(lxz \supset px)$$

$$\mathbf{P}p \equiv \exists x\,(lzx \,\&\, px) \qquad\qquad \mathbf{H}p \equiv \forall x\,(lzx \supset px)$$

The above mentioned proposal of interpreting tense logic within the logic of earlier and later is regarded by Goldblatt (2005) as a reason for numbering Prior among the precursors of relational semantics, the birth of which is usually attributed to Kripke (1959). As observed by Goldblatt (2005), Prior did not however pursue the implicit relational model theory related to the l-calculus, "and would not have thought it philosophically worthwhile to do so"[8]. On the contrary, Prior argued that

> If there is to be any 'interpretation' of our calculi in the metaphysical sense, it will probably need to be the other way round; that is, the l-calculus should be exhibited as a logical construction out of the PF-calculus rather then *vice versa*. [Prior (1958), p. 116. Author's italics]

[6]See Prior (1958), pp. 112–113.
[7]Prior (1958), pp. 113.
[8]Goldblatt (2005), p. 27.

In later works, this idea has been further developed as a major programme: in analogy with Quine's 'three grades of modal involvement'[9], Prior (1968) discussed four grades of tense-logical involvement "by presenting a series of calculi involving the notion of being true [...] at an instant, making more and more controversial assumptions at each main stage"[10].

In the first and lower grade, tense operators are not assumed as primitive, but are defined as metalinguistic abbreviations for expressions in the language of an earlier-later calculus with a binary relation $T(a,p)$, 'it is the case at the instant a that p'. The latter calculus is named U-calculus after the symbol U used for the earlier-later relation, with Uab being read as 'the instant a is earlier than the instant b'. Prior (1968) uses a prefix notation; however, for ease of reading, we shall adopt the infix notation and the usual symbol $<$ for the precedence relation. The temporal operators are thus defined as follows:

$$T(a, \mathbf{G}p) \equiv_{def} \forall b\,(a < b \supset T(b,p))$$

$$T(a, \mathbf{H}p) \equiv_{def} \forall b\,(b < a \supset T(b,p))$$

Only formulas of the form $a < b$ and $T(a,p)$ are complete propositions, the truth value of which is independent of time; tensed propositions, denoted for instance by the variable p, are nothing but predicates of instants, and "tense logic, we might say, is a logic of pure predicates which are artificially torn away from their subjects and given a spurious indipendence"[11]. In a sense, the logic for the earlier-later relation is considered as a proper characterisation of time: in its framework the basic tense logic (the above mentioned K_t) can be proved as a mere by-product of the definitions above, whereas special conditions can be imposed on the flow of time by adding to the U-calculus supplementary

[9] Quine (1953).
[10] Prior (1968), p. 116.
[11] Prior (1968), p. 117.

axioms corresponding to the properties of the earlier-later relation. In Table 1.1 below the special axioms considered above are compared with the corresponding properties for the earlier-later relation.

Tense-logical axiom	Property for the earlier-later relation
$\mathbf{FF}p \supset \mathbf{F}p$	$\forall a \forall b \forall c\, ((a < b \,\&\, b < c) \supset a < c)$
$\mathbf{F}p \supset \mathbf{FF}p$	$\forall a \forall b\, (a < b \supset \exists c\, (a < c \,\&\, c < b))$
$\mathbf{FP}p \supset (\mathbf{P}p \vee p \vee \mathbf{F}p)$	$\forall a \forall b \forall c\, ((b < a \,\&\, c < a) \supset (b < c \vee b = c \vee c < b))$
$\mathbf{PF}p \supset (\mathbf{P}p \vee p \vee \mathbf{F}p)$	$\forall a \forall b \forall c\, ((a < b \,\&\, a < c) \supset (b < c \vee b = c \vee c < b))$
$\mathbf{H}p\mathbf{P}p$	$\forall a \exists b\, (b < a)$
$\mathbf{G}p\mathbf{F}p$	$\forall a \exists b\, (a < b)$

Table 1.1: Tense-logical axioms and properties of time flow

Nonetheless, the first grade presupposes the existence of a very odd kind of entities, namely the instants of time, and the time-series they are supposed to constitute. On the contrary, Prior would have preferred to consider them "as mere logical constructions out of tensed facts"[12]: for this reason, he proposed to go further towards the second grade.

In the second grade of tense-logical involvement atemporal statements of the form $a < b$ and $T(a, p)$ are treated on a par with the tensed propositions, thus becoming admissible values for the propositional variables $p, q, \ldots$, and the latters are not viewed as incomplete propositions anymore, but are admitted as propositions on their own right. Both $T(b, T(a, p))$ and p are thus well-formed, and of the same sort as $T(a, p)$. The equalisation of tensed propositions with atemporal statements leads to important results: in particular, if the necessity modal operator $\mathbf{L}p$ is defined as $\forall a\, T(a, p)$, Gödel's postulates for the modal system S5 become provable for $\mathbf{L}$. However, the main conceptual consequence of the second grade is that

> we no longer have merely *parallel* tense logics and U-calculi; the tense logics now appear as *parts* of the U-calculi, and this may prepare

[12]Prior (1967), p. 118.

> the way for treating the U-calculi as parts of the tense logics. [Prior (1968), p. 121. Author's italics]

In the third grade of tense-logical involvement the instant-variables represent propositions: any instant is equated with the conjunction of all the propositions which are true at that instant. These 'world-state propositions' are meant to supply an exhaustive and unique description of the world in a given instant, and are denoted by the variables $a, b, \ldots$ formerly used for instants. Prior admits that this can seem a "highly artificial procedure"[13], but he claims that

> 'instants' are artificial entities anyhow, i.e. that all talk which appears to be about them, and about the 'time-series' which they are supposed to constitute, is just disguised talk about what is and has been and will be the case. [Prior (1968), p. 123]

The binary relation $T(a, p)$ is redefined in terms of the necessity operator $\mathbf{L}$, which is now assumed as primitive; since it does not make sense to say that a proposition is true at another proposition, the formula $T(a, p)$ is considered as an abridgement for $\mathbf{L}(a \supset p)$, and a proposition of the form $T(a, b)$ is held to be true if and only if $a \equiv b$. Analogously, the earlier later relation is redifined in terms of the tense operators $\mathbf{F}$ and $\mathbf{P}$, thanks to the provability in the new systems of the theses

$$a < b \equiv T(a, \mathbf{F}b)$$

$$b < a \equiv T(a, \mathbf{P}b)$$

Furthermore, special conditions on the earlier-later relation (such as transitivity, density, and backwards linearity) become provable in the third system if the corresponding tense-logical axioms are added to the basic temporal calculus. As a consequence, the U-calculi turns out to be a by-product of the interaction

[13] Prior (1968), p. 123.

of the tense-logical operators for past and future with a modal operator for necessity. All we still have to do for reducing the U-calculi to the tense logics is to supply a tense-logical definition for the necessity operator $\mathbf{L}$.

We thus reach the fourth and last grade of tense-logical involvement, in which the only primitive operators are the tense-logical ones. A whole hierarchy of modal operators $\mathbf{L}^n$ is inductively defined as follows

$\mathbf{L}^0 p \equiv p$

$\mathbf{L}^{n+1} p \equiv \mathbf{HL}^n p \,\&\, \mathbf{GL}^n p$

Furthermore, quantifiers are introduced for binding the n's, and the necessity operator $\mathbf{L}p$ becomes a shorthand for $\forall n\, \mathbf{L}^n p$. All the postulates of the modal system S5 can be proved for the above defined operator $\mathbf{L}$, though the derivations are usually complicated and require induction on n. Most importantly, the U-calculi are now completely derivable within the purely tense-logical system corresponding to the fourth grade.

As a final remark, we observe that Prior did not claim to have found the definitive solution to the problems raised by the concept of time; on the contrary, he believed that the logical analysis of time was not only useful but even indispensable for showing the hidden presuppositions and the remote consequences, which the proponent of any given conception of time should consider not to fall into contradiction. With this motivation, Prior formulated tense logic as the suitable logical theory, in which the necessary analyses can be carried out while avoiding the most undesirable metaphysical commitments on the nature of time.

1.2 Sequent calculi for temporal logics

Early work in proof theory for temporal logic has been developed by means of the tools offered by Hilbert-style axiomatic calculi: proof systems were constituted by several axioms and a small number of rules of inference. However, axiomatic calculi, although being very useful for a formal presentation of what has already been proved, are inappropriate for the actual search of proofs. To carry out a derivation, in fact, we first have to find the proper instances of axioms to begin with, but this work can be very hard, since no guidance is offered by the thesis to be proved.

At the beginning of the Thirties, Gerhard Gentzen formulated sequent calculus as an alternative formal deductive system, which allows more natural inferences. Sequent calculi consist of several rules of inference, that introduce logical constants to the right or to the left of the sequents. Furthermore, Gentzen calculi have a special notation, which compensates the lack of guidance of Hilbert-style calculi: at any step of the derivation the antecedent of a sequent shows the open assumptions on which the formulas in the consequent depend.

Unfortunately, it seems that the logic of time does not fit in the framework of sequent calculi, insomuch as the chapter on 'Temporal Logic' written by Hodkinson and Reynolds for the recent *Handbook of Modal Logic* (Blackburn, van Benthem, and Wolter 2006) does not even consider sequent calculi in any detail[14]. It is also noticed that natural deduction and semantic tableau systems are generally preferred to sequent calculi because "for efficient automation, the cut rule is problematic"[15].

Moreover, the use of the logic of time in computer science introduced a rather

[14] See Blackburn, van Benthem, and Wolter (2006), p. 702.
[15] Blackburn, van Benthem, and Wolter (2006), p. 702.

exasperate fragmentation of temporal systems. The application of temporal logics to the specification and verification of reactive systems has the consequence that only very specific fragments are considered, which are approporiate for computer implementation, and different flows of time are dealt with separately, not as modular extensions of a basic temporal calculus.

Gentzen-style calculi for temporal logics have been proposed in Nishimura (1980), but the systems do not enjoy syntactic cut elimination, and the temporal rules there formulated contain some additional difficulties. Two sequent systems are presented, GK_t for the basic temporal logic, and GK_t4 for the transitive time flow. The operators **G** and **H** are assumed as the only primitive temporal operators. The basic system GK_t has the following rules for **G** and **H**:

$$\frac{\Gamma \Rightarrow A, \mathbf{H}\Delta}{\mathbf{G}\Gamma \Rightarrow \mathbf{G}A, \Delta}\,{\scriptstyle(\Rightarrow\mathbf{G})} \qquad\qquad \frac{\Gamma \Rightarrow A, \mathbf{G}\Delta}{\mathbf{H}\Gamma \Rightarrow \mathbf{H}A, \Delta}\,{\scriptstyle(\Rightarrow\mathbf{H})}$$

where $\mathbf{G}\Gamma = \{\mathbf{G}A | A \in \Gamma\}$ and $\mathbf{H}\Gamma = \{\mathbf{H}A | A \in \Gamma\}$.

The system GK_t4 is obtained by replacing the rules above with

$$\frac{\mathbf{G}\Gamma, \Gamma \Rightarrow A, \mathbf{H}\Delta, \mathbf{H}\Sigma}{\mathbf{G}\Gamma \Rightarrow \mathbf{G}A, \Delta, \mathbf{H}\Sigma}\,{\scriptstyle(\Rightarrow\mathbf{G})_4} \qquad\qquad \frac{\mathbf{H}\Gamma, \Gamma \Rightarrow A, \mathbf{G}\Delta, \mathbf{G}\Sigma}{\mathbf{H}\Gamma \Rightarrow \mathbf{H}A, \Delta, \mathbf{G}\Sigma}\,{\scriptstyle(\Rightarrow\mathbf{H})_4}$$

Nishimura observes that the rules $(\Rightarrow \mathbf{G})$ and $(\Rightarrow \mathbf{H})$ are admissibile in GK_t4; nonetheless, the latter system is not obtained as a modular extension of the basic modal calculus GK_t, but through a complete reformulation of the temporal rules. Moreover, no rule is formulated for other properties of time flow, as for example the already mentioned[16] density and backwards linearity.

Further calculi for specific temporal systems are proposed for example in Gudzhinskas (1982), Kawai (1987), Paech (1988), Szalas (1995).

The problem of finding a uniform method which allows to deal with different

[16]See Section 1.1.

systems at once, while preserving important structural properties, does not concern only temporal logic. Gentzen-style formalisation of modal logics in general has encountered analogous difficulties: for example, sequent calculi for the modal system S5 usually lack cut elimination as in Ohnishi and Matsumoto (1957), or, when cut elimination is preserved, either subformula property is lost as in Mints (1968), or non-local rules are required as in Braüner (2000). As pointed out by Wansing, in fact

> many normal modal and temporal logics are presentable as ordinary Gentzen calculi, [...] However, no uniform way of presenting only the most important normal modal and temporal propositional logics as ordinary Gentzen calculi is known. Further, the standard approach fails to be *modular*: in general it is not the case that a single axiom schema is captured by a single sequent rule (or a finite set of such rules). [Wansing (2002), p. 68. Author's italics]

This situation also led one of the most recent textbooks on modal logic (Blackburn, de Rijke, and Venema 2001) to complain about the lack of a general solution:

> modal proof theory and automated reasoning are still relatively youthful enterprises; they are exciting and active fields, but as yet there is little consensus about methods and few general results. [Blackburn, de Rijke, and Venema (2001), p. xvi]

In order to overcome the limitations of ordinary sequent calculi, several generalisations have been considered in the form of higher-level, higher-dimensional, higher-arity, multiple sequent systems, hypersequents, display logic: Wansing (2002) offers a survey on those different attempts. In the very last years, deep sequent system and tree-hypersequents have been formulated by Brünnler (2006) and Poggiolesi (2008), respectively. It is worth noting that none of the systems proposed, with the remarkable exception of display logic, has been systematically applied to the analysis of temporal logics.

The temporal and modal calculi proposed in Kashima (1994) and Cerrato (1996) show an approach which is similar to some extent to that of display logic; however, the latter has a greater tradition and a broader development. In the following we shall recall the main features of display calculi for temporal logics.

In addition to the logical operators, display logic considers some structural connectives, $\mathbf{I}, *, \bullet, \circ$, the modal interpretations of which depend on their being in the antecedent or in the consequent. For the sake of clarity, let us use X, Y as variables for structures. Display structures are defined by

$$X \ ::= \ A \,|\, \mathbf{I} \,|\, *X \,|\, \bullet X \,|\, X \circ Y$$

A display sequent is an expression of the form $X \Rightarrow Y$, the intuitive meaning of which is given by the following translations τ_1 for the antecedent and τ_2 for the consequent:

$$\begin{array}{ll}
(A)^{\tau_1} = A & (A)^{\tau_2} = A \\
(\mathbf{I})^{\tau_1} = \top & (\mathbf{I})^{\tau_2} = \bot \\
(*X)^{\tau_1} = \neg(X)^{\tau_2} & (*X)^{\tau_2} = \neg(X)^{\tau_1} \\
(\bullet X)^{\tau_1} = \mathbf{P}(X)^{\tau_1} & (\bullet X)^{\tau_2} = \mathbf{G}(X)^{\tau_2} \\
(X \circ Y)^{\tau_1} = (X)^{\tau_1} \,\&\, (Y)^{\tau_1} & (X \circ Y)^{\tau_2} = (X)^{\tau_2} \vee (Y)^{\tau_2}
\end{array}$$

Display logic rules for temporal operators are then formulated as follows, by means of the structural connectives $*$ and $\bullet$

$$\frac{A \Rightarrow Y}{\mathbf{G}A \Rightarrow \bullet Y}\,(\mathbf{G}\Rightarrow) \qquad \frac{\bullet X \Rightarrow A}{X \Rightarrow \mathbf{G}A}\,(\Rightarrow\mathbf{G})$$

$$\frac{* \bullet * A \Rightarrow Y}{\mathbf{F}A \Rightarrow Y}\,(\mathbf{F}\Rightarrow) \qquad \frac{X \Rightarrow A}{* \bullet * X \Rightarrow \mathbf{F}A}\,(\Rightarrow\mathbf{F})$$

$$\frac{A \Rightarrow Y}{\mathbf{H}A \Rightarrow * \bullet * Y}\,(\mathbf{H}\Rightarrow) \qquad \frac{X \Rightarrow * \bullet * A}{X \Rightarrow \mathbf{H}A}\,(\Rightarrow\mathbf{H})$$

$$\frac{A \Rightarrow \bullet Y}{\mathbf{P}A \Rightarrow Y}\,(\mathbf{P}\Rightarrow) \qquad \frac{X \Rightarrow A}{\bullet X \Rightarrow \mathbf{P}A}\,(\Rightarrow\mathbf{P})$$

We are not examining display logic in detail; the interested reader is referred to the above cited Wansing (2002). We observe here that the calculus enjoy cut elimination; furthermore, Kracht's algorithm (Kracht 1996) gives a general method for obtaining richer temporal systems by modular additions of special structural rules corresponding to the different properties of time flow. For instance, the above mentioned axioms

$\mathbf{FF}A \supset \mathbf{F}A$ for transitivity

$\mathbf{F}A \supset \mathbf{FF}A$ for density

$\mathbf{FP}A \supset (\mathbf{P}A \vee A \vee \mathbf{F}A)$ for backwards linearity

$\mathbf{PF}A \supset (\mathbf{P}A \vee A \vee \mathbf{F}A)$ for forwards lineatity

$\mathbf{H}A \supset \mathbf{P}A$ for left seriality

$\mathbf{G}A \supset \mathbf{F}A$ for right seriality

correspond to the following display rules:

$$\frac{* \bullet * X \Rightarrow Y}{* \bullet \bullet * X \Rightarrow Y}\,trans \qquad \frac{* \bullet \bullet * X \Rightarrow Y}{* \bullet * X \Rightarrow Y}\,dens$$

$$\frac{X \Rightarrow Y \quad \bullet X \Rightarrow Y \quad * \bullet * X \Rightarrow Y}{* \bullet * \bullet X \Rightarrow Y}\,linp \qquad \frac{X \Rightarrow Y \quad \bullet X \Rightarrow Y \quad * \bullet * X \Rightarrow Y}{\bullet * \bullet * X \Rightarrow Y}\,linf$$

$$\frac{\bullet \mathbf{I} \Rightarrow Y}{\mathbf{I} \Rightarrow Y}\,serp \qquad \frac{* \bullet * \mathbf{I} \Rightarrow Y}{\mathbf{I} \Rightarrow Y}\,serf$$

However, the calculi do not enjoy subformula property, since the structural connectives can disappear while moving from the premises to the conclusions of the rules.

Display logic and the other generalisations cited above require a, sometimes rather complicated, modification of the syntax of sequent calculi. In addition to such attempts, a recent approach has been proposed, which exploits the internalisation of the relational semantics into the syntax of sequent calculi: semantic information is thus available at the same level as the syntactic information.

In our treatment of temporal logics we will adopt the method of labelled sequent calculi proposed in Negri (2005) for the basic modal system K and its normal extensions. The present chapter is dedicated to the illustration of the general methodology.

1.3 Relational semantics for modal logic

Relational semantics is generally attributed to Saul Kripke, who, in its first paper on modal logic (1959), proved the completeness of S5 system with respect to an appropriate semantics based on relatively possible worlds. Anticipations of Kripke inventions can be found in the work of many authors (Hintikka, Meredith, Prior[17], among the others), and questions about originality were raised in several circumstances (see for example Bull and Segerberg 1984, Goldblatt 2005). We are not discussing the problem of attribution here, neither we are explaining in detail the widely known issues of relational semantics: on the contrary, we will introduce here and in the following, whenever necessary, useful notions for a better understanding. The reader who is interested in going into depth is referred to the cited works, and to Hughes and Cresswell (1984, 1996).

A *Kripke frame* is a set of possible worlds $\mathcal{K}$, together with a binary relation R, saying which worlds can be seen (or are accessible) from every world $k \in \mathcal{K}$. A Kripke model is a Kripke frame, togeher with an evaluation determining which atomic formulas are true at what world. Evaluation are then extended to propositional connectives in the usual way, whereas modal operators essentially

[17] See above, Section 1.1.

appeal to relational features:

Definition 1.3.1. *Let* $\mathcal{F} = (\mathcal{K}, R^{\mathcal{K}})$ *be a Kripke frame with a binary accessibility relation* $R^{\mathcal{K}}$. *An* evaluation *of atomic formulas in a frame is a map* $\mathcal{V} : AtFrm \rightarrow \wp(\mathcal{K})$, *assigning to any atom* P *the set of instants in which* P *holds. The standard notation for* $k \in \mathcal{V}(P)$ *is* $k \Vdash P$. *Evaluations are extended to arbitrary formulas by the following inductive clauses:*

For all $k \in \mathcal{K}$, *it is not the case that* $k \Vdash \bot$ *(abbr.* $k \nVdash \bot$*);*

$k \Vdash A \& B$ *if* $k \Vdash A$ *and* $k \Vdash B$*;*

$k \Vdash A \vee B$ *if* $k \Vdash A$ *or* $k \Vdash B$*;*

$k \Vdash A \supset B$ *if* $k \Vdash A$ *implies* $k \Vdash B$*;*

$k \Vdash \Box A$ *if for all* k', $kR^{\mathcal{K}}k'$ *implies* $k' \Vdash A$*;*

$k \Vdash \Diamond A$ *if there exists* k' *such that* $kR^{\mathcal{K}}k'$ *and* $k' \Vdash A$

The principle of distributivity of necessity over implication

$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$

is the characteristic axiom of the basic modal logic K, and it is valid in any arbitrary frame, with no condition imposed on the accessibility relation R.

	Axiom	Frame property
T	$\Box A \supset A$	$\forall x\ xRx$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall x \forall y \forall z\,(xRy \,\&\, yRz \supset xRz)$ transitivity
B	$A \supset \Box\Diamond A$	$\forall x \forall y\,(xRy \supset yRx)$ symmetry
E	$\Diamond A \supset \Box\Diamond A$	$\forall x \forall y \forall z\,(xRy \,\&\, xRz \supset yRz)$ euclideanness
D	$\Box A \supset \Diamond A$	$\forall x \exists y\ xRy$ seriality
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall x \forall y \forall z\,(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$ directedness

Table 1.2: Modal axioms and frames properties

Characteristic axioms for different modal systems correspond to specific frame properties: Table 1.2 gives a list of Hilbert-style axioms for the principal modal systems with the corresponding frame properties.

The view, developed after Prior's work, of temporal logic as a special modal logic suggests to interpret temporal formulas on appropriate relational frames $\mathcal{F} = (\mathcal{K}, <^{\mathcal{K}}, >^{\mathcal{K}})$, called *Prior frames*, characterized by the presence of a pair of accessibility relations satisfying the equivalence: $k <^{\mathcal{K}} k'$ iff $k' >^{\mathcal{K}} k$. Different temporal axioms correspond to special properties imposed on the order relation, as well as richer systems of modal logic correspond to special properties of the accessibility relation.

1.4 Sequent calculi for modal logics with internalised semantics

The idea of internalising semantical notions into the syntax of proof systems made a fleeting appearance in Kanger (1957), but has been gaining an increasing pervasiveness in recent years. Deductive systems enriched by the accessibility relation for possible-world semantics are proposed in the form of tableaux in Fitting (1983), in Catach (1991) and in Goré (1999), of natural deduction in Simpson (1994), in Basin *et al.* (1998) and in Indrzejczak (2003), of sequent calculi in Mints (1997) and in Castellini (2005). General features of labelled deductive systems are studied in Gabbay (1996) and in Viganò (2000). As anticipated, we recall here the method proposed in Negri (2005).

1.4.1 The basic modal calculus

The starting point is the cut- and contraction-free sequent calculus G3 that was introduced by Ketonen (1944) and recently systematically presented in Troelstra and Schwichtenberg (2000). All the rules of G3 are height-preserving invertible, that is whenever the conclusion is derivable so are the premises, with the same derivation height. Furthermore, structural rules of weakening and contraction are height-preserving admissible: their conclusion is derivable whenever the premise is, without increasing the height of derivation. Finally, cut is eliminated in a purely syntactical way.

A labelled sequent calculus for the basic modal logic K is formulated through the internalisation of Kripke semantics into the syntax: every formula in a sequent $\Gamma \Rightarrow \Delta$ is either a relational atomic formula xRy, or a labelled formula $x : A$. Intuitively, relational atoms and labelled formulas are the syntactic counterpart of the accessibility relation and of the forcing relation $x \Vdash A$ of temporal frames, respectively. The rules for the propositional connectives are analogous to the standard rules, with the active and principal formulas all marked by the same label x. The rules for the temporal operators are obtained from their meaning explanations in terms of the relational semantics:

$x \Vdash \Box A$ *iff for all* y, xRy *implies* $y \Vdash A$

$x \Vdash \Diamond A$ *iff for some* y, xRy *and* $y \Vdash A$

The left-to-right direction in the inductive definitions of validity for modal formulas justifies the left rules, the right-to-left direction the right rules. Arbitrariness of y becomes the variable conditions that y is not in the conclusion of rules $R\Box$ and $L\Diamond$. The propositional and modal rules for the basic calculus G3K are given in Table 1.3.

Initial sequents and $L\bot$:

$$x : P, \Gamma \Rightarrow \Delta, x : P \qquad\qquad xRy, \Gamma \Rightarrow \Delta, xRy$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \& B, \Gamma \Rightarrow \Delta}\ L\& \qquad\qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \& B}\ R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad\qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\ R\supset$$

Modal rules

$$\frac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\ L\Box \qquad\qquad \frac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ R\Box$$

$$\frac{xRy, y : A, \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\ L\Diamond \qquad\qquad \frac{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A}\ R\Diamond$$

Rules $R\Box$ and $L\Diamond$ have the condition that y is not in the conclusion.

Table 1.3: The basic modal calculus G3K

Observe that initial sequents are restricted to labelled atomic formulas $x : P$ or relational atoms. This feature, common to all G3 systems of sequent calculus, is needed to ensure admissibility of structural properties. A further remark concerns initial sequents of the form $xRy, \Gamma \Rightarrow \Delta, xRy$. As observed by Negri:

> no rule removes an atom of the form xRy from the right-hand side of sequents, and such atoms are never active in the logical rules. [...] As a consequence, initial sequents of the form $xRy, \Gamma \Rightarrow \Delta, xRy$ are needed only for deriving properties of accessibility relation, namely, the axioms corresponding to the rules for R [...] Thus such initial sequent can as well be left out from the calculus without impairing the completeness of the system. [Negri (2005), p. 513]

The calculus G3K corresponds to the basic modal logic K, which is characterized by arbitrary frames: no property is imposed on the accessibility relation R. Sequent calculi for normal extensions of K, namely the systems T, B, S4, S5 and so on, are obtained by adding to G3K one or more rules for the relational atoms of the form xRy.

1.4.2 Mathematical rules

In Negri and von Plato (1998, 2001) and in Negri (2003) a general method was presented for extending G3-style logical sequent calculus without losing its remarkable structural properties: axioms for specific theories, expressible by universal axioms or geometric implications, are suitably converted into inference rules to be added to the logical sequent calculus while preserving all the structural properties of the basic sequent system.

Universal axioms are sentences of the form $\forall x_1 \ldots \forall x_q A$, where A is quantifier free. They can be expressed in *conjunctive normal form* as the universal closure of the conjunction of formulas of the form $P_1 \,\&\, \ldots \,\&\, P_m \supset Q_1 \vee \cdots \vee Q_n$, where P_i, Q_j are atomic formulas and the consequent is equal to $\perp$ if $n = 0$. Each conjunct can be converted into a rule, called regular rule scheme, of the form

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}\,_{Reg}$$

The formulas $P_1, \ldots, P_m$ in the conclusion are the principal formulas of the rule, whereas $Q_1, \ldots, Q_n$ in the premises are the active formulas. Observe that the repetition of the principal formulas in the premises is required for obtaining admissibility of contraction in the extended calculus (see Section 2.3 below).

The general rule scheme can be specialised into the following rules

$$\frac{P, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,_1 \qquad \frac{P_1, P_2, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,_2 \qquad \frac{Q_1, \Gamma \Rightarrow \Delta \quad Q_2, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,_3$$

$$\frac{Q, P, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta}\,_4 \qquad \frac{}{P, \Gamma \Rightarrow \Delta}\,^5 \qquad \frac{}{P_1, P_2, \Gamma \Rightarrow \Delta}\,^6$$

The rules above correspond, respectively, to the axioms P, $P_1 \,\&\, P_2$, $Q_1 \vee Q_2$, $P \supset Q$, $\neg P$, and $\neg(P_1 \& P_2)$.

The above mentioned properties of reflexivity, transitivity, symmetry, and euclideannes are expressed by universal axioms, and can be converted into the following mathematical rules

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, _{Ref} \qquad \frac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\, _{Trans}$$

$$\frac{yRx, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\, _{Sym} \qquad \frac{yRz, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}\, _{Eucl}$$

Table 1.4: Regular rules for the accessibility relation R

A *geometric implication* is a formula of the form $\forall \overline{x}(A \supset B)$ where A and B do not contain $\supset$ or $\forall$ and $\overline{x}$ indicates a vector of variables $x_1, \ldots, x_p$. Any geometric implication can be converted into a conjuction of formulas

$$\forall \overline{x}\, (P_1 \,\&\, \ldots \,\&\, P_m \supset (\exists \overline{y}_1\, M_1 \vee \cdots \vee \exists \overline{y}_n\, M_n))$$

where M_j is the conjunction of atomic formulas $Q_{j_1}, \ldots, Q_{j_{k_j}}$ and the variables $\overline{y}_i$ are not free in $P_1, \ldots, P_m$.

By using the vector notation for multisets of formulas we write $\overline{P}$ for $P_1, \ldots,$ P_m and $\overline{Q}_j$ for $Q_{j_1}, \ldots, Q_{j_{k_j}}$. A susbstitution $\overline{Q}_j(\overline{y}_j/\overline{x}_j)$ denotes the substitution of variables in each of the Q_{j_l}, i.e. $Q_{j_1}(\overline{y}_j/\overline{x}_j), \ldots, Q_{j_{k_j}}(\overline{y}_j/\overline{x}_j)$. Clearly, universal axioms can be considered as special cases of geometric implications.

The rule scheme corresponding to geometric implications is

$$\frac{\overline{Q}_1(\overline{y}_1/\overline{x}_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n(\overline{y}_n/\overline{x}_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\, _{GRS}$$

where $\overline{P}$ and $\overline{Q}_j$ are multisets of atomic formulas $P_1, \ldots, P_m$ and $Q_{j_1}, \ldots, Q_{j_{k_j}}$, respectively, and the conclusion satisfies the condition that $\overline{y}_1, \ldots, \overline{y}_n$ are not in $\overline{P}, \Gamma, \Delta$. As in the case of the regular rule scheme, also the geometric rule scheme has the principal formulas repeated in the premises in order to guarantee admissibility of contraction.

The frame properties of seriality and directedness are geometric implications. These are converted into the following mathematical rules

$$\frac{xRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ser} \qquad\qquad \frac{yRw, zRw, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}\,{\scriptstyle Dir}$$

The rules have the condition that w is not in the conclusion.

Table 1.5: Geometric rules for the accessibility relation R

We observe here that, in order to guarantee admissibility of contraction in case substitution in the atoms produces a duplication among the principal formulas of a mathematical rule, we have to add to the system the contracted rule by ensuring that the following closure condition is satisfied:

Closure condition. *Given a system with geometric rules, if it has a rule with an instance of the form*

$$\frac{\overline{Q}_1(\overline{y}_1/\overline{x}_1), P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n(\overline{y}_n/\overline{x}_n), P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}$$

then also the rule

$$\frac{\overline{Q}_1(\overline{y}_1/\overline{x}_1), P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n(\overline{y}_n/\overline{x}_n), P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta}$$

has to be included in the system.

The condition is not problematic, since the number of rules to be added to a given system is finite and often the closure condition is even superfluous, because the contracted rule is already a rule of the system or admissible in it.

Theorem 1.4.1. (Soundness and Completeness) *The calculus* G3K *and its extensions with mathematical rules for accessibility relation are sound and complete with respect to the intended class of frames.*

Proof. Proof of soundness is straightforward. Completeness is proved by showing that Hilbert-style calculi for different modal systems can be embedded into the corresponding extensions of G3K. □

A direct proof of completeness for G3K and its extensions, based on the construction of a countermodel from a failed proof search, has been recently presented in Negri and von Plato (2008), pp. 201–207.

All the extensions are obtained in a modular way. Therefore, the remarkable structural properties summarized in the following theorem are established at once for all systems:

Theorem 1.4.2. *All the rules of* G3K *and of its extensions with mathematical rules for accessibility relation are height-preserving invertible. Rules of substitution, left and right weakening, left and right contraction are height-preserving admissible. The calculus* G3K *and its extensions enjoy cut elimination.*

We will prove similar results for our basic temporal calculus and its extensions with mathematical rules for order relation: the reader who is interested in the details is referred below, Section 2.3.

Negri (2005) also considers the interesting case of Gödel-Löb provability logic GL, which is characterized by the axiom

$$\Box(\Box A \supset A) \supset \Box A$$

The latter corresponds to the frame property that the accessibility relation is transitive and there are no infinite R-chains. This property is not expressible as a first-order sentence: as a consequence, it cannot be turned into a mathematical rule to be added to the basic calculus. However, a contraction- and cut-free sequent calculus for GL is obtained by a slight modification of the rules for the modal operator $\Box$. We are not discussing this result further: we refer to the cited paper for in-depth explanations.

Although cut elimination generally has subformula property as one of its immediate consequences, the presence of modal and mathematical rules that remove relational atoms from the left-hand side of the sequents prevents G3K and its extensions from having a full subformula property. The following section is completely dedicated to the discussion of this problem.

1.4.3 Subformula property and subterm property

A given sequent calculus enjoys the *subformula property* if the following condition is satisfied: every formula in a derivation is a subformula of the formulas in the endsequent. Such a property was the main aim of Gentzen's *Hauptsatz*:

> Intuitively speaking, these properties of derivation without cuts may be expressed as follows: the S-formulae [formulas in sequents] become longer as we descend lower down in derivation, never shorter. The final result is, as it were, gradually built up from its constituent elements. The proof represented by the derivation is not roundabout in that it contains only concepts which recur in the final result [Gentzen (1935), p. 88]

A suitable version of subformula property is formulated in Negri (2005, p. 21) for system G3K and its extensions with non logical rules. First, the notion of subformula is modified in order to match the context of labelled calculi:

Definition 1.4.3. *For every propositional connective $\circ$, the* subformulas *of $x : A \circ B$ are $x : A \circ B$ and all the subformulas of $x : A$ and $x : B$. For every modal or temporal operator $\mathbf{M}$, the subformulas of $x : \mathbf{M}A$ are $x : \mathbf{M}A$ and all the subformulas of $y : A$ for arbitrary y.*

The *weak subformula property* is then stated as follows: every formula in a derivation is either a subformula of the formulas in the endsequent or a relational atomic formula. The weak subformula property holds for system G3K and its extension as a consequence of Theorem 1.4.2: in fact only the logical rules for the modal operators and the mathematical rules for the accessibility relation can remove formulas from a derivation, and all such formulas are atoms of the form xRy.

The lack of a full subformula property constitutes a serious obstacle to the possibility of ensuring decidability of labelled calculi, and some concerns can be raised about "variables that are an essential part of the formulas and that disappear from the premises to the conclusion in the special logical rules [mathematical rules for the accessibility relation]"[18].

However, a more refined result is given in the form of the *subterm property*: all labels in a derivation in G3K and its extension with mathematical rules for accessibility relation are either eigenvariables or labels in the endsequent. No variable, except for eigenvariables, disappears, and the number of the latter is bound by the labels in the endsequents and by the number of applications of modal or mathematical rules with a variable condition.

As a consequence, the subterm property, together with the structural properties stated in Theorem 1.4.2, makes G3K and its extensions with mathematical rules suitable for proving decidability in a purely syntactical way, by calculating an effective bound on proof search[19].

[18]See Poggiolesi (2008), p. 82.
[19]See Negri (2005), pp. 529–538.

1.5 Methodological remarks about the internalisation of semantics

It is generally recognised that labelled inference systems are more successful and easy to use than unlabelled approaches, and allow to deal modularly a wide class of modal logics[20]. Furthermore, they are much more expressive and enjoy remarkable structural properties[21]. However, some objections have recently been raised about the internalisation of the semantics into the syntax of sequent calculi.

The main concerns regard the purity of methods, in the sense that the internalisation of Kripke semantics contaminates the syntactic purity of sequent calculi. In our opinion, this question appears a quite otiose one, and it does not take into account the very origin of modal and temporal logics. The possibility of a commixture of modal logics with a formal calculus corresponding to their semantics was already precognised by Prior, in the context of tense logic[22]:

> [...] the laws of PF-calculus will be not only interpretable but provable in the l-calculus if the latter contains
>
> (i) the usual laws and rules for truth operators and quantifiers; and
>
> (ii) a set of special axioms expressing the properties of 'l', e. g. ClxyClyzlxz (the law of transitivity for 'l'), ClxyNlyx (the law of asymmetry for 'l'), AAIxylxylyx ('Either the date x is identical to the date y or it is later than y or it is earlier'- the law of trichotomy for dates). [Prior (1958), p. 113]

Here PF-calculus stands for tense logic, and l-calculus for a monadic first-order logic with equality and a binary order relation l ('later than'). Observe that condition (i) is rendered in the labelled modal calculus G3K and in the

[20]See Brünnler (2006), p. 107.
[21]See Poggiolesi (2008), p. 94.
[22]See also above, Section 1.1

temporal calculus[23] G3K$_t$ by the usual rules for the propositional connectives and by the variable conditions on modal/temporal rules, which reflect the rôle of quantifiers in the semantical explanation of the modal/temporal operators; whereas the mathematical rules for accessibility/order relation allow to obtain extensions of the basic calculi similarly to the special axioms for the relation l of condition (ii).

Careful to avoid any metaphysical commitment about the nature of time, Prior defended the predominance of tense logic with respect to the logic of earlier and later. Nevertheless, he admitted the usefulness of the method:

> This [the possibility of proving of tense-logical theses in the calculus for earlier-later relation] is what I call the first or lowest grade of tense-logical involvement. Philosophers who are uneasy about tense logic will almost certainly find little in this amount of it to worry about. And there is a *nice economy* about it; it reduces the minimal tense logic to a by-product of the introduction of four definitions [of temporal operators] into an ordinary first-order theory, and richer systems to by-products of conditions imposed on a relation in that theory. [Prior (1968), p. 118. Italics mine]

From this point of view, the risk of a metaphysical commitment is somehow neutralised, in a sense acceptable by Prior, by the internalisation of the logic of earlier and later as a part of the syntax, since it allows to consider the relational semantics only as a helpful formal tool, "a device of considerable metalogical utility"[24], and not as "an 'interpretation' in the sense of a metaphysical explanation of what we mean by 'is', 'has been' and 'will be' "[25].

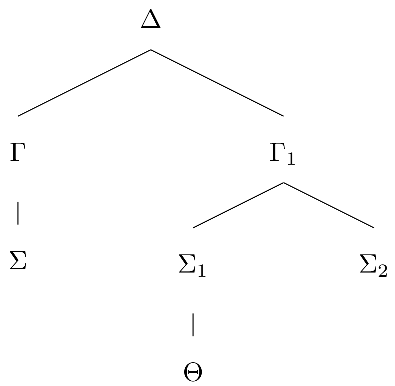
On the other hand, the methods proposed as alternatives, namely tree-hypersequents calculi (Poggiolesi 2008) and deep sequent systems (Brünnler 2006), although absolutely legitimated on their formal base, do not seem to be

[23] See below, Section 2.1.
[24] Prior (1958), p. 115.
[25] Prior (1958), p. 115.

unaffected by the same criticism. Whereas the semantics notions are explicitly internalised into the labelled calculi in the form of the syntactical counterparts of forcing ($x : A$) and accessibility relation (xRy), tree-hypersequents and deep sequent systems hide their relational semantics under a more complex syntax, as shown by the examples below (with Greek capital letters standing for sequents), which also show a substantial equivalence between the two methods.

Tree-hypersequents	*Deep sequent systems*
$\Delta/(\Gamma/\Sigma);(\Gamma_1/(\Sigma_1/\Theta);\Sigma_2)$	$\Delta,[\Gamma,[\Sigma]],[\Gamma_1,[\Sigma_1,[\Theta]],[\Sigma_2]]$

Both the tree-hypersequent on the left and the deep sequent on the right have the following intuitive explanation on a tree frame, with every node labelled by a sequent



Clearly, the root corresponds to the *actual world* (the world of utterance of a modal formula), and each child represents a world accessible from its parent. Observe that this correspondence is explicitly stated by the authors: see in particular Brünnler (2006, p. 109), and Poggiolesi (2008, pp. 101-102).

Another generalisation of sequents was given in Cerrato (1993), where modal sequent systems work explicitly on tree frames, which are graphic objects of the same kind of the figure above. Given the above mentioned intuitive interpretation of Poggiolesi's and Brünnler's formal entities into the intended tree structures, we do not see any significant difference between tree-hypersequent and deep sequent systems on the one hand, and Cerrato's sequent calculi on the other; nonetheless, the latter are charged with being syntactically impure[26]. The formal improvements reached through the methods of Brünnler and Poggiolesi are undeniable, but in our opinion their calculi contain explicit Kripke semantic elements in so far as Cerrato's ones.

In the final analysis, the problem at stake reduces to state how much of semantics can be introduced into a formal calculus without compromising the purity of the calculus. This question does not seem a solvable one, and the only possible answer concerns the effectiveness of the calculus: from this point of view, it is worth noting that also in the framework of tree-hypersequents the incisive presence of semantical elements allows to obtain important results through a staightforward adaptation[27] of the results from Negri (2005).

A further concern about labelled calculi is related to the fact that "a labelled sequent [...] does not generally have an equivalent modal formula"[28]. However, we can easily single out the class of *purely logical sequents* as those sequents containing no relational atoms and in which every formula is labelled by one and the same variable x. A correspondence between a purely logical sequent $\Gamma \Rightarrow \Delta$ and its associated formula A is then defined by putting $A \equiv \wedge\Gamma^x \supset \vee\Delta^x$, where Γ^x (resp. Δ^x) consists of the formulas in Γ (resp. Δ) labelled by x. In general, root-first proof search starts from a purely logical sequents, and the introduction of sequents without a plain corrispondence with modal formulas is

[26]See Poggiolesi (2008), p. 65-66.
[27]See Poggiolesi (2008), Sections 3.6 and 3.7.
[28]See Brünnler (2006), p. 108

completely governed by the rules of the calculus.

On the other hand, if we accept the interpretation of modal logic within the logic of accessibility relation as a useful formal device, then we can yield a formal meaning to every labelled sequent, although not in the terms of its corresponding modal formula, but rather in the sense of Prior's first grade of tense-logical involvement[29].

As a final remark, we observe that, whereas "the application of the tree-hypersequent method to temporal logics seems quite complicated because of the tree shape of such a syntactic object"[30], the presence in labelled calculi of sequents with a richer syntax, which cannot be plainly interpreted as modal formulas, turns out to be an advantage, rather than a drawback, with respect to their handiness and expressiveness.

[29] See above, Section 1.1.

[30] Poggiolesi (2008), p. 184.

Chapter 2

A labelled calculus for temporal logic

2.1 The basic temporal calculus

Our labelled system for basic temporal logic is a temporal adaptation of the basic modal calculus G3K introduced in Negri (2005) and described above in Section 1.4: every formula in a sequent $\Gamma \Rightarrow \Delta$ is either a labelled formula $x : A$, intuitively corresponding to the forcing relation $x \Vdash A$ of Kripke models, or an atomic formula $x < y$, standing for the syntactic counterpart of the accessibility relation.

The rules for the propositional connectives are analogous to standard propositional rules, with active and principal formulas all marked by the same label x. Modal rules for the standard temporal operators **G**, **F**, **H**, and **P** are obtained from the meaning explanations in terms of their relational semantics:

$x \Vdash \mathbf{G}A$ *iff for all* y, $x < y$ *implies* $y \Vdash A$

$x \Vdash \mathbf{F}A$ *iff for some* y, $x < y$ *and* $y \Vdash A$

$x \Vdash \mathbf{H}A$ *iff for all* y, $y < x$ *implies* $y \Vdash A$

$x \Vdash \mathbf{P}A$ *iff for some* y, $y < x$ *and* $y \Vdash A$

As for the modalities $\Box$ and $\Diamond$, the left-to-right direction in the explanations above justifies the left rules, the right-to-left direction the right rules. The rôle of the quantifiers is reflected in the variable conditions for rules $R\mathbf{G}$, $L\mathbf{F}$, $R\mathbf{H}$, and $L\mathbf{P}$ below.

Initial sequents and $L\bot$:

$x : P, \Gamma \Rightarrow \Delta, x : P$ $\qquad$ $x < y, \Gamma \Rightarrow \Delta, x < y$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\bot}$$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\&} \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\, {\scriptstyle R\&}$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, {\scriptstyle R\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\supset} \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, {\scriptstyle R\supset}$$

Temporal rules

$$\frac{y : A, x : \mathbf{G}A, x < y, \Gamma \Rightarrow \Delta}{x : \mathbf{G}A, x < y, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\mathbf{G}} \qquad \frac{x < y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\, {\scriptstyle R\mathbf{G}}$$

$$\frac{x < y, y : A, \Gamma \Rightarrow \Delta}{x : \mathbf{F}A, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\mathbf{F}} \qquad \frac{x < y, \Gamma \Rightarrow \Delta, x : \mathbf{F}A, y : A}{x < y, \Gamma \Rightarrow \Delta, x : \mathbf{F}A}\, {\scriptstyle R\mathbf{F}}$$

$$\frac{y : A, x : \mathbf{H}A, y < x, \Gamma \Rightarrow \Delta}{x : \mathbf{H}A, y < x, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\mathbf{H}} \qquad \frac{y < x, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{H}A}\, {\scriptstyle R\mathbf{H}}$$

$$\frac{y < x, y : A, \Gamma \Rightarrow \Delta}{x : \mathbf{P}A, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\mathbf{P}} \qquad \frac{y < x, \Gamma \Rightarrow \Delta, x : \mathbf{P}A, y : A}{y < x, \Gamma \Rightarrow \Delta, x : \mathbf{P}A}\, {\scriptstyle R\mathbf{P}}$$

Rules $R\mathbf{G}$, $L\mathbf{F}$, $R\mathbf{H}$ and $L\mathbf{P}$ have the condition that y is not in the conclusion.

Table 2.1: The basic temporal calculus $\mathrm{G3K}_t$

The logical rules of the basic temporal calculus $\mathrm{G3K}_t$ are given in Table 2.1. Observe that initial sequents are restricted to labelled atomic formulas $x : P$ or relational atoms. This feature, common to all G3 systems of sequent calculus, is needed to ensure height-preserving invertibility of the rules and height-preserving admissibility of contraction (see Definition 2.3.3, Lemma 2.3.7, and Theorem 2.3.8). The rules for negation are special cases of the rules for implication, by the definition $\neg A \equiv A \supset \bot$.

Because of the internalisation of the semantics, most labelled sequents cannot be directly interpreted as temporal formulas. However, we can single out a class of sequents with a plain correspondence to their associated formulas[1]:

Definition 2.1.1. *A* purely logical *sequent is a sequent that contains no relational atoms and in which every formula is labelled by the same variable x.*

Observe that no temporal rule removes a relational atom from the right-hand side of sequents, and such atoms are never active in the propositional rules. As a consequence, initial sequents of the form $x < y, \Gamma \Rightarrow \Delta, x < y$ cannot be used in deriving purely logical sequents, and can be left out from the system[2].

2.2 Mathematical rules

The semantics for temporal logic is based on *Prior frames*. These are special Kripke frames[3] $\mathcal{F} = (\mathcal{K}, <, >)$ with two accessibility relations inverse of each other, that is $x < y \equiv y > x$. The calculus G3K$_t$ corresponds to the logic for arbitrary Prior frames: no frame property is imposed on the accessibility relation $<$. However, we often assume certain properties for the flow of time as, for example, transitivity - if an instant a precedes an instant b and b precedes c, then a precedes c -, left and right linearity - if a follows (resp. precedes) b and a follows (resp. precedes) c, then b precedes c or b is equal to c or c precedes

[1]A detailed discussion can be found in Section 1.5.
[2]See also Section 1.4
[3]See Section 1.4.

b -, and left and right seriality - every instant is preceded (resp. followed) by another instant.

Frame properties are added to the system G3K$_t$ in the form of appropriate *mathematical rules* that act on relational atoms and follow the mathematical rule schemes described in Negri and von Plato (1998, 2001) and Negri (2003) for universal axioms and geometric implications, respectively (see Section 1.4.2).

The above mentioned properties of transitivity, and left and right linearity for the flow of time are expressed by universal axioms:

Transitivity: $\forall x \forall y \forall z\, ((x < y \,\&\, y < z) \supset x < z)$

Left linearity: $\forall x \forall y \forall z\, ((y < x \,\&\, z < x) \supset (y < z \vee y = z \vee z < y))$

Right linearity: $\forall x \forall y \forall z\, ((x < y \,\&\, x < z) \supset (y < z \vee y = z \vee z < y))$

These are converted into the following mathematical rules

$$\frac{x < z, x < y, y < z, \Gamma \Rightarrow \Delta}{x < y, y < z, \Gamma \Rightarrow \Delta}\, {\scriptstyle Trans}$$

$$\frac{y < z, y < x, z < x, \Gamma \Rightarrow \Delta \quad y = z, y < x, z < x, \Gamma \Rightarrow \Delta \quad z < y, y < x, z < x, \Gamma \Rightarrow \Delta}{y < x, z < x, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\text{-}Lin}$$

$$\frac{y < z, x < y, x < z, \Gamma \Rightarrow \Delta \quad y = z, x < y, x < z, \Gamma \Rightarrow \Delta \quad z < y, x < y, x < z, \Gamma \Rightarrow \Delta}{x < y, x < z, \Gamma \Rightarrow \Delta}\, {\scriptstyle R\text{-}Lin}$$

Table 2.2: The rules for transitivity, left linearity, and right linearity

The following frame properties are geometric implications:

Density: $\forall x \forall y (x < y \supset \exists z\, (x < z \,\&\, z < y))$

Left seriality (no first instant): $\forall x \exists y\, (y < x)$

Right seriality (no last instant): $\forall x \exists y\, (x < y)$

Left directedness: $\forall x \forall y \forall z\, ((y < x \,\&\, z < x) \supset \exists w\, (w < y \,\&\, w < z))$

Right directedness: $\forall x \forall y \forall z\, ((x < y \,\&\, x < z) \supset \exists w\, (y < w \,\&\, z < w))$

These are converted into the mathematical rules of Table 2.3

$$\frac{x < w, w < y, x < y, \Gamma \Rightarrow \Delta}{x < y, \Gamma \Rightarrow \Delta}\ {\scriptstyle Dens}$$

$$\frac{w < x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle L\text{-}Ser} \qquad \frac{x < w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle R\text{-}Ser}$$

$$\frac{w < y, w < z, y < x, z < x, \Gamma \Rightarrow \Delta}{y < x, z < x, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\text{-}Dir} \qquad \frac{y < w, z < w, x < y, x < z, \Gamma \Rightarrow \Delta}{x < y, x < z, \Gamma \Rightarrow \Delta}\ {\scriptstyle R\text{-}Dir}$$

All the rules have the condition that w is not in the conclusion.

Table 2.3: The rules for density, left and right seriality, left and right directedness

As observed in Section 1.4.2, in order to guarantee admissibility of contraction, a closure condition must be satisfied stating that, if substitution in the atoms produces a duplication among the principal formulas of a mathematical rule, then also the contracted instance of the rule has to be added to the system. For example, whenever a system contains the rule for left directedness, it contains also the following instance

$$\frac{w < y, w < y, y < x, \Gamma \Rightarrow \Delta}{y < x, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\text{-}Dir}$$

Unfortunately, not every frame property can be converted into a mathematical rule by means of the methodology illustrated in Section 1.4.2: for instance, the properties of discreteness towards the future and towards the past as presented in van Benthem (1983, p. 161) for a frame without end points

$\forall x \exists y\, (x < y \,\&\, \forall z\, (z < y \supset (x = z \vee z < x)))$

$\forall x \exists y\, (y < x \,\&\, \forall z\, (y < z \supset (x = z \vee x < z)))$

are expressed neither by a universal axiom nor by a geometric implication.

However, the addition of the immediate successor relation $\prec$ allows to reformulate these properties as a pair of universal axiom stating that any instant smaller (greater) than the immediate successor (predecessor) of x is equal to or smaller (greater) than x

$\forall x \forall y \forall z\, ((x \prec y \,\&\, z < y) \supset (x = z \vee z < x))$

$\forall x \forall y \forall z\, ((y \prec x \,\&\, y < z) \supset (x = z \vee x < z))$

In Chapter 3, we shall show that the corresponding mathematical rules, together with the rules for the immediate successor relation, added to G3K$_t$+Eq (see below, Section 2.4) permit to derive the purely logical sequents equivalent to Hamblin's formulas for discreteness:

$$\mathbf{GP}A \supset (A \vee \mathbf{P}A) \qquad\qquad \mathbf{HF}A \supset (A \vee \mathbf{F}A)$$

2.3 Structural properties

In what follows we use G3K$_t$* to denote an arbitrary temporal calculus obtained by extending the basic temporal calculus with mathematical rules for the accessibility relation $<$ described in Section 2.2. We prove here that the calculus G3K$_t$* enjoys important structural properties.

Definition 2.3.1. *The* length $l(A)$ *of a temporal formula* A *is defined inductively as follows:*

$l(\bot) = 0$*;*

$l(P) = 1$ *for propositional atoms* P*;*

$l(B \circ C) = l(B) + l(C) + 1$ *for conjunction, disjunction, and implication;*

$l(\mathbf{M}B) = l(B) + 1$ *for temporal operators* $\mathbf{M}$*.*

The length of a labelled formula $x : A$ *is defined as the length of* A*.*

Finally, $l(xRy) = 1$ *for arbitrary relational atoms.*

Observe that $l(\neg A) = l(A \supset \bot) = l(A) + 1$

Definition 2.3.2. *A* derivation *is either an initial sequent, or an instance of* $L\bot$*, or an application of a logical or mathematical rule to the derivation(s) concluding its premise(s). A sequent* $\Gamma \Rightarrow \Delta$ *is* derivable *if there exists a derivation* *for it. The* height of a derivation *is the greatest number of successive applications of rules in it, where initial sequents and* $L\bot$ *have height 0.*

Definition 2.3.3. *A rule in* $\mathrm{G3K}_t{}^*$ *is* height-preserving admissible *if, whenever its premise(s) is (are) derivable, also the conclusion is derivable with the same derivation height. A rule is* height-preserving invertible *if, whenever its conclusion is derivable, also the premise(s) is (are) derivable with the same derivation height.*

Substitution of labels is defined in the obvious way as follows for relational atoms and labelled formulas.

$$
\begin{aligned}
xRy(z/w) &\equiv xRy \text{ if } w \neq x \text{ and } w \neq y \\
xRy(z/x) &\equiv zRy \text{ if } x \neq y \\
xRy(z/y) &\equiv xRz \text{ if } x \neq y \\
xRx(z/x) &\equiv zRz \\
x : A(z/y) &\equiv x : A \text{ if } y \neq x \\
x : A(z/x) &\equiv z : A
\end{aligned}
$$

The definition of substitution is extended to multisets of formulas componentwise. We have:

Lemma 2.3.4. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathrm{G3K}_t{}^*$*, then* $\Gamma(y/x) \Rightarrow \Delta(y/x)$ *is also derivable with the same derivation height.*

Proof. By induction on the height h of the derivation of the sequent $\Gamma \Rightarrow \Delta$. If $h = 0$, the sequent $\Gamma \Rightarrow \Delta$ is an initial sequent or conclusion of $L\bot$: in either case the sequent $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is also an initial sequent or conclusion of $L\bot$. Suppose that $\Gamma \Rightarrow \Delta$ is derivable with $h = n + 1$ and that the claim holds for $h = n$, and consider the last rule applied in the derivation. If it is a propositional rule, or a temporal or mathematical rule without variable condition, apply the inductive hypothesis to the premise(s) of the rule, and then the rule. If the last rule is a temporal or a mathematical rule with variable condition and x is the eigenvariable of the rule, then the substitution is vacuous and there is nothing to prove. If neither x nor y is an eigenvariable, we consider here the case of rule $R\mathbf{G}$, all the other case being analogous. If the principal formula is not labelled by x, we simply apply the inductive hypothesis to the premise of the rule, and then the rule. If the principal formula is $x : \mathbf{G}A$, we have

$$\frac{\overset{\vdots}{x < z, \Gamma \Rightarrow \Delta', z : A}}{\Gamma \Rightarrow \Delta', x : \mathbf{G}A} R\mathbf{G}$$

where $\Delta \equiv \Delta', x : \mathbf{G}A$.

We apply the inductive hypothesis to the shorter derivation of the premise and then the rule

$$\frac{\overset{\vdots}{y < z, \Gamma(y/x) \Rightarrow \Delta'(y/x), z : A}}{\Gamma(y/x) \Rightarrow \Delta'(y/x), y : \mathbf{G}A} R\mathbf{G}$$

If y is the eigenvariable, the derivation ends with

$$\frac{\overset{\vdots}{x < y, \Gamma \Rightarrow \Delta', y : A}}{\Gamma \Rightarrow \Delta', x : \mathbf{G}A} R\mathbf{G}$$

We first apply the inductive hypothesis in order to replace the eigenvariable y with a fresh variable w. By the variable condition, the substitution does not affect Γ, Δ' and we obtain a height-preserving derivation of the premise $x < w, \Gamma \Rightarrow \Delta', w : A$. Then we apply again the inductive hypothesis to replace x with y, and then rule $R\mathbf{G}$

$$\frac{\overset{\vdots}{y < w, \Gamma(y/x) \Rightarrow \Delta'(y/x), w : A}}{\Gamma(y/x) \Rightarrow \Delta'(y/x), y : \mathbf{G}A} R\mathbf{G}$$

$\square$

Lemma 2.3.5. *Sequents of the form $x : A, \Gamma \Rightarrow \Delta, x : A$ are derivable in* G3K$_t$* *for arbitrary formulas A, arbitrary contexts Γ, Δ, and arbitrary labels x.*

Proof. By induction on the length of the formula A. If $A \equiv \bot$, the sequent $x : \bot, \Gamma \Rightarrow \Delta, x : \bot$ is a conclusion of $L\bot$. If $A \equiv P$, then $x : P, \Gamma \Rightarrow \Delta, x : P$ is an initial sequent. If $A \equiv \bot \supset \bot \equiv \top$, we have the derivation

$$\dfrac{\dfrac{}{x : \bot, x : \bot \supset \bot, \Gamma \Rightarrow \Delta, x : \bot}\,{\scriptstyle L\bot}}{x : \bot \supset \bot, \Gamma \Rightarrow \Delta, x : \bot \supset \bot}\,{\scriptstyle R\supset}$$

If $A \equiv B\&C$, by inductive hypothesis the sequents $x : B, x : C, \Gamma \Rightarrow \Delta, x : B$ and $x : B, x : C, \Gamma \Rightarrow \Delta, x : C$ are derivable, so we have the derivation

$$\dfrac{\dfrac{x : B, x : C, \Gamma \Rightarrow \Delta, x : B}{x : B\&C, \Gamma \Rightarrow \Delta, x : B}\,{\scriptstyle L\&} \qquad \dfrac{x : B, x : C, \Gamma \Rightarrow \Delta, x : C}{x : B\&C, \Gamma \Rightarrow \Delta, x : C}\,{\scriptstyle L\&}}{x : B\&C, \Gamma \Rightarrow \Delta, x : B\&C}\,{\scriptstyle R\&}$$

If $A \equiv B \vee C$, by inductive hypothesis the sequents $x : B, \Gamma \Rightarrow \Delta, x : B, x : C$ and $x : C, \Gamma \Rightarrow \Delta, x : B, x : C$ are derivable, so we have the derivation

$$\dfrac{\dfrac{x : B, \Gamma \Rightarrow \Delta, x : B, x : C}{x : B, \Gamma \Rightarrow \Delta, x : B \vee C}\,{\scriptstyle R\vee} \qquad \dfrac{x : C, \Gamma \Rightarrow \Delta, x : B, x : C}{x : C, \Gamma \Rightarrow \Delta, x : B \vee C}\,{\scriptstyle R\vee}}{x : B \vee C, \Gamma \Rightarrow \Delta, x : B \vee C}\,{\scriptstyle L\vee}$$

If $A \equiv B \supset C$, by inductive hypothesis the sequents $x : B, \Gamma \Rightarrow \Delta, x : B, x : C$ and $x : C, \Gamma \Rightarrow \Delta, x : B, x : C$ are derivable, thus we have the derivation

$$\dfrac{\dfrac{x : B, \Gamma \Rightarrow \Delta, x : C, x : B \quad x : C, x : B, \Gamma \Rightarrow \Delta, x : C}{x : B \supset C, x : B, \Gamma \Rightarrow \Delta, x : C}\,{\scriptstyle L\supset}}{x : B \supset C, \Gamma \Rightarrow \Delta, x : B \supset C}\,{\scriptstyle R\supset}$$

If $A \equiv \mathbf{G}B$, by inductive hypothesis $x < y, y : B, x : \mathbf{G}B, \Gamma \Rightarrow \Delta, y : B$ is derivable, thus we have the derivation

$$\dfrac{\dfrac{x < y, y : B, x : \mathbf{G}B, \Gamma \Rightarrow \Delta, y : B}{x < y, x : \mathbf{G}B, \Gamma \Rightarrow \Delta, y : B}\,{\scriptstyle L\mathbf{G}}}{x : \mathbf{G}B, \Gamma \Rightarrow \Delta, x : \mathbf{G}B}\,{\scriptstyle R\mathbf{G}}$$

where we choose y different from x and not in Γ, Δ.

If $A \equiv \mathbf{F}B$, by inductive hypothesis $x < y, y : B, \Gamma \Rightarrow \Delta, x : \mathbf{F}B, y : B$ is derivable, and we have

$$\dfrac{\dfrac{x < y, y : B, \Gamma \Rightarrow \Delta, x : \mathbf{F}B, y : B}{x < y, y : B, \Gamma \Rightarrow \Delta, x : \mathbf{F}B} \; R\mathbf{F}}{x : \mathbf{F}B, \Gamma \Rightarrow \Delta, x : \mathbf{F}B} \; L\mathbf{F}$$

where we choose y different from x and not in Γ, Δ.

If $A \equiv \mathbf{H}B$, by inductive hypothesis $y < x, y : B, x : \mathbf{H}B, \Gamma \Rightarrow \Delta, y : B$ is derivable, and we have

$$\dfrac{\dfrac{y < x, y : B, x : \mathbf{H}B, \Gamma \Rightarrow \Delta, y : B}{y < x, x : \mathbf{H}B, \Gamma \Rightarrow \Delta, y : B} \; L\mathbf{H}}{x : \mathbf{H}B, \Gamma \Rightarrow \Delta, x : \mathbf{H}B} \; R\mathbf{H}$$

where we choose y different from x and not in Γ, Δ.

If $A \equiv \mathbf{P}B$, by inductive hypothesis $y < x, y : B, \Gamma \Rightarrow \Delta, x : \mathbf{P}B, y : B$ is derivable; consider the following derivation

$$\dfrac{\dfrac{y < x, y : B, \Gamma \Rightarrow \Delta, x : \mathbf{P}B, y : B}{y < x, y : B, \Gamma \Rightarrow \Delta, x : \mathbf{P}B} \; R\mathbf{P}}{x : \mathbf{P}B, \Gamma \Rightarrow \Delta, x : \mathbf{P}B} \; L\mathbf{P}$$

where we choose y different from x and not in Γ, Δ. $\square$

In what follows, Greek lower case is used for denoting labelled formulas or relational atoms.

Theorem 2.3.6. *The rules of left and right weakening*

$$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \; LWk \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \; RWk$$

are height-preserving admissible in $\mathrm{G3K}_t{}^*$.

Proof. By induction on the height of the derivation of the premise. If $\Gamma \Rightarrow \Delta$ is an initial sequent or conclusion of $L\bot$, so are $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi$. The cases of rules without variable condition are straightforward. If the last step is a rule with a variable condition, we first apply Lemma 2.3.4 to avoid a clash of variables and then the inductive hypothesis and the rule in question. Let us consider for instance the case with $\Gamma \Rightarrow \Delta$ concluded by $R\mathbf{G}$ and $\varphi \equiv y : B$

$$\frac{x < y, \Gamma \Rightarrow \Delta', y : A}{\Gamma \Rightarrow \Delta', x : \mathbf{G}A} \, R\mathbf{G}$$

where $\Delta \equiv \Delta', x : \mathbf{G}A$.

We first apply Lemma 2.3.4 to replace the eigenvariable y with a fresh variable w; then we apply the inductive hypothesis and the rule $R\mathbf{G}$

$$\frac{y : B, x < w, \Gamma \Rightarrow \Delta', w : A}{y : B, \Gamma \Rightarrow \Delta', x : \mathbf{G}A} \, R\mathbf{G} \qquad \frac{x < w, \Gamma \Rightarrow \Delta', w : A, y : B,}{\Gamma \Rightarrow \Delta', x : \mathbf{G}A, y : B} \, R\mathbf{G}$$

□

Lemma 2.3.7. *All the rules of* G3K$_t$* *are height-preserving invertible.*

Proof. By induction on the height of derivation. For the propositional rules we consider in detail only $L \supset$ and $R \supset$, all the other cases being analogous.

If $x : A \supset B, \Gamma \Rightarrow \Delta$ is an initial sequent or conclusion of $L\bot$, then $x : A \supset B$ is not principal in it and also $\Gamma \Rightarrow \Delta, x : A$ and $x : B, \Gamma \Rightarrow \Delta$ are initial sequents or conclusions of $L\bot$. Let us suppose that we have a derivation with height $h = n + 1$ and that the claim holds for $h = n$, and consider the last rule applied. If the sequent $x : A \supset B, \Gamma \Rightarrow \Delta$ is the conclusion of $L \supset$ with principal formula $x : A \supset B$, then $\Gamma \Rightarrow \Delta, x : A$ and $x : B, \Gamma \Rightarrow \Delta$ are derived with $h \leq n$. If $x : A \supset B, \Gamma \Rightarrow \Delta$ is conclusion of a rule different from $L \supset$ or with principal formula other than $x : A \supset B$, we apply the inductive hypothesis to the premise(s) $x : A \supset B, \Gamma' \Rightarrow \Delta'$ (and $x : A \supset B, \Gamma'' \Rightarrow \Delta''$) thus obtaining derivations with $h \leq n$ of $\Gamma' \Rightarrow \Delta', x : A$ and $x : B, \Gamma' \Rightarrow \Delta'$

($\Gamma'' \Rightarrow \Delta'', x : A$ and $x : B, \Gamma'' \Rightarrow \Delta''$). Next we apply the rule to $\Gamma' \Rightarrow \Delta', x : A$ (and $\Gamma'' \Rightarrow \Delta'', x : A$) and to $x : B, \Gamma' \Rightarrow \Delta'$ (and $x : B, \Gamma'' \Rightarrow \Delta''$) to conclude $\Gamma \Rightarrow \Delta, x : A$ and $x : B, \Gamma \Rightarrow \Delta$ in at most $n + 1$ step.

If $\Gamma \Rightarrow \Delta, x : A \supset B$ is an initial sequent or conclusion of $L\perp$, then $x : A \supset B$ is not principal in it and also $x : A, \Gamma \Rightarrow \Delta, x : B$ is an initial sequent or conclusion of $L\perp$. Let us suppose now that we have a derivation with height $h = n + 1$ and that the claim holds for $h = n$. If $\Gamma \Rightarrow \Delta, x : A \supset B$ is conclusion of $R\supset$ with principal formula $x : A \supset B$, then $x : A, \Gamma \Rightarrow \Delta, x : B$ is derived with $h = n$. If the last rule is different from $R\supset$ or it is $R\supset$ with principal formula other than $x : A \supset B$, we apply the inductive hypothesis to the premise(s) $\Gamma' \Rightarrow \Delta', x : A \supset B$ (and $\Gamma'' \Rightarrow \Delta'', x : A \supset B$) thus obtaining a derivation with $h \leq n$ of $x : A, \Gamma' \Rightarrow \Delta', x : B$ (and $x : A, \Gamma'' \Rightarrow \Delta'', x : B$). Next we apply the rule to conclude $x : A, \Gamma \Rightarrow \Delta, x : B$ in at most $n + 1$ step.

If the last rule is a temporal rule with a variable condition, we consider in detail only $R\mathbf{G}$, all the other cases being analogous. If $\Gamma \Rightarrow \Delta, x : \mathbf{G}A$ is an initial sequent or conclusion of $L\perp$, then $x : \mathbf{G}A$ is not principal in it and also $x < y, \Gamma \Rightarrow \Delta, y : A$ is an initial sequent or conclusion of $L\perp$. Let us suppose now that we have a derivation with height $h = n + 1$ and that the claim holds for $h = n$, and consider the last rule applied. If $\Gamma \Rightarrow \Delta, x : \mathbf{G}A$ is conclusion of $R\mathbf{G}$ with principal formula $x : \mathbf{G}A$, then $x < y, \Gamma \Rightarrow \Delta, y : A$ has a derivation with $h = n$. If $\Gamma \Rightarrow \Delta, x : \mathbf{G}A$ is the conclusion of a rule without variable condition, we apply the inductive to the premise(s) $\Gamma' \Rightarrow \Delta', x : \mathbf{G}A$ (and $\Gamma'' \Rightarrow \Delta'', x : \mathbf{G}A$) thus obtaining a derivation with $h \leq n$ of the sequent $x < y, \Gamma' \Rightarrow \Delta', y : A$ (and $x < y, \Gamma'' \Rightarrow \Delta'', y : A$). Next, we apply the rule to conclude $x < y, \Gamma \Rightarrow \Delta, y : A$ in at most $n + 1$ step. If the last step is a rule with a variable condition, we apply first Lemma 2.3.4 in order to avoid a clash of variables and then the inductive hypothesis and the rule in question. Consider the following instance with $\Gamma \Rightarrow \Delta, x : \mathbf{G}A$ concluded by $L\mathbf{P}$ and with $\Gamma \equiv w : \mathbf{P}B, \Gamma'$

$$\dfrac{y < w, y : B, \Gamma' \Rightarrow \Delta, x : \mathbf{G}A}{w : \mathbf{P}B, \Gamma' \Rightarrow \Delta, x : \mathbf{G}A}\,L\mathbf{P} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{y < w, y : B, \Gamma' \Rightarrow \Delta, x : \mathbf{G}A}{v < w, v : B, \Gamma' \Rightarrow \Delta, x : \mathbf{G}A}\,H.\text{-}p.Subst}{x < y, v < w, v : B, \Gamma' \Rightarrow \Delta, y : A}\,Ind.Hyp.}{x < y, w : \mathbf{P}B, \Gamma' \Rightarrow \Delta, y : A}\,L\mathbf{P}$$

where v has been chosen different from x, y, w and not in Γ', Δ.

Finally, the temporal rules $L\mathbf{G}$, $R\mathbf{F}$, $L\mathbf{H}$, $R\mathbf{P}$, and the mathematical rules for the accessibility relation are trivially height-preserving invertible since their respective premises are obtained by height-preserving weakening from the conclusion. As usual, clash of variables is avoided through Lemma 2.3.4. $\square$

Theorem 2.3.8. *The rules of left and right contraction*

$$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,LCtr \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\,RCtr$$

are height-preserving admissible in $\mathrm{G3K}_t{}^*$.

Proof. By simultaneous induction on the height of derivation for left and right contraction. For $h = 0$, note that if $\varphi, \varphi, \Gamma \Rightarrow \Delta$ (resp. $\Gamma \Rightarrow \Delta, \varphi, \varphi$) is an initial sequent or conclusion of $L\bot$, so is $\varphi, \Gamma \Rightarrow \Delta$ (resp. $\Gamma \Rightarrow \Delta, \varphi$). For $h = n + 1$, we distinguish two cases: if none of the contraction formulas is principal in the last rule, then both occurrences are in the premise(s) and we apply the inductive hypothesis to the premise(s) and then the rule. If one of the contraction formulas is principal, we first apply Lemma 2.3.7 to the premise(s), the inductive hypothesis and then the rule. For the details, we distinguish three cases: (i) active formulas are proper subformulas of principal formulas, as in propositional rules; (ii) the principal formula(s) of the rule appear(s) in the premise(s), as in $L\mathbf{G}$, $R\mathbf{F}$, $L\mathbf{H}$, $R\mathbf{P}$ and in the mathematical rules; (iii) active formulas are relational atoms and proper subformulas of principal formulas, as in $R\mathbf{G}$, $L\mathbf{F}$, $R\mathbf{H}$, $L\mathbf{P}$.

In case (i) we consider in detail the rule $L\&$

$$\dfrac{\dfrac{x : A, x : B, x : A\&B, \Gamma \Rightarrow \Delta}{x : A\&B, x : A\&B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\&}}{x : A\&B, \Gamma \Rightarrow \Delta}\,{\scriptstyle LCtr}$$

We apply Lemma 2.3.7 to obtain $x : A, x : B, x : A, x : B, \Gamma \Rightarrow \Delta$ from the premise of $L\&$. Next, we apply twice the inductive hypothesis to obtain $x : A, x : B, \Gamma \Rightarrow \Delta$, and finally $L\&$ to conclude $x : A\&B, \Gamma \Rightarrow \Delta$ in at most $n + 1$ step. The cases of $R\&$, $L\vee$ and $R\vee$ are analogoous.

Observe that if the last rule is a rule for $\supset$, simultaneous induction on left and right contraction is required. In the case of left contraction on $x : A \supset B$, we have

$$\dfrac{\dfrac{x : A \supset B, \Gamma \Rightarrow \Delta, x : A \quad x : B, x : A \supset B, \Gamma \Rightarrow \Delta}{x : A \supset B, x : A \supset B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\supset}}{x : A \supset B, \Gamma \Rightarrow \Delta}\,{\scriptstyle LCtr}$$

We apply Lemma 2.3.7 to obtain $\Gamma \Rightarrow \Delta, x : A, x : A$ and $x : B, x : B, \Gamma \Rightarrow \Delta$ with $h \leq n$ from the premises of $L \supset$. Next, we apply the inductive hypotheses to obtain $\Gamma \Rightarrow \Delta, x : A$ and $x : B, \Gamma \Rightarrow \Delta$, and finally $L \supset$ to conclude $x : A \supset B, \Gamma \Rightarrow \Delta$ in at most $n + 1$ step.

Analogously, in the case of right contraction on $x : A \supset B$, we have

$$\dfrac{\dfrac{x : A, \Gamma \Rightarrow \Delta, x : A \supset B, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B, x : A \supset B}\,{\scriptstyle R\supset}}{\Gamma \Rightarrow \Delta, x : A \supset B}\,{\scriptstyle RCtr}$$

We apply Lemma 2.3.7 to obtain $x : A, x : A, \Gamma \Rightarrow \Delta, x : B, x : B$ with $h = n$ from the premise of $R \supset$. Next, we apply twice the inductive hypothesis to obtain $x : A, \Gamma \Rightarrow \Delta, x : B$, and finally $R\supset$ to conclude $\Gamma \Rightarrow \Delta, x : A \supset B$ in at most $n+1$ step.

In case (ii) we consider in detail only the rule $L\mathbf{G}$

$$\dfrac{\dfrac{x < y, y : A, x : \mathbf{G}A, x : \mathbf{G}A, \Gamma' \Rightarrow \Delta}{x < y, x : \mathbf{G}A, x : \mathbf{G}A, \Gamma' \Rightarrow \Delta}\, L\mathbf{G}}{x < y, x : \mathbf{G}A, \Gamma' \Rightarrow \Delta}\, LCtr$$

with $\Gamma \equiv x < y, \Gamma'$

By applying the inductive hypothesis on the premise of $L\mathbf{G}$, we obtain the sequent $x < y, y : A, x : \mathbf{G}A, \Gamma \Rightarrow \Delta$. Next, we apply rule $L\mathbf{G}$ to conclude $x < y, x : \mathbf{G}A, \Gamma \Rightarrow \Delta$ in at most $n+1$ step. The cases of $R\mathbf{F}$, $L\mathbf{H}$ and $R\mathbf{P}$ are analogous.

In case (iii) we consider in detail only the rule $L\mathbf{F}$

$$\dfrac{\dfrac{x < y, y : A, x : \mathbf{F}A, \Gamma \Rightarrow \Delta}{x : \mathbf{F}A, x : \mathbf{F}A, \Gamma \Rightarrow \Delta}\, L\mathbf{F}}{x : \mathbf{F}A, \Gamma \Rightarrow \Delta}\, LCtr$$

We apply Lemma 2.3.7 to obtain $x < y, x < y, y : A, y : A, \Gamma \Rightarrow \Delta$ from the premise of $L\mathbf{F}$. Next, we apply twice the inductive hypothesis to the latter sequent to obtain $x < y, y : A, \Gamma \Rightarrow \Delta$, and finally $L\mathbf{F}$ to conclude $x : \mathbf{F}A, \Gamma \Rightarrow \Delta$ in at most $n+1$ step. The cases of $R\mathbf{G}$, $R\mathbf{H}$ and $L\mathbf{P}$ are analogous.

Observe that the case with both contraction formulas principal in a mathematical rule is taken care by the closure condition (see Section 1.4.2). $\square$

Definition 2.3.9. *The* height of a cut *is the sum of the heights of the derivations of its premises.*

Theorem 2.3.10. *The rule of cut*

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut$$

is admissible in G3K$_t$*.

Proof. The proof is by induction on the length of the cut formula, with subinduction on cut-height. We consider first the case in which at least one of the premise of cut is an initial sequent or conclusion of $L\bot$. Then, we consider three cases: (i) the cut formula is not principal in the left premise of cut; (ii) the cut formula is not principal in the right premise; (iii) the cut formula is principal in both premises.

If the left premise is an initial sequent, and φ is not principal in it, also the conclusion of cut is an initial sequent. Otherwise, we have

$$\dfrac{\varphi, \Gamma'' \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\varphi, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

with $\Gamma \equiv \varphi, \Gamma''$

The conclusion of cut is obtained by weakening on the right premise.

If the left premise is conclusion of $L\bot$, we have

$$\dfrac{\dfrac{}{x : \bot, \Gamma'' \Rightarrow \Delta, \varphi}\,{\scriptstyle L\bot} \quad \varphi, \Gamma' \Rightarrow \Delta'}{x : \bot, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

with $\Gamma \equiv x : \bot, \Gamma'$

The conclusion of cut is also conclusion of $L\bot$.

If the right premise is an initial sequent, and φ is not principal in it, also the conclusion of cut is an initial sequent. Otherwise, we have

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'', \varphi}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', \varphi}\,{\scriptstyle Cut}$$

with $\Delta' \equiv \Delta'', \varphi$

The conclusion of cut is obtained by weakening on the left premise.

If the right premise is the conclusion of an instance of $L\bot$, and φ is not principal, also the conclusion of cut is conclusion of $L\bot$. Otherwise, we have

$$\dfrac{\Gamma \Rightarrow \Delta, x : \bot \quad \dfrac{}{x : \bot, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle L\bot}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

However, $x : \bot$ cannot be principal in the left premise, and the corresponding transformation is a special case of (i) below.

(i) If the cut formula is not principal in the left premise of cut, cut is permuted up with respect to the rule concluding it. We consider here only the case of $L\mathbf{G}$, all the other cases being analogous.

$$\dfrac{\dfrac{x < y, y : A, x : \mathbf{G}A, \Gamma'' \Rightarrow \Delta, \varphi}{x < y, x : \mathbf{G}A, \Gamma'' \Rightarrow \Delta, \varphi}\, {\scriptstyle L\mathbf{G}} \qquad \varphi, \Gamma' \Rightarrow \Delta'}{x < y, x : \mathbf{G}A, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Cut}$$

with $\Gamma \equiv x < y, x : \mathbf{G}A, \Gamma''$

We perform the following transformation

$$\dfrac{\dfrac{x < y, y : A, x : \mathbf{G}A, \Gamma'' \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{x < y, y : A, x : \mathbf{G}A, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Cut}}{x < y, x : \mathbf{G}A, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\, {\scriptstyle L\mathbf{G}}$$

thus obtaining a cut with less height. Note that this transformation applies to all the cases in which φ is a relational atom, since no rule of $\mathrm{G3K}_t$* can introduce a relational atom as a principal formula in the succedent. If the rule concluding the left premise of cut is a rule with a variable condition, we apply Lemma 2.3.4 in order to avoid a clash of variables.

(ii) If the cut formula is not principal in the right premise of cut, cut is permuted up with respect to the rule that concludes it. We consider here only the case of $L\mathbf{F}$, all the other cases being analogous. We have

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \dfrac{\varphi, x < y, y : A, \Gamma'' \Rightarrow \Delta'}{\varphi, x : \mathbf{F}A, \Gamma'' \Rightarrow \Delta'}\, {\scriptstyle L\mathbf{F}}}{x : \mathbf{F}A, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Cut}$$

with $\Gamma' \equiv x : \mathbf{F}A, \Gamma''$

We perform the following transformation

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, x < y, y : A, \Gamma'' \Rightarrow \Delta'}{x < y, y : A, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Cut}}{x : \mathbf{F}A, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\, {\scriptstyle L\mathbf{F}}$$

thus obtaining a cut with less height. As usual, clash of variables is avoided through Lemma 2.3.4.

(iii) If the cut formula is principal in both premises, we have to consider several cases, according to the form of the cut formula. As noticed before, relational atoms cannot be principal in the succedent, so this case is excluded.

If $\varphi \equiv A\&B$, we have

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}{\scriptstyle R\&} \quad \dfrac{x : A, x : B, \Gamma' \Rightarrow \Delta'}{x : A\&B, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\&}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}$$

We perform the following transformation

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : B \quad \dfrac{\Gamma \Rightarrow \Delta, x : A \quad x : A, x : B, \Gamma' \Rightarrow \Delta'}{x : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}{\scriptstyle Cut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Ctr^*}$$

Thus obtaining two cuts on smaller cut formulas. We use Ctr^* to denote several contractions.

If $\varphi \equiv A \vee B$, we have

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}{\scriptstyle R\vee} \quad \dfrac{x : A, \Gamma' \Rightarrow \Delta' \quad x : B, \Gamma' \Rightarrow \Delta'}{x : A \vee B, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\vee}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}$$

We perform the following transformation

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A, x : B \quad x : B, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x : A}{\scriptstyle Cut} \quad x : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}{\scriptstyle Cut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Ctr^*}$$

Thus obtaining two cuts on smaller cut formulas.

If $\varphi \equiv A \supset B$, we have

$$\dfrac{\dfrac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}{\scriptstyle R\supset} \quad \dfrac{\Gamma' \Rightarrow \Delta', x : A \quad x : B, \Gamma' \Rightarrow \Delta'}{x : A \supset B, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\supset}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}$$

We perform the following transformation

$$\dfrac{\Gamma' \Rightarrow \Delta', x : A \qquad \dfrac{x : A, \Gamma \Rightarrow \Delta, x : B \quad x : B, \Gamma' \Rightarrow \Delta'}{x : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Cut}}{\dfrac{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Ctr^*}}\, {\scriptstyle Cut}$$

Thus obtaining two cuts on smaller cut formulas.

If $\varphi \equiv \mathbf{G}A$, we have

$$\dfrac{\dfrac{x < z, \Gamma \Rightarrow \Delta, z : A}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\, {\scriptstyle R\mathbf{G}} \qquad \dfrac{x < y, y : A, x : \mathbf{G}A, \Gamma'' \Rightarrow \Delta'}{x < y, x : \mathbf{G}A, \Gamma'' \Rightarrow \Delta'}\, {\scriptstyle L\mathbf{G}}}{x < y, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Cut}$$

with $\Gamma' \equiv x < y, \Gamma''$

We perform the following transformation

$$\dfrac{\dfrac{x < z, \Gamma \Rightarrow \Delta, z : A}{x < y, \Gamma \Rightarrow \Delta, y : A}\, {\scriptstyle H.\text{-}p.Subst} \qquad \dfrac{\dfrac{x < z, \Gamma \Rightarrow \Delta, z : A}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\, {\scriptstyle R\mathbf{G}} \quad x < y, y : A, x : \mathbf{G}A, \Gamma'' \Rightarrow \Delta'}{x < y, y : A, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Cut}}{\dfrac{x < y, x < y, \Gamma, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta, \Delta'}{x < y, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Ctr^*}}\, {\scriptstyle Cut}$$

The first cut has reduced cut-height and the second is on a smaller cut formula.

If $\varphi \equiv \mathbf{F}A$, we have

$$\dfrac{\dfrac{x < y, \Gamma'' \Rightarrow \Delta, x : \mathbf{F}A, y : A}{x < y, \Gamma'' \Rightarrow \Delta, x : \mathbf{F}A}\, {\scriptstyle R\mathbf{F}} \qquad \dfrac{x < z, z : A, \Gamma' \Rightarrow \Delta'}{x : \mathbf{F}A, \Gamma' \Rightarrow \Delta'}\, {\scriptstyle L\mathbf{F}}}{x < y, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Cut}$$

with $\Gamma \equiv x < y, \Gamma''$

We perform the following transformation

$$\dfrac{\dfrac{x < y, \Gamma'' \Rightarrow \Delta, x : \mathbf{F}A, y : A \qquad \dfrac{x < z, z : A, \Gamma' \Rightarrow \Delta'}{x : \mathbf{F}A, \Gamma' \Rightarrow \Delta'}\, {\scriptstyle L\mathbf{F}}}{x < y, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta', y : A}\, {\scriptstyle Cut} \qquad \dfrac{x < z, z : A, \Gamma' \Rightarrow \Delta'}{x < y, y : A, \Gamma' \Rightarrow \Delta'}\, {\scriptstyle H.\text{-}p.Subst}}{\dfrac{x < y, x < y, \Gamma'', \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}{x < y, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\, {\scriptstyle Ctr^*}}\, {\scriptstyle Cut}$$

The first cut has reduced cut-height and the second is on a smaller cut formula.

If $\varphi \equiv \mathbf{H}A$, we have

$$\dfrac{\dfrac{z < x, \Gamma \Rightarrow \Delta, z : A}{\Gamma \Rightarrow \Delta, x : \mathbf{H}A}\,{\scriptstyle R\mathbf{H}} \qquad \dfrac{y < x, y : A, x : \mathbf{H}A, \Gamma'' \Rightarrow \Delta'}{y < x, x : \mathbf{H}A, \Gamma'' \Rightarrow \Delta'}\,{\scriptstyle L\mathbf{H}}}{y < x, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

with $\Gamma' \equiv y < x, \Gamma''$

We perform the following transformation

$$\dfrac{\dfrac{\dfrac{z < x, \Gamma \Rightarrow \Delta, z : A}{y < x, \Gamma \Rightarrow \Delta, y : A}\,{\scriptstyle H.\text{-}p.Subst} \qquad \dfrac{\dfrac{z < x, \Gamma \Rightarrow \Delta, z : A}{\Gamma \Rightarrow \Delta, x : \mathbf{H}A}\,{\scriptstyle R\mathbf{H}} \quad y < x, y : A, x : \mathbf{H}A, \Gamma'' \Rightarrow \Delta'}{y < x, y : A, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}}{y < x, y < x, \Gamma, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta, \Delta'}\,{\scriptstyle Cut}}{y < x, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Ctr^*}$$

The first cut has reduced cut-height and the second is on a smaller cut formula.

If $\varphi \equiv \mathbf{P}A$, we have

$$\dfrac{\dfrac{y < x, \Gamma'' \Rightarrow \Delta, x : \mathbf{P}A, y : A}{y < x, \Gamma'' \Rightarrow \Delta, x : \mathbf{P}A}\,{\scriptstyle R\mathbf{P}} \qquad \dfrac{z < x, z : A, \Gamma' \Rightarrow \Delta'}{x : \mathbf{P}A, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle L\mathbf{P}}}{y < x, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

with $\Gamma \equiv y < x, \Gamma''$

We perform the following transformation

$$\dfrac{\dfrac{\dfrac{y < x, \Gamma'' \Rightarrow \Delta, x : \mathbf{P}A, y : A \qquad \dfrac{z < x, z : A, \Gamma' \Rightarrow \Delta'}{x : \mathbf{P}A, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle L\mathbf{P}}}{y < x, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta', y : A}\,{\scriptstyle Cut} \qquad \dfrac{z < x, z : A, \Gamma' \Rightarrow \Delta'}{y < x, y : A, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle H.\text{-}p.Subst}}{y < x, y < x, \Gamma'', \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}\,{\scriptstyle Cut}}{y < x, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Ctr^*}$$

The first cut has reduced cut-height and the second is on a smaller cut formula.

□

As a consequence of cut elimination, every formula in a derivation of the sequent $\Gamma \Rightarrow \Delta$ is either a subformula of the formulas in Γ, Δ or a relational atomic formula:

Corollary 2.3.11. *The calculus* G3K$_t$* *enjoys the weak subformula property.*

2.4 Equality

In some cases we need the relation of equality to express certain frame properties; this is the case when the accessibility relation is left and/or right linear

$\forall x \forall y \forall z\, ((y < x \,\&\, z < x) \supset (y < z \vee y = z \vee z < y))$

$\forall x \forall y \forall z\, ((x < y \,\&\, x < z) \supset (y < z \vee y = z \vee z < y))$

or left and/or right discrete

$\forall x \exists y\, (x < y \,\&\, \forall z\, (z < y \supset (x = z \vee z < x)))$

$\forall x \exists y\, (y < x \,\&\, \forall z\, (y < z \supset (x = z \vee x < z)))$

Therefore G3K$_t$* should be extended with initial sequents, and appropriate rules corresponding to reflexivity of equality and substitution of equals in relational atoms and as labels for propositional atoms.

$$x = y, \Gamma \Rightarrow \Delta, x = y \qquad \frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, {\scriptstyle EqRef}$$

$$\frac{At(y), x = y, At(x), \Gamma \Rightarrow \Delta}{x = y, At(x), \Gamma \Rightarrow \Delta}\, {\scriptstyle EqSubst_{At}} \qquad \frac{y : P, x = y, x : P, \Gamma \Rightarrow \Delta}{x = y, x : P, \Gamma \Rightarrow \Delta}\, {\scriptstyle EqSubst}$$

$At(y)$ stands for an equality or an arbitrary relational atom yRz or zRy, and P is a propositional atom.

Table 2.4: The rules for equality

Nonlogical rules for equality were introduced in Negri and von Plato (2001), and extended to labelled modal calculi in Negri (2005).

Let G3K$_t$*+Eq be the calculus obtained by adding the rules of Table 2.4 to G3K$_t$*. As an application of the results in Section 2.3, we have:

Theorem 2.4.1. *All the rules of* G3K$_t$*+Eq *are height-preserving invertible. The rules of substitution, left and right weakening, left and right contraction are height-preserving admissible. The calculus* G3K$_t$*+Eq *enjoys cut elimination.*

The relation of equality enjoys the properties of reflexivity, symmetry, and transitivity. Reflexivity has been transformed into the rule $EqRef$ above, whereas symmetry and transitivity can be turned into the corresponding rules

$$\frac{y = x, x = y, \Gamma \Rightarrow \Delta}{x = y, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSym} \qquad\qquad \frac{x = y, x = z, z = y, \Gamma \Rightarrow \Delta}{x = z, z = y, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqTrans}$$

However, the following theorem states that we do not need to assume these rules explicitly:

Proposition 2.4.2. *The rules of symmetry and transitivity for equality are admissible in* G3K$_t$*+Eq.

Proof. The proof consists of the following derivations:

$$\dfrac{\dfrac{\dfrac{y = x, x = y, \Gamma \Rightarrow \Delta}{y = x, x = x, x = y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk}}{x = x, x = y, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst_{At}}}{x = y, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqRef} \qquad \dfrac{\dfrac{\dfrac{x = y, x = z, z = y, \Gamma \Rightarrow \Delta}{z = x, x = y, x = z, z = y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk}}{z = x, x = z, z = y, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst_{At}}}{x = z, z = y, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSym}$$

□

The rule of substitution of equals $EqSubst$ is restricted to atomic formulas for the purposes of proof analysis, but the generalisation to arbitrary temporal formulas

$$\frac{y : A, x = y, x : A, \Gamma \Rightarrow \Delta}{x = y, x : A, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst}$$

is admissible in G3K$_t$*+Eq.

Lemma 2.4.3. *The sequent $x = y, x : A \Rightarrow y : A$ is derivable in* G3K$_t$*+Eq *for arbitrary temporal formula A.*

Proof. By induction on the length of A. If $A \equiv \bot$, then $x = y, x : \bot \Rightarrow y : \bot$ is an instance of $L\bot$. If $A \equiv \bot \supset \bot$, we have the following derivation

$$\dfrac{\dfrac{}{x = y, y : \bot, x : \bot \supset \bot \Rightarrow y : \bot}\ {\scriptstyle L\bot}}{x = y, x : \bot \supset \bot \Rightarrow y : \bot \supset \bot}\ {\scriptstyle R\supset}$$

If A is a propositional atom, then $x = y, x : P \Rightarrow y : P$ is given by the following derivation

$$\dfrac{x = y, y : P, x : P \Rightarrow y : P}{x = y, x : P \Rightarrow y : P}\ {\scriptstyle EqSubst}$$

If $A \equiv B\&C$ we apply the inductive hypothesis to B and C, and then left and right rules

$$\dfrac{\dfrac{\dfrac{x = y, x : B \Rightarrow y : B}{x = y, x : B, x : C \Rightarrow y : B}\ {\scriptstyle LWk} \qquad \dfrac{x = y, x : C \Rightarrow y : C}{x = y, x : B, x : C \Rightarrow y : C}\ {\scriptstyle LWk}}{x = y, x : B, x : C \Rightarrow y : B\&C}\ {\scriptstyle R\&}}{x = y, x : B\&C \Rightarrow y : B\&C}\ {\scriptstyle L\&}$$

If $A \equiv B \vee C$, we have

$$\dfrac{\dfrac{\dfrac{x = y, x : B \Rightarrow y : B}{x = y, x : B \Rightarrow y : B, y : C}\ {\scriptstyle RWk} \qquad \dfrac{x = y, x : C \Rightarrow y : C}{x = y, x : C \Rightarrow y : B, y : C}\ {\scriptstyle LWk}}{x = y, x : B \vee C \Rightarrow y : B, y : C}\ {\scriptstyle L\vee}}{x = y, x : B \vee C \Rightarrow y : B \vee C}\ {\scriptstyle R\vee}$$

If $A \equiv B \supset C$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y = x, y : B \Rightarrow x : B}{x = y, y = x, y : B \Rightarrow x : B}\ {\scriptstyle LWk}}{x = y, y : B \Rightarrow x : B}\ {\scriptstyle EqSym}}{x = y, y : B \Rightarrow y : C, x : B}\ {\scriptstyle RWk} \qquad \dfrac{x = y, x : C \Rightarrow y : C}{x = y, y : B, x : C \Rightarrow y : C}\ {\scriptstyle LWk}}{x = y, y : B, x : B \supset C \Rightarrow y : C}\ {\scriptstyle L\supset}}{x = y, x : B \supset C \Rightarrow y : B \supset C}\ {\scriptstyle R\supset}$$

If $A \equiv \mathbf{G}B$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{y=x, x=y, x<z, y<z, z:B, x:\mathbf{G}B \Rightarrow z:B}{y=x, x=y, x<z, y<z, x:\mathbf{G}B \Rightarrow z:B}\,{\scriptstyle L\mathbf{G}}}{y=x, x=y, y<z, x:\mathbf{G}B \Rightarrow z:B}\,{\scriptstyle EqSubst_{At}}}{x=y, y<z, x:\mathbf{G}B \Rightarrow z:B}\,{\scriptstyle EqSym}}{x=y, x:\mathbf{G}B \Rightarrow y:\mathbf{G}B}\,{\scriptstyle R\mathbf{G}}$$

The sequent $y = x, x = y, x < z, y < z, z : B, x : \mathbf{G}B \Rightarrow z : B$ is derivable by Lemma 2.3.5.

If $A \equiv \mathbf{F}B$, we have

$$\dfrac{\dfrac{\dfrac{x=y, y<z, x<z, z:B \Rightarrow y:\mathbf{F}B, z:B}{x=y, y<z, x<z, z:B \Rightarrow y:\mathbf{F}B}\,{\scriptstyle R\mathbf{F}}}{x=y, x<z, z:B \Rightarrow y:\mathbf{F}B}\,{\scriptstyle EqSubst_{At}}}{x=y, x:\mathbf{F}B \Rightarrow y:\mathbf{F}B}\,{\scriptstyle L\mathbf{F}}$$

The sequent $x = y, y < z, x < z, z : B \Rightarrow y : \mathbf{F}B, z : B$ is derivable by Lemma 2.3.5.

If $A \equiv \mathbf{H}B$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{y=x, x=y, z<x, z<y, z:B, x:\mathbf{H}B \Rightarrow z:B}{y=x, x=y, z<x, z<y, x:\mathbf{H}B \Rightarrow z:B}\,{\scriptstyle L\mathbf{H}}}{y=x, x=y, z<y, x:\mathbf{H}B \Rightarrow z:B}\,{\scriptstyle EqSubst_{At}}}{x=y, z<y, x:\mathbf{H}B \Rightarrow z:B}\,{\scriptstyle EqSym}}{x=y, x:\mathbf{H}B \Rightarrow y:\mathbf{H}B}\,{\scriptstyle R\mathbf{H}}$$

The sequent $y = x, x = y, z < x, z < y, z : B, x : \mathbf{H}B \Rightarrow z : B$ is derivable by Lemma 2.3.5.

If $A \equiv \mathbf{P}B$, we have

$$\dfrac{\dfrac{\dfrac{x=y, z<y, z<x, z:B \Rightarrow y:\mathbf{P}B, z:B}{x=y, z<y, z<x, z:B \Rightarrow y:\mathbf{P}B}\,{\scriptstyle R\mathbf{P}}}{x=y, z<x, z:B \Rightarrow y:\mathbf{P}B}\,{\scriptstyle EqSubst_{At}}}{x=y, x:\mathbf{P}B \Rightarrow y:\mathbf{P}B}\,{\scriptstyle L\mathbf{P}}$$

The sequent $x = y, z < y, z < x, z : B \Rightarrow y : \mathbf{P}B, z : B$ is derivable by Lemma 2.3.5.

$\square$

Proposition 2.4.4. *The generalised rule of substitution of equals for arbitrary temporal formula A is admissible in* G3K$_t$*+Eq.

Proof. By Lemma 2.4.3, the sequent $x = y, x : A \Rightarrow y : A$ is derivable. A cut with the premise of the generalised rule *EqSubst* and contractions give the conclusion $x = y, x : A, \Gamma \Rightarrow \Delta$. The result follows by admissibility of cut and contraction in G3K$_t$*+Eq. □

Chapter 3

An infinitary calculus for Priorean linear time

3.1 Linear discrete time

In Chapter 2, we have introduced the basic calculus for temporal logic and its extensions with mathematical rules for the accessibility relation(s). The present chapter aims at applying the general results achieved for G3K$_t$*+Eq to a specific temporal logic.

Among the different versions of temporal logic proposed by Prior (1967), we choose here the system 7.3 (p. 178) that characterises linear discrete frames without a first and a last instant, which are isomorphic to the set of the integers $\mathbb{Z}$. Our choice is determined both by the intrinsic interest of this class of temporal frames, and by the importance that this logic has gained in computer science logic for the specification and verification of reactive systems (see for example Gabbay *et al.* 1980, Manna and Pnueli 1981).

In addition to the standard temporal operators $\mathbf{G}$ and $\mathbf{H}$, the next-time operator[1] $\mathbf{T}$ and the previous-time operator $\mathbf{Y}$ are considered, with the following semantical readings (with $\prec$ standing for immediate successor relation):

$x \Vdash \mathbf{T}A$ *iff for all* y, $x \prec y$ *implies* $y \Vdash A$

$x \Vdash \mathbf{Y}A$ *iff for all* y, $y \prec x$ *implies* $y \Vdash A$

The corresponding temporal rules are given similarly to those for $\mathbf{G}$ and $\mathbf{H}$

$$\frac{y : A, x : \mathbf{T}A, x \prec y, \Gamma \Rightarrow \Delta}{x : \mathbf{T}A, x \prec y, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\mathbf{T}} \qquad \frac{x \prec y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{T}A}\,{\scriptstyle R\mathbf{T}}$$

$$\frac{y : A, x : \mathbf{Y}A, y \prec x, \Gamma \Rightarrow \Delta}{x : \mathbf{Y}A, y \prec x, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\mathbf{Y}} \qquad \frac{y \prec x, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{Y}A}\,{\scriptstyle R\mathbf{Y}}$$

Rules $R\mathbf{T}$ and $R\mathbf{Y}$ have the condition that y is not in the conclusion.

Table 3.1: The rules for $\mathbf{T}$ and $\mathbf{Y}$

The relation of immediate precedence, $\prec$, is characterized by the properties:

Left seriality: $\forall x \exists y\, y \prec x$

Right seriality: $\forall x \exists y\, x \prec y$

Uniqueness of the immediate predecessor: $\forall x \forall y \forall z((y \prec x \,\&\, z \prec x) \supset y = z)$

Uniqueness of the immediate successor: $\forall x \forall y \forall z((x \prec y \,\&\, x \prec z) \supset y = z)$

which are turned into the following initial sequents and mathematical rules

$$x \prec y, \Gamma \Rightarrow \Delta, x \prec y$$

$$\frac{y \prec x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle L\text{-}Ser} \qquad \frac{x \prec y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle R\text{-}Ser}$$

$$\frac{y = z, y \prec x, z \prec x, \Gamma \Rightarrow \Delta}{y \prec x, z \prec x, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnPred} \qquad \frac{y = z, x \prec y, x \prec z, \Gamma \Rightarrow \Delta}{x \prec y, x \prec z, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnSucc}$$

Rules *L-Ser* and *L-Ser* have the condition that y is not in the conclusion.

Table 3.2: The rules for immediate successor

[1] Temporal operators for the next moment were first studied by Scott (1965) and von Wright (1965). See also Segerberg (1967).

The order relation $x < y$ is defined as the transitive closure of the immediate successor relation $x \prec y$, that is,

$$x < y \;\equiv\; \exists n \in \mathbb{N}^{+} \, (x \prec^{n} y) \tag{$*$}$$

The notation means that if $x < y$, then y is reachable from x by iterating finitely many times the immediate successor relation: If x is less than y, then x is the predecessor of y, or it is the predecessor of the predecessor of y, or ... and so on.

The iterated successor relation is defined inductively by the following clauses:

$x \prec^{1} y \;\equiv x \prec y$;

$x \prec^{n+1} y \;\equiv\; \exists z(x \prec^{n} z \,\&\, z \prec y)$ for $n \geq 1$.

The iterated successor relation requires mathematical rules working both on the left- and on the right-hand side of a sequent

$$\frac{x \prec^{n} y, y \prec z, \Gamma \Rightarrow \Delta}{x \prec^{n+1} z, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef} \qquad \frac{\Gamma \Rightarrow \Delta, x \prec^{n+1} z, x \prec^{n} y \quad \Gamma \Rightarrow \Delta, x \prec^{n+1} z, y \prec z}{\Gamma \Rightarrow \Delta, x \prec^{n+1} z}\,{\scriptstyle RDef}$$

Rule $LDef$ has the condition that y is not in the conclusion.

Table 3.3: The rules for iterated successor

The left-to-right direction of (*) gives a rules with infinitely many premises, whereas the right-to-left direction gives rules Inc_n for every $n \geq 1$

$$\frac{\{x \prec^{n} y, x < y, \Gamma \Rightarrow \Delta\}_{n \in \mathbb{N}+}}{x < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle T^{\omega}} \qquad \frac{x < y, x \prec^{n} y, \Gamma \Rightarrow \Delta}{x \prec^{n} y, \Gamma \Rightarrow \Delta}\,{\scriptstyle Inc_n}$$

Table 3.4: The rules for transitive closure

However, in Section 3.2 we will show that we do not need to assume all the rules of inclusion Inc_n as primitive and that $Inc = Inc_1$ is sufficient in the presence of *Trans*.

The calculus G3LT for Priorean linear time is obtained by adding to the basic calculus G3K$_t$ the rules for equality of Table 2.4, rule of transitivity for $<$ from Table 2.2, and the temporal and mathematical rules of Tables 3.1-3.4.

3.2 Structural properties

Next, we show that the system G3LT enjoys the same structural properties that hold for the ground system G3K$_t$ and for its extensions G3K$_t$* as recalled in Chapter 2. The proofs are omitted when they are straightforward adaptations of the proofs thereof.

Substitution of labels is defined as in Section 2.3; here observe that the clauses for the generic relation R can be instantiated to $=$, $\prec$, $\prec^n$, or $<$.

Lemma 3.2.1. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* G3LT*, then also* $\Gamma(y/x) \Rightarrow \Delta(y/x)$ *is derivable, with the same derivation height.*

Proof. By induction on the height h of the derivation (see Lemma 2.3.4). If $h = 0$, the sequent $\Gamma \Rightarrow \Delta$ is either an initial sequent or conclusion of $L\bot$, in either case the sequent $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is also an initial sequent or conclusion of $L\bot$. Suppose that $\Gamma \Rightarrow \Delta$ is derivable with $h = n + 1$ and that the claim holds for $h = n$, and consider the last rule applied in the derivation. If it is a propositional rule or a temporal or mathematical rule without variable condition, apply the inductive hypothesis to the premise(s) and then the rule. If the last rule is a rule with a variable condition, we need to avoid a clash with the eigenvariable: in that case, we apply twice the inductive hypothesis to the premise(s) first to replace the eigenvariable with a fresh variable not appearing in the derivation and then to perform the desired substitution. $\square$

Lemma 3.2.2. *Sequents of the form $x : A, \Gamma \Rightarrow \Delta, x : A$ are derivable in* G3LT *for arbitrary formulas A, arbitrary contexts Γ, Δ, and arbitrary labels x.*

Proof. By induction on the length of the formula A. For propositional connectives and for the temporal operators $\mathbf{G}$, $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{P}$ the proof is analogous to the proof of Lemma 2.3.5 for G3K$_t$*: we consider here only the cases of $A \equiv \mathbf{T}B$ and $A \equiv \mathbf{Y}B$. If $A \equiv \mathbf{T}B$, by inductive hypothesis, we have that $x \prec y, y : B, x : \mathbf{T}B, \Gamma \Rightarrow \Delta, y : B$ is derivable. Consider then the following derivation

$$\cfrac{\cfrac{y : B, x \prec y, x : \mathbf{T}B, \Gamma \Rightarrow \Delta, y : B}{x \prec y, x : \mathbf{T}B, \Gamma \Rightarrow \Delta, y : B}\,{\scriptstyle L\mathbf{T}}}{x : \mathbf{T}B, \Gamma \Rightarrow \Delta, x : \mathbf{T}B}\,{\scriptstyle R\mathbf{T}}$$

where we choose y different from x and not in Γ, Δ.

If $A \equiv \mathbf{Y}B$, by inductive hypothesis and arbitrariness of the contexts and of the label, we have that $y \prec x, y : B, x : \mathbf{Y}B, \Gamma \Rightarrow \Delta, y : B$ is derivable. Consider then the following derivation

$$\cfrac{\cfrac{y : B, y \prec x, x : \mathbf{Y}B, \Gamma \Rightarrow \Delta, y : B}{y \prec x, x : \mathbf{Y}B, \Gamma \Rightarrow \Delta, y : B}\,{\scriptstyle L\mathbf{Y}}}{x : \mathbf{Y}B, \Gamma \Rightarrow \Delta, x : \mathbf{Y}B}\,{\scriptstyle R\mathbf{Y}}$$

where we choose y different from x and not in Γ, Δ. □

Lemma 3.2.3. *Sequents of the form*

$$x \prec^n y, \Gamma \Rightarrow \Delta, x \prec^n y$$

are derivable in G3LT *for all $n \in \mathbb{N}^+$.*

Proof. By induction on n. For $n = 1$, the sequent $x \prec y, \Gamma \Rightarrow \Delta, x \prec y$ is initial. For $n + 1$, assume a derivation of $x \prec^n z, z \prec y, \Gamma \Rightarrow \Delta, x \prec^{n+1} y, x \prec^n z$ with z different from x, y and not in Γ, Δ, and derive the claim as follows

$$\dfrac{\dfrac{x \prec^n z, z \prec y, \Gamma \Rightarrow \Delta, x \prec^{n+1} y, x \prec^n z \quad x \prec^n z, z \prec y, \Gamma \Rightarrow \Delta, x \prec^{n+1} y, z \prec y}{x \prec^n z, z \prec y, \Gamma \Rightarrow \Delta, x \prec^{n+1} y}\, RDef}{x \prec^{n+1} y, \Gamma \Rightarrow \Delta, x \prec^{n+1} y}\, LDef$$

□

In what follows, Greek lower case is used for denoting labelled formulas or relational atoms.

Theorem 3.2.4. *The rules of left and right weakening are height-preserving admissible in* G3LT.

Proof. By induction on the height of the derivation of the premise (see Theorem 2.3.6). If $\Gamma \Rightarrow \Delta$ is an initial sequent or conclusion of $L\bot$, so are $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi$. The cases of rules without variable condition are straightforward. If the last step is a rule with a variable condition, we apply first Lemma 3.2.1 in order to avoid a clash of variables and then the inductive hypothesis and the rule in question. □

Lemma 3.2.5. *All the rules of* G3LT *are height-preserving invertible.*

Proof. The cases of propositional and temporal rules and of the rules for the accessibility relations are dealt with analogously to the proof of height-preserving invertibility for G3K$_t$* (see Lemma 2.3.7): the proof of height-preserving invertibility of the rules for **T** and for **Y** goes analogously to that of the rules for **G** and **H**, respectively. We consider here the essentially new cases of rules $LDef$ and $RDef$. Rule $RDef$ is trivially height-preserving invertible by height-preserving admissibility of weakening. For the rule $LDef$, the proof is by induction on the height of derivation. If $h = 0$, then $x \prec^{m+1} y, \Gamma \Rightarrow \Delta$ is an initial sequent or conclusion of $L\bot$: in both cases $x \prec^m z, z \prec y, \Gamma \Rightarrow \Delta$ is also an initial sequent or conclusion of $L\bot$, since initial sequents of the form $x \prec^p y, \Gamma \Rightarrow \Delta, x \prec^p y$ were not allowed for $p > 1$. For $h = n + 1$, we simply apply the inductive hypothesis to the premise(s) and then the rule; clash of variables is avoided through Lemma 3.2.1. □

Theorem 3.2.6. *The rules of left and right contraction are height-preserving admissible in* G3LT.

Proof. Analogous to the proof of height-preserving admissibility of contraction for G3K$_t$* (see Theorem 2.3.8). By simultaneous induction on the height of derivation for left and right contractions. For $h = 0$, note that if $\varphi, \varphi, \Gamma \Rightarrow \Delta$ (resp. $\Gamma \Rightarrow \Delta, \varphi, \varphi$) is an initial sequent or conclusion on $L\bot$, so is $\varphi, \Gamma \Rightarrow \Delta$ (resp. $\Gamma \Rightarrow \Delta, \varphi$). For $h = n + 1$, we distinguish two cases: if none of the contraction formulas is principal in the last rule, we apply the inductive hypothesis to the premise(s) and then the rule; if one of the contraction formulas is principal, we first apply height-preserving inversion to the premise(s), the inductive hypothesis and then the rule. □

The system G3LT has mathematical rules that act both on the left- and on the right-hand side of sequents, and a measure of complexity for relational atoms is needed in the proof of cut elimination, as in Boretti and Negri (2006).

The length of a labelled formula $x : A$ is defined as in Definition 2.3.1. In addition we have the following:

Definition 3.2.7. *The length of relational or equality atoms is defined by:*

$l(x \prec y) = l(x < y) = l(x = y) = 1$ *and* $l(x \prec^n y) = n$.

Theorem 3.2.8. *The rule of cut is admissible in* G3LT.

Proof. By induction on the length of the cut formula and subinduction on the sum of the heights of the derivations of the premises of cut. The proof has the structure of the proof of cut elimination for G3K$_t$* (see Theorem 2.3.10). We consider in detail the cases of cut formula $x : \mathbf{T}A$ or $x : \mathbf{Y}A$ principal in both premises of cut. If the original derivation contains the following instance of cut

$$\dfrac{\dfrac{x \prec z, \Gamma \Rightarrow \Delta, z : A}{\Gamma \Rightarrow \Delta, x : \mathbf{T}A}{\scriptstyle R\mathbf{T}} \quad \dfrac{x \prec y, y : A, x : \mathbf{T}A, \Gamma'' \Rightarrow \Delta'}{x \prec y, x : \mathbf{T}A, \Gamma'' \Rightarrow \Delta'}{\scriptstyle L\mathbf{T}}}{x \prec y, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}$$

with $\Gamma' \equiv x \prec y, \Gamma''$

we perform the following transformation

$$\dfrac{\dfrac{\dfrac{x \prec z, \Gamma \Rightarrow \Delta, z : A}{x \prec y, \Gamma \Rightarrow \Delta, y : A}{\scriptstyle H.\text{-}p.Subst} \quad \dfrac{\dfrac{x \prec z, \Gamma \Rightarrow \Delta, z : A}{\Gamma \Rightarrow \Delta, x : \mathbf{T}A}{\scriptstyle R\mathbf{T}} \quad x \prec y, y : A, x : \mathbf{T}A, \Gamma'' \Rightarrow \Delta'}{x \prec y, y : A, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}}{x \prec y, x \prec y, \Gamma, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta, \Delta'}{\scriptstyle Cut}}{x \prec y, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}{\scriptstyle Ctr^*}$$

If the original derivation contains the following instance of cut

$$\dfrac{\dfrac{z \prec x, \Gamma \Rightarrow \Delta, z : A}{\Gamma \Rightarrow \Delta, x : \mathbf{Y}A}{\scriptstyle R\mathbf{Y}} \quad \dfrac{y \prec x, y : A, x : \mathbf{Y}A, \Gamma'' \Rightarrow \Delta'}{y \prec x, x : \mathbf{Y}A, \Gamma'' \Rightarrow \Delta'}{\scriptstyle L\mathbf{Y}}}{y \prec x, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}$$

with $\Gamma' \equiv y \prec x, \Gamma''$

we perform the following transformation

$$\dfrac{\dfrac{\dfrac{z \prec x, \Gamma \Rightarrow \Delta, z : A}{y \prec x, \Gamma \Rightarrow \Delta, y : A}{\scriptstyle H.\text{-}p.Subst} \quad \dfrac{\dfrac{z \prec x, \Gamma \Rightarrow \Delta, z : A}{\Gamma \Rightarrow \Delta, x : \mathbf{Y}A}{\scriptstyle R\mathbf{Y}} \quad y \prec x, y : A, x : \mathbf{Y}A, \Gamma'' \Rightarrow \Delta'}{y \prec x, y : A, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}}{y \prec x, y \prec x, \Gamma, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta, \Delta'}{\scriptstyle Cut}}{y \prec x, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}{\scriptstyle Ctr^*}$$

Furthermore, we have to consider an essentially new case, given by the simultaneous presence of mathematical rules that act both on the left- and on the right-hand side of sequents: this is the case with cut formula $x \prec^{n+1} y$ principal in both premises of cut

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x \prec^{n+1} y, x \prec^{n} z \quad \Gamma \Rightarrow \Delta, x \prec^{n+1} y, z \prec y}{\Gamma \Rightarrow \Delta, x \prec^{n+1} y}\,{\scriptstyle RDef} \quad \dfrac{x \prec^{n} w, w \prec y, \Gamma' \Rightarrow \Delta'}{x \prec^{n+1} y, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle LDef}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

This derivation is transformed as follows: we first cut the left premise of $RDef$ with the conclusion of $LDef$

1. $$\dfrac{\Gamma \Rightarrow \Delta, x \prec^{n+1} y, x \prec^{n} z \quad \dfrac{x \prec^{n} w, w \prec y, \Gamma \Rightarrow \Delta}{x \prec^{n+1} y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x \prec^{n} z}\,{\scriptstyle Cut}$$

thus obtaining a cut with shorter height. Then we cut the right premise of $RDef$ with the conclusion of $LDef$

2. $$\dfrac{\Gamma \Rightarrow \Delta, x \prec^{n+1} y, z \prec y \quad \dfrac{x \prec^{n} w, w \prec y, \Gamma' \Rightarrow \Delta'}{x \prec^{n+1} y, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle LDef}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', z \prec y}\,{\scriptstyle Cut}$$

thus obtaining another cut with shorter height. Finally, we use the sequents thus obtained and the premise of $LDef$ as follows

$$\dfrac{\dfrac{\Gamma, \Gamma' \overset{2.}{\Rightarrow} \Delta, \Delta', z \prec y \quad \dfrac{\Gamma, \Gamma' \overset{1.}{\Rightarrow} \Delta, \Delta', x \prec^{n} z \quad \dfrac{x \prec^{n} w, w \prec y, \Gamma' \Rightarrow \Delta'}{x \prec^{n} z, z \prec y, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle H.\text{-}P.Subst}}{z \prec y, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}\,{\scriptstyle Cut}}{\Gamma, \Gamma, \Gamma', \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta', \Delta'}\,{\scriptstyle Cut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Ctr^{*}}$$

where the two cuts are on formulas with smaller length. □

Corollary 3.2.9. *The calculus* G3LT *enjoys the weak subformula property.*

Even if the latter is not as strong as the full subformula property, it can be further refined: the relational atomic formulas that can appear in a derivation are bounded by the labels appearing in the endsequent. A detailed discussion on this point can be found in Section 1.4.3.

Definition 3.2.10. *In an instance of rule* R-Ser *(resp.* L-Ser*) with active formula $x \prec y$ (resp. $y \prec x$), the label x is called* side label*.*

Lemma 3.2.11. *A derivation in* G3LT *can be transformed into a derivation with all instances of* R-Ser *and* L-Ser *applied on side labels that appear in the conclusion of the rule.*

Proof. Suppose that we have an application of *L-Ser* or *R-Ser* on a side label z not in its conclusion: we can perform the following transformations

$$\frac{y \prec z, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle L\text{-}Ser} \quad \rightsquigarrow \quad \frac{\dfrac{y \prec z, \Gamma \Rightarrow \Delta}{y \prec x, \Gamma \Rightarrow \Delta}\,{\scriptstyle H.\text{-}p.Subst(x/z)}}{\Gamma \Rightarrow \Delta}\,{\scriptstyle L\text{-}Ser}$$

$$\frac{z \prec y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle R\text{-}Ser} \quad \rightsquigarrow \quad \frac{\dfrac{z \prec y, \Gamma \Rightarrow \Delta}{x \prec y, \Gamma \Rightarrow \Delta}\,{\scriptstyle H.\text{-}p.Subst(x/z)}}{\Gamma \Rightarrow \Delta}\,{\scriptstyle R\text{-}Ser}$$

Here x has been chosen among the variables in Γ, Δ. □

Definition 3.2.12. *A derivation is* minimal *when it is not possible to shorten it through height-preserving admissibility of contraction or other modifications of the derivation.*

Lemma 3.2.13. *In a minimal derivation of a sequent* $\Gamma \Rightarrow \Delta$ *in* G3LT*, all the labels in atoms of the form* $x = x$ *removed by* $EqRef$ *are labels in* Γ, Δ.

Proof. Consider a minimal derivation of a sequent $\Gamma \Rightarrow \Delta$ and suppose there is a variable x in an atom $x = x$ removed by $EqRef$. Consider the last occurrence of x and the step of $EqRef$ removing it

$$\frac{\begin{array}{c}\vdots\\ x = x, \Gamma' \Rightarrow \Delta'\end{array}}{\Gamma' \Rightarrow \Delta'}\,{\scriptstyle EqRef}$$

Trace the atom $x = x$ up in the derivation, by following its occurrences from the sequent $x = x, \Gamma' \Rightarrow \Delta'$ to the leaves of the derivation. If $x = x$ is never principal in a rule, we trace it up to the initial sequents of the derivation tree. If it is principal in an initial sequent, the latter has the form $x = x, \Gamma'' \Rightarrow \Delta'', x = x$, and we find $x = x$ in the succedent: since no rule removes $x = x$ from the right-hand side of a sequent, we find $x = x$ in Δ, contrary to the hypothesis. If $x = x$ is principal in none of the initial sequents, then it can be removed all along the derivation together with the instance of $EqRef$, thus shortening the derivation, contrary to the hypothesis of minimality.

If $x = x$ is principal in a rule, this can be only $EqSubst$ or $EqSubst_{At}$. Thus, we have an inference of one of the following forms

$$\dfrac{\begin{matrix}\vdots\\ x = x, x : P, x : P, \Gamma'' \Rightarrow \Delta''\end{matrix}}{x = x, x : P, \Gamma'' \Rightarrow \Delta''}\,{\scriptstyle EqSubst} \qquad \dfrac{\begin{matrix}\vdots\\ x = x, At(x), At(x), \Gamma'' \Rightarrow \Delta''\end{matrix}}{x = x, At(x), \Gamma'' \Rightarrow \Delta''}\,{\scriptstyle EqSubst_{At}}$$

that can be shortened by height-preserving admissibility of contraction, contrary to the hypothesis of minimality. $\square$

Proposition 3.2.14. *Let us suppose that* $\Gamma \Rightarrow \Delta$ *does not contain relational atoms in its succedent. All the labels in a minimal derivation of* $\Gamma \Rightarrow \Delta$ *in* G3LT *are eigenvariables or labels in* Γ, Δ.

Proof. Immediate for the logical rules and the mathematical rules $UnSucc$, $UnPred$, $LDef$, T^{ω}, Inc, $Trans$, $EqSubst$ and $EqSubst_{At}$. Rule $RDef$ is excluded by the condition that no relational atom is in Δ. The cases of rules *L-Ser* and *R-Ser* are dealt with in Lemma 3.2.11. The case of rule $EqRef$ is considered in Lemma 3.2.13. $\square$

We will refer to the property stated by Proposition 3.2.14 above as *subterm property*.

Observe that the restriction to minimal derivations is no way a limiting one, since, by Definition 3.2.12, those are exactly the kind of derivations usually aimed to. Furthermore, the condition that no relational atom is in Δ is not restrictive, when considering derivations of purely logical sequents, since, as noticed in Section 2.1, sequents $xRy, \Gamma \Rightarrow \Delta, xRy$ (for arbitrary relational atoms xRy) and even rule $RDef$ are needed only for deriving properties of accessibility relations, and thus can be left out from the system.

In the following we prove the admissibility of some useful rules.

Lemma 3.2.15. *The sequent $x = y, x : A \Rightarrow y : A$ is derivable in* G3LT *for arbitrary temporal formulas A.*

Proof. By induction on the length of A. For the propositional connectives and the temporal operators $\mathbf{G}$, $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{P}$ the proof is analogous to the proof of Lemma 2.4.3 for G3K$_t$*+Eq. We consider here only the cases of $A \equiv \mathbf{T}B$ and $A \equiv \mathbf{Y}B$.

If $A \equiv \mathbf{T}B$, we have

$$\cfrac{\cfrac{\cfrac{\cfrac{y = x, x = y, x \prec z, y \prec z, z : B, x : \mathbf{T}B \Rightarrow z : B}{y = x, x = y, x \prec z, y \prec z, x : \mathbf{T}B \Rightarrow z : B}\, L\mathbf{T}}{y = x, x = y, y \prec z, x : \mathbf{T}B \Rightarrow z : B}\, EqSubst_{At}}{x = y, y \prec z, x : \mathbf{T}B \Rightarrow z : B}\, EqSym}{x = y, x : \mathbf{T}B \Rightarrow y : \mathbf{T}B}\, R\mathbf{T}$$

If $A \equiv \mathbf{Y}B$, we have

$$\cfrac{\cfrac{\cfrac{\cfrac{y = x, x = y, z \prec x, z \prec y, z : B, x : \mathbf{Y}B \Rightarrow z : B}{y = x, x = y, z \prec x, z \prec y, x : \mathbf{Y}B \Rightarrow z : B}\, L\mathbf{Y}}{y = x, x = y, z \prec y, x : \mathbf{Y}B \Rightarrow z : B}\, EqSubst_{At}}{x = y, z \prec y, x : \mathbf{Y}B \Rightarrow z : B}\, EqSym}{x = y, x : \mathbf{Y}B \Rightarrow y : \mathbf{Y}B}\, R\mathbf{Y}$$

□

Proposition 3.2.16. *The generalised rule of substitution of equals for arbitrary temporal formulas A*

$$\frac{y : A, x = y, x : A, \Gamma \Rightarrow \Delta}{x = y, x : A, \Gamma \Rightarrow \Delta}\, EqSubst$$

is admissible in G3LT.

Proof. By Lemma 3.2.15, the sequent $x = y, x : A \Rightarrow y : A$ is derivable. A cut with the premise of the generalised rule *EqSubst* and contractions give the conclusion $x = y, x : A, \Gamma \Rightarrow \Delta$. The result follows by admissibility of cut and contraction in G3LT. □

Mathematical rules for immediate successor relation can be generalised to the relation of iterated successor for every $n \in \mathbb{N}^+$.

$$\frac{y = z, x \prec^n y, x \prec^n z, \Gamma \Rightarrow \Delta}{y = z, x \prec^n y, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst_n} \qquad \frac{y = z, y \prec^n x, z \prec^n x, \Gamma \Rightarrow \Delta}{y = z, y \prec^n x, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst_n}$$

$$\frac{x < y, x \prec^n y, \Gamma \Rightarrow \Delta}{x \prec^n y, \Gamma \Rightarrow \Delta}\,{\scriptstyle Inc_n} \qquad \frac{x \prec^m y, y \prec^n z, \Gamma \Rightarrow \Delta}{x \prec^{m+n} z, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef_n}$$

$$\frac{y = z, y \prec^n x, z \prec^n x, \Gamma \Rightarrow \Delta}{y \prec^n x, z \prec^n x, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnPred_n} \qquad \frac{y = z, x \prec^n y, x \prec^n z, \Gamma \Rightarrow \Delta}{x \prec^n y, x \prec^n z, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnSucc_n}$$

$$\frac{y \prec^n x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle L\text{-}Ser_n} \qquad \frac{x \prec^n y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle R\text{-}Ser_n}$$

Rules $LDef_n$, L-Ser_n, and R-Ser_n have the variable condition that y is not in the conclusion.

Table 3.5: The generalised rules for iterated successor

Proposition 3.2.17. *For all $n \in \mathbb{N}^+$, the rules $EqSubst_n$ of substitution of equals in the relation of iterated successor are admissible in* G3LT.

Proof. By induction on n. If $n = 1$ we simply apply $EqSubst_{At}$. Let us assume that the claim holds for n; the result is given by the following inferences:

Case 1

$$
\begin{array}{cl}
y = z, x \prec^{n+1} y, x \prec^{n+1} z, \Gamma \Rightarrow \Delta & \\
\hline
y = z, x \prec^{n} w, w \prec z, x \prec^{n} v, v \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle H.\text{-}p.Inv} \\
\hline
w = v, y = z, x \prec^{n} w, w \prec z, w \prec y, x \prec^{n} v, v \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle LWk^*} \\
\hline
w = v, y = z, x \prec^{n} w, w \prec z, w \prec y, v \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle EqSubst_n} \\
\hline
y = z, x \prec^{n} w, w \prec z, w \prec y, v \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle UnPred} \\
\hline
y = z, x \prec^{n} w, w \prec z, w \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle L\text{-}Ser} \\
\hline
y = z, x \prec^{n} w, w \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle EqSubst_{At}} \\
\hline
y = z, x \prec^{n+1} y, \Gamma \Rightarrow \Delta & {\scriptstyle LDef}
\end{array}
$$

We choose two new labels v, w different from x, y, z and not in Γ, Δ.

Case 2

$$
\begin{array}{cl}
y = z, y \prec^{n+1} x, z \prec^{n+1} x, \Gamma \Rightarrow \Delta & \\
\hline
y = z, y \prec^{n} v, v \prec x, z \prec^{n} w, w \prec x, \Gamma \Rightarrow \Delta & {\scriptstyle H.\text{-}p.Inv} \\
\hline
w = v, y = z, y \prec^{n} w, y \prec^{n} v, v \prec x, z \prec^{n} w, w \prec x, \Gamma \Rightarrow \Delta & {\scriptstyle LWk^*} \\
\hline
w = v, y = z, y \prec^{n} w, v \prec x, z \prec^{n} w, w \prec x, \Gamma \Rightarrow \Delta & {\scriptstyle EqSubst_n} \\
\hline
y = z, y \prec^{n} w, v \prec x, z \prec^{n} w, w \prec x, \Gamma \Rightarrow \Delta & {\scriptstyle UnPred} \\
\hline
y = z, y \prec^{n} w, z \prec^{n} w, w \prec x, \Gamma \Rightarrow \Delta & {\scriptstyle L\text{-}Ser} \\
\hline
y = z, y \prec^{n} w, w \prec x, \Gamma \Rightarrow \Delta & {\scriptstyle EqSubst_n} \\
\hline
y = z, y \prec^{n+1} x, \Gamma \Rightarrow \Delta & {\scriptstyle LDef}
\end{array}
$$

Again, we choose two new labels v, w different from x, y, z and not in Γ, Δ. $\square$

Proposition 3.2.18. *For all $n \in \mathbb{N}^+$, the generalised rule Inc_n is admissible in* G3LT *if Inc is assumed as primitive.*

Proof. For $n = 1$, Inc_n is just Inc. Let us assume admissibility of Inc_n and prove admissibility of Inc_{n+1} as follows

$$
\begin{array}{cl}
x < y, x \prec^{n+1} y, \Gamma \Rightarrow \Delta & \\
\hline
x < y, x \prec^{n} z, z \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle H.\text{-}p.Inv} \\
\hline
x < y, x < z, z < y, x \prec^{n} z, z \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle LWk^*} \\
\hline
x < z, z < y, x \prec^{n} z, z \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle Trans} \\
\hline
z < y, x \prec^{n} z, z \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle Inc_n} \\
\hline
x \prec^{n} z, z \prec y, \Gamma \Rightarrow \Delta & {\scriptstyle Inc} \\
\hline
x \prec^{n+1} y, \Gamma \Rightarrow \Delta & {\scriptstyle LDef}
\end{array}
$$

where z has been chosen different from x, y and not in Γ, Δ. $\square$

Proposition 3.2.19. *For all $n \in \mathbb{N}^+$, the generalised rule $LDef_n$ is admissible in* G3LT.

Proof. By induction on n. For $n = 1$, the rule is just $LDef$. Admissibility of $LDef_{n+1}$ is reduced to admissibility of $LDef_n$ by the following derivation

$$\dfrac{\dfrac{\dfrac{x \prec^m z, z \prec^{n+1} y, \Gamma \Rightarrow \Delta}{x \prec^m z, z \prec^n w, w \prec y, \Gamma \Rightarrow \Delta}\,{\scriptstyle H.\text{-}p.Inv.}}{x \prec^{m+n} w, w \prec y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef_n}}{x \prec^{m+n+1} y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef}$$

where we choose w different from x, y, z and not in Γ, Δ and z was not in the conclusion by hypothesis. □

Proposition 3.2.20. *For all $n \in \mathbb{N}^+$, rules $UnPred_n$ and $UnSucc_n$ are admissible in* G3LT.

Proof. By induction on n. If $n = 1$, we simply have $UnPred$ and $UnSucc$. Admissibility of $UnPred_{n+1}$ and $UnSucc_{n+1}$ is reduced to admissibility of $UnPred_n$ and $UnSucc_n$ by the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y = z, y \prec^{n+1} x, z \prec^{n+1} x, \Gamma \Rightarrow \Delta}{y = x, y \prec^n v, v \prec x, z \prec^n w, w \prec x, \Gamma \Rightarrow \Delta}\,{\scriptstyle H.\text{-}p.Inv}}{y = x, v = w, y \prec^n w, y \prec^n v, v \prec x, z \prec^n w, w \prec x, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk^*}}{v = w, y \prec^n w, y \prec^n v, v \prec x, z \prec^n w, w \prec x, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnPred_n}}{v = w, y \prec^n v, v \prec x, z \prec^n w, w \prec x, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst_n}}{y \prec^n v, v \prec x, z \prec^n w, w \prec x\Gamma \Rightarrow \Delta}\,{\scriptstyle UnPred}}{y \prec^{n+1} x, y \prec^{n+1} x, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef,LDef}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y = z, x \prec^{n+1} y, x \prec^{n+1} z, \Gamma \Rightarrow \Delta}{y = x, x \prec^n v, v \prec y, x \prec^n w, w \prec z\Gamma \Rightarrow \Delta}\,{\scriptstyle H.\text{-}p.Inv}}{y = x, v = w, x \prec^n v, v \prec y, w \prec y, x \prec^n w, w \prec z\Gamma \Rightarrow \Delta}\,{\scriptstyle LWk^*}}{v = w, x \prec^n v, v \prec y, w \prec y, x \prec^n w, w \prec z\Gamma \Rightarrow \Delta}\,{\scriptstyle UnSucc}}{v = w, x \prec^n v, v \prec y, x \prec^n w, w \prec z\Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst_{At}}}{x \prec^n v, v \prec y, x \prec^n w, w \prec z\Gamma \Rightarrow \Delta}\,{\scriptstyle UnSucc_n}}{x \prec^{n+1} y, x \prec^{n+1} z, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef,LDef}$$

□

Proposition 3.2.21. *For all* $n \in \mathbb{N}^+$*, the generalised rules* $L\text{-}Ser_n$ *and* $L\text{-}Ser_n$ *are admissible in* G3LT.

Proof. By induction on n. For $n = 1$, we simply have *L-Ser* and *R-Ser*. Admissibility of $L\text{-}Ser_{n+1}$ and $R\text{-}Ser_{n+1}$ is reduced to admissibility of $L\text{-}Ser_n$ and $R\text{-}Ser_n$ by the following derivations:

$$\dfrac{\dfrac{\dfrac{y \prec^{n+1} x, \Gamma \Rightarrow \Delta}{y \prec^{n} z, z \prec x, \Gamma \Rightarrow \Delta}\, {\scriptstyle H.\text{-}p.Inv}}{z \prec x, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\text{-}Ser_n}}{\Gamma \Rightarrow \Delta}\, {\scriptstyle L\text{-}Ser} \qquad \dfrac{\dfrac{\dfrac{x \prec^{n+1} y, \Gamma \Rightarrow \Delta}{x \prec^{n} z, z \prec y, \Gamma \Rightarrow \Delta}\, {\scriptstyle H.\text{-}p.Inv}}{x \prec^{n} z, \Gamma \Rightarrow \Delta}\, {\scriptstyle R\text{-}Ser}}{\Gamma \Rightarrow \Delta}\, {\scriptstyle R\text{-}Ser_n}$$

Choose z different from x, y and not in Γ, Δ. The label y was not in the conclusion by hypothesis. $\square$

The rules of seriality can be generalised also to the order relation $<$:

$$\dfrac{y < x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, {\scriptstyle L\text{-}Ser_<} \qquad \dfrac{x < y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, {\scriptstyle R\text{-}Ser_<}$$

Both rules have the variable condition that y is not in the conclusion.

Table 3.6: The rules of seriality for $<$

Proposition 3.2.22. *The rules of left and right seriality for* $<$ *are admissible in* G3LT.

Proof. By weakening with $y \prec x$ and $x \prec y$, respectively, and then applying *Inc* and *L-Ser* or *R-Ser*. $\square$

Proposition 3.2.23. *The rules of left and right linearity of Table 2.2 are admissible in* G3LT.

Proof. By means of two applications of the infinitary rule T^ω, with principal formulas $z < x, y < x$ and $x < z, x < y$, respectively, and derivability of the sequents

$$z \prec^m x, z < x, y \prec^n x, y < x, \Gamma \Rightarrow \Delta$$

$$x \prec^m z, x < z, x \prec^n y, x < y, \Gamma \Rightarrow \Delta$$

for every $m, n \in \mathbb{N}^+$, whenever the premises of *L-Lin* and *R-Lin* are derivable.

□

Further useful mathematical rules connect immediate successor and order relation:

$$\frac{x \prec y, x < y, \Gamma \Rightarrow \Delta \quad x \prec z, z < y, x < y, \Gamma \Rightarrow \Delta}{x < y, \Gamma \Rightarrow \Delta}\, {\scriptstyle Mix_1}$$

$$\frac{x \prec y, x < y, \Gamma \Rightarrow \Delta \quad x < z, z \prec y, x < y, \Gamma \Rightarrow \Delta}{x < y, \Gamma \Rightarrow \Delta}\, {\scriptstyle Mix_2}$$

$$\frac{x = z, y \prec x, y < z, \Gamma \Rightarrow \Delta \quad x < z, y \prec x, y < z, \Gamma \Rightarrow \Delta}{y \prec x, y < z, \Gamma \Rightarrow \Delta}\, {\scriptstyle Discr_1}$$

$$\frac{x = z, x \prec y, z < y, \Gamma \Rightarrow \Delta \quad z < x, x \prec y, z < y, \Gamma \Rightarrow \Delta}{x \prec y, z < y, \Gamma \Rightarrow \Delta}\, {\scriptstyle Discr_2}$$

Rules Mix_1 and Mix_2 have the variable condition that z is not in the conclusion.

Table 3.7: The rules Mix_1, Mix_2, and the rules for left and right discreteness

Proposition 3.2.24. *Rules* Mix_1 *and* Mix_2 *are admissible in* G3LT.

Proof. For *Mix*$_1$ we have the following inferences

$$\frac{x \prec y, x < y, \Gamma \Rightarrow \Delta \qquad \dfrac{\dfrac{\dfrac{x \prec z, z < y, x < y, \Gamma \Rightarrow \Delta}{x \prec z, z \prec y, z < y, x < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk}}{x \prec z, z \prec y, x < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle Inc}}{x \prec^2 y, x < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef} \qquad \dfrac{\dfrac{\dfrac{x \prec z, z < y, x < y, \Gamma \Rightarrow \Delta}{x \prec z, z \prec^2 y, z < y, x < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk}}{x \prec z, z \prec^2 y, x < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle Inc_2}}{x \prec^3 y, x < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef_2} \qquad \cdots}{x < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle T^\omega}$$

where by hypothesis z is different from x, y and not in Γ, Δ.

Note that in the derivations of the premises $x \prec^n y, x < y, \Gamma \Rightarrow \Delta$ for $n > 2$ applications of $LDef_n$ and Inc_n are needed. The case of *Mix*$_2$ is analogous. □

Proposition 3.2.25. *The rules for left and right discreteness are admissible in* G3LT.

Proof. The proof consists of the following inferences

$$
\dfrac{
\dfrac{
\dfrac{x = z, y \prec x, y < z, \Gamma \Rightarrow \Delta}{x = z, y \prec z, y \prec x, y < z, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk}
}{y \prec z, y \prec x, y < z, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnSucc}
\qquad
\dfrac{
\dfrac{
\dfrac{x < z, y \prec x, y < z, \Gamma \Rightarrow \Delta}{w = x, y \prec w, w < z, x < z, y \prec x, y < z, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk^*}
}{w = x, y \prec w, w < z, y \prec x, y < z, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst_{At}}
}{y \prec w, w < z, y \prec x, y < z, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnSucc}
}{y \prec x, y < z, \Gamma \Rightarrow \Delta}\,{\scriptstyle Mix_1}
$$

$$
\dfrac{
\dfrac{
\dfrac{x = z, x \prec y, z < y, \Gamma \Rightarrow \Delta}{x = z, z \prec y, x \prec y, z < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk}
}{z \prec y, x \prec y, z < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnPred}
\qquad
\dfrac{
\dfrac{
\dfrac{z < x, x \prec y, z < y, \Gamma \Rightarrow \Delta}{w = x, z < w, w \prec y, z < x, x \prec y, z < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk^*}
}{w = x, z < w, w \prec y, x \prec y, z < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle EqSubst_{At}}
}{z < w, w \prec y, x \prec y, z < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle UnPred}
}{x \prec y, z < y, \Gamma \Rightarrow \Delta}\,{\scriptstyle Mix_2}
$$

In both derivations we choose w different from x, y, z and not in Γ, Δ $\square$

3.3 Adequateness of the calculus G3LT

In order to prove soundness of G3LT, we recall here the notions of Kripke semantics for temporal logic of Section 1.3.

Definition 3.3.1. *A* discrete linear temporal frame $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ *is a linearly ordered set, with the order relation* $<^{\mathcal{K}}$ *defined as the transitive closure of the immediate successor relation* $\prec^{\mathcal{K}}$*, functional and unbounded in both directions.*

Definition 3.3.2. *Let* $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ *be a discrete linear temporal frame. An* evaluation *of atomic formulas in a frame is a map* $\mathcal{V} : AtFrm \to \wp(\mathcal{K})$*, assigning to any atom* P *the set of instants in which* P *holds. The standard notation for* $k \in \mathcal{V}(P)$ *is* $k \Vdash P$*. Evaluations are extended to arbitrary formulas by the following inductive clauses:*

For all $k \in \mathcal{K}$*, it is not the case that* $k \Vdash \perp$ *(abbr.* $k \nVdash \perp$*);*

$k \Vdash A\&B$ *if* $k \Vdash A$ *and* $k \Vdash B$*;*

$k \Vdash A \vee B$ if $k \Vdash A$ or $k \Vdash B$;

$k \Vdash A \supset B$ if $k \Vdash A$ implies $k \Vdash B$;

$k \Vdash \mathbf{G}A$ (resp. $k \Vdash \mathbf{H}A$) if for all k', $k <^{\mathcal{K}} k'$ (resp. $k' <^{\mathcal{K}} k$) implies $k' \Vdash A$;

$k \Vdash \mathbf{F}A$ (resp. $k \Vdash \mathbf{P}A$) if there exists k' such that $k <^{\mathcal{K}} k'$ (resp. $k' <^{\mathcal{K}} k$) and $k' \Vdash A$

$k \Vdash \mathbf{T}A$ (resp. $k \Vdash \mathbf{Y}A$) if for all k', $k \prec^{\mathcal{K}} k'$ (resp. $k' \prec^{\mathcal{K}} k$) implies $k' \Vdash A$

Definition 3.3.3. *Let $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ be a frame with accessibility relations $<^{\mathcal{K}}$ and $\prec^{\mathcal{K}}$ satisfying the properties corresponding to mathematical rules of* G3LT*. Let W be the set of labels used in derivations in* G3LT*. An* interpretation *of W in $\mathcal{K}$ is a function $[\![\cdot]\!] : W \to \mathcal{K}$. A sequent $\Gamma \Rightarrow \Delta$ is valid for a given interpretation of labels and evaluation of propositional variables in a frame, if for all labelled formulas $z : A$ and relational atoms $x < y$, $x \prec y$, $x \prec^{n+1} y$, $x = y$ in Γ, if $[\![z]\!] \Vdash A$ and $[\![x]\!] <^{\mathcal{K}} [\![y]\!]$, $[\![x]\!] \prec^{\mathcal{K}} [\![y]\!]$, $\exists k_1 \dots k_n\,([\![x]\!] \prec k_1 \,\&\, \dots \,\&\, k_n \prec [\![y]\!])$, $[\![x]\!] = [\![y]\!]$, then for some $w : B$ or relational atom $v < s$, $v \prec s$, $v \prec^{m+1} s$, $v = s$ in Δ, $[\![w]\!] \Vdash B$ or $[\![v]\!] <^{\mathcal{K}} [\![s]\!]$, $[\![v]\!] \prec^{\mathcal{K}} [\![s]\!]$, $\exists k'_1 \dots k'_m\,([\![v]\!] \prec k'_1 \,\&\, \dots \,\&\, k'_m \prec [\![s]\!])$, $[\![v]\!] = [\![s]\!]$. A sequent is valid if it is valid for every interpretation and every evaluation of propositional variables in a frame. A rule is sound if whenever the premises are valid then the conclusion is valid.*

Theorem 3.3.4. (Soundness) *If sequent $\Gamma \Rightarrow \Delta$ is derivable in* G3LT*, then it is valid.*

Proof. By induction on the height of the derivation of $\Gamma \Rightarrow \Delta$. We simply have to prove the soundness of G3LT rules. The case of initial sequents and propositional rules is straightforward. The rules for the temporal operators are justified by their meaning explanations in terms of the intended relational semantics. The mathematical rules correspond to the frame properties for $<$ and $\prec$, and to the definition of the iterated successor relation $\prec^n$. $\square$

We prove completeness of G3LT with respect to Priorean linear time logic indirectly, by showing that the Hilbert-style system for Priorean linear time logic can be embedded into our calculus: the purely logical sequents[2] that correspond to the temporal axioms are derivable, and *modus ponens* and the temporal generalisation rules are admissible in our calculus.

Proposition 3.3.5. *The following purely logical sequents*

$x : \mathbf{G}(A \supset B), x : \mathbf{G}A \Rightarrow x : \mathbf{G}B$	$x : \mathbf{H}(A \supset B), x : \mathbf{H}A \Rightarrow x : \mathbf{H}B$
$x : A \Rightarrow x : \mathbf{GP}A$	$x : A \Rightarrow x : \mathbf{HF}A$
$x : \mathbf{G}A \Rightarrow x : \mathbf{GG}A$	$x : \mathbf{H}A \Rightarrow x : \mathbf{HH}A$
$x : \mathbf{G}A \Rightarrow x : \mathbf{F}A$	$x : \mathbf{H}A \Rightarrow x : \mathbf{P}A$
$x : \mathbf{H}A, x : A, x : \mathbf{G}A \Rightarrow x : \mathbf{HG}A$	$x : \mathbf{H}A, x : A, x : \mathbf{G}A \Rightarrow x : \mathbf{GH}A$
$x : \mathbf{G}A \Rightarrow x : \mathbf{T}A$	$x : \mathbf{H}A \Rightarrow x : \mathbf{Y}A$
$x : \neg\mathbf{T}\neg A \Rightarrow x : \mathbf{T}A$	$x : \neg\mathbf{Y}\neg A \Rightarrow x : \mathbf{Y}A$
$x : \mathbf{T}A \Rightarrow x : \neg\mathbf{T}\neg A$	$x : \mathbf{Y}A \Rightarrow x : \neg\mathbf{Y}\neg A$
$x : \mathbf{T}(A \supset B), x : \mathbf{T}A \Rightarrow x : \mathbf{T}B$	$x : \mathbf{Y}(A \supset B), x : \mathbf{Y}A \Rightarrow x : \mathbf{Y}B$
$x : A \Rightarrow x : \mathbf{TY}A$	$x : A \Rightarrow x : \mathbf{YT}A$
$x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$	$x : \mathbf{Y}A, x : \mathbf{H}(A \supset \mathbf{Y}A) \Rightarrow x : \mathbf{H}A$

are derivable in G3LT.

Proof. We consider here only the future axioms, the cases of their temporal mirror images[3] being analogous. The proof consists of the following derivations,

[2]See Definition 2.1.1

[3]The *temporal mirror image* of a purely logical sequent is obtained by replacing each occurrence of a future (resp. past) operator by its past (resp. future) analogue. For example the temporal mirror image of $x : \mathbf{G}P \Rightarrow x : P \,\&\, \mathbf{TG}P$ is $x : \mathbf{H}P \Rightarrow x : P \,\&\, \mathbf{YH}P$.

found by root-first proof search from the sequent to be derived together with Lemma 3.2.2.

$$\dfrac{\dfrac{\dfrac{\dfrac{x<y,y:A,\dots\Rightarrow y:A,y:B \qquad x<y,y:B,y:A,\dots\Rightarrow y:B}{x<y,y:A\supset B,y:A,x:\mathbf{G}(A\supset B),x:\mathbf{G}A\Rightarrow y:B}\,L{\supset}}{x<y,y:A\supset B,x:\mathbf{G}(A\supset B),x:\mathbf{G}A\Rightarrow y:B}\,L\mathbf{G}}{x<y,x:\mathbf{G}(A\supset B),x:\mathbf{G}A\Rightarrow y:B}\,L\mathbf{G}}{x:\mathbf{G}(A\supset B),x:\mathbf{G}A\Rightarrow x:\mathbf{G}B}\,R\mathbf{G}$$

$$\dfrac{\dfrac{x<y,x:A\Rightarrow y:\mathbf{P}A,x:A}{x<y,x:A\Rightarrow y:\mathbf{P}A}\,R\mathbf{P}}{x:A\Rightarrow x:\mathbf{GP}A}\,R\mathbf{G}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x<z,x<y,y<z,z:A,x:\mathbf{G}A\Rightarrow z:A}{x<z,x<y,y<z,x:\mathbf{G}A\Rightarrow z:A}\,L\mathbf{G}}{x<y,y<z,x:\mathbf{G}A\Rightarrow z:A}\,Trans}{x<y,x:\mathbf{G}A\Rightarrow y:\mathbf{G}A}\,R\mathbf{G}}{x:\mathbf{G}A\Rightarrow x:\mathbf{GG}A}\,R\mathbf{G}$$

$$\dfrac{\dfrac{\dfrac{x<y,y:A,x:\mathbf{G}A\Rightarrow x:\mathbf{F}A,y:A}{x<y,y:A,x:\mathbf{G}A\Rightarrow x:\mathbf{F}A}\,R\mathbf{F}}{x<y,x:\mathbf{G}A\Rightarrow x:\mathbf{F}A}\,L\mathbf{G}}{x:\mathbf{G}A\Rightarrow x:\mathbf{F}A}\,R\text{-}Ser_{<}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{z<x,z:A,x:\mathbf{H}A,\dots\Rightarrow z:A}{z<x,x:\mathbf{H}A,\dots\Rightarrow z:A}\,L\mathbf{H} \quad \dfrac{x=z,\dots,z:A,x:A\Rightarrow z:A}{x=z,\dots,x:A\Rightarrow z:A}\,EqSubst \quad \dfrac{x<z,\dots,z:A,x:\mathbf{G}A\Rightarrow z:A}{x<z,\dots,x:\mathbf{G}A\Rightarrow z:A}\,L\mathbf{G}}{y<z,y<x,x:\mathbf{H}A,x:A,x:\mathbf{G}A\Rightarrow z:A}\,R\text{-}Lin}{y<x,x:\mathbf{H}A,x:A,x:\mathbf{G}A\Rightarrow y:\mathbf{G}A}\,R\mathbf{G}}{x:\mathbf{H}A,x:A,x:\mathbf{G}A\Rightarrow x:\mathbf{HG}A}\,R\mathbf{H}$$

$$\dfrac{\dfrac{\dfrac{x<y,x\prec y,y:A,x:\mathbf{G}A\Rightarrow y:A}{x<y,x\prec y,x:\mathbf{G}A\Rightarrow y:A}\,L\mathbf{G}}{x\prec y,x:\mathbf{G}A\Rightarrow y:A}\,Inc}{x:\mathbf{G}A\Rightarrow x:\mathbf{T}A}\,R\mathbf{T}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y=z,x\prec y,x\prec z,z:A,y:A\Rightarrow z:A,y:\bot}{y=z,x\prec y,x\prec z,y:A\Rightarrow z:A,y:\bot}\,EqSubst}{x\prec y,x\prec z,y:A\Rightarrow z:A,y:\bot}\,UnSucc}{x\prec y,x\prec z\Rightarrow z:A,y:\neg A}\,R{\supset}}{x\prec y\Rightarrow x:\mathbf{T}A,y:\neg A}\,R\mathbf{T}}{\Rightarrow x:\mathbf{T}A,x:\mathbf{T}\neg A}\,R\mathbf{T} \qquad \dfrac{}{x:\bot\Rightarrow x:\mathbf{T}A}\,L\bot}{x:\neg\mathbf{T}\neg A\Rightarrow x:\mathbf{T}A}\,L{\supset}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x\prec y,y:A,x:\mathbf{T}A,x:\mathbf{T}\neg A\Rightarrow x:\bot,y:A \qquad \dfrac{}{x\prec y,y:A,x:\mathbf{T}A,y:\bot,x:\mathbf{T}\neg A\Rightarrow x:\bot}\,L\bot}{x\prec y,y:A,x:\mathbf{T}A,y:\neg A,x:\mathbf{T}\neg A\Rightarrow x:\bot}\,L{\supset}}{x\prec y,y:A,x:\mathbf{T}A,x:\mathbf{T}\neg A\Rightarrow x:\bot}\,L\mathbf{T}}{x\prec y,x:\mathbf{T}A,x:\mathbf{T}\neg A\Rightarrow x:\bot}\,L\mathbf{T}}{x:\mathbf{T}A,x:\mathbf{T}\neg A\Rightarrow x:\bot}\,R\text{-}Ser}{x:\mathbf{T}A\Rightarrow x:\neg\mathbf{T}\neg A}\,R{\supset}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x \prec y, y : A, \dots \Rightarrow y : A, y : B \quad x \prec y, y : B, y : A, \dots \Rightarrow y : B}{x \prec y, y : A \supset B, y : A, x : \mathbf{T}(A \supset B), x : \mathbf{T}A \Rightarrow y : B}\,{\scriptstyle L\supset}}{x \prec y, y : A \supset B, x : \mathbf{T}(A \supset B), x : \mathbf{T}A \Rightarrow y : B}\,{\scriptstyle L\mathbf{T}}}{x \prec y, x : \mathbf{T}(A \supset B), x : \mathbf{T}A \Rightarrow y : B}\,{\scriptstyle L\mathbf{T}}}{x : \mathbf{T}(A \supset B), x : \mathbf{T}A \Rightarrow x : \mathbf{T}B}\,{\scriptstyle R\mathbf{T}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x = z, z \prec y, x \prec y, x : A, z : A \Rightarrow z : A}{x = z, z \prec y, x \prec y, x : A \Rightarrow z : A}\,{\scriptstyle EqSubst}}{z \prec y, x \prec y, x : A \Rightarrow z : A}\,{\scriptstyle UnPred}}{x \prec y, x : A \Rightarrow y : \mathbf{Y}A}\,{\scriptstyle R\mathbf{Y}}}{x : A \Rightarrow x : \mathbf{TY}A}\,{\scriptstyle R\mathbf{T}}$$

The derivation of the sequent $x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$ (and of its temporal mirror image) requires an application of the rule T^ω:

$$\dfrac{\dfrac{\{x \prec^n y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A\}_{n \in \mathbb{N}^+}}{x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A}\,{\scriptstyle T^\omega}}{x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A}\,{\scriptstyle R\mathbf{G}}$$

We show therefore derivability of the premises for every $n \in \mathbb{N}^+$:

$n = 1$

$$\dfrac{x \prec y, x < y, y : A, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A}{x \prec y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A}\,{\scriptstyle R\mathbf{T}}$$

Next, we show that $x \prec^{n+1} y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A$ is derivable whenever $x \prec^n y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A$ is derivable

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \prec^n y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A}{x \prec^n z, x < z, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow z : A}\,{\scriptstyle H.P.Subst}}{x < z, x \prec^n z, z \prec y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A, z : A}\,{\scriptstyle Wk^*} \quad \dfrac{z \prec y, y : A, z : \mathbf{T}A \Rightarrow y : A}{z \prec y, z : \mathbf{T}A \Rightarrow y : A}\,{\scriptstyle L\mathbf{T}}}{x < z, x \prec^n z, z \prec y, x < y, z : A \supset \mathbf{T}A, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A}\,{\scriptstyle L\supset}}{x < z, x \prec^n z, z \prec y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A}\,{\scriptstyle L\mathbf{G}}}{x \prec^n z, z \prec y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A}\,{\scriptstyle Inc_n}}{x \prec^{n+1} y, x < y, x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow y : A}\,{\scriptstyle LDef}$$

Observe that some side formulas have been omitted in the right premise of $L \supset$ for ease of writing. $\square$

Proposition 3.3.6. *The rule of* modus ponens

$$\frac{\Rightarrow x : A \quad \Rightarrow x : A \supset B}{\Rightarrow x : B}\,{\scriptstyle MP}$$

is admissible in G3LT*.*

Proof. Let us consider the following derivation:

$$\frac{\Rightarrow x : A \quad \dfrac{\Rightarrow x : A \supset B}{x : A \Rightarrow x : B}\,{\scriptstyle H.\text{-}p.Inv}}{\Rightarrow x : B}\,{\scriptstyle Cut}$$

The result follows by admissibility of cut. □

Proposition 3.3.7. *The temporal generalisation rules for* **G***,* **H***,* **T** *and* **Y**

$$\frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{G}A}\,{\scriptstyle \mathbf{G}Nec} \qquad \frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{H}A}\,{\scriptstyle \mathbf{H}Nec} \qquad \frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{T}A}\,{\scriptstyle \mathbf{T}Nec} \qquad \frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{Y}A}\,{\scriptstyle \mathbf{Y}Nec}$$

are admissible in G3LT*.*

Proof. Let us suppose that we have a derivation of $\Rightarrow x : A$. By Lemma 3.2.1 we obtain a derivation of $\Rightarrow y : A$ and by admissibility of weakening we obtain the sequents $x < y \Rightarrow y : A$, $y < x \Rightarrow y : A$, $x \prec y \Rightarrow y : A$ and $y \prec x \Rightarrow y : A$. We finally obtain $\Rightarrow x : \mathbf{G}A$, $\Rightarrow x : \mathbf{H}A$, $\Rightarrow x : \mathbf{T}A$ and $\Rightarrow x : \mathbf{Y}A$ by a single step of $R\mathbf{G}$, $R\mathbf{H}$, $R\mathbf{T}$ and $R\mathbf{Y}$, respectively. □

Corollary 3.3.8. (Completeness) *The calculus* G3LT *is complete with respect to Priorean linear time logic.*

We also prove the following useful propositions:

Proposition 3.3.9. *The following purely logical sequents*

$$x : \mathbf{G}A \Rightarrow x : \mathbf{T}A \,\&\, \mathbf{TG}A \qquad x : \mathbf{H}A \Rightarrow x : \mathbf{Y}A \,\&\, \mathbf{YH}A$$

are derivable in G3LT.

Proof. The proof consists of the following inferences together with Lemma 3.2.2

$$
\begin{prooftree}
\AxiomC{$x < w, w : A, x \prec w, x : \mathbf{G}A \Rightarrow w : A$}
\RightLabel{\scriptsize $L\mathbf{G}$}
\UnaryInfC{$x < w, x \prec w, x : \mathbf{G}A \Rightarrow w : A$}
\RightLabel{\scriptsize Inc}
\UnaryInfC{$x \prec w, x : \mathbf{G}A \Rightarrow w : A$}
\RightLabel{\scriptsize $R\mathbf{T}$}
\UnaryInfC{$x : \mathbf{G}A \Rightarrow x : \mathbf{T}A$}
\AxiomC{$x \prec y, x < z, x < y, y < z, z : A, x : \mathbf{G}A \Rightarrow z : A$}
\RightLabel{\scriptsize $L\mathbf{G}$}
\UnaryInfC{$x \prec y, x < z, x < y, y < z, x : \mathbf{G}A \Rightarrow z : A$}
\RightLabel{\scriptsize $Trans$}
\UnaryInfC{$x \prec y, x < y, y < z, x : \mathbf{G}A \Rightarrow z : A$}
\RightLabel{\scriptsize Inc}
\UnaryInfC{$x \prec y, y < z, x : \mathbf{G}A \Rightarrow z : A$}
\RightLabel{\scriptsize $R\mathbf{G}$}
\UnaryInfC{$x \prec y, x : \mathbf{G}A \Rightarrow y : \mathbf{G}A$}
\RightLabel{\scriptsize $R\mathbf{T}$}
\UnaryInfC{$x : \mathbf{G}A \Rightarrow x : \mathbf{TG}A$}
\RightLabel{\scriptsize $R\&$}
\BinaryInfC{$x : \mathbf{G}A \Rightarrow x : \mathbf{T}A \,\&\, \mathbf{TG}A$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$w < x, w : A, w \prec x, x : \mathbf{H}A \Rightarrow w : A$}
\RightLabel{\scriptsize $L\mathbf{H}$}
\UnaryInfC{$w < x, w \prec x, x : \mathbf{H}A \Rightarrow w : A$}
\RightLabel{\scriptsize Inc}
\UnaryInfC{$w \prec x, x : \mathbf{H}A \Rightarrow w : A$}
\RightLabel{\scriptsize $R\mathbf{Y}$}
\UnaryInfC{$x : \mathbf{H}A \Rightarrow x : \mathbf{Y}A$}
\AxiomC{$y \prec x, z < x, z < y, y < x, z : A, x : \mathbf{H}A \Rightarrow z : A$}
\RightLabel{\scriptsize $L\mathbf{H}$}
\UnaryInfC{$y \prec x, z < x, z < y, y < x, x : \mathbf{H}A \Rightarrow z : A$}
\RightLabel{\scriptsize $Trans$}
\UnaryInfC{$y \prec x, z < y, y < x, x : \mathbf{H}A \Rightarrow z : A$}
\RightLabel{\scriptsize Inc}
\UnaryInfC{$y \prec x, z < y, x : \mathbf{H}A \Rightarrow z : A$}
\RightLabel{\scriptsize $R\mathbf{H}$}
\UnaryInfC{$y \prec x, x : \mathbf{H}A \Rightarrow y : \mathbf{H}A$}
\RightLabel{\scriptsize $R\mathbf{Y}$}
\UnaryInfC{$x : \mathbf{H}A \Rightarrow x : \mathbf{YH}A$}
\RightLabel{\scriptsize $R\&$}
\BinaryInfC{$x : \mathbf{H}A \Rightarrow x : \mathbf{Y}A \,\&\, \mathbf{YH}A$}
\end{prooftree}
$$

□

Proposition 3.3.10. *The following purely logical sequents*

$$x : \mathbf{T}A, x : \mathbf{TG}A \Rightarrow x : \mathbf{G}A \qquad\qquad x : \mathbf{Y}A, x : \mathbf{YH}A \Rightarrow x : \mathbf{H}A$$

are derivable in G3LT.

Proof. By means of rules Mix_1 and Mix_2, respectively. Consider the following inferences, where the leaves are obtained by Lemma 3.2.2

$$
\begin{prooftree}
\AxiomC{$x \prec y, x < y, x : \mathbf{T}A, y : A, \ldots \Rightarrow y : A$}
\RightLabel{\scriptsize $L\mathbf{T}$}
\UnaryInfC{$x \prec y, x < y, x : \mathbf{T}A, \ldots \Rightarrow y : A$}
\AxiomC{$x \prec z, z < y, x < y, x : \mathbf{T}A, y : A, z : \mathbf{G}A, \ldots \Rightarrow y : A$}
\RightLabel{\scriptsize $L\mathbf{G}$}
\UnaryInfC{$x \prec z, z < y, x < y, x : \mathbf{T}A, z : \mathbf{G}A, x : \mathbf{TG}A \Rightarrow y : A$}
\RightLabel{\scriptsize $L\mathbf{T}$}
\UnaryInfC{$x \prec z, z < y, x < y, x : \mathbf{T}A, x : \mathbf{TG}A \Rightarrow y : A$}
\RightLabel{\scriptsize Mix_1}
\BinaryInfC{$x < y, x : \mathbf{T}A, x : \mathbf{TG}A \Rightarrow y : A$}
\RightLabel{\scriptsize $R\mathbf{G}$}
\UnaryInfC{$x : \mathbf{T}A, x : \mathbf{TG}A \Rightarrow x : \mathbf{G}A$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$y \prec x, y < x, y : A, x : \mathbf{Y}A, \ldots \Rightarrow y : A$}
\RightLabel{\scriptsize $L\mathbf{Y}$}
\UnaryInfC{$y \prec x, y < x, x : \mathbf{Y}A, \ldots \Rightarrow y : A$}
\AxiomC{$y < z, z \prec x, y < x, x : \mathbf{Y}A, y : A, z : \mathbf{H}A, \ldots \Rightarrow y : A$}
\RightLabel{\scriptsize $L\mathbf{H}$}
\UnaryInfC{$y < z, z \prec x, y < x, x : \mathbf{Y}A, z : \mathbf{H}A, x : \mathbf{YH}A \Rightarrow y : A$}
\RightLabel{\scriptsize $L\mathbf{Y}$}
\UnaryInfC{$y < z, z \prec x, y < x, x : \mathbf{Y}A, x : \mathbf{YH}A \Rightarrow y : A$}
\RightLabel{\scriptsize Mix_2}
\BinaryInfC{$y < x, x : \mathbf{Y}A, x : \mathbf{YH}A \Rightarrow y : A$}
\RightLabel{\scriptsize $R\mathbf{H}$}
\UnaryInfC{$x : \mathbf{Y}A, x : \mathbf{YH}A \Rightarrow x : \mathbf{H}A$}
\end{prooftree}
$$

□

Proposition 3.3.11. *The purely logical sequents that correspond to Hamblin's formulas for discreteness*[4] *are derivable in* G3LT.

Proof. Consider the following inferences, with leaves obtained by Lemma 3.2.2 (some repetitions are omitted)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z = x, x \prec y, z < y, x < y, x : A, \ldots \Rightarrow x : A}{z = x, x \prec y, z < y, x < y, z : A, \ldots \Rightarrow x : A}{\scriptstyle EqSubst}}{x = z, x \prec y, z < y, x < y, z : A, \ldots \Rightarrow x : A}{\scriptstyle EqSym} \qquad \dfrac{z < x, x \prec y, z < y, x < y, z : A, \ldots \Rightarrow \ldots, z : A}{z < x, x \prec y, z < y, x < y, z : A \ldots \Rightarrow x : \mathbf{P}A}{\scriptstyle R\mathbf{P}}}{x \prec y, z < y, x < y, z : A, x : \mathbf{GP}A \Rightarrow x : A, x : \mathbf{P}A}{\scriptstyle Discr_2}}{x \prec y, x < y, y : \mathbf{P}A, x : \mathbf{GP}A \Rightarrow x : A, x : \mathbf{P}A}{\scriptstyle L\mathbf{P}}}{x \prec y, x < y, x : \mathbf{GP}A \Rightarrow x : A, x : \mathbf{P}A}{\scriptstyle L\mathbf{G}}}{x \prec y, x : \mathbf{GP}A \Rightarrow x : A, x : \mathbf{P}A}{\scriptstyle Inc}}{x : \mathbf{GP}A \Rightarrow x : A, x : \mathbf{P}A}{\scriptstyle R\text{-}Ser}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z = x, y \prec x, y < z, y < x, x : A, \ldots \Rightarrow x : A}{z = x, y \prec x, y < z, y < x, z : A, \ldots \Rightarrow x : A}{\scriptstyle EqSubst}}{x = z, y \prec x, y < z, y < x, z : A, \ldots \Rightarrow x : A}{\scriptstyle EqSym} \qquad \dfrac{x < z, y \prec x, y < z, y < x, z : A, \ldots \Rightarrow \ldots, z : A}{x < z, y \prec x, y < z, y < x, z : A, \ldots \Rightarrow x : \mathbf{F}A}{\scriptstyle R\mathbf{F}}}{y \prec x, y < z, y < x, z : A, x : \mathbf{HF}A \Rightarrow x : A, x : \mathbf{F}A}{\scriptstyle Discr_1}}{y \prec x, y < x, y : \mathbf{F}A, x : \mathbf{HF}A \Rightarrow x : A, x : \mathbf{F}A}{\scriptstyle L\mathbf{F}}}{y \prec x, y < x, x : \mathbf{HF}A \Rightarrow x : A, x : \mathbf{F}A}{\scriptstyle L\mathbf{H}}}{y \prec x, x : \mathbf{HF}A \Rightarrow x : A, x : \mathbf{F}A}{\scriptstyle Inc}}{x : \mathbf{HF}A \Rightarrow x : A, x : \mathbf{F}A}{\scriptstyle L\text{-}Ser}$$

□

3.4 Partial finitisations

As we have shown in Section 3.2, the calculus G3LT enjoys important structural properties, most importantly it admits syntactical cut elimination. However, the presence of an infinitary rule in the system is harmful for the purposes of proof theory, in particular for proof search.

Proof search is a procedure that permits to construct a derivation starting from the conclusion: the endsequent is analysed in order to determine a last possible rule of inference and thus its premise(s), the latters are then analysed in the same way, and so on. Therefore, we construct a proof-search tree, the root

[4] See Section 2.2.

of which is the endsequent and the nodes of which are the sequents successively introduced as premises: if every leaf is an initial sequent or a conclusion of $L\bot$, then the proof search succeeds and we obtain a derivation. On the contrary, the procedure fails if at least one of the leaves is not an initial sequent or a conclusion of $L\bot$ and cannot be further analysed, or if the proof search does not stop.

Unfortunately, in the presence of an infinitary rule it is in principle impossible to distinguish whether the procedure does not stop because the endsequent is underivable or simply because we have to derive the infinitely many premises of T^ω.

Several attempts have been done in the literature in order to obtain a finitary cut-free calculus for LTL, but the inherent presence of induction makes the development of a finitary proof system problematic: for istance, the natural deduction calculus for linear time logic described in Bolotov *et al.* (2006) is not normalizable, whereas the proof-systems proposed by Brotherston and Simspon (2007) for inductive definitions require non-local rules, in the form of global correctness conditions for derivations. A significant indirect contribution is found in Jäger *et al.* (2007), where the finite model property is used to give an upper bound to the number of premises of an infinitary rule in an unlabelled sequent calculus for the logic of common knowledge; however, the whole approach appears quite factitious because it relies on model-theoretical rather than proof-theoretical arguments.

In what follows we consider two finitary versions of the calculus G3LT. The system G3LT$_{n\text{-}s}$ is obtained by replacing the infinitary rule with two finitary counterparts that permit the splitting of an interval $[x, y]$ with an immediate successor of x and an immediate predecessor of y, respectively. The calculus G3LT$_{n\text{-}s}$ is weaker than G3LT, but we identify a fragment of G3LT for which conservativity with respect to G3LT$_{n\text{-}s}$ is proved.

Next, we consider the future-oriented fragment of G3LT and give a finite bound to the number of premises in the infinitary rule in a purely syntactical way, by simply counting the number of occurences of the operator **T** in the negative part of its premises. We then show that the finitary rule T^δ thus obtained is as strong as the infinitary one for derivations of purely logical sequents that do not contain **G** in the negative nor **F** in the positive part.

Somehow related results were obtained in a different, but qualitatively similar case, namely in epistemic logic: a conservativity result, parallel to the one of Section 3.4.1, is presented by Antonakos (2007) and Artemov (2006). A finite bound analogous to that of Section 3.4.2 was obtained in Alberucci and Jäger (2005) for an unlabelled Tait-style calculus with an infinitary rule for the common knowledge operator. However, our calculus allows for syntactical cut elimination, whereas the latter work only shows through a semantical argument that the rule of cut is not needed.

3.4.1 A non-standard system for Linear Time

We define the system G3LT$_{n\text{-}s}$ by substituting, in the calculus G3LT, the rules T^ω, $LDef$ and $RDef$ with the rules Mix_1, Mix_2, *L-Lin* and *R-Lin* as primitive.

The standard frame for Priorean linear time logic corresponds to the set of the integers $\mathbb{Z}$: every instant greater (smaller) than x can be reached from x by finitely many iterations of the immediate successor (predecessor) relation. This condition corresponds to the infinitary rule T^ω of the calculus G3LT.

Because of the absence of T^ω, the systems G3LT$_{n\text{-}s}$ allows non-standard linear discrete frames that consist of several, possibly infinite, consecutive copies of the integers, $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$: even though every point is the unique immediate successor of its unique immediate predecessor (and viceversa), it is not always true that between any two points x, y such that $x < y$, there are finitely many other points.

If the rule of right linearity is dropped from the calculus G3LT$_{n\text{-}s}$, a calculus for the so-called branching time gaps is obtained, the frames of which are trees made of copies of $\mathbb{Z}$. The first-order temporal logic that corresponds to branching time gaps (well-founded trees of copies of $\mathbb{N}$) has been investigated from a model-theoretic point of view by Baaz *et al.* (1996). Its proof-theoretic aspects have been investigated through a G3-style unlabelled sequent calculus by Alonderis (2000).

It is easy to verify that the system G3LT$_{n\text{-}s}$ can be embedded in G3LT:

Theorem 3.4.1. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* G3LT$_{n\text{-}s}$*, then* $\Gamma \Rightarrow \Delta$ *is derivable in* G3LT.

Proof. Every rule of G3LT$_{n\text{-}s}$, except Mix_1, Mix_2, *L-Lin*, and *R-Lin*, is a rule of G3LT, and Mix_1, Mix_2, *L-Lin*, and *R-Lin* are admissible in G3LT, by Proposition 3.2.24 and Proposition 3.2.23, respectively. □

The converse fails because of the infinitary rule: for instance, any proof search for the induction principle $x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$ would require infinitely many applications of rule Mix_1. Nevertheless, we identify a conservative fragment for which derivability in G3LT implies derivability in G3LT$_{n\text{-}s}$. Our result is confined to purely logical sequents, but this condition is not restrictive, since, as we noticed before, only purely logical sequents can be interpreted as the corresponding modal formulas.

Definition 3.4.2. *A labelled formula* occurs positively (negatively) *in a sequent if it is contained in the succedent (antecedent).* Positive (negative) occurrences *of subformulas of labelled formulas in a sequent are defined as follows:*

- *Both A and B occur positively (negatively) if $A \circ B$ occurs positively (negatively), for conjunction and disjunction;*
- *A occurs positively (negatively) if $A \supset B$ occurs negatively (positively) and B occurs positively (negatively) if $A \supset B$ occurs positively (negatively);*
- *A occurs positively (negatively) if $\mathbf{M}A$ occurs positively (negatively), for any temporal operator $\mathbf{M}$.*

The positive (negative) part of a sequent *consists of the positive (negative) occurrences of subformulas in it.*

Theorem 3.4.3. *If a purely logical sequent $\Gamma \Rightarrow \Delta$ is derivable in* G3LT*, and the operators $\mathbf{G}$, $\mathbf{H}$ do not appear in its positive part nor $\mathbf{F}$, $\mathbf{P}$ in its negative part, then $\Gamma \Rightarrow \Delta$ is derivable without the use of the infinitary rule.*

Proof. We show that all the applications of the infinitary rule can be dispensed with. Without loss of generality, we assume that the given derivation is minimal (see Definition 3.2.12). Observe that all the relational atoms $x < y$, in particular those concluded by T^{ω}, have to disappear before the conclusion. We consider one such downmost atom and the rule that makes it disappear: rules $R\mathbf{G}$, $R\mathbf{H}$, $L\mathbf{F}$ and $L\mathbf{P}$ are excluded because they would introduce $\mathbf{G}$, $\mathbf{H}$ in the positive part or $\mathbf{F}$, $\mathbf{P}$ in the negative part. Thus, the atom can disappear only by means of *Inc* or $EqSubst_{At}$.

If the atom concluded by T^ω is removed by Inc, we have

$$\cfrac{\cfrac{\{x \prec^n y, x < y, \Gamma' \Rightarrow \Delta'\}_{n \in \mathbb{N}^+}}{x < y, \Gamma' \Rightarrow \Delta'}\, T^\omega}{\vdots} \qquad \cfrac{x < y, x \prec y, \Gamma'' \Rightarrow \Delta''}{x \prec y, \Gamma'' \Rightarrow \Delta''}\, Inc$$

The first premise of T^ω has the form $x \prec y, x < y, x \prec y, \Gamma''' \Rightarrow \Delta'$, with $\Gamma' \equiv x \prec y, \Gamma'''$. By height-preserving contraction we obtain $x < y, \Gamma' \Rightarrow \Delta'$ and proceed with the derivation until we reach $x \prec y, x < y, \Gamma'' \Rightarrow \Delta''$. Then we conclude $x \prec y, \Gamma'' \Rightarrow \Delta''$ by an application of Inc. Note that the derivation is shortened, contrary to the assumption of minimality.

If the atom concluded by T^ω is removed by applications of $Trans$ followed by applications of Inc, we have the derivation

$$\dfrac{\{x \prec^n y, x < y, z_1 < y, \ldots, z_{m-1} < y, x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'\}_{n \in \mathbb{N}^+}}{x < y, z_1 < y, \ldots, z_{m-1} < y, x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, T^\omega$$
$$\vdots$$
$$\dfrac{x < y, z_1 < y, \ldots, z_{m-1} < y, x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}{x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}\, Trans{\times}m$$
$$\vdots$$
$$\dfrac{x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}{x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}\, Inc{\times}(m+1)$$

Let us consider the $(m+1)$-st premise of T^ω, and transform the derivation as follows

$$\dfrac{\dfrac{x \prec^{m+1} y, x < y, z_1 < y, \ldots, z_{m-1} < y, x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}{x \prec z_1, \ldots, z_m \prec y, x < y, z_1 < y, \ldots, z_{m-1} < y, x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, \mathcal{I}}{x < y, z_1 < y, \ldots, z_{m-1} < y, x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, \mathcal{C}$$
$$\vdots$$
$$\dfrac{x < y, z_1 < y, \ldots, z_{m-1} < y, x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}{x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}\, Trans{\times}m$$
$$\vdots$$
$$\dfrac{x < z_1, \ldots, z_m < y, x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}{x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}\, Inc{\times}(m+1)$$

Here $\mathcal{I}$ stands for m applications of height-preserving invertibility of rule $LDef$ and $\mathcal{C}$ for several application of height preserving contraction. Again, the derivation is shortened, contrary to the assumption.

If the atom concluded by T^ω is removed by an application of $EqSubst_{At}$, we have the following derivation:

$$\cfrac{\cfrac{\{x \prec^n y, z = y, x < y, x < z, \Gamma' \Rightarrow \Delta'\}_{n\in\mathbb{N}^+}}{z = y, x < y, x < z, \Gamma' \Rightarrow \Delta'}\, T^\omega}{\cfrac{\vdots}{\cfrac{z = y, x < y, x < z, \Gamma'' \Rightarrow \Delta''}{z = y, x < z, \Gamma'' \Rightarrow \Delta''}\, EqSubst_{At}}}$$

It is possible to permute up rule $EqSubst_{At}$ with respect to rule T^ω. We modify each premise of T^ω as follows:

$$\cfrac{x \prec^n y, x < y, z = y, x < z, \Gamma' \Rightarrow \Delta'}{x \prec^n y, x < y, x \prec^n z, z = y, x < z, \Gamma' \Rightarrow \Delta'}\, LWk$$
$$\vdots$$
$$\cfrac{\cfrac{x \prec^n y, x < y, x \prec^n z, z = y, x < z, \Gamma'' \Rightarrow \Delta''}{x \prec^n y, x \prec^n z, z = y, x < z, \Gamma' \Rightarrow \Delta'}\, Inc_n}{x \prec^n z, z = y, x < z, \Gamma'' \Rightarrow \Delta''}\, EqSubst_n$$

We can now apply the above considered modifications on the rule removing $x < z$. The case of $EqSubst_{At}$ with active formulas $z = x, x < y, z < y$ is analogous. □

Corollary 3.4.4. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathrm{G3LT}_{n\text{-}s}$ *and* $\Gamma \Rightarrow \Delta$ *is as in the previous theorem, then* $\Gamma \Rightarrow \Delta$ *is derivable in* G3LT.

Proof. By Theorem 3.4.3, $\Gamma \Rightarrow \Delta$ is derivable in G3LT without using the infinitary rule T^ω. □

3.4.2 A finite bound for the infinitary rule

Let us now consider the system $\mathrm{G3LT}^f$, the future-oriented fragment of the calculus G3LT, obtained from G3LT by dropping the rules for the past operators (**H**, **P** and **Y**), and the mathematical rules *L-Ser* and *UnPred*.

The calculus $\mathrm{G3LT}^\delta$ is obtained by substituting in $\mathrm{G3LT}^f$ the infinitary rule T^ω with the following finitary version T^δ

$$\frac{\{x \prec^m y, x < y, \Gamma \Rightarrow \Delta\}_{1 \leq m \leq \delta(\Gamma,\Delta)+1}}{x < y, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, T^\delta$$

where, $\delta(\Gamma, \Delta)$ is defined as the number of occurrences of the temporal operator **T** in the negative part[5] of the sequent $x \prec^m y, x < y, \Gamma \Rightarrow \Delta$, and, in order to preserve admissibility of weakening, Γ' and Δ' are arbitrary multisets.

Lemma 3.4.5. *Let*

$$x \prec^m z, z \prec y, x < y, \Gamma \Rightarrow \Delta$$

be a sequent derivable in $\mathrm{G3LT}^f$ *not containing* **G** *in the negative part, nor* **F** *in the positive part, nor relational atoms in* Δ*, and with* z *different from* x, y *and not in* Γ, Δ*, and let* m *be* $\delta(\Gamma, \Delta)$*. Then also the sequent*

$$x \prec^m z, z \prec^n y, x < y, \Gamma \Rightarrow \Delta$$

is derivable in $\mathrm{G3LT}^f$ *for all* $n \geq 1$*.*

Proof. (Sketch). We are interested in minimal derivations. Trace up the atom $z \prec y$ along the derivation. If it is never principal, it can be replaced by $z \prec^n y$ all along the derivation. It cannot be principal in an axiomatic sequent because of the condition that no relational atoms are in Δ. Therefore the possibilities for it to be principal are $L\mathbf{T}$, $EqSubst_{At}$, $UnSucc$, and Inc.

[5] See Definition 3.4.2.

If $z \prec y$ were principal in $L\mathbf{T}$ we would have the derivation

$$\frac{z \prec y, y : A, z : \mathbf{T}A, \Gamma' \Rightarrow \Delta'}{z \prec y, z : \mathbf{T}A, \Gamma' \Rightarrow \Delta'}\, L\mathbf{T}$$

observe that the principal formula $z : \mathbf{T}A$ has to disappear because of the assumption on the label z. It can do so only going through applications of $L\mathbf{T}$ and/or instances of $L\mathbf{G}$ or $R\mathbf{F}$ with an active relational atom from the decomposition of the chain $x \prec^m z$ (possibly combined with rule Inc). However, $L\mathbf{G}$ and $R\mathbf{F}$ are excluded by the condition that $\mathbf{G}$ is not in the negative part and $\mathbf{F}$ is not in the positive part of the sequent; on the other hand, applications of $L\mathbf{T}$ would introduce in the negative part of the sequent a number of operators $\mathbf{T}$ greater than m, contrary to the hypothesis that $m = \delta(\Gamma, \Delta)$. Note that by the same reason for no label t an atom of the form $z \prec t$ can be principal in a left rule for $\mathbf{T}$, whereas the case of $z \prec t$ active in an instance of $R\mathbf{T}$ would introduce a formula $z : \mathbf{T}A$ in the consequent that cannot disappear without violating the variable conditions on temporal rules or the hypothesis that z is not in Γ, Δ.

For no label t the atom $z \prec t$ ($z \prec y$ included) can be principal in $EqSubst_{At}$ with equality on z

$$\frac{w \prec t, z = w, z \prec t, \Gamma' \Rightarrow \Delta'}{z = w, z \prec t, \Gamma' \Rightarrow \Delta'}\, EqSubst_{At}$$

In fact, since z is not in Γ, Δ, the atom $z = w$ should disappear from the derivation. It cannot be removed by rule $EqRef$, because otherwise we could shorten the derivation by means of height-preserving admissibility of contraction on formulas $w \prec t, z \prec t$ (by $z \equiv w$), contrary to the hypothesis of minimality. The atom $z = w$ could disappear by $UnSucc$ with principal formulas $v \prec z$, $v \prec w$, but both formulas cannot disappear without introducing new relational atoms with variable z or logical formulas labelled by z, contrary to the hypothesis that z is not in Γ, Δ.

For the cases with $z \prec y$ principal in $UnSucc$ or Inc or $EqSubst_{At}$, we observe that we can replace all along the derivation relational formulas of the form $z \prec t$ for every label t ($z \prec y$ included) with $z \prec^n t$ and apply the admissible rules Inc_n, $UnSucc_n$, $EqSubst_n$ and $R\text{-}Ser_n$ (see Table 3.5) whenever Inc, $UnSucc$, $EqSubst_{At}$ and $R\text{-}Ser$ are applied in the original derivation. $\square$

Theorem 3.4.6. *If the purely logical sequent* $\Gamma \Rightarrow \Delta$ *does not contain* $\mathbf{G}$ *in the negative part nor* $\mathbf{F}$ *in the positive part, then* $\Gamma \Rightarrow \Delta$ *is derivable in* G3LT^f *iff* $\Gamma \Rightarrow \Delta$ *is derivable in* G3LT$^\delta$.

Proof. The left-to-right implication is obvious: we simply note here that every rule of G3LT^f except for T^ω is a rule of G3LT$^\delta$, and we need to consider the first $\delta(\Gamma, \Delta) + 1$ premises of T^ω in order to apply T^δ whenever T^ω is applied in G3LT^f. The right-to-left direction is proved by induction on the height of the derivation in G3LT$^\delta$; we assume that the given derivation is minimal. We need to consider only the case of T^δ, all the other rules being common to G3LT^f and G3LT$^\delta$. If T^δ is used, the rightmost premise is of the form

$$x \prec^{m+1} y, x < y, \Gamma' \Rightarrow \Delta'$$

where $m = \delta(\Gamma, \Delta)$.

By induction on the height of derivation $x \prec^{m+1} y, x < y, \Gamma' \Rightarrow \Delta'$ is derivable in G3LT, and thanks to invertibility of $LDef$ we obtain

$$x \prec^m z, z \prec y, x < y, \Gamma' \Rightarrow \Delta'$$

with z not in Γ, Δ.

By hypothesis **G** is not in the negative part of the sequent and **F** is not in the positive part, and no relational atom is in Δ: all the conditions of the Lemma 3.4.5 are satisfied. So, every sequent of the form $x \prec^m z, z \prec^n y, x < y, \Gamma' \Rightarrow \Delta'$ is derivable in G3LT^f for all $n \geq 1$, and the premises of T^ω are obtained by admissibility of $LDef_n$. $\square$

Unfortunately, the results concerning the calculus G3LT$^\delta$ cannot be extended to the whole system with past operators because of the possibility of going back and forth along the chain $x \prec^m z, z \prec y$ by means of the rules for **T** and **Y**.

3.5 Adding Until and Since

In Kamp (1968) linear time logic was enriched with two further temporal operators, intuitively called *Until* and *Since*, with the following semantic readings in irreflexive frames:

$x \Vdash A\mathcal{U}B$ *iff there exists* y *such that* $x < y$ *and* $y \Vdash B$

and for all z*, if* $x < z$ *and* $z < y$*, then* $z \Vdash A$

$x \Vdash A\mathcal{S}B$ *iff there exists* y *such that* $y < x$ *and* $y \Vdash B$

and for all z*, if* $y < z$ *and* $z < x$*, then* $z \Vdash A$

Kamp proved that linear temporal logic with until and since is considerably more expressive than the traditional Priorean logic with operators **T**, **G**, **Y** and **H**. Sequent calculus rules for Until and Since can be formulated along the lines of the method employed so far on the base of the following recursive definitions:

$$A\mathcal{U}B \;\equiv\; \mathbf{T}B \vee (\mathbf{T}A \,\&\, \mathbf{T}(A\mathcal{U}B))$$

$$A\mathcal{S}B \;\equiv\; \mathbf{Y}B \vee (\mathbf{Y}A \,\&\, \mathbf{Y}(A\mathcal{S}B))$$

or, equivalently, by distributivity of $\vee$ over &

$$A\mathcal{U}B \;\equiv\; (\mathbf{T}B \vee \mathbf{T}A)\,\&\,(\mathbf{T}B \vee \mathbf{T}(A\mathcal{U}B))$$

$$A\mathcal{S}B \;\equiv\; (\mathbf{Y}B \vee \mathbf{Y}A)\,\&\,(\mathbf{Y}B \vee \mathbf{Y}(A\mathcal{S}B))$$

In addition to initial sequents for propositional and relational atomic formulas, initial sequents for compound formulas of the form $x : A\mathcal{U}B$ and $x : A\mathcal{S}B$ are required by the recursive definition. Moreover, a further condition, $x : \mathbf{F}B$ (resp. $x : \mathbf{P}B$), has to be added to the rules, in order to guarantee that B will be (has been) satisfied at some point.

Initial Sequents:

$x : A\mathcal{U}B, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B$

$x : A\mathcal{S}B, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B$

Rules for Until:

$$\frac{x \prec y, y : B, \Gamma \Rightarrow \Delta \quad x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta}{x \prec y, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\mathcal{U}}$$

$$\frac{\begin{array}{l} x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A \\ x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A\mathcal{U}B \\ x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, x : \mathbf{F}B \end{array}}{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B}\, {\scriptstyle R\mathcal{U}}$$

Rules for Since

$$\frac{y \prec x, y : B, \Gamma \Rightarrow \Delta \quad y \prec x, y : A, y : A\mathcal{S}B, x : \mathbf{P}B, \Gamma \Rightarrow \Delta}{y \prec x, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\mathcal{S}}$$

$$\frac{\begin{array}{l} y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A \\ y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A\mathcal{S}B \\ y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, x : \mathbf{P}B \end{array}}{\Gamma \Rightarrow \Delta, x : A\mathcal{S}B}\, {\scriptstyle R\mathcal{S}}$$

Rules $R\mathcal{U}$ and $R\mathcal{S}$ have the condition that y is not in the conclusion.

Table 3.8: The rules for Until and Since

The calculus G3LT+$\mathcal{U}$+$\mathcal{S}$ is obtained by adding the rules of Table 3.8 to G3LT.

The rules for $\mathcal{U}$ and $\mathcal{S}$ are of a peculiar form, since the complexity of the active formulas is not strictly less than the complexity of the principal formulas. A similar situation occurs in the sequent rules for Gödel-Löb logic presented in Negri (2005) and here, as well, admissibility of the structural rules is shown through a refined inductive measure based on the notion of range.

The right rules for $\mathcal{U}$ and $\mathcal{S}$ are height-preserving invertible by height-preserving admissibility of weakening. By the following lemmas, also the left rules are invertible, but invertibility does not preserve derivation height.

Lemma 3.5.1. *If the sequent* $x \prec y, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta$ *is derivable in the calculus* G3LT+$\mathcal{U} + \mathcal{S}$*, then also the sequents*

$$x \prec y, y : B, \Gamma \Rightarrow \Delta \qquad\qquad x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta$$

are derivable.

Proof. By induction on the height h of the given derivation. If $h = 0$ and $x : A\mathcal{U}B, \Gamma \Rightarrow \Delta$ is an initial sequent with $x : A\mathcal{U}B$ not principal, or conclusion of $L\bot$, then also the corresponding premises of $L\mathcal{U}$ are. If it is an initial sequent with $x : A\mathcal{U}B$ principal, we derive the sequents

$x \prec z, x \prec y, y : B, \Gamma \Rightarrow \Delta, z : B, z : A$

$x \prec z, x \prec y, y : B, \Gamma \Rightarrow \Delta, z : B, z : A\mathcal{U}B$

$x \prec z, x \prec y, y : B, \Gamma \Rightarrow \Delta, z : B, x : \mathbf{F}B$

by means of Lemma 3.2.2, *EqSubst* and *UnSucc*, and then apply $R\mathcal{U}$ to obtain the sequent $x \prec y, y : B, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B$. Analogously, we derive the sequents

$x \prec z, x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta, z : B, z : A$

$x \prec z, x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta, z : B, z : A\mathcal{U}B$

$$x \prec z, x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta, z : B, x : \mathbf{F}B$$

and then apply $R\mathcal{U}$ to obtain $x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B$. In both cases, z is chosen different from x, y and not in Γ, Δ.

If $x : A\mathcal{U}B, \Gamma \Rightarrow \Delta$ is the conclusion of a rule different from $L\mathcal{U}$, apply the inductive hypothesis to the premise(s) and then the rule. Similarly if it is conclusion of $L\mathcal{U}$ and $x : A\mathcal{U}B$ is not principal. If it is conclusion of $L\mathcal{U}$ and $A\mathcal{U}B$ is principal we have two cases: (i) if $x \prec y$ is principal too, simply delete the last rule to obtain the sequent $x \prec y, y : B, \Gamma \Rightarrow \Delta$ as left premise and the sequent $x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta$ as right premise. (ii) If $x \prec y$ is not principal we have the following derivation

$$\dfrac{x \prec y, z : B, x \prec z, \Gamma' \Rightarrow \Delta \quad x \prec y, z : A, z : A\mathcal{U}B, x : \mathbf{F}B, x \prec z, \Gamma' \Rightarrow \Delta}{x \prec y, x : A\mathcal{U}B, x \prec z, \Gamma' \Rightarrow \Delta}\, L\mathcal{U}$$

with $\Gamma \equiv x \prec z, \Gamma'$.

The desired sequents are obtained by the following derivations

$$\dfrac{\dfrac{\dfrac{x \prec y, z : B, x \prec z, \Gamma' \Rightarrow \Delta}{y = z, x \prec y, z : B, y : B, x \prec z, \Gamma' \Rightarrow \Delta}\, LWk^*}{y = z, x \prec y, y : B, x \prec z, \Gamma' \Rightarrow \Delta}\, EqSubst}{x \prec y, y : B, x \prec z, \Gamma' \Rightarrow \Delta}\, UnSucc$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x \prec y, z : A, z : A\mathcal{U}B, x : \mathbf{F}B, x \prec z, \Gamma' \Rightarrow \Delta}{y = z, x \prec y, z : A, y : A, z : A\mathcal{U}B, y : A\mathcal{U}B, x : \mathbf{F}B, x \prec z, \Gamma' \Rightarrow \Delta}\, LWk^*}{y = z, x \prec y, z : A, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, x \prec z, \Gamma' \Rightarrow \Delta}\, EqSubst}{y = z, x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, x \prec z, \Gamma' \Rightarrow \Delta}\, EqSubst}{x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, x \prec z, \Gamma' \Rightarrow \Delta}\, UnSucc$$

□

Lemma 3.5.2. *If the sequent $y \prec x, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta$ is derivable in the calculus* G3LT+$\mathcal{U} + \mathcal{S}$, *then the sequents*

$$y \prec x, y : B, \Gamma \Rightarrow \Delta \qquad\qquad y \prec x, y : A, y : A\mathcal{S}B, x : \mathbf{P}B, \Gamma \Rightarrow \Delta$$

are derivable.

Proof. Analogous to Lemma 3.5.1. □

The length of labelled formulas is defined as in Definition 2.3.1 with the following additional clauses:

Definition 3.5.3. *The length of Until and Since formulas is defined as follows:*

$l(A\mathcal{U}B) = l(A) + l(B) + 1$

$l(A\mathcal{S}B) = l(A) + l(B) + 1$

Observe that $l(\bot\mathcal{U}B) = l(\bot\mathcal{S}B) = 0 + 1 + l(B) = l(\mathbf{F}B) = l(\mathbf{P}B)$.

Definition 3.5.4. *The* right range *of x in a derivation $\mathcal{D}$ is the (finite) set of instants y such that either $x \prec y$ or for some $n \geq 1$ and for some $x_1, \ldots, x_n$, the atoms $x \prec x_1, x_1 \prec x_2, \ldots, x_n \prec y$ appear in the sequents of $\mathcal{D}$. The* left range *of x is defined analogously as the set of instants y such that either $y \prec x$ or for some $n \geq 1$ and for some $y_1, \ldots, y_n$, the atoms $y \prec y_1, y_1 \prec y_2, \ldots, y_n \prec x$ appear in the sequents of $\mathcal{D}$. Ranges of variables are ordered by set inclusion.*

Theorem 3.5.5. *The rules of contraction are admissible in* G3LT+$\mathcal{U} + \mathcal{S}$ *and preserve left and right range of derivation.*

Proof. By simultaneous induction for left and right contraction, with induction on the range and on the length of contraction formula, and subinduction on derivation height. We consider here in detail only the cases arising from the addition of the rules for Until and Since. Note that the case of right contraction with one of the contraction formulas principal in $R\mathcal{U}$ or $R\mathcal{S}$ is taken care of by the repetition of the principal formula in the premises: contraction is applied to the shorter derivation of the premises of the rule, and then the rule is applied.

For left contraction with contraction formula $x : A\mathcal{U}B$ we distinguish the following cases: if $x : A\mathcal{U}B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta$ is an initial sequent or conclusion of $L\bot$, so is $x : A\mathcal{U}B, \Gamma \Rightarrow \Delta$. If it is conclusion of a rule different from $L\mathcal{U}$ or with other principal formula than the contraction formula, we apply the inductive hypothesis to the premise(s) and then the rule.

If the last rule is $L\mathcal{U}$ with principal formula $x : A\mathcal{U}B$ and A is not $\bot$, we have the sequent $x \prec y, y : B, x : A\mathcal{U}B, \Gamma' \Rightarrow \Delta$ as left premise and the sequent $x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, x : A\mathcal{U}B, \Gamma' \Rightarrow \Delta$ as right premise, where $\Gamma \equiv x \prec y, \Gamma'$. By Lemma 3.5.1, we derive $x \prec y, y : B, y : B, \Gamma \Rightarrow \Delta$ and $x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta$, and we obtain derivations of $x \prec y, y : B, \Gamma \Rightarrow \Delta$ and $x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta$ by means of the inductive hypotheses applied to shorter formulas ($y : B$, $y : A$, and $x : \mathbf{F}B$) and smaller right range ($y : A\mathcal{U}B$). If $A \equiv \bot$, we cannot apply the induction on the length of formulas to obtain the right premise of $L\mathcal{U}$: in fact, because of the equivalence $l(\bot\mathcal{U}B) = l(\mathbf{F}B)$, contraction on $x : \mathbf{F}B$ neither reduces the length nor the right range of of the formula. However, the contracted instance of the right premise can be obtained by the following derivation

$$\cfrac{}{x \prec y, y : \bot, y : \bot\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta}\, L\bot$$

Analogously, if $x : A\mathcal{S}B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta$ is an initial sequent or conclusion of $L\bot$, so is $x : A\mathcal{S}B, \Gamma \Rightarrow \Delta$. If it is conclusion of a rule different from $L\mathcal{S}$ or with other principal formula than the contraction formula, we apply the inductive hypothesis to the premise(s) and then the rule.

If the last rule is $L\mathcal{S}$ with principal formula $x : A\mathcal{S}B$ and A is not $\bot$, we have the sequent $y \prec x, y : B, x : A\mathcal{S}B, \Gamma' \Rightarrow \Delta$ as left premise, and the sequent

$y \prec x, y : A, y : A\mathcal{S}B, x : \mathbf{P}B, x : A\mathcal{S}B, \Gamma' \Rightarrow \Delta$ as right premise, where $\Gamma \equiv y \prec x, \Gamma'$. By Lemma 3.5.2, we derive $y \prec x, y : B, y : B, \Gamma \Rightarrow \Delta$ and $y \prec x, y : A, y : A\mathcal{S}B, x : \mathbf{P}B, y : A, y : A\mathcal{S}B, x : \mathbf{P}B, \Gamma \Rightarrow \Delta$, and we obtain derivations of $y \prec x, y : B, \Gamma \Rightarrow \Delta$ and $y \prec x, y : A, y : A\mathcal{S}B, x : \mathbf{P}B, \Gamma \Rightarrow \Delta$ by means of the inductive hypothesis applied to shorter formulas ($y : B$, $y : A$, and $x : \mathbf{P}B$) and smaller left range ($y : A\mathcal{S}B$). If $A \equiv \bot$, we cannot apply the induction on the length of formulas to obtain the right premise of $L\mathcal{S}$: in fact, because of the equivalence $l(\bot\mathcal{S}B) = l(\mathbf{P}B)$, contraction on $x : \mathbf{P}B$ neither reduces the length nor the left range of the formula. However, the contracted instance of the right premise can be obtained by the following derivation

$$\frac{}{y \prec x, y : \bot, y : \bot\mathcal{S}B, x : \mathbf{P}B, \Gamma \Rightarrow \Delta} L\bot$$

Finally, we observe that all the other cases are range preserving. The cases of propositional rules and of the rules with repetition of the principal formula in the premise(s) are obvious. As for the rules with variable condition, let us consider for instance the case of contraction formula $x : \mathbf{F}B$ introduced by $L\mathbf{F}$:

$$\cfrac{\cfrac{x < y, y : B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta}{x : \mathbf{F}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta} L\mathbf{F}}{x : \mathbf{F}B, \Gamma \Rightarrow \Delta} LCtr$$

By applying height-preserving invertibility of rule $L\mathbf{F}$, we obtain the sequent $x < y, y : B, x < y, y : B, \Gamma \Rightarrow \Delta$; we then apply the inductive hypothesis to obtain $x < y, y : B, \Gamma \Rightarrow \Delta$, and the rule $L\mathbf{F}$ to conclude $x : \mathbf{F}B, \Gamma \Rightarrow \Delta$. Note that although, in general, invertibility of $R\mathbf{G}$, $R\mathbf{H}$, $R\mathbf{T}$, $R\mathbf{Y}$, $L\mathbf{F}$, $L\mathbf{P}$, and $LDef$ introduces a new world, the special instance of invertibility used here does not, as the world needed in the inversion is already a label used in the derivation. It follows that the rules of left and right contraction are range-preserving admissible in G3LT+$\mathcal{U}+\mathcal{S}$. $\Box$

Theorem 3.5.6. *The rule of cut is admissible in the calculus* G3LT+$\mathcal{U}+\mathcal{S}$.

Proof. By induction on the right and left range of x, on the length of the cut formula and on the sum of the derivation heights of the two premises of cut. The proof is organized as the proof of Theorem 2.3.10. We consider here only the critical cases in which induction on range is needed. Let us suppose that A is different from $\bot$ and the sequent $x \prec z, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is obtained by a cut on the conclusions of the following instances of $R\mathcal{U}$ and $L\mathcal{U}$

$$\frac{\begin{array}{c} x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A \\ x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A\mathcal{U}B \\ x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, x : \mathbf{F}B \end{array}}{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B} R\mathcal{U}$$

$$\frac{x \prec z, z : B, \Gamma' \Rightarrow \Delta' \quad x \prec z, z : A, z : A\mathcal{U}B, x : \mathbf{F}B, \Gamma' \Rightarrow \Delta'}{x \prec z, x : A\mathcal{U}B, \Gamma' \Rightarrow \Delta'} L\mathcal{U}$$

First we apply height-preserving substitution on the premises of $R\mathcal{U}$ and obtain

$$x \prec z, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, z : B, z : A$$

$$x \prec z, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, z : B, z : A\mathcal{U}B$$

$$x \prec z, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, z : B, x : \mathbf{F}B$$

since by the variable condition y is not in the conclusion of $R\mathcal{U}$. Then, the sequents

$$x \prec z, x \prec z, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : B, z : A$$

$$x \prec z, x \prec z, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : B, z : A\mathcal{U}B$$

$$x \prec z, x \prec z, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : B, x : \mathbf{F}B$$

are obtained through lower-height cuts with the conclusion of $L\mathcal{U}$. Next, we use the latter sequents to obtain

$$x \prec z, x \prec z, x \prec z, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta', z : A \tag{1}$$

$$x \prec z, x \prec z, x \prec z, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta', z : A\mathcal{U}B \tag{2}$$

$$x \prec z, x \prec z, x \prec z, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta', x : \mathbf{F}B \tag{3}$$

through cuts on reduced cut formula with the left premise of rule $L\mathcal{U}$, that is $x \prec z, z : B, \Gamma' \Rightarrow \Delta'$. The sequent

$$x \prec z, \ldots, x \prec z, z : A\mathcal{U}B, x : \mathbf{F}B, \Gamma, \ldots, \Gamma' \Rightarrow \Delta, \ldots, \Delta' \tag{4}$$

is obtained through a cut on a smaller formula, with (1) and the right premise of $L\mathcal{U}$, $x \prec z, z : A, z : A\mathcal{U}B, x : \mathbf{F}B, \Gamma' \Rightarrow \Delta'$, as premises. Next, we obtain

$$x \prec z, \ldots, x \prec z, z : A\mathcal{U}B, \Gamma, \ldots, \Gamma' \Rightarrow \Delta, \ldots, \Delta' \tag{5}$$

by a cut on a smaller formula, with premises (3) and (4). Finally, the sequent

$$x \prec z, \ldots, x \prec z, \Gamma, \ldots, \Gamma' \Rightarrow \Delta, \Delta, \ldots, \Delta' \tag{6}$$

is obtained through a cut on a formula with smaller right range, with premises (2) and (5). The conclusion is obtained by admissibility of contraction.

If $A \equiv \bot$, the equivalence $l(\bot\mathcal{U}B) = l(\mathbf{F}B)$ prevents from applying the induction on the length of formulas, since cut on $x : \mathbf{F}B$ neither reduces the length nor the right range of the formula. However, by pruning the original derivation, we can assume without loss of generality that the left premise of $L\mathcal{U}$, namely $x \prec z, z : \bot, z : \bot\mathcal{U}B, x : \mathbf{F}B, \Gamma' \Rightarrow \Delta'$, is a conclusion of $L\bot$. We have the following transformation

$$\dfrac{\dfrac{x \prec z, x \prec z, x \prec z, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta', z : \bot \quad \dfrac{}{z : \bot \Rightarrow}{\scriptstyle L\bot}}{x \prec z, x \prec z, x \prec z, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}{\scriptstyle Cut}}{x \prec z, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Ctr^*}$$

where $x \prec z, x \prec z, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta', z : \bot$ is obtained as (1) above and the derivation of $x \prec z, z : \bot, z : \bot\mathcal{U}B, x : \mathbf{F}B, \Gamma' \Rightarrow \Delta'$ is substituted with the derivation of $z : \bot \Rightarrow$, which has the same derivation height.

The case of cut formula $x : A\mathcal{S}B$ principal in both premises of cut is dealt with similarly, with induction on the left range. $\square$

Soundness of the calculus G3LT+$\mathcal{U}$+$\mathcal{S}$ follows by Theorem 3.3.4 and validity of the recursive definition of Until and Since in any standard frame for linear discrete time.

Completeness is given by Corollary 3.3.8 and the following result:

Proposition 3.5.7. *The following purely logical sequents*

$x : A\mathcal{U}B \Rightarrow x : \mathbf{F}B$	$x : A\mathcal{S}B \Rightarrow x : \mathbf{P}B$
$x : \mathbf{F}B \Rightarrow x : \top\mathcal{U}B$	$x : \mathbf{P}B \Rightarrow x : \top\mathcal{S}B$
$x : \mathbf{T}B \Rightarrow x : \bot\mathcal{U}B$	$x : \mathbf{Y}B \Rightarrow x : \bot\mathcal{S}B$
$x : \bot\mathcal{U}B \Rightarrow x : \mathbf{T}B$	$x : \bot\mathcal{S}B \Rightarrow x : \mathbf{Y}B$
$x : A\mathcal{U}B \Rightarrow x : \mathbf{T}B, x : \mathbf{T}A \,\&\, \mathbf{T}(A\mathcal{U}B)$	$x : A\mathcal{S}B \Rightarrow x : \mathbf{Y}B, x : \mathbf{Y}A \,\&\, \mathbf{Y}(A\mathcal{S}B)$
$x : \mathbf{T}B \vee (\mathbf{T}A \,\&\, \mathbf{T}(A\mathcal{U}B)) \Rightarrow x : A\mathcal{U}B$	$x : \mathbf{Y}B \vee (\mathbf{Y}A \,\&\, \mathbf{Y}(A\mathcal{S}B)) \Rightarrow x : A\mathcal{S}B$

are derivable in G3LT+$\mathcal{U} + \mathcal{S}$.

Proof. By root-first proof search from the sequent to be derived. We only observe that for the derivation of $x : \mathbf{F}B \Rightarrow x : \top\mathcal{U}B$ and of $x : \mathbf{P}B \Rightarrow x : \top\mathcal{S}B$ an application of the infinitary rule T^ω is needed. $\square$

Theorem 3.5.8. (Completeness) *A sequent* $\Gamma \Rightarrow \Delta$ *is derivable in the calculus* G3LT+$\mathcal{U} + \mathcal{S}$ *iff it is valid.*

Chapter 4

Decidability through terminating proof search

4.1 A fixed-point proof system

In Section 3.4, we already noticed that the presence of an infinitary rule in G3LT constitutes an intrinsic obstacle to the possibility of establishing decidability of Priorean linear time logic through a terminating proof-search procedure. In this chapter we present a different labelled calculus G3LT$_{cl}$ for Priorean linear time. All the rules of the system are finitary, but proofs generally require arguments by infinite descent in the sense of Brotherston and Simpson (2007). In a temporal frame for Priorean linear time, between any two points there are only finitely many other points, therefore any model that appeals to an infinite increasing or decreasing sequence of points between two instants can be ignored as contradictory: this situation is reflected in a proof, for instance, when root-first applications of the rules do never realise a future formula $x : \mathbf{F}B$ in the antecedent with a labelled formula $y : B$ and a finite chain $x \prec y_0, \ldots, y_n \prec y$.

A particular class of sequents, that correspond to the syntactic counterparts of countermodels for unprovable purely logical sequents, is identified and used for giving a sound and complete definition of proof in G3LT$_{cl}$. Termination of a proof search is then obtained thanks to the analogy of the rules of the calculus to the algorithm that produces saturated subsets of formulas.

The basic idea is to formulate a labelled calculus G3LT$_{cl}$, the rules of which reflect a natural closure algorithm that exploits the fixed point properties of temporal operators

$$\mathbf{G}A \supset\subset \mathbf{T}A \,\&\, \mathbf{TG}A \qquad\qquad \mathbf{F}A \supset\subset \mathbf{T}A \vee \mathbf{TF}A$$

$$\mathbf{H}A \supset\subset \mathbf{Y}A \,\&\, \mathbf{YH}A \qquad\qquad \mathbf{P}A \supset\subset \mathbf{Y}A \vee \mathbf{YP}A$$

In Coquand (2007), a similar closure algorithm is given for LTL (see Remark 4.8.22, below): a countermodel for an invalid sentence is constructed as a relational structure where a saturated set of closure formulas Δ is the immediate successor of a saturated set of closure formulas Γ if $A \in \Delta$ whenever $\mathbf{T}A \in \Gamma$, and a fairness condition is satisfied, namely that all the eventualities of the form $\mathbf{F}A$ are fulfilled at some point. In this chapter, the notion of ($\prec$-)saturated label (see Definitions 4.1.5, 4.1.7) will be defined in order to identify the class of sequents which correspond to syntactical counterparts of countermodels for invalid sequents.

In initial sequents, ϕ is either an atomic formula or a formula prefixed by $\mathbf{T}$ or $\mathbf{Y}$. Observe that initial sequents of the form $x \prec y, \Gamma \Rightarrow \Delta, x \prec y$ are not in G3LT$_{cl}$; this is not problematic, as explained in Section 1.4.1.

Initial sequents and $L\bot$:

$$x : \phi, \Gamma \Rightarrow \Delta, x : \phi \qquad\qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\, L\bot$$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\, L\& \qquad\qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\, R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\, L\supset \qquad\qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, R\supset$$

Fixed point rules:

$$\frac{x : \mathbf{T}A, x : \mathbf{TG}A, \Gamma \Rightarrow \Delta}{x : \mathbf{G}A, \Gamma \Rightarrow \Delta}\, L\mathbf{G}_{cl} \qquad \frac{\Gamma \Rightarrow \Delta, x : \mathbf{G}A, x : \mathbf{T}A \quad \Gamma \Rightarrow \Delta, x : \mathbf{G}A, x : \mathbf{TG}A}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\, R\mathbf{G}_{cl}$$

$$\frac{x : \mathbf{T}A, x : \mathbf{F}A, \Gamma \Rightarrow \Delta \quad x : \mathbf{TF}A, x : \mathbf{F}A, \Gamma \Rightarrow \Delta}{x : \mathbf{F}A, \Gamma \Rightarrow \Delta}\, L\mathbf{F}_{cl} \qquad \frac{\Gamma \Rightarrow \Delta, x : \mathbf{T}A, x : \mathbf{TF}A}{\Gamma \Rightarrow \Delta, x : \mathbf{F}A}\, R\mathbf{F}_{cl}$$

$$\frac{x : \mathbf{Y}A, x : \mathbf{YH}A, \Gamma \Rightarrow \Delta}{x : \mathbf{H}A, \Gamma \Rightarrow \Delta}\, L\mathbf{H}_{cl} \qquad \frac{\Gamma \Rightarrow \Delta, x : \mathbf{H}A, x : \mathbf{Y}A \quad \Gamma \Rightarrow \Delta, x : \mathbf{H}A, x : \mathbf{YH}A}{\Gamma \Rightarrow \Delta, x : \mathbf{H}A}\, R\mathbf{H}_{cl}$$

$$\frac{x : \mathbf{Y}A, x : \mathbf{P}A, \Gamma \Rightarrow \Delta \quad x : \mathbf{YP}A, x : \mathbf{P}A, \Gamma \Rightarrow \Delta}{x : \mathbf{P}A, \Gamma \Rightarrow \Delta}\, L\mathbf{P}_{cl} \qquad \frac{\Gamma \Rightarrow \Delta, x : \mathbf{Y}A, x : \mathbf{YP}A}{\Gamma \Rightarrow \Delta, x : \mathbf{P}A}\, R\mathbf{P}_{cl}$$

Tomorrow and Yesterday rules:

$$\frac{x \prec y, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}{x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}\, L\mathbf{T} \qquad\qquad \frac{x \prec y, \Gamma \Rightarrow \Delta, x : \mathbf{T}A, y : A}{x \prec y, \Gamma \Rightarrow \Delta, x : \mathbf{T}A}\, R\mathbf{T}_{cl}$$

$$\frac{y \prec x, y : A, x : \mathbf{Y}A, \Gamma \Rightarrow \Delta}{y \prec x, x : \mathbf{Y}A, \Gamma \Rightarrow \Delta}\, L\mathbf{Y} \qquad\qquad \frac{y \prec x, \Gamma \Rightarrow \Delta, x : \mathbf{Y}A, y : A}{y \prec x, \Gamma \Rightarrow \Delta, x : \mathbf{Y}A}\, R\mathbf{Y}_{cl}$$

Mathematical rules:

$$\frac{y \prec x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, L\text{-}Ser \qquad\qquad \frac{x \prec y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, R\text{-}Ser$$

Rules *L-Ser* and *R-Ser* have the condition that y is not in the conclusion.

Table 4.1: The rules of the system G3LT$_{cl}$

Repetition of the principal formula in the premises of $R\mathbf{G}_{cl}$, $L\mathbf{F}_{cl}$, $R\mathbf{H}_{cl}$ and $L\mathbf{P}_{cl}$ is required for the definition of fulfilling sequent (see Definition 4.3.5). The propositional rules, the left rules for $\mathbf{T}$ and $\mathbf{Y}$, and the rules of left and right seriality are identical to those of G3LT. The right rules for $\mathbf{T}$ and $\mathbf{Y}$ are instead different from those of G3LT. If the flow of time is linear and unbounded, the next-time operator $\mathbf{T}$ and the previous-time operator $\mathbf{Y}$ satisfy

$x \Vdash \mathbf{T}A$ *iff, for all* y, $x \prec y$ *implies* $y \Vdash A$

iff there exists y *such that* $x \prec y$ *and* $y \Vdash A$

$x \Vdash \mathbf{Y}A$ *iff, for all* y, $y \prec x$ *implies* $y \Vdash A$

iff there exists y *such that* $y \prec x$ *and* $y \Vdash A$

Therefore the right rules of G3LT$_{cl}$ can be modified into the form given in Table 4.1, without variable conditions.

The notion of derivability in the calculus G3LT$_{cl}$ is defined in the standard way, as in Definition 2.3.2. In Section 4.3 we shall introduce a generalised notion of provability in G3LT$_{cl}$, which admits derivation trees with infinite branches.

We recall here that every purely logical sequent[1] $\Gamma \Rightarrow \Delta$ with all its formulas labelled by x corresponds to a formula $\wedge\Gamma^x \supset \vee\Delta^x$, where $\Gamma^x = \{A|x : A \in \Gamma\}$, and similarly Δ^x. With this identification, the rules of the system G3LT$_{cl}$, read root first, correspond to the algorithm for producing the saturated subsets of closure formulas from a given formula.

Definition 4.1.1. *The* set of closure formulas *of a formula* A, $cl(A)$, *of Priorean linear time logic is defined inductively as follows:*

- $B \in cl(A)$ *for every subformula* B *of* A;
- $\mathbf{T}B$ *and* $\mathbf{TG}B \in cl(A)$ *if* $\mathbf{G}B \in cl(A)$;
- $\mathbf{T}B$ *and* $\mathbf{TF}B \in cl(A)$ *if* $\mathbf{F}B \in cl(A)$;
- $\mathbf{Y}B$ *and* $\mathbf{YH}B \in cl(A)$ *if* $\mathbf{H}B \in cl(A)$;
- $\mathbf{Y}B$ *and* $\mathbf{YP}B \in cl(A)$ *if* $\mathbf{H}B \in cl(A)$.

Lemma 4.1.2. *Let* $|A|$ *be the number of subformulas of* A. *The cardinality of* $cl(A)$ *is linearly bounded by* $|A|$, *namely* $|cl(A)| \leq 3 \cdot |A|$.

Proof. By induction on the length of A:

[1] See Definition 2.1.1.

1. $|cl(\bot)| = |\{\bot\}| = 1 \leq 3$;

2. $|cl(P)| = |\{P\}| = 1 \leq 3$;

3. $|cl(B{\circ}C)| = |\{B{\circ}C\}{\cup}cl(B){\cup}cl(C)| \leq 1{+}3|B|{+}3|C| \leq 3(|B|{+}|C|{+}1) = 3|B{\circ}C|$;

4. $|cl(\mathbf{T}B)| = |\{\mathbf{T}B\} \cup cl(B)| \leq 1 + 3|B| \leq 3(|B| + 1) = 3|\mathbf{T}B|$;

5. $|cl(\mathbf{Y}B)| = |\{\mathbf{Y}B\} \cup cl(B)| \leq 1 + 3|B| \leq 3(|B| + 1) = 3|\mathbf{Y}B|$;

6. $|cl(\mathbf{G}B)| = |\{\mathbf{G}B, \mathbf{T}B, \mathbf{TG}B\} \cup cl(B)| \leq 3 + 3|B| = 3(|B| + 1) = 3|\mathbf{G}B|$;

7. $|cl(\mathbf{F}B)| = |\{\mathbf{F}B, \mathbf{T}B, \mathbf{TF}B\} \cup cl(B)| \leq 3 + 3|B| = 3(|B| + 1) = 3|\mathbf{F}B|$;

8. $|cl(\mathbf{H}B)| = |\{\mathbf{H}B, \mathbf{Y}B, \mathbf{YH}B\} \cup cl(B)| \leq 3 + 3|B| = 3(|B| + 1) = 3|\mathbf{H}B|$;

9. $|cl(\mathbf{P}B)| = |\{\mathbf{P}B, \mathbf{Y}B, \mathbf{YP}B\} \cup cl(B)| \leq 3 + 3|B| = 3(|B| + 1) = 3|\mathbf{P}B|$.

□

Corollary 4.1.3. *The number of subsets of* $cl(A)$ *is at most* $2^{3|A|}$.

The notion of a saturated set of Priorean linear time formulas is defined as follows:

Definition 4.1.4. *A set* S *of formulas is* saturated *if:*

- $\bot$ *is not in* S*;*
- *For every formula* B*, it is not possible that both* B *and* $\neg B$ *are in* S*;*
- $\neg\neg B$ *in* S *implies that* B *is in* S*;*
- $B\&C$ *in* S *implies that both* B *and* C *are in* S*;*
- $\neg(B\&C)$ *in* S *implies that either* $\neg B$ *or* $\neg C$ *is in* S*;*
- $B \vee C$ *in* S *implies that* B *or* C *is in* S*;*

- $\neg(B \vee C)$ *in* S *implies both* $\neg B$ *and* $\neg C$ *are in* S*;*
- $B \supset C$ *in* S *implies that either* $\neg B$ *or* C *is in* S*;*
- $\neg(B \supset C)$ *in* S *implies that both* B *and* $\neg C$ *are in* S*;*
- $\mathbf{G}B$ *in* S *implies that both* $\mathbf{T}B$ *and* $\mathbf{TG}B$ *are in* S*;*
- $\neg\mathbf{G}B$ *in* S *implies that either* $\neg\mathbf{T}B$ *or* $\neg\mathbf{TG}B$ *is in* S*;*
- $\mathbf{F}B$ *in* S *implies that* $\mathbf{T}B$ *or* $\mathbf{TF}B$ *is in* S*;*
- $\neg\mathbf{F}B$ *in* S *implies that both* $\neg\mathbf{T}B$ *and* $\neg\mathbf{TF}B$ *are in* S*;*
- $\mathbf{H}B$ *in* S *implies that both* $\mathbf{Y}B$ *and* $\mathbf{YH}B$ *are in* S*;*
- $\neg\mathbf{H}B$ *in* S *implies that either* $\neg\mathbf{Y}B$ *or* $\neg\mathbf{YH}B$ *is in* S*;*
- $\mathbf{P}B$ *in* S *implies that* $\mathbf{Y}B$ *or* $\mathbf{YP}B$ *is in* S*;*
- $\neg\mathbf{P}B$ *in* S *implies that both* $\neg\mathbf{Y}B$ *and* $\neg\mathbf{YP}B$ *are in* S*.*

We can now introduce the notion of saturated label in a sequent:

Definition 4.1.5. *Let* x *be a label in* $\Gamma \Rightarrow \Delta$*. We say that* x *is* saturated *if:*

- $x : \bot$ *is not in* Γ*;*
- *For every formula* B*, it is not possible that* $x : B$ *is both in* Γ *and in* Δ*;*
- $x : B\&C$ *in* Γ *(resp.* $x : B \vee C$ *in* Δ*) implies that both* $x : B$ *and* $x : C$ *are in* Γ *(resp. in* Δ*);*
- $x : B \vee C$ *in* Γ *(resp.* $x : B\&C$ *in* Δ*) implies that* $x : B$ *or* $x : C$ *is in* Γ *(resp. in* Δ*);*
- $x : B \supset C$ *in* Γ *implies that* $x : B$ *is in* Δ *or* $x : C$ *is in* Γ*;*

- *$x : B \supset C$ in Δ implies that $x : B$ is in Γ and $x : C$ is in Δ;*
- *$x : \mathbf{G}B$ in Γ (resp. $x : \mathbf{F}B$ in Δ) implies that both $x : \mathbf{T}B$ and $x : \mathbf{TG}B$ (resp. $x : \mathbf{TF}B$) are in Γ (resp. in Δ);*
- *$x : \mathbf{F}B$ in Γ (resp. $x : \mathbf{G}B$ in Δ) implies that $x : \mathbf{T}B$ or $x : \mathbf{TF}B$ (resp. $x : \mathbf{TG}B$) is in Γ (resp. in Δ);*
- *$x : \mathbf{H}B$ in Γ (resp. $x : \mathbf{P}B$ in Δ) implies that both $x : \mathbf{Y}B$ and $x : \mathbf{YH}B$ (resp. $x : \mathbf{YP}B$) are in Γ (resp. in Δ);*
- *$x : \mathbf{P}B$ in Γ (resp. $x : \mathbf{H}B$ in Δ) implies that $x : \mathbf{Y}B$ or $x : \mathbf{YP}B$ (resp. $x : \mathbf{YH}B$) is in Γ (resp. in Δ).*

Note that, by the equivalences $\mathbf{F} \equiv \neg\mathbf{G}\neg$ (resp. $\mathbf{P} \equiv \neg\mathbf{H}\neg$), a formula $x : \mathbf{G}B$ (resp. $x : \mathbf{H}B$) in the succedent behaves like a formula $x : \mathbf{F}B$ (resp. $x : \mathbf{P}B$) in the antecedent and viceversa.

Saturated sets of formulas and saturated labels in a sequent are linked by the following lemma:

Lemma 4.1.6. *The label x in a sequent $\Gamma \Rightarrow \Delta$ is saturated iff the set $\Gamma^x \cup \overline{\Delta^x}$ is saturated, where $\Gamma^x\{B | x : B \in \Gamma\}$, $\overline{\Delta^x} = \{\overline{B} | x : B \in \Delta\}$, and $\overline{B} \equiv \neg B$ if $B \neq \neg C$, $\overline{B} \equiv C$ otherwise.*

Proof. Straightforward. □

Definition 4.1.7. *A label x in $\Gamma \Rightarrow \Delta$ is* $\prec$-saturated *if it is saturated and:*

- *$x : \mathbf{T}B$ in Γ (resp. in Δ) implies that, if $x \prec y$ is in Γ, then $y : B$ is in Γ (resp. in Δ);*
- *$x : \mathbf{Y}B$ in Γ (resp. in Δ) implies that, if $y \prec x$ is in Γ, then $y : B$ is in Γ (resp. in Δ).*

4.2 Structural properties

Lemma 4.2.1. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathrm{G3LT}_{cl}$*, then also* $\Gamma(y/x) \Rightarrow \Delta(y/x)$ *is derivable, with the same derivation height.*

Proof. By induction on the height h of the derivation. The proof is analogous to the proof of proof of Lemma 2.3.4. If $h = 0$, then $\Gamma \Rightarrow \Delta$ is either an initial sequent or a conclusion of $L\bot$. In both cases, the sequent $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is also an initial sequent or a conclusion of $L\bot$. Suppose that $\Gamma \Rightarrow \Delta$ is derivable with $h = n + 1$ and that the claim holds for $h = n$, and consider the last rule applied in the derivation. If it is a propositional or a temporal rule, apply the inductive hypothesis to the premise(s) and then the rule. If the last rule is a rule with a variable condition (*L-Ser* or *R-Ser*), we need to avoid a clash with the eigenvariable: in that case, we apply twice the inductive hypothesis to the premise, first to replace the eigenvariable with a fresh variable not appearing in the derivation, and then to perform the desired substitution. □

Lemma 4.2.2. *Sequents of the form* $x : A, \Gamma \Rightarrow \Delta, x : A$*, with* A *an arbitrary temporal formula, are derivable in* $\mathrm{G3LT}_{cl}$.

Proof. By induction on the length of the formula A. Note that if A is a temporal formula prefixed by $\mathbf{T}$ or $\mathbf{Y}$, then the sequent $x : A, \Gamma \Rightarrow \Delta, x : A$ is initial. □

In what follows, Greek lower case is used for labelled and relational formulas.

Theorem 4.2.3. *The rules of left and right weakening*

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,{\scriptstyle LWk} \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\,{\scriptstyle RWk}$$

are height-preserving admissible in $\mathrm{G3LT}_{cl}$.

Proof. By induction on the height of the derivation (see Theorem 2.3.6). If $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of $L\bot$, also $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi$ are. The cases of rules without variable condition are straightforward. If the last step is a rule with a variable condition, we apply first Lemma 4.2.1 to avoid a clash of variables, and then the inductive hypothesis and the rule in question. $\square$

Lemma 4.2.4. *All rules of* G3LT$_{cl}$ *are height-preserving invertible.*

Proof. The proof of height-preserving invertibility for the propositional rules and for the fixed-point temporal rules is by induction on the height of derivation (see Lemma 2.3.7). The temporal rules for **T** and **Y**, and the rules for seriality are trivially invertible, since their premise(s) are obtained by weakening from the conclusion and weakening is height-preserving admissible by Theorem 4.2.3. As usual, clashes of variables are avoided through the substitution lemma. $\square$

Theorem 4.2.5. *The rules of left and right contraction*

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,LCtr \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\,RCtr$$

are height-preserving admissible in G3LT$_{cl}$.

Proof. Analogous to the proof of Theorem 2.3.8. $\square$

Contrary to G3LT, the calculus G3LT$_{cl}$ does not permit syntactic cut elimination. This is because the rules for **T** and **Y** are given in a non-harmonious way, that is, the left and the right rules are justified by different semantical explanations. However, it is precisely because of this particular choice of rules that the essential properties of G3LT$_{cl}$ hold. We will show, however, that the system without cut is complete, and thus prove that $\mathrm{G3LT}_{cl}$ is closed with respect to cut.

Lemma 4.2.6. *A derivation in* $\mathrm{G3LT}_{cl}$ *can be transformed into a derivation with all instances of* R-Ser *and* L-Ser *applied on side labels*[2] *that appear in the conclusion of the rule.*

Proof. Analogous to the proof of Lemma 3.2.11. □

When considering minimal derivations[3], applications of rules that produce duplications of atoms when read from conclusion to premises can be dispensed with by height-preserving admissibility of contraction. The same holds if a duplication occurs modulo fresh replacement of eigenvariables, in particular we have:

Lemma 4.2.7. *In* $\mathrm{G3LT}_{cl}$*, rule* R-Ser *(resp.* L-Ser*) need not be applied on a relational atom* $x \prec y$ *(resp.* $y \prec x$*) if its conclusion contains an atom* $x \prec z$ *(resp.* $z \prec x$*) in the antecedent.*

Proof. We consider a minimal derivation and suppose that we have an application of *R-Ser* and an atom $x \prec z$ in its conclusion: we can perform the following transformation

$$\dfrac{x \prec y, x \prec z, \Gamma \Rightarrow \Delta}{x \prec z, \Gamma \Rightarrow \Delta}\,{\scriptstyle R\text{-}Ser} \quad \rightsquigarrow \quad \dfrac{\dfrac{x \prec y, x \prec z, \Gamma \Rightarrow \Delta}{x \prec z, x \prec z, \Gamma \Rightarrow \Delta}\,{\scriptstyle H.\text{-}p.Subst(z/y)}}{x \prec z, \Gamma \Rightarrow \Delta}\,{\scriptstyle H.\text{-}p.LCtr}$$

A shorter derivation is obtained, contrary to the hypothesis. We proceed similarly for *L-Ser*. □

[2] See Definition 3.2.10.
[3] See Definition 3.2.12.

Lemma 4.2.8. *The rules for left and right seriality permute up with respect to all the rules of* G3LT$_{cl}$ *in the case their eigenvariable is not contained in the active formula(s) of the latter.*

Proof. We consider here only the case of rule *R-Ser*, the case of *L-Ser* being analogous. If *L-Ser* is applied after *L*&, we can perform the following transformation

$$\dfrac{\dfrac{x \prec y, z : B, z : C, \Gamma \Rightarrow \Delta}{x \prec y, z : B\&C, \Gamma \Rightarrow \Delta}{\scriptstyle L\&}}{z : B\&C, \Gamma \Rightarrow \Delta}{\scriptstyle R\text{-}Ser} \quad \rightsquigarrow \quad \dfrac{\dfrac{x \prec y, z : B, z : C, \Gamma \Rightarrow \Delta}{z : B, z : C, \Gamma \Rightarrow \Delta}{\scriptstyle R\text{-}Ser}}{z : B\&C, \Gamma \Rightarrow \Delta}{\scriptstyle L\&}$$

with z different from x, y and not in Γ, Δ. The permutation is similar for all the other cases. $\square$

Lemma 4.2.9. *Temporal rules permute down with respect to all the rules of* G3LT$_{cl}$ *in the case their principal formulas are not active in the latter.*

Proof. We consider here only rule $L\mathbf{T}$, all the other cases being analogous. The permutation is straightforward in the case of one-premise rule. For instance, for *L*& we have

$$\dfrac{\dfrac{x \prec y, y : A, x : \mathbf{T}A, z : B, z : C, \Gamma \Rightarrow \Delta}{x \prec y, x : \mathbf{T}A, z : B, z : C, \Gamma \Rightarrow \Delta}{\scriptstyle L\mathbf{T}}}{x \prec y, x : \mathbf{T}A, z : B\&C, \Gamma \Rightarrow \Delta}{\scriptstyle L\&}$$

that can be transformed into

$$\dfrac{\dfrac{x \prec y, y : A, x : \mathbf{T}A, z : B, z : C, \Gamma \Rightarrow \Delta}{x \prec y, y : A, x : \mathbf{T}A, z : B\&C, \Gamma \Rightarrow \Delta}{\scriptstyle L\&}}{x \prec y, x : \mathbf{T}A, z : B\&C, \Gamma \Rightarrow \Delta}{\scriptstyle L\mathbf{T}}$$

The permutation is similar for all the other one-premise rules. In the case of a two-premise rule, use of height-preserving admissibility of weakening is needed: for instance, the derivation

$$\dfrac{\dfrac{x \prec y, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : B}{x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : B}{\scriptstyle L\mathbf{T}} \qquad x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : C}{x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : B\&C}{\scriptstyle R\&}$$

is transformed into

$$\dfrac{\dfrac{x \prec y, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : B \quad \dfrac{x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : C}{x \prec y, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : C}\,{\scriptstyle LWk}}{x \prec y, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : B\&C}\,{\scriptstyle R\&}}{x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta, z : B\&C}\,{\scriptstyle L\mathbf{T}}$$

The permutation is similar for all the other two-premise rules. ☐

Lemma 4.2.10. *On any branch of a minimal derivation in* G3LT$_{cl}$*, a given temporal rule with the repetition of the principal formula(s) in the premise(s) need not be applied more than once on the same formulas.*

Proof. We consider here only rule $L\mathbf{T}$, all the other cases being analogous. If rule $L\mathbf{T}$ has been applied twice with principal formulas $x \prec y$, $x : \mathbf{T}A$, by Lemma 4.2.9 we have without loss of generality a derivation of the following form, that can be transformed as indicated:

$$\dfrac{\dfrac{x \prec y, y : A, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}{x \prec y, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\mathbf{T}}}{x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\mathbf{T}} \quad \rightsquigarrow \quad \dfrac{\dfrac{x \prec y, y : A, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}{x \prec y, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle H.\text{-}p.Ctr}}{x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\mathbf{T}}$$

☐

4.3 Proofs in G3LT$_{cl}$

In this Section we shall define G3LT$_{cl}$ proofs through the identification of a particular class of sequents, as finite syntactical counterparts of countermodels for invalid sequents.

Given a purely logical sequent $\Gamma \Rightarrow \Delta$, we start a proof search by applying root-first the rules of G3LT$_{cl}$ for the propositional connectives and for **G**, **F**, **H**, and **P**, whenever possible. When x becomes saturated, we apply once *R-Ser* and *L-Ser* with side label x, thus introducing new labels y and y' and the accessibility relations $x \prec y$ and $y' \prec x$: by Lemma 4.2.8 we are allowed to postpone the application of the rules for seriality until no more logical rule can be applied, and by Lemmas 4.2.6 and 4.2.7 we do not need to apply a seriality rule with side label z, if z is not a label in the sequent or the antecedent already contains an atom $z \prec z'$ (resp. $z' \prec z$). Next, we apply the rules $L\mathbf{T}$ and $R\mathbf{T}_{cl}$ (resp. $L\mathbf{Y}$ and $R\mathbf{Y}_{cl}$) on the formulas with $\mathbf{T}$ (resp. $\mathbf{Y}$) as their outermost operator until x becomes $\prec$-saturated. Note that by Lemma 4.2.10, we need not apply more than once a temporal rule on the same principal formula(s). We repeat the procedure with the formulas marked by y and y'. We continue as before with all the labels possibly introduced by *R-Ser* and *L-Ser*, and so on. In what follows, we use x to denote the label that marks all the formulas in the purely logical sequent the proof search starts with.

Definition 4.3.1. *A* pre-proof *of a purely logical sequent in* G3LT$_{cl}$ *is a (possibly infinite) tree obtained by applying root-first the logical and mathematical rules of the calculus, whenever possible.*

Before giving the definition of a proof in G3LT$_{cl}$, we need some preliminary definitions. As said, our aim is to construct the syntactic counterpart of a countermodel from a failed proof search. For the sake of clarity, we define syntactic entities through their intuitive correspondence to the semantic features of a model for Priorean linear time.

The notions of Kripke semantics for temporal logic were introduced in Section 3.3. The definition of evaluation of formulas is as in Definition 3.3.2; the definition of validity for labelled formulas and relational atoms in a discrete linear temporal frame is as in Definition 3.3.3. The notion of a countermodel for a sequent is defined as follows:

Definition 4.3.2. *A countermodel to a sequent* $\Gamma \Rightarrow \Delta$ *is a discrete linear temporal frame* $(\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ *together with an interpretation and an evaluation that validates all the formulas and relational atoms in* Γ *and no formula in* Δ.

The semantic explanations for possibility-like temporal operators **F** and **P** and the definition of the order relation $<^{\mathcal{K}}$ as the transitive closure of the immediate successor relation $\prec^{\mathcal{K}}$ justify the following notion of future and past witness.

Definition 4.3.3. *Given a labelled formula* $z : \mathbf{F}B$ *in the antecedent of a sequent* $\Gamma \Rightarrow \Delta$ *(resp.* $z : \mathbf{G}B$ *in the succedent), we say that a label* z' *is a* future witness *for* $z : \mathbf{F}B$ *(resp.* $z : \mathbf{G}B$*) if* $z' : B$ *is in* Γ *(resp.* $z' : B$ *is in* Δ*) and the relational atoms* $z \prec z_0, \ldots, z_{n-1} \prec z_n \equiv z'$ *are in* Γ *for some* n.
Given a labelled formula $z : \mathbf{P}B$ *in the antecedent of a sequent* $\Gamma \Rightarrow \Delta$ *(resp.* $z : \mathbf{H}B$ *in the succedent), we say that a label* z' *is a* past witness *for* $z : \mathbf{P}B$ *(resp.* $z : \mathbf{H}B$*) if* $z' : B$ *is in* Γ *(resp.* $z' : B$ *is in* Δ*) and the relational atoms* $z' \prec z_0, \ldots, z_{n-1} \prec z_n \equiv z$ *are in* Γ *for some* n.

In costructing a Priorean linear time model from our syntactic object, we have to ensure that every possibility-like formulas in it is realised by some label:

Definition 4.3.4. *A chain* $z_i \prec z_{i+1}, \ldots, z_{j-1} \prec z_j$ *(with* $j \geq i+1$*) in a sequent* $\Gamma \Rightarrow \Delta$ *is a* future loop *if* z_j *marks exactly the same formulas as the label* z_i*, for no label* y *the relational atom* $z_j \prec y$ *is in* Γ*, and, for every labelled formula* $z_q : \mathbf{F}B$ *in* Γ *(resp.* $z_q : \mathbf{G}B$ *in* Δ*) with* $i \leq q \leq j$*, there exists* z_k *such that* $i \leq k \leq j$ *and* $z_k : B$ *is in* Γ *(resp. in* Δ*). We call* z_j *the* future looping label *with respect to* z_i.
A chain $z_i \prec z_{i+1}, \ldots, z_{j-1} \prec z_j$ *(with* $j \geq i + 1$*) in a sequent* $\Gamma \Rightarrow \Delta$ *is a*

past loop *if* z_i *marks exactly the same formulas as the label* z_j*, for no label* y *the relational atom* $y \prec z_i$ *is in* Γ*, and, for every labelled formula* $z_q : \mathbf{P}B$ *in* Γ *(resp.* $z_q : \mathbf{H}B$ *in* Δ*) with* $i \leq q \leq j$*, there exists some variable* z_k *such that* $i \leq k \leq j$ *and* $z_k : B$ *is in* Γ *(resp. in* Δ*). We call* z_i *the* past looping label *with respect to* z_j*.*

A root-first proof search succeeds when a derivation is found, namely all the leaves of the derivation tree are initial sequents or instances of $L\bot$. However, a failed proof search does not in general assure that an endsequent $\Gamma \Rightarrow \Delta$ is invalid unless an adequate countermodel can be constructed from it. Here comes into play the notion of fulfilling sequent for a purely logical sequent $\Gamma \Rightarrow \Delta$:

Definition 4.3.5. *Let us suppose that the sequent* $\Gamma^* \Rightarrow \Delta^*$ *has been obtained by root-first proof search from the purely logical sequent* $\Gamma \Rightarrow \Delta$ *(with all its formulas labelled by* x*). Then,* $\Gamma^* \Rightarrow \Delta^*$ *is a* fulfilling sequent *if the following conditions are satisfied:*

(i) Every label in it is $\prec$*-saturated;*

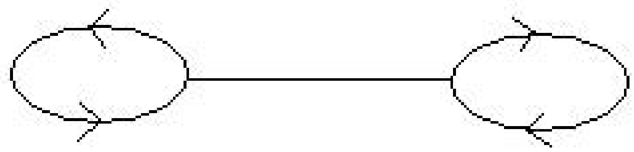
(ii) It contains a chain of relational atoms $z_{-m} \prec z_{-(m-1)}, \dots, z_{-1} \prec z_0 \equiv x$, $z_0 \prec z_1, \dots, z_{n-1} \prec z_n$*, such that for some* $-m < i \leq 0$ *the subchain* $z_{-m} \prec z_{-(m-1)}, \dots, z_{i-1} \prec z_i$ *is a past loop, and for some* $0 \leq j < n$*, the subchain* $z_j \prec z_{j+1}, \dots, z_{n-1} \prec z_n$ *is a future loop;*

(iii) Every labelled formula $z : \mathbf{F}B$ *in the antecedent (resp.* $z : \mathbf{G}B$ *in the succedent) is either witnessed by a future witness label* z'*, or has* z *inside a future loop;*

(iv) Every labelled formula $z : \mathbf{P}B$ *in the antecedent (resp.* $z : \mathbf{H}B$ *in the*

succedent) is either witnessed by a past witness label z', or has z inside a past loop.

Intuitively, a fulfilling sequent corresponds to a structure constituted by a (possibly empty) linear chain with two simple loops at the ends, as in the following figure, where the left and the right loop are obtained by identifying the first and the last label of the past and of the future loop, respectively.



In Section 4.4 we shall prove that, given a model for Priorean linear time, it is possible to extract the corresponding fulfilling sequent, and in Section 4.5 we shall show how to linearise the future and the past loop in order to obtain an appropriate model.

Proposition 4.3.6. *Let $\Gamma' \Rightarrow \Delta'$ be obtained by applying root-first the rules of G3LT$_{cl}$ from the purely logical sequent $\Gamma \Rightarrow \Delta$ with x as the uniform label that marks all the formulas in the latter. Then $\Gamma' \Rightarrow \Delta'$ contains a unique chain $z_{-m} \prec z_{-(m-1)}, \ldots, z_{-1} \prec z_0 \equiv x$, $z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$ with z_i different from z_j for $i \neq j$.*

Proof. Since the root sequent $\Gamma \Rightarrow \Delta$ is purely logical, the result follows by Lemmas 4.2.6, 4.2.7 and the fact that only seriality rules can introduce relational atoms. $\square$

While searching for a fulfilling sequent, we want to find one as small as possible. Therefore we should try to avoid useless circles, namely those exploring instants reachable as well through a more direct path. This motivates the following definition:

Definition 4.3.7. *Let $\Gamma' \Rightarrow \Delta'$ be obtained by applying root-first the rules of G3LT$_{cl}$ from the purely logical sequent $\Gamma \Rightarrow \Delta$ with x as the uniform label that marks all the formulas in the latter. A chain $x \equiv y_0 \prec y_1, \ldots, y_{n-1} \prec y_n$ (resp. $y_{-n} \prec y_{-(n-1)}, \ldots, y_{-1} \prec y_0 \equiv x$) in $\Gamma' \Rightarrow \Delta'$ is* roundabout *if it contains labels y_i, y_j with $0 \leq i < j \leq n$ such that y_i and y_j mark the same formulas, $y_i \prec y_{i+1}, \ldots, y_{j-1} \prec y_j$ is not the future loop (resp. the past loop) and either $j = i + 1$ or for every y_k with $i < k < j$ there exists some y_l such that $l > j$ (resp. $l < i$) and y_k and y_l mark the same formulas. We say that the subchain $y_i \prec y_{i+1}, \ldots, y_{j-1} \prec y_j$ is* dispensable. *A fulfilling sequent is* reduced *if it does not contain dispensable subchains.*

Note that by Definition 4.3.7 a chain can be roundabout also in the case that y_i and y_j mark no formulas.

Theorem 4.3.8. *If a proof search for a purely logical sequent $\Gamma \Rightarrow \Delta$ (with all its formulas labelled by x) leads to a fulfilling sequent $\Gamma^* \Rightarrow \Delta^*$, then it also leads to a reduced fulfilling sequent.*

Proof. (*Sketch*) Note that for every label z introduced by *R-Ser* (resp. *L-Ser*) a labelled formula $z : C$ in $\Gamma^* \Rightarrow \Delta^*$ either is introduced by applying root-first rules $L\mathbf{T}$ and $R\mathbf{T}_{cl}$ (resp. $L\mathbf{Y}$ and $R\mathbf{Y}_{cl}$) or is the result of root-first application of the other rules on a formula thus introduced. If the chain $x \equiv z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$ contains a dispensable subchain $z_i \prec z_{i+1}, \ldots, z_{j-1} \prec z_j$, then the labels z_i and z_j mark the same formulas: therefore $z_{j+1} : B$ is introduced by $L\mathbf{T}$ (resp. $R\mathbf{T}_{cl}$) with principal formulas $z_j \prec z_{j+1}, z_j : \mathbf{T}B$ iff $z_{i+1} : B$ can be introduced by $L\mathbf{T}$ (resp. $R\mathbf{T}_{cl}$) with principal formulas $z_i \prec z_{i+1}, z_i : \mathbf{T}B$. Given a set of formulas marked by a label z, the rules of G3LT$_{cl}$ explore different subsets of closure formulas that possibly $\prec$-saturate z: while applying root-first the rules of G3LT$_{cl}$ we have to continue along the branch in which the label z_{i+1} is $\prec$-saturated by the same subset of closure formulas that $\prec$-saturates z_{j+1} in the original fulfilling sequent. By choosing the appropriate premise of a branching rule whenever a roundabout chain is met, we finally reach the desired reduced fulfilling sequent. Let us consider for instance the following simple case of a proof-search tree

$$
\dfrac{\begin{matrix}\vdots\\ x : \mathbf{T}A, x : \mathbf{F}A \Rightarrow\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y' \prec x, x \prec y, y \prec z, z : A, y : \mathbf{T}A, \ldots \Rightarrow}{y' \prec x, x \prec y, y \prec z, y : \mathbf{T}A, \ldots \Rightarrow}\,{\scriptstyle L\mathbf{T}}}{y' \prec x, x \prec y, y : \mathbf{T}A, y : \mathbf{F}A, \ldots \Rightarrow}\,{\scriptstyle R\text{-}Ser} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y' \prec x, x \prec y, y \prec v, v \prec w, w : A, v : \mathbf{T}A, \ldots \Rightarrow}{y' \prec x, x \prec y, y \prec v, v \prec w, v : \mathbf{T}A, \ldots \Rightarrow}\,{\scriptstyle L\mathbf{T}}}{y' \prec x, x \prec y, y \prec v, v : \mathbf{T}A, v : \mathbf{F}A, \ldots \Rightarrow}\,{\scriptstyle R\text{-}Ser} \qquad \ldots}{y' \prec x, x \prec y, y \prec v, v : \mathbf{F}A, y : \mathbf{TF}A, \ldots \Rightarrow}\,{\scriptstyle L\mathbf{F}_{cl}}}{y' \prec x, x \prec y, y \prec v, y : \mathbf{TF}A, \ldots \Rightarrow}\,{\scriptstyle L\mathbf{T}}}{y' \prec x, x \prec y, y : \mathbf{TF}A, y : \mathbf{F}A, \ldots \Rightarrow}\,{\scriptstyle R\text{-}Ser}}{y' \prec x, x \prec y, y : \mathbf{F}A, x : \mathbf{TF}A, \ldots \Rightarrow}\,{\scriptstyle L\mathbf{F}_{cl}}}{y' \prec x, x \prec y, x : \mathbf{TF}A, x : \mathbf{F}A \Rightarrow}\,{\scriptstyle L\mathbf{T}}}{x : \mathbf{TF}A, x : \mathbf{F}A \Rightarrow}\,{\scriptstyle R\text{-}Ser, L\text{-}Ser}}{x : \mathbf{F}A \Rightarrow}\,{\scriptstyle L\mathbf{F}_{cl}}
$$

Clearly, the fulfilling sequent

$$y' \prec x, x \prec y, y \prec v, v \prec w, w : A, v : \mathbf{T}A, y : \mathbf{TF}A, x : \mathbf{TF}A, v : \mathbf{F}A, y : \mathbf{F}A, x : \mathbf{F}A \Rightarrow$$

contains the dispensable subchain $x \prec y$: we can reach the reduced fulfilling sequent

$$y' \prec x, x \prec y, y \prec z, z : A, y : \mathbf{T}A, x : \mathbf{TF}A, y : \mathbf{F}A, x : \mathbf{F}A \Rightarrow$$

by following the branch in which y is $\prec$-saturated by the subset of closure formulas that $\prec$-saturates v in the former sequent. The case of roundabout chain $z_{-n} \prec z_{-(n-1)}, \ldots, z_{-1} \prec z_0 \equiv x$ is analogous. $\square$

We are now in possess of all the notions required to give a definition of proof in the calculus G3LT$_{cl}$.

Definition 4.3.9. *A pre-proof of a purely logical sequent* $\Gamma \Rightarrow \Delta$ *is a* proof *if no branch in it leads to a fulfilling sequent. A sequent* $\Gamma \Rightarrow \Delta$ *is* provable *if there exists a proof for it.*

Clearly, every G3LT$_{cl}$ derivation is a G3LT$_{cl}$ proof, but the converse does not hold.

We observe here that, contrary to the definition of cyclic calculi for induction and infinite descent of Brotherston and Simpson (2007), our definition of proof in G3LT$_{cl}$ is completely local, i.e. there is no need of keeping information on previous parts of the derivation tree: at any step of the proof search we simply have to consider the sequents introduced by root-first application of the rules and check if they are initial sequents, fulfilling sequents, or neither.

4.4 Soundness

Contrary to the calculus G3LT, soundness cannot be proved for G3LT$_{cl}$ by showing that the initial sequents and the rules of the system are sound: by Definition 4.3.9, proofs in G3LT$_{cl}$ can contain infinitely long branches, so the validity of a sequent in a tree cannot, in general, be founded on the validity of initial sequents or instances of $L\bot$.

Therefore, we prove soundness by contraposition: if there exists a countermodel for $\Gamma \Rightarrow \Delta$, then the corresponding proof search should contain a fulfilling sequent and so $\Gamma \Rightarrow \Delta$ is unprovable in G3LT$_{cl}$. As a consequence, the absence of any fulfilling sequent from a derivation tree turns to be a global soundness condition for the proofs.

Before proving soundness, some preliminary results concerning standard models are needed. In particular, we have to prove that, given a countermodel $\mathcal{M}$ for A, it is possible to extract a fulfilling sequent all the labels of which mark $\prec$-saturated sets of closure formulas of A: the Lemmas below show how to construct a future and a past loop from $\mathcal{M}$. In the following, we use $s \leqslant^{\mathcal{K}} s'$ to denote the fact that $s = s'$ or $s <^{\mathcal{K}} s'$ in a model $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$.

Lemma 4.4.1. *Let $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$ be a model for Priorean linear time and suppose that, for some instant w, $w \nVdash A$. Then for some s such that $w \leqslant^{\mathcal{K}} s$, there exists s' such that $s <^{\mathcal{K}} s'$, s and s' satisfy the same subset $H \subset cl(A)$, and for every t if $s \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s'$ and $t \Vdash \mathbf{F}B$ and $\mathbf{F}B \in cl(A)$ (resp. $t \nVdash \mathbf{G}B$ and $\mathbf{G}B \in cl(A)$) there exists u such that $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s'$ and $u \Vdash B$ (resp. $u \nVdash B$).*

Proof. Since every model for Priorean linear time is isomorphic to the integers, we can assume without loss of generality that $\mathcal{M}$ is the standard model, namely $\mathcal{K}$ corresponds to $\mathbb{Z}$. So, there are infinitely many instants greater than w. However, by Corollary 4.1.3, there are only $2^{3|A|}$ subsets of $cl(A)$: as an application of Ramsey's Theorem, for some instant(s) greater than w there exist infinitely many instants satisfying the same subset H of closure formulas of A. Let s be the first instant of the infinite set of instants

$$s_0 <^{\mathcal{K}} s_1 <^{\mathcal{K}} s_2 <^{\mathcal{K}} s_3 <^{\mathcal{K}} \dots$$

all satisfying the same subset $H \subseteq cl(A)$ and such that $w \leqslant^{\mathcal{K}} s$. Let $s \leqslant^{\mathcal{K}} t$ and $t \Vdash \mathbf{F}B$ and $\mathbf{F}B \in cl(A)$ (resp. $t \nVdash \mathbf{G}B$ and $\mathbf{G}B \in cl(A)$). If there exists a u such that $u \Vdash B$ and $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} t$, we are done. Otherwise, since $t \Vdash \mathbf{F}B$ (resp. $t \nVdash \mathbf{G}B$), there exists some u such that $t <^{\mathcal{K}} u$ and $u \Vdash B$ (resp. $u \nVdash B$). Since, by hypothesis, there are infinitely many instants greater than s satisfying H, but u can be reached from t by finitely many iterations of the relation $\prec^{\mathcal{K}}$, for some $i = 1, 2, \ldots$, we have $s <^{\mathcal{K}} u \leqslant^{\mathcal{K}} s_i$. For every i there are only finitely many closure formulas of A of the form $\mathbf{F}B$ (resp. $\mathbf{G}B$) validated (resp. invalidated) by an instant t such that $s \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s_i$, and for every such t we can find a k and a u such that $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s_{i+k}$ and $u \Vdash B$ (resp. $u \nVdash B$). Since the set of closure formulas of A is finite, the process eventually ends with the determination of a s' such that $s <^{\mathcal{K}} s'$ and for every t if $s \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s'$ and $t \Vdash \mathbf{F}B$ and $\mathbf{F}B \in cl(A)$ (resp. $t \nVdash \mathbf{G}B$ and $\mathbf{G}B \in cl(A)$) there exists u such that $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s'$ and $u \Vdash B$ (resp. $u \nVdash B$). $\square$

Lemma 4.4.2. *Let $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$ be a model for Priorean linear time such that for some instant w, $w \nVdash A$. Then for some instant s such that $s \leqslant^{\mathcal{K}} w$, there exists s' such that $s' <^{\mathcal{K}} s$, s and s' satisfy the same subset $H \subset cl(A)$ and for every t if $s' \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s$ and $t \Vdash \mathbf{P}B$ and $\mathbf{P}B \in cl(A)$ (resp. $t \nVdash \mathbf{H}B$ and $\mathbf{H}B \in cl(A)$) there exists u such that $s' \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s$ and $u \Vdash B$ (resp. $u \nVdash B$).*

Proof. Analogous to the proof of Lemma 4.4.1. $\square$

Lemma 4.4.3. *All the rules of* $\mathrm{G3LT}_{cl}$ *are sound.*

Proof. The case of the initial sequents and the propositional rules is straightforward. The rules for $\mathbf{G}$, $\mathbf{F}$, $\mathbf{H}$ and $\mathbf{P}$ are sound by definition, since they are justified by their fixed point interpretations. Similarly, the rules for the next-time operator $\mathbf{T}$ and the previous-time operator $\mathbf{Y}$ are justified by their semantic explanations, and the mathematical rules correspond to the frame properties of left and right seriality for $\prec$. $\square$

Theorem 4.4.4. *If a purely logical sequent* $\Gamma \Rightarrow \Delta$ *(with all its formulas labelled by* x*) has a countermodel, then it is not provable in* $\mathrm{G3LT}_{cl}$.

Proof. Let us suppose that there exists a countermodel $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$ for the purely logical sequent $\Gamma \Rightarrow \Delta$, namely there exists $w \in \mathcal{K}$ such that $[\![x]\!] = w$ and $w \nVdash \wedge\Gamma \supset \vee\Delta$. By Lemma 4.4.3, every countermodel for the conclusion of any of the rules of $\mathrm{G3LT}_{cl}$ is a countermodel for (at least one of) the premise(s): by choosing the appropriate branch we eventually find a sequent with a chain

$$z_{-m} \prec z_{-(m-1)}, \ldots, z_{-1} \prec z_0 \equiv x,\, z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$$

every label of which matches an instant in the corresponding position in $\mathcal{M}$. We want to show that such sequent is a fulfilling sequent for $\Gamma \Rightarrow \Delta$. We have to consider several cases according to the conditions characterising a fulfilling sequent in Definition 4.3.5:

(i) We show that evey label z appearing in the tree can be $\prec$-saturated by applying root-first the rules of the calculus. First observe that for no formula B, $z : B$ can be both in the antecedent and in the succedent, otherwise by definition of countermodel $[\![z]\!] \Vdash B$ and $[\![z]\!] \nVdash B$, which is impossible. Analogously, if $z : \bot$ were in the antecedent, $[\![z]\!] \Vdash \bot$, which is impossible. If the formula $z : A\&B$ (resp. $z : A \vee B$) is in the antecedent (resp. succedent), a single application of $L\&$ (resp. $R\vee$) introduces both $z : A$ and $z : B$ in the antecedent (resp. succedent) of the premise. If the formula $z : A \vee B$ (resp. $z : A\&B$) is in the antecedent (resp. succedent), a single application of $L\vee$ (resp. $R\&$) introduces $z : A$ in the antecedent (resp. succedent) of the left premise and $z : B$ in the antecedent (resp. succedent) of the right premise. If the formula $z : A \supset B$ is in the antecedent, a single application of $L\supset$ introduces $z : A$ in the succedent of the left premise and $z : B$ in the antecedent of the right premise. If the formula $z : A \supset B$ is in the succedent a single application of $R\supset$ introduces both $z : A$ in the antecedent and $z : B$ in the succedent of the premise. If the formula $z : \mathbf{G}A$ (resp. $\mathbf{H}A$) is in the antecedent, a single application of $L\mathbf{G}$ (resp. $L\mathbf{H}$) introduces both $z : \mathbf{T}A$ and $z : \mathbf{TG}A$ (resp. $z : \mathbf{Y}A$ and $z : \mathbf{YH}A$) in the antecedent of the premise. If the formula $z : \mathbf{G}A$ (resp. $\mathbf{H}A$) is in the succedent, a single application of $R\mathbf{G}$ (resp. $R\mathbf{H}$) introduces $z : \mathbf{T}A$ (resp. $z : \mathbf{Y}A$) in the succedent of the left premise and $z : \mathbf{TG}A$ (resp. $z : \mathbf{YH}A$) in the succedent of the right premise. If the formula $z : \mathbf{F}A$ (resp. $z : \mathbf{P}A$) is in the antecedent, a single application of $L\mathbf{F}$ (resp. $L\mathbf{P}$) introduces $z : \mathbf{T}A$ (resp. $z : \mathbf{Y}A$) in the antecedent of the left premise and $z : \mathbf{TF}A$ (resp. $z : \mathbf{YP}A$) in the antecedent of the right premise. If the formula $z : \mathbf{F}A$ (resp. $z : \mathbf{P}A$) is in the succedent, a single application of $R\mathbf{F}$ (resp. $R\mathbf{P}$) introduces both $z : \mathbf{T}A$ and $z : \mathbf{TF}A$ (resp. $z : \mathbf{Y}A$ and $z : \mathbf{YP}A$) in the succedent of the premise. If the relational atom $z \prec z'$ is in the antecedent and $z : \mathbf{T}A$ is in the antecedent (resp. succedent), a single application of $L\mathbf{T}$ (resp. $R\mathbf{T}_{cl}$) introduces $z' : A$ in the antecedent (resp. succedent) of the premise. If the relational atom $z' \prec z$ is in the antecedent and $z : \mathbf{Y}A$ is in the antecedent (resp. succedent), a single application of $L\mathbf{Y}$ (resp. $R\mathbf{Y}_{cl}$) introduces $z' : A$ in the antecedent (resp. succedent) of the premise.

(ii) The presence of a future and a past loop follows from Lemmas 4.4.1 and 4.4.2, and the fact that we can go on applying right and left seriality rules and introduce new labels until a future and a past loop are met;

(iii) If the formula $z : \mathbf{F}B$ (resp. $z : \mathbf{G}B$) is in the antecedent (resp. succedent), then $[\![z]\!] \Vdash \mathbf{F}B$ (resp. $[\![z]\!] \nVdash \mathbf{G}B$). Therefore, either there exists an instant s such that $[\![z]\!] <^{\mathcal{K}} s$ and $s \Vdash B$ (resp. $s \nVdash B$), and for some z', $[\![z']\!] = s$

and z' is the future witness of $z : \mathbf{F}B$ (resp. $z : \mathbf{G}B$), or $[\![z]\!]$ falls under the conditions of Lemma 4.4.1, and thus z is inside a future loop;

(iv) If the formula formula $z : \mathbf{P}B$ (resp. $z : \mathbf{H}B$) is in the antecedent (resp. succedent), then $[\![z]\!] \Vdash \mathbf{P}B$ (resp. $[\![z]\!] \nVdash \mathbf{H}B$). Therefore, either there exists an instant s such that $s <^{\mathcal{K}} [\![z]\!]$ and $s \Vdash B$ (resp. $s \nVdash B$), and for some z', $[\![z']\!] = s$ and z' is the past witness of $z : \mathbf{P}B$ (resp. $z : \mathbf{H}B$), or $[\![z]\!]$ falls under the conditions of Lemma 4.4.2, and so z is inside a past loop. □

4.5 Completeness

We prove completeness by contraposition: if the sequent $\Gamma \Rightarrow \Delta$ is not provable in G3LT$_{cl}$, that is if the root-first proof search leads to a fulfilling sequent, then a countermodel for $\Gamma \Rightarrow \Delta$ can be constructed.

Our completeness proof has been suggested by the proof presented in Negri and von Plato (2008, pp. 201–207). However, the definition of fulfilling sequents allows to consider only finite objects, and not (possibly) infinite reduction tree; furthemore, the presence of the fixed-point rules for the temporal operators requires additional work in proving the inductive steps for temporal formulas, since we cannot appeal directly to the semantic explanations for the corresponding operators.

Let us consider the standard frame $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ for Priorean linear time, with $\mathcal{K} = \{s_i | i \in \mathbb{Z}\}$, $s_i \prec^{\mathcal{K}} s_{i+1}$ and $s_i <^{\mathcal{K}} s_j$ for $i < j$. Given a fulfilling sequent $\Gamma^* \Rightarrow \Delta^*$ for the purely logical sequent $\Gamma \Rightarrow \Delta$, we construct a countermodel $\mathcal{M}$ by defining an appropriate interpretation for the set of labels in $\Gamma^* \Rightarrow \Delta^*$ into the domain $\mathcal{K}$ as follows: we put $[\![x]\!] = s_0$ if x is the label that marks all the formulas in $\Gamma \Rightarrow \Delta$, and for every label z if the relational atoms $x \equiv z_0 \prec z_1, \ldots, z_{n-1} \prec z_n \equiv z$ are in Γ, we put $[\![z]\!] = s_n$. Analogously, if $z \equiv z_{-n} \prec z_{-(n-1)}, \ldots, z_{-1} \prec z_0 \equiv x$ are in Γ, we put $[\![z]\!] = s_{-n}$. We evaluate the atomic formulas by putting $[\![z]\!] \Vdash P$ if $z : P$ is in Γ^* and $[\![z]\!] \nVdash P$ if $z : P$ is in Δ^*. Furthermore, if z_{n+l} is the future looping label with respect to z_n, $[\![z_{n+l}]\!] = s_{n+l}$ and $[\![z_n]\!] = s_n$, then for every instant $s_{n+m \cdot l+q}$ (with $m \geq 0$ and $0 \leq q \leq l-1$) we put $s_{n+m \cdot l+q} \Vdash P$ if $z_{n+q} : P$ is in Γ^* and $s_{n+m \cdot l+q} \nVdash P$ if $z_{n+q} : P$ is in Δ^*. Analogously, if $z_{-(n+l)}$ is the past looping label with respect to z_{-n}, $[\![z_{-(n+l)}]\!] = s_{-(n+l)}$ and $[\![z_{-n}]\!] = s_{-n}$, then for every instant $s_{-(n+m \cdot l+q)}$ (with $m \geq 0$ and $0 \leq q \leq l-1$) we put $s_{-(n+m \cdot l+q)} \Vdash P$ if $z_{-(n+q)} : P$ is in Γ^* and $s_{-(n+m \cdot l+q)} \nVdash P$ if $z_{-(n+q)} : P$ is in Δ^*. Observe that this interpretation can be made consistently because a fulfilling sequent is neither initial nor contains $\perp$ in the antecedent.

Lemma 4.5.1. *$\mathcal{M}$ is a countermodel for* $\Gamma^* \Rightarrow \Delta^*$.

Proof. By definition, if $z \prec z'$ is in Γ^*, then $[\![z]\!] \prec^{\mathcal{K}} [\![z']\!]$. We have to show that, for arbitrary formulas B, if $z : B$ is in Γ^*, then $[\![z]\!] \Vdash B$, and if $z : B$ is in Δ^*, then $[\![z]\!] \nVdash B$. We proceed by induction on the length of the formula B.

1. If B is an atomic formula P and $z : P$ is in Γ^*, then $[\![z]\!] \Vdash P$ by construction. If $z : P$ is in Δ^*, then $[\![z]\!] \nVdash P$ by construction. Since z is $\prec$-saturated, $z : P$ cannot be both in Γ^* and in Δ^*.

2. If $B \equiv \perp$, then it cannot be in Γ^* by definition of fulfilling sequent. If $z : \perp$ is in Δ^*, then $[\![z]\!] \nVdash \perp$ by Definition 3.3.2.

3. If $B \equiv C \& D$ and $z : C \& D$ is in Γ^*, then, since z is $\prec$-saturated, both $z : C$ and $z : D$ are in Γ^*. By inductive hypothesis $[\![z]\!] \Vdash C$ and $[\![z]\!] \Vdash D$, and therefore $[\![z]\!] \Vdash C \& D$. If $z : C \& D$ is in Δ^*, then, since z is $\prec$-saturated, either $z : C$

or $z : D$ is in Δ^*. By inductive hypothesis $[\![z]\!] \nVdash C$ or $[\![z]\!] \nVdash D$, and therefore $[\![z]\!] \nVdash C\&D$.

4. If $B \equiv C \vee D$ and $z : C \vee D$ is in Γ^*, then, since z is $\prec$-saturated, either $z : C$ or $z : D$ is in Γ^*. By inductive hypothesis $[\![z]\!] \Vdash C$ or $[\![z]\!] \Vdash D$, and therefore $[\![z]\!] \Vdash C \vee D$. If $z : C \vee D$ is in Δ^*, then, since z is $\prec$-saturated, both $z : C$ and $z : D$ are in Δ^*. By inductive hypothesis $[\![z]\!] \nVdash C$ and $z \nVdash D$, and therefore $[\![z]\!] \nVdash C \vee D$.

5. If $B \equiv C \supset D$ and $z : C \supset D$ is in Γ^*, then, since z is $\prec$-saturated, either $z : C$ is in Δ^* or $z : D$ is in Γ^*. By inductive hypothesis $[\![z]\!] \nVdash C$ or $[\![z]\!] \Vdash D$, and therefore $[\![z]\!] \Vdash C \supset D$. If $z : C \supset D$ is in Δ^*, then, since z is $\prec$-saturated, we have $z : C$ in Γ^* and $z : D$ in Δ^*. By inductive hypothesis $[\![z]\!] \Vdash C$ and $[\![z]\!] \nVdash D$, and therefore $[\![z]\!] \nVdash C \supset D$.

6. If $B \equiv \mathbf{T}C$ and $z : \mathbf{T}C$ is in Γ^* (resp. Δ^*), then we have two cases: (i) if the label z is not the future looping label z_f, then it is connected to it by a chain $z \equiv z_{n+l-i} \prec z_{n+l-(i-1)}, \ldots, z_{n+l-1} \prec z_{n+l} \equiv z_f$ and, since the label z_{n+l-i} is $\prec$-saturated, we have $z_{n+l-(i-1)} : C$ in Γ^* (resp. Δ^*). Therefore, by construction, we have $[\![z_{n+l-i}]\!] \prec^{\mathcal{K}} [\![z_{n+l-(i-1)}]\!]$ and by inductive hypothesis $[\![z_{n+l-(i-1)}]\!] \Vdash C$ (resp. $[\![z_{n+l-(i-1)}]\!] \nVdash C$). So $[\![z]\!] \Vdash \mathbf{T}C$ (resp. $[\![z]\!] \nVdash \mathbf{T}C$). (ii) If z is the future looping label, then by definition for no label z' the atom $z \prec z'$ is in Γ^*. However, we have some label z_n such that $x \equiv z_0 \prec z_1$, $\ldots, z_{n-1} \prec z_n, z_n \prec z_{n+1}, \ldots, z_{n+l-1} \prec z_{n+l} \equiv z$ are in Γ^* for $l > 0$ and z_n marks the same formulas as z; in particular $z_n : \mathbf{T}C$ is in Γ^* (resp. Δ^*). Since z_n is $\prec$-saturated, $z_{n+1} : C$ is in Γ^* (resp. Δ^*). By construction $[\![z]\!] = s_{n+l}$, so $[\![z]\!] \prec^{\mathcal{K}} s_{n+l+1}$ and, by construction and inductive hypothesis, $s_{n+l+1} \Vdash C$ (resp. $s_{n+l+1} \nVdash C$). Therefore $[\![z_{n+l}]\!] \Vdash \mathbf{T}C$ (resp. $[\![z_{n+l}]\!] \nVdash \mathbf{T}C$).

7. If $B \equiv \mathbf{G}C$ and $z : \mathbf{G}C$ is in Γ^*, then, since z is $\prec$-saturated, both $z : \mathbf{T}C$ and $z : \mathbf{TG}C$ are in Γ^*, and, if the label $z \prec z'$ is in Γ^*, both $z' : C$ and $z' : \mathbf{G}C$ are in Γ^*. Therefore, by repeating this argument, we have that for every z'', if $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z''$ are in Γ^* for some $i, j \geq 0$, then $z'' : C$ and $z'' : \mathbf{G}C$ are in Γ^*. Note that, if z is the future looping label or z'' is inside a future loop $z_m \prec z_{m+1}, \ldots, z_{n-1} \prec z_n$ (with $n > m$) both $z_k : C$ and $z_k : \mathbf{G}C$ are in Γ^* for every $m \leq k \leq n$. By inductive hypothesis for every s if $[\![z]\!] <^{\mathcal{K}} s$ then $s \Vdash C$, therefore $[\![z]\!] \Vdash \mathbf{G}C$.

If $z : \mathbf{G}C$ is in Δ^* then, by Definitions 4.3.4 and 4.3.5, we have two cases: (i) there exists some future witness label z' such that $z' : C$ is in Δ^* and the atoms $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z'$ are in Γ^* for some $i, j \geq 0$. So, by construction and inductive hypothesis there is some $s = [\![z']\!]$ such that $[\![z]\!] <^{\mathcal{K}} s$ and $s \nVdash C$, therefore $[\![z]\!] \nVdash \mathbf{G}C$. (ii) label z is inside a future loop $z_n \prec z_{n+1}$, $\ldots, z_{n+i-1} \prec z_{n+i} \equiv z$, $z_{n+i} \prec z_{n+i+1}, \ldots, z_{n+l-1} \prec z_{n+l}$ (with $l \geq i$). Then there exists some label z' such that either $z_n \equiv z'$ or the relational atoms $z_n \prec z_{n+1}, \ldots, z_{n+q-1} \prec z_{n+q} \equiv z'$ are in Γ^* for $0 \leq q \leq i$ and the labelled formula $z' : C$ is in Δ^*. By construction $[\![z']\!] = s_{n+q}$, so $[\![z]\!] <^{\mathcal{K}} s_{n+l+q}$ and, by inductive hypothesis, $s_{n+l+q} \nVdash C$. Therefore $[\![z]\!] \nVdash \mathbf{G}C$.

8. If $z : \mathbf{F}C$ is in Γ^* then, by Definitions 4.3.4 and 4.3.5, we have two cases: (i) there exists some future witness label z' such that $z' : C$ is in Δ^* and the atoms $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z'$ are in Γ^* for some $i, j \geq 0$. So, by construction and inductive hypothesis there is some $s = [\![z']\!]$ such that $[\![z]\!] <^{\mathcal{K}} s$ and $s \Vdash C$, then $[\![z]\!] \Vdash \mathbf{F}C$. (ii) z is inside a future loop $z_n \prec z_{n+1}$, $\ldots, z_{n+i-1} \prec z_{n+i} \equiv z$, $z_{n+i} \prec z_{n+i+1}, \ldots, z_{n+l-1} \prec z_{n+l}$ (with $l \geq i$). Then there exists z' such that either $z_n \equiv z'$ or the atoms $z_n \prec z_{n+1}, \ldots, z_{n+q-1} \prec z_{n+q} \equiv z'$ are in

Γ^* for $0 \leq q \leq i$ and the labelled formula $z' : C$ is in Γ^*. By construction $[\![z']\!] = s_{n+q}$, so $[\![z]\!] <^{\mathcal{K}} s_{n+l+q}$ and, by inductive hypothesis, $s_{n+l+q} \Vdash C$. Therefore $[\![z]\!] \Vdash \mathbf{F}C$.

If $z : \mathbf{F}C$ is in Δ^*, then, since z is $\prec$-saturated, both $z : \mathbf{T}C$ and $z : \mathbf{TF}C$ are in Δ^*, and, if $z \prec z'$ is in Γ^*, both $z' : C$ and $z' : \mathbf{F}C$ are in Δ^*. Then, by repeating this argument, we have that for every z'', if $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z''$ are in Γ^* for some $i, j \geq 0$, then $z'' : C$ and $z'' : \mathbf{F}C$ are in Δ^*. Note that, if z is the future looping label or z'' is inside a future loop $z_m \prec z_{m+1}, \ldots, z_{n-1} \prec z_n$ (with $n > m$) both $z_k : C$ and $z_k : \mathbf{F}C$ are in Δ^* for every $m \leq k \leq n$. By inductive hypothesis for every s if $[\![z]\!] <^{\mathcal{K}} s$ then $s \nVdash C$, therefore $[\![z]\!] \nVdash \mathbf{F}C$.

9. If $z : \mathbf{Y}C$ is in Γ^* (resp. Δ^*), then we have two cases: (i) if z is not the past looping label z_p, it is connected by a chain $z_p \equiv z_{-(n+l)} \prec z_{-(n+l-1)}$, $\ldots, z_{-(n+l-(i-1))} \prec z_{-(n+l-i)} \equiv z$ and, since $z_{-(n+l-i)}$ is $\prec$-saturated, we have $z_{-(n+l-(i-1))} : C$ in Γ^* (resp. Δ^*). Therefore, by construction and inductive hypothesis, we have $[\![z_{-(n+l-(i-1))}]\!] \prec^{\mathcal{K}} [\![z_{-(n+l-i)}]\!]$ and $[\![z_{-(n+l-(i-1))}]\!] \Vdash C$ (resp. $[\![z_{-(n+l-(i-1))}]\!] \nVdash C$). Therefore $[\![z]\!] \Vdash \mathbf{Y}C$ (resp. $[\![z]\!] \nVdash \mathbf{Y}C$). (ii) If z is the past looping label, then by definition for no label z' the atom $z' \prec z$ is in Γ^*. However, we have some label z_{-n} such that $z \equiv z_{-(n+l)} \prec z_{-(n+l-1)}, \ldots, z_{-(n+1)} \prec z_{-n}$, $z_{-n} \prec z_{-(n-1)}, \ldots, z_{-1} \prec z_0 \equiv x$ are in Γ^* for $l > 0$ and z_{-n} marks the same formulas as z; in particular $z_{-n} : \mathbf{Y}C$ is in Γ^* (resp. Δ^*). Since the label z_{-n} is $\prec$-saturated, $z_{-(n+1)} : C$ is in Γ^* (resp. Δ^*). By construction $[\![z]\!] = s_{-(n+l)}$, therefore $s_{-(n+l+1)} \prec^{\mathcal{K}} [\![z]\!]$ and, by construction and inductive hypothesis, $s_{-(n+l+1)} \Vdash C$ (resp. $s_{-(n+l+1)} \nVdash C$). So $[\![z_{-(n+l)}]\!] \Vdash \mathbf{Y}C$ (resp. $[\![z_{-(n+l)}]\!] \nVdash \mathbf{Y}C$).

10. If $B \equiv \mathbf{H}C$ and $z : \mathbf{H}C$ is in Γ^*, then, since z is $\prec$-saturated, both $z : \mathbf{Y}C$

and $z : \mathbf{YH}C$ are in Γ^*, and, if $z' \prec z$ is in Γ^*, both $z' : C$ and $z' : \mathbf{H}C$ are in Γ^*. Then, by repeating this argument, we have that for every z'', if $z'' \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z$ are in Γ^* for some $i, j \geq 0$, then $z'' : C$ and $z'' : \mathbf{H}C$ are in Γ^*. Note that, if z is the past looping label or z'' is inside a past loop $z_m \prec z_{m+1}, \ldots, z_{n-1} \prec z_n$ (with $n > m$) both $z_k : C$ and $z_k : \mathbf{H}C$ are in Γ^* for every $m \leq k \leq n$. By inductive hypothesis for every s if $s <^{\mathcal{K}} [\![z]\!]$ then $s \Vdash C$, therefore $[\![z]\!] \Vdash \mathbf{H}C$.

If $z : \mathbf{H}C$ is in Δ^* then, by Definitions 4.3.4 and 4.3.5, we have two cases: (i) there exists some past witness label z' such that $z' : C$ is in Δ^* and the atoms $z' \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z$ are in Γ^* for some $i, j \geq 0$. By construction and inductive hypothesis there is some s such that $s <^{\mathcal{K}} [\![z]\!]$ and $s \nVdash C$, therefore $[\![z]\!] \nVdash \mathbf{H}C$. (ii) the label z is inside a past loop $z_{-(n+l)} \prec z_{-(n+l-1)}$, $\ldots, z_{-(n+i+1)} \prec z_{-(n+i)} \equiv z$, $z_{-(n+i)} \prec z_{-(n-(i-1))}$, $\ldots, z_{-(n-1)} \prec z_{-n}$ (with $l \geq i$). Then there exists a label z' such that $z_{-n} \equiv z'$ or the relational atoms $z' \equiv z_{-(n+q)} \prec z_{-(n+(q-1))}, \ldots, z_{-(n+1)} \prec z_{-n}$ are in Γ^* for $0 \leq q \leq i$ and the labelled formula $z' : C$ is in Δ^*. By construction $[\![z']\!] = s_{-(n+q)}$, and therefore $s_{-(n+l+q)} <^{\mathcal{K}} [\![z]\!]$ and, by inductive hypothesis, $s_{-(n+l+q)} \nVdash C$. So, $[\![z]\!] \nVdash \mathbf{H}C$.

11. If $z : \mathbf{P}C$ is in Γ^* then, by Definitions 4.3.4 and 4.3.5, we have two cases: (i) there exists some past witness label z' such that $z' : C$ is in Δ^* and the atoms $z' \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z$ are in Γ^* for some $i, j \geq 0$. So, by construction and inductive hypothesis there is some s such that $s <^{\mathcal{K}} [\![z]\!]$ and $s \Vdash C$, therefore $[\![z]\!] \Vdash \mathbf{P}C$. (ii) z is inside a past loop $z_{-(n+l)} \prec z_{-(n+l-1)}$, $\ldots, z_{-(n+i+1)} \prec z_{-(n+i)} \equiv z$, $z_{-(n+1)} \prec z_{-(n-(i-1))}$, $\ldots, z_{-(n-1)} \prec z_{-n}$ (with $l \geq i$). Then there exists a label z' such that $z_{-n} \equiv z'$ or the relational atoms $z' \equiv z_{-(n+q)} \prec z_{-(n+(q-1))}$, $\ldots, z_{-(n+1)} \prec z_{-n}$ are in Γ^* for $0 \leq q \leq i$ and

the labelled formula $z' : C$ is in Γ^*. By construction $[\![z']\!] = s_{-(n+q)}$, therefore we have $s_{-(n+l+q)} <^{\mathcal{K}} [\![z]\!]$ and, by inductive hypothesis, $s_{-(n+l+q)} \Vdash C$. So, $[\![z]\!] \Vdash \mathbf{P}C$.

If $B \equiv \mathbf{P}C$ and $z : \mathbf{P}C$ is in Δ^*, then, since z is $\prec$-saturated, both $z : \mathbf{Y}C$ and $z : \mathbf{YP}C$ are in Δ^*, and, if $z' \prec z$ is in Γ^*, both $z' : C$ and $z' : \mathbf{P}C$ are in Δ^*. Then, by repeating this argument, we have that for every z'', if $z'' \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z$ are in Γ^* for some $i, j \geq 0$, then $z'' : C$ and $z'' : \mathbf{P}C$ are in Δ^*. Note that, if z is the past looping label or z'' is inside a past loop $z_m \prec z_{m+1}, \ldots, z_{n-1} \prec z_n$ (with $n > m$) both $z_k : C$ and $z_k : \mathbf{P}C$ are in Δ^* for every $m \leq k \leq n$. By inductive hypothesis for every s if $s <^{\mathcal{K}} [\![z]\!]$ then $s \nVdash C$, therefore $[\![z]\!] \nVdash \mathbf{P}C$. □

By the following result, every countermodel for the fulfilling sequent $\Gamma^* \Rightarrow \Delta^*$ is a countermodel for the corresponding endsequent $\Gamma \Rightarrow \Delta$:

Lemma 4.5.2. *All the rules of* G3LT$_{cl}$ *preserve countermodels, that is, a countermodel for (at least one of) the premises is also a countermodel for the conclusion.*

Proof. Immediate for the rules for **T** and **Y** and for the rules of seriality. For the propositional rules, by definition of validity for the propositional connectives. For the rules for **G**, **F**, **H** and **P**, by their fixed-point interpretation. □

Theorem 4.5.3. *If the purely logical sequent* $\Gamma \Rightarrow \Delta$ *has no standard countermodels, then it is provable in* G3LT$_{cl}$.

Corollary 4.5.4. *Provability of purely logical sequents in* G3LT$_{cl}$ *is closed with respect to cut.*

Proof. By soundness of the cut rule and completeness of G3LT$_{cl}$. □

4.6 Termination of proof search

While applying root-first the rules of G3LT$_{cl}$ along a branch, there can be two cases: (i) the proof search terminates because we find a fulfilling sequent or every branch leads to an initial sequent or an instance of $L\bot$; otherwise (ii) the proof search does not terminate and, by König's Lemma, there is at least one infinite branch.

However, we show how to truncate a potentially infinite proof search. By Theorem 4.3.8, if $\Gamma \Rightarrow \Delta$ is not provable, then the proof search leads to a reduced fulfilling sequent. Whenever a branch leads to a sequent with a roundabout chain, we can drop that branch and start a new one: if every branch in the proof search for $\Gamma \Rightarrow \Delta$ leads either to an initial sequent or to a sequent with a roundabout chain, then $\Gamma \Rightarrow \Delta$ is provable in G3LT$_{cl}$.

Lemma 4.6.1. *Consider a purely logical sequent $\Gamma \Rightarrow \Delta$ with all the formulas in it labelled by x. Let us suppose that the proof search for $\Gamma \Rightarrow \Delta$ leads to a sequent $\Gamma' \Rightarrow \Delta'$: if the chain $y_{-m} \prec y_{-(m-1)}, \dots, y_{-1} \prec y_0 \equiv x$ and the chain $x \equiv y_0 \prec y_1, \dots, y_{n-1} \prec y_n$ in it are not roundabout then the number of labels has an exponential bound on the order of the length of $A \equiv \wedge\Gamma^x \supset \vee\Delta^x$, namely $m, n \leq \sum_{i=1}^{2^{3|A|}} i$.*

Proof. (*Sketch*) We recall here that the rules of G3LT$_{cl}$ reflect the closure algorithm that from a formula A gives the set of its closure formulas and, by Corollary 4.1.3, the number of subsets of closure formulas of A is at most $2^{3|A|}$. Let us consider the longest case of a non-roundabout chain $y_0 \prec y_1, \dots, y_{n-1} \prec y_n$ such that for every k with $0 \leq k \leq n$, y_k labels a subset of closure formulas of A. It contains a first subchain $y_0 \prec y_1, \dots, y_{i-2} \prec y_{i-1}$ such that $i = 2^{3|A|}$ and every subset of closure formulas of A is labelled by some y_k, for $0 \leq k \leq i-1$. Then we have a second subchain $y_i \prec y_{i+1}, \ldots, y_{i+j-2} \prec y_{i+j-1}$, such that $j = 2^{3|A|} - 1$ and every subset of closure formulas of A except one is marked by y_k for $i \leq k \leq i+j-1$. Analogously, the subchain in the $l+1$st position contains $j = 2^{3|A|} - l$ labels, that mark the same subsets of closure formula marked by the members of the chain in the lth position, except one. Summing up the numbers of the members of each subchain, we finally obtain that $n = \sum_{i=1}^{2^{3|A|}} i$. The same argument applies to the chain $y_{-m} \prec y_{-(m-1)}, \ldots, y_{-1} \prec y_0$, therefore $m = \sum_{i=1}^{2^{3|A|}} i$. □

Theorem 4.6.2. *Proof search for* G3LT$_{cl}$ *terminates.*

Proof. Let us suppose that the proof search for the purely logical sequent $\Gamma \Rightarrow \Delta$ (with all its formulas labelled by x) does not terminate. Since every rule of G3LT$_{cl}$ has a finite number of premises, any derivation tree is finitely branching, so by König's Lemma there is at least one infinite branch. Obviously it cannot lead to an initial sequent, nor to a conclusion of L$\perp$, nor to a fulfilling sequent, because otherwise it would be finite. We have to show that it contains a sequent with a roundabout chain. Note that the endsequent contains a finite number of formulas: the logical rules for connectives and for temporal operators can introduce only a finite number of new formulas, and by Lemma 4.2.10 temporal rules cannot be applied more than once with the same principal formula(s). Furthermore, by Lemmas 4.2.6 and 4.2.7 we need not apply a seriality rule with side label z, if z is not a label in the sequent or the antecedent already contains an atom $z \prec z'$ (resp. $z' \prec z$). Consequently, an infinite branch should contain a sequent with an infinite $\prec$-chain. However, by Lemma 4.6.1 if a chain is not roundabout, then it is finite and exponentially bounded on the order of the length of the formula corresponding to the endsequent $\Gamma \Rightarrow \Delta$. Therefore, any potentially infinite branch can be truncated as soon as a sequent is met containing a chain $z_{-m} \prec z_{-(m-1)}, \ldots, z_{-1} \prec z_0 \equiv x, z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$ with $m > \sum_{i=1}^{2^{3|\wedge\Gamma^x \supset \vee\Delta^x|}} i$ or $n > \sum_{i=1}^{2^{3|\wedge\Gamma^x \supset \vee\Delta^x|}} i$. $\square$

4.7 Comparison of $\mathbf{G3LT}_{cl}$ with G3LT

Lemma 4.7.1. *All the rules of* $\mathrm{G3LT}_{cl}$ *are admissible in* G3LT.

Proof. The rules of seriality, $L\bot$, the propositional rules and the left rules for $\mathbf{T}$ and $\mathbf{Y}$ are identical to those of G3LT. The initial sequents $x : \phi, \Gamma \Rightarrow \Delta, x : \phi$ are derivable in G3LT by Theorem 2.3.5. For the other rules, the result follows by admissibility of cut and contraction in G3LT and the following derivations.

Rule $L\mathbf{G}_{cl}$:

$$\dfrac{\dfrac{x : \mathbf{G}A \Rightarrow x : \mathbf{T}A \qquad \dfrac{x : \mathbf{G}A \Rightarrow x : \mathbf{TG}A \quad x : \mathbf{T}A, x : \mathbf{TG}A, \Gamma \Rightarrow \Delta}{x : \mathbf{T}A, x : \mathbf{G}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle Cut}}{x : \mathbf{G}A, x : \mathbf{G}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle Cut}}{x : \mathbf{G}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle LCtr^*}$$

The sequents $x : \mathbf{G}A \Rightarrow x : \mathbf{TG}A$ and $x : \mathbf{G}A \Rightarrow x : \mathbf{T}A$ are derivable in G3LT (see Proposition 3.3.9).

Rule $R\mathbf{G}_{cl}$:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : \mathbf{G}A, x : \mathbf{TG}A \qquad \dfrac{\Gamma \Rightarrow \Delta, x : \mathbf{G}A, x : \mathbf{T}A \quad x : \mathbf{T}A, x : \mathbf{TG}A \Rightarrow x : \mathbf{G}A}{x : \mathbf{TG}A, \Gamma \Rightarrow \Delta, x : \mathbf{G}A, x : \mathbf{G}A}\,{\scriptstyle Cut}}{\Gamma, \Gamma \Rightarrow \Delta, \Delta, x : \mathbf{G}A, x : \mathbf{G}A, x : \mathbf{G}A}\,{\scriptstyle Cut}}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\,{\scriptstyle Ctr^*}$$

The sequent $x : \mathbf{T}A, x : \mathbf{TG}A \Rightarrow x : \mathbf{G}A$ is derivable in G3LT (see Proposition 3.3.10).

Rule $L\mathbf{F}_{cl}$:

$$\dfrac{\dfrac{x:\mathbf{F}A \Rightarrow x:\mathbf{T}A, x:\mathbf{TF}A \quad x:\mathbf{TF}A, x:\mathbf{F}A, \Gamma \Rightarrow \Delta}{x:\mathbf{F}A, x:\mathbf{F}A, \Gamma \Rightarrow \Delta, x:\mathbf{T}A}\,{\scriptstyle Cut} \quad x:\mathbf{T}A, x:\mathbf{F}A, \Gamma \Rightarrow \Delta}{\dfrac{x:\mathbf{F}A, x:\mathbf{F}A, x:\mathbf{F}A, \Gamma, \Gamma \Rightarrow \Delta, \Delta}{x:\mathbf{F}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle Ctr^*}}\,{\scriptstyle Cut}$$

The sequent $x:\mathbf{F}A \Rightarrow x:\mathbf{T}A, x:\mathbf{TF}A$ is obtained by the following derivation (the repetition of the principal formulas is omitted)

$$\dfrac{\dfrac{\dfrac{\dfrac{y=z, z:A, y:A \Rightarrow z:A, x:\mathbf{TF}A}{y=z, y:A \Rightarrow z:A, x:\mathbf{TF}A}\,{\scriptstyle EqSubst}}{x \prec z, x \prec y, y:A \Rightarrow z:A, x:\mathbf{TF}A}\,{\scriptstyle UnSucc}}{x \prec y, y:A \Rightarrow x:\mathbf{T}A, x:\mathbf{TF}A}\,{\scriptstyle R\mathbf{T}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{w<y, y:A \Rightarrow x:\mathbf{T}A, w:\mathbf{F}A, y:A}{w<y, y:A \Rightarrow x:\mathbf{T}A, w:\mathbf{F}A}\,{\scriptstyle R\mathbf{F}}}{v=w, v<y, y:A \Rightarrow x:\mathbf{T}A, w:\mathbf{F}A}\,{\scriptstyle EqSubst_{At}}}{x \prec w, x \prec v, v<y, y:A \Rightarrow x:\mathbf{T}A, w:\mathbf{F}A}\,{\scriptstyle UnSucc}}{x \prec v, v<y, y:A \Rightarrow x:\mathbf{T}A, x:\mathbf{TF}A}\,{\scriptstyle R\mathbf{T}}}{\dfrac{x<y, y:A \Rightarrow x:\mathbf{T}A, x:\mathbf{TF}A}{x:\mathbf{F}A \Rightarrow x:\mathbf{T}A, x:\mathbf{TF}A}\,{\scriptstyle L\mathbf{F}}}\,{\scriptstyle Mix_1}$$

Rule Mix_1 is admissible in G3LT by Proposition (see Proposition 3.2.24).

Rule $R\mathbf{F}_{cl}$:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x:\mathbf{T}A, x:\mathbf{TF}A \quad x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A}{\Gamma \Rightarrow \Delta, x:\mathbf{F}A, x:\mathbf{T}A}\,{\scriptstyle Cut} \quad x:\mathbf{T}A \Rightarrow x:\mathbf{F}A}{\dfrac{\Gamma \Rightarrow \Delta, x:\mathbf{F}A, x:\mathbf{F}A}{\Gamma \Rightarrow \Delta, x:\mathbf{F}A}\,{\scriptstyle RCtr^*}}\,{\scriptstyle Cut}$$

The sequents $x:\mathbf{T}A \Rightarrow x:\mathbf{F}A$ and $x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A$ are obtained by the following derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{x<y, x \prec y, y:A, x:\mathbf{T}A \Rightarrow x:\mathbf{F}A, y:A}{x<y, x \prec y, y:A, x:\mathbf{T}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle R\mathbf{F}}}{x \prec y, y:A, x:\mathbf{T}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle Inc}}{x \prec y, x:\mathbf{T}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle L\mathbf{T}}}{x:\mathbf{T}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle R\text{-}Ser}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \prec y, x<z, x<y, y<z, z:A, x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A, z:A}{x \prec y, x<z, x<y, y<z, z:A, x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle R\mathbf{F}}}{x \prec y, x<y, y<z, z:A, x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle Trans}}{x \prec y, y<z, z:A, x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle Inc}}{x \prec y, y:\mathbf{F}A, x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle L\mathbf{F}}}{x \prec y, x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle L\mathbf{T}}}{x:\mathbf{TF}A \Rightarrow x:\mathbf{F}A}\,{\scriptstyle R\text{-}Ser}$$

Rule $L\mathbf{H}_{cl}$:

$$\dfrac{\dfrac{x:\mathbf{H}A \Rightarrow x:\mathbf{Y}A \qquad \dfrac{x:\mathbf{H}A \Rightarrow x:\mathbf{YH}A \quad x:\mathbf{Y}A, x:\mathbf{YH}A, \Gamma \Rightarrow \Delta}{x:\mathbf{Y}A, x:\mathbf{H}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle Cut}}{x:\mathbf{H}A, x:\mathbf{H}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle Cut}}{x:\mathbf{H}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle LCtr^*}$$

The sequents $x:\mathbf{H}A \Rightarrow x:\mathbf{YH}A$ and $x:\mathbf{H}A \Rightarrow x:\mathbf{Y}A$ are derivable in G3LT (see Proposition 3.3.9).

Rule $R\mathbf{H}_{cl}$:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x:\mathbf{H}A, x:\mathbf{YH}A \qquad \dfrac{\Gamma \Rightarrow \Delta, x:\mathbf{H}A, x:\mathbf{Y}A \quad x:\mathbf{Y}A, x:\mathbf{YH}A \Rightarrow x:\mathbf{H}A}{x:\mathbf{YH}A, \Gamma \Rightarrow \Delta, x:\mathbf{H}A, x:\mathbf{H}A}\,{\scriptstyle Cut}}{\Gamma, \Gamma \Rightarrow \Delta, \Delta, x:\mathbf{H}A, x:\mathbf{H}A, x:\mathbf{H}A}\,{\scriptstyle Cut}}{\Gamma \Rightarrow \Delta, x:\mathbf{H}A}\,{\scriptstyle Ctr^*}$$

The sequent $x:\mathbf{Y}A, x:\mathbf{YH}A \Rightarrow x:\mathbf{H}A$ is derivable in G3LT (see Proposition 3.3.10).

Rule $L\mathbf{P}_{cl}$:

$$\dfrac{\dfrac{\dfrac{x:\mathbf{P}A \Rightarrow x:\mathbf{Y}A, x:\mathbf{YP}A \quad x:\mathbf{YP}A, x:\mathbf{P}A, \Gamma \Rightarrow \Delta}{x:\mathbf{P}A, x:\mathbf{P}A, \Gamma \Rightarrow \Delta, x:\mathbf{Y}A}\,{\scriptstyle Cut} \qquad x:\mathbf{Y}A, x:\mathbf{P}A, \Gamma \Rightarrow \Delta}{x:\mathbf{P}A, x:\mathbf{P}A, x:\mathbf{P}A, \Gamma, \Gamma \Rightarrow \Delta, \Delta}\,{\scriptstyle Cut}}{x:\mathbf{P}A, \Gamma \Rightarrow \Delta}\,{\scriptstyle Ctr^*}$$

The sequent $x:\mathbf{P}A \Rightarrow x:\mathbf{Y}A, x:\mathbf{YP}A$ is obtained by the following derivation (the repetition of the principal formulas is omitted)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y = z, z \prec x, y \prec x, z : A \Rightarrow z : A, x:\mathbf{YP}A}{y = z, z \prec x, y \prec x, y : A \Rightarrow z : A, x:\mathbf{YP}A}\,{\scriptstyle EqSubst}}{z \prec x, y \prec x, y : A \Rightarrow z : A, x:\mathbf{YP}A}\,{\scriptstyle UnPred}}{y \prec x, y : A \Rightarrow x:\mathbf{Y}A, x:\mathbf{YP}A}\,{\scriptstyle R\mathbf{T}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{y < w, y : A \Rightarrow x:\mathbf{Y}A, w:\mathbf{P}A, y : A}{y < w, y : A \Rightarrow x:\mathbf{Y}A, w:\mathbf{P}A}\,{\scriptstyle R\mathbf{F}}}{v = w, y < v, y : A \Rightarrow x:\mathbf{Y}A, w:\mathbf{P}A}\,{\scriptstyle EqSubst_{At}}}{y < v, w \prec x, v \prec x, y : A \Rightarrow x:\mathbf{Y}A, w:\mathbf{P}A}\,{\scriptstyle UnPred}}{y < v, v \prec x, y : A \Rightarrow x:\mathbf{Y}A, x:\mathbf{YP}A}\,{\scriptstyle R\mathbf{Y}}}{y < x, y : A \Rightarrow x:\mathbf{Y}A, x:\mathbf{YP}A}\,{\scriptstyle Mix_2}}{x:\mathbf{P}A \Rightarrow x:\mathbf{Y}A, x:\mathbf{YP}A}\,{\scriptstyle L\mathbf{P}}$$

Rule Mix_2 is admissible in G3LT by Proposition (see Proposition 3.2.24).

Rule $R\mathbf{P}_{cl}$:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : \mathbf{Y}A, x : \mathbf{YP}A \quad x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A}{\Gamma \Rightarrow \Delta, x : \mathbf{P}A, x : \mathbf{Y}A}\,{\scriptstyle Cut} \qquad x : \mathbf{Y}A \Rightarrow x : \mathbf{P}A}{\Gamma \Rightarrow \Delta, x : \mathbf{P}A, x : \mathbf{P}A}\,{\scriptstyle Cut}}{\Gamma \Rightarrow \Delta, x : \mathbf{P}A}\,{\scriptstyle RCtr^*}$$

The sequents $x : \mathbf{Y}A \Rightarrow x : \mathbf{P}A$ and $x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A$ are obtained by the following derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{y < x, y \prec x, y : A, x : \mathbf{Y}A \Rightarrow x : \mathbf{P}A, y : A}{y < x, y \prec x, y : A, x : \mathbf{Y}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle R\mathbf{P}}}{y \prec x, y : A, x : \mathbf{Y}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle Inc}}{y \prec x, x : \mathbf{Y}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle L\mathbf{Y}}}{x : \mathbf{Y}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle L\text{-}Ser}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z < x, z < y, y < x, y \prec x, z : A, x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A, z : A}{z < x, z < y, y < x, y \prec x, z : A, x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle R\mathbf{P}}}{z < y, y < x, y \prec x, z : A, x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle Trans}}{z < y, y \prec x, z : A, x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle Inc}}{y \prec x, y : \mathbf{P}A, x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle L\mathbf{P}}}{y \prec x, x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle L\mathbf{Y}}}{x : \mathbf{YP}A \Rightarrow x : \mathbf{P}A}\,{\scriptstyle L\text{-}Ser}$$

Rule $R\mathbf{T}_{cl}$:

$$\dfrac{\dfrac{x \prec y, \Gamma \Rightarrow \Delta, x : \mathbf{T}A, y : A \qquad \dfrac{\dfrac{\dfrac{y = z, x \prec z, x \prec y, y : A, z : A \Rightarrow z : A}{y = z, x \prec z, x \prec y, y : A \Rightarrow z : A}\,{\scriptstyle EqSubst}}{x \prec z, x \prec y, y : A \Rightarrow z : A}\,{\scriptstyle UnSucc}}{x \prec y, y : A \Rightarrow x : \mathbf{T}A}\,{\scriptstyle R\mathbf{T}}}{x \prec y, x \prec y, \Gamma \Rightarrow \Delta, x : \mathbf{T}A, x : \mathbf{T}A}\,{\scriptstyle Cut}}{x \prec y, \Gamma \Rightarrow \Delta, x : \mathbf{T}A}\,{\scriptstyle Ctr^*}$$

Rule $R\mathbf{Y}_{cl}$:

$$\dfrac{\dfrac{y \prec x, \Gamma \Rightarrow \Delta, x : \mathbf{Y}A, y : A \qquad \dfrac{\dfrac{\dfrac{y = z, z \prec x, y \prec x, z : A, y : A \Rightarrow z : A}{y = z, z \prec x, y \prec x, y : A \Rightarrow z : A}\,{\scriptstyle EqSubst}}{z \prec x, y \prec x, y : A \Rightarrow z : A}\,{\scriptstyle UnPred}}{y \prec x, y : A \Rightarrow x : \mathbf{Y}A}\,{\scriptstyle R\mathbf{Y}}}{y \prec x, y \prec x, \Gamma \Rightarrow \Delta, x : \mathbf{Y}A, x : \mathbf{Y}A}\,{\scriptstyle Cut}}{y \prec x, \Gamma \Rightarrow \Delta, x : \mathbf{Y}A}\,{\scriptstyle Ctr^*}$$

□

Theorem 4.7.2. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathrm{G3LT}_{cl}$ *then it is derivable in* G3LT*.*

The converse does not hold: let us consider, in fact, the following proof-search tree for the purely logical sequent $x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$ corresponding to the induction principle towards the future

$$
\dfrac{\dfrac{x : \mathbf{T}A \Rightarrow x : \mathbf{T}A \qquad \dfrac{\dfrac{\dfrac{y : A, \ldots \Rightarrow \ldots y : A \qquad \begin{array}{c}\vdots\\ x \prec y, z \prec x, y : A, y : \mathbf{T}A, y : \mathbf{G}(A \supset \mathbf{T}A), \cdots \Rightarrow \ldots, y : \mathbf{G}A\end{array}}{x \prec y, z \prec x, y : A, y : A \supset \mathbf{T}A, y : \mathbf{G}(A \supset \mathbf{T}A), \ldots \Rightarrow \ldots, y : \mathbf{G}A}{\scriptstyle L\supset}}{x \prec y, z \prec x, x : \mathbf{T}A, x : \mathbf{T}(A \supset \mathbf{T}A), x : \mathbf{TG}(A \supset \mathbf{T}A) \Rightarrow \ldots, x : \mathbf{TG}A}{\scriptstyle R\mathbf{T}_{cl}, L\mathbf{T}}}{x : \mathbf{T}A, x : \mathbf{T}(A \supset \mathbf{T}A), x : \mathbf{TG}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A, x : \mathbf{TG}A}{\scriptstyle R\text{-}Ser, L\text{-}Ser}}{x : \mathbf{T}A, x : \mathbf{T}(A \supset \mathbf{T}A), x : \mathbf{TG}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A}{\scriptstyle R\mathbf{G}_{cl}}}{x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A}{\scriptstyle L\mathbf{G}_{cl}}
$$

Clearly, $x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$ is not derivable in G3LT$_{cl}$, since the proof search produces an infinite derivation tree, which is incompatible with Definition 2.3.2. However, by proceeding along the open branch containing

$$
\begin{aligned}
&x \prec y, z \prec x, y : A, y : \mathbf{T}A, y : \mathbf{G}(A \supset \mathbf{T}A), x : \mathbf{T}A, x : \mathbf{T}(A \supset \mathbf{T}A), x : \mathbf{TG}(A \supset \mathbf{T}A) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow x : \mathbf{G}A, x : \mathbf{TG}A, y : \mathbf{G}A
\end{aligned}
$$

we eventually find a proof (see Definition 4.3.9): the derivation tree, in fact, cannot contain a fulfilling sequent for $x : \mathbf{T}A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$, since every branch either terminates in an initial sequent or leads to a sequent in which the formula $x : \mathbf{G}A$ in the succedent is not witnessed, contrary to Definition 4.3.5.

Soundness and completeness of G3LT together with Theorems 4.4.4 and 4.5.3 give the following result:

Theorem 4.7.3. *The purely logical sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* G3LT *iff it is provable in* G3LT$_{cl}$.

4.8 Adding Until and Since

The calculus G3LT$_{cl}$+$\mathcal{U}$+$\mathcal{S}$ is obtained by adding to G3LT$_{cl}$ the initial sequents and the rules of Table 4.2.

Initial sequents:

$x : A\mathcal{U}B, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B \qquad\qquad x : A\mathcal{S}B, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B$

Rules for Until:

$$\frac{x : \mathbf{T}B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta \quad x : \mathbf{T}A, x : \mathbf{T}(A\mathcal{U}B), x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}{x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\mathcal{U}_{cl}}$$

$$\frac{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B, x : \mathbf{T}B, x : \mathbf{T}A \quad \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, x : \mathbf{T}B, x : \mathbf{T}(A\mathcal{U}B)}{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B}\, {\scriptstyle R\mathcal{U}_{cl}}$$

Rules for Since

$$\frac{x : \mathbf{Y}B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta \quad x : \mathbf{Y}A, x : \mathbf{Y}(A\mathcal{S}B), x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\mathcal{S}_{cl}}$$

$$\frac{\Gamma \Rightarrow \Delta, x : A\mathcal{S}B, x : \mathbf{Y}B, x : \mathbf{Y}A \quad \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, x : \mathbf{Y}B, x : \mathbf{Y}(A\mathcal{S}B)}{\Gamma \Rightarrow \Delta, x : A\mathcal{S}B}\, {\scriptstyle R\mathcal{S}_{cl}}$$

Table 4.2: Fixed point rules for Until and Since

As in Section 3.5, the rules for Until and Since are justified by following recursive definitions

$A\mathcal{U}B \equiv \mathbf{T}B \vee (\mathbf{T}A \& \mathbf{T}(A\mathcal{U}B))$ (equivalently, $A\mathcal{U}B \equiv (\mathbf{T}B \vee \mathbf{T}A) \& (\mathbf{T}B \vee \mathbf{T}(A\mathcal{U}B)))$

$A\mathcal{S}B \equiv \mathbf{Y}B \vee (\mathbf{Y}A \& \mathbf{Y}(A\mathcal{S}B)$ (equivalently, $A\mathcal{S}B \equiv (\mathbf{Y}B \vee \mathbf{Y}A) \& (\mathbf{Y}B \vee \mathbf{Y}(A\mathcal{S}B)))$

Observe, however, that in the calculus G3LT$_{cl}$ there is no need of the additional conditions $x : \mathbf{F}B$ and $x : \mathbf{P}B$.

A straightforward adaptation of the proofs in Section 4.2 gives the following results:

Theorem 4.8.1. *All the rules of* G3LT$_{cl}$ $+ \mathcal{U} + \mathcal{S}$ *are height-preserving invertible. The rules of substitution, left and right weakening, and left and right* *contraction are height-preserving admissible.*

Lemma 4.8.2. *The temporal rules permute down with respect to all the rules of* $\mathrm{G3LT}_{cl} + \mathcal{U} + \mathcal{S}$ *in the case their principal formulas are not active in the latter.*

Corollary 4.8.3. *On any branch of a minimal derivation in* $\mathrm{G3LT}_{cl} + \mathcal{U} + \mathcal{S}$*, a given temporal rule need not be applied more than once on the same principal formula(s).*

Definition 4.8.4. *The Definition 4.1.1 of* a set of closure formulas *is augmented with the following inductive clauses when the until and since operators are added:*

- $\mathbf{T}A$*,* $\mathbf{T}B$ *and* $\mathbf{T}(A\mathcal{U}B) \in cl(A\mathcal{U}B)$*;*
- $\mathbf{Y}A$*,* $\mathbf{Y}B$ *and* $\mathbf{Y}(A\mathcal{S}B) \in cl(A\mathcal{S}B)$*.*

Correspondingly, the notion of saturated label in a sequent $\Gamma \Rightarrow \Delta$ has to be generalised to until and since formulas:

Definition 4.8.5. *A label* x *in a sequent* $\Gamma \Rightarrow \Delta$ *is* saturated *if it is as in Definition 4.1.5 and the following clauses are satisfied:*

- $x : A\mathcal{U}B$ *(resp.* $x : A\mathcal{S}B$*) in* Γ *implies that* $x : \mathbf{T}B$ *or both* $x : \mathbf{T}A$ *and* $\mathbf{T}(A\mathcal{U}B)$ *(resp.* $x : \mathbf{Y}B$ *or both* $x : \mathbf{Y}A$ *and* $x : \mathbf{Y}(A\mathcal{S}B)$*) are in* Γ*;*
- $x : A\mathcal{U}B$ *(resp.* $x : A\mathcal{S}B$*) in* Δ *implies that both* $x : \mathbf{T}B$ *and* $x : \mathbf{T}A$ *or both* $x : \mathbf{T}B$ *and* $\mathbf{T}(A\mathcal{U}B)$ *(resp. both* $x : \mathbf{Y}B$ *and* $x : \mathbf{Y}A$ *or both* $x : \mathbf{Y}B$ *and* $x : \mathbf{Y}(A\mathcal{S}B)$*) are in* Δ*.*

The notion of $\prec$-saturated label is as in Definition 4.1.7.

Lemma 4.8.6. *Let* $|A|$ *be the number of subformulas of* A*. The cardinality of* $cl(A)$ *is linearly bounded by* $|A|$*, namely* $|cl(A)| \leq 3 \cdot |A| + 1$*.*

Proof. By Lemma 4.1.2 and the following

10. $|cl(B\mathcal{U}C)| = |\{\mathbf{T}C, \mathbf{T}B, \mathbf{T}(B\mathcal{U}C)\} \cup \{B\mathcal{U}C\} \cup cl(B) \cup cl(C)| \leq 3+1+3|B|+3|C| = 3(|B|+|C|+1)+1 = 3|B\mathcal{U}C|+1$;

11. $|cl(B\mathcal{S}C)| = |\{\mathbf{Y}C, \mathbf{Y}B, \mathbf{Y}(B\mathcal{S}C)\} \cup \{B\mathcal{S}C\} \cup cl(B) \cup cl(C)| \leq 3+1+3|B|+3|C| = 3(|B|+|C|+1)+1 = 3|B\mathcal{S}C|+1$.

□

Corollary 4.8.7. *The number of subsets of* $cl(A)$ *is at most* $2^{3|A|+1}$.

Definition 4.8.8. Evaluation *in a model for linear discrete time logic (see Definition 3.3.2) is extended to until and since formulas by the following inductive clauses:*

$$k \Vdash A\mathcal{U}B \text{ iff there exists } k' \text{ such that } k <^{\mathcal{K}} k' \text{ and } k' \Vdash B$$
$$\text{and for all } k'', \text{ if } k <^{\mathcal{K}} k'' \text{ and } k'' <^{\mathcal{K}} k', \text{ then } k'' \Vdash A;$$

$$k \Vdash A\mathcal{S}B \text{ iff there exists } k' \text{ such that } k' <^{\mathcal{K}} k \text{ and } k' \Vdash B$$
$$\text{and for all } k'', \text{ if } k' <^{\mathcal{K}} k'' \text{ and } k'' <^{\mathcal{K}} k, \text{ then } k'' \Vdash A.$$

Note that the semantic explanations for $A\mathcal{U}B$ and $A\mathcal{S}B$ require that the subformula B is satisfied at some point, in analogy to $\mathbf{F}B$ and $\mathbf{P}B$ respectively. This justifies the definitions below of future and past witness for until and since formulas:

Definition 4.8.9. *Given a labelled formula* $z : A\mathcal{U}B$ *in the antecedent of a sequent* $\Gamma \Rightarrow \Delta$, *we say that a label* z' *is a* future witness *for* $z : A\mathcal{U}B$ *if* $z' : B$ *is in* Γ *and the atoms* $z \prec z_0, \ldots, z_{n-1} \prec z_n \equiv z'$ *are in* Γ *for some* n.

Given a labelled formula $z : A\mathcal{S}B$ in the antecedent of a sequent $\Gamma \Rightarrow \Delta$, we say that a label z' is a past witness *for $z : A\mathcal{S}B$ if $z' : B$ is in Γ and the atoms $z' \prec z_0, \ldots, z_{n-1} \prec z_n \equiv z$ are in Γ for some n.*

We modify correspondingly the notion of future and past loop:

Definition 4.8.10. *A chain $z_i \prec z_{i+1}, \ldots, z_{j-1} \prec z_j$ (with $j \geq i+1$) in a sequent $\Gamma \Rightarrow \Delta$ is a* future loop *if z_j marks exactly the same formulas as the label z_i, for no label y the relational atom $z_j \prec y$ is in Γ, and, for every labelled formula $z_q : \mathbf{F}B, z_q : A\mathcal{U}B$ in Γ (resp. $z_q : \mathbf{G}B$ in Δ) with $i \leq q \leq j$, there exists z_k such that $i \leq k \leq j$ and $z_k : B$ is in Γ (resp. in Δ). We call z_j the* future looping label *with respect to z_i.*
A chain $z_i \prec z_{i+1}, \ldots, z_{j-1} \prec z_j$ (with $j \geq i+1$) in a sequent $\Gamma \Rightarrow \Delta$ is a past loop *if z_i marks exactly the same formulas as the label z_j, for no label y the relational atom $y \prec z_i$ is in Γ, and, for every labelled formula $z_q : \mathbf{P}B, z_q : A\mathcal{S}B$ in Γ (resp. $z_q : \mathbf{H}B$ in Δ) with $i \leq q \leq j$, there exists some variable z_k such that $i \leq k \leq j$ and $z_k : B$ is in Γ (resp. in Δ). We call z_i the* past looping label *with respect to z_j.*

Similarly, we modify also the notion of fulfilling sequents:

Definition 4.8.11. *Let us suppose that the sequent $\Gamma^* \Rightarrow \Delta^*$ has been obtained by root-first proof search from the purely logical sequent $\Gamma \Rightarrow \Delta$ (with all its formulas labelled by x). Then, $\Gamma^* \Rightarrow \Delta^*$ is a* fulfilling sequent *if the following conditions are satisfied:*

(i) Every label in it is $\prec$-saturated;

(ii) It contains a chain of relational atoms $z_{-m} \prec z_{-(m-1)}, \ldots, z_{-1} \prec z_0 \equiv x$, $z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$, such that for some $-m < i \leq 0$ the subchain

$z_{-m} \prec z_{-(m-1)}, \ldots, z_{i-1} \prec z_i$ is a past loop, and for some $0 \leq j < n$, the subchain $z_j \prec z_{j+1}, \ldots, z_{n-1} \prec z_n$ is a future loop;

(iii) Every labelled formula $z : \mathbf{F}B, z : A\mathcal{U}B$ in the antecedent (resp. $z : \mathbf{G}B$ in the succedent) is either witnessed by a future witness label z', or has z inside a future loop;

(iv) Every labelled formula $z : \mathbf{P}B, z : A\mathcal{S}B$ in the antecedent (resp. $z : \mathbf{H}B$ in the succedent) is either witnessed by a past witness label z', or has z inside a past loop.

Definition 4.8.12. *A* pre-proof *of a purely logical sequent in* $\mathrm{G3LT}_{cl} + \mathcal{U} + \mathcal{S}$ *is a (possibly infinite) proof-search tree obtained by applying root-first the logical and mathematical rules of the calculus, whenever possible. A pre-proof of a purely logical sequent* $\Gamma \Rightarrow \Delta$ *is a* proof *if no branch in it leads to a fulfilling sequent. A sequent* $\Gamma \Rightarrow \Delta$ *is* provable *if there exists a proof for it.*

The proof of soundness of $\mathrm{G3LT}_{cl} + \mathcal{U} + \mathcal{S}$ is given by a straightforward adaptation of the proofs of Lemmas 4.4.1 and 4.4.2 and of Theorem 4.4.4:

Lemma 4.8.13. *Let $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$ be a model such that for some instant s_0, $s_0 \nVdash A$. Then for some s such that $s_0 \leqslant^{\mathcal{K}} s$, there exists an instant s' such that $s \equiv s_i \prec^{\mathcal{K}} s_{i+1}, \ldots, s_{j-1} \prec^{\mathcal{K}} s_j \equiv s'$ (with $j \geq i+1$), s and s' satisfy the same subset of closure formulas of A and for every t if $s \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s'$ and $t \Vdash \mathbf{F}B$ or $t \Vdash C\mathcal{U}B$, (resp. $t \nVdash \mathbf{G}B$) there exists u such that $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s'$ and $u \Vdash B$ (resp. $u \nVdash B$).*

Proof. Analogously to the proof of Lemma 4.4.1. □

Lemma 4.8.14. *Let $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$ be a model such that for some s_0, $s_0 \Vdash A$. Then for some s such that $s \leqslant^{\mathcal{K}} s_0$, there exists an instant s' such that*

$s' \equiv s_i \prec^{\mathcal{K}} s_{i+1}, \ldots, s_{j-1} \prec^{\mathcal{K}} s_j \equiv s$ (with $j \geq i+1$), s and s' satisfy the same subset of closure formulas of A and for every t if $s' \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s$ and $t \Vdash \mathbf{P}B$ or $t \Vdash C\mathcal{S}B$ (resp. $t \nVdash \mathbf{H}B$) there exists u such that $s' \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s$ and $u \Vdash B$ (resp. $u \nVdash B$).

Proof. Analogously to the proof of Lemma 4.4.2. □

Theorem 4.8.15. *If the purely logical sequent $\Gamma \Rightarrow \Delta$ is provable in the calculus* G3LT$_{cl}$ $+\mathcal{U}+\mathcal{S}$*, then it has no standard countermodels.*

Proof. Analogously to the proof of Theorem 4.4.4. □

The costruction of the countermodel $\mathcal{M}$ is obtained by defining an appropriate interpretation of the fulfilling sequent into the standard frame for Priorean linear time $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$, as in Section 4.5.

Lemma 4.8.16. *$\mathcal{M}$ is a countermodel for $\Gamma^* \Rightarrow \Delta^*$.*

Proof. By the proof of Lemma 4.5.1, and the following inductive clauses:

12. If $B \equiv C\mathcal{U}D$ and $z : C\mathcal{U}D$ is in Γ^*, by definition of fulfilling sequent we have two cases: (i) there exists some future witness label z' such that $z' : D$ is in Γ^* and the atoms $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z'$ are in Γ^* for some $i, j \geq 0$. Let us consider the leftmost future witness z': By construction and inductive hypothesis there is some $s = [\![z']\!]$ such that $[\![z]\!] <^{\mathcal{K}} s$ and $s \Vdash D$. Furthermore, since z is $\prec$-saturated, either $z : \mathbf{T}D$ or both $z : \mathbf{T}C$ and $z : \mathbf{T}(C\mathcal{U}D)$ are Γ^*, and, if $z \prec z''$ is in Γ^*, then either $z'' : D$ or both $z'' : C$ and $z'' : C\mathcal{U}D$ are Γ^*. By repeating this argument, we have that for every label y, if the atoms $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv y$, $y \prec z_{i+j+1}, \ldots, z_{i+j+l-1} \prec z_{i+j+l} \equiv z'$ are in Γ^* for some $i, j, l \geq 0$, then $y : C$ and $y : C\mathcal{U}D$ are in Γ^*. By construction

and inductive hypothesis, for every instant s' if $[\![z]\!] <^{\mathcal{K}} s' <^{\mathcal{K}} s$, then $s' \Vdash C$. Therefore, $[\![z]\!] \Vdash C\mathcal{U}D$. (ii) the label z is inside a future loop $z_n \prec z_{n+1}$, $\ldots, z_{n+i-1} \prec z_{n+i} \equiv z$, $z_{n+i} \prec z_{n+i+1}, \ldots, z_{n+l-1} \prec z_{n+l}$ (with $l \geq i$). Then there exists z' such that either $z_n \equiv z'$ or $z_n \prec z_{n+1}, \ldots, z_{n+q-1} \prec z_{n+q} \equiv z'$ are in Γ^* for $0 \leq q \leq i$ and the labelled formula $z' : D$ is in Γ^*: Let us consider the leftmost such z'. By construction $[\![z']\!] = s_{n+q}$, so $[\![z]\!] <^{\mathcal{K}} s_{n+l+q}$ and, by inductive hypothesis, $s_{n+l+q} \Vdash D$. Furthermore, since z is $\prec$-saturated, either $z : \mathbf{T}D$ or both $z : \mathbf{T}C$ and $z : \mathbf{T}(C\mathcal{U}D)$ are Γ^*, and, if $z \prec z''$ is in Γ^*, then either $z'' : D$ or both $z'' : C$ and $z'' : C\mathcal{U}D$ are Γ^*. By repeating this argument, we have that for every label y, if the atoms $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv y$ are in Γ^* for some $i, j, l \geq 0$, then $y : C$ and $y : C\mathcal{U}D$ are in Γ^*. Since z_n and z_{n+l} mark the same formula, $z_n : C$ and $z_n : C\mathcal{U}D$ are in Γ^* and, from the previous argument follows that for every i if $n \leq i < n+q$, then $z_i : C$ and $z_i : C\mathcal{U}D$ are in Γ^*. By construction and inductive hypothesis, for every s', if $[\![z]\!] <^{\mathcal{K}} s' <^{\mathcal{K}} s_{n+l+q}$, then $s' \Vdash C$. Therefore, $[\![z]\!] \Vdash C\mathcal{U}D$.

If $B \equiv C\mathcal{U}D$ and $z : C\mathcal{U}D$ is in Δ^*, then, since z is $\prec$-saturated, both $z : \mathbf{T}C$ and $z : \mathbf{T}D$ or both $z : \mathbf{T}D$ and $z : \mathbf{T}(C\mathcal{U}D)$ are in Δ^*. In the former case, by point 6 of Lemma 4.5.1, there exists an instant s s.t. $[\![z]\!] \prec^{\mathcal{K}} s$, $s \nVdash C$, $s \nVdash D$ and, by discreteness, for every s' such that $[\![z]\!] <^{\mathcal{K}} s'$ and $s' \Vdash D$, if any, $s <^{\mathcal{K}} s'$. Therefore, $[\![z]\!] \nVdash C\mathcal{U}D$. In the latter case, since z is $\prec$-saturated, if $z \prec z'$ is in Γ^*, then both $z' : D$ and $z' : C\mathcal{U}D$ are in Δ^*. By repeating this argument, we have that for every label z'', such that $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z''$ are in Γ^* for $i, j \geq 0$, both $z'' : D$ and $z'' : C\mathcal{U}D$ are in Δ^*. Note that, if z is the future looping label or z'' is inside a future loop $z_m \prec z_{m+1}, \ldots, z_{n-1} \prec z_n$ (with $n > m$) both $z_k : D$ and $z_k : C\mathcal{U}D$ are in Δ^* for every $m \leq k \leq n$. By inductive hypothesis for every s if $[\![z]\!] <^{\mathcal{K}} s$ then $s \nVdash D$, therefore $[\![z]\!] \nVdash C\mathcal{U}D$.

13. If $B \equiv C\mathcal{S}D$ and $z : C\mathcal{S}D$ is in Γ^*, by definition of fulfilling sequent we have two cases: (i) there exists some past witness label z' such that $z' : D$ is in Γ^* and the atoms $z' \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z$ are in Γ^* for some $i, j \geq 0$. Let us consider the rightmost past witness z': By construction and inductive hypothesis there is some $s = [\![z']\!]$ such that $s <^{\mathcal{K}} [\![z]\!]$ and $s \Vdash D$. Moreover, since z is $\prec$-saturated, either $z : \mathbf{Y}D$ or both $z : \mathbf{Y}C$ and $z : \mathbf{Y}(C\mathcal{S}D)$ are Γ^*, and, if $z'' \prec z$ is in Γ^*, then either $z'' : D$ or both $z'' : C$ and $z'' : C\mathcal{S}D$ are Γ^*. By repeating this argument, we have that for every y, if the relational atoms $z' \prec z_i$, $\ldots, z_{i+j-1} \prec z_{i+j} \equiv y, y \prec z_{i+j+1}, \ldots, z_{i+j+l-1} \prec z_{i+j+l} \equiv z$ are in Γ^* for some $i, j, l \geq 0$, then $y : C$ and $y : C\mathcal{S}D$ are in Γ^*. By construction and inductive hypothesis, for every s' if $s <^{\mathcal{K}} s' <^{\mathcal{K}} [\![z]\!]$, then $s' \Vdash C$. Then, $[\![z]\!] \Vdash C\mathcal{S}D$. (ii) label z is inside a past loop $z_{-(n+l)} \prec z_{-(n+l-1)}, \ldots, z_{-(n+i+1)} \prec z_{-(n+i)} \equiv z$, $z_{-(n+1)} \prec z_{-(n-(i-1))}, \ldots, z_{-(n-1)} \prec z_{-n}$ (with $l \geq i$). There exists z' s.t. either $z_{-n} \equiv z'$ or the atoms $z' \equiv z_{-(n+q)} \prec z_{-(n+(q-1))}, \ldots, z_{-(n+1)} \prec z_{-n}$ are in Γ^* for $0 \leq q \leq i$ and the labelled formula $z' : D$ is in Γ^*: Let us consider the rightmost such z'. By construction $[\![z']\!] = s_{-(n+q)}$, so $s_{-(n+l+q)} <^{\mathcal{K}} [\![z]\!]$ and, by inductive hypothesis, $s_{-(n+l+q)} \Vdash D$. Furthermore, since z is $\prec$-saturated, either $z : \mathbf{Y}D$ or both $z : \mathbf{Y}C$ and $z : \mathbf{Y}(C\mathcal{S}D)$ are Γ^*, and, if $z'' \prec z$ is in Γ^*, then either $z'' : D$ or both $z'' : C$ and $z'' : C\mathcal{S}D$ are Γ^*. By repeating this argument, we have that for every label y, if $y \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z$ are in Γ^* for some $i, j, l \geq 0$, then $y : C$ and $y : C\mathcal{S}D$ are in Γ^*. Since z_n and z_{n+l} mark the same formula, $z_{-n} : C$ and $z_{-n} : C\mathcal{S}D$ is in Γ^* and, from the previous argument follows that for every i, if $-(n+q) < i \leq -n$, then $z_i : C$ and $z_i : C\mathcal{S}D$ are in Γ^*. Therefore, by construction and inductive hypothesis,

for every s if $s_{-(n+l+q)} <^{\mathcal{K}} s <^{\mathcal{K}} [\![z]\!]$, then $s \Vdash C$. Therefore, $[\![z]\!] \Vdash C\mathcal{S}D$.

If $B \equiv C\mathcal{S}D$ and $z : C\mathcal{S}D$ is in Δ^*, then, since z is $\prec$-saturated, both $z : \mathbf{Y}C$ and $z : \mathbf{Y}D$ or both $z : \mathbf{Y}D$ and $z : \mathbf{Y}(C\mathcal{S}D)$ are in Δ^*. In the former case, by point 9 of Lemma 4.5.1, we have that, there exists an instant s s.t. $s \prec^{\mathcal{K}} [\![z]\!]$, $s \nVdash C$, $s \nVdash D$ and, by discreteness, for every s' such that $s' <^{\mathcal{K}} [\![z]\!]$ and $s' \Vdash D$, if any, $s' <^{\mathcal{K}} s$. Therefore, $[\![z]\!] \nVdash C\mathcal{S}D$. In the latter case, since z is $\prec$-saturated, if $z' \prec z$ is in Γ^*, then both $z' : D$ and $z' : C\mathcal{S}D$ are in Δ^*. By repeating this argument, we have that for every label z'', such that $z'' \prec z_i, \dots, z_{i+j-1} \prec z_{i+j} \equiv z$ are in Γ^* for $i, j \geq 0$, $z'' : D$ and $z'' : C\mathcal{S}D$ in Δ^*. Note that, if z is the past looping label or z'' is inside a past loop $z_m \prec z_{m+1}, \dots, z_{n-1} \prec z_n$ (with $n > m$) both $z_k : D$ and $z_k : C\mathcal{S}D$ are in Δ^* for every $m \leq k \leq n$. By inductive hypothesis for every s if $[\![z]\!] <^{\mathcal{K}} s$ then $s \nVdash D$, therefore $[\![z]\!] \nVdash C\mathcal{U}D$. □

Again, a straightforward adaptation of the results in Section 4.6 gives the following results:

Lemma 4.8.17. *Consider a purely logical sequent $\Gamma \Rightarrow \Delta$ with all the formulas in it labelled by x. Let us suppose that the proof search for $\Gamma \Rightarrow \Delta$ leads to a sequent $\Gamma' \Rightarrow \Delta'$: if the chain $y_{-m} \prec y_{-(m-1)}, \dots, y_{-1} \prec y_0 \equiv x$ and the chain $x \equiv y_0 \prec y_1, \dots, y_{n-1} \prec y_n$ in it are not roundabout then the number of labels has a bound on the order of the length of $A \equiv \wedge\Gamma^x \supset \vee\Delta^x$, namely $m, n \leq \sum_{i=1}^{2^{3|A|+1}} i$.*

Theorem 4.8.18. *Proof search for* G3LT$_{cl}$ *terminates.*

We conclude this section with a comparison of G3LT$_{cl} + \mathcal{U} + \mathcal{S}$ with the calculus G3LT$+\mathcal{U} + \mathcal{S}$.

Lemma 4.8.19. *The rules $L\mathcal{U}_{cl}$, $R\mathcal{U}_{cl}$, $L\mathcal{S}_{cl}$ and $R\mathcal{S}_{cl}$ are admissible in the calculus* G3LT+$\mathcal{U}+\mathcal{S}$.

Proof. The proof consists in the following derivations, together with admissibility of cut in G3LT+$\mathcal{U}+\mathcal{S}$. Notice that some repetitions are omitted and y is chosen different from x and not in Γ, Δ.

Rule $L\mathcal{U}_{cl}$:

$$\dfrac{\dfrac{\dfrac{x \prec y, y : B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta \quad x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}{x \prec y, x : A\mathcal{U}B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\;{\scriptstyle L\mathcal{U}}}{x \prec y, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\;{\scriptstyle LCtr^*}}{x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\;{\scriptstyle R\text{-}Ser}$$

where the premises of $L\mathcal{U}$ are derived as follows

$$\dfrac{x \prec y, y : B \Rightarrow x : \mathbf{T}B \quad x : \mathbf{T}B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}{x \prec y, y : B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\;{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\dfrac{x \prec y, y : A\mathcal{U}B \Rightarrow x : \mathbf{T}(A\mathcal{U}B) \quad \dfrac{x \prec y, y : A \Rightarrow x : \mathbf{T}A \quad x : \mathbf{T}A, x : \mathbf{T}(A\mathcal{U}B), x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}{x \prec y, y : A, x : \mathbf{T}(A\mathcal{U}B), x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\;{\scriptstyle Cut}}{x \prec y, x \prec y, y : A, y : A\mathcal{U}B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\;{\scriptstyle Cut}}{x \prec y, y : A, y : A\mathcal{U}B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\;{\scriptstyle LCtr^*}}{x \prec y, y : A, y : A\mathcal{U}B, x : \mathbf{F}B, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta}\;{\scriptstyle LWk}$$

Notice that the sequents $x \prec y, y : B \Rightarrow x : \mathbf{T}B$ and $x \prec y, y : A \Rightarrow x : \mathbf{T}A$, and the sequent $x \prec y, y : A\mathcal{U}B \Rightarrow x : \mathbf{T}(A\mathcal{U}B)$ are obtained by rules $EqSubst$, $UnSucc$ and $R\mathbf{T}$.

Rule $R\mathcal{U}_{cl}$:

$$\dfrac{\begin{array}{c} x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A \\ x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A\mathcal{U}B \\ x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, x : \mathbf{F}B \end{array}}{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B}\;{\scriptstyle R\mathcal{U}}$$

where the premises of $R\mathcal{U}$ are derived as follows

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B, x : \mathbf{T}B, x : \mathbf{T}A \quad \dfrac{x \prec y, y : B \Rightarrow y : B}{x \prec y, x : \mathbf{T}B \Rightarrow y : B}\;{\scriptstyle L\mathbf{T}}}{x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, x : \mathbf{T}A}\;{\scriptstyle Cut} \quad \dfrac{x \prec y, y : A \Rightarrow y : A}{x \prec y, x : \mathbf{T}A \Rightarrow y : A}\;{\scriptstyle L\mathbf{T}}}{x \prec y, x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A}\;{\scriptstyle Cut}}{x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A}\;{\scriptstyle LCtr^*}$$

$$
\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B, x : \mathbf{T}B, x : \mathbf{T}(A\mathcal{U}B) \quad \dfrac{x \prec y, y : B \Rightarrow y : B}{x \prec y, x : \mathbf{T}B \Rightarrow y : B}{\scriptstyle L\mathbf{T}}}{x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, x : \mathbf{T}(A\mathcal{U}B)}{\scriptstyle Cut} \quad \dfrac{x \prec y, y : A\mathcal{U}B \Rightarrow y : A\mathcal{U}B}{x \prec y, x : \mathbf{T}(A\mathcal{U}B) \Rightarrow y : A\mathcal{U}B}{\scriptstyle L\mathbf{T}}}{\dfrac{x \prec y, x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A\mathcal{U}B}{x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A\mathcal{U}B}{\scriptstyle LCtr^*}}{\scriptstyle Cut}
$$

$$
\dfrac{x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, y : A\mathcal{U}B \quad \dfrac{y : A\mathcal{U}B \Rightarrow y : \mathbf{F}B \quad \dfrac{\dfrac{\dfrac{\dfrac{x < z, z : B \Rightarrow z : B}{x < z, z : B \Rightarrow x : \mathbf{F}B}{\scriptstyle R\mathbf{F}}}{x < y, y < z, z : B \Rightarrow x : \mathbf{F}B}{\scriptstyle Trans}}{x < y, y : \mathbf{F}B \Rightarrow x : \mathbf{F}B}{\scriptstyle L\mathbf{F}}}{x \prec y, y : \mathbf{F}B \Rightarrow x : \mathbf{F}B}{\scriptstyle Inc}}{x \prec y, y : A\mathcal{U}B \Rightarrow x : \mathbf{F}B}{\scriptstyle Cut}}{\dfrac{x \prec y, x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, x : \mathbf{F}B}{x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : B, x : \mathbf{F}B}{\scriptstyle LCtr^*}}{\scriptstyle Cut}
$$

and the sequent $y : A\mathcal{U}B \Rightarrow y : \mathbf{F}B$ is derivable in G3LT+$\mathcal{U}+\mathcal{S}$ by Proposition 3.5.7.

Rule $L\mathcal{S}_{cl}$:

$$
\dfrac{\dfrac{\dfrac{y \prec x, y : B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta \quad y \prec x, y : A, y : A\mathcal{S}B, x : \mathbf{P}B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{y \prec x, x : A\mathcal{S}B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{\scriptstyle L\mathcal{S}}}{y \prec x, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{\scriptstyle LCtr^*}}{x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{\scriptstyle L\text{-}Ser}
$$

where the premises of $L\mathcal{S}$ are derived as follows

$$
\dfrac{y \prec x, y : B \Rightarrow x : \mathbf{Y}B \quad x : \mathbf{Y}B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{y \prec x, y : B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{\scriptstyle Cut}
$$

$$
\dfrac{y \prec x, y : A\mathcal{S}B \Rightarrow x : \mathbf{Y}(A\mathcal{S}B) \quad \dfrac{y \prec x, y : A \Rightarrow x : \mathbf{Y}A \quad x : \mathbf{Y}A, x : \mathbf{Y}(A\mathcal{S}B), x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{y \prec x, y : A, x : \mathbf{Y}(A\mathcal{S}B), x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{\scriptstyle Cut}}{\dfrac{\dfrac{y \prec x, y \prec x, y : A, y : A\mathcal{S}B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{y \prec x, y : A, y : A\mathcal{S}B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{\scriptstyle LCtr^*}}{y \prec x, y : A, y : A\mathcal{S}B, x : \mathbf{P}B, x : A\mathcal{S}B, \Gamma \Rightarrow \Delta}{\scriptstyle LWk}}{\scriptstyle Cut}
$$

note that the sequents $y \prec x, y : B \Rightarrow x : \mathbf{Y}B$ and $x \prec y, y : A \Rightarrow x : \mathbf{Y}A$, and the sequent $y \prec x, y : A\mathcal{S}B \Rightarrow x : \mathbf{Y}(A\mathcal{S}B)$ are obtained by rules $EqSubst$, $UnPred$ and $R\mathbf{Y}$.

Rule $R\mathcal{S}_{cl}$:

$$
\dfrac{\begin{array}{c} y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A \\ y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A\mathcal{S}B \\ y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, x : \mathbf{P}B \end{array}}{\Gamma \Rightarrow \Delta, x : A\mathcal{S}B}{\scriptstyle R\mathcal{S}}
$$

where the premises of $R\mathcal{S}$ are derived as follows

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A\mathcal{S}B, x : \mathbf{Y}B, x : \mathbf{Y}A \quad \dfrac{y \prec x, y : B \Rightarrow y : B}{y \prec x, x : \mathbf{Y}B \Rightarrow y : B}{\scriptstyle L\mathbf{Y}}}{y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, x : \mathbf{Y}A}{\scriptstyle Cut} \quad \dfrac{y \prec x, y : A \Rightarrow y : A}{y \prec x, x : \mathbf{Y}A \Rightarrow y : A}{\scriptstyle L\mathbf{Y}}}{\dfrac{y \prec x, y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A}{y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A}{\scriptstyle LCtr^*}}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A\mathcal{S}B, x : \mathbf{Y}B, x : \mathbf{Y}(A\mathcal{S}B) \quad \dfrac{y \prec x, y : B \Rightarrow y : B}{y \prec x, x : \mathbf{Y}B \Rightarrow y : B}{\scriptstyle L\mathbf{Y}}}{y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, x : \mathbf{Y}(A\mathcal{S}B)}{\scriptstyle Cut} \quad \dfrac{y \prec x, y : A\mathcal{S}B \Rightarrow y : A\mathcal{S}B}{y \prec x, x : \mathbf{Y}(A\mathcal{S}B) \Rightarrow y : A\mathcal{S}B}{\scriptstyle L\mathbf{Y}}}{\dfrac{y \prec x, y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A\mathcal{S}B}{y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A\mathcal{S}B}{\scriptstyle LCtr^*}}{\scriptstyle Cut}$$

$$\dfrac{y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A\mathcal{S}B \quad \dfrac{y : A\mathcal{S}B \Rightarrow y : \mathbf{P}B \quad \dfrac{\dfrac{\dfrac{\dfrac{z < x, z : B \Rightarrow z : B}{z < x, z : B \Rightarrow x : \mathbf{P}B}{\scriptstyle R\mathbf{P}}}{z < y, y < x, z : B \Rightarrow x : \mathbf{P}B}{\scriptstyle Trans}}{y < x, y : \mathbf{P}B \Rightarrow x : \mathbf{P}B}{\scriptstyle L\mathbf{P}}}{y \prec x, y : \mathbf{P}B \Rightarrow x : \mathbf{P}B}{\scriptstyle Inc}}{y \prec x, y : A\mathcal{S}B \Rightarrow x : \mathbf{P}B}{\scriptstyle Cut}}{\dfrac{y \prec x, y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, y : A\mathcal{S}B}{y \prec x, \Gamma \Rightarrow \Delta, x : A\mathcal{S}B, y : B, x : \mathbf{P}B}{\scriptstyle LCtr^*}}{\scriptstyle Cut}$$

and the sequent $y : A\mathcal{S}B \Rightarrow y : \mathbf{P}B$ is derivable in G3LT+$\mathcal{U} + \mathcal{S}$ by Proposition 3.5.7. □

Theorem 4.8.20. *If the purely logical sequent* $\Gamma \Rightarrow \Delta$ *is derivable in the calculus* G3LT$_{cl}$ $+\mathcal{U} + \mathcal{S}$*, then it is derivable in* G3LT+$\mathcal{U} + \mathcal{S}$*.*

Finally, soundness and completeness of derivability in G3LT+$\mathcal{U} + \mathcal{S}$ and of provability in G3LT$_{cl}$ $+\mathcal{U} + \mathcal{S}$ give the following result:

Theorem 4.8.21. *The purely logical sequent* $\Gamma \Rightarrow \Delta$ *is derivable in the calculus* G3LT+$\mathcal{U} + \mathcal{S}$ *iff it is provable in* G3LT$_{cl}$ $+\mathcal{U} + \mathcal{S}$*.*

Remark 4.8.22. What is commonly known as propositional linear time logic (LTL) is the future-oriented reflexive version of Priorean linear time logic: only the future operators **G**, **F**, **T**, and $\mathcal{U}$ are considered in LTL, and **G** and **F** have the intuitive meanings of 'it is and will always be the case' and 'it is or will be the case', respectively. Note that the last condition corresponds to the substitution of the irreflexive relation $<$ with the reflexive relation $\leqslant$ in the intended explanations in terms of the relational semantics.

Linear time logic is known to be decidable (Sistla and Clarke 1985). Decidability has been stated in several papers (Wolper 1985, Kesten *et al.* 1993, Lichtenstein and Pnueli 2000) through 2-phase tableau systems: in such systems, after the construction of the tableau graph, a second phase is required in order to check whether every eventuality formula has been satisfied. In Schmitt and Goubalt-Larrecq (1997) a tableau system has been proposed, in which the termination of the proof-search procedure can be determined locally, but the system covers only a limited fragment of LTL. In Schwendimann (1998) a decision procedure for the whole logic has been achieved through a tableau calculus in which the second phase is local and incorporated into the rules by annotating sets of formulas with history information. However, the system contains a non-local closing rule, which terminates a branch whenever a loop is met: thus, information should be kept on previous parts of a derivation in order to check if an earlier node (prestate) is reachable from the current one.

As noticed above in Section 4.3, the definition of proofs in $\mathrm{G3LT}_{cl}$ is completely local, and termination can be determined with no need of keeping information on previous parts of derivations. Furthermore, the use of labels supplies an immediate and simple construction of a countermodel for an unprovable sequent. Finally, the calculus $\mathrm{G3LT}_{cl}$ and its extension (Section 4.8) contains also past temporal operators, analogously to the LTL counterpart considered in Lichtenstein and Pnueli (2000), but our decision procedure is given in the stronger form of an explicit bound on proof search, although the absence of a global condition on derivations imposes an exponential size on it.

Conclusion and further work

In this thesis, we applied the method of the internalisation of Kripke-style semantics into the syntax of sequent calculus to the proof-theoretical analysis of temporal logics.

The choice of the methodology was motivated both by philosophical reasons, connected to the very birth of tense logic at the hand of Prior, and by the consideration of its generality with respect to several temporal systems. From the first point of view, the adopted method exploits the natural interpretation of temporal logics into a first-order monadic logic for the accessibility relation, and we claimed that, if the elements from the latter are considered as parts of the syntax, relational semantics can be considered as a useful formal device rather than a metaphysical commitment on the real nature of time.

As far as effectiveness is concerned, we showed that labelled calculi with internalised semantics allow to deal with uniformly a wide range of temporal logics for different flows of time: the logical rules for the temporal operators **G**, **F**, **H**, and **P** were formulated on the base of their meaning explanations in terms of the intended relational semantics, and several different systems were obtained modularly as extensions of the basic temporal calculus G3K$_t$ by means of the mathematical rules corresponding to the frame properties for the accessibility relation $<$. As a consequence, important structural properties were proved at once for all such systems: all the calculi enjoy height-preserving admissibility of the substitution of labels, all the rules are height-preserving invertible, the structural rules of weakening and contraction are height-preserving admissible, and cut elimination is proved in a purely syntactical way.

The calculus G3K$_t$ and its extensions enjoy the weak subformula property, that is, every formula in a derivation is a subformula of the endsequent or a relational atomic formula. The lack of a full subformula property is somehow compensated by the subterm property, that states that every label in a derivation is a label in the endsequent or an eigenvariable, and thus guarantees a strict control over the labels progressively introduced in a derivation by root-first application of the rules.

The case of Priorean linear discrete time was considered in detail through the analysis of two infinitary calculi. The calculus G3LT is an extension of G3K$_t$ with the logical rules for temporal operators $\mathbf{T}$ and $\mathbf{Y}$ for the next and the previous moment, and with the mathematical rules for equality and for the accessibility relations $\prec$ and $<$. In particular, an infinitary rule was required, which defines the order relation $<$ as the transitive closure of the immediate successor relation $\prec$: if x is less than y, then y is the immediate successor of x, or the immediate successor of the immediate successor of x, or ... and so on. As an application of the general results, G3LT has all the remarkable structural properties cited above: since the calculus contains mathematical rules that act on atomic formulas both in the left- and in the right-hand side of the sequents, syntactic cut elimination was obtained by means of a measure of complexity for relational atoms.

The calculus G3LT was proved sound with respect to Kripke semantics thanks to the correspondence of the logical rules with the notion of validity in a Kripke model, and of the mathematical rules with the properities of the intended class of frames. Completeness followed from the fact that the Hilbert-style system for Priorean linear time can be embedded into the calculus G3LT.

Two partial finitisations for G3LT were achieved through conservativity results. A weaker system G3LT$_{n\text{-}s}$ was formulated by substituting the infinitary rule with a pair of finitary rules that permit the splitting of an interval $[x, y]$ with an immediate successor of x, and an immediate predecessor of y, respectively. Proof-theoretical analysis allowed to identify an appropriate fragment of G3LT for which conservativity with respect to G3LT$_{n\text{-}s}$ was proved: if a sequent does not contain relational atoms and the operators $\mathbf{G}$, $\mathbf{H}$ do not appear in its positive part, nor $\mathbf{F}$, $\mathbf{P}$ in its negative part, then it is derivable in G3LT if and only if it is derivable in G3LT$_{n\text{-}s}$.

Next, the system G3LT$^{\delta}$ was obtained by replacing in the future fragment G3LT^{f} of G3LT the infinitary rule with a finitary one, in which the number of the premises is bounded in a purely syntactical way, by simply counting the occurrences of the operator $\mathbf{T}$ in them. We showed that the finitised rule is as strong as the infinitary rule for the derivation of a sequent, if the latter does not contain relational atoms and the operator $\mathbf{G}$ does not appear in its negative part nor $\mathbf{F}$ in its positive part.

The second infinitary calculus, G3LT$_{cl}$, was obtained through a different formulation of the rules for temporal operators, reflecting a natural closure algorithm that exploits the fixed-point properties of **G**, **F**, **H**, and **P**. All the rules of the system G3LT$_{cl}$ are finitary, however, proofs are generally constituted by derivation trees containing (at least) an infinite branch.

Then, we stated soundness and completeness of G3LT$_{cl}$ with respect to an appropriate notion of provability, defined by imposing an adequate condition on derivation trees: a sequent is provable in G3LT$_{cl}$ if and only if no branch leads to a so-called 'fulfilling sequent', which is the syntactical counterpart of a countermodel for an invalid sequent. Thanks to the use of labels such condition is completely local, and there is no need of keeping information on previous parts of the derivation tree.

Decidability was also proved through a terminating proof-search procedure, in the form of an exponential upper bound to the branches of derivation trees for valid sequents, calculated on the length of the temporal formula corresponding to the endsequent.

Finally, extensions with logical rules for the temporal operators Until and Since were considered both for G3LT and for G3LT$_{cl}$, and the main results adapted to the calculi thus obtained.

Linear discrete time unbounded in both directions was considered in the present work as the privileged case study because of its interest and importance in the literature. Further temporal systems are worth being analised through the methodology illustrated. For example, the idea that time had a first instant is maintained for metaphysical reason by those who believe in Creationism, but has technical application in the specification and verification of reactive systems.

The temporal formula describing this situation is

$$\mathbf{H}\bot \vee \mathbf{PH}\bot$$

Since no instant can force falsity, the formula above means that a given instant either has no predecessor or is preceded by an instant without any predecessor. In a left-linear time flow, the axiom corresponds to the frame property

$$\exists x \forall y\, (x = y \vee x < y)$$

which is not expressible through a universal axiom, and not even through a geometric implication. However, we can denote the initial instant by a constant 0 and reformulate this frame property by means of a universal axiom and a geometric implication:

1. $\forall x \forall y \neg\, (x = 0 \,\&\, y < x)$
2. $\forall x\, (x = 0 \vee \exists y\, (y = 0 \,\&\, y < x))$

Condition 1 states that no instant precedes the first one, whereas condition 2 is a weak form of left seriality: every instant, except for the first one, is preceded by at least another instant (namely, the initial one).

The corresponding mathematical rules are formulated as follows:

$$\frac{}{x = 0, y < x, \Gamma \Rightarrow \Delta}\,{\scriptstyle Initial} \qquad \frac{x = 0, \Gamma \Rightarrow \Delta \quad y = 0, y < x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle WkL\text{-}Ser}$$

The rule WkL-Ser has the condition that y is not in the conclusion.

The characteristic sequent $\Rightarrow x : \mathbf{H}\bot, x : \mathbf{PH}\bot$ is easily derivable if the rules for the initial instant are added to G3K$_t$

$$\frac{\dfrac{\dfrac{}{z < x, x = 0 \Rightarrow z : \bot, x : \mathbf{PH}\bot}\,{\scriptstyle Initial}}{x = 0 \Rightarrow x : \mathbf{H}\bot, x : \mathbf{PH}\bot}\,{\scriptstyle R\mathbf{H}} \qquad \dfrac{\dfrac{\dfrac{}{y = 0, w < y, y < x \Rightarrow x : \mathbf{H}\bot, x : \mathbf{PH}\bot, w : \bot}\,{\scriptstyle Initial}}{y = 0, y < x \Rightarrow x : \mathbf{H}\bot, x : \mathbf{PH}\bot, y : \mathbf{H}\bot}\,{\scriptstyle R\mathbf{H}}}{y = 0, y < x \Rightarrow x : \mathbf{H}\bot, x : \mathbf{PH}\bot}\,{\scriptstyle R\mathbf{P}}}{\Rightarrow x : \mathbf{H}\bot, x : \mathbf{PH}\bot}\,{\scriptstyle WkL\text{-}Ser}$$

If linear discrete time is considered, we can modify the axioms expressing the properties for the initial instant by replacing the order relation $<$ with the immediate successor relation $\prec$:

1′. $\forall x \forall y \neg (x = 0 \,\&\, y \prec x)$

2′. $\forall x\, (x = 0 \vee \exists y\, (y \prec x))$

These are turned into the mathematical rules

$$\frac{}{x = 0, y \prec x, \Gamma \Rightarrow \Delta}\, Initial' \qquad \frac{x = 0, \Gamma \Rightarrow \Delta \quad y \prec x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, WkL\text{-}Ser'$$

The rule WkL-Ser' has the condition that y is not in the conclusion.

The rule WkL-Ser' typically produces proofs by infinite descent: an interesting question is then whether results similar to those of Chapter 4 can be proved for linear discrete time with a first instant (temporal frames isomorphic to $\mathbb{N}$) through the calculus obtained by substituting in G3LT$_{cl}$ the rule for left seriality L-Ser with the rules $Initial'$ and WkL-Ser'.

Riassunto in italiano

La logica del tempo

La fecondità dell'analisi logica del tempo è testimoniata dalla lunga tradizione filosofica che, a partire dalle riflessioni aristoteliche sui futuri contingenti (il cosiddetto argomento della battaglia navale) e dall'*Argomento vittorioso* di Diodoro Crono a sostegno del determinismo, si è poi dipanata nelle riflessioni che su questi e altri problemi connessi alla concezione del tempo e del suo scorrere sono state condotte a più riprese con motivazioni e soluzioni diverse da pensatori medievali come Pietro da Rivo, Buridano e Ockham.

È proprio dallo studio della filosofia e della logica medievale che il padre della moderna logica temporale, Arthur N. Prior, trasse impulso e ispirazione per lo sviluppo della nuova disciplina. Il giovane Prior, educato alla religione protestante (metodista prima, presbiteriana in seguito), credeva che la dottrina della prescienza divina fosse incompatibile con l'indeterminismo, ma non era per questo disposto a rinunciare al libero arbitrio e aveva cercato nella teologia medievale una soluzione al suo dilemma. Prior riteneva che, sebbene la scelta tra determinismo e indeterminismo fosse in qualche modo una questione di inclinazione personale, tuttavia l'analisi logica del tempo costituisse uno strumento indispensabile per esplicitare i presupposti nascosti e le più remote conseguenze, che i sostenitori dell'una o dell'altra posizione dovevano essere disposti ad accettare per non cadere in contraddizione.

In quei testi riscoprì l'idea che il valore di verità di una proposizione come 'Socrate è seduto' può mutare al variare dei riferimenti di tempo. Questa posizione era unanimemente (anche se implicitamente) condivisa dai pensatori antichi e medievali, ma lo stretto legame della logica moderna con la matematica, in cui le proposizioni sono valutate in maniera atemporale, aveva prodotto in tempi moderni l'emergere di una diversa concezione, secondo la quale una proposizione è considerata incompleta ai fini della determinazione del suo valore di verità, a meno di specificare eventuali riferimenti temporali tali da garantire che sia inalterabilmente vera o inalterabilmente falsa.

Il lavoro di Findlay (1941) suggerì, inoltre, a Prior di studiare la logica del tempo per mezzo degli strumenti offerti dalla nascente logica modale, cosa che puntualmente fece a partire dal libro del 1957, che non a caso reca il titolo di *Time and Modality*. In analogia con la logica modale del possibile e del necessario, Prior introdusse un operatore temporale futuro **F**, 'si darà il caso che' e un operatore temporale passato **P**, 'si è dato il caso che', con i rispettivi duali **G** e **H** ('sarà sempre il caso che' ed 'è sempre stato il caso che'). Dal punto di vista semantico, tali operatori vengono interpretati in opportune strutture di tipo kripkiano, chiamate *Prior Frames*, in cui i mondi possibili rappresentano gli istanti temporali e le relazioni di accessibilità sono relazioni d'ordine parziale.

La logica temporale è pertanto una logica bimodale, in cui cioè sono presenti due sistemi di operatori modali, ciascuno dotato di una propria relazione di accessibilità. D'altra parte, una caratteristica fondamentale della semantica temporale è rappresentata dal fatto che le due relazioni non sono indipendenti, ma sono l'una la conversa dell'altra. Tale caratteristica è espressa dall'equivalenza $x < y \equiv y > x$ e si traduce nella validità in tutte le strutture di Prior delle formule $p \supset \mathbf{GP}p$ e $p \supset \mathbf{HF}p$.

Come nel caso della logica modale, infatti, anche nella logica temporale è possibile trovare una corrispondenza tra (molte delle) proprietà della relazione d'ordine e assiomi caratteristici della logica corrispondente: ad esempio la proprietà della transitività corrisponde alla formula $\mathbf{FF}p \supset \mathbf{F}p$ (o, equivalentemente, $\mathbf{PP}p \supset \mathbf{P}p$). Esistono tuttavia notevoli eccezioni, come nel caso dell'ariflessività, $\forall x \neg (x < x)$, che non può essere espressa per mezzo di una formula temporale.

Lo studio della logica temporale ha ricevuto in tempi recenti un enorme impulso a seguito del suo impiego in ambito informatico nella specificazione e nella verificazione dei sistemi reattivi. Tuttavia, l'enfasi sull'aspetto applicativo in questo contesto favorisce in genere l'analisi del solo tempo futuro, a discapito di un approccio più completo che tenga in considerazione anche gli eventi passati.

Il crescente interesse manifestato nei confronti della logica del tempo ha fatto sì che l'analisi semantica della logica temporale sia stata approfondita sia nei suoi fondamenti filosofici (Schindler 1970, van Benthem 1984, Goldblatt 1992) che nelle sue potenzialità applicative (Gabbay *et al.* 1980, Manna and Pnueli 1981, Lichtenstein and Pnueli 2000, Huth and Ryan 2004).

Al contrario, una soddisfacente analisi sintattica della logica temporale si segnala tuttora per la sua assenza nell'ambito della teoria della dimostrazione e i sistemi formali che sono stati finora proposti (Nishimura 1980, Schmitt and

Goubault-Larrecq 1997, Schwendimann 1998, Bolotov *et al.* 2006, per fare qualche esempio) trattano i diversi flussi temporali separatamente e non come estensioni modulari di un calcolo temporale di base. L'unica eccezione degna di nota è costituita dalla *display logic* (Belnap 1982, Wansing 1998), la quale però deve fare i conti con una sintassi decisamente complessa.

Per questo motivo, ho deciso di dedicare la mia tesi allo sviluppo di calcoli dei sequenti per la logica temporale tramite una metodologia generale che consenta di trattare in maniera uniforme una molteplicità di sistemi per i diversi flussi temporali. Per amor di chiarezza, ricordo brevemente di seguito le caratteristiche principali del calcolo dei sequenti.

Il calcolo dei sequenti è stato formulato intorno al 1930 da Gerhard Gentzen: esso utilizza una notazione particolare, che consente di segnalare ad ogni passo della derivazione l'insieme delle assunzioni aperte da cui dipende un certo numero di conclusioni. Un sequente è un'espressione del tipo

$$A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$$

il cui significato informale è identico a quello della formula

$$(A_1 \,\&\, \ldots \,\&\, A_m) \supset (B_1 \vee \cdots \vee B_n)$$

L'antecedente $A_1, \ldots, A_m$ e il conseguente $B_1, \ldots, B_n$ del sequente (spesso abbreviati con le maiuscole greche Γ e Δ, rispettivamente) sono da intendersi come multiinsiemi di formule, ovvero liste in cui conta il numero di occorrenze di una stessa formula ma non l'ordine in cui le formule compaiono.

Il calcolo dei sequenti è costituito dalle regole logiche, che introducono le costanti logiche nella parte sinistra e destra dei sequenti, e dalle regole strutturali di *indebolimento*, *contrazione* e *taglio*, che, come il nome stesso suggerisce, non riguardano le costanti logiche, ma agiscono sulla struttura dei sequenti. Tra queste regole la più importante è senza dubbio quella del taglio (*Cut*), per mezzo della quale un teorema complesso viene scomposto in due lemmi più semplici da dimostrare, che vengono poi riuniti secondo lo schema della regola stessa:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

Si tratta di un procedimento molto comune nella pratica matematica in quanto permette di ottenere dimostrazioni più corte e, in un certo senso, più comprensibili. Tuttavia tale procedura non è del tutto innocua dal punto di vista dell'analisi della dimostrazione, in quanto introduce nella derivazione un elemento in qualche modo estraneo (la formula di taglio φ), che non appartiene cioè alla conclusione, e perciò non può essere pienamente controllato. Al contrario, da un punto di vista puramente logico, è auspicabile che in una derivazione compaiano esclusivamente i concetti richiesti dalla sua conclusione: questa istanza corrisponde, in termini formali, alla *proprietà della sottoformula*, secondo cui ogni formula che compare nella derivazione deve essere sottoformula del sequente conclusivo. Per questo motivo l'*Hauptsatz*, il (meta)teorema fondamentale del calcolo dei sequenti, stabilisce precisamente che di tale regola si può fare a meno e fornisce un procedimento effettivo per trasformare una derivazione che fa uso del taglio in una derivazione che non lo contiene.

Sfortunatamente al di fuori dell'ambito della logica pura classica e intuizionista la formulazione di calcoli dei sequenti che soddisfino buone proprietà strutturali (prima tra tutte l'eliminazione del taglio) si configura come un auspicio spesso disatteso: per molto tempo infatti è stata opinione comune che "l'*Hauptsatz* fallisce nel caso di sistemi dotati di assiomi propri"[4]. Tuttavia

[4] Girard (1987), p. 125. Traduzione mia.

Negri e von Plato (1998) e Negri (2003) hanno proposto un metodo generale per trasformare i sistemi assiomatici, che godono di determinate proprietà, in sistemi di regole non-logiche (o matematiche), che preservano l'*Hauptsatz*.

Tale metodo è stato in seguito generalizzato in Negri (2005) al fine di trattare le logiche modali come calcoli dei sequenti indicizzati, eventualmente estesi con regole matematiche, in cui la semantica di Kripke diviene parte del formalismo. In base alle considerazioni precedenti, la metodologia dell'internalizzazione della semantica relazionale nella sintassi del calcolo dei sequenti può essere applicata all'analisi della dimostrazione nella logica temporale.

Sequenti indicizzati per la logica temporale

Il punto di partenza del mio lavoro è costituito dalla formulazione di un calcolo dei sequenti per la logica temporale di base $\mathrm{G3K}_t$ a partire dal noto calcolo G3 per la logica classica proposizionale. La sintassi del calcolo viene arricchita da indici e relazioni: ciascuna formula in un sequente $\Gamma \Rightarrow \Delta$ è una formula indicizzata (*labelled*) $x : A$ o una formula atomica relazionale $x < y$. Intuitivamente, gli indici corrispondono agli istanti di tempo e gli atomi relazionali rappresentano a livello della sintassi la relazione d'ordine tra di essi.

Le regole per i connettivi proposizionali possono agire soltanto su formule indicizzate dalla stessa variabile e non coinvolgono gli atomi relazionali, mentre le regole per gli operatori temporali sono giustificate sulla base della corrispondente interpretazione semantica:

$x \Vdash \mathbf{G}A$ *sse per ogni* y, $x < y$ *implica* $y \Vdash A$

$x \Vdash \mathbf{F}A$ *sse per qualche* y, $x < y$ *e* $y \Vdash A$

$x \Vdash \mathbf{H}A$ *sse per ogni* y, $y < x$ *implica* $y \Vdash A$

$x \Vdash \mathbf{P}A$ *sse per qualche* y, $y < x$ *e* $y \Vdash A$

La tabella seguente riporta le regole del calcolo $\mathrm{G3K}_t$:

Sequenti iniziali e $L\bot$:

$$x : P, \Gamma \Rightarrow \Delta, x : P \qquad\qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\, L\bot$$

Regole proposizionali:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\, L\& \qquad\qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\, R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\, L\supset \qquad\qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, R\supset$$

Regole temporali

$$\frac{y : A, x : \mathbf{G}A, x < y, \Gamma \Rightarrow \Delta}{x : \mathbf{G}A, x < y, \Gamma \Rightarrow \Delta}\, L\mathbf{G} \qquad\qquad \frac{x < y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\, R\mathbf{G}$$

$$\frac{x < y, y : A, \Gamma \Rightarrow \Delta}{x : \mathbf{F}A, \Gamma \Rightarrow \Delta}\, L\mathbf{F} \qquad\qquad \frac{x < y, \Gamma \Rightarrow \Delta, x : \mathbf{F}A, y : A}{x < y, \Gamma \Rightarrow \Delta, x : \mathbf{F}A}\, R\mathbf{F}$$

$$\frac{y : A, x : \mathbf{H}A, y < x, \Gamma \Rightarrow \Delta}{x : \mathbf{H}A, y < x, \Gamma \Rightarrow \Delta}\, L\mathbf{H} \qquad\qquad \frac{y < x, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{H}A}\, R\mathbf{H}$$

$$\frac{y < x, y : A, \Gamma \Rightarrow \Delta}{x : \mathbf{P}A, \Gamma \Rightarrow \Delta}\, L\mathbf{P} \qquad\qquad \frac{y < x, \Gamma \Rightarrow \Delta, x : \mathbf{P}A, y : A}{y < x, \Gamma \Rightarrow \Delta, x : \mathbf{P}A}\, R\mathbf{P}$$

Le regole $R\mathbf{G}$, $L\mathbf{F}$, $R\mathbf{H}$ e $L\mathbf{P}$ sono soggette alla condizione che y non compaia nella conclusione.

Quando una classe di strutture gode di proprietà della relazione d'ordine esprimibili per mezzo di assiomi universali o di implicazioni geometriche, tali proprietà possono essere trasformate in opportune regole di inferenza matematiche secondo gli schemi introdotti in Negri e von Plato (2001) e Negri (2003).

Ad esempio il calcolo per la logica del tempo lineare senza inizio né fine si ottiene aggiungendo al calcolo $\mathrm{G3K}_t$ le regole per la transitività, la linearità verso il passato e verso il futuro e la serialità in entrambe le direzioni:

$$\frac{x < z, x < y, y < z, \Gamma \Rightarrow \Delta}{x < y, y < z, \Gamma \Rightarrow \Delta}\, Trans$$

$$\frac{y < z, y < x, z < x, \Gamma \Rightarrow \Delta \quad y = z, y < x, z < x, \Gamma \Rightarrow \Delta \quad z < y, y < x, z < x, \Gamma \Rightarrow \Delta}{y < x, z < x, \Gamma \Rightarrow \Delta}\, L\text{-}Lin$$

$$\frac{y < z, x < y, x < z, \Gamma \Rightarrow \Delta \quad y = z, x < y, x < z, \Gamma \Rightarrow \Delta \quad z < y, x < y, x < z, \Gamma \Rightarrow \Delta}{x < y, x < z, \Gamma \Rightarrow \Delta}\, R\text{-}Lin$$

$$\frac{y < x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, L\text{-}Ser \qquad \frac{x < y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, R\text{-}Ser$$

con la condizione per L-Ser e R-Ser che y non compaia nella conclusione.

Si noti in particolare che le prime tre regole corrispondono ad assiomi universali, mentre le ultime due ad implicazioni geometriche:

$\forall x \forall y \forall z\, ((x < y \,\&\, y < z) \supset x < z)$

$\forall x \forall y \forall z\, ((y < x \,\&\, z < x) \supset (y < z \vee y = z \vee z < y))$

$\forall x \forall y \forall z\, ((x < y \,\&\, x < z) \supset (y < z \vee y = z \vee z < y))$

$\forall x \exists y\, (y < x)$

$\forall x \exists y\, (x < y)$

Le regole matematiche agiscono nella parte sinistra del sequente esclusivamente sulle formule atomiche di relazione, introducendo l'atomo principale e rimuovendo quelli attivi; inoltre, gli atomi principali delle regole matematiche (e delle regole per gli operatori temporali) sono ripetuti nelle premesse. La particolare formulazione di tali regole garantisce che i calcoli estesi per loro tramite godano delle proprietà strutturali di ammissibilità (*height-preserving admissibility*) delle regole di sostituzione degli indici, di indebolimento e di contrazione. L'eliminazione sintattica della regola del taglio garantisce una forma indebolita della proprietà della sottoformula, secondo cui ogni formula che compare nella derivazione è sottoformula del sequente finale oppure un atomo relazionale. Quest'ultima è in un certo senso rafforzata dalla proprietà del sottotermine, che stabilisce che ogni indice che appare in una derivazione ricompare come indice nella conclusione oppure è un'*eigenvariable* (cioè un indice che scompare in seguito all'applicazione di una regola soggetta alla condizione sulla variabile).

Qualora si considerino proprietà che, come la linerità sinistra e destra, richiedono la relazione di uguaglianza, occorre aggiungere anche le regole matematiche per la relazione di uguaglianza, corrispondenti alla riflessività e alla sostituibilità degli identici in atomi relazionali e in quanto indici di formule:

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, {\scriptstyle EqRef}$$

$$\frac{At(y), x = y, At(x), \Gamma \Rightarrow \Delta}{x = y, At(x), \Gamma \Rightarrow \Delta}\, {\scriptstyle EqSubst_{At}} \qquad \frac{y : P, x = y, x : P, \Gamma \Rightarrow \Delta}{x = y, x : P, \Gamma \Rightarrow \Delta}\, {\scriptstyle EqSubst}$$

La regola *EqSubst* è ristretta agli atomi proposizionali per gli scopi dell'analisi della dimostrazione, ma si dimostra facilmente che può essere generalizzata a formule arbitrarie.

Il tempo lineare discreto

La logica del tempo lineare discreto senza inizio né fine (corrispondente al Sistema 7.3 di Prior 1967, p. 178) è il caso di studio privilegiato del mio lavoro. Le proprietà della discretezza

$$\forall x \exists y\, (y < x \,\&\, \forall z (y < z \supset (x = z \vee x < z)))$$

$$\forall x \exists y\, (x < y \,\&\, \forall z (z < y \supset (x = z \vee z < x))$$

non sono tuttavia esprimibili né mediante assiomi universali né mediante implicazioni geometriche e pertanto non possono essere trasformate direttamente in regole matematiche. Introducendo l'ulteriore relazione di successore immediato $x \prec y$ ho potuto trattare la logica del tempo lineare discreto con la metodologia sopra descritta. La relazione $x \prec y$ rappresenta la relazione di accessibilità degli operatori di Scott per l'istante immediatamente successivo e immediatamente precedente, $\mathbf{T}$ (*tomorrow*) e $\mathbf{Y}$ (*yesterday*):

$x \Vdash \mathbf{T}A$ *sse per ogni* y, $x \prec y$ *implica* $y \Vdash A$

$x \Vdash \mathbf{Y}A$ *sse per ogni* y, $y \prec x$ *implica* $y \Vdash A$

È pertanto possibile formulare le regole corrispondenti giustificandole in maniera analoga a quelle per $\mathbf{G}$ e $\mathbf{H}$, rispettivamente:

$$\frac{y : A, x : \mathbf{T}A, x \prec y, \Gamma \Rightarrow \Delta}{x : \mathbf{T}A, x \prec y, \Gamma \Rightarrow \Delta}\, L\mathbf{T} \qquad \frac{x \prec y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{T}A}\, R\mathbf{T}$$

$$\frac{y : A, x : \mathbf{Y}A, y \prec x, \Gamma \Rightarrow \Delta}{x : \mathbf{Y}A, y \prec x, \Gamma \Rightarrow \Delta}\, L\mathbf{Y} \qquad \frac{y \prec x, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{Y}A}\, R\mathbf{Y}$$

con la condizione per $R\mathbf{T}$ e $R\mathbf{Y}$ che y non compaia nella conclusione.

La relazione di successore immediato $x \prec y$ gode delle proprietà della serialità sinistra e destra ed è funzionale in entrambe le direzioni (ogni istante ha un unico predecessore e un unico successore):

$\forall x \exists y\, (y \prec x)$

$\forall x \exists y\, (x \prec y)$

$\forall x \forall y \forall z\, ((y \prec x \,\&\, z \prec x) \supset y = z)$

$\forall x \forall y \forall z\, ((x \prec y \,\&\, x \prec z) \supset y = z)$

Inoltre la relazione d'ordine "minore di" è definita come la chiusura transitiva della relazione di predecessore immediato:

$$x < y \;\equiv\; \exists n \in \mathbb{N}^+\, (x \prec^n y) \qquad (*)$$

dove la relazione di successore iterato $x \prec^n y$ è definita induttivamente come segue:

$x \prec^1 y \;\equiv\; x \prec y;$

$x \prec^{n+1} y \;\equiv\; \exists z\, (x \prec^n z \,\&\, z \prec y)$ per $n \geq 1$.

Il verso da sinistra a destra dell'equivalenza $(*)$ può essere ricondotto al fatto che x è minore del suo successore:

$\forall x \forall y\, (x \prec y \supset x < y)$

mentre il verso opposto corrisponde alla condizione che, se x è minore di y, allora x è predecessore di y, oppure x è predecessore del predecessore di y, oppure ... e così via. La proprietà della chiusura transitiva nasconde però una difficoltà sostanziale, costituita dal fatto che il conseguente della seconda implicazione è una disgiunzione infinita, e può essere tradotta in una regola di inferenza matematica solo ammettendo che essa abbia un numero infinito di premesse:

$$\frac{\{x \prec^n y, x < y, \Gamma \Rightarrow \Delta\}_{n \in \mathbb{N}^+}}{x < y, \Gamma \Rightarrow \Delta}\; T^{\omega}$$

Il calcolo G3LT si ottiene aggiungendo al calcolo di base G3K$_t$ le regole per $\mathbf{T}$ e $\mathbf{Y}$, la regola infinitaria T^{ω}, la regola della transitività per $<$ e le regole seguenti per la relazione $\prec$ e per il successore iterato:

$$\frac{y \prec x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; \textit{L-Ser} \qquad \frac{x \prec y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\; \textit{R-Ser} \qquad \frac{x < y, x \prec y, \Gamma \Rightarrow \Delta}{x \prec y, \Gamma \Rightarrow \Delta}\; \textit{Inc}$$

$$\frac{y = z, y \prec x, z \prec x, \Gamma \Rightarrow \Delta}{y \prec x, z \prec x, \Gamma \Rightarrow \Delta}\; \textit{UnPred} \qquad \frac{y = z, x \prec y, x \prec z, \Gamma \Rightarrow \Delta}{x \prec y, x \prec z, \Gamma \Rightarrow \Delta}\; \textit{UnSucc}$$

$$\frac{x \prec^n y, y \prec z, \Gamma \Rightarrow \Delta}{x \prec^{n+1} z, \Gamma \Rightarrow \Delta}\; \textit{LDef} \qquad \frac{\Gamma \Rightarrow \Delta, x \prec^{n+1} z, x \prec^n y \quad \Gamma \Rightarrow \Delta, x \prec^{n+1} z, y \prec z}{\Gamma \Rightarrow \Delta, x \prec^{n+1} z}\; \textit{RDef}$$

con la condizione per *L-Ser*, *R-Ser* e $LDef$ che y non compaia in Γ, Δ.

Si noti che ai fini dell'eliminazione del taglio, la presenza di regole matematiche che agiscono su entrambi i lati del sequente rende necessaria una misura di complessità delle formule atomiche definita in maniera analoga a Boretti e Negri (2006).

Il calcolo G3LT caratterizza la classe di strutture standard per il tempo lineare discreto, isomorfe all'insieme dei numeri interi $\mathbb{Z}$, e permette di derivare i due principi di induzione sugli istanti futuri e passati:

$\mathbf{T}A \supset (\mathbf{G}(A \supset \mathbf{T}A) \supset \mathbf{G}A)$

$\mathbf{Y}A \supset (\mathbf{H}(A \supset \mathbf{Y}A) \supset \mathbf{H}A)$

La validità del calcolo segue dal fatto che le regole logiche sono state giustificate sulla base della nozione di validità in un modello relazionale, e che le regole matematiche corrispondono alle proprietà godute dalle relazioni di accessibilità nelle corrispettive strutture. La completezza è stata dimostrata indirettamente dimostrando che gli assiomi del calcolo hilbertiano per il tempo lineare discreto sono derivabili in G3LT, e le regole della generalizzazione temporale e del *modus ponens* sono ammissibili.

Sebbene il calcolo G3LT goda di buone proprietà strutturali, prima tra tutte l'eliminazione sintattica del taglio, tuttavia la presenza di una regola infinitaria non è innocua per gli scopi dell'analisi della dimostrazione, e in particolare per la ricerca delle dimostrazioni. La ricerca delle dimostrazioni è una procedura cosiddetta *bottom-up*, mediante la quale si analizza il sequente che si desidera dimostrare al fine di individuare l'ultima regola di inferenza applicata nella sua derivazione; una volta rintracciata, la regola viene applicata a ritroso ricavandone le premesse. Ripetendo la procedura su queste ultime, si costruisce un albero la cui radice è il sequente di partenza e i nodi sono i sequenti ricavati di volta in volta come premesse: se tutti i rami dell'albero conducono a sequenti iniziali, la ricerca ha successo e il risultato, letto *top-down*, corrisponde alla derivazione desiderata. Al contrario la procedura fallisce se una delle foglie è un sequente non ulteriormente scomponibile diverso da un sequente iniziale, oppure se la ricerca non si arresta.

Ovviamente, in un tempo finito possono essere sottoposti a questa procedura solo un numero finito di sequenti, e in particolare solo un numero finito di premesse della regola della chiusura transitiva: la ricerca di una dimostrazione che coinvolga tale regola è di conseguenza destinata a continuare all'infinito, senza che si possa a priori distinguere il caso in cui la procedura non si arresta perché il sequente di partenza non è derivabile, dal caso in cui la procedura non si arresta perché semplicemente necessita di un numero infinito di passaggi. A livello metateorico, è naturalmente possibile ricorrere al principio di induzione per ottenere un'eventuale dimostrazione della derivabilità delle infinite premesse di T^ω, tuttavia le metaderivazioni così introdotte non possono essere a loro volta formalizzate nel calcolo senza dover rinunciare alle proprietà strutturali del sistema.

Per questo motivo ho giudicato interessante individuare un frammento della logica del tempo lineare discreto, che sia sufficientemente significativo, e in un certo senso naturale, ma che possa essere trattato per mezzo di un calcolo finitario.

La condizione che, se x è minore di y, allora y possa essere raggiunto a partire da x iterando un numero finito di volte la relazione $x \prec y$ è stata sostituita dalle due condizioni più deboli che se x è minore di y, allora x è predecessore immediato di y oppure x è minore del predecessore immediato di y (rispettivamente, il successore di x è minore di y)

$$\forall x \forall y\, (x < y \supset (x \prec y \vee \exists z\, (x < z \,\&\, z \prec y)))$$

$$\forall x \forall y\, (x < y \supset (x \prec y \vee \exists z\, (x \prec z \,\&\, z < y)))$$

Il sistema G3LT$_{n\text{-}s}$ si ottiene sostituendo in G3LT la regola infinitaria e le regole per il successore iterato con le regole corrispondenti alle proprietà precedenti, unite alle regole per la linearità destra e sinistra. Tale calcolo ammette strutture discrete non-standard costituite da molteplici copie di interi giustapposte le une alle altre, $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$.

Ho potuto tuttavia dimostrare un interessante risultato di conservatività parziale, secondo il quale, se un sequente non contiene l'operatore **G** nella parte positiva né **F** nella parte negativa e non contiene atomi relazionali, allora è derivabile nel calcolo infinitario G3LT se e solo se è derivabile nel calcolo G3LT$_{n\text{-}s}$.

Inoltre, ho dimostrato che nel frammento di G3LT orientato solo al futuro (privo cioè delle regole per **H**, **P** e **Y**, e delle regole matematiche *L-Ser* e *UnPred*), è possibile limitare ad un numero finito le premesse della regola della chiusura transitiva nel caso in cui il sequente che si intende derivare non contenga atomi relazionali e non presenti **G** nella parte negativa né **F** nella parte positiva. Tale limite è calcolato in maniera puramente sintattica, contando semplicemente il numero di **T** che compaiono nella parte negativa dei contesti delle premesse.

Una differente formulazione delle regole per gli operatori temporali mi ha permesso inoltre di studiare il tempo lineare discreto per mezzo del calcolo G3LT$_{cl}$. Quest'ultimo riflette un naturale algoritmo di chiusura, che sfrutta le proprietà di punto fisso degli operatori **G**, **F**, **H**, and **P**:

$$\mathbf{G}A \equiv \mathbf{T}A \,\&\, \mathbf{TG}A \qquad\qquad \mathbf{F}A \equiv \mathbf{T}A \vee \mathbf{TF}A$$

$$\mathbf{H}A \equiv \mathbf{Y}A \,\&\, \mathbf{YH}A \qquad\qquad \mathbf{P}A \equiv \mathbf{Y}A \vee \mathbf{YP}A$$

Inoltre le regole per **T** e **Y** sono giustificate sulla base del fatto che, nel caso del tempo lineare discreto senza estremi, valgono le equivalenze seguenti:

$x \Vdash \mathbf{T}A$ *sse per ogni* y, $x \prec y$ *implica* $y \Vdash A$

sse esiste y, *tale che* $x \prec y$ *e* $y \Vdash A$

$x \Vdash \mathbf{Y}A$ *sse per ogni* y, $y \prec x$ *implica* $y \Vdash A$

sse esiste y, *tale che* $y \prec x$ *e* $y \Vdash A$

Le regole proposizionali per il calcolo G3LT$_{cl}$ sono le stesse che per G3K$_t$, mentre le formule principali dei sequenti iniziali sono atomi proposizionali o formule prefisse da **G**, **F**, **H** e **P**. Le regole per gli operatori temporali e per la relazione di successore immediato sono illustrate nella tabella seguente:

Regole temporali di punto fisso:

$$\frac{x : \mathbf{T}A, x : \mathbf{TG}A, \Gamma \Rightarrow \Delta}{x : \mathbf{G}A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathbf{G}_{cl}} \qquad \frac{\Gamma \Rightarrow \Delta, x : \mathbf{G}A, x : \mathbf{T}A \quad \Gamma \Rightarrow \Delta, x : \mathbf{G}A, x : \mathbf{TG}A}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\ {\scriptstyle R\mathbf{G}_{cl}}$$

$$\frac{x : \mathbf{T}A, x : \mathbf{F}A, \Gamma \Rightarrow \Delta \quad x : \mathbf{TF}A, x : \mathbf{F}A, \Gamma \Rightarrow \Delta}{x : \mathbf{F}A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathbf{F}_{cl}} \qquad \frac{\Gamma \Rightarrow \Delta, x : \mathbf{T}A, x : \mathbf{TF}A}{\Gamma \Rightarrow \Delta, x : \mathbf{F}A}\ {\scriptstyle R\mathbf{F}_{cl}}$$

$$\frac{x : \mathbf{Y}A, x : \mathbf{YH}A, \Gamma \Rightarrow \Delta}{x : \mathbf{H}A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathbf{H}_{cl}} \qquad \frac{\Gamma \Rightarrow \Delta, x : \mathbf{H}A, x : \mathbf{Y}A \quad \Gamma \Rightarrow \Delta, x : \mathbf{H}A, x : \mathbf{YH}A}{\Gamma \Rightarrow \Delta, x : \mathbf{H}A}\ {\scriptstyle R\mathbf{H}_{cl}}$$

$$\frac{x : \mathbf{Y}A, x : \mathbf{P}A, \Gamma \Rightarrow \Delta \quad x : \mathbf{YP}A, x : \mathbf{P}A, \Gamma \Rightarrow \Delta}{x : \mathbf{P}A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathbf{P}_{cl}} \qquad \frac{\Gamma \Rightarrow \Delta, x : \mathbf{Y}A, x : \mathbf{YP}A}{\Gamma \Rightarrow \Delta, x : \mathbf{P}A}\ {\scriptstyle R\mathbf{P}_{cl}}$$

Regole per T e Y:

$$\frac{x \prec y, y : A, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}{x \prec y, x : \mathbf{T}A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathbf{T}} \qquad \frac{x \prec y, \Gamma \Rightarrow \Delta, x : \mathbf{T}A, y : A}{x \prec y, \Gamma \Rightarrow \Delta, x : \mathbf{T}A}\ {\scriptstyle R\mathbf{T}_{cl}}$$

$$\frac{y \prec x, y : A, x : \mathbf{Y}A, \Gamma \Rightarrow \Delta}{y \prec x, x : \mathbf{Y}A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathbf{Y}} \qquad \frac{y \prec x, \Gamma \Rightarrow \Delta, x : \mathbf{Y}A, y : A}{y \prec x, \Gamma \Rightarrow \Delta, x : \mathbf{Y}A}\ {\scriptstyle R\mathbf{Y}_{cl}}$$

Regole per $\prec$:

$$\frac{y \prec x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{L-Ser} \qquad \frac{x \prec y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{R-Ser}$$

Con la condizione per *L-Ser* e *R-Ser* che y non sia nella conclusione.

Tutte le regole di G3LT$_{cl}$ sono finitarie, ma le dimostrazioni sono in genere costituite da alberi di derivazioni che contengono (almeno) un ramo infinito.

Imponendo un'adeguata condizione di validità sugli alberi di derivazione infiniti, ho definito una nuova nozione di dimostrabilità all'interno del calcolo: un sequente è dimostrabile in G3LT$_{cl}$ qualora nessun ramo conduca ad un particolare tipo di sequente, chiamato 'sequente realizzante' (*fulfilling sequent*), che può essere considerato la controparte sintattica di un contromodello per un sequente non valido. L'uso degli indici fa sì che tale condizione sia squisitamente locale: non occorre cioè tenere conto delle parti precedenti dell'albero di derivazione, ma ad ogni passo è sufficiente prendere in considerazione solo il sequente in esame.

Il calcolo $\mathrm{G3LT}_{cl}$ è valido e completo rispetto alla suddetta condizione di dimostrabilità, nel senso che una formula è valida rispetto alla classe di strutture temporali isomorfe a $\mathbb{Z}$ se e solo se nessun ramo dell'albero di derivazione per il sequente corrispondente conduce ad un sequente realizzante. Ho inoltre dimostrato che è possibile imporre un limite superiore alla procedura di ricerca delle dimostrazioni per i sequenti validi, calcolata sulla base della lunghezza della formula corrispondente al sequente conclusivo (*terminating proof-search procedure*). Da quest'ultima la decidibilità di $\mathrm{G3LT}_{cl}$ segue come conseguenza principale.

Infine, ho preso in considerazione anche gli operatori temporali binari *Until* e *Since*, che, nel caso in cui la relazione $<$ sia ariflessiva, hanno il seguente significato intuitivo:

$$x \Vdash A\mathcal{U}B \textit{ sse esiste } y \textit{ tale che } x < y \textit{ e } y \Vdash B,$$

$$\textit{e per ogni } z, \textit{ se } x < z \textit{ e } z < y, \textit{ allora } z \Vdash A$$

$$x \Vdash A\mathcal{S}B \textit{ sse esiste } y \textit{ tale che } y < x \textit{ e } y \Vdash B,$$

$$\textit{e per ogni } z, \textit{ se } y < z \textit{ e } z < x, \textit{ allora } z \Vdash A$$

Come si può notare, le interpretazioni semantiche per $\mathcal{U}$ e $\mathcal{S}$ non sono esprimibili per mezzo di assiomi universali o implicazioni geometriche; tuttavia le seguenti definizioni ricorsive

$$A\mathcal{U}B \equiv \mathbf{T}B \vee (\mathbf{T}A \,\&\, \mathbf{T}(A\mathcal{U}B))$$

$$A\mathcal{S}B \equiv \mathbf{Y}B \vee (\mathbf{Y}A \,\&\, \mathbf{Y}(A\mathcal{S}B))$$

hanno permesso di formulare opportune regole logiche da aggiungere ai calcoli G3LT e G3LT$_{cl}$. Sfruttando l'internalizzazione della semantica relazionale, ho facilmente adattato i risultati citati precedentemente al caso dei calcoli G3LT+$\mathcal{U} + \mathcal{S}$ e G3LT$_{cl} + \mathcal{U} + \mathcal{S}$.

In conclusione, l'internalizzazione della semantica relazionale nel formalismo del calcolo dei sequenti mi ha permesso di trattare in maniera uniforme una varietà di sistemi temporali. I calcoli sono ottenuti come estensioni modulari di un calcolo temporale di base per mezzo di regole matematiche corrispondenti alle proprietà delle relazioni di accessibilità esprimibili sotto forma di assiomi universali o implicazioni geometriche.

Tutti i calcoli godono di notevoli proprietà strutturali e di una forma indebolita della proprietà della sottoformula, mentre la proprietà del sottotermine garantisce un buon controllo degli indici introdotti nelle derivazioni dall'applicazione a ritroso delle regole del calcolo.

L'efficacia della metodologia adottata nell'analisi della dimostrazione è stata verificata nello studio del tempo lineare discreto. Quest'ultimo è stato analizzato per mezzo due calcoli infinitari, che richiedono rispettivamente una regola con infinite premesse e una definizione di dimostrabilità che ammette alberi di derivazione che contengono rami infiniti.

Nel primo caso, ho ottenuto una finitizzazione parziale identificando due diversi sistemi finitari, per ciascuno dei quali ho dimostrato un risultato di conservatività rispetto ad un opportuno frammento del calcolo originale.

Nel secondo caso, ho dimostrato la finitizzazione del calcolo sotto forma della terminazione della procedura di ricerca delle dimostrazioni e, quindi, di una adeguata procedura di decisione.

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