Distributed Intervals:

A Formal Framework for Information Granulation

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***Abstract*—The notion of distributed interval, as a formal framework for information granulation, represented by the collection of finite number of general-intervals is introduced. Operations on distributed intervals are defined based on the corresponding general-intervals'. Distributed intervals provide a bi-criteria framework for information granulation that can be used as a conceptually rich structure in granular computing.**

Keywords-granular computing; distributed interval; general interval

I. INTRODUCTION

In general, as the name stipulates, granular computing is about information granulation, i.e. aggregating entities based on their similarity, functionality, proximity, coherency and indistinguishability, and their processing. It is argued that "[information granules] are central to processes of abstraction guiding our intellectual pursuits" [1]. Although granularity is an intrinsic part of human endeavors, like many other cases, it is quite difficult to come up with its comprehensive definition everybody could agree upon [11]. Given this, it is worth considering here a multidisciplinary approach to this paradigm.

Not all criteria and structures used by humans are well-known, abstracted and implemented. Moreover, beside the fact that "human perception and understanding of real world depends, to a large extent, on [such] nested and hierarchical structures", [12] for any real world entity, more than a single hierarchy may exist even from the same perspective. To this end, it is clear that other formal frameworks than those yet known, for information granulation and their processing are required.

The main objective of this study is to introduce a formal framework for information granulation in terms of the criteria of coherency and proximity. The framework originated from the well-known theory of interval analysis, which enables tolerating computational errors by indicating an interval, "known in advanced, to contain the desired exact result" [7]. However, we enhanced the theory to draw a suitable framework in granular computing through relinquishing the constraint of crisp bounds in the first level – similar to the concept of multi-intervals [5]- and also by integrating the concept of distribution into the theory of intervals. The result, coming under the name of distributed intervals, provides as at least bi-criteria framework –due to the combination of the natures of intervals and distributed entities - that can be used as a conceptually rich structure in granular computing.

Discussing coherency and coherence classes, we will explain how distributed intervals support for approximation of coherent relations, when the determination of the exact coherence classes is not viable. Moreover, we discuss ways of approximation of entities based on the coherence classes.

II. GENERAL INTERVAL

Interval is defined as a compact bounded subset of the real numbers denoted by a pair of real numbers identifying its bounds (lower and upper bound, respectively). Formally, we have $A=[\underline{a},\overline{a}]=\{x\in R \mid \underline{a}\le x\le \overline{a}\}$ where $\underline{a},\overline{a}\in R$ and $\underline{a}\le\overline{a}$. "Intervals on the real line have a dual nature, as set (of real numbers) with the usual set operations, and as a new kind of number represented by pairs of real numbers with an arithmetic"[8].

In this paper, these constructs will be referred to as conventional intervals. Degenerate intervals of the form $[a,a]$ are equivalent to real numbers. Given that *IR* denotes the set of all real intervals and $[\,]$ indicates null interval, then for $A,B\in IR$ the following notions are defined [3, 7].

- $w(A)=\overline{a}-\underline{a}$, width of A.
- $r(A)=(\overline{a}-\underline{a})/2$, radius of A
- $m(A)=(\overline{a}+\underline{a})/2$, mid-point of A.
- $|A|=\max\{|\underline{a}|,|\overline{a}|\}$, magnitude of A.
- $A\subseteq B$ iff $\underline{b}\le\underline{a}\le\overline{a}\le\overline{b}$.
- $A=B$ iff $\underline{a}=\underline{b}$ *and* $\overline{a}=\overline{b}$
- $A<B$ iff $\overline{a}<\underline{b}$.
- $A\bigcap B=\{x\in R \mid x\in A \text{ and } x\in B\}$
- $A\bigcup B=\{x\in R \mid x\in A \text{ or } x\in B\}$ if $A\bigcap B\ne[\,]$, otherwise the union is undefined. In [4] instead e.g., interval hull is considered.

Using the symbol $\circ$ to denote these four basic operations +, -, * and /, the arithmetic operations on intervals are defined as $A\circ B=\{a\circ b \mid a\in A \text{ and } b\in B\}$ where if $0\in B$ then / is undefined. It could be proved that the four operations are continuous mapping from R^2 to R and $A\circ B\in IR$.

By defining a proper distance function for example, $d(A,B)=\max(|\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|)$ [7], IR could be made into a metric space.

Definition: Given $A,B \in IR$, then $A \preceq B$ -A is loosely smaller than or equal to B- if and only if $\overline{a} < \overline{b}$ or $A \subseteq B$.

Definition: Given $A = [\underline{a},\overline{a}]$, $B = [\underline{b},\overline{b}]$, we define $Inf(A,B) = [\min(\underline{a},\underline{b}), \min(\overline{a},\overline{b})]$, $Sup(A,B) = [\max(\underline{a},\underline{b}), \max(\overline{a},\overline{b})]$.

Definition: General Interval is defined to be $[A_L, A_R]$ where $A_L =_{def} [\underline{a}, \underline{a}+\varepsilon_L]$, $A_R =_{def} [\overline{a}-\varepsilon_R, \overline{a}]$, $\underline{a},\overline{a},\varepsilon_L,\varepsilon_R \in R$, $\varepsilon_L,\varepsilon_R \geq 0$ and $\underline{a}+\varepsilon_L \leq \overline{a}-\varepsilon_R$. A_L and A_R denote the left and right bounds of the general interval respectively. If $\varepsilon_L,\varepsilon_R$ are known, we show general interval as $⁅\underline{a},\overline{a}⁆$. It is clear that if $\varepsilon_L = \varepsilon_R = 0$, then the concept of the general interval reduces to the conventional interval and consequently real numbers are shown as degenerated conventional intervals. However, if either $\varepsilon_L = 0$ *or* $\varepsilon_R = 0$, we may present the interval as $[\underline{a},\overline{a}⁆$ and $⁅\underline{a},\overline{a}]$ respectively.

Definition: Given $\mathbb{IR}$ denotes the family of all real general intervals, we define unary operators $\uparrow$ and $\downarrow$ on any $A \in \mathbb{IR}$ as $A\uparrow = [\underline{a},\overline{a}]$ and $A\downarrow = [\underline{a}+\varepsilon_L, \overline{a}-\varepsilon_R]$ where $A\uparrow, A\downarrow \in IR$ and $A\downarrow \subseteq A\uparrow$.

Definition: Given $A,B \in \mathbb{IR}$, we define the following notions:

- $w(A) = A_R - A_L = [\overline{a}-\underline{a}-\varepsilon_L-\varepsilon_R\ ,\ \overline{a}-\underline{a}]$ $= [w(A\downarrow), w(A\uparrow)]$, width of A.
- $r(A) = w(A)/2 = (A_R - A_L)/2 = [w(A\downarrow), w(A\uparrow)]/2$, radius of A.
- $m(A) = (A_L + A_R)/2 = [\overline{a}+\underline{a}-\varepsilon_R, \overline{a}+\underline{a}+\varepsilon_L]/2$, mid-point of A.
- $|A| = [|A\downarrow|, |A\uparrow|]$, magnitude of A.
- $A < B$ iff $A_R < B_L$.
- $A = B$ iff $A_L = B_L$ *and* $A_R = B_R$.
- $A \subset B$ iff $A\downarrow \subseteq B\downarrow$ *and* $A\uparrow \subseteq B\uparrow$.
- $A \cap B =$ $[Sup(A_L,B_L), Inf(A_R,B_R)]$
- $A \cup B =$
  - $[Inf(A_L,B_L), Sup(A_R,B_R)]$ if $A\uparrow \cap B\uparrow \neq ⁅⁆$,
  - Otherwise the union is undefined in the framework of general intervals.
- $A \setminus B$
  - $= ⁅⁆$, if $A \subseteq B$,
  - $= [A_L, Inf(A_R,B_L)] \cup [Sup(A_L,B_R), A_R]$, otherwise.

    Be noticed that in the framework of general intervals, the result may be undefined regarding to the union operation.

Definition: Given $A,B \in \mathbb{IR}$, then $A \preceq B$ -A is said to be loosely smaller than or equal to B- if and only if $(A \setminus B)_R \preceq (B \setminus A)_R$.

Operational definition of the arithmetic operations reveals the relationships shown below. Consider that $A,B \in \mathbb{IR}$ then we have

$⁅\underline{a},\overline{a}⁆$ + $⁅\underline{b},\overline{b}⁆$ = $⁅\underline{a}+\underline{b},\overline{a}+\overline{b}⁆$. In special cases we have $⁅\underline{a},\overline{a}⁆$ + $[\underline{b},\overline{b}]$ = $⁅\underline{a}+\underline{b},\overline{a}+\overline{b}⁆$ and likewise $⁅\underline{a},\overline{a}]$ + $[\underline{b},\overline{b}⁆$ = $⁅\underline{a}+\underline{b},\overline{a}+\overline{b}⁆$ and so on. Similarly, $⁅\underline{a},\overline{a}⁆$ - $⁅\underline{b},\overline{b}⁆$ = $⁅\underline{a}-\overline{b},\overline{a}-\underline{b}⁆$ where in special cases we have $⁅\underline{a},\overline{a}⁆$ - $[\underline{b},\overline{b}]$ = $⁅\underline{a}-\overline{b},\overline{a}-\underline{b}⁆$ likewise $⁅\underline{a},\overline{a}]$ - $[\underline{b},\overline{b}⁆$ = $⁅\underline{a}-\overline{b},\overline{a}-\underline{b}]$ and $⁅\underline{a},\overline{a}]$ - $⁅\underline{b},\overline{b}]$= $⁅\underline{a}-\overline{b},\overline{a}-\underline{b}⁆$ and so on. Multiplication and division are expressed as follows.

$⁅\underline{a},\overline{a}⁆ * ⁅\underline{b},\overline{b}⁆ = ⁅\min(\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b}),\ \max(\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b})⁆$,

$⁅\underline{a},\overline{a}⁆ / ⁅\underline{b},\overline{b}⁆ = ⁅\underline{a},\overline{a}⁆ * ⁅1/\overline{b}, 1/\underline{b}⁆$ whilst

$⁅1/\overline{b}, 1/\underline{b}⁆ =_{def} [[1/\overline{b}\ ,\ 1/(\overline{b}-\varepsilon_R^B)]\ ,\ [1/(\underline{b}+\varepsilon_L^B)\ ,\ 1/\underline{b}]]$

given $0 \notin B\uparrow$. The resulting $\varepsilon_L,\varepsilon_R$ depends on the terms involved and would be calculated accordingly.

Theorem: Given $A,B \in \mathbb{IR}$ and $\circ$ stands for the four operations +, -, * and /, then $A \circ B \in \mathbb{IR}$.

Consider $\mathbb{IR}$ it is possible to construct a metric space by forming a suitable distance function. As a simple example, given $A,B \in \mathbb{IR}$, we may consider $\mathbb{d}: \mathbb{IR} * \mathbb{IR} \to R$ defined as $\mathbb{d}(A,B) = \max(d(A\downarrow,B\downarrow), d(A\uparrow,B\uparrow))$. It can be easily observed that $\forall A,B,C \in \mathbb{IR}$ we have:

$\mathbb{d}(A,A) = 0$,

$\mathbb{d}(A,B) = \mathbb{d}(B,A)$,

$\mathbb{d}(A,B) \geq 0$,

$\mathbb{d}(A,C) \leq \mathbb{d}(A,B) + \mathbb{d}(B,C)$.

III. Distributed Interval

Given the definition of general intervals, we define distributed interval as the collection of finite number of general intervals. More specifically we have

$$\widetilde{A} = A_1 \cup A_2 \cup \ldots\ldots \cup A_n = \bigcup_{i=1}^{n} A_i,$$ where $A_i \in \mathbb{IR}$. It is clear that for $A_i, A_j, i \neq j, i,j \in \{1,\ldots,n\}$ if $A_i\uparrow \cap A_j\uparrow \neq ⁅⁆$ the general-interval union may take place and the result would be substituted, otherwise they are left unchanged. $\widetilde{A}$ would be referred to as a reduced distributed interval if $A_i\uparrow \cap A_j\uparrow = ⁅⁆$ *for* $i \neq j$, $i,j \in \{1,\ldots,n\}$. Through the paper by distributed interval we mean reduced one unless otherwise specified explicitly. Without any loss of generality, we assume that $A_i < A_{i+1}$, $i \in \{2,\ldots,n\}$. Distributed interval $\widetilde{A}$ when $n = 1$, reduces to general interval A. Be aware that the operations $\cup$ and $\setminus$ on general intervals is always meaningful in the framework of distributed intervals. Given $\widetilde{\mathbb{IR}}$ denotes the set of all real distributed intervals and $\widetilde{A},\widetilde{B} \in \widetilde{\mathbb{IR}}$ be

defined as $\widetilde{A}=\bigcup_{i=1}^{n} A_i$ and $\widetilde{B}=\bigcup_{j=1}^{m} B_j$ then we arrive at the following notions:

- $w(\widetilde{A})=\sum_{i=1}^{n} w(A_i)$, which in general is an interval number.
- $r(\widetilde{A})=w(\widetilde{A})/2$, radius of A.
- $|\widetilde{A}|=[\max(|A_1\downarrow|,|A_n\downarrow|),\max(|A_1\uparrow|,|A_n\uparrow|)]$, the magnitude of $\widetilde{A}\in\widetilde{IR}$.
- $\widetilde{A}\subset\widetilde{B}$ iff $\forall A_i, \exists B_j$ $s.t.$ $A_i\subseteq B_j$.
- $\widetilde{A}=\widetilde{B}$ iff $m=n,\ A_k=B_k$, $k\in\{1,.....,n\}$.
- $\widetilde{A}<\widetilde{B}$ iff $A_n<B_1$ we say that $\widetilde{A}$ is smaller than $\widetilde{B}$.
- $\widetilde{A}\cap\widetilde{B}=\bigcup_{i=1}^{n}(\bigcup_{j=1}^{m}(A_i\cap B_j))$
- $\widetilde{A}\cup\widetilde{B}=\bigcup_{i=1}^{n}(\bigcup_{j=1}^{m}(A_i\cup B_j))$
- $\widetilde{A}\setminus\widetilde{B}=\bigcup_{i=1}^{n}(\bigcap_{j=1}^{m}(A_i\setminus B_j))$

Definition: $\widetilde{A}$ is loosely smaller than or equal to $\widetilde{B}$, denoted by $\widetilde{A}\preceq\widetilde{B}$, if and only if, given $\widetilde{A}'=\widetilde{A}\setminus\widetilde{B}=\bigcup_{i=1}^{n'} A_i'$ and $\widetilde{B}'=\widetilde{B}\setminus\widetilde{A}=\bigcup_{j=1}^{m'} B_i'$ then $A_{n'}'\preceq B_{m'}'$. This indicates that if $\widetilde{A}<\widetilde{B}$ then $\widetilde{A}\prec\widetilde{B}$ but the reverse does not always hold. It becomes clear that $(\widetilde{IR},\preceq)$ is a lattice, while $\forall\widetilde{A},\widetilde{B}\in\widetilde{IR}$ then join is defined to be the union i.e. $\widetilde{A}\vee\widetilde{B}=\widetilde{A}\cup\widetilde{B}$ and meet is intersection that is, $\widetilde{A}\wedge\widetilde{B}=\widetilde{A}\cap\widetilde{B}$.

We also note that if the universe of discourse is finite then we can define the inverse of a given distributed interval. Given $A=[\underline{a},\overline{a}]=[A_L,A_R]$ defined on bounded universe U, then $\exists\varepsilon>0$, $\overline{A}=[\underline{U},\ A_L-\varepsilon]\cup[A_R+\varepsilon,\overline{U}]$. Consequently the inverse of the distributed interval $\widetilde{A}=A_1\cup A_2\cup.......\cup A_n=\bigcup_{i=1}^{n} A_i$ would be defined as $\overline{\widetilde{A}}=\bigcup_i(\bigcap\overline{A}_i)$.

If $\circ$ denotes the four basic operations +, -, * and /, then arithmetic operations on distributed intervals would be defined as $\widetilde{A}\circ\widetilde{B}=\bigcup_{i=1}^{n}(\bigcup_{j=1}^{m}(A_i\circ B_j))$. However, offering a fully operational definition of the arithmetic operations is not straight forward.

Theorem: Given $\widetilde{A},\widetilde{B}\in\widetilde{IR}$ and $\circ$ stands for the four operations +, -, * and /, then $\widetilde{A}\circ\widetilde{B}\in\widetilde{IR}$.

It is also possible to construct a metric space over $\widetilde{IR}$. Given $\widetilde{A},\widetilde{B}\in\widetilde{IR}$ we may define distance function $\widetilde{d}:\widetilde{IR}*\widetilde{IR}\to R$ as $\widetilde{d}(A,B)=\sum_{i=1}\min_j(\not{d}(A_i,B_j))$ where $d:IR*IR\to R$ is defined as before. It can be easily proved that $\forall\ \widetilde{A},\widetilde{B},\widetilde{C}\in\widetilde{IR}$,

$\widetilde{d}(\widetilde{A},\widetilde{A})=0$,

$\widetilde{d}(\widetilde{A},\widetilde{B})=\widetilde{d}(\widetilde{B},\widetilde{A})$,

$\widetilde{d}(\widetilde{A},\widetilde{B})\geq 0$,

$\widetilde{d}(\widetilde{A},\widetilde{C})\leq\widetilde{d}(\widetilde{A},\widetilde{B})+\widetilde{d}(\widetilde{B},\widetilde{C})$.

Based on the above definitions, the following properties hold for the $\subseteq$ relation, given $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}\in\widetilde{IR}$.

$\widetilde{A}\subseteq\widetilde{B}\Leftrightarrow\widetilde{A}\cap\widetilde{B}=\widetilde{A}$,

$\widetilde{B}\subseteq\widetilde{A}\Leftrightarrow\widetilde{A}\cap\widetilde{B}=\widetilde{B}$,

$\widetilde{A}\subseteq\widetilde{B}\ and\ \widetilde{C}\subseteq\widetilde{D}\Leftrightarrow\widetilde{A}\cap\widetilde{C}\subseteq\widetilde{B}\cap\widetilde{D}$,

$\widetilde{A}\subseteq\widetilde{B}\ and\ \widetilde{C}\subseteq\widetilde{D}\Leftrightarrow\widetilde{A}\cup\widetilde{C}\subseteq\widetilde{B}\cup\widetilde{D}$,

$\widetilde{A}\cap\widetilde{B}\subseteq\widetilde{A}$, $\widetilde{A}\cap\widetilde{B}\subseteq\widetilde{B}$,

$\widetilde{A}\subseteq\widetilde{A}\cup\widetilde{B}$, $\widetilde{B}\subseteq\widetilde{A}\cup\widetilde{B}$.

A. Distributed Intervals and Interval Sets

In effect $[\underline{a},\overline{a}]$ could be interpreted as an interval whose lower bound and upper bound are known to be in A_L and A_R respectively. In other words, the notion of the general interval arises as a consequence of our inability to precisely characterize an interval. In essence the general interval A denotes a family of intervals in the form of $A=[\underline{a},\overline{a}]=[[\underline{a},\underline{a}+\varepsilon_L],[\overline{a}-\varepsilon_R,\overline{a}]]=[A_L,\ A_R]=\{[\alpha,\beta]\in IR|\alpha\in A_L\ and\ \beta\in A_R\}$ that is $\widetilde{IR}\subset 2^{IR}$ where degenerated general interval becomes a *conventional interval* $[\underline{a},\overline{a}]$ where $\varepsilon_L=\varepsilon_R=0$. Clearly, $[A_L,\ A_R]$ as a general interval is equal to the interval set [9] $[A\downarrow,\ A\uparrow]=\{A\in IR|A\downarrow\subseteq A\subseteq A\uparrow\}$. Consequently, we can say, $\widetilde{A}\in\widetilde{IR}$ defined as $\widetilde{A}=\bigcup_{i=1}^{n} A_i$ would be rewritten as

$\widetilde{A}=\bigcup_{i=1}^{n} A_i=\bigcup_{i=1}^{n}[A_i\downarrow,\ A_i\uparrow]=[\widetilde{A}\downarrow,\ \widetilde{A}\uparrow]$

$=\{A\ |A\downarrow\subseteq A\subseteq A_i\uparrow\}$.

IV. DISSCUSSION

Distributed interval, based on the concept of general interval provide a rich structure for granulation, based on coherency and proximity. This is due to the combination of the nature and properties of intervals and distributed entities which permit granulation exploiting these two paradigms. To further explain functionality of distributed interval we consider the notion of rough sets [6].

Given is E forming an equivalence relation defined over the universe U that partitions it to equivalence classes $U \setminus E = \{E_1, E_2, ..., E_n\}$. The pair (U, E) is then called approximation space that permits approximating a set based on a collection of elements that *definitely* belong to the set, called lower bound, and by elements that *possibly* belong to the set, which constitutes an upper bound. The main challenge concerns rough set is defining E and $U \setminus E$ in a precise manner.

For the sake of simplicity, we consider U to be linearly ordered. Then each equivalence class could be modeled by a distributed interval and consequently we can express any rough set in terms of the distributed intervals. Being endowed with distributed intervals, we can approximate each equivalence class in the form of $\widetilde{E}_i = \bigcup [\underline{a_i}, \overline{a_i}]$ where if $a_i \in E_i$ then $a_i \in [\underline{a_i}, \overline{a_i}]$. Be reminded that elements in $[\underline{a_i}, \overline{a_i}]$ e.g. regarding their proximity are not distinguishable from each other. It may be argued that in this case the resulted classes may overlap which put E out of the equivalence relation space.

We name the surjective symmetric relation $\hbar \subseteq U * U$ coherence relation, that is $\{b \mid (a,b) \in \hbar\} = U$ and consequently $\{a \mid (a,b) \in \hbar\} = U$. Coherence class is to be defined as $[a]_\hbar = \{b \in U \mid a\hbar b\}$. It is clear that in general $[a]_\hbar$ and $[b]_\hbar$ where $a \neq b$ are not necessarily equal or disjoint. As a specific case, $\hbar$ would be a tolerance relation [2] or equivalence relation. We used the term coherence relation to stress that what we mean is more general than similarity. Equivalence and even tolerance relation, are mainly used to show *similarity* [10] however there are entities that would be allocated to the same granule just because of their coherency, and not necessarily according to similarity. A simple example could be words constituting a sentence; they are not similar except coherent that has forced them to be put in the same granule, here sentence. As another example, we sometimes highlight some parts of the same paragraph, i.e. put the highlighted parts of the paragraph as "aim of the paragraph". The highlighted parts are grouped mainly according to their coherency.

We may claim that coherent relation $\hbar$ in U implemented in terms of distributed intervals, as an approximation of so to say covering relations like equivalence relation, provide covering coherent classes $U \setminus \hbar = \{\hbar_1, \hbar_2, ..., \hbar_m\}$. Consequently, any given set H would be e.g. said to be in $\left(\bigcup_{H \subseteq \hbar_i} \hbar_i, \bigcup_{H \cap \hbar_i \neq \{\}} \hbar_i \right)$ or be approximated based on its distance to coherent classes. In the former case, for instance, approximated rough sets would be reformulated based on the approximated *granules*- approximated equivalence classes. On the other hand it is possible to approximate coherent relation $\hbar$ through implementing it by distributed intervals. When determining the exact members of coherence class is not viable or classes are not covering, distributed interval may come into picture.

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