ON THE ORDERS OF ZEROS OF IRREDUCIBLE CHARACTERS

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Abstract. Let G be a finite group and p a prime number. We say that an element g in G is a *vanishing element* of G if there exists an irreducible character χ of G such that $\chi(g) = 0$. The main result of this paper shows that, if G does not have any vanishing element of p-power order, then G has a normal Sylow p-subgroup. Also, we prove that this result is a generalization of some classical theorems in Character Theory of finite groups.

Keywords: finite groups, characters, zeros of characters.

1. Introduction

It is well-known that a great deal of information on the structure of a finite group G is encoded in the set $\mathrm{Irr}(G)$ of irreducible complex characters of G, which are the trace functions of the irreducible representations of G over $\mathbb{C}$. Many results in literature show that even the relatively small set of values $\mathrm{cd}(G) = \{\chi(1) \ : \ \chi \in \mathrm{Irr}(G)\}$ has a strong influence on the group structure of G.

The aim of this paper is to provide some evidence that also the zeros of the functions in $\mathrm{Irr}(G)$ encode nontrivial information of G. Following [10], we shall say that an element x of a group G is a *vanishing element* if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$. If this is not the case, we shall call x a *nonvanishing element*. The main result of this paper is the following.

Theorem A. *Let G be a finite group and p a prime number. If all the p-elements of G are nonvanishing, then G has a normal Sylow p-subgroup.*

We recall that one striking instance of the connection between $\mathrm{cd}(G)$ and the local structure of G is the following classical result, known as Ito-Michler's theorem (see [8, 19.10 and 19.11]): if a prime p does not divide any element in $\mathrm{cd}(G)$, then G has an abelian normal Sylow p-subgroup. Now, since a character whose degree is not divisible by p cannot vanish on any p-element, it follows that Theorem A implies Ito-Michler's theorem, whereas the converse does not hold (see Section 4).

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It is worth remarking that, as well as the (known) proof of Ito-Michler's theorem, our proof of Theorem A relies on the classification of finite simple groups.

Other remarkable consequences of Theorem A are presented in Section 4. In particular, immediate applications of Theorem A yield an analogue of a well-known theorem proved by J. Thompson in [15] (see Corollary C), and a generalization of Burnside's $p^\alpha q^\beta$ theorem (see Corollary D).

To conclude, we mention that various questions related to the zeros of the irreducible characters of a finite group have been investigated by several authors (see for instance [2], [3], [4], [10], [11], [13]). In particular, nonvanishing elements were first introduced and studied in [10]. Indeed, Theorem D in [10] shows that if x is a nonvanishing element of odd order of a solvable group G, then x lies in the Fitting subgroup of G. So, in the case of solvable groups and for an odd prime, Theorem A follows at once from the results in [10].

2. Preliminaries

Since in this paper we are interested in vanishing elements, we introduce the following notation. If G is a finite group, then we denote by $\mathrm{Van}(G)$ the set $\{g \in G \,:\, \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}(G)\}$.

Let p be a prime number. Recall that a character χ in $\mathrm{Irr}(G)$ is said to be of *p-defect zero* if p does not divide $|G|/\chi(1)$. By a fundamental result of R. Brauer (see [9, Theorem 8.17]), if $\chi \in \mathrm{Irr}(G)$ is of p-defect zero then, for every element $g \in G$ such that p divides $o(g)$, we have $\chi(g) = 0$.

It is very often the case that a nonabelian simple group has irreducible characters of p-defect zero for every prime p (see [7, Corollary 2]).

Proposition 2.1. *Let S be a nonabelian simple group and p a prime number. If S is of Lie type, or if $p \geq 5$, then there exists $\theta \in \mathrm{Irr}(S)$ of p-defect zero.*

Proof. By results of G. Michler and W. Willems (see [12] and [16]), it is known that a simple group of Lie type has irreducible characters of p-defect zero for every prime p. The main result in [7] yields that an alternating group has irreducible characters of p-defect zero for every $p \geq 5$ (see [7, Corollary 1]). Finally, it can be checked in [5] that a sporadic simple group has irreducible characters of p-defect zero for every $p \geq 5$. □

The following proposition, which appears as Lemma 5 in [1], will turn out to be useful in handling the case when a nonabelian simple group fails to have irreducible characters of p-defect zero.

Proposition 2.2. *Let $M = S_1 \times \cdots \times S_k$ be a minimal normal subgroup of G, where $S_i \simeq S$, a nonabelian simple group. If $\theta \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\theta \times \cdots \times \theta \in \mathrm{Irr}(M)$ extends to G.*

In order to apply Proposition 2.2, we need to gather some information concerning characters of sporadic simple groups and alternating groups. This will be done through Lemma 2.3 and Proposition 2.4, respectively. Although we use Lemma 2.3 only when the prime p is 2 or 3, we state and prove it for a general p.

Lemma 2.3. *Let S be a sporadic simple group and p a prime divisor of $|S|$. Then there exist a p-element x of S and a character $\theta \in \mathrm{Irr}(S)$, such that θ extends to $\mathrm{Aut}(S)$ and $\theta(x) = 0$. If $S \not\simeq M_{22}$ or $p \neq 2$, then x can be chosen to have order p.*

Proof. Assume either $S \not\simeq M_{22}$ or p odd. A direct inspection of [5] shows that there exist an element x of S with $o(x) = p$ and $\theta \in \mathrm{Irr}(S)$, such that $\theta(x) = 0$ and θ is invariant under $\mathrm{Aut}(S)$. On the other hand, if $S \simeq M_{22}$ and $p = 2$, then there exist an element x of S with $o(x) = 8$ and $\theta \in \mathrm{Irr}(S)$, such that $\theta(x) = 0$ and θ is invariant under $\mathrm{Aut}(S)$. In any case, since $\mathrm{Out}(S)$ is cyclic, the character θ extends to $\mathrm{Aut}(S)$ by [9, Corollary 11.22]. $\square$

We now move to alternating groups. The character table of the symmetric groups and of the alternating groups is well-known (see [14]). In particular, every irreducible character of $\mathrm{Sym}(n)$ corresponds naturally to a partition of n. We recall that if π is the partition of n corresponding to the character $\chi \in \mathrm{Irr}(\mathrm{Sym}(n))$, then $\chi_{\mathrm{Alt}(n)}$ is irreducible if and only if the Young diagram corresponding to the partition π is not symmetric (note that in [14] symmetric Young diagrams are called "self-associated"). If χ is an irreducible character of $\mathrm{Sym}(n)$ and g is an element of $\mathrm{Sym}(n)$, then in Proposition 2.4 we use the Murnaghan-Nakayama formula to compute $\chi(g)$ (see [14, 2.4.7]). Also, we identify an element of $\mathrm{Sym}(n)$ with a partition of n, i.e. its *cycle type* (see [14, 1.2.4]). For instance, we identify an n-cycle with the partition (n), an $(n-1)$-cycle with $(n-1, 1)$ and a 2-cycle with $(2, 1^{n-2})$.

Proposition 2.4. *If $n \geq 7$, then $\mathrm{Alt}(n)$ has two irreducible characters χ_2, χ_3 such that χ_2 vanishes on a 2-element x and χ_3 vanishes on an element y of order 3. Furthermore, both χ_2 and χ_3 extend to $\mathrm{Aut}(\mathrm{Alt}(n)) = \mathrm{Sym}(n)$.*

Proof. If $n \equiv 0 \mod 4$, then use $\chi_2 = (n-3, 2, 1)$ and $x = (2^{n/2})$.

If $n \equiv 1 \mod 4$, then use $\chi_2 = (n-1, 1)$ and $x = (2^{(n-1)/2}, 1)$.

If $n \equiv 2 \mod 4$, then use $\chi_2 = (n-3, 2, 1)$ and $x = (2^{(n-2)/2}, 1^2)$.

If $n \equiv 3 \mod 4$, then use $\chi_2 = (n-1, 1)$ and $x = (4, 2^{(n-5)/2}, 1)$.

If $n \equiv 0 \mod 3$, then use $\chi_3 = (n-4, 2, 1^2)$ and $y = (3^{n/3})$.

If $n \equiv 1 \mod 3$, then use $\chi_3 = (n-2, 1^2)$ and $y = (3^{(n-1)/3}, 1)$.

If $n \equiv 2 \mod 3$, then use $\chi_3 = (n-2, 1^2)$ and $y = (3^{(n-2)/3}, 1^2)$.

Furthermore, the Young diagrams of the partitions corresponding to all of the characters mentioned above are not symmetric. $\square$

Note that, in the proof of Proposition 2.4, the element x is chosen to be of order 2 provided $n \not\equiv 3 \bmod 4$. For the sake of completeness, we point out that the same can be done also if $n \equiv 3 \bmod 4$, except when n is 7 or 15. Indeed, if $n \equiv 3 \mod 4$ and $n \geq 19$, then $\chi_2 = (5,4,3,2,1^{n-14})$ is an irreducible character of $\mathrm{Alt}(n)$, that extends to $\mathrm{Sym}(n)$, vanishing on the element $x = (2^{(n-3)/2}, 1^3)$ of order 2 of $\mathrm{Alt}(n)$. Similarly, if $n = 11$, then $\chi_2 = (6,3,1^2)$ is an irreducible character of $\mathrm{Alt}(11)$, that extends to $\mathrm{Sym}(11)$, vanishing on the element $x = (2^2, 1^7)$ of order 2 of $\mathrm{Alt}(11)$. Moreover, it can be checked by direct calculation that both $\mathrm{Alt}(7)$ and $\mathrm{Alt}(15)$ do not have any irreducible character vanishing on an element of order 2. Finally we point out that $\mathrm{Alt}(6)$ has irreducible characters vanishing on elements of order 2, but none of these characters extend to $\mathrm{Aut}(\mathrm{Alt}(6))$ (see [5]).

Remark 2.5. Let S be a nonabelian simple group and p a prime divisor of $|S|$. From Proposition 2.1, Lemma 2.3, Proposition 2.4 and the previous paragraph, we have that if $p \geq 3$ or if $p = 2$ and S is not isomorphic to M_{22}, $\mathrm{Alt}(7)$ or $\mathrm{Alt}(15)$, then either there exists a $\chi \in \mathrm{Irr}(S)$ of p-defect zero or there exist an irreducible character $\chi \in \mathrm{Irr}(S)$ that extends to $\mathrm{Aut}(S)$ and an element x of order p of S such that $\chi(x) = 0$. (Note that $\mathrm{Alt}(5) \simeq \mathrm{PSL}(2,5)$ and $\mathrm{Alt}(6) \simeq \mathrm{PSL}(2,9)$ are considered in Proposition 2.1.)

We need to establish the existence, in finite groups of Lie type, of suitable elements that are not fixed by certain field automorphisms.

Given integers $q \geq 2$ and $n \geq 2$, we say that a prime r is a *primitive prime divisor* of $q^n - 1$ if r divides $q^n - 1$ and r does not divide $q^j - 1$ for $1 \leq j < n$. We denote by $\mathrm{ppd}(q^n - 1)$ the set (depending on q and n) of the primitive prime divisors of $q^n - 1$. By a classical result by Zsigmondy, we have that the set $\mathrm{ppd}(q^n - 1)$ is empty if and only if either $(n, q) = (6, 2)$ or $n = 2$ and q is a Mersenne prime.

Lemma 2.6. *Let S be a simple group. Then there exists a prime divisor r of $|S|$ such that $(r, |\mathbf{C}_S(\alpha)|) = 1$ for every nontrivial $\alpha \in \mathrm{Aut}(S)$ of order coprime to $|S|$.*

Proof. We can clearly assume that S is nonabelian and that there exists a nontrivial automorphism α of $|S|$ with $(o(\alpha), |S|) = 1$, since otherwise the statement is clearly true. Since $\mathbf{C}_S(\alpha) \leq \mathbf{C}_S(\alpha^k)$ for every positive integer k, by taking a suitable power we can also assume that α has prime order p. It is known that, if there exists a prime divisor p of $|\mathrm{Aut}(S)|$ such that p does not divide $|S|$, then S is a simple group of Lie type and every $\alpha \in \mathrm{Aut}(S)$ of order p is conjugate to a field automorphism of S. So, S is isomorphic to ${}^dL_n(q^f)$, where ${}^dL_n(q^f)$ denotes the group of Lie type L of rank n (untwisted if $d = 1$, twisted if $d = 2$, and ${}^3D_4(q^f)$ if $d = 3$), q being a suitable prime, and f a multiple of p.

Let $\Phi_i(x)$ be the i-th cyclotomic polynomial. Observe that the order of any simple group of Lie type ${}^dL_n(t)$, where t is a prime power, can be decomposed as a product of factors of the form $\Phi_i(t)$ and a t-power. In fact, if ${}^dL_n(t) \neq {}^3D_4(t)$, then $|{}^dL_n(t)|$ is, up to a t-power, a product of factors $t^k \pm 1$, for some positive integers k, and each of those is a product of suitable $\Phi_i(t)$'s; the case ${}^3D_4(t)$ is checked directly.

Let $m = \max\{i : \Phi_i(q^f) \text{ divides } |{}^dL_n(q^f)|\}$. Observe that m does not depend on the order q^f of the ground field and that $m \geq 2$. We claim that $\mathrm{ppd}(q^{fm}-1)$ is nonempty. Since p does not divide $|S|$, by the Odd Order Theorem, we get $f \neq 2$. Assume $f = 3$. The only simple groups of Lie type of order not divisible by 3 are the Suzuki groups ${}^2B_2(2^{2s+1})$. Checking the order of ${}^2B_2(2^{2s+1})$, we have $m = 4$. In particular, $fm = 12$ and $\mathrm{ppd}(q^{12}-1) \neq \emptyset$. Now assume $f > 3$. In this case clearly $fm \neq 2, 6$, therefore $\mathrm{ppd}(q^{fm}-1) \neq \emptyset$.

Consider $r \in \mathrm{ppd}(q^{fm}-1)$. Note that r divides $\Phi_m(q^f)$ and so r divides $|S|$.

Let now α be an automorphism of order p of S. By the fact that α is conjugate to a field automorphism of S, and by [6, Theorem 9.1], we see that $|\mathbf{C}_S(\alpha)| = |{}^dL_n(q^{f/p})|$ and hence, by the choice of r, it follows that r does not divide $|\mathbf{C}_S(\alpha)|$. □

The investigation on zeros of irreducible characters often leads to questions concerning orbits in coprime group actions. We recall some well-known facts and some standard notation in this context. Let M be a group and, for x in M, denote by x^M the conjugacy class of x in M. If a group H acts by automorphisms on M, then there is a natural action of H on $\mathrm{Irr}(M)$ and on $\mathrm{Cl}(M) = \{x^M : x \in M\}$ given by $(x^M)^h = (x^h)^M$ and $\phi^h(m) = \phi(m^{h^{-1}})$ for $h \in H$, $x, m \in M$ and $\phi \in \mathrm{Irr}(M)$. Also, we denote by $I_H(\phi)$ and $\mathrm{Stab}_H(x^M)$, respectively, the stabilizer of ϕ (i.e. the inertia subgroup of ϕ) and the stabilizer of x^M in the relevant actions.

Proposition 2.7. *Let H be a group that acts by automorphisms on a group M. Assume that H and M have coprime orders. Then the actions of H on $\mathrm{Irr}(M)$ and on $\mathrm{Cl}(M)$ are permutation isomorphic. In particular, for every $x \in M$ there exists $\phi \in \mathrm{Irr}(M)$ such that $I_H(\phi) = \mathrm{Stab}_H(x^M)$.*

Proof. See [8, Theorem 18.9]. □

It is not true in general that in a faithful coprime action of H on M there exists a regular orbit of H on $\mathrm{Irr}(M)$, i.e. a $\theta \in \mathrm{Irr}(M)$ such that $I_H(\theta) = 1$. Nevertheless, in some cases relevant to the proof of Theorem A, we prove that such an orbit does exist. We recall that a group M is said to be characteristically simple if M has no proper nontrivial $\mathrm{Aut}(M)$-invariant subgroups. A group M is characteristically simple exactly when M is the direct product of isomorphic simple groups.

Lemma 2.8. *Let A be an abelian group that acts faithfully by automorphisms on a group M. Assume that $|A|$ and $|M|$ are coprime. If M is characteristically simple, then there exists $\theta \in \mathrm{Irr}(M)$ such that $I_A(\theta) = 1$.*

Proof. Write $M = S_1 \times S_2 \times \cdots \times S_n$ with $S_i \simeq S$ for every i, S a simple group.

If S is nonabelian, then every normal subgroup of M is the product of the subgroups belonging to a subset of $\Omega = \{S_1, S_2, \ldots, S_n\}$. In particular, the subgroups S_i are the only minimal normal subgroups of M and A acts on Ω.

Assume that there exists a proper nontrivial A-invariant subgroup $M_1 \trianglelefteq M$. We claim that there exists an A-invariant $M_2 \trianglelefteq M$ such that $M = M_1 \times M_2$. In fact, if M is abelian we can apply Maschke's theorem. If M is nonabelian, this follows by the previous paragraph. Working by induction on $|M|$, for $i = 1, 2$ there exists $\theta_i \in \mathrm{Irr}(M_i)$ such that $I_A(\theta_i) = \mathbf{C}_A(M_i)$. Let θ denote the irreducible character $\theta_1 \times \theta_2$ of M. Then

$$I_A(\theta) = I_A(\theta_1) \cap I_A(\theta_2) = \mathbf{C}_A(M) = 1.$$

We can hence assume that 1 and M are the only A-invariant normal subgroups of M. By Proposition 2.7, it is enough to show that there exists $x \in M$ such that $\mathrm{Stab}_A(x^M) = 1$.

Assume first that M is abelian. For $a, b \in A$, we have $\mathbf{C}_M(a)^b = \mathbf{C}_M(a^b) = \mathbf{C}_M(a)$, so $\mathbf{C}_M(a)$ is an A-invariant normal subgroup of M. If $a \neq 1$, then $\mathbf{C}_M(a) < M$ and hence $\mathbf{C}_M(a) = 1$. Thus $\mathrm{Stab}_A(x^M) = \mathbf{C}_A(x) = 1$ for every nontrivial $x \in M$.

Assume now that M is nonabelian. Then $S = S_1$ is a nonabelian simple group and A acts transitively on Ω. By Lemma 2.6 there exists a prime divisor r of $|S|$ such that r does not divide $|\mathbf{C}_S(\alpha)|$ for every nontrivial $\alpha \in \mathrm{Aut}(S)$ with $(o(\alpha), |S|) = 1$. Let $x \in S$ be an element of order r. We show that $\mathrm{Stab}_A(x^M) = 1$. As $x^M \subseteq S$ and $x \neq 1$, we see that $\mathrm{Stab}_A(x^M) \leq N$, where $N = \mathbf{N}_A(S)$. For every $T \in \Omega$ there exists $a \in A$ such that $T = S^a$, therefore we get $\mathbf{C}_A(T) = \mathbf{C}_A(S)^a = \mathbf{C}_A(S)$, because A is abelian. Thus $\mathbf{C}_A(S) = \mathbf{C}_A(M) = 1$, so that N acts faithfully on S. Let now $b \in \mathrm{Stab}_A(x^M)$. By Glauberman Permutation Lemma (see [9, Lemma 13.8]) there is a $y \in x^M \cap \mathbf{C}_M(b)$. But $b \in N$ acts as an automorphism of coprime order on S and $o(y) = r$ divides $|\mathbf{C}_S(b)|$. Then, by the choice of r, we get $b = 1$ and hence $\mathrm{Stab}_A(x^M) = 1$. □

The next result will provide some control on the set $\mathrm{Van}(G)$.

Lemma 2.9. *Let $M \leq N \leq G$, with M and N normal in G and $(|M|, |N/M|) = 1$. If M is minimal normal in G, $\mathbf{C}_N(M) \leq M$ and N/M is abelian, then $N \setminus M \subseteq \mathrm{Van}(G)$.*

Proof. Since $(|M|, |N/M|) = 1$, by the Schur-Zassenhaus theorem there exists a complement A of M in N. Then $A \simeq N/M$ is an abelian group that acts faithfully by automorphisms on M, because $\mathbf{C}_A(M) = \mathbf{C}_N(M) \cap A \leq M \cap A = 1$.

Since M is minimal normal in G, we get that M is characteristically simple and hence, by Lemma 2.8, there exists $\theta \in \mathrm{Irr}(M)$ such that $I_A(\theta) = 1$. Now, set $I = I_G(\theta)$. Take an irreducible character ψ of I which lies over θ, and consider the character $\chi = \psi^G$ of G. Clifford Correspondence yields that χ is irreducible.

For every $g \in G$, since $M \leq I$ and $N, M \trianglelefteq G$, we have

$$I^g \cap N = (I \cap N)^g = (I \cap AM)^g = ((I \cap A)M)^g = M^g = M$$

because $I \cap A = I_A(\theta) = 1$. Therefore, $N \cap (\bigcup_{g \in G} I^g) = M$. Since χ vanishes on $G \setminus \bigcup_{g \in G} I^g$, it follows that $\chi(x) = 0$ for every $x \in N \setminus M$. □

3. Proof of Theorem A

We are now ready to prove Theorem A, which we state again.

Theorem 3.1. *Let G be a group and p a prime number. If all the p-elements of G are nonvanishing, then G has a normal Sylow p-subgroup.*

Proof. Let G be a minimal counterexample and P a Sylow p-subgroup of G. Observe that $P \neq 1$.

Consider a minimal normal subgroup M of G and assume that p divides $|M|$. If M is solvable then M is a p-group, $\mathrm{Van}(G/M)$ contains no p-elements and so, by minimality, $P = PM \trianglelefteq G$, a contradiction. Therefore M is nonsolvable. Write $M = S_1 \times \cdots \times S_k$, where $S_i \simeq S$ and S is a nonabelian simple group with $|S|$ divisible by p.

Suppose first that S is a simple group of Lie type or that $p \geq 5$. By Proposition 2.1, there exists an irreducible character θ of S of p-defect zero. Let $\psi = \theta \times \cdots \times \theta \in \mathrm{Irr}(M)$. Observe that ψ is a character of p-defect zero of M. Consider $\chi \in \mathrm{Irr}(G)$ lying over ψ. By Clifford's theorem, χ_M is a sum of G-conjugates ψ_i of ψ. As $\psi_i(1) = \psi(1)$, every ψ_i is a character of p-defect zero of M. Hence, by [9, Theorem 8.17], the character ψ_i vanishes on every element of M of order multiple of p. Let x be an element of order p in M. Then $\chi(x) = \sum_i \psi_i(x) = 0$, and hence $x \in \mathrm{Van}(G)$, a contradiction.

Assume now that $p \in \{2, 3\}$ and that S is either a sporadic simple group or $S \simeq \mathrm{Alt}(n)$ for some $n \geq 7$ (observe that $\mathrm{Alt}(5) \simeq \mathrm{PSL}(2,5)$ and $\mathrm{Alt}(6) \simeq \mathrm{PSL}(2,9)$ have already been considered as groups of Lie type). By Lemma 2.3 and Proposition 2.4, there exists a $\theta \in \mathrm{Irr}(S)$ such that θ extends to $\mathrm{Aut}(S)$ and $\theta(x) = 0$ for some p-element $x \in S$. Let $g = x \times \cdots \times x \in M$. Now, by Proposition 2.2 there exists a $\chi \in \mathrm{Irr}(G)$ that extends $\theta \times \cdots \times \theta \in \mathrm{Irr}(M)$. So $\chi(g) = (\theta(x))^k = 0$ and hence the p-element g belongs to $\mathrm{Van}(G)$, a contradiction.

We conclude that p does not divide the order of every minimal normal subgroup of G. In particular, $\mathbf{O}_p(G) = 1$.

Consider again a minimal normal subgroup M of G. We claim that $\mathrm{Van}(G/M)$ does not contain any p-element. Assume that there exist a $gM \in G/M$ and a $\chi \in \mathrm{Irr}(G/M)$ such that gM has p-power order in G/M and $\chi(gM) = 0$. We may assume that g is a p-element. By inflation, χ is an irreducible character of G whose kernel contains M. So, $\chi(g) = \chi(gM) = 0$. Hence g is an element of p-power order in $\mathrm{Van}(G)$, a contradiction.

Our assumption of minimality on G yields now that PM/M is a normal Sylow p-subgroup of G/M. Let $N/M = \mathbf{Z}(PM/M)$. Observe that $N \trianglelefteq G$ and that $N \neq M$. Further, $\mathbf{C}_N(M) \leq M$ because $\mathbf{O}_p(N) \leq \mathbf{O}_p(G) = 1$. Hence, by Lemma 2.9 we obtain $N \setminus M \subseteq \mathrm{Van}(G)$. Since $\mathrm{Van}(G)$ contains no p-elements, it follows that $N = M$, the final contradiction. □

Remark 3.2. We note that Theorem A cannot be strengthened by replacing "p-elements" with "elements of order p". Consider, for example, a group G where a Sylow p-subgroup of G is not normal and where every element of order p in G lies in $\mathbf{Z}(G)$. Also, another example is given by the alternating group on 7 objects. In fact, $\mathrm{Alt}(7)$ does not have any vanishing element of order 2. See also Theorem 4.3.

4. Consequences of Theorem A

In this section we point out some interesting consequences of Theorem A.

We let $\mathrm{Vo}(G)$ denote the set $\{o(g) \ : \ g \in \mathrm{Van}(G)\}$ consisting of the orders of the elements in $\mathrm{Van}(G)$.

The following corollary, whose proof follows at once by Theorem A, shows some formal similarities between Theorem A and Ito-Michler's theorem.

Corollary B. *Let G be a finite group, and p a prime number. If p does not divide any element in $\mathrm{Vo}(G)$, then G has a normal Sylow p-subgroup.*

The next remark is needed to get some deeper relations between Theorem A and Ito-Michler's theorem.

Remark 4.1. Let G be a group, p a prime number, and g a p-element of G. We claim that if χ is a character of G such that $\chi(g) = 0$, then the degree of χ is divisible by p. Observe that $\chi(g)$ is a sum of $\chi(1)$ m-th roots of unity, where $m = o(g)$ is a power of p. Let ε be a primitive m-th root of unity and let $\chi(g) = \sum_{i=1}^{\chi(1)} \varepsilon^{k_i}$, where $0 \leq k_i < m$. Now, ε is a root of the polynomial $h(x) = \sum_{i=1}^{\chi(1)} x^{k_i}$. Whence $h(x)$ is divisible by the m-th cyclotomic polynomial $\Phi_m(x)$. In particular, $p = \Phi_m(1)$ divides $h(1) = \chi(1)$, as desired.

A consequence of Remark 4.1, as mentioned in the Introduction, is that if p does not divide any element in $\mathrm{cd}(G)$, then all p-elements of G are nonvanishing. In particular, the hypotheses of Ito-Michler's theorem imply the hypotheses of Theorem A. So, Theorem A yields Ito-Michler's theorem (the fact that the Sylow p-subgroup is abelian follows easily).

On the other hand, the following example shows that in general the hypotheses of Theorem A do not imply those of Ito-Michler's theorem. At the same time, it shows that the (normal) Sylow p-subgroup of a group G such that all p-elements are nonvanishing may be nonabelian. In this respect, Theorem A can be regarded as an improvement of Ito-Michler's theorem.

Example 4.2. Let G be the normalizer of a Sylow 2-subgroup in the Suzuki group $\mathrm{Suz}(8)$. Then G is a Frobenius group with complement of order 7 and nonabelian Frobenius kernel. It turns out that $\mathrm{Vo}(G) = \{7\}$. On the other hand, $\mathrm{cd}(G) = \{1, 7, 14\}$.

More generally, we note that [2, Example 1] shows that there exists no bound on the derived length of the Sylow p-subgroup of a group G such that p does not divide any element in $\mathrm{Vo}(G)$.

While Ito-Michler's theorem is an "if and only if" theorem, at the time of this writing we do not know of any condition on a (normal) Sylow p-subgroup P of G guaranteeing the converse of Theorem A. However, we note that such a condition should involve the action of G on P as well as the group structure of P.

In a sort of complementary setting to Ito-Michler's theorem, a theorem by Thompson states that, given a prime number p, if every number greater than 1 in $\mathrm{cd}(G)$ is a multiple of p, then the group G has a normal p-complement (see [15]). From Theorem A, we derive the following analogue of Thompson's theorem (observe that 1 is not an element of $\mathrm{Vo}(G)$).

Corollary C. *Let G be a finite group and p a prime number. If every number in $\mathrm{Vo}(G)$ is a multiple of p, then G has a nilpotent normal p-complement.*

Proof. Let π be the set of prime divisors of $|G|$ different from p. Our assumption implies that, for every q in π, there does not exist any q-element of G lying in $\mathrm{Van}(G)$. Theorem A yields that G has a normal Sylow q-subgroup Q. Therefore, $H = \prod_{q\in\pi} Q$ is a nilpotent normal p-complement of G. $\square$

The following corollary of Theorem A can be regarded as an extension of Burnside's $p^\alpha q^\beta$ theorem (see [9, 3.10]).

Corollary D. *Let p and q be prime numbers. If every element in $\mathrm{Van}(G)$ is a $\{p, q\}$-element, then G is solvable.*

Proof. Let σ be the set of prime divisors of $|G|$ different from p and q. Arguing as in the proof of Corollary C, we have that G has a nilpotent normal Hall σ-subgroup T. Now, G/T is a $\{p, q\}$-group, whence G/T is solvable by Burnside's $p^\alpha q^\beta$ theorem. Thus, G is solvable. $\square$

We conclude by stating a result (Theorem 4.3) which can be easily obtained adapting the proof of Theorem A and taking into account Remark 2.5. The details are left to the reader. We note that, for groups satisfying the assumptions of Theorem 4.3 (in particular, for solvable groups), this is actually a strengthening of Theorem A. In fact, Theorem A follows by an easy argument which uses induction on the order of the group.

Theorem 4.3. *Let G be a group and p a prime divisor of $|G|$. Assume that either $p > 2$ or that $p = 2$ and G has no composition factor isomorphic to M_{22}, $\mathrm{Alt}(7)$ or $\mathrm{Alt}(15)$. If $p \notin \mathrm{Vo}(G)$, then $\mathbf{O}_p(G) \neq 1$.*

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References

[1] M. Bianchi, D. Chillag, M.L. Lewis, E. Pacifici, *Character degree graphs that are complete graphs*, Proc. Amer. Math. Soc. **135** (2007), 671–676.

[2] D. Bubboloni, S. Dolfi, P. Spiga, *Finite groups whose irreducible characters vanish only on p-elements*, J. Pure Appl. Algebra, to appear.

[3] D. Chillag, *Finite groups with restrictions on the zero sets of their irreducible characters*, Algebra Colloq. **11** (2004), 387–398.

[4] D. Chillag, *On zeros of characters of finite groups*, Proc. Amer. Math. Soc. **127** (1999), 977–983.

[5] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[6] D. Gorenstein, R. Lyons, *The local structure of finite simple groups of characteristic 2-type*, Memoires of the American Math. Soc. **42** n. 276 (1983).

[7] A. Granville, K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. **38** (1996), 331–347.

[8] B. Huppert, *Character Theory of finite groups*, De Gruyter, Berlin, 1998.

[9] I.M. Isaacs, *Character Theory of Finite Groups*, Dover, New York, 1976.

[10] I.M. Isaacs, G. Navarro, T.R. Wolf, *Finite group elements where no irreducible character vanishes*, J. Algebra **222** (1999), 413–423.

[11] G. Malle, G. Navarro, J.B. Olsson, *Zeros of characters of finite groups*, J. Group Theory **3** (2000), 353–368.

[12] G. Michler, *Finite simple group of Lie type has p-blocks with different defects if $p = 2$*, J. Algebra **104** (1986), 220–230.

[13] G. Navarro, *Irreducible restrictions and zeros of characters*, Proc. Amer. Math. Soc. **129** (2001), 1643–1645.

[14] G. James, A. Kerber, *The representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its applications Vol. 16, Addison-Wesley Publishing Co. 1981.

[15] J.G. Thompson, *Normal p-complements and irreducible characters*, J. Algebra **33** (1975), 129–134.

[16] W. Willems, *Blocks of defect zero in finite simple groups of Lie type*, J. Algebra **113** (1988), 511–522.

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