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GLOBAL ATTRACTORS FOR A THREE-DIMENSIONAL CONSERVED PHASE-FIELD SYSTEM WITH MEMORY

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Abstract. We consider a conserved phase-field system on a tridimensional bounded domain. The heat conduction is characterized by memory effects depending on the past history of the (relative) temperature ϑ. These effects are represented through a convolution integral whose relaxation kernel k is a summable and decreasing function. Therefore the system consists of a linear integrodifferential equation for ϑ which is coupled with a viscous Cahn-Hilliard type equation governing the order parameter χ. The latter equation contains a nonmonotone nonlinearity ϕ and the viscosity effects are taken into account by the term $-\alpha\Delta\chi_t$, for some $\alpha \geq 0$. Thus, we formulate a Cauchy-Neumann problem depending on α. Assuming suitable conditions on k, we prove that this problem generates a dissipative strongly continuous semigroup $S^\alpha(t)$ on an appropriate phase space accounting for the past histories of ϑ as well as for the conservation of the spatial means of the enthalpy $\vartheta + \chi$ and of the order parameter. We first show, for any $\alpha \geq 0$, the existence of the global attractor $\mathcal{A}_\alpha$. Also, in the viscous case ($\alpha > 0$), we prove the finiteness of the fractal dimension and the smoothness of $\mathcal{A}_\alpha$.

1. **Introduction.** This paper is concerned with the study of the large time behavior of a phase-separation model with memory. The mathematical formulation that we are going to describe is basically a modification of the well known phase separation model proposed by G. Caginalp [5] (see also Brokate and Sprekels [4]). Let $\Omega \subset \mathbb{R}^3$ be a given bounded domain with regular boundary $\partial\Omega$. Suppose that Ω is occupied by an isotropic, rigid and homogeneous heat conductor, free of mechanical stresses. Let us denote by θ the absolute temperature, and assume that at a specific temperature value θ_c the phase-transition occurs. Then, we define the *temperature variation field*, by setting

$$\vartheta = \frac{\theta - \theta_c}{\theta_c}.$$

The variable that accounts for the presence of two phases (e.g. the concentrations of two chemical substances) is called *order parameter* χ (or *phase-field*). Both the state variables ϑ and χ depend on $x \in \Omega$ and on $t \in [0, \infty)$.

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In absence of heat sources and mechanical stresses, the *energy balance equation* reads

$$\partial_t \mathcal{E} + \nabla \cdot \mathbf{q} = 0 \quad \text{in} \quad \Omega \times [0, \infty),$$

$\mathcal{E}$ being the *internal energy* and $\mathbf{q}$ the *heat flux vector*.

If we consider only small variations of the absolute temperature and its gradient, the equation for the relative temperature can be formulated by introducing two suitable *constitutive assumptions*. Concerning the internal energy, we consider the linear relation

$$\mathcal{E}(x,t) = \mathcal{E}_{\mathrm{eq}} + c_v \theta_c \vartheta(x,t) + \ell \chi(x,t) \quad \text{for} \quad (x,t) \in \Omega \times [0, \infty),$$

$\mathcal{E}_{\mathrm{eq}}$ being the *temperature at equilibrium*, c_v the *specific heat* and ℓ the *latent heat* constant. Regarding the heat flux vector, different choices can be made. The most simple and well known is the classical Fourier law

$$(\boldsymbol{F}) \quad \mathbf{q}(x,t) = -K_{\mathrm{diff}} \nabla \vartheta(x,t) \quad \text{for} \quad (x,t) \in \Omega \times [0, \infty),$$

where K_{diff} is the (positive) *instantaneous diffusivity* coefficient. On the other hand, if we consider a linearized version of the Coleman-Gurtin law [7], we have

$$(\boldsymbol{CG}) \quad \mathbf{q}(x,t) = -K_{\mathrm{diff}} \nabla \vartheta(x,t) - \int_0^\infty k(s) \nabla \vartheta(x, t-s) ds \quad \text{for} \quad (x,t) \in \Omega \times [0, \infty).$$

Here k is a nonnegative and summable *memory kernel*. When $K_{\mathrm{diff}} = 0$ we obtain the law proposed by Gurtin and Pipkin [24],

$$(\boldsymbol{GP}) \quad \mathbf{q}(x,t) = -\int_0^\infty k(s) \nabla \vartheta(x, t-s) ds \quad \text{for} \quad (x,t) \in \Omega \times [0, \infty).$$

The latter two choices have been widely discussed in the literature. In particular, Herrera and Pavòn in [26], analyze some relevant dissipative processes (e.g. the telegraph equation and transport equation), claiming that a hyperbolic description leads to a deeper understanding of the transient states. Also, Jäckle in [27], following the phenomenological theory of thermoviscoelasticity, discusses the heat conduction processes in some materials (e.g., high viscosity liquids), pointing out how, in the frequency domain, the conduction coefficient may depend on the frequency itself. In this regard, we also refer the reader to the work of Jou and Casas-Vázquez [30] and Joseph and Preziosi [28, 29], where the concept of heat wave propagation is reviewed and interpreted in great detail. In this paper we consider the $(\boldsymbol{GP})$ heat conduction law.

Therefore, substituting the constitution laws for the internal energy and $(\boldsymbol{GP})$ into the energy balance equation, we obtain

$$(\boldsymbol{e\vartheta}) \ \partial_t \left[c_v \theta_c \vartheta(x,t) + \ell \chi(x,t) \right] - \int_0^\infty k(s) \Delta \vartheta(x, t-s) ds = 0 \quad \text{for} \quad (x,t) \in \Omega \times [0, \infty).$$

This is the first equation of our system. Note that it entails a finite speed of propagation of ϑ, provided that k is smooth enough.

We now define an evolution equation for the order parameter, following the Landau-Ginzburg approach (see, for instance, [4, Section 4.4]). Let us consider the free energy functional defined by

$$\mathcal{F}_\vartheta\{\chi\} = \int_\Omega \left[\frac{1}{2} |\nabla \chi|^2 + \Phi(\chi) - \ell \vartheta \chi \right] d\Omega,$$

where the nonlinear term Φ is usually a double well potential which accounts for the presence of two different phases (typically, $\Phi(r) = (r^2 - 1)^2$, $r \in \mathbb{R}$). The main feature of this approach to nonequilibrium processes is based on the phenomenological assumption that the equation for χ has the form

$$\tau \partial_t \chi = -\Delta \left(\frac{\delta \mathcal{F}_\vartheta\{\chi\}}{\delta \chi} \right),$$

τ being a positive relaxation time. On account of the definition of the functional, we thus derive the following Cahn-Hilliard type equation

$$(\boldsymbol{CH}) \qquad \tau \partial_t \chi - \Delta \left(-\Delta \chi + \phi(\chi) - \ell \vartheta \right) = 0 \quad \text{in} \quad \Omega \times [0, \infty),$$

having set $\Phi' = \phi$. More precisely, we shall consider a generalized version of $(\boldsymbol{CH})$, which describes the influence of viscosity effects, by means of a viscous term $\alpha \partial_t \chi$ inside the Laplace operator (see, [33] for the physical justification, cf. also [12, 13, 14]). The modified equation then reads

$$(\boldsymbol{e\chi}) \qquad \tau \partial_t \chi - \Delta \left(-\Delta \chi + \alpha \partial_t \chi + \phi(\chi) - \ell \vartheta \right) = 0 \quad \text{in} \quad \Omega \times [0, \infty),$$

where the constant $\alpha \geq 0$ is the *viscosity* parameter. We point out that the assumption $\alpha > 0$ will play a basic role in some of our results. Without loss of generality, in sequel we shall suppose $\alpha \in [0, 1]$.

Collecting equations $(\boldsymbol{e\vartheta})$-$(\boldsymbol{e\chi})$, we deduce the following nonlinear integro-partial differential system

$$\partial_t \left(c_v \theta_c \vartheta + \ell \chi \right) - \int_0^\infty k(s) \Delta \vartheta(t - s) ds = 0, \tag{1.1}$$

$$\tau \partial_t \chi - \Delta \left(-\Delta \chi + \alpha \partial_t \chi + \phi(\chi) - \ell \vartheta \right) = 0, \tag{1.2}$$

endowed with the *adiabatic* boundary conditions

$$\partial_{\boldsymbol{n}} \vartheta \;=\; \partial_{\boldsymbol{n}} \chi \;=\; \partial_{\boldsymbol{n}} \left(-\Delta \chi + \alpha \partial_t \chi + \phi(\chi) - \ell \vartheta \right) \;=\; 0 \tag{1.3}$$

on $\partial\Omega$, for $t \in \mathbb{R}$, where $\partial_{\boldsymbol{n}}$ represents the outward normal derivative to $\partial\Omega$, and initial conditions

$$\vartheta(0) = \vartheta_0 \qquad \text{in } \Omega, \tag{1.4}$$

$$\chi(0) = \chi_0 \qquad \text{in } \Omega, \tag{1.5}$$

$$\vartheta(-s) = \vartheta_1(s) \qquad \text{in } \Omega, \;\; s \in (0, \infty). \tag{1.6}$$

Here $\vartheta_0, \chi_0 : \Omega \to \mathbb{R}$ and $\vartheta_1 : \Omega \times (0, \infty) \to \mathbb{R}$ are given functions, whose properties will be discussed later on.

Systems like (1.1)-(1.2), endowed with various boundary and initial conditions, have been studied by many authors. In particular, the first existence result was provided by Novick-Cohen in [32]. There, uniqueness was proven as well under a smallness assumption on the latent heat ℓ and for a smooth kernel k. Then, Colli, Gilardi, Laurençot and Novick-Cohen in [9] improved the uniqueness result by removing the smallness of ℓ, and showed the stabilization of solutions for large times. Colli, Gilardi, Grasselli, Schimperna in [8] instead proven well-posedness and regularity for a more general class of ϕ.

The common feature of all the results quoted above is that the contributions to the convolution integrals due to the past history of the temperature up to $t = 0$ are always considered as given data, and therefore regarded as external sources. Thus, the only convolution integrals appearing in the equations are over $(0,t)$. Notice that such a formulation forces the system to become non autonomous, even if the original system is not. Therefore, in this case, the given past history approach seems not convenient to study the problem within the framework of dynamical systems (cf., for instance, [1], [25] and [40] for a general overview). This point shall be discussed in details in the next subsection.

1.1. **The past history formulation.** From now on, for the sake of simplicity, we let

$$c_v = \theta_c = \ell = \tau = 1.$$

In order to prove that our problem generates a dynamical system, we follow an approach based on an idea contained in [11], and then developed by several of authors in the context of dynamical systems (see the review papers [19, 22]). This idea consists in introducing an additional variable, usually called the *summed past history*, which in our case is

$$\eta^t(s) = -\int_0^s \Delta\left(e(t-y) - \chi(t-y)\right) dy \qquad \text{in } \Omega,\ \ (t,s) \in [0,\infty) \times (0,\infty),$$

where

$$e = \vartheta + \chi$$

is the *enthalpy density*.

It is immediate to check that η^t formally satisfies the first order hyperbolic equation

$$\partial_t \eta = -\partial_s \eta - \Delta\left(e - \chi\right) \qquad \text{in } \Omega,\ \ (t,s) \in (0,\infty) \times (0,\infty). \tag{1.7}$$

Concerning the boundary and initial conditions to associate with equation (1.7), on account of (1.3) and (1.6), we deduce

$$\eta^t(0) = 0 \qquad \text{on } \Omega,\ \ t \in (0,\infty), \tag{1.8}$$

$$\eta^0(s) = \eta_0(s) \qquad \text{in } \Omega,\ \ s \in (0,\infty) \tag{1.9}$$

with

$$\eta_0(s) = -\int_0^s \Delta\vartheta_1(y)dy, \quad s \in (0,\infty).$$

Considering then equation (1.1), and making physically reasonable assumptions on the past history and the memory kernel, we observe that a formal integration by parts yields

$$-\int_0^\infty k(s)\Delta\left(e(t-s) - \chi(t-s)\right) ds = \int_0^\infty \mu(s)\eta^t(s)ds \quad \text{in } \Omega,\ \ s \in (0,\infty),$$

where we have set

$$\mu(s) = -k'(s), \quad \forall s \in (0,\infty).$$

Then we can reformulate the original boundary and initial value problem as the following integro-partial differential system in the variables (e,χ,η).

Problem P. Find a solution (e,χ,η) to the system

$$\partial_t e + \int_0^\infty \mu(s)\eta(s)ds = 0,$$

$$\partial_t \chi - \Delta\left(-\Delta\chi + \alpha\partial_t\chi + \phi(\chi) - e + \chi\right) = 0,$$

$$\partial_t \eta = -\partial_s \eta - \Delta\left(e - \chi\right),$$

in $\Omega \times (0,\infty)$, subjected to the boundary and initial conditions

$$\begin{aligned}
\partial_{\boldsymbol{n}} e = \partial_{\boldsymbol{n}} \chi = 0, &\qquad \text{on } \partial\Omega \times (0,\infty),\\
\partial_{\boldsymbol{n}}\left(-\Delta\chi + \alpha\partial_t\chi + \phi(\chi) - e + \chi\right) = 0, &\qquad \text{on } \partial\Omega \times (0,\infty),\\
e(0) = e_0 = \vartheta_0 + \chi_0, &\qquad \text{in } \Omega,\\
\chi(0) = \chi_0 &\qquad \text{in } \Omega,\\
\eta^0 = \eta_0, &\qquad \text{in } \Omega \times (0,\infty).
\end{aligned}$$

Observe that, according to the original boundary conditions, a formal application of the Green formula yields immediately the following identities

$$\int_\Omega e(t)d\Omega = \int_\Omega e_0 d\ \Omega, \quad \int_\Omega \chi(t)d\ \Omega = \int_\Omega \chi_0\ d\Omega, \quad \int_\Omega \eta^t(s)\ d\Omega = 0,$$

for any $t \in (0,\infty)$ and any $s > 0$. The conservation of such quantities is a structural feature of our system. Such a feature also explains the reason why the system under consideration is usually called the *conserved* phase-field model (cf. [4, 5]). This feature will be taken into special account when we shall face the problem of constructing a suitable phase-space in order to study **Problem P** as a dissipative dynamical system (see Section 2). To be more precise, the constraints introduced above, though they play no role as far as well-posedness is concerned, will be essential in order to prove the existence of a bounded absorbing set (see Section 4).

The first formulation of **Problem P** can be found in [41], where existence and dissipativity of the dynamical system were proven in the nonviscous case and with mixed Dirichlet and Neumann boundary conditions for the temperature field and for the order parameter, respectively. Subsequently, in [21], the existence of the global attractor was demonstrated in the same setting as [41]. Such a task was achieved by means of sharp interpolation techniques. Nevertheless, regularity properties of the attractor as well as the existence of an exponential attractor were left as open problems. This paper is to deepen the results provided in [21] and [41] in the case of adiabatic boundary conditions.

It is also worth recalling that, in the simpler case of Coleman-Gurtin heat conduction law ($\boldsymbol{CG}$), well-posedness, dissipativity and the existence of a global attractor were proven in [23]. Further regularity properties, such as the existence of an exponential attractors were shown in [15]. Note that, in this case, the presence of an instantaneous diffusive term improves the dissipative nature of the system.

More precisely, in the mentioned results, a more general constitutive law for the internal energy was considered, namely, by assuming a thermal memory effects, i.e.,

$$\mathcal{E}_h(x,t) = \mathcal{E}_{\rm eq} + c_v\theta_c\vartheta(x,t) + \ell\chi(x,t) + \int_0^\infty h(s)\vartheta(x,t-s)ds \quad \text{for} \quad (x,t) \in \Omega\times[0,\infty),$$

h being a positive smooth function such that h' and h'' are summable, and $h(0) > 0$. This law can also be handled by means of the techniques we shall apply, but we chose to neglect h for the sake of clarity. However, note that the presence of h would also improve the dissipative nature of the system (compare with [18] and [21]).

The goal of the present paper is to investigate the existence of the global attractor both for the viscous case ($\alpha > 0$) and the nonviscous one ($\alpha = 0$). In the latter case we shall also demonstrate that the global attractor is smooth and possesses finite fractal dimension. We conclude by observing that it seems particularly hard to establish analogous results in the nonviscous case.

2. Preliminary tools.

This section is devoted to the construction of the functional setting which will be used to treat the equations, and to recall many results that will be useful in the sequel. Since most of the tools that we need are well known, we shall omit the proofs, providing appropriate references when necessary.

2.1. Function spaces and operators.

Let H be the Hilbert space $L^2(\Omega)$ of the measurable functions which are square summable on Ω, endowed with the usual scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$.

Given any $\omega \in H$ we define the *spatial mean value* of ω on Ω as the real number

$$m_\omega = \frac{1}{|\Omega|} \langle \omega, 1 \rangle.$$

We then introduce

$$H_0 = \{ \omega \in H : m_\omega = 0 \}.$$

Denoting, as usual, by Δ the spatial Laplacian, we now define the (unbounded) operators

$$B : \mathcal{D}(B) \to H_0 \quad \text{and} \quad B_0 : \mathcal{D}(B_0) \to H_0$$

by setting

$$B = -\Delta, \quad \mathcal{D}(B) = \{ \omega \in H^2(\Omega) : \partial_{\boldsymbol{n}} \omega = 0 \text{ a.e. on } \partial\Omega \},$$

$$B_0 = -\Delta, \quad \mathcal{D}(B_0) = \mathcal{D}(B) \cap H_0.$$

Here the symbol $\partial_{\boldsymbol{n}}$ denotes the outward normal derivative. Since B_0 is a strictly positive operator, we can define

$$V_0^r = \mathcal{D}(B_0^{r/2}), \quad \forall r \in \mathbb{R},$$

as well as the shorthands

$$V_0 = V_0^1 \quad \text{and} \quad W_0 = V_0^2.$$

For further use, we also introduce the Hilbert spaces

$$V = H^1(\Omega) \quad \text{and} \quad W = \mathcal{D}(B),$$

endowed with the norms

$$\|\omega\|_V^2 = \|\omega\|^2 + \|P\omega\|_{V_0}^2 \quad \text{and} \quad \|\omega\|_W^2 = \|\omega\|_V^2 + \|P\omega\|_{W_0}^2;$$

it is easy to realize that the norms defined above are equivalent, respectively, to the usual norms in $H^1(\Omega)$ and $H^2(\Omega)$.

By means of the Poincaré inequality (see, for instance, [40]), we deduce

$$\|\omega\| \le c_P \|\nabla \omega\|, \quad \forall\, \omega \in V_0,$$

$c_P > 0$ being the Poincaré-Wirtinger constant, and $\nabla : V \to H^3$ the gradient operator. Here and by, we replace $\|\cdot\|_{X^3}$ with the shorter notation $\|\cdot\|_X$, for any vector valued Banach space $X^3 = X \times X \times X$, for the sake of convenience.

In the sequel, we shall need to work in nonzero mean function spaces (see, in particular, Sections 5 and 6). To this purpose, on account to the definitions and notation introduced in the previous subsection for any $\sigma > 0$, we can consider the Banach space V_σ equipped with the norm

$$\|\omega\|^2_{V_\sigma} = \|\omega\|^2 + \|P\omega\|^2_{V_0^\sigma}.$$

Notice immediately that there holds

$$\|P\omega\|_{V_0^\sigma} \leq \|\omega\|_{V_\sigma}, \quad \forall \omega \in H.$$

Throughout the rest of the paper, in order to derive suitable controls on the physically relevant quantities discussed in the introduction, we shall often need to estimate functions in the norms introduced above. Therefore, we need to recall some consequences of the Sobolev embedding theorems (see, e.g., in [40]).

Making the identification $H \equiv H^*$ (here and by, the superposed asterisk denotes the topological dual of a Banach space), we have the compact and dense embeddings

$$W \hookrightarrow V \hookrightarrow H \hookrightarrow V^* \hookrightarrow W^* \tag{2.1}$$

and

$$W_0 \hookrightarrow V_0 \hookrightarrow H_0 \hookrightarrow V_0^* \hookrightarrow W_0^*. \tag{2.2}$$

Notice that, according to the notation introduced above, we have

$$V_0^* = V_0^{-1} \quad \text{and} \quad W_0^* = V_0^{-2}.$$

Moreover, there holds

$$V \hookrightarrow L^p(\Omega),\ \forall p \in [2,6], \quad W \hookrightarrow C(\overline{\Omega}), \quad \text{and} \quad V_0 \hookrightarrow V, \quad W_0 \hookrightarrow W, \tag{2.3}$$

and, for any $\sigma \in (0, 3/2)$

$$V_0^\sigma \hookrightarrow V_\sigma \quad \hookrightarrow \quad L^{6/(3-2\sigma)}(\Omega), \tag{2.4}$$

$$W^{1,6/(3-2\sigma)}(\Omega) \quad \hookrightarrow \quad V_\sigma. \tag{2.5}$$

Here we denote by $W^{n,p}(\Omega)$ the usual Sobolev space of order n and exponent $p \in [1, \infty]$.

2.2. **Assumptions on the ϕ and μ.** In order to state our results, we need to make some structural assumptions on the nonlinearity and on the memory kernel. Concerning the former one, the assumptions that we consider include (and slightly generalize) the case of the derivative of a double-well potential. Concerning the latter, the key property to ensure the dissipativity of our system (cf. Section 4) is the exponential decay of μ.

Conditions on ϕ. Let $\phi \in C^2(\mathbb{R})$ and assume that there exist $c_0 > 0$ and $c_1, c_2 \geq 0$ such that

$$(\boldsymbol{H1}) \quad r\phi(r) \geq c_0 r^4 - c_1, \quad \forall r \in \mathbb{R},$$

$$(\boldsymbol{H2}) \quad |\phi''(r)| \leq c_2(1 + |r|), \quad \forall r \in \mathbb{R}.$$

Remark 1. As outlined in [20], under assumptions $(\boldsymbol{H1})$ and $(\boldsymbol{H2})$, the function ϕ admits a decomposition of the form $\phi = \phi_0 + \phi_1$, with $\phi_0, \phi_1 \in C^2(\mathbb{R})$ such that

$$(\boldsymbol{H1_0}) \quad r\phi_0(r) \geq 0, \quad \forall r \in \mathbb{R},$$

$$(\boldsymbol{H2_0}) \quad |\phi''(r)| \leq c_3(1+|r|), \quad \forall r \in \mathbb{R},$$

and

$$(\boldsymbol{H1_1}) \quad \liminf_{|r|\to\infty} \frac{\phi_1(r)}{r} > -\frac{1}{c_P},$$

$$(\boldsymbol{H2_1}) \quad |\phi_1(r)| \leq c_4(1+|r|^\theta), \quad \forall r \in \mathbb{R},$$

for some constants $c_3, c_4 \geq 0$ and $\theta \in [0,3)$. Without loss of generality we can suppose $\theta \in [2,3)$.

Conditions on μ. Let $\mu : (0,\infty) \to (0,\infty)$ be a summable function such that

$$(\boldsymbol{K1}) \quad \mu \in C^1((0,\infty)) \cap L^1(0,\infty),$$

$$(\boldsymbol{K2}) \quad \mu(s) \geq 0, \quad \mu'(s) \leq 0, \quad \forall s \in (0,\infty),$$

$$(\boldsymbol{K3}) \quad k_0 = \int_0^\infty \mu(s) ds > 0, \quad \forall s \in (0,\infty),$$

$$(\boldsymbol{K4}) \quad \mu'(s) + \lambda\mu(s) \leq 0, \quad \forall s \in (0,\infty),$$

for some $\lambda > 0$.

Remark 2. Notice that μ is decreasing and the Gronwall Lemma entails the exponential decay

$$\mu(s) \leq \mu(s_0)e^{-\lambda(s-s_0)}, \quad \forall s \geq s_0 > 0. \tag{2.6}$$

Notice also that μ is allowed to be unbounded in a right neighborhood of 0.

2.3. **The past history function space.** The presence of memory effects in our phase-field system requires the introduction of suitable past history spaces.
Let $r \in \mathbb{R}$. On account of assumptions $(\boldsymbol{K1})$-$(\boldsymbol{K2})$, we consider the family of weighted Hilbert spaces

$$\mathcal{M}_r = L^2_\mu(0,\infty; V_0^{r-1}),$$

endowed with the inner product defined by

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{M}_r} = \int_0^\infty \mu(s) \langle \eta_1(s), \eta_2(s) \rangle_{V_0^{r-1}} ds, \quad \forall\ \eta_1, \eta_2 \in \mathcal{M}_r.$$

For the sake of clarity, from now on we will use the shorthand $\mathcal{M}$ in place of $\mathcal{M}_0$, and $\mathcal{N}$ in place of $\mathcal{M}_1$. In these cases, the norms become, respectively,

$$\|\eta\|^2_{\mathcal{M}} = \int_0^\infty \mu(s)\|\eta(s)\|^2_{V_0^*} ds \quad \text{and} \quad \|\eta\|^2_{\mathcal{N}} = \int_0^\infty \mu(s)\|\eta(s)\|^2 ds.$$

Notice that the embeddings $\mathcal{M}_{r_1} \subset \mathcal{M}_{r_2}$, for $r_1 > r_2$, are clearly continuous but not compact. To construct a functional space which is compactly embedded in $\mathcal{M}$, we proceed as follows (see [15] and [16] for details). Let T be the linear operator on $\mathcal{M}$ with domain

$$\mathcal{D}(T) = \{\eta \in \mathcal{M} : \partial_s \eta \in \mathcal{M},\ \eta(0) = 0\},$$

defined by

$$T\eta = -\partial_s \eta,$$

where $\partial_s \eta$ is the distributional derivative of η with respect to the internal variable s. We also define, for any $\eta \in \mathcal{M}$, the *tail* of η in $\mathcal{M}$, that is the function

$$\mathcal{T}_\eta : [1,\infty) \rightarrow [0,\infty)$$

given by

$$\mathcal{T}_\eta(x) = \int_{(0,1/x)\cup(x,\infty)} \mu(s)\|\eta(s)\|^2_{V_0^*}\, ds.$$

Thus, on account of an immediate generalization of [37, Lemma 5.5], we have

Lemma 2.1. *Let $\mathcal{C} \subset \mathcal{M}$ such that*

$$\sup_{\eta\in\mathcal{C}} \left[\|\eta\|_{\mathcal{M}_r} + \|T\eta\|_{\mathcal{M}_{-1}}\right] < \infty \quad \text{and} \quad \lim_{x\to\infty}\left[\sup_{\eta\in\mathcal{C}} \mathcal{T}_\eta(x)\right] = 0.$$

for some $r > 0$. Then $\mathcal{C}$ is relatively compact in $\mathcal{M}$.

The representation formula. On account of notation and notion introduced so far in this section, we can now come back to equation (1.7) and provide its rigorous formulation within the theory of strongly continuous linear semigroups, which has been detailed in [19]. Let us recall the following

Theorem 2.2. *The operator $T : \mathcal{D}(T) \to \mathcal{M}$ is the generator of the right-translation (strongly continuous) linear semigroup of operators on the space $\mathcal{M}$.*

Moreover, we have

Corollary 1. *Let $T > 0$ to be fixed. Then, for $f \in L^1(0,T;V_0^*)$ and $\eta_0 \in \mathcal{M}$, the Cauchy problem*

$$\begin{cases} \partial_t \eta^t = T\eta^t + f, & t \in (0,\infty), \\ \eta^0 = \eta_0, \end{cases}$$

admits a unique solution $\eta \in C([0,\infty);\mathcal{M})$ which has the explicit representation formula

$$\eta^t(s) = \begin{cases} \displaystyle\int_0^s f(t-\tau)d\tau, & 0 < s \le t, \\ \displaystyle\eta_0(s-t) + \int_0^t f(t-\tau)d\tau, & s > t. \end{cases} \tag{2.7}$$

Remark 3. By means of Theorem 2.2, we learn also that the inclusion $\mathcal{D}(T) \subset \mathcal{M}$ is dense.

2.4. **The phase-space.** We are now in a position to define the functional setup of our investigation, namely, the phase-space for our dynamical system. We then define the product space

$$\mathcal{H} = H \times V \times \mathcal{M}.$$

Proposition 1. *$\mathcal{H}$ is a Hilbert space, if endowed with the inner product*

$$\langle (e_1,\chi_1,\eta_1),(e_2,\chi_2,\eta_2)\rangle_{\mathcal{H}} = \langle e_1,e_2\rangle + \langle \chi_1,\chi_2\rangle_V + \langle \eta_1,\eta_2\rangle_{\mathcal{M}},$$

for all $(e_1,\chi_1,\eta_1),(e_2,\chi_2,\eta_2) \in \mathcal{H}$.

From now on, we agree to denote by $B_{\mathcal{H}}(R,z)$ the ball of radius R and center z in $\mathcal{H}$. In the case $z=0$, we shall use the shorthand $B_{\mathcal{H}}(R)$.

We also recall that the *Hausdorff semidistance* between subsets $\mathcal{A}$ and $\mathcal{B}$ of a metric space $\mathcal{X}$ endowed with a distance d is defined as

$$\operatorname{dist}_{\mathcal{X}}(\mathcal{A},\mathcal{B}) = \sup_{a\in\mathcal{A}} d\,(a,\mathcal{B}) = \sup_{a\in\mathcal{A}} \inf_{b\in\mathcal{B}} d(a,b).$$

Notice that the Hausdorff semidistance is not symmetric (so it is not a distance). In particular, $\operatorname{dist}_{\mathcal{X}}(\mathcal{A},\mathcal{B})=0$ if and only if $\mathcal{A}\subset\overline{\mathcal{B}}$.

Finally, on account of the fact that the spatial means e and χ are constant in time, we also consider the function space

$$\mathcal{H}_{\beta,\gamma} = \left\{(e,\chi,\eta)\in\mathcal{H} : \left|m_e\right| \le \beta \text{ and } \left|m_\chi\right| \le \gamma\right\}$$

for some fixed $\beta,\gamma\ge 0$. Notice that $\mathcal{H}_{\beta,\gamma}$ is not linear spaces. Nevertheless, it has a metric structure, as stated in the next

Proposition 2. *Let $\beta,\gamma\ge 0$ be fixed. Then $\mathcal{H}_{\beta,\gamma}$ is a complete metric space with respect to the topology induced by the Hilbert structure of $\mathcal{H}$.*

Remark 4. In the case $\beta=\gamma=0$ the space $\mathcal{H}_{0,0}$ turns out to be Hilbert.

Even in this case, we agree to denote by $B_{\mathcal{H}_{\beta,\gamma}}(R,z)$ the ball of radius R and center z in $\mathcal{H}_{\beta,\gamma}$. Moreover, if $z=0$, we shall use the shorthand $B_{\mathcal{H}_{\beta,\gamma}}(R)$.

3. Well-posedness.

On account of the previous section, we can now introduce the rigorous operator formulation of our problem, namely,

Problem $\boldsymbol{P_\alpha}$. Given $(e_0,\chi_0,\eta_0)\in\mathcal{H}$, find $(e,\chi,\eta)\in C\left(\mathcal{H};\mathcal{H}\right)$ satisfying the equations

$$\partial_t e + \int_0^\infty \mu(s)\eta(s)ds = 0, \tag{3.1}$$

$$\partial_t\chi + B_0\left(B_0 P\chi + \alpha\partial_t\chi + P\phi(\chi) - P(e_c-\chi_c)\right) = 0, \tag{3.2}$$

$$\partial_t\eta = T\eta + B_0 P(e-\chi), \tag{3.3}$$

$$(e(0),\chi(0),\eta(0)) = (e_0,\chi_0,\eta_0), \tag{3.4}$$

where equation (3.3) has to be interpreted in a distributional sense.

3.1. Semigroup generation.

The next theorem ensures the existence of such a solution, and the proofs can be obtained by approximating problem $\boldsymbol{P_\alpha}$ by means of a suitable Faedo-Galerkin scheme. Details go exactly like in [41] (see also [23]).

Theorem 3.1. *Let assumptions (**H1**)-(**H2**) and (**K1**)-(**K3**) hold true. Then, for any $T>0$, there exists a solution to problem $\boldsymbol{P_\alpha}$ such that*

$$(e,\chi,\eta)\in C\left([0,T];\mathcal{H}\right).$$

Moreover, the further regularity properties hold

$$\begin{aligned} \partial_t e &\in C([0,T];V_0^*), \\ \chi &\in L^2(0,T;W)\cap H^1(0,T;V^*), \\ \alpha\,\partial_t\chi &\in L^2(0,T;H_0), \\ \omega &\in L^2(0,T;V). \end{aligned}$$

Remark 5. From now on, whenever necessary, we shall always assume to work within a *regularization scheme* provided by the constructive Faedo-Galerkin method, devised in [41]. Such a remark is crucial, in order to make rigorous the formal derivation of most of the estimates in the following.

The next theorem provides the continuous dependence estimate for the solutions of problem $\boldsymbol{P_\alpha}$.

Theorem 3.2. *For any $T>0$ and any $R>0$, there exists a positive and increasing continuous function $C_0=C_0(R):[0,T]\to[0,\infty)$, independent of $\alpha\in[0,1]$, such that, given $z_{0,1},z_{0,2}\in\mathcal{H}$ with $\|z_{0,i}\|_{\mathcal{H}}\le R$, the following continuous dependence estimate holds*

$$\begin{aligned} &\|e_1(t)-e_2(t)\|^2+\|\chi_1(t)-\chi_2(t)\|_V^2+\|\eta_1^t-\eta_2^t\|_{\mathcal{M}}^2 \\ &+\int_0^t\Big[\|\chi_1(\tau)-\chi_2(\tau)\|_W^2+\|\partial_t\chi_1(\tau)-\partial_t\chi_2(\tau)\|_{V_0^*}^2+\alpha\|\partial_t\chi_1(\tau)-\partial_t\chi_2(\tau)\|^2\Big]\,d\tau \\ &\le C_0(t)\|z_{0,1}-z_{0,2}\|_{\mathcal{H}}^2,\quad \forall t\in[0,T], \end{aligned}$$

where we call $(e_i,\chi_i,\eta_i)\in C([0,T];\mathcal{H})$ the solution to $\boldsymbol{P_\alpha}$ with initial datum $z_{0,i}$, for $i=1,2$.

As an immediate consequence of Theorems 3.1 and 3.2, we have the generation theorem

Theorem 3.3. *Problem $\boldsymbol{P_\alpha}$ generates a strongly continuous semigroup $S^\alpha(t)$, both on the phase-space $\mathcal{H}$ and on the phase-space $\mathcal{H}_{\beta,\gamma}$, for any fixed $\beta,\gamma\ge0$.*

Let us observe that, for any $\alpha\in[0,1]$, $S^\alpha(t)$ is injective on $\mathcal{H}$. This is an immediate consequence of the backward uniqueness property (cf., e.g., [40, Chap.III, Sec.6] and references therein).

Proposition 3. *For any given $z_{0,1},z_{0,2}\in\mathcal{H}$, if $S^\alpha(\tau)z_{0,1}=S^\alpha(\tau)z_{0,2}$ for some $\tau>0$, then $S^\alpha(t)z_{0,1}=S^\alpha(t)z_{0,2}$, for all $t\in[0,\infty)$.*

Proof. Denoting by $(\widetilde{e}_0,\widetilde{\chi}_0,\widetilde{\eta}_0)$ the difference of two initial data in $\mathcal{H}$, let $(\widetilde{e},\widetilde{\chi},\widetilde{\eta})$ be the difference of the corresponding solutions, that coincide at time τ. Since $\widetilde{\eta}^\tau=0$, we get directly from (2.7) (with $f=B_0P(e-\chi)$) that $\widetilde{\eta}_0=0$ and so $\widetilde{\eta}^t=0$ in $[0,\tau]$. Owing to equation (3.3), we deduce that $\widetilde{e}-\widetilde{\chi}=0$ in $[0,\tau]$. Then, equation (3.1), written for the differences, yields $\partial_t\widetilde{\chi}=0$ almost everywhere in $\Omega\times[0,\tau]$, so that $\widetilde{\chi}=0$ in $[0,\tau]$. Equality for $t>\tau$ follows from uniqueness. □

3.2. **Proof of Theorem 3.2.** In the course of this proof, let c and c_* be generic positive constants depending on the structural data of the problem and on T only. We also define, for all $t\in[0,T]$, the function

$$\Theta(t)=1+\|\chi_1(t)\|_W^2+\|\chi_2(t)\|_W^2.$$

On account of Theorem 3.1 we have $\Theta \in L^1(0,T)$ and, besides,

$$\|\chi_1(t)\|_V^2 + \|\chi_2(t)\|_V^2 \le c, \quad \forall t \in [0,T]. \tag{3.5}$$

For $z_{0,1}, z_{0,2} \in \mathcal{H}$, we set

$$\widetilde{z}_0 = (\widetilde{e}_0, \widetilde{\chi}_0, \widetilde{\eta}_0) = (e_{0,1}, \chi_{0,1}, \eta_{0,1}) - (e_{0,2}, \chi_{0,2}, \eta_{0,2}).$$

Then the difference of trajectories, defined as

$$\widetilde{z}(t) = (\widetilde{e}(t), \widetilde{\chi}(t), \widetilde{\eta}^t) = (e_1(t), \chi_1(t), \eta_1(t)) - (e_2(t), \chi_2(t), \eta_2(t))$$

fulfills the system

$$\partial_t \widetilde{e} + \int_0^\infty \mu(s)\widetilde{\eta}(s)ds = 0, \tag{3.6}$$

$$\partial_t \widetilde{\chi} + B_0\big(B_0 P\widetilde{\chi} + \alpha\partial_t\widetilde{\chi} + P\left(\phi(\chi_1) - \phi(\chi_2)\right) - P(\widetilde{e} - \widetilde{\chi})\big) = 0, \tag{3.7}$$

$$\partial_t \widetilde{\eta} = T\widetilde{\eta} + B_0 P(\widetilde{e} - \widetilde{\chi}), \tag{3.8}$$

$$\widetilde{z}(0) = \widetilde{z}_0. \tag{3.9}$$

We consider the product of (3.8) by $\widetilde{\eta}$ in $\mathcal{M}$. This yields

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\eta}\|_{\mathcal{M}}^2 = \int_0^\infty \mu(s)\langle B_0^{-1/2}T\widetilde{\eta}(s), B_0^{-1/2}\widetilde{\eta}(s)\rangle ds + \int_0^\infty \mu(s)\langle P\left(\widetilde{e} - \widetilde{\chi}\right), \widetilde{\eta}(s)\rangle ds.$$

Since we work in a regularization scheme, on account of assumption (***K2***), it is possible to prove that (see [21, Theorem 3.1])

$$\int_0^\infty \mu(s)\langle B_0^{-1/2}T\widetilde{\eta}(s), B_0^{-1/2}\widetilde{\eta}(s)\rangle ds \le 0.$$

Henceforth, we have

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\eta}\|_{\mathcal{M}}^2 \le \int_0^\infty \mu(s)\langle P\left(\widetilde{e} - \widetilde{\chi}\right), \widetilde{\eta}(s)\rangle ds. \tag{3.10}$$

Consider the product of equation (3.6) by $P(\widetilde{e} - \widetilde{\chi})$. Keeping (3.10) into account, we get

$$\frac{1}{2}\frac{d}{dt}\left[\|P\left(\widetilde{e} - \widetilde{\chi}\right)\|^2 + \|\widetilde{\eta}\|_{\mathcal{M}}^2\right] + \langle \partial_t\chi, P\left(\widetilde{e} - \widetilde{\chi}\right)\rangle \le 0. \tag{3.11}$$

We now perform the following products of equation (3.7) by suitable test functions.

- By $B_0^{-1}\partial_t\widetilde{\chi}$, to get

$$\frac{1}{2}\frac{d}{dt}\|P\widetilde{\chi}\|_{V_0}^2 + \|\partial_t\widetilde{\chi}\|_{V_0^*}^2 + \alpha\|\partial_t\widetilde{\chi}\|^2 - \langle \partial_t\widetilde{\chi}, P(\widetilde{e} - \widetilde{\chi})\rangle = -\langle \phi(\chi_1) - \phi(\chi_2), \partial_t\widetilde{\chi}\rangle.$$

Notice that, by the Young inequality, we have

$$-\langle \phi(\chi_1) - \phi(\chi_2), \partial_t\widetilde{\chi}\rangle \le \|\phi(\chi_1) - \phi(\chi_2)\|_V^2 + \frac{1}{2}\|\partial_t\widetilde{\chi}\|_{V_0^*}^2. \tag{3.12}$$

On the other hand, there holds

$$\|\phi(\chi_1) - \phi(\chi_2)\|_V^2 \le \|\phi(\chi_1) - \phi(\chi_2)\|^2 + \|\left(\phi'(\chi_1) - \phi'(\chi_2)\right)\nabla\chi_1\|^2 + \|\phi'(\chi_1)\nabla\widetilde{\chi}\|^2. \tag{3.13}$$

In order to estimate terms on the right-hand side of (3.13), we shall use assumption (***H2***) and bound (3.5). Concerning the first summand, it is immediate to check that

$$\|\phi(\chi_1)-\phi(\chi_2)\|^2 \quad \leq \quad c\left(\|\chi_1\|_V^4+\|\chi_2\|_V^4\right)\|\widetilde{\chi}\|_V^2 \leq c\|\widetilde{\chi}\|_V^2. \qquad (3.14)$$

Concerning the second and third summands, we use the generalized Hölder inequality with suitable exponents, to get

$$\begin{aligned}\|\left(\phi'(\chi_1)-\phi'(\chi_2)\right)\nabla\chi_1\|^2 &\leq c\left(\|\chi_1\|_V^2+\|\chi_2\|_V^2\right)\|\nabla\chi_1\|_V^2\|\widetilde{\chi}\|_V^2 \qquad (3.15)\\ &\leq c\|\chi_1\|_W^2\|\widetilde{\chi}\|_V^2 \leq c\Theta\|\widetilde{\chi}\|_V^2,\end{aligned}$$

and

$$\|\phi(\chi_1)-\phi(\chi_2)\|_V^2 \quad \leq \quad c\left(1+\|\chi_1\|_V^4\right)\|\nabla\widetilde{\chi}\|_V^2 \leq c_*\|\widetilde{\chi}\|_W^2. \qquad (3.16)$$

Substituting controls (3.14)-(3.16) in (3.13), the above differential inequality yields

$$\frac{1}{2}\frac{d}{dt}\|P\widetilde{\chi}\|_{V_0}^2+\frac{1}{2}\|\partial_t\widetilde{\chi}\|_{V_0^*}^2+\alpha\|\partial_t\widetilde{\chi}\|^2+\langle\partial_t\widetilde{\chi},P(\widetilde{e}-\widetilde{\chi})\rangle \leq c\Theta\|\widetilde{\chi}\|_V^2+c_*\|\widetilde{\chi}\|_W^2. \qquad (3.17)$$

- By $v=P\widetilde{\chi}$, to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|P\widetilde{\chi}\|^2+\alpha\|P\widetilde{\chi}\|_{V_0}^2\right]+\|P\widetilde{\chi}\|_{W_0}^2 &= -\langle P\left(\phi(\chi_1)-\phi(\chi_2)\right),B_0P\widetilde{\chi}\rangle\\ &\quad+\langle P(\widetilde{e}-\widetilde{\chi}),B_0P\widetilde{\chi}\rangle.\end{aligned}$$

Recalling (3.14), we deduce the inequalities

$$\begin{aligned}&-\langle P\left(\phi(\chi_1)-\phi(\chi_2)\right),B_0P\widetilde{\chi}\rangle+\langle P(\widetilde{e}-\widetilde{\chi}),B_0P\widetilde{\chi}\rangle\\ \leq\;& \|\phi(\chi_1)-\phi(\chi_2)\|^2+\frac{1}{2}\|P\widetilde{\chi}\|_{W_0}^2+\|P(\widetilde{e}-\widetilde{\chi})\|^2+\|P\widetilde{\chi}\|^2\\ \leq\;& c\left(\|\widetilde{e}-\widetilde{\chi}\|^2+\|\widetilde{\chi}\|_V^2\right)+\frac{1}{2}\|P\widetilde{\chi}\|_{W_0}^2.\end{aligned}$$

We thus infer

$$\frac{1}{2}\frac{d}{dt}\left[\|P\widetilde{\chi}\|^2+\alpha\|P\widetilde{\chi}\|_{V_0}^2\right]+\frac{1}{2}\|P\widetilde{\chi}\|_{W_0}^2 \leq c\left(\|\widetilde{e}-\widetilde{\chi}\|^2+\|\widetilde{\chi}\|_V^2\right). \qquad (3.18)$$

Adding together inequalities (3.11) and (3.17), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|P\left(\widetilde{e}-\widetilde{\chi}\right)\|^2+\|P\widetilde{\chi}\|_{V_0}^2+\|\widetilde{\eta}\|_{\mathcal{M}}^2\right] \qquad (3.19)\\ &+\frac{1}{2}\|\partial_t\widetilde{\chi}\|_{V_0^*}^2+\alpha\|\partial_t\widetilde{\chi}\|^2 \leq c\Theta\|\widetilde{\chi}\|_V^2+c_*\|\widetilde{\chi}\|_W^2.\end{aligned}$$

Let us now introduce the positive functional, defined for all $t\in[0,\infty)$,

$$\widetilde{\Phi}(t) \quad = \quad \|P\left(\widetilde{e}(t)-\widetilde{\chi}(t)\right)\|^2+4c_*\|P\widetilde{\chi}(t)\|^2+(1+4\alpha c_*)\|P\widetilde{\chi}(t)\|_{V_0}^2+\|\widetilde{\eta}^t\|_{\mathcal{M}}^2.$$

Using the control

$$\|\widetilde{e}-\widetilde{\chi}\|^2+\|\chi\|_V^2 \leq \|P\left(\widetilde{e}-\widetilde{\chi}\right)\|^2+\|P\chi\|_{V_0}^2+c\left(m_{\widetilde{e}-\widetilde{\chi}}^2+m_{\widetilde{\chi}}^2\right) \leq c\left(\|\widetilde{e}-\widetilde{\chi}\|^2+\|\chi\|_V^2\right),$$

we can find some positive $\widetilde{c}>1$ such that there holds

$$\frac{1}{\widetilde{c}}\|\widetilde{z}(t)\|_{\mathcal{H}}^2 \leq \widetilde{\Phi}(t) \leq \widetilde{c}\|\widetilde{z}(t)\|_{\mathcal{H}}^2, \quad \forall t\in[0,+\infty). \qquad (3.20)$$

Since

$$\|\widetilde{\chi}\|_W^2=\|P\widetilde{\chi}\|_{W_0}^2+m_{\widetilde{\chi}}^2 \leq \|P\widetilde{\chi}\|_{W_0}^2+c\|\widetilde{\chi}\|_V^2,$$

then, adding together inequality (3.18) multiplied by $4c_*$ and (3.19), we infer

$$\frac{d}{dt}\widetilde{\Phi}\,(t) + \widetilde{K}\left(\|\widetilde{\chi}(t)\|_W^2 + \|\partial_t\widetilde{\chi}(t)\|_{V_0^*}^2 + \alpha\|\partial_t\widetilde{\chi}(t)\|^2\right) \leq c\Theta(t)\widetilde{\Phi}\,(t)\,, \quad \forall t \in [0,T], \tag{3.21}$$

for some positive $\widetilde{K}$. Recalling now that $\Theta \in L^1(0,T)$, we can apply [3, Lemma A.5], to get

$$\widetilde{\Phi}\,(t) \leq c\|\widetilde{z}_0\|_{\mathcal{H}}^2 e^{c\int_0^t \Theta(\tau)d\tau}, \quad \forall t \in [0,T]. \tag{3.22}$$

On account of (3.20) the thesis is then proven by integrating both sides of (3.21) on the time interval $[0,T]$, and using (3.22), provided that we choose

$$C_0(t) = ce^{c\int_0^t \Theta(\tau)d\tau}, \quad \forall t \in [0,T]. \tag{3.23}$$

4. **Absorbing sets.** We begin our asymptotic analysis by proving the existence of a bounded absorbing set for $S^\alpha(t)$.

In the course of this section, $c \geq 0$ will denote a generic constant depending on the structural data of the problem. By $\widetilde{c} \geq 0$ we shall indicate a generic constant which also depends on β, γ and on ϕ, such that, if $\beta = \gamma = 0$ and $c_1 = 0$ in assumption ($\boldsymbol{H1}$), then $\widetilde{c} = 0$. We point out that c and $\widetilde{c}$ may vary even within the same formula and are both independent of $\alpha \in [0,1]$. Further dependencies will be pointed out in the sequel, if needed. We shall also use the Hölder and the Young inequalities repeatedly, avoiding to stress it out each time.

The crucial assumption to prove the following dissipation result is ($\boldsymbol{K4}$), that will be always assumed from now on.

Theorem 4.1. *There exists $R_0 > 0$ such that the ball $\mathcal{B}_0 = \mathcal{B}_0(\beta,\gamma)$ of $\mathcal{H}_{\beta,\gamma}$ of radius $2R_0$ centered in zero is a bounded absorbing set for the restriction of $S^\alpha(t)$ to $\mathcal{H}_{\beta,\gamma}$, uniformly with respect to $\alpha \in [0,1]$. Namely, given any $R > 0$, there exists a time $t_0 = t_0(R) \geq 0$ such that*

$$S^\alpha(t)B_{\mathcal{H}_{\beta,\gamma}}(R) \subset \mathcal{B}_0, \qquad \forall t \geq t_0.$$

Moreover, for every $R > 0$, there exists a constant $C_0 = C_0(R,\beta,\gamma) \geq 0$ independent of $\alpha \in [0,1]$, such that, for any $z_0 \in \mathcal{B}_0$,

$$\sup_{t\in[0,\infty)} \|S^\alpha(t)z_0\|_{\mathcal{H}} \leq C_0. \tag{4.1}$$

In addition, for any fixed $r > 0$, the following bound holds

$$\sup_{z_0\in\mathcal{B}_0}\sup_{t\in[0,\infty)} \int_t^{t+r} \left[\|\chi(\tau)\|_W^2 + \|\partial_t\chi(\tau)\|_{V_0^*}^2 + \alpha\|\partial_t\chi(\tau)\|^2\right] d\tau \leq c, \tag{4.2}$$

uniformly with respect to $\alpha \in [0,1]$.

Remark 6. It is worth noting that, up to choosing R_0 large enough in Theorem 4.1, the resulting absorbing set $\mathcal{B}_0$ is independent of $\alpha \in [0,1]$.

4.1. **Proof of inequality** (4.1)**.** We divide the proof into several steps.

As a first step, in order to handle the possible integrable singularity of μ at 0, following [34], we introduce, for any fixed $s_0 \in [0,\infty)$, the function $\psi = \psi_{s_0} : [0,\infty) \to [0,\infty)$, defined by

$$\psi(s) = \mu(s_0)\chi_{(0,s_0]}(s) + \mu(s)\chi_{[s_0,\infty)}(s),$$

where χ_{I} denotes the indicator function of any interval $\mathrm{I} \subset [0,\infty)$. Notice that conditions $(\boldsymbol{K1})$-$(\boldsymbol{K4})$ imply

$$\psi(s) \le \mu(s) \quad \text{and} \quad -\psi'(s) \le -\mu'(s) \quad \text{for a.e.} \quad s \in (0,\infty). \tag{4.3}$$

Moreover, by choosing s_0 large enough,

$$\int_0^\infty \psi(s)ds = \frac{k_0}{2}, \tag{4.4}$$

and, without loss of generality, we can also suppose

$$\mu(s_0) \le 1.$$

Next, to overcome the lack of an instantaneous diffusion term in equation (3.1), along the lines of [17], we first introduce, for all $z = (e,\chi,\eta) \in \mathcal{H}_{\beta,\gamma}$, the functional

$$L_0(z) = -\int_0^\infty \psi(s)\langle B_0^{-1/2}\eta(s), B_0^{-1/2}P\,(e-\chi)\rangle ds.$$

Notice first that, using $\boldsymbol{(K3)}$, there holds

$$|L_0(z)| \le c\left(\|P(e-\chi)\|^2 + \|\eta\|^2_{\mathcal{M}}\right). \tag{4.5}$$

Taking the time-derivative of L_0, by means of equation (3.3), we get

$$\begin{aligned}\frac{d}{dt}L_0(z) + \frac{k_0}{2}\|P\,(e-\chi)\,\|^2 &= \int_0^\infty \psi(s)\langle B_0^{-1/2}\eta(s), B_0^{-1/2}\partial_t\chi\rangle ds \\ + \int_0^\infty \psi(s)\mu(s)\,\|\eta(s)\|^2_{V_0^*}\,ds &- \int_0^\infty \mu(s)\langle B_0^{-1/2}T\eta(s), B_0^{-1/2}P\,(e-\chi)\rangle ds,\end{aligned} \tag{4.6}$$

where the last equality is obtained by taking the product of equation (3.1) with $B_0^{-1}\eta(s)$ and integrating over $(0,\infty)$ with respect to $\psi(s)ds$.

Using now assumptions $\boldsymbol{(K2)} - \boldsymbol{(K4)}$, we see that

$$\int_0^\infty \psi(s)\langle B_0^{-1/2}\eta(s), B_0^{-1/2}\partial_t\chi\rangle ds \quad \le \quad \|\eta\|^2_{\mathcal{M}} + c\|\partial_t\chi\|^2_{V_0^*}, \tag{4.7}$$

$$\int_0^\infty \psi(s)\mu(s)\,\|\eta(s)\|^2_{V_0^*}\,ds \quad \le \quad \mu(s_0)\|\eta\|^2_{\mathcal{M}} \le \|\eta\|^2_{\mathcal{M}}. \tag{4.8}$$

Observe now that, since we work in a regularization scheme, there holds

$$\lim_{s\to 0}\psi(s)\langle B_0^{-1/2}P(e-\chi), B_0^{-1/2}\eta(s)\rangle = \lim_{s\to\infty}\psi(s)\langle B_0^{-1/2}P(e-\chi), B_0^{-1/2}\eta(s)\rangle = 0.$$

Then, we have

$$\begin{aligned}
& - \int_0^\infty \mu(s)\langle B_0^{-1/2} T\eta(s), B_0^{-1/2} P\,(e-\chi)\rangle ds \qquad (4.9)\\
= \;& - \int_0^\infty \mu'(s)\langle B_0^{-1/2} \eta(s), B_0^{-1/2} P\,(e-\chi)\rangle ds \\
\leq \;& \frac{k_0}{4}\|P\,(e-\chi)\,\|^2 - c\int_0^\infty \mu'(s)\|\eta(s)\|^2_{V_0^*} ds.
\end{aligned}$$

Substituting (4.7)-(4.9) into (4.6), we are thus led to

$$\begin{aligned}
\frac{d}{dt} L_0(z) + \frac{k_0}{4}\|P\,(e-\chi)\,\|^2 \quad \leq & \quad 2\|\eta\|^2_{\mathcal{M}} + c\|\partial_t\chi\|^2_{V_0^*} \qquad (4.10)\\
- & \quad c\int_0^\infty \mu'(s)\|\eta(s)\|^2_{V_0^*} ds.
\end{aligned}$$

We now consider the product of equation (3.3) by η in $\mathcal{M}$. As a consequence of (**K4**) we get

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\eta\|^2_{\mathcal{M}} + \frac{\lambda}{2}\|\eta\|^2_{\mathcal{M}} - \frac{1}{2}\int_0^\infty \mu'(s)\|\eta(s)\|^2_{V_0^*} ds \qquad (4.11)\\
\leq \;& \int_0^\infty \mu(s)\langle P\,(e-\chi)\,, \eta(s)\rangle ds.
\end{aligned}$$

On the other hand, considering the product of equation (3.1) by $P\,(e-\chi)$, and taking (4.11) into account, we obtain

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left[\|P\,(e-\chi)\,\|^2 + \|\eta\|^2_{\mathcal{M}}\right] + \frac{\lambda}{2}\|\eta\|^2_{\mathcal{M}} \qquad (4.12)\\
& -\frac{1}{2}\int_0^\infty \mu'(s)\|\eta(s)\|^2_{V_0^*} ds + \langle \partial_t\chi, P\,(e-\chi)\rangle \leq 0.
\end{aligned}$$

Let us perform the following products of equation (3.2).

• By $B_0^{-1}\partial_t\chi$, to get

$$\frac{1}{2}\frac{d}{dt}\left[\|P\chi\|^2_{V_0} + 2\langle\Phi(\chi), 1\rangle\right] + \|\partial_t\chi\|^2_{V_0^*} + \alpha\|\partial_t\chi\|^2 - \langle P(e-\chi), \partial_t\chi\rangle = 0, \qquad (4.13)$$

where we define

$$\Phi(x) = \int_0^x \phi(y)dy, \quad \forall x\in\mathbb{R}.$$

Here we have used the fact that, since $\partial_t\chi$ has null average, there holds

$$\langle P\phi(\chi), \partial_t\chi\rangle = \langle \phi(\chi), \partial_t\chi\rangle.$$

• By $B_0^{-1}P\chi$, to get

$$\frac{1}{2}\frac{d}{dt}\left[\|P\chi\|^2_{V_0^*} + \alpha\|P\chi\|^2\right] + \|P\chi\|^2_{V_0} + \langle P\phi(\chi), P\chi\rangle = \langle P(e-\chi), P\chi\rangle.$$

Notice first that, by the continuous embedding $V_0 \hookrightarrow H_0$, we have

$$\langle P(e-\chi), P\chi\rangle \leq c\|P\,(e-\chi)\,\|^2 + \frac{1}{2}\|P\chi\|^2_{V_0}.$$

Concerning the nonlinear term, there holds

$$\langle P\phi(\chi), P\chi\rangle = \langle\phi(\chi), \chi\rangle + (2+|\Omega|)m_{\phi(\chi)}m_\chi, \qquad (4.14)$$

where we denote by $|\Omega|$ the Lebesgue measure of Ω. It is then immediate to check

$$c_0\|\chi\|^4_{L^4(\Omega)} - \widetilde{c} \leq \langle \phi(\chi), \chi \rangle,$$

c_0 being the same constant appearing in ($\boldsymbol{H1}$). Furthermore, by means of ($\boldsymbol{H2}$) and the bound on the mean value of χ, we infer

$$(2+|\Omega|) m_{\phi(\chi)} m_\chi \leq \widetilde{c}\left(1+\|\chi\|^3_{L^3(\Omega)}\right) \leq \widetilde{c} + \frac{c_0}{2}\|\chi\|^4_{L^4(\Omega)}.$$

We are therefore led to

$$\frac{1}{2}\frac{d}{dt}\left[\|P\chi\|^2_{V_0^*} + \alpha\|P\chi\|^2\right] + \frac{1}{2}\|P\chi\|^2_{V_0} \leq c\|P(e-\chi)\|^2 + \widetilde{c}. \tag{4.15}$$

Adding together (4.12) and (4.13), we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|P(e-\chi)\|^2 + \|\eta\|^2_{\mathcal{M}} + \|P\chi\|^2_{V_0} + 2\langle \Phi(\chi), 1\rangle\right] + \frac{\lambda}{2}\|\eta\|^2_{\mathcal{M}} \tag{4.16}$$
$$-\frac{1}{2}\int_0^\infty \mu'(s)\|\eta(s)\|^2_{V_0^*} ds + \|\partial_t\chi\|^2_{V_0^*} + \alpha\|\partial_t\chi\|^2 \leq 0.$$

Let $\nu > 0$ be a small constant, and add together inequalities ν times (4.10) and (4.16). By choosing

$$\nu < \min\left\{\frac{\lambda}{2}, \frac{1}{2c}\right\},$$

we get

$$\frac{1}{2}\frac{d}{dt}\left[\|P(e-\chi)\|^2 + \|\eta\|^2_{\mathcal{M}} + \|P\chi\|^2_{V_0} + 2\langle \Phi(\chi), 1\rangle + \nu L_0(z)\right] \tag{4.17}$$
$$+c\left[\|P(e-\chi)\|^2 + \|\eta\|^2_{\mathcal{M}}\right] + K_0\left[\|\partial_t\chi\|^2_{V_0^*} + \alpha\|\partial_t\chi\|^2\right] \leq 0,$$

for some positive K_0 independent of $\alpha \in [0,1]$.

We now introduce the functional, defined for all $z = (e,\chi,\eta) \in \mathcal{H}_{\beta,\gamma}$ and for all $\alpha \in [0,1]$

$$\begin{aligned}\Phi_0(z) \quad = \quad & \|P(e-\chi)\|^2 + \xi\|P\chi\|^2_{V_0^*} + \xi\alpha\|P\chi\|^2 \\ & + \|P\chi\|^2_{V_0} + \|\eta\|^2_{\mathcal{M}} + 2\langle \Phi(\chi), 1\rangle + \nu L_0(z),\end{aligned}$$

where $\xi > 0$ is a small constant to be chosen in the sequel. Since the structural bounds on the averages imply

$$\begin{aligned}\|e-\chi\|^2 + \|\chi\|^2_V - \widetilde{c} \quad & \leq \quad \|P(e-\chi)\|^2 + \xi\|P\chi\|^2_{V_0^*} + \xi\alpha\|P\chi\|^2 + \|P\chi\|^2_{V_0} \\ & \leq \quad \leq \|e-\chi\|^2 + \|\chi\|^2_V + \widetilde{c},\end{aligned}$$

by means of (4.5) and assumptions ($\boldsymbol{H1}$) and ($\boldsymbol{H2}$), we infer that there exists a constant $\omega_0 \in (0,1)$, independent of $\alpha \in [0,1]$, such that

$$\omega_0\|z\|^2_{\mathcal{H}} - \widetilde{c} \leq \Phi_0(z) \leq c\left(1 + \|z\|^4_{\mathcal{H}}\right). \tag{4.18}$$

Now we add ξ times (4.15) to (4.17). By setting $\xi = c/2$, we can find $\kappa_0 > 0$, independent of $\alpha \in [0,1]$, such that

$$\frac{d}{dt}\Phi_0\left(S^\alpha(t)z_0\right) + \kappa_0\|S^\alpha(t)z_0\|^2_{\mathcal{H}} + K_0\left[\|\partial_t\chi\|^2_{V_0^*} + \alpha\|\partial_t\chi\|^2\right] \leq \widetilde{c}. \tag{4.19}$$

We now proceed to prove inequality (4.1). Since, by assumption, $\|z_0\|_{\mathcal{H}} \le R$, by (4.18) we infer the bounds

$$\sup_{t\in[0,\infty)} \Phi_0\left(S^\alpha(t)z_0\right) \ge -\omega R - \widetilde{c} \quad \text{and} \quad \Phi_0\left(z_0\right) \le c(1+R^4), \tag{4.20}$$

so that we are in a position to apply the Gronwall-type lemma [2, Lemma 2.7], which implies the existence of $t_0 = t_0(R) > 0$ such that

$$\Phi_0\left(S^\alpha(t)z_0\right) \le \sup_{\zeta\in\mathcal{H}_{\beta,\gamma}} \Big\{\Phi_0(\zeta) : \kappa_0\|\zeta\|^2_{\mathcal{H}} \le \widetilde{c}\Big\}, \qquad \forall t \ge t_0.$$

Together with (4.18) we then have

$$\sup_{t\in[t_0,\infty)} \|z(t)\|_{\mathcal{H}} \le \widetilde{c}.$$

In order to prove the inequality we are left to prove the same control on $[0,t_0)$. To this purpose, integrate (4.19) on $[0,t_0)$ and use bounds (4.18) and (4.20). This yields

$$\sup_{t\in[0,t_0)} \|z(t)\|_{\mathcal{H}} \le c.$$

By choosing $C_0 = \min\{c, \widetilde{c}\}$ we reach the desired conclusion.

Remark 7. If we work in a null average space for the order parameter (i.e., in the space $\mathcal{H}_{\beta,0}$), condition (***H1***) can be replaced by the weaker

$$(\boldsymbol{H1'}) \quad \liminf_{|r|\to\infty} \frac{\phi(r)}{r} > -\frac{1}{c_P},$$

where c_P is the Poincaré-Wirtinger constant for null-average functions of V. This is enough in order to deduce inequality (4.15).

4.2. **Proof of inequality** (4.2)**.** Taking the product of equation (3.2) with $P\chi$ in H, we deduce

$$\frac{1}{2}\frac{d}{dt}\left[\|P\chi\|^2 + \alpha\|P\chi\|^2_{V_0}\right] + \|P\chi\|^2_{W_0} \le \langle B_0P(e-\chi), P\chi\rangle - \langle B_0P\phi(\chi), P\chi\rangle.$$

Concerning the first summand term on the right-hand side of the above inequality, it is easy to get

$$\langle B_0P(e-\chi), P\chi\rangle = \langle P(e-\chi), B_0P\chi\rangle \le c\|P\left(e-\chi\right)\|^2 + \frac{1}{4}\|P\chi\|^2_{W_0}.$$

On the other hand, by assumption (***H2***), we have

$$\langle B_0P\phi(\chi), P\chi\rangle = \langle P\phi(\chi), B_0P\chi\rangle \le c + c\|\chi\|^6_{L^6(\Omega)} + \frac{1}{4}\|P\chi\|^2_{W_0}.$$

Thanks to the continuous embedding $V \hookrightarrow L^6(\Omega)$ and (4.1), we finally deduce

$$\frac{d}{dt}\left[\|P\chi\|^2 + \alpha\|P\chi\|^2_{V_0}\right] + \|P\chi\|^2_{W_0} \le c \tag{4.21}$$

Adding together inequalities (4.19) and (4.21), and invoking the bound on the mean values of e and χ, we infer

$$\frac{d}{dt}\Phi_0\left(S^\alpha(t)z_0\right) + K_0\left[\|\chi\|^2_W + \|\partial_t\chi\|^2_{V_0^*} + \alpha\|\partial_t\chi\|^2\right] \le c. \tag{4.22}$$

Inequality (4.2) is then proven by integrating both members of (4.22) over $(t, t+r)$ and using (4.1) and (4.18) once again.

5. **Global attractors.** In this section we consider a further asymptotic property for $S^\alpha(t)$, namely the existence of the global attractor $\mathcal{A}_\alpha$. We recall that the global attractor is the (unique) compact subset of $\mathcal{H}_{\beta,\gamma}$ which is fully invariant under the action of $S^\alpha(t)$ and attracts bounded sets with respect to the Hausdorff semidistance (cf., e.g., [40]).

Therefore, the main result of this section is

Theorem 5.1. *The strongly continuous semigroup $S^\alpha(t)$ acting on the phase-space $\mathcal{H}_{\beta,\gamma}$ possesses a connected global attractor $\mathcal{A}_\alpha = \mathcal{A}_\alpha(\beta,\gamma)$, which is given by*

$$\mathcal{A}_\alpha = \{z^\alpha(0) : z^\alpha(t) \textit{ is a complete bounded trajectory of } S^\alpha(t)\}.$$

Furthermore, since $\mathcal{A}_\alpha$ is by definition fully invariant under the action of the semigroup $S^\alpha(t)$, as an immediate consequence of the backward uniqueness property given by Proposition 3, we have

Proposition 4. *The semigroup $S^\alpha(t)$ on $\mathcal{A}_\alpha$ uniquely extends to a continuous group of operators $\tilde{S}^\alpha(t)$ on $\mathcal{A}_\alpha$.*

For any $z_0 = (e_0, \chi_0, \eta_0) \in \mathcal{B}_0$, consider the decomposition of

$$z(t) = (e(t), \chi(t), \eta^t) = S^\alpha(t) z_0$$

into the sum

$$z(t) = z_d(t) + z_c(t),$$

where

$$z_d(t) = (e_d(t), \chi_d(t), \eta_d^t) \quad \text{and} \quad z_c(t) = (e_c(t), \chi_c(t), \eta_c^t)$$

are the solutions at time $t \in [0, \infty)$ to the following problems, respectively,

$$\partial_t e_d + \int_0^\infty \mu(s) \eta_d(s) ds = 0, \tag{5.1}$$

$$\partial_t \chi_d + B_0\big(B_0 \chi_d + \alpha \partial_t \chi_d + P\phi_0(\chi_d) - (e_d - \chi_d)\big) = 0, \tag{5.2}$$

$$\partial_t \eta_d = T\eta_d + B_0(e_d - \chi_d), \tag{5.3}$$

$$z_d(0) = (Pe_0, P\chi_0, \eta_0), \tag{5.4}$$

and

$$\partial_t e_c + \int_0^\infty \mu(s) \eta_c(s) ds = 0, \tag{5.5}$$

$$\partial_t \chi_c + B_0\big(B_0 P \chi_c + \alpha \partial_t \chi_c + P\left(\phi(\chi) - \phi(\chi_d) + \phi_1(\chi_d)\right) - P(e_c - \chi_c)\big) = 0, \tag{5.6}$$

$$\partial_t \eta_c = T\eta_c + B_0 P(e_c - \chi_c), \tag{5.7}$$

$$z_c(0) = (m_{e_0}, m_{\chi_0}, 0), \tag{5.8}$$

where the nonlinearities ϕ_0 and ϕ_1 are defined as in Remark 1.

The next Lemmas provide basic asymptotic properties for z_d and z_c, respectively.

Lemma 5.2. *There exist $\kappa_d > 0$ and $c_d \geq 0$, independent of $\alpha \in [0,1]$, such that*

$$\|z_d(t)\|_{\mathcal{H}} \leq c_d e^{-\kappa_d t}\|z_0\|_{\mathcal{H}}, \quad \forall t \in [0,\infty), \tag{5.9}$$

for all $z_0 \in \mathcal{B}_0$. In addition, for any fixed $r > 0$, the following bound holds

$$\sup_{z_0 \in \mathcal{B}_0} \sup_{t \in [0,\infty)} \int_t^{t+r} \left[\|\chi_d(\tau)\|_W^2 + \|\partial_t \chi_d(\tau)\|_{V_0^*}^2 + \alpha\|\partial_t \chi_d(\tau)\|^2\right] d\tau \leq c, \tag{5.10}$$

uniformly with respect to $\alpha \in [0,1]$.

Lemma 5.3. *For all $t \in [0,\infty)$, there exists a compact set $\mathcal{K}(t) = \mathcal{K}(t,\beta,\gamma) \subset \mathcal{H}_{\beta,\gamma}$ such that $z_c(t) \in \mathcal{K}(t)$, for all $z_0 \in \mathcal{B}_0$.*

5.1. **Proof of Lemma 5.2.** Arguing as in the proof of Theorem 4.1, we can easily deduce the inequality (analogous to (4.19))

$$\frac{d}{dt}\Phi_d(t) + \kappa_d\|z_d(t)\|_{\mathcal{H}}^2 \leq 0, \quad \forall t \in [0,\infty) \tag{5.11}$$

for some positive κ_d, where we have defined, for all $t \in [0,\infty)$ and for all $\alpha \in [0,1]$, the functional

$$\begin{aligned} \Phi_d(t) \quad &= \quad \|P\left(e_d(t) - \chi_d(t)\right)\|^2 + \xi\|P\chi_d(t)\|_{V_0^*}^2 + \xi\alpha\|P\chi_d(t)\|^2 \\ &\quad + \|P\chi_d(t)\|_{V_0}^2 + \|\eta_d(t)\|_{\mathcal{M}}^2 + 2\langle\Phi_0(\chi_d(t)), 1\rangle + \nu L_0(z_d(t)), \end{aligned}$$

with

$$\Phi_0(x) = \int_0^x \phi_0(y)dy, \quad \forall x \in \mathbb{R},$$

and ξ small enough. Note that in this case the constant $\widetilde{c}$ in inequality (4.19) is null, as a consequence of $(\boldsymbol{H1_0})$ and of (5.4), which also implies the lower bound

$$\omega_d\|z_d(t)\|_{\mathcal{H}}^2 \leq \Phi_d(t), \quad \forall t \in [0,\infty) \tag{5.12}$$

for some $\omega_d > 0$. Therefore, [3, Lemma A.5] applied to inequality (5.11) yields the exponential decay provided in (5.9).

On the other hand, the integral control (5.10) follows immediately by integrating over $(t, t+1)$ the inequality

$$\frac{d}{dt}\Phi_d(t) + K_d\|\chi_d(t)\|_W^2 \leq c, \quad \forall t \in [0,\infty),$$

with $K_d > 0$, which can be deduced arguing as to get (4.22), and using (5.9) and the bounds on $\partial_t \chi_d$ again.

5.2. **Proof of Lemma 5.3.** For any $\sigma > 0$ we define the product space

$$\mathcal{H}_\sigma = V_\sigma \times V_{1+\sigma} \times \mathcal{M}_\sigma,$$

V_σ, $V_{1+\sigma}$ and $\mathcal{M}_\sigma$ having been introduced in section 2.

In the course of the proof we shall make use the following continuous Sobolev embeddings, which can be deduced by (2.4) and (2.5), under the limitation $\sigma \in (0, 1/2)$:

$$V_0^{1-\sigma} \hookrightarrow V_{1-\sigma} \quad \hookrightarrow \quad L^{6/(1+2\sigma)}(\Omega) \tag{5.13}$$

$$V_0^{1+\sigma} \hookrightarrow V_{1+\sigma} \quad \hookrightarrow \quad L^{6/(1-2\sigma)}(\Omega) \tag{5.14}$$

$$W^{1,6/(3+2\sigma)}(\Omega) \quad \hookrightarrow \quad V_{1-\sigma}. \tag{5.15}$$

Moreover (see [31])

$$\nabla : V_{1+\sigma} \to (V_\sigma)^3 \text{ is a continuous linear operator.} \tag{5.16}$$

In the rest of the proof $\sigma \in (0, 1/2)$ will be fixed. Moreover, the next set of controls, which are straightforward consequence of Theorem 4.1 and Lemma 5.2, will play a basic role.

$$\sup_{t\in[0,\infty)} \left[\|e-\chi\|^2 + \|e_d - \chi_d\|^2 + \|\chi(t)\|_V^2 + \|\chi_d(t)\|_V^2\right] \le c, \tag{5.17}$$

and

$$\int_0^t \Theta_c(\tau)d\tau \le ct, \quad \forall t \in [0,\infty), \tag{5.18}$$

where we define

$$\Theta_c(t) = 1 + \|\chi(t)\|_W^2 + \|\partial_t\chi(t)\|_{V_0^*}^2 + \|\chi_d(t)\|_W^2 + \|\partial_t\chi_d(t)\|_{V_0^*}^2. \tag{5.19}$$

Observe also that, on account of the definition of V_σ and bound (5.17), we have

$$\|\chi_c\|_{V_\sigma} \le c + \|P\chi_c\|_{V_0^\sigma}, \quad \forall \sigma > 0. \tag{5.20}$$

We stress out the fact that the above estimates hold uniformly with respect to $z_0 \in \mathcal{B}_0$ and $\alpha \in [0,1]$.

We now consider the product of (5.7) by η_c in $\mathcal{M}_\sigma$. As a consequence of ($\boldsymbol{K4}$) we get

$$\frac{1}{2}\frac{d}{dt}\|\eta_c\|_{\mathcal{M}_\sigma}^2 \le \int_0^\infty \mu(s)\langle B_0^{\sigma/2}P\left(e_c - \chi_c\right), B_0^{\sigma/2}\eta_c(s)\rangle ds. \tag{5.21}$$

Consider the product of equation (5.5) by $B_0^\sigma P(e_c - \chi_c)$; keeping (5.21) into account, we get

$$\frac{1}{2}\frac{d}{dt}\left[\|P(e_c-\chi_c)\|_{V_0^\sigma}^2 + \|\eta_c\|_{\mathcal{M}_\sigma}^2\right] \le -\langle B_0^{\sigma/2}\partial_t\chi_c, B_0^{\sigma/2}P(e_c-\chi_c)\rangle. \tag{5.22}$$

Now we perform the following products of equation (5.6).

- By $B_0^{-1+\sigma}\partial_t\chi_c$, to get

$$\frac{1}{2}\frac{d}{dt}\left[\|P\chi_c\|_{V_0^{1+\sigma}}^2 + 2\langle P\left(\phi(\chi) - \phi(\chi_d) + \phi_1(\chi_d)\right), B_0^\sigma P\chi_c\rangle\right] \tag{5.23}$$

$$+\|\partial_t\chi_c\|_{V_0^{-1+\sigma}}^2 + \alpha\|\partial_t\chi_c\|_{V_0^\sigma}^2 - \langle B_0^{\sigma/2}\partial_t\chi_c, B_0^{\sigma/2}P(e_c-\chi_c)\rangle = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

having set

$$\mathcal{I}_1 = \langle\phi'(\chi)B_0^\sigma P\chi_c, \partial_t\chi_c\rangle, \tag{5.24}$$

$$\mathcal{I}_2 = \langle\left(\phi'(\chi) - \phi'(\chi_d)\right)B_0^\sigma P\chi_c, \partial_t\chi_d\rangle, \tag{5.25}$$

$$\mathcal{I}_3 = \langle\phi_1'(\chi_d)B_0^\sigma P\chi_c, \partial_t\chi_d\rangle. \tag{5.26}$$

- By $B_0^\sigma P\chi_c$, to get

$$\frac{1}{2}\frac{d}{dt}\left[\|P\chi_c\|_{V_0^\sigma}^2 + \alpha\|P\chi_c\|_{V_0^{1+\sigma}}^2\right] + \|P\chi_c\|_{V_0^{2+\sigma}}^2 \tag{5.27}$$

$$= \langle B_0^{\sigma/2}P(e_c-\chi_c), B_0^{(2+\sigma)/2}\chi_c\rangle + \mathcal{I}_4 + \mathcal{I}_5,$$

having set

$$\mathcal{I}_4 = -\langle B_0^{\sigma/2} P\left(\phi(\chi)-\phi(\chi_d)\right), B_0^{(2+\sigma)/2} P\chi_c\rangle, \tag{5.28}$$

$$\mathcal{I}_5 = -\langle B_0^{\sigma/2} P\phi_1(\chi_d), B_0^{(2+\sigma)/2} P\chi_c\rangle; \tag{5.29}$$

We need to estimate the nonlinear terms $\mathcal{I}_i$ $(i=1,..,5)$, defined by (5.24)-(5.26) and (5.28)-(5.29). Details appear to be rather cumbersome and therefore will be outlined in the appendix. More precisely, we will prove

$$\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3 \le c\Theta_c + c\Theta_c\|\chi_c\|^2_{V_{1+\sigma}} + c_*\|P\chi_c\|^2_{V_0^{2+\sigma}} + \frac{1}{2}\|\partial_t\chi_c\|^2_{V_0^{-1+\sigma}}, \tag{5.30}$$

$$\mathcal{I}_4+\mathcal{I}_5 \le c\Theta_c + \frac{1}{4}\|P\chi_c\|^2_{V_0^{2+\sigma}}, \tag{5.31}$$

for some (generic) constant $c_*>0$, Θ_c being defined in (5.19).

Adding together inequalities (5.22) and (5.23), and exploiting (5.30), we have that

$$\frac{1}{2}\frac{d}{dt}\left[\|P\left(e_c-\chi_c\right)\|^2_{V_0^\sigma} + \|P\chi_c\|^2_{V_0^{1+\sigma}} + \|\eta_c\|^2_{\mathcal{M}_\sigma} + \langle P\left(\phi(\chi)-\phi(\chi_d)\right), B_0^\sigma P\chi_c\rangle\right]$$
$$\le c\Theta_c + c\Theta_c\|\chi_c\|^2_{V_{1+\sigma}} + c_*\|P\chi_c\|^2_{V_0^{2+\sigma}}. \tag{5.32}$$

By (5.31) and owing to the immediate inequality

$$\langle B_0^{\sigma/2}P(e_c-\chi_c), B_0^{(2+\sigma)/2}\chi_c\rangle \le \|P(e_c-\chi_c)\|^2_{V_0^\sigma} + \frac{1}{4}\|P\chi_c\|^2_{V_0^{2+\sigma}},$$

equation (5.27) implies

$$\frac{1}{2}\frac{d}{dt}\left[\|P\chi_c\|^2_{V_0^\sigma} + \alpha\|P\chi_c\|^2_{V_0^{1+\sigma}}\right] + \frac{1}{2}\|P\chi_c\|^2_{V_0^{2+\sigma}} \le c\Theta_c + c\|e_c-\chi_c\|^2_{V_0^\sigma}. \tag{5.33}$$

For all $t\in[0,\infty)$, for all $\alpha\in[0,1]$ and for all fixed $\sigma\in(0,1/2)$, we define the functional

$$\Phi_c(t) = \frac{1}{2c_*}\|P\left(e_c(t)-\chi_c(t)\right)\|^2_{V_0^\sigma} + \|P\chi_c(t)\|^2_{V_0^\sigma} + \left(\frac{1}{2c_*}+\alpha\right)\|P\chi_c(t)\|^2_{V_0^{1+\sigma}}$$
$$+\|\eta_c^t\|^2_{\mathcal{M}_\sigma} + \frac{1}{2c_*}\langle P\left(\phi(\chi(t))-\phi(\chi_d(t))\right), B_0^\sigma P\chi_c(t)\rangle + k_c,$$

where k_c is some positive constant to be properly chosen. Indeed, since $\sigma\in(0,1/2)$, then the continuous embedding $V_0\hookrightarrow V_0^{2\sigma}$ holds. Therefore, by means of assumption (***H2***) and bound (5.17), we have

$$\langle P\left(\phi(\chi)-\phi(\chi_d)\right), B_0^\sigma P\chi_c(t)\rangle \le c\left(\|\chi\|^3_V + \|\chi_d\|^3_V\right)\left(\|\chi_c\|_V + \|\chi_d\|_V\right) \le c,$$

so that it is possible to choose k_c large enough to ensure that $\Phi_c(t)\ge 0$, for all $t\in[0,\infty)$. In addition, it is easy to realize that there exists a constant $\kappa_c>1$, independent of $\alpha\in[0,1]$, such that

$$\frac{1}{\kappa_c}\|z_c(t)\|^2_{\mathcal{H}_\sigma} \le \Phi_c(t) \le \kappa_c\|z_c(t)\|^2_{\mathcal{H}_\sigma} + c, \quad \forall t\in[0,\infty). \tag{5.34}$$

Adding together $1/2c_*$ times (5.32) to (5.33), by means of (5.34), we obtain the following differential inequality

$$\frac{d}{dt}\Phi_c(t) \le c\Theta_c(t) + c\Theta_c(t)\Phi_c(t), \quad \forall t\in[0,\infty). \tag{5.35}$$

Recall that, by means of (5.8), there holds

$$\Phi_c(0) = 0.$$

Then, applying a Gronwall-type lemma (see, e.g., [36]), and using (5.18), we find

$$\Phi_c(t) \le 2ce^{\int_0^t \Theta(\tau)d\tau} \int_0^t \Theta(\tau)d\tau \le ce^{ct}, \quad \forall t \in [0,\infty),$$

which yields (cf. (5.34))

$$\|z_c(t)\|^2_{\mathcal{H}_\sigma} \le ce^{ct}, \quad \forall t \in [0,\infty). \tag{5.36}$$

Compactness of the past history variable. We now need to overcome the problem rising from the lack of compactness of the embedding $\mathcal{M}_\sigma \hookrightarrow \mathcal{M}$. Exploiting the assumptions of Lemma 2.1, we prove that the set

$$\mathcal{C}(t) = \bigcup_{z_0 \in \mathcal{B}_0} \eta_c^t \subset \mathcal{M}_\sigma, \tag{5.37}$$

is relatively compact in $\mathcal{M}$, for any fixed nonnegative time. Let also c_σ denote a constant depending on the structural data of the problem and on t, but independent of $\alpha \in [0,1]$.

Concerning the first condition, note that, by the representation formula (2.7) and (5.8)

$$T\eta_c(s) = B_0 P(e_c(t-s) - \chi_c(t-s)) + B_0 P(e_c(t) - \chi_c(t))$$

so that, by means of (4.1)

$$\begin{aligned} \|T\eta_c\|^2_{\mathcal{M}_{-1}} &\le \int_0^\infty \mu(s) \left[\|P(e_c(t-s) - \chi_c(t-s))\|^2 + \|P(e_c(t) - \chi_c(t))\|^2\right] ds \\ &\le c\int_0^\infty \mu(s) ds \le c, \end{aligned}$$

where the constant appearing on the right-hand side does not depend on η_c. Thus, invoking also (5.36), we have

$$\sup_{\eta_c^t \in \mathcal{C}(t)} \left[\|\eta_c^t\|_{\mathcal{M}_\sigma} + \|T\eta_c^t\|_{\mathcal{M}_{-1}}\right] < \infty,$$

which is the first condition of Lemma 2.1.

In order to control the tails, let us fix $x \ge 1$. We first observe that, by interpolation, there holds

$$\|f\|^2_{V_0} \le \|f\|^{2(1-\vartheta)}_{V_0^{1+\sigma}} \|f\|^{2\vartheta}_{V_0^\sigma}, \quad \forall f \in V_0^{1+\sigma},$$

where $\vartheta = \sigma/(1+2\sigma)$. The above inequality, with $f = B_0^{-1}\eta_c(s)$, yields

$$\|\eta_c(s)\|^2_{V_0^*} \le \|\eta_c(s)\|^{2(1-\vartheta)}_{V_0^{-1+\sigma}} \|\eta_c(s)\|^{2\vartheta}_{V_0^{-2+\sigma}}. \tag{5.38}$$

By means of the representation formula (2.7), and inequality (5.36), we have

$$\|\eta_c(s)\|_{V_0^{-2+\sigma}} \le \int_0^s \|P\left(e_c(t-\tau) - \chi_c(t-\tau)\right)\|_{V_0^\sigma} d\tau \le \varphi(s), \quad \forall s \in [0,\infty),$$

having set $\varphi(s) = c_\sigma(1+s)$. Here we used also the null initial condition (5.8), so that, integrating (5.38) over $(0,1/x)\cup(x,\infty)$ with respect to $\mu(s)ds$, we obtain

$$\begin{aligned}\mathcal{T}_{\eta_c}(x) &= \int_{(0,1/x)\cup(x,\infty)} \mu(s)^{1-\vartheta}\|\eta_c(s)\|^{2(1-\vartheta)}_{V_0^{-1+\sigma}}\ \mu(s)^{\vartheta}\|\eta_c(s)\|^{2\vartheta}_{V_0^{-2+\sigma}} ds \\ &\le \int_{(0,1/x)\cup(x,\infty)} \Big(\mu(s)\|\eta_c(s)\|^2_{V_0^{-1+\sigma}}\Big)^{1-\vartheta} \big(\mu(s)\varphi(s)^2\big)^{\vartheta} ds.\end{aligned}$$

The Hölder inequality, with exponents $1/(1-\vartheta)$ and $1/\vartheta$, respectively, implies

$$\begin{aligned}\mathcal{T}_{\eta_c}(x) &\le \|\eta_c\|^{2(1-\vartheta)}_{\mathcal{M}_\sigma}\left(\int_{(0,1/x)\cup(x,\infty)}\mu(s)\varphi(s)^2 ds\right)^{\vartheta} \\ &\le c_\sigma\left(\int_{(0,1/x)\cup(x,\infty)}\mu(s)\varphi(s)^2 ds\right)^{\vartheta}.\end{aligned} \tag{5.39}$$

By (***K*3**) and (2.6), we infer the immediate inequalities

$$\begin{aligned}\int_0^{1/x}\mu(s)\varphi(s)^2 ds &\le \varphi(1)^2\|\mu\|_{L^1(0,\infty)} \le c_\sigma k_0, \\ \int_x^\infty \mu(s)\varphi(s)^2 ds &\le \mu(s_0)e^{\lambda s_0}\int_x^\infty e^{-\lambda s}\varphi(s)^2 ds,\end{aligned}$$

for all $x \ge 1$. Than we go back to (5.39) and we obtain

$$\lim_{x\to\infty}\left[\sup_{\eta_c^t\in\mathcal{C}(t)}\mathcal{T}_{\eta_c^t}(x)\right] = 0$$

is proven. Thus the thesis follows by Lemma 2.1.

Conclusion of the proof of Lemma 5.3. Define $\mathcal{B}^\sigma_{K(t)}$ as the ball of $V_\sigma \times V_{1+\sigma}$ of radius $K(t) = ce^{ct}$, centered at zero, and set

$$\mathcal{K}(t) = \overline{\Big(\mathcal{B}^\sigma_{K(t)}\times\mathcal{C}(t)\Big)}\cap\mathcal{H}_{\beta,\gamma}, \quad \forall t\in[0,\infty),$$

where the closure is taken with respect to the $\mathcal{H}$-norm. Since $V_\sigma\times V_{1+\sigma}$ is compactly embedded into $H\times V$, then, by (5.36), it turns out that $\mathcal{K}(t)$ is relatively compact in $\mathcal{H}$. Moreover, as a consequence of (5.36), it is apparent that $z_c(t)\in\mathcal{K}(t)$. The proof of Lemma 5.3 is thus complete.

5.3. **Proof of Theorem 5.1.** Collecting lemmas 5.2 and 5.3, it is readily seen that

$$\lim_{t\to\infty}\delta_{\mathcal{H}}\left[S^\alpha(t)\mathcal{B}_0\right] = 0,$$

$\delta_{\mathcal{H}}$ being the Kuratowski measure of noncompactness in $\mathcal{H}$. The thesis of Theorem 5.1 is thus proven by invoking standard arguments of the theory of dynamical systems (cf., for instance, [25]).

6. **Fractal dimension in the viscous case.** In this section we prove that in the viscous case (i.e., more precisely, there holds $\alpha > 0$), the fractal dimension of the global attractor $\mathcal{A}_\alpha \subset \mathcal{H}_\sigma$, $\sigma \in (0, \sigma_0]$, is finite. Here we take $\sigma_0 = (3-\theta)/2$, θ being defined in assumption ($\boldsymbol{H2_1}$) (see the appendix for details). More precisely, there holds

Theorem 6.1. *For any fixed $\alpha > 0$, the fractal dimension of the global attractor $\mathcal{A}_\alpha$ is finite.*

A well known condition to ensure the finite fractal dimension of the global attractor of a dynamical system, namely, the α-contractivity, can be found in [6] (see also [25, Theorem 2.8.1]), that is,

Theorem 6.2. *Let $\mathcal{X}$ be a Banach space and let $S(t)$ to be a strongly continuous semigroup on $\mathcal{X}$ which possesses a bounded absorbing set $\mathcal{B}$. Assume that there exists $\eta < 1$ and a time $t^* > 0$ such that*

$$\|S(t^*)z_1 - S(t^*)z_2\| \leq \eta\|z_1 - z_2\| + \rho\,(z_1, z_2)\,, \quad \forall\ z_1, z_2 \in \mathcal{B},$$

where $\rho(\cdot,\cdot)$ is a compact pseudometric on $\mathcal{X}$. Then $S(t)$ possesses a global attractor $\mathcal{A} = \omega(\mathcal{B})$ of finite fractal dimension.

Recall that a pseudometric ρ on $\mathcal{X}$ is said to be compact if and only if, for any bounded set $\mathcal{B} \subset \mathcal{X}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{B}$ such that $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$ (cf. [25]).

Remark 8. We point out that Theorem 6.1 also yields the existence of the global attractor already proven in Section 5.

If we set

$$z(t) = (e(t), \chi(t), \eta^t) = S^\alpha(t)z_0$$

with $z_0 \in \mathcal{H}$, then the following preliminary result can be established

Lemma 6.3. *For any given $T > 0$, the function*

$$\rho_\sigma^T : C\left([0,\infty);\mathcal{H}\right) \times C\left([0,\infty);\mathcal{H}\right) \to [0,\infty)$$

defined by

$$\rho_\sigma^T(z_1, z_2) = \sup_{t\in[0,T]} \|\chi_1(t) - \chi_2(t)\|_{V_{1-\sigma}} + |m_{e_1-e_2}|\,, \quad \forall\ z_1, z_2 \in C([0,\infty);\mathcal{H}),$$

is a precompact pseudometric both on the space $C([0,T];\mathcal{H})$, and on the space $\mathcal{H}$.

Proof. It is apparent that ρ_σ^T defines a pseudometric on $\mathcal{H}$. To prove the precompactness, let us set

$$z_{0,n} = (e_{0,n}, \chi_{0,n}, \eta_{0,n}) \in \mathcal{H} \quad \text{s.t.} \quad \|\,(e_{0,n}, \chi_{0,n}, \eta_{0,n})\,\|_{\mathcal{H}} \leq c,$$

for some $c > 0$ independent of $n \in \mathbb{N}$, and define

$$z_n(t) = S^\alpha(t)z_{0,n} = (e_n(t), \chi_n(t), \eta_n^t), \quad \forall n \in \mathbb{N}.$$

In this proof, c_T shall denote a positive constant independent of n, but possibly depending on T. Consider now the product of equation (3.2) by $B_0^{-1}\partial_t\chi_n$. We have

$$\frac{1}{2}\frac{d}{dt}\|P\chi_n\|_{V_0}^2 + \|\partial_t\chi_n\|_{V_0^*}^2 + \alpha\|\partial_t\chi_n\|^2 = -\langle\phi(\chi_n), \partial_t\chi_n\rangle + \langle P(e_n - \chi_n), \partial_t\chi_n\rangle. \quad (6.1)$$

So that, by ($\boldsymbol{H2}$) and the bound on the $\mathcal{H}$-norm on $z_{0,n}$, we have

$$\begin{aligned} -\langle \phi(\chi_n), \partial_t \chi_n \rangle &\leq \frac{1}{\alpha}\|\phi(\chi_n)\|^2 + \frac{\alpha}{4}\|\partial_t \chi_n\|^2 \leq \frac{1}{\alpha}\left(1 + c\|\chi_n\|_V^2\right) + \frac{\alpha}{4}\|\partial_t \chi_n\|^2 \\ &\leq \frac{c}{\alpha} + \frac{\alpha}{2}\|\partial_t \chi_n\|^2 \\ \langle P(e_n - \chi_n), \partial_t \chi_n \rangle &\leq \frac{1}{\alpha}\|e_n - \chi_n\|^2 + \frac{\alpha}{4}\|\partial_t \chi_n\|^2 \leq \frac{c}{\alpha} + \frac{\alpha}{4}\|\partial_t \chi_n\|^2. \end{aligned}$$

Therefore (6.1) yields

$$\frac{d}{dt}\|P\chi_n\|_{V_0}^2 + \|\partial_t \chi_n\|_{V_0^*}^2 + \frac{\alpha}{2}\|\partial_t \chi_n\|^2 \leq \frac{c}{\alpha}. \tag{6.2}$$

Applying [3, Lemma A.5] to (6.2), we infer

$$\|P\chi_n(t)\|_{V_0}^2 \leq \frac{c_T}{\alpha}, \quad \forall t \in [0, T].$$

Therefore, since we have

$$m_{\chi_n} \leq \|\chi_n\|_V \leq c,$$

we deduce

$$\|\chi_n\|_{L^\infty(0,T;V)} \leq \frac{c_T}{\alpha}.$$

Then, integrating (6.2) over $[0, T]$

$$\|\partial_t \chi_n\|_{L^2(0,T;V_0^*)} \leq \frac{c_T}{\alpha}.$$

On account of a well-known compactness result (see [39, Corollary 4]), we conclude that there exists a subsequence of χ_n which converges in $C([0, T]; V_{1-\sigma})$, for any $\sigma \leq \sigma_0$. This proves the precompactness of the first summand, ρ_σ^T being constant on the first and the third components. Concerning the second summand, the thesis follows easily, since it belongs to a finite dimensional space. □

Remark 9. It is worth noting that, by slightly modifying the above proof, it is possible to remove the assumption $\alpha > 0$.

The following lemma is crucial to prove Theorem 6.1.

Lemma 6.4. *For any fixed $\alpha > 0$, there exist two nonnegative continuous functions*

$$f \in L^1(0, \infty) \cap C([0, \infty)) \quad \textit{and} \quad g_\alpha \in L^\infty_{\text{loc}}(0, \infty),$$

such that

$$\|S^\alpha(t) z_{0,1} - S^\alpha(t) z_{0,2}\|_{\mathcal{H}}^2 \leq f(t)\|z_{0,1} - z_{0,2}\|_{\mathcal{H}}^2 + g_\alpha(t)\rho_\sigma^t(z_{0,1}, z_{0,2})^2 \tag{6.3}$$

for any $t \in [0, \infty)$ and any $z_{0,1}, z_{0,2} \in \mathcal{B}_0$.

Proof. The proof of this lemma will be carried out by sharply refining the continuous dependence estimates provided by Section 3.2. Once again, for $z_{0,1}, z_{0,2} \in \mathcal{B}_0$, we set

$$\widetilde{z}_0 = (\widetilde{e}_0, \widetilde{\chi}_0, \widetilde{\eta}_0) = (e_{0,1}, \chi_{0,1}, \eta_{0,1}) - (e_{0,2}, \chi_{0,2}, \eta_{0,2}),$$

$$z_1(t) = (e_1(t), \chi_1(t), \eta_1^t) = S^\alpha(t) z_{0,1} \quad \text{and} \quad z_2(t) = (e_2(t), \chi_2(t), \eta_2^t) = S^\alpha(t) z_{0,2}.$$

Then the difference of trajectories, defined by

$$\widetilde{z}(t) = (\widetilde{e}(t), \widetilde{\chi}(t), \widetilde{\eta}^t) = S^\alpha(t) z_{0,1} - S^\alpha(t) z_{0,2},$$

fulfills system (3.6)-(3.9).

By linearity, inequality (4.10) can be extended to the difference of trajectories. This yields

$$\frac{d}{dt}L_0(\widetilde{z})+\frac{k_0}{2}\|P\left(\widetilde{e}-\widetilde{\chi}\right)\|^2 \leq 2k_0\|\widetilde{\eta}\|^2_{\mathcal{M}}+\|\partial_t\widetilde{\chi}\|^2_{V_0^*}-c\int_0^\infty \mu'(s)\|\widetilde{\eta}(s)\|^2_{V_0^*}ds. \quad (6.4)$$

We now consider the product of (3.8) by $\widetilde{\eta}$ in $\mathcal{M}$. As a consequence of ($\boldsymbol{K4}$) we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\widetilde{\eta}\|^2_{\mathcal{M}}+\frac{\lambda}{2}\|\widetilde{\eta}\|^2_{\mathcal{M}}-\frac{1}{2}\int_0^\infty \mu'(s)\|\widetilde{\eta}(s)\|^2_{V_0^*}ds\\ \leq\ &\int_0^\infty \mu(s)\langle P\left(\widetilde{e}-\widetilde{\chi}\right),\widetilde{\eta}(s)\rangle ds. \end{aligned}\quad (6.5)$$

Take the product of equation (3.6) by $P(\widetilde{e}-\widetilde{\chi})$. Keeping (6.5) into account, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|P\left(\widetilde{e}-\widetilde{\chi}\right)\|^2+\|\widetilde{\eta}\|^2_{\mathcal{M}}\right]+\frac{\lambda}{2}\|\widetilde{\eta}\|^2_{\mathcal{M}} \quad (6.6)\\ &-\frac{1}{2}\int_0^\infty \mu'(s)\|\widetilde{\eta}(s)\|^2_{V_0^*}ds+\langle\partial_t\chi, P\left(\widetilde{e}-\widetilde{\chi}\right)\rangle \leq 0.\end{aligned}$$

We now perform the following products of equation (3.7) by suitable test functions.

- By $B_0^{-1}\partial_t\widetilde{\chi}$, to get

$$\frac{1}{2}\frac{d}{dt}\|P\widetilde{\chi}\|^2_{V_0}+\|\partial_t\widetilde{\chi}\|^2_{V_0^*}+\alpha\|\partial_t\widetilde{\chi}\|^2-\langle\partial_t\widetilde{\chi},P(\widetilde{e}-\widetilde{\chi})\rangle=-\langle\phi(\chi_1)-\phi(\chi_2),\partial_t\widetilde{\chi}\rangle.$$

Since

$$-\langle\phi(\chi_1)-\phi(\chi_2),\partial_t\widetilde{\chi}\rangle\leq\frac{1}{2\alpha}\|\phi(\chi_1)-\phi(\chi_2)\|^2+\frac{\alpha}{2}\|\partial_t\widetilde{\chi}\|^2,$$

then

$$\frac{1}{2}\frac{d}{dt}\|P\widetilde{\chi}\|^2_{V_0}+\|\partial_t\widetilde{\chi}\|^2_{V_0^*}+\frac{\alpha}{2}\|\partial_t\widetilde{\chi}\|^2+\langle\partial_t\widetilde{\chi},P(\widetilde{e}-\widetilde{\chi})\rangle\leq\frac{1}{\alpha}\|\phi(\chi_1)-\phi(\chi_2)\|^2. \quad (6.7)$$

- By $v=B_0^{-1}P\widetilde{\chi}$, to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|P\widetilde{\chi}\|^2_{V_0^*}+\alpha\|P\widetilde{\chi}\|^2\right]+\|P\widetilde{\chi}\|^2_{V_0}\\ =\ &\langle P(\widetilde{e}-\widetilde{\chi}),P\widetilde{\chi}\rangle-\langle P\left(\phi(\chi_1)-\phi(\chi_2)\right),P\widetilde{\chi}\rangle.\end{aligned}$$

As a consequence of the immediate inequalities

$$\begin{aligned}\langle P(\widetilde{e}-\widetilde{\chi}),P\widetilde{\chi}\rangle &\leq \|P(\widetilde{e}-\widetilde{\chi})\|\|P\widetilde{\chi}\|\leq c\|\widetilde{e}-\widetilde{\chi}\|\|\widetilde{\chi}\|_{V_{1-\sigma}}\\ -\langle P\left(\phi(\chi_1)-\phi(\chi_2)\right),P\widetilde{\chi}\rangle &\leq c\|\phi(\chi_1)-\phi(\chi_2)\|^2+\frac{1}{2}\|P\widetilde{\chi}\|^2_{V_0},\end{aligned}$$

we infer

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|P\widetilde{\chi}\|^2_{V_0^*}+\alpha\|P\widetilde{\chi}\|^2\right]+\frac{1}{2}\|P\widetilde{\chi}\|^2_{V_0} \quad (6.8)\\ \leq\ &c\|\widetilde{e}-\widetilde{\chi}\|\|\widetilde{\chi}\|_{V_{1-\sigma}}+c\|\phi(\chi_1)-\phi(\chi_2)\|^2.\end{aligned}$$

The term to be controlled is

$$\mathcal{J}=c\|\phi(\chi_1)-\phi(\chi_2)\|^2.$$

First note that, for all $\sigma \in (0,1/2)$, the following embeddings hold

$$\begin{aligned} L^{6/(1-2\sigma)}(\Omega) &\hookrightarrow L^{12/(2-\sigma)}(\Omega), \\ V_{1-\sigma/2} &\hookrightarrow L^{6/(1+\sigma)}(\Omega). \end{aligned}$$

Now we use assumption ($\boldsymbol{H2}$) and then apply the Hölder inequality choosing the exponents $\frac{3}{2-\sigma}$ and $\frac{3}{1+\sigma}$, respectively

$$\begin{aligned} \mathcal{J} &\le c\|(1+|\chi_1|^2+|\chi_2|^2)\widetilde{\chi}\|^2 \\ &\le c\left(1+\|\chi_1\|^4_{L^{12/(2-\sigma)}(\Omega)}+\|\chi_2\|^4_{L^{12/(2-\sigma)}(\Omega)}\right)\|\widetilde{\chi}\|^2_{L^{6/(1+\sigma)}(\Omega)} \\ &\le c\left(1+\|\chi_1\|^4_{L^{6/(1-2\sigma)}(\Omega)}+\|\chi_2\|^4_{L^{6/(1-2\sigma)}(\Omega)}\right)\|\widetilde{\chi}\|^2_{V_{1-\sigma/2}} \\ &\le c\left(1+\|\chi_1\|^4_{V_{1+\sigma}}+\|\chi_2\|^4_{V_{1+\sigma}}\right)\|\widetilde{\chi}\|^2_{V_{1-\sigma/2}}. \end{aligned}$$

Moreover, since, by interpolation

$$\|f\|_{V_{1+\sigma}} \le \|f\|^{1/2}_{V_{1+2\sigma}}\|f\|^{1/2}_{V} \le c\|f\|^{1/2}_{W}\|f\|^{1/2}_{V}, \quad \forall f \in W,$$

and

$$\|g\|_{V_{1-\sigma/2}} \le \|g\|^{1/2}_{V}\|g\|^{1/2}_{V_{1-\sigma}}, \quad \forall g \in V,$$

then we deduce

$$\mathcal{J} \le c\left(\|\chi_1\|^2_W+\|\chi_2\|^2_W\right)\|\widetilde{\chi}\|_V\|\widetilde{\chi}\|_{V_{1-\sigma}}. \tag{6.9}$$

Combining (6.6) with (6.7) and taking (6.9) into account, we obtain the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|P\left(\widetilde{e}-\widetilde{\chi}\right)\|^2+\|\widetilde{\eta}\|^2_{\mathcal{M}}+\|\widetilde{\chi}\|^2_{V_0}\right]+\frac{\lambda}{2}\|\widetilde{\eta}\|^2_{\mathcal{M}} \qquad (6.10)\\ &-\frac{1}{2}\int_0^\infty \mu'(s)\|\widetilde{\eta}(s)\|^2_{V_0^*}ds+\|\partial_t\chi\|^2_{V_0^*} \le \frac{c}{\alpha}\left(\|\chi_1\|^2_W+\|\chi_2\|^2_W\right)\|\widetilde{\chi}\|_V\|\widetilde{\chi}\|_{V_{1-\sigma}}. \end{aligned}$$

Let $\nu>0$ be a small constant, and add together inequalities ν times (6.6) to (6.10). By choosing

$$\nu < \min\left\{\frac{\lambda}{2k_0},\frac{1}{2c}\right\},$$

we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|P\left(\widetilde{e}-\widetilde{\chi}\right)\|^2+\|\widetilde{\eta}\|^2_{\mathcal{M}}+\|P\chi\|^2_{V_0}+\nu L_0(z)\right] \qquad (6.11)\\ &+c\left[\|P\left(\widetilde{e}-\widetilde{\chi}\right)\|^2+\|\widetilde{\eta}\|^2_{\mathcal{M}}\right] \le \frac{c}{\alpha}\left(\|\chi_1\|^2_W+\|\chi_2\|^2_W\right)\|\widetilde{\chi}\|_V\|\widetilde{\chi}\|_{V_{1-\sigma}}. \end{aligned}$$

Let us introduce the following functional, defined for all $t \in [0,\infty)$

$$\begin{aligned} \widetilde{\Phi}_0(t) &= \|P\left(\widetilde{e}(t)-\widetilde{\chi}(t)\right)\|^2+\|P\widetilde{\chi}(t)\|^2_{V_0^*}+\alpha\|P\widetilde{\chi}(t)\|^2+\|P\widetilde{\chi}(t)\|^2_{V_0} \\ &\quad +\|\widetilde{\eta}^t\|^2_{\mathcal{M}}+\nu L_0(\widetilde{z}(t))+\widetilde{k}_0, \end{aligned}$$

for a positive constant $\widetilde{k}_0$ to be properly chosen. Observe that

$$\|\widetilde{e}-\widetilde{\chi}\|^2+\|\chi\|^2_V \le \|P\left(\widetilde{e}-\widetilde{\chi}\right)\|^2+\|P\chi\|^2_{V_0}+c\left(m^2_{\widetilde{e}-\widetilde{\chi}}+m^2_{\widetilde{\chi}}\right), \tag{6.12}$$

so that we can find $\widetilde{k}_0$ large enough to have

$$\widetilde{\omega}_0\|\widetilde{z}(t)\|^2_{\mathcal{H}} \le \widetilde{\Phi}_0(t) \le c\|\widetilde{z}(t)\|^2_{\mathcal{H}}+c\left(m^2_{\widetilde{e}-\widetilde{\chi}}+m^2_{\widetilde{\chi}}\right), \quad \forall t \in [0,+\infty), \tag{6.13}$$

for some positive $\widetilde{\omega}_0$.

We now add (6.8) to (6.11). By (6.12) and (6.13), there exists $\widetilde{\kappa}_0 > 0$ such that

$$\begin{aligned}\frac{d}{dt}\widetilde{\Phi}_0(t) + \widetilde{\kappa}_0\|\widetilde{z}(t)\|^2_{\mathcal{H}} \quad &\leq \quad \frac{c}{\alpha}\left(\|\chi_1(t)\|^2_W + \|\chi_2(t)\|^2_W\right)\sqrt{\widetilde{\Phi}_0(t)}\|\widetilde{\chi}(t)\|_{V_{1-\sigma}} \\ &\quad + c\left(m^2_{\widetilde{e}-\widetilde{\chi}} + m^2_{\widetilde{\chi}}\right), \quad \forall t \in [0,\infty).\end{aligned}$$

Since

$$m_{\widetilde{\chi}} \leq c\|\widetilde{\chi}\|_{V_{1-\sigma}},$$

then [3, Lemma A.5] yields

$$\widetilde{\Phi}_0(t) \quad \leq \quad 2e^{-\widetilde{\kappa}_0 t}\widetilde{\Phi}_0(\widetilde{z}_0) + C_\alpha(t)\left[\sup_{\tau\in[0,t]}\|\widetilde{\chi}(\tau)\|^2_{V_{1-\sigma}} + m^2_{\widetilde{e}}\right], \quad \forall t \in [0,\infty),$$

having set (see (4.2))

$$C_\alpha(t) = \frac{c}{\alpha}\left[\int_0^t \left(\|\chi_1(\tau)\|^2_W + \|\chi_2(\tau)\|^2_W\right)d\tau\right]^2 \leq \frac{c}{\alpha}(1+t^2).$$

Therefore, recalling (6.13), Lemma 6.4 is proven, provided that we choose

$$f(t) = ce^{-\widetilde{\kappa}t} \quad \text{and} \quad g_\alpha(t) = \frac{c}{\alpha}(1+t^2).$$

□

7. **Smoothness of the attractor in the viscous case.** In this section we aim to investigate the regularity properties of $\mathcal{A}_\alpha$ in the viscous case. Under a further assumption on the nonlinearity ϕ that reads, namely,

$$(\boldsymbol{H3}) \quad \phi'(r) \geq -\ell, \quad \forall r \in \mathbb{R},$$

for some $\ell \geq 0$, we shall prove the following

Theorem 7.1. *For any fixed $\alpha > 0$, $\mathcal{A}_\alpha$ is a bounded subset of the higher order phase space*

$$\mathcal{V}_{\beta,\gamma} = \mathcal{H}_{\beta,\gamma} \cap (V \times W \times \mathcal{N}).$$

Remark 10. We immediately stress out that the inclusion $\mathcal{V} = V \times W \times \mathcal{N} \subset \mathcal{H}$ is clearly continuous but not compact. Nevertheless, we know already that $\mathcal{A}_\alpha$ is compact by definition.

In order to prove Theorem 7.1 we shall exploit the decomposition technique devised in [38]. As a consequence of the assumption ($\boldsymbol{H3}$) and bound (4.1), we can choose $\theta \geq \ell$ large enough such that the inequality holds

$$\frac{1}{2}\|z\|^2_V + (\theta - 2\ell)\|z\|^2 - \langle\phi'(\chi(t))z, z\rangle \geq 0 \tag{7.1}$$

holds for every $z \in V$ and every $t \in [0,\infty)$. Then we define

$$\psi(r) = \phi(r) + \theta r, \quad \forall r \in \mathbb{R}.$$

To this purpose we consider the further decomposition

$$z(t) = z^d(t) + z^c(t),$$

where

$$z^d(t) = (e^d(t), \chi^d(t), \eta^{d,t}) \quad \text{and} \quad z^c(t) = (e^c(t), \chi^c(t), \eta^{c,t})$$

are the solutions at time $t \in [0,\infty)$ to the following problems, respectively,

$$\partial_t e^d + \int_0^\infty \mu(s)\eta^d(s)ds = 0, \tag{7.2}$$

$$\partial_t \chi^d + B_0\big(B_0\chi^d + \alpha\partial_t\chi^d + P(\psi(\chi) - \psi(\chi^c)) - (\eta^d - \chi^d)\big) = 0, \tag{7.3}$$

$$\partial_t \eta^d = T\eta^d + B_0(\eta^d - \chi^d), \tag{7.4}$$

$$z^d(0) = (Pe_0, P\chi_0, \eta_0), \tag{7.5}$$

and

$$\partial_t e^c + \int_0^\infty \mu(s)\eta^c(s)ds = 0, \tag{7.6}$$

$$\partial_t \chi^c + B_0\big(B_0 P\chi^c + \alpha\partial_t\chi^c + P\psi(\chi^c) - P(e^c - \chi^c)\big) = \theta B_0 P\chi, \tag{7.7}$$

$$\partial_t \eta^c = T\eta^c + B_0 P(\eta^c - \chi^c), \tag{7.8}$$

$$z^c(0) = (m_{e_0}, m_{\chi_0}, 0). \tag{7.9}$$

The next technical lemmas provide asymptotic properties of z^c and z^d.

Lemma 7.2. *There holds*

$$\sup_{z_0\in\mathcal{B}_0}\sup_{t\in[0,\infty)} \|z^c\|_{\mathcal{H}} < \infty. \tag{7.10}$$

Moreover, the following integral bound holds

$$\sup_{z_0\in\mathcal{B}_0}\sup_{t\in[0,\infty)} \int_s^t \Big[\|\partial_t\chi^c\|^2_{V_0^*} + \alpha\|\partial_t\chi^c\|^2\Big] d\tau \le \omega(t-s) + \frac{c}{\omega}, \tag{7.11}$$

for all $t \ge s \ge 0$ and all $\omega > 0$.

Proof. The proof of inequality (7.10) goes exactly like the one of (4.1) (cf. Section 4.1), noticing that $B_0P\chi \in L^\infty(0,\infty;V_0^*)$. In order to prove (7.11), we consider the product of equation (7.8) byη^c in $\mathcal{M}$, equation (7.6) with $P(e^c - \chi^c)$ and equation (7.7) by $B_0^{-1}\partial_t\chi^c$. Summing up the resulting equalities, and exploiting assumption (***K*4**) once again, we end up with

$$\frac{1}{2}\frac{d}{dt}\left[\|P\left(e^c - \chi^c\right)\|^2 + \|P\chi^c\|^2_{V_0} + \|\eta^c\|^2_{\mathcal{M}} + 2\langle\Psi(\chi),1\rangle\right] \tag{7.12}$$
$$+\|\partial_t\chi^c\|^2_{V_0^*} + \alpha\|\partial_t\chi^c\|^2 \le \theta\langle\chi,\partial_t\chi^c\rangle,$$

where we have defined

$$\Psi(x) = \int_0^x \psi(y)dy, \quad \forall x \in \mathbb{R}.$$

Since

$$\langle\chi,\partial_t\chi^c\rangle = \frac{d}{dt}\langle\chi,\chi^c\rangle - \langle\partial_t\chi,\chi^c\rangle,$$

and, by means of (7.10), we have, for all $\omega > 0$,

$$-\theta\langle\partial_t\chi,\chi^c\rangle \le \frac{\omega}{2}\|\chi^c\|^2_{V_0} + \frac{c}{\omega}\|\partial_t\chi\|^2_{V_0^*} \le \frac{\omega}{2} + \frac{c}{\omega}\|\partial_t\chi\|^2_{V_0^*},$$

then we get from (7.12)

$$\frac{d}{dt}\Phi_0^c(t) + \|\partial_t \chi^c\|^2_{V_0^*} + \alpha\|\partial_t \chi^c\|^2 \leq \frac{\omega}{2} + \frac{c}{\omega}\|\partial_t \chi\|^2_{V_0^*}, \tag{7.13}$$

having set, for all $t \in [0, \infty)$,

$$\Phi_0^c(t) = \|P\left(e^c(t) - \chi^c(t)\right)\|^2 + \|\eta^{c,t}\|^2_{\mathcal{M}} + \|P\chi^c(t)\|^2_{V_0} + 2\langle \Psi(\chi(t)), 1\rangle + \theta\langle \chi(t), \chi^c(t)\rangle.$$

Thanks to assumption (***H*2**) and to bounds (4.1) and (7.10), we infer $\Phi_0^c(t) \leq c$ for all $t \in [0, \infty)$. Therefore the thesis is reached by integrating (7.13) on the time-interval (s, t), and by invoking integral bound (4.2). □

Theorem 7.1 is a consequence of the following

Lemma 7.3. *There exist $\kappa > 0$, independent of $\alpha \in (0, 1]$, and $c_\alpha > 0$, such that*

$$\|z^d(t)\|_{\mathcal{H}} \leq c_\alpha e^{-\kappa t}\|z_0\|_{\mathcal{H}}, \quad \forall t \in [0, \infty), \tag{7.14}$$

for all $z_0 \in \mathcal{B}_0$.

Lemma 7.4. *There exists $C_\alpha > 0$ such that*

$$\sup_{z_0 \in \mathcal{B}_0} \sup_{t \in [0,\infty)} \|z^c\|_{\mathcal{V}} \leq C_\alpha. \tag{7.15}$$

In fact, Lemma 7.3 and 7.4 yield the existence of an exponentially attracting (closed and bounded) set in $\mathcal{V}_{\beta,\gamma}$. This entails the thesis of Theorem 7.1, since $\mathcal{A}_\alpha$ is, by definition, the minimal (closed) attracting set in $\mathcal{H}_{\beta,\gamma}$.

7.1. **Proof of Lemma 7.3.** First notice that, by linearity, the functional L_0 defined in Section 4.1 fulfills the estimates

$$|L_0(z^d)| \leq c\left(\|e^d - \chi^d\|^2 + \|\eta^d\|^2_{\mathcal{M}}\right) \tag{7.16}$$

and

$$\frac{d}{dt}L_0(z^d) + \frac{k_0}{4}\|e^d - \chi^d\|^2 \leq c\left(\|\eta^d\|^2_{\mathcal{M}} + \|\partial_t \chi^d\|^2_{V_0^*}\right). \tag{7.17}$$

Now multiply equation (7.3) by $B_0^{-1}\chi^d$, to get

$$\frac{1}{2}\frac{d}{dt}\left[\|\chi^d\|^2_{V_0^*} + \alpha\|\chi^d\|^2\right] + \|\chi^d\|^2_{V_0} + \langle \psi(\chi) - \psi(\chi^c), \chi^d\rangle = \langle e^d - \chi^d, \chi^d\rangle.$$

Since assumption (***H*3**) entails $\psi'(r) \geq 0$ for any $r \in \mathbb{R}$, then

$$\langle \psi(\chi) - \psi(\chi^c), \chi^d\rangle = \int_\Omega \psi'(\xi)|\chi^d|^2 d\Omega \geq 0,$$

where, for all $t \in [0, \infty)$, we have set

$$\min\{\chi(t), \chi^d(t)\} \leq \xi(t) \leq \max\{\chi(t), \chi^d(t)\}.$$

By the immediate inequality

$$\langle e^d - \chi^d, \chi^d\rangle \leq \frac{1}{2}\|\chi^d\|^2_{V_0} + \|e^d - \chi^d\|^2,$$

we get

$$\frac{d}{dt}\left[\|\chi^d\|^2_{V_0^*} + \alpha\|\chi^d\|^2\right] + \|\chi^d\|^2_{V_0} \leq 2\|e^d - \chi^d\|^2. \tag{7.18}$$

Next, we multiply equation (7.4) by η^d in $\mathcal{M}$, equation (7.2) by $e^d - \chi^d$ and equation (7.3) by $B_0^{-1}\partial_t \chi^d$. Analogously the the previous cases, adding together the resulting equalities, and exploiting assumption ($\boldsymbol{K4}$) once again, we end up with the following inequality

$$\begin{aligned}&\frac{d}{dt}\left[\|e^d-\chi^d\|^2+\|\chi^d\|^2_{V_0}+\|\eta^d\|^2_{\mathcal{M}}\right]\\ +\quad &2\lambda\|\eta^d\|^2_{\mathcal{M}}+2\langle\psi(\chi)-\psi(\chi^c),\partial_t\chi^d\rangle\leq 0. \qquad (7.19)\end{aligned}$$

Concerning the nonlinear term, observe that

$$\begin{aligned}2\langle\psi(\chi)-\psi(\chi^c),\partial_t\chi^d\rangle \;=\;& \frac{d}{dt}\left[2\langle\psi(\chi)-\psi(\chi^c),\chi^d\rangle-\langle\psi'(\chi),|\chi^d|^2\rangle\right]\\ &+2\langle[\psi'(\chi)-\psi'(\chi^c)]\,\chi^d,\partial_t\chi^c\rangle-\langle\psi''(\chi)|\chi^d|^2,\partial_t\chi^d\rangle.\end{aligned}$$

Moreover, by means of assumption ($\boldsymbol{H2}$), (4.1) and (7.10), we have

$$\begin{aligned}&2\langle[\psi'(\chi)-\psi'(\chi^c)]\,\chi^d,\partial_t\chi^c\rangle-\langle\psi''(\chi)|\chi^d|^2,\partial_t\chi^d\rangle\\ \leq\;& c\|(|\chi|^2+|\chi^c|^2)|\partial_t\chi^c||\chi^d|^2\|+c\||\chi|^2|\partial_t\chi^d||\chi^d|^2\|\\ \leq\;& c(\|\chi\|^2_V+\|\chi^c\|^2_V)\|\partial_t\chi^c\|\|\chi^d\|^2_{V_0}\\ \leq\;& \varepsilon\|\chi^d\|^2_{V_0}+c(\|\partial_t\chi\|^2+\|\partial_t\chi^d\|^2)\|\|\chi^d\|^2_{V_0},\end{aligned}$$

being $\varepsilon>0$ to be chosen small enough in the sequel. Therefore, from (7.19), we infer

$$\begin{aligned}&\frac{d}{dt}\left[\|e^d-\chi^d\|^2+\|\chi^d\|^2_{V_0}+\|\eta^d\|^2_{\mathcal{M}}+2\langle\psi(\chi)-\psi(\chi^c),\chi^d\rangle\right.\\ &\left.-\langle\psi'(\chi),|\chi^d|^2\rangle\right]\leq a\|\chi^d\|^2_{V_0}+c(\|\partial_t\chi\|^2+\|\partial_t\chi^d\|^2)\|\chi^d\|^2_{V_0}. \qquad (7.20)\end{aligned}$$

Adding together γ times (7.17), δ times (7.18) and (7.20), for some $\gamma,\delta>0$ to be chosen in the sequel, there holds

$$\begin{aligned}&\frac{d}{dt}\Phi^d(t)+\left(\frac{\gamma k_0}{4}-2\delta\right)\|e^d-\chi^d\|^2+\left(\frac{\delta}{2}-\varepsilon\right)\|\chi^d\|^2_{V_0} \qquad (7.21)\\ &+(2\lambda-\gamma c)\,\|\eta^d\|^2_{\mathcal{M}}+(1-\gamma)\,\|\partial_t\chi^d\|^2_{V_0^*}\leq c\left(\|\partial_t\chi^d\|^2_{V_0^*}+\|\partial_t\chi^d\|^2\right)\|\chi^d\|^2_{V_0},\end{aligned}$$

having set, for all $t\in[0,\infty)$,

$$\begin{aligned}\Phi^d(t)\;=\;&\|e^d(t)-\chi^d(t)\|^2+\|\chi^d(t)\|^2_{V_0}+\|\eta^{d,t}\|^2_{\mathcal{M}}+\delta\|\chi^d(t)\|^2_{V_0^*}+\delta\alpha\|\chi^d(t)\|^2\\ &+2\langle[\psi'(\chi)-\psi'(\chi^c)]\,\chi^d,\partial_t\chi^c\rangle-\langle\psi''(\chi)|\chi^d|^2,\partial_t\chi^d\rangle+\gamma L_0(z^d(t)).\end{aligned}$$

Since assumption ($\boldsymbol{H3}$) and (7.1) entail

$$2\langle\psi(\chi)-\psi(\chi^c),\chi^d\rangle-\langle\psi'(\chi),|\chi^d|^2\rangle\geq(\theta-2\ell)\|\chi^d\|^2-\langle\phi'(\chi)\chi^d,\chi^d\rangle\geq-\frac{1}{2}\|\chi^d\|^2_{V_0},$$

by choosing

$$\gamma<\min\{2\lambda/c,1/c,1\},\quad \delta<k_0/8\quad\text{and}\quad\varepsilon<\delta/2,$$

we deduce the inequality

$$\frac{1}{2}\|z^d(t)\|^2_{\mathcal{H}}\leq\Phi^d(t)\leq c\|z^d(t)\|^2_{\mathcal{H}}. \qquad (7.22)$$

Therefore, for some $\kappa > 0$

$$\frac{d}{dt}\Phi^d(t) + 2\kappa\Phi^d(t) \le \Lambda(t)\Phi^d(t),$$

being

$$\Lambda(t) = c\left(\|\partial_t\chi(t)\|^2_{V_0^*} + \|\partial_t\chi(t)\|^2 + \|\partial_t\chi^c(t)\|^2_{V_0^*} + \|\partial_t\chi^c(t)\|^2\right).$$

Since, on account of (4.17), we easily deduce

$$\int_0^\infty \left(\|\partial_t\chi(t)\|^2_{V_0^*} + \alpha\|\partial_t\chi(t)\|^2\right) d\tau \le c,$$

keeping in mind (7.11), and choosing ω small enough, we are led to

$$\int_s^t \Lambda(\tau)d\tau \le \kappa(t-s) + \frac{c}{\alpha},$$

for all $t \ge s \ge 0$ and all $\omega > 0$. We are now in a position to apply [38, Lemma 5], which yields

$$\Phi^d(t) \le \Phi^d(0)e^{c/\alpha}e^{-\kappa t}, \qquad \forall t \in [0,\infty).$$

The thesis is then achieved by invoking (7.22) and setting $c_\alpha = e^{c/2\alpha}$.

7.2. **Proof of Lemma 7.4.** We first derive a further integral bound for χ^c. Let us multiply equation (7.7) by $P\chi^c$. Arguing as in Subsection 4.2, we easily deduce the inequality (analogous to (4.21))

$$\begin{aligned}\frac{d}{dt}\left[\|P\chi^c\|^2 + \alpha\|P\chi^c\|^2_{V_0}\right] + \|P\chi^c\|^2_{W_0} &\le \theta\langle B_0^{1/2}P\chi, B_0^{1/2}P\chi^c\rangle + c \qquad (7.23)\\ &\le \|\chi\|_V\|\chi^c\|_V + c \le c,\end{aligned}$$

where in the last inequality we have used bounds (4.1) and (7.10). Integrating (7.23) on the time-interval $(t, t+r)$ and using (7.10) once again, we have

$$\sup_{z_0\in\mathcal{B}_0}\sup_{t\in[0,\infty)}\int_t^{t+r}\|\chi^c(\tau)\|^2_W d\tau \le c. \qquad (7.24)$$

Let us now introduce, for all $z = (e, \chi, \eta) \in \mathcal{V}_{\beta,\gamma}$, the functional

$$L_1(z) = -\int_0^\infty \psi(s)\langle\eta^c(s), e^c - \chi^c\rangle ds,$$

where ψ is the truncated kernel defined in Subsection 4.1. Using inequality (4.3) and (4.4) and bound (7.10) we get at once

$$|L_1(z^c)| \le \|P(e^c - \chi^c)\|^2_{V_0} + c. \qquad (7.25)$$

Taking the time derivative of $L_1(z^c)$ and using (7.8) we obtain

$$\begin{aligned}\frac{d}{dt}L_1(z^c) + \frac{k_0}{2}\|P(e^c - \chi^c)\|^2_{V_0} &= \int_0^\infty \psi(s)\langle\eta^c(s), \partial_t\chi^c\rangle ds \qquad (7.26)\\ + \int_0^\infty \psi(s)\mu(s)\|\eta^c(s)\|^2 ds &+ \int_0^\infty \psi(s)\langle P(e^c - \chi^c), T\eta^c(s)\rangle ds,\end{aligned}$$

Using once more (4.3), (4.4) and (7.10), we deduce

$$\int_0^\infty \psi(s)\langle\eta^c(s),\partial_t\chi^c\rangle ds \le \left(\int_0^\infty \psi(s)ds\right)\|\partial_t\chi^c\|^2+\int_0^\infty \psi(s)\|\eta^c(s)\|^2 ds$$
$$\le c\|\partial_t\chi\|^2+\|\eta\|^2_{\mathcal{N}}, \tag{7.27}$$

$$\int_0^\infty \psi(s)\mu(s)\|\eta^c(s)\|^2 ds \le \mu(s_0)\|\eta^c\|^2_{\mathcal{N}} \le \|\eta^c\|^2_{\mathcal{N}}. \tag{7.28}$$

Observe now that

$$\lim_{s\to 0}\psi(s)\langle P(e^c-\chi^c),\eta^c(s)\rangle=\lim_{s\to\infty}\psi(s)\langle P(e^c-\chi^c),\eta^c(s)\rangle=0.$$

Then, integrating by parts with respect to s the third summand in (7.26), we obtain

$$\int_0^\infty \psi(s)\langle P(e^c-\chi^c),T\eta^c(s)\rangle ds=-\int_0^\infty \psi'(s)\langle P(e^c-\chi^c),\eta^c(s)\rangle ds$$
$$\le \frac{k_0}{4}\mu(s_0)\|P(e^c-\chi^c)\|^2_{V_0}-\frac{1}{k_0}\int_0^\infty \mu'(s)\|\eta^c(s)\|^2_{V_0^*}ds$$
$$\le \frac{k_0}{4}\|P(e^c-\chi^c)\|^2_{V_0}+\frac{\lambda}{k_0}\|\eta^c\|^2_{\mathcal{M}} \le \frac{k_0}{4}\|P(e^c-\chi^c)\|^2_{V_0}+c. \tag{7.29}$$

Substituting (7.27)-(7.29) into (7.26), we deduce

$$\frac{d}{dt}L_1(z^c)+\frac{k_0}{4}\|P(e^c-\chi^c)\|^2_{V_0} \le c\|\partial_t\chi^c\|^2+2\|\eta^c\|^2_{\mathcal{N}}+c. \tag{7.30}$$

Next, we multiply equation (7.8) by η^c in $\mathcal{N}$, equation (7.6) by $B_0P(e^c-\chi^c)$ and equation (7.7) by $\partial_t\chi^c$. Analogously to the previous cases, summing up the resulting equalities, and exploiting assumption (***K*4**) once again, we end up with the following inequality

$$\frac{1}{2}\frac{d}{dt}\left[\|P(e^c-\chi^c)\|^2_{V_0}+\|P\chi^c\|^2_{W_0}+\|\eta^c\|^2_{\mathcal{N}}\right]+\lambda\|\eta^c\|^2_{\mathcal{N}} \tag{7.31}$$
$$+\|\partial_t\chi^c\|^2+\alpha\|\partial_t\chi^c\|^2_{V_0} \le -\langle B_0P\psi(\chi^c),\partial_t\chi^c\rangle+\theta\langle B_0P\chi,\partial_t\chi^c\rangle.$$

By means of assumption (***H*2**) and bounds (4.1) and (7.10), we see that

$$\langle B_0P\psi(\chi^c),\partial_t\chi^c\rangle+\theta\langle B_0P\chi,\partial_t\chi^c\rangle \le \frac{c}{\alpha}\|\psi'(\chi^c)\nabla\chi^c\|^2+\frac{c}{\alpha}\|\chi\|^2_V+\alpha\|\partial_t\chi^c\|^2_{V_0}$$
$$\le \frac{c}{\alpha}+\frac{c}{\alpha}\left(1+\|\chi^c\|^2_V\right)\|\nabla\chi^c\|^2_V+\alpha\|\partial_t\chi^c\|^2_{V_0}$$
$$\le \frac{c}{\alpha}\left(1+\|\chi^c\|^2_W\right)+\alpha\|\partial_t\chi^c\|^2_{V_0},$$

so that, in (7.31),

$$\frac{d}{dt}\left[\|P(e^c-\chi^c)\|^2_{V_0}+\|P\chi^c\|^2_{W_0}+\|\eta^c\|^2_{\mathcal{N}}\right]+2\lambda\|\eta^c\|^2_{\mathcal{N}}+\le\frac{c}{\alpha}\left(1+\|\chi^c\|^2_W\right). \tag{7.32}$$

Now we add together equation (7.23), δ times equation (7.30) and (7.32), to get

$$\frac{d}{dt}\Phi^c(t)+\frac{k_0}{2}\|P(e^c-\chi^c)\|^2_{V_0} + \|P\chi^c\|^2_{W_0}+2(\lambda-\delta)\|\eta^c\|^2_{\mathcal{N}} \tag{7.33}$$
$$\le \frac{c}{\alpha}\left(1+\|\chi^c\|^2_W+\alpha\|\partial_t\chi^c\|^2\right),$$

having set, for all $t \in [0,\infty)$,

$$\begin{aligned}\Phi^c(t) &= \|P\left(e^c(t)-\chi^c(t)\right)\|^2_{V_0} + \|\eta^{c,t}\|^2_{\mathcal{N}} + \|P\chi^c(t)\|^2_{W_0} + \alpha\|P\chi^c\|^2_{V_0} \\ &\quad +\delta L_1(z^c(t)) + k,\end{aligned}$$

being k a positive constant. Indeed, due to (7.25), it is possible to choose k large enough so as to ensure $\Phi^c(t) \geq 0$. Moreover, as a consequence of the bound on the averages, we realize that

$$\frac{1}{2}\|z^c(t)\|^2_{\mathcal{V}} \leq \Phi^c(t) \leq c\|z^c(t)\|^2_{\mathcal{V}} + c. \tag{7.34}$$

Therefore, for some $\kappa > 0$,

$$\frac{d}{dt}\Phi^c(t) + 2\kappa\Phi^c(t) \leq \frac{c}{\alpha}\left(1 + \|\chi^c\|^2_W + \alpha\|\partial_t\chi^c\|^2\right).$$

Thanks to (7.11) and (7.24), it is possible to apply [3, Lemma A.5], which yields

$$\Phi^c(t) \leq \Phi^c(0)e^{-\kappa t} + \frac{c}{\alpha}\int_0^t \left(1 + \|\chi^c(\tau)\|^2_W + \alpha\|\partial_t\chi^c(\tau)\|^2\right)e^{-\kappa(t-\tau)}d\tau \leq \frac{c}{\alpha}.$$

The thesis is then reached, by invoking (7.34) and setting $C_\alpha = \sqrt{c/\alpha}$.

8. **Appendix: control of the nonlinear terms** (5.24)-(5.26) **and** (5.28)-(5.29).

In the sequel we shall indicate by $c_* \geq 0$ a generic constant, independent of $\alpha \in [0,1]$, while Θ_c is defined by (5.19).

Control of $\mathcal{I}_1$ (cf. (5.24)*).* We have

$$\mathcal{I}_1 \quad \leq \quad \|\phi'(\chi)B_0^\sigma P\chi_c\|_{V_{1-\sigma}}\|\partial_t\chi_c\|_{V_0^{-1+\sigma}}. \tag{8.1}$$

By means of embedding (5.15), we get

$$\begin{aligned}&\|\phi'(\chi)B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \leq c\|\phi'(\chi)B_0^\sigma P\chi_c\|_{L^{6/(3+2\sigma)}(\Omega)} \\ &+c\|\phi'(\chi)\nabla B_0^\sigma P\chi_c\|_{L^{6/(3+2\sigma)}(\Omega)} + c\|\phi''(\chi)\nabla\chi B_0^\sigma P\chi_c\|_{L^{6/(3+2\sigma)}(\Omega)}.\end{aligned}$$

In order to estimate the terms appearing on the right-hand side of the above inequality, we use embeddings (5.14)-(5.16), controls (5.17)-(5.20) and assumption (***H2***). Applying the generalized Hölder inequality with exponents $\frac{3+2\sigma}{2}$ and $\frac{3+2\sigma}{1+2\sigma}$, respectively, we obtain

$$\begin{aligned}\|\phi'(\chi)B_0^\sigma P\chi_c\|_{L^{6/(3+2\sigma)}(\Omega)} &\leq c\left(1+\|\chi\|^2_{L^6(\Omega)}\right)\|B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)} \\ &\leq c\left(1+\|\chi\|^2_V\right)\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \leq c\|\chi_c\|_{V_{1+\sigma}},\end{aligned}$$

$$\begin{aligned}\|\phi'(\chi)\nabla B_0^\sigma P\chi_c\|_{L^{6/(3+2\sigma)}(\Omega)} &\leq c\left(1+\|\chi\|^2_{L^6(\Omega)}\right)\|\nabla B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)} \\ &\leq c\left(1+\|\chi\|^2_V\right)\|\nabla B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \leq c\|\nabla B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \\ &\leq c\|B_0^\sigma P\chi_c\|_{V_{2-\sigma}} \leq c\|\chi_c\|_{V_{2+\sigma}} \leq c + c\|P\chi_c\|_{V_0^{2+\sigma}}.\end{aligned}$$

By choosing the Hölder exponents $3+2\sigma$, $3+2\sigma$ and $\frac{3+2\sigma}{1+2\sigma}$, respectively, we get

$$\begin{aligned}
&\|\phi''(\chi)\nabla\chi B_0^\sigma P\chi_c\|_{L^{6/(3+2\sigma)}(\Omega)}\\
\leq\;& c\left(1+\|\chi\|_{L^6(\Omega)}\right)\|\nabla\chi\|_{L^6(\Omega)}\|B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)}\\
\leq\;& c\left(1+\|\chi\|_V\right)\|\nabla\chi\|_V\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}}\\
\leq\;& c\|\chi\|_W\|\chi_c\|_{V_{1+\sigma}}.
\end{aligned}$$

Back to (8.1), a further application of the Young inequality yields

$$\mathcal{I}_1 \leq c + c\Theta_c\|\chi_c\|^2_{V_{1+\sigma}} + c_*\|P\chi_c\|^2_{V_0^{2+\sigma}} + \frac{1}{2}\|\partial_t\chi_c\|^2_{V_0^{-1+\sigma}}.$$

Control of $\mathcal{I}_2$ (cf. (5.25)*).* We have

$$\mathcal{I}_2 \leq \|\left(\phi'(\chi)-\phi'(\chi_d)\right)B_0^\sigma P\chi_c\|_V\|\partial_t\chi_d\|_{V_0^*}, \tag{8.2}$$

with

$$\begin{aligned}
&\|\left(\phi'(\chi)-\phi'(\chi_d)\right)B_0^\sigma P\chi_c\|_V\\
\leq\;& c\|\left(\phi'(\chi)-\phi'(\chi_d)\right)B_0^\sigma P\chi_c\| + c\|\left(\phi'(\chi)-\phi'(\chi_d)\right)\nabla B_0^\sigma P\chi_c\|\\
&+c\|\phi''(\chi)\nabla\chi_c B_0^\sigma P\chi_c\| + c\|\left(\phi''(\chi)-\phi''(\chi_d)\right)\nabla\chi_d B_0^\sigma P\chi_c\|.
\end{aligned}$$

Invoking embeddings (5.14)-(5.16) and controls (5.17)-(5.20), we use again the generalized Hölder inequality. On account of assumption (***H2***), we now set the Hölder exponents equal to 3, $\frac{3}{1-2\sigma}$ and $\frac{3}{1+2\sigma}$, respectively. We find

$$\begin{aligned}
&\|\left(\phi'(\chi)-\phi'(\chi_d)\right)B_0^\sigma P\chi_c\| \leq c\|\left(1+|\chi|+|\chi_d|\right)\chi_c B_0^\sigma P\chi_c\|\\
\leq\;& c\left(1+\|\chi\|_{L^6(\Omega)}+\|\chi_d\|_{L^6(\Omega)}\right)\|\chi_c\|_{L^{6/(1-2\sigma)}(\Omega)}\|B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)}\\
\leq\;& c\left(1+\|\chi\|_V+\|\chi_d\|_V\right)\|\chi_c\|_{V_{1+\sigma}}\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \leq c\|\chi_c\|^2_{V_{1+\sigma}}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\|\left(\phi'(\chi)-\phi'(\chi_d)\right)\nabla B_0^\sigma P\chi_c\| \leq c\|\left(1+|\chi|+|\chi_d|\right)\chi_c\nabla B_0^\sigma P\chi_c\|\\
\leq\;& c\left(1+\|\chi\|_{L^6(\Omega)}+\|\chi_d\|_{L^6(\Omega)}\right)\|\chi_c\|_{L^{6/(1-2\sigma)}(\Omega)}\|\nabla B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)}\\
\leq\;& c\left(1+\|\chi\|_V+\|\chi_d\|_V\right)\|\chi_c\|_{V_{1+\sigma}}\|\nabla B_0^\sigma P\chi_c\|_{V_{1-\sigma}}\\
\leq\;& c\|\chi_c\|_{V_{1+\sigma}}\|B_0^\sigma P\chi_c\|_{V_{2-\sigma}} \leq c\|\chi_c\|_{V_{1+\sigma}}\|\chi_c\|_{V_{2+\sigma}}\\
\leq\;& c\|\chi_c\|_{V_{1+\sigma}} + c\|\chi_c\|_{V_{1+\sigma}}\|P\chi_c\|_{V_0^{2+\sigma}},
\end{aligned}$$

and

$$\begin{aligned}
&\|\phi''(\chi)\nabla\chi_c B_0^\sigma P\chi_c\| \leq c\|\left(1+|\chi|\right)\nabla\chi_c B_0^\sigma P\chi_c\|\\
\leq\;& c\left(1+\|\chi\|_{L^6(\Omega)}\right)\|\nabla\chi_c\|_{L^{6/(1-2\sigma)}(\Omega)}\|B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)}\\
\leq\;& c\left(1+\|\chi\|_V\right)\|\nabla\chi_c\|_{V_{1+\sigma}}\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}}\\
\leq\;& c\|\nabla\chi_c\|_{V_{1+\sigma}}\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \leq c\|\chi_c\|_{V_{2+\sigma}}\|\chi_c\|_{V_{1+\sigma}}\\
\leq\;& c\|\chi_c\|_{V_{1+\sigma}} + c\|\chi_c\|_{V_{1+\sigma}}\|P\chi_c\|_{V_0^{2+\sigma}}.
\end{aligned}$$

Moreover, there holds

$$
\begin{aligned}
&\| (\phi''(\chi) - \phi''(\chi_d)) \nabla\chi_d B_0^\sigma P\chi_c\| \le c\|\nabla\chi_d \chi_c B_0^\sigma P\chi_c\| \\
\le\ & c\|\nabla\chi_d\|_{L^{3/2}(\Omega)}\|\chi_c\|_{L^{6/(1-2\sigma)}(\Omega)}\|B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)} \\
\le\ & c\|\nabla\chi_d\|_{L^6(\Omega)}\|\chi_c\|_{V_{1+\sigma}}\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \le c\|\nabla\chi_d\|_V\|\chi_c\|_{V_{1+\sigma}}\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \\
\le\ & c\|\chi_d\|_W\|\chi_c\|_{V_{1+\sigma}}\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \le c\|\chi_d\|_W\|\chi_c\|^2_{V_{1+\sigma}}.
\end{aligned}
$$

Summing up, we get

$$
\| (\phi'(\chi) - \phi'(\chi_d)) B_0^\sigma P\chi_c\|_V \le c\,(1 + \|\chi_d\|_W)\,\|\chi_c\|^2_{V_{1+\sigma}} + c\|\chi_c\|_{V_{1+\sigma}}\|P\chi_c\|_{V_0^{2+\sigma}}.
$$

Thus, from (8.2) we infer

$$
\mathcal{I}_2 \quad \le \quad c\Theta_c + c\Theta_c\|\chi_c\|^2_{V_{1+\sigma}} + c_*\|P\chi_c\|^2_{V_0^{2+\sigma}}.
$$

Control of $\mathcal{I}_3$ *(cf.* (5.28)*).* We have

$$
\mathcal{I}_3 \quad \le \quad \|\phi_1'(\chi_d)B_0^\sigma P\chi_c\|_V\|\partial_t\chi_d\|_{V_0^*}. \tag{8.3}
$$

Observe that

$$
\begin{aligned}
\|\phi_1'(\chi_d)B_0^\sigma P\chi_c\|_V \quad \le \quad & c\|\phi_1'(\chi_d)B_0^\sigma P\chi_c\| + c\|\phi_1'(\chi_d)\nabla B_0^\sigma P\chi_c\| \\
& + c\|\phi_1''(\chi_d)\nabla\chi_d B_0^\sigma P\chi_c\|.
\end{aligned}
$$

Applying assumption ($\boldsymbol{H2_1}$), and subsequently the generalized Hölder inequality with exponents, $\frac{3}{2}$ and 3, respectively, we obtain

$$
\begin{aligned}
&\|\phi_1'(\chi_d)B_0^\sigma P\chi_c\| \\
\le\ & c\|(1 + |\chi_d|^{\theta-1})B_0^\sigma P\chi_c\| \le c\left(1 + \|\chi_d\|^{\theta-1}_{L^{3(\theta-1)}(\Omega)}\right)\|B_0^\sigma P\chi_c\|_{L^6(\Omega)} \\
\le\ & c\left(1 + \|\chi_d\|^{\theta-1}_{L^{3(\theta-1)}(\Omega)}\right)\|B_0^\sigma P\chi_c\|_{L^6(\Omega)} \le c(1 + \|\chi_d\|^{\theta-1}_V)\|B_0^\sigma P\chi_c\|_V \\
\le\ & c\|\chi_c\|_{V_{1+2\sigma}} \le c\|\chi_c\|_{V_{2+\sigma}} \le c + c\|\chi_c\|_{V_0^{2+\sigma}}.
\end{aligned}
$$

Similarly, by choosing the Hölder exponents $\frac{3}{2-2\sigma}$ and $\frac{3}{1+2\sigma}$, we deduce

$$
\begin{aligned}
&\|\phi_1'(\chi_d)\nabla B_0^\sigma P\chi_c\| \\
\le\ & c\|(1 + |\chi_d|^{\theta-1})\nabla B_0^\sigma P\chi_c\| \\
\le\ & c\left(1 + \|\chi_d\|^{\theta-1}_{L^{3(\theta-1)/(1-\sigma)}(\Omega)}\right)\|\nabla B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)} \\
\le\ & c\left(1 + \|\chi_d\|^{\theta-1}_{L^6(\Omega)}\right)\|\nabla B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \\
\le\ & c\left(1 + \|\chi_d\|^{\theta-1}_V\right)\|B_0^\sigma P\chi_c\|_{V_{2-\sigma}} \le c\|\chi_c\|_{V_{2+\sigma}} \le c + c\|\chi_c\|_{V_0^{2+\sigma}};
\end{aligned}
$$

Then, choosing the Hölder exponents as $\frac{3}{1-2\sigma}$, 3 and $\frac{3}{1+2\sigma}$, we infer

$$
\begin{aligned}
& \|\phi_1''(\chi_d)\nabla\chi_d B_0^\sigma P\chi_c\| \\
\leq\ & c\|(1+|\chi_d|^{\theta-2})\nabla\chi_d B_0^\sigma P\chi_c\| \\
\leq\ & c\left(1+\|\chi_d\|^{\theta-2}_{L^{6(\theta-2)/(1-2\sigma)}(\Omega)}\right)\|\nabla\chi_d\|_{L^6(\Omega)}\|B_0^\sigma P\chi_c\|_{L^{6/(1+2\sigma)}(\Omega)} \\
\leq\ & c\left(1+\|\chi_d\|^{\theta-2}_{L^6(\Omega)}\right)\|\nabla\chi_d\|_V\|B_0^\sigma P\chi_c\|_{V_{1-\sigma}} \\
\leq\ & c\left(1+\|\chi_d\|_V^{\theta-2}\right)\|\chi_d\|_W\|\chi_c\|_{V_{1+\sigma}} \leq c\|\chi_d\|_W\|\chi_c\|_{V_{1+\sigma}}.
\end{aligned}
$$

Notice that in the estimates of the last two terms we have used the embeddings

$$
\begin{aligned}
L^6(\Omega) &\hookrightarrow L^{3(\theta-1)/(1-\sigma)}(\Omega), \\
L^6(\Omega) &\hookrightarrow L^{6(\theta-2)/(1-2\sigma)}(\Omega),
\end{aligned}
$$

which hold provided that $\sigma \leq \sigma_0 = (3-\theta)/2$.

Therefore, on account of the above controls, (8.3) yields

$$\mathcal{I}_3 \leq c\Theta_c + c\Theta_c\|\chi_c\|^2_{V_{1+\sigma}} + c_*\|P\chi_c\|^2_{V_0^{2+\sigma}},$$

and adding together the controls on $\mathcal{I}_i$ $(i = 1, .., 3)$, we eventually get inequality (5.30).

Control of $\mathcal{I}_4$ *(cf.* (5.28)*).* To obtain the desired control note first that, by assumption (***H2***), there holds

$$\|B_0^{\sigma/2}P\phi(\chi)\|^2 + \|B_0^{\sigma/2}P\phi(\chi_d)\|^2 \leq c\left(1+\|\chi\|_W^2+\|\chi_d\|_W^2\right),$$

for all $\sigma \in (0, 1/2)$. Hence we find

$$\mathcal{I}_4 \leq c\Theta_c + \frac{1}{8}\|P\chi_c\|^2_{V_0^{2+\sigma}}.$$

Control of $\mathcal{I}_5$ *(cf.* (5.29)*).* Similarly to the control provided for $\mathcal{I}_4$, by ($\boldsymbol{H2_1}$), it is readily seen that

$$\|B_0^{\sigma/2}P\phi_1(\chi_d)\|^2 \leq c\left(1+\|\chi_d\|_W^2\right),$$

which entails

$$\mathcal{I}_4 \leq c\Theta_c + \frac{1}{8}\|P\chi_c\|^2_{V_0^{2+\sigma}}.$$

Adding together the controls on $\mathcal{I}_4$ and $\mathcal{I}_5$, we infer inequality (5.31).

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REFERENCES

[1] A.V. Babin, M.I. Vishik, *Attractors of evolution equations*, North-Holland, Amsterdam 1992

[2] V. Belleri, V. Pata, *Attractors for semilinear strongly damped wave equation on* $\mathbb{R}^3$, Discrete Contin. Dynam. Systems **7**, 719–735 (2001)

[3] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam 1973

[4] M. Brokate, J. Sprekels, *Hysteresis and phase transitions*, Springer-Verlag, New York 1996

[5] G. Caginalp, *The dynamics of a conserved phase-field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math. 44 **1**, 77–94 (1990)

[6] S. Ceron, O. Lopes, *α-contractions and attractors for dissipative semilinear hyperbolic equations and systems*, Ann. Mat. Pura Appl. (IV) **160**, 193–206 (1992)

[7] B.D. Coleman, M.E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys. **18**, 199–208 (1967)

[8] P. Colli, G. Gilardi, M. Grasselli, G. Schimperna, *Global existence for the conserved phase-field model with memory and quadratic nonlinearity*, Port. Math. (N.S.) **58**, 159–170 (2001)

[9] P. Colli, G. Gilardi, P. Laurençot, A. Novick-Cohen, *Uniqueness and long-time behavior for the conserved phase-field system with memory*, Discrete Contin. Dynam. Systems **5**, 375–390 (1999)

[10] M. Conti, V. Pata, M. Squassina, *Singular limit of differential system with memory*, Indiana Univ. Math. J. 55 **1**, 169–216 (2006)

[11] C.M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37**, 297–308 (1970)

[12] M. Efendiev, A. Miranville, S. Zelik, *Exponential attractors for a singularly perturbed Cahn-Hilliard system*, Math. Nachr. **272**, 11–31 (2004)

[13] C.M. Elliot, I.N. Kostin, *Lower semicontinuity of a non-hyperbolic attractor for the viscous Cahn-Hilliard equation*, Nonlinearity **3**, 687–702 (1996)

[14] C.M. Elliot, A.M. Stuart, *Viscous Cahn-Hilliard equation. II. Analysis*, J. Differential Equations **2**, 387–414 (1996)

[15] S. Gatti, M. Grasselli, V. Pata, *Exponential Attractors for a conserved phase-field system with memory*, Phys. D **189**, 31–48 (2004)

[16] S. Gatti, M. Grasselli, V. Pata, *Exponential Attractors for a phase-field model with memory and quadratic nonlinearities*, Indiana Univ. Math. J. **53**, 719–754 (2004)

[17] C. Giorgi, J.E. Muñoz Rivera, V. Pata, *Global attractors for a semilinear hyperbolic equation in viscoelasticity*, J. Math. Anal. Appl. **260**, 83–99 (2001)

[18] M. Grasselli, V. Pata, *On the dissipativity of a hyperbolic phase-field with memory*, Nonlinear Anal. **47**, 3157–3169 (2001)

[19] M. Grasselli, V. Pata, *Uniform attractors of nonautonomous systems with memory*, in "Evolution Equations, Semigroups and Functional Analysis" (A. Lorenzi and B. Ruf, Eds.) , pp.155–178, Progr. Nonlinear Differential Equations Appl. no. 50, Birkhäuser, Boston 2002

[20] M. Grasselli, V. Pata, *Asymptotic behavior of a parabolic-hyperbolic system*, Commun. Pure Appl. Anal. **4**, 849–881 (2004)

[21] M. Grasselli, V. Pata, *Attractor for a conserved phase-field system with hyperbolic heat conduction*, Math. Methods Appl. Sci. **27**, 1917–1934 (2004)

[22] M. Grasselli, V. Pata, *Attractors of phase-field systems with memory*, Chap. 7 of "Mathematical methods and models in phase transitions" (A. Miranville, Ed.), Nova Science Publishers, Inc., 157–175 (2005)

[23] M. Grasselli, V. Pata, F.M. Vegni, *Longterm dynamics of a conserved phase-field model with memory*, Asymptot. Anal. **33**, 261–320 (2003)

[24] M.E. Gurtin, A.C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31**, 113-126 (1968)

[25] J.K. Hale, *Asymptotic behaviour of dissipative systems*, Amer. Math. Soc., Providence 1988

[26] L. Herrera, D. Pavòn, *Hyperbolic theories of dissipation: Why and when do we need them?*, Phys. A **307**, 121–130 (2002)

[27] J. Jäckle, *Heat conduction and relaxation in liquids of high viscosity*, Phys. A **162**, 377–404 (1990)

[28] D.D. Joseph, L. Preziosi, *Heat waves*, Rev. Modern Phys. **61**, 4173 (1989)

[29] D.D. Joseph, L. Preziosi, *Addendum to the paper: Heat waves [Rev. Modern Phys. 61 (1989), no. 1, 4173]*, Rev. Modern Phys. **62**, 375391 (1990)

[30] D. Jou, J. Casás-Vasquez, G. Lebon, *Extended irreversible thermodynamics*, Rep. Progr. Phys. **51**, 1105-1179 (1988)

[31] J.-L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications*, I Springer-Verlag, Berlin 1972

[32] A. Novick-Cohen, *Conserved phase-field equations with memory* in Curvature flows and related topics (Levico, 1994), 179–197 GAKUTO Internat. Ser. Math. Sci. Appl., Gakkōtosho **5**, Tokyo 1995

[33] A. Novick-Cohen, *On the viscous Cahn-Hilliard equation. Material instabilities in continuum mechanics* (Edinburgh, 1985–1986), 329–342, Oxford Sci. Publ., Oxford Univ. Press, New York 1988

[34] V. Pata, *Exponential stability in linear viscoelasticity*, Quart. Appl. Math., **64**, 499–513 (2006)

[35] V. Pata, *Hyperbolic limit of parabolic semilinear heat equations with fading memory* Z. Anal. Anwendungen. **20**, 359–377 (2001)

[36] V. Pata, G. Prouse, M.I. Vishik, *Traveling waves of dissipative nonautonomous hyperbolic equations in a strip*, Adv. Differential Equations **2**, 249–270 (1998)

[37] V. Pata, A. Zucchi, *Attractors for a damped hyperbolic equation with linear memory*, Adv. Math. Sci. Appl. **11**, 505–529 (2001)

[38] V. Pata, S. Zelik, *A remark on the weakly damped wave equation*, Comm. Pure. Appl. Anal. **5**, 609–614 (2006)

[39] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (IV), **146**, 65–96 (1987)

[40] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York 1988

[41] F.M. Vegni, *Dissipativity of a conserved phase-field system with memory*, Discrete Contin. Dynam. Systems **9**, 949–968 (2003)

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