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ACHILLE VERNIZZI MARIA MONTI

MAREK KOSNY

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Università degli Studi di Milano

Dipartimento di
Scienze Economiche, Aziendali e Statistiche

Via Conservatorio 7 -- 20122 Milano
tel. ++39 02 503 21501 (21522) fax ++39 02 503 21450 (21505)
http://www.economia.unimi.it E Mail: dipeco@unimi.it

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AN OVERALL INEQUALITY REDUCING AND HORIZONTALLY EQUITABLE TAX SYSTEM WITH APPLICATION TO POLISH DATA

Achille Vernizzi Maria Monti ♦ Marek Kosny ♥

In the case of homogeneous[1] populations progressive taxation is usually associated with a reduction of inequality in after tax income distribution. In the real world, populations of tax payers are not homogeneous and households differ not only with respect to income but also in other social, demographic and economic characteristics. The aim of the paper is to propose an inequality reducing tax system taking into account, in some way, differences in needs between tax payers.

A population of households is considered and each household is distinguished by two attributes: pre-tax income and family size. To compare incomes in this heterogeneous environment an affine transformation is proposed as equivalent income function, from which income dependent equivalence scales are derived.

By first, the horizontal equity concern of a tax system has to be faced. Ebert and Lambert (2004) define horizontal equity requirements for income dependent scales. Applying the proposed scales a tax system can be derived and we show that it satisfies the horizontal equity conditions as defined by two authors.

Then, we consider the redistributive effects of a tax system by introducing an inequality parameterized measure as in Ebert and Moyes (2000). We show that, when the reference type family is fixed, the application of the suggested equivalence income function is such that the tax system can be overall inequality reducing provided that the reference type tax function is average rate or minimal progressive.

As it is known, conditions ensuring that a tax system is overall inequality reducing are not sufficient to guarantee inequality reduction within each set of households with the same size. We derive restrictions which allow the equivalent income function to generate a tax system which is overall inequality reducing and within type inequality reducing. These restrictions concern the tax function for the reference family type and the domain of incomes.

To see where these restrictions come from, it is important to note that a tax system which is overall inequality reducing for a fixed reference may well increase inequality (or keep it unaltered) when another reference type is chosen. But, if a tax system is within type inequality reducing, it is also overall inequality reducing for any reference type. Ebert and Moyes (2000) show that only two particular form of equivalence function can be adopted when reference independence is required. The first function yields an income independent absolute scale; the second function yields an income independent relative. The former has to be used when inequality is considered from the absolute point of view, the latter has to be used when an intermediate between absolute and relative concept of inequality is adopted or when relative inequality is considered.

♦ Department of Economics, Business and Statistics, Universita' degli Studi di Milano; achille.vernizzi@unimi.it, maria.monti@unimi.it.
♥ Department of Operational Research, Wroclaw University of Economics; marek.kosny@ae.wroc.pl.
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[1] Population of income units of the same type, that is with the same relevant social and demographic characteristics.

Here we suggest only an income transformation function which originates income depending scales. On considering inequality either in absolute or in relative sense, using this function we obtain a tax system which is overall inequality reducing and reference independent if some restrictions hold on income domain, on the set of taxable income and on the tax function for the reference type: we think that these restrictions are not too limitative as it can be understood from the applicative part.

The paper is organized as follows. In the first part, following Ebert-Lambert (2004), horizontal equity (HE) conditions are defined in both cases when equivalence scales are income dependent or income independent. Then, following Ebert-Moyes (2000), we summarize what has to be meant for an overall inequality reducing tax system (OIR), for a within type inequality reducing (WIR) tax system and for a reference independent tax system (RI).

In the second part the paper we describe the particular affine income transformation function that we adopt and we investigate on the conditions allowing to this function to generate a tax system which is OIR and WIR.

In the third part we simulate modifications to the present personal income tax in Poland, with the aim to design a more family oriented system.

The paper is completed by an appendix which analyzes the relation among the most important instruments used in the real world to take into account horizontal equity. Quotient, exemption and tax credit are discussed under different hypotheses on tax function.

1 NOTATIONS AND PRELIMINARY DEFINITIONS

In § 1.1 we introduce the notion of equivalent income function: this function can be used to evaluate the purchasing capability of households which, being heterogeneous in their size, face different cost levels to reach the same living standards. The equivalent income function converts the income of a generic family into the income necessary to the reference family to enjoy the same welfare as the former and vice versa. The choice of the reference family is arbitrary: usually it falls on the single or on the couple without children. In this paragraph we reassume the general characteristics and properties of equivalent income functions. In § 1.2 the horizontal equity principle is recalled. Remembering that, from the classical point of view, horizontal equity calls for equal tax treatment for equals, one considers two different households as equal, when they both present the same equivalent income. In accordance with Ebert and Lambert (2004) we report conditions intended to secure horizontal equity with respect to relative or absolute equivalence scales, which may be either independent or dependent on family incomes. The main technical instruments adopted in real world to pursue horizontal equity in personal income tax systems are summarized at the end of § 1.2.

Paragraph 1.3 describes how tax progression reduces income distribution inequality. As Ebert and Moyes (2000) show, on considering homogeneous population, the definition of progression is strictly linked to the adopted definition of inequality. They introduce a parameterized criterion which allows to evaluate either inequality of income distributions or progression of a tax system when different concepts of inequality hold. Under some conditions, the relation between progression and inequality can be extended to heterogeneous population. As it is well known, in heterogeneous environments, the evaluation about income distribution inequality can be performed only when nominal incomes are transformed into equivalent incomes: the transformation is carried on with respect to a specific household type which is chosen as the reference one. When there is a social agreement on the reference household, one can state conditions insuring that the post-tax income distribution is not more unequal than the pre-tax distribution. If a tax system respects these conditions, it reduces *overall* inequality: that is, the tax system reduces inequality whatever pre-tax income distribution and family sizes. But two questions have to be remarked. The first is that overall inequality reduction does not ensure inequality reduction in income distributions of each type of household. The second point is that the redistributive impact of the tax system depends on the chosen reference type, i.e same conditions cannot guarantee income inequality reduction if another reference family type is selected. More severe conditions are requested in order a tax system to be both overall inequality reducing and within type inequality reducing, or, equivalently, reference independent.[2] These conditions are summarized the last part of § 1.3.

1.1 Equivalent income function

In this section we define the meaning of the equivalent income function and we recall a class of functions proposed by Donaldson and Pendakur (1999) to which the function adopted in the second part of the paper belongs.

Let us consider a population consisting of *N* households, where each household is distinguished by two attributes: pre-tax income and family size[3]. Supposing that there exists a given and finite number of family sizes, let $H=\{1, 2, i,.. n\}$ represent the set of possible family sizes and let *D* pre-tax income domain. The set *H* is typically assumed to be a subset of positive integers, and *D* is the set, or a subset, of real number.

In order to measure welfare and to make comparisons of living standards among households with different number of components, we assume that there is a decision maker able to propose a suitable utility function[4]. Let us denote as $y \in D$ and $i \in H$, respectively, the pre-tax income and the size of a household. The welfare function of a representative member of the household of size *i* is $U(y,i)$, here abbreviated with $U_i(y)$. The function $U_i(y)$ is assumed to be *a*) continuous, increasing and invertible in *y*; *b*) non increasing in *i*.

Given a reference type household *r*, the equivalent income function $S_{r,i}(y)$ specifies the transformation that converts the income of household *i* into the reference household equivalent income. The equivalent income is the income needed by the reference household in order to be as well off as household *i*, when the latter's income is *y*.
Formally, we can write[5]

$$U_i(y) = U_r(S_{r,i}(y)) \quad \text{for all } y \in D \text{ and } r,\, i \in H \tag{1}$$

Taking as reference type the *r* size household, the welfare index $U_i(\cdot)$, for each family of type *i* (i=1,2,…,n), may be expressed as function of the welfare index of household *r*. For any value of *y*, $S_{r,i}(y)$ is equal to the income the reference household *r* needs in order to be as well off as household *i* possessing income *y*. For example let be y_i type *i* household's income, then

$$y_{r,i}=S_{r,i}(y_i)$$

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[2] As Ebert and Moyes (2000, page 137) write, a tax system is reference independent when it is both overall inequality reducing and within type inequality reducing.

[3] For the sake of simplicity, we consider only a socio-demographic characteristic: the family dimension.

[4] In order to derive an equalizing procedure by a social welfare function, an individual utility function *U* has to be:

(a) continuous;
(b) strictly increasing in *y*;
(c) non-increasing in *i*
(d) having the same image in *D* for all *i*;
(e) concave;
(f) U and $\tilde{U}$ are informationally equivalent if there is a function $F(y) = a + by$, $b>0$, such that $\tilde{U}= a+b\,U$

Properties (b) and (d) imply that $U(\cdot)$ is invertible.
Property (f) implies that $U(\cdot)$ is a cardinal full comparable utility function.
For more on this argument see Ebert (2000).

[5] We remark that in expression (1), the equivalent income function *S*(.) depends on the welfare function *U*(.): the specification of *S*(.), as well as the specification of the welfare function *U*(.), depends on the decision maker value judgments.

is r type household's equivalent income.

Employing $S(.)$, household's living standards can be compared and a household j is (weakly) better off than a household i if and only if $S_{r,i}(y_i) \leq S_{r,j}(y_j)$. In other words, welfare is defined as that yielded by the reference household at $S(.)$: the equivalent income is a welfare measure.

From (1) and from the properties of the welfare function U (see note 4), the following properties[6] hold for $S(.)$:

(a) $S_{r,r}(y) = S_{r,r}(y) = U_r^{-1} \circ U_r(y) = y$
(b) $S_{r,i}(y)$ is continuous and increasing in y for all $r,i \in H$;
(c) $S_{r,i}(y) = S_{r,l}(y) \circ S_{l,i}(y)$;
(d) $S_{r,i}(y)$ is non increasing in i for all $i \in H$ and $y \in D$.

Having defined[7] the equivalent income function[8], it can be observed that the conventional equivalence scale approach assumes a proportional transformation of income. In this case, using a deflator $sd_i(y)$, that may or not depend on income, one writes

$$S_{r,i}(y_i) = \frac{y_i}{sd_i(y_i)} \quad \text{for all } i \tag{2}$$

if the deflator varies with income and

$$S_{r,i}(y_i) = \frac{y_i}{sd_i} \quad \text{for all } i \tag{2a}$$

if the deflator is income independent.

We define $sd_i(y)$ in expression (2) as the relative equivalence scale, whilst the general form for an absolute equivalence scale, say $sa(y)$, satisfies the following expression:

$$S_{r,i}(y_i) = y_i - sa_i(y_i) \tag{3}$$

If the absolute scale is income independent, that is if $sa(y) = sa$, expression (3) becomes:

$$S_{r,i}(y_i) = y_i - sa_i \tag{3a}$$

Ebert and Lambert (2004) observe that $sd_i(y)$ can be turned into the form of an absolute scale, and, conversely, $sa_i(y)$ can be turned into the form of a relative scale. Specifically:

$$sa_i(y_i) = y_i - \frac{y_i}{sd_i(y_i)} = y_i\left(1 - \frac{1}{sd_i(y_i)}\right) = y_i\left(\frac{sd_i(y_i) - 1}{sd_i(y_i)}\right) \tag{4}$$

and

$$sd_i(y_i) = \frac{y_i}{y_i - sa_i(y_i)} \tag{5}$$

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[6] We write $S_{r,i}(y_i) = S_{r,l} \circ S_{l,i}(y_i)$ instead of $S_{r,i}(y_i) = S_{r,l}(S_{l,i}(y_i))$.

[7] Ebert and Moyes (2000, page. 129) list a further property: $S_{r,i}(D)=D$. They explain that this property is a direct consequence of the previous conditions. This property imposes that the nominal income interval D is mapped in the same interval irrespective of the type of the household.

[8] For more details on this function, see Donaldson Pendakur (1999), Ebert (2000) and Ebert and Moyes (2000).

We stress that a relative income independent scale *sd*, when turned into an absolute scale, becomes income dependent and, vice versa, an absolute income independent scale *sa*, when turned into a relative scale, becomes income dependent.

An improved specification of expression (3) is proposed by Donaldson and Pendakur (1999). Their starting point is the remark that expression (3a) implies that equal absolute increases of income preserve utility equality across household types. In fact, from: $y_{r,i} = S_{r,i}(y)$ one has

$$U_i(y_i) = U_r(S_{r,i}) = U_r(y_i - sa_i) = U_r(y_{r,i}) \tag{6}$$

that is, the i type household with income y_i and the reference type with income y_i-da_i have the same welfare. It follows that, if y_i varies, the equality

$$U_i(y_i + dy_i) = U_r(y_{r,i} + dy_{r,i}) \tag{7}$$

is verified if and only if incomes increase exactly by the same amount (dy_i=$dy_{r,i}$).
This is a quite restrictive assumption, alternatively, one can require the weaker condition that the change in the reference household's income that preserves utility equality is income independent. This implies the existence of a function $\rho(i)>0$ such that

$$dy_{r,i} = \rho(i)dy_i \tag{8}$$

i.e. a one unity money increase in household income requires an increase of $\rho(i)$ unity of money to preserve equality of well-being, whatever the income value is.
Integrating, one obtains, for $\rho(r)=1$ and some function $\alpha(i)$ with $\alpha(r)=0$, the equivalent income function

$$y_{r.i} = \rho(i)y_i + \alpha(i) \tag{8a}$$

Expression (8a) may be rewritten in the more usual form

$$y_{r,i} = \frac{y_i - A(i)}{R(i)} \tag{8b}$$

with $A(i) = -\alpha(i)/\rho(i)$ and $R(i) = 1/\rho(i)$. From (8b), an affine absolute equivalence scale, depending on the income value, can be obtained

$$sa_i(y_i) = \frac{(R(i)-1)y_i + A(i)}{R(i)} \tag{9}$$

The absolute scale (9) is increasing (decreasing) in y_i, if $R(i)>1$ ($R(i)<1$)
The relative equivalence scale $sd_i(y_i)$ is given by

$$sd(y_i) = \frac{R(i)y_i}{y_i - A(i)} \tag{9a}$$

which is increasing (decreasing) in y if $A(i)<0$ ($A(i)>0$).

1.2 Horizontal equity defined using equivalent income function

Let **Y** represent the income distribution and for a household, with pre-tax income *y*, let the tax liability *t*(*y*) be a continuous[9] and non decreasing function in *y*. If the population is homogeneous, the tax system may be represented by the net income schedule $v(y)=y-t(y)$ which is, by assumption, continuous and non decreasing in pre-tax income *y*. In a non homogeneous population, let $t(y,i)=t_i(y_i)$ represent the tax liability faced by the *i* type household with income y_i and let $v_i(y_i)=y_i-t_i(y_i)$ represent *i*'s net income schedule.

Horizontal equity may be expressed, in accordance with Feldstein (1976, page 83) remarking that "if two individuals would be equally well-off in absence of taxation, they should be equally well-of if there is taxation". That is, choosing *r* type as reference household and $S_{r,i}(y_i)$ as equivalent income function, if before taxation it holds (see expression 1)

$$U_i(y_i)=U_r\left(S_{r,i}(y_i)\right)=U_r(y_r)$$

the tax system is horizontal equitable if and only if relation (10) holds on net income

$$U_i\left[y_i-t_i(y_i)\right]=U_r\left[y_{r,i}-t_r(y_r)\right] \tag{10}$$

In other words, H.E requires that if

$$y_{r,i}=S_{r,i}(y_i) \quad \text{for all } i \tag{11}$$

then it must be verified that[10]

$$v_r(y_{r,i})=S_{r,i}[v_i(y_i)] \quad \text{for all } i \tag{12}$$

If the equivalent income function takes the form of the relative scale (see expression (2)), from (12) it follows that, to satisfy H.E. requirement[11], one must have

$$v_i(y_i)=sd_i\left(v_i(y_i)\right)v_r\left(\frac{y_i}{sd_i(y_i)}\right) \tag{13}$$

Rearranging expression (13) one obtains H.E. condition on tax function

$$t_i(y_i)=sd_i(y_i)\cdot t_r\left(\frac{y_i}{sd_i(y_i)}\right)+\left[sd_i(y_i)-sd_i\left(v_i(y_i)\right)\right]\cdot v_r\left(\frac{y_i}{sd_i(y_i)}\right) \tag{14}$$

It is easy to see that, if the relative scale is income independent, expression (13) and expression (14) assume, respectively, the simpler form

$$v_i(y_i)=sd_i\cdot v_r\left(\frac{y_i}{sd_i}\right) \tag{13a}$$

$$t_i(y_i)=sd_i\cdot t\left(\frac{y_i}{sd_i}\right) \tag{14a}$$

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[9] Or almost continuous admitting a piecewise linear tax function.

[10] One may observe that (12) rewrites as $v_i(y_i)=S_{r,i}^{-1}\circ v_r\circ S_{r,i}(y_i)=S_{r,i}^{-1}\left\{S_{r,i}(y_i)-t_r\left[S_{r,i}(y_i)\right]\right\}$.

[11] For more on this argument see Ebert and Lambert (2004).

When an absolute income dependent scale is applied (expression 3), the H.E requirement (11) specializes into the following form:

$$v_i(y_i) = v_r(y_i - sa_i(y_i)) + sa_i(v_i(y_i)) \tag{15}$$

and from expression (15) the H.E. condition on tax function requires that:

$$t_i(y_i) = t_r(y_i - sa_i(y_i)) + \left[sa_i(y_i) - sa_i(v_i(y_i))\right] \tag{16}$$

If the absolute scale is income independent, the H.E. condition on net income becomes:

$$v_i(y_i) = v_r(y_i - sa_i) \tag{15a}$$

and condition on taxation yields:

$$t_i(y_i) = t_r(y_i - sa_i) \tag{16a}$$

1.3 Horizontal equity instruments

In actual tax systems, various procedures are adopted in order to keep into account the fact that tax liability should be affected by difference in social and demographic characteristics of households with the same nominal income. Among the most important we list: income splitting, allowances, tax credits and the so called *quotient familial*, adopted in France and Luxemburg which extends the income splitting rules to other members of households. These procedures are the real world concession to the principle of horizontal equity: we shortly describe three of them[12]

(a) quotients or splitting;
(b) exemptions;
(c) tax credit.

The easiest and simplest way to apply the above instruments is to consider these instruments depending on household demographic characteristics only, taking them invariant with respect to income.

Quotient: A tax calculated according to quotient has the following expression:

$$t_i(y_i) = q_i t_r\left(\frac{y_i}{q_i}\right) \tag{17}$$

We observe that expression (17) is exactly the tax function given in (14a) if we put sd_i equal to q_i, that is (17) is the H.E. tax with a relative scale invariant with respect to income if $q_i = sd_i$.

Income exemption A tax calculated according to exemption has the following expression:

$$t_i(y_i) = t_r(y_i - e_i) \tag{18}$$

(18) coincides with (15a) if we put $sa_i = e_i$, that is (18) is the H.E. tax function with an absolute scale invariant with respect to income if $sa_i = e_i$.

[12] Actual tax systems often use mixtures of the three specified instruments. For instance, the German tax system adopts the splitting for spouses and exemptions for children. More on this argument is discussed in the Appendix

Tax credit A tax calculated according to tax credit has the following expression:

$$t_i(y_i) = t_r(y_i) - c_i \tag{19}$$

At the end of the last century, there was in economic literature a debate about the research of a rational background for a tax system applying tax credit. Lambert and Yitzhaki (1997) conclude that perhaps tax credit emerged as an administrative convenience but, from an utilitarian point of view, cannot be explained as a mean to implement horizontal and vertical equity.
Here, we only observe that a tax credit satisfies horizontal equity commands when:

$$\begin{aligned} c_i &= t_r(y_i) - t_r\left(y_i - sa_i(y_i)\right) - \left[sa_i(y_i) - sa_i\left(v_i(y_i)\right)\right] = \\ &= t_r(y_i) - sd_i(y_i)\, t_r\left(\frac{y_i}{sd_i(y_i)}\right) - \left[sd_i(y_i) - sd_i\left(v_i(y_i)\right)\right] v_r\left(\frac{y_i}{sd_i(y_i)}\right) \end{aligned} \tag{20}$$

That is, an income independent tax credit can be horizontally equitable, if the functional form of the scale combines with the tax schedule in such a way to keep the difference between $t_i(y_i)$ and $t_r(y_i)$ constant whatever y may be.
It follows immediately that while the horizontal equitability of either an income independent quotient or of an income independent exemption does not depend on the tax schedule, the horizontal equitability of a tax credit probably is no longer verified when the tax schedule changes, with a scale that remains the same.
A tax credit $c_i(y_i)$ may be interpreted as the application of (16), if it is a function of income: we observe that in general expression (20) is a non decreasing function of income, it follows that in any case, $c_i(y_i)$ should not be a decreasing function of y. The Appendix describes some relations between tax credit and family quotient or exemptions, when either the former or the latter is constant.

In actual tax systems these three instruments can be applied in conjunction. Exemptions may be allowed to exclude from taxable income costs which should not be considered in equalized incomes. Tax credit can be introduced to subsidize particular costs[13] for merit goods[14].

A tax system that uses an income dependent quotient as horizontal equity instrument, may adopt the following tax function:

$$t_i(y_i) = q_i(y_i)\, t_r\left(\frac{y_i}{q_i(y_i)}\right) \tag{21}$$

It is important to note that to satisfy horizontal equity principle from (14) the quotient $q_i(y_i)$ should be such that:

$$q_i(y_i)\, t_r\left(\frac{y_i}{q_i(y_i)}\right) = sd_i(y_i)\, t_r\left(\frac{y_i}{sd_i(y_i)}\right) + \left[sd_i(y_i) - sd_i\left(v_i(y_i)\right)\right] v_r\left(\frac{y_i}{sd_i(y_i)}\right) \tag{21a}$$

Analogously, if there is an exemption which depends on income, the tax function assumes the form

[13] For instance, the French system allows exemptions for income production expenses and then applies a family quotient. Tax credits are allowed for instruction costs.
[14] In the Italian the fiscal law subsidizes a percentage of building costs through tax credits.

$$t_i(y_i) = t_r(y_i - e_i(y_i)) \tag{22}$$

from (16) the horizontal equity principle is satisfied, if

$$t_r(y_i - e_i(y_i)) = t_r(y_i - sa_i(y_i)) + \left[sa_i(y_i) - sa_i(v_i(y_i))\right] \tag{22a}$$

1.4 Relation between inequality and progression in homogeneous and non homogeneous frameworks

As it is well known, when a homogeneous population is considered, a progressive taxation system is characterized by two distributive features. First, tax liabilities are distributed more unequally than pre-tax incomes; second, post-tax incomes are distributed more equally than pre-tax income. The problem is how to make inequality comparisons in a non homogeneous framework. Ebert and Moyes (2000) answer this question in two steps. First they derive the relation existing between inequality and progression for homogeneous population and second they extend their results to the case of heterogeneous populations. In doing so, they state conditions that guarantee that the after-tax distribution is always no more unequal than the before tax distribution when there is agreement on a particular reference type and a particular equivalent function is chosen. In this case equivalent incomes are computed with respect to the chosen reference type and *equivalent* income distributions are compared. They observe that, when a tax system is required to be inequality reducing for each family size income distribution, this tax system is also reference independent. This is an important remark which will prove to be very useful to derive our results. Before doing that we shall summarize here the main results obtained by Ebert and Moyes

When making inequality evaluations, following Atkinson (1970) one can appeal to the relative Lorenz criterion or, following Kolm (1976), to the absolute Lorenz criterion[15]. In this paper, following Ebert and Moyes (2000), we adopt a parametric approach that encompasses both absolute and relative Lorenz criterion.[16] This is based on the θ Lorenz criterion.

Given a parameter $\theta \in [0,1]$

$$L_\theta(k, \mathbf{y}/\mathbf{w}) = \sum_{j=1}^{k} \frac{w_{(j)}}{\sum_{i=1}^{n} w_{(i)}} \left[\frac{y_{(j)} - \mu(\mathbf{y}/\mathbf{w})}{\theta\mu(\mathbf{y}/\mathbf{w}) + 1 - \theta} + \theta \right];\ k=1,2\ldots\ldots,N \tag{23}$$

is the θ Lorenz curve for the income distribution **Y;** in (23) w_i is the sample weight assigned to household i (i=1,2,…N) and $\mu(\mathbf{y}/\mathbf{w})$ is the weighted mean of the distribution. According to this criterion, a distribution is less unequal than another distribution, if the θ Lorenz curve of the former dominates the θ Lorenz curve of the latter[17].

It is easy to see that when θ=0, expression (23) reduces to the absolute Lorenz curve and, when θ=1, (23) becomes the relative Lorenz curve.

[15] Adopting the relative criterion, a distribution is no less unequal than another distribution if its relative curve is nowhere below the relative curve of the other distribution. The absolute Lorenz criterion plays a similar role, drawing upon the comparison of the absolute Lorenz curves of the distributions. In current practice the relative criterion is the most used, because of its connection with relative inequality measures.

[16] For more on this approach see Bossert and Pfigsten (1990).

[17] Let us introduce the function $(1-\theta)/\theta = \xi(\theta)$ decreasing in θ. The θ Lorenz criterion is well defined for $\mu(\mathbf{y}/\mathbf{w}) > -\xi(\theta)$.

Adopting the θ Lorenz criterion to measure inequality, the same parameter can be introduced to evaluate the progression of the tax system[18]. Following Ebert and Moyes (2000) a tax function $t(y)$ is θ progressive if

$$\frac{t(y)}{\theta y+1-\theta} \tag{24}$$

is non decreasing[19] in income y. Or, equivalently, the net income function $v(y)$ is θ progressive if

$$\frac{v(y)-y}{\theta y+1-\theta} \tag{25}$$

is non increasing in income[20] y.

If one supposes θ=0, by definition θ progression implies a tax function $t(y)$ increasing w.r.t. income. It means that when absolute inequality is considered minimal progression[21] of tax function is required.

If θ=1, definition of θ progression implies $\frac{d[t(y)/y]}{dy} \geq 0$ and then $t^{'}(y) \geq \frac{t(y)}{y}$. It follows that, when relative inequality is considered, marginal tax rate cannot be lower than average tax rate[22]. When relative inequality is considered, average rate progression of the tax function is required.

If 0<θ<1, θ progression implies $\frac{d\left[t(y)/\theta y+1-\theta\right]}{dy}$ and then $t^{'}(y) \geq \frac{t(y)}{y+(1-\theta)/\theta}$.

Being

$$\frac{1-\theta}{\theta} > 0, \quad \lim_{\theta\to 0}\frac{1-\theta}{\theta} = \infty, \quad \lim_{\theta\to 1}\frac{1-\theta}{\theta} = 0$$

one has

$$\lim_{\theta\to 0}\frac{t(y)}{y+(1-\theta)/\theta} = 0 \qquad \lim_{\theta\to 1}\frac{t(y)}{y+(1-\theta)/\theta} = \frac{t(y)}{y}$$

and then, noting that $(1-\theta)/\theta = \xi(\theta)$ is decreasing in θ (see note 17), one has:

$$0 < \frac{t(y)}{y+(1-\theta)/\theta} < \frac{t(y)}{y}$$

It follows that, when θ varies inside its range, θ progression requires that marginal rate must not be lower than a quantity smaller than average tax rate.

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[18] For more on progression concept see Pfingsten (1988).

[19] We remember that if θ=1, strict progression requires $t(y)/y$ increases with y. Relaxing the condition and considering a non decreasing function, means to admit into the analysis the case of a tax which is zero below a threshold, as well as taxes with linear portion and fully proportional taxes. For more on this question see Lambert (2001).

[20] The set of income values has to be restricted to $y > -\xi(\theta)$ to satisfy the requirement of θ progression. Let us suppose θ=0.2: substituting into expression (25) we have $(y-t(y)-y)/(0.2y+1-0.2) = -t(y)/\left[0.2(y+4)\right]$. The ratio conserves its sign and the function its behaviour only if $y > -(1-0,2)/0,2 = -4$ .Obviously this is true for all value of θ.

[21] The minimal concept of progression is due to Fei (1981): it requires that $t(y)$ is non decreasing in y.

[22] That is the so called Fellman (1976) and Jacobsson (1976) concept of progression.

Ebert and Moyes show that, in homogeneous environment, if a tax system is θ progressive, than the post-tax θ Lorenz curve dominates the pre-tax θ Lorenz curve. They state this as:

PROPOSITION 1 (Ebert and Moyes, 2000). *θ progression is both necessary and sufficient condition for θ inequality reducing.*

The value of the parameter θ stresses the connection between inequality and progression. Inequality concept determines the progression concept, and the choice of an inequality concept is a value judgment. Now this result has to be extended to heterogeneous population.

When facing inequality comparisons for an heterogeneous population, first of all an equivalent income function $S_{r,i}$ has to be introduced to transform nominal incomes for non homogeneous families into equivalent incomes with respect to reference family *r*. Then, weighting the obtained equivalent incomes with a suitable system of weights, the θ Lorenz criterion is applied to the obtained distributions.

On considering equivalent incomes, calculated with respect to a particular reference family, the tax system is *overall inequality reducing* if θ Lorenz curve of the post-tax equivalent incomes lies above the θ Lorenz curve of pre-tax equivalent incomes, whatever the distribution of pre-tax income is.

Ebert and Moyes show that a tax system is overall inequality reducing, if the following necessary and sufficient conditions are satisfied

PROPOSITION 2 (Ebert and Moyes, 2000). *A tax system is overall (θ) inequality reducing if and only if*

a2. $v_i(y) = S_{r,i}^{-1} \circ v_r \circ S_{r,i}(y)$

b2. $v_r(y) = (y - t_r(y))$ is θ progressive.

In other words, provided that the equivalence function $S(\cdot)$ is the right way to compare the tax capability among household with different size and that the tax function is θ inequality reducing for the reference family, the tax system is overall inequality reducing.

It is important to observe that *any* tax system is overall inequality reducing if it satisfies requirements *a2* and *b2*. We emphasize that in proposition 2 there is not any particular requirement either on the form of the equivalent income function or on the form of the tax function. One can choose any tax function, provided that it is θ progressive for the reference family, and any equivalent income function provided that *a2* is satisfied. It means that functional separability holds between the considerations underlying the tax system for the reference family and the evaluations of needs among families possessing different characteristics, expressed by the equivalent income function.

We underline that the model, we are discussing about, is founded on two main value judgments: the former (see note 5) concerns the utility function from which the equivalent income function is derived and the latter involves the inequality concept which reflects on the progression of the tax system. As conditions *a2* and *b2* are not related, the two value judgments are independent.

One has to observe that, on fixing the equivalent income function for pre-tax income distribution and asking condition *a2* to be satisfied, we require that the tax system satisfies horizontal equity condition[23]. Then, for a given reference family, it seems possible to say that a tax system is overall θ inequality reducing, if it is horizontally equitable and if the tax function is θ progressive for the reference type.

But it is not granted that inequality is reduced within each homogeneous family type. Moreover, the redistributive effects of the tax system depends on the particular family chosen as the reference one.

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[23] See expression (12) and note 10.

This means that if a tax system is inequality reducing when family r is selected as the reference one, the same tax system may be not necessarily inequality reducing when another family typology is selected as reference.

To grant that inequality is reduced within each homogeneous family type, it must hold what follows

PROPOSITION 3 (Ebert and Moyes, 2000). *A tax system is within type reference independent (WIR), if and only if*

(i) *the tax schedule for a particular reference family r, $t_r(y)$, is θ progressive*[24]

(ii) there exist an equivalent income function $S_{r,i}$ such that:

$$v_i(y) = S_{r,i}^{-1} \circ v_r \circ S_{r,i} \quad \text{is } \theta \text{ progressive } \underline{\text{for all}} \quad i \in H$$

As observed by Ebert and Moyes (2000, page 137), within type inequality reduction and reference independence are equivalent, provided that the tax system is overall inequality reducing. But imposing reference independence has a consequence: it is no longer possible to design overall inequality reducing tax system, whatever the chosen reference type for any arbitrary equivalent income function.

More precisely, the authors state the following proposition:

PROPOSITION 4 (Ebert and Moyes, 2000). *Given an equivalent income function S an inequality concept θ and a system of weight* ***w****, suppose that there exists a reference type r such that the tax system is θ overall inequality reducing. Then the tax system is reference independent if and only if*

a3. $S_{r,i}(y) = \alpha_{r,i} + y$ *for all $r,i \in H$ when $\theta=0$*

b3. $S_{r,i}(y) = \beta_{r,i}\left(y - \zeta(\theta)\right) - \xi(\theta)$ *for all $r,i \in H$ when $0<\theta \leq 1$*

In other words, the price to pay in order to obtain either within type inequality reduction or reference independence consists in giving up the independence between inequality concept and equivalent income function.

But this drastic consequence has to be interpreted in a right way. “Proposition 4 does not mean that a tax system verifying condition in Proposition 2 cannot at all be reference independent but rather that, without introducing further restrictions, there are no guarantee that *any* tax system defined by Proposition 2 will be overall inequality reducing whatever the reference type” (Ebert and Moyes, 2000, page 141).

In the second part of the paper a particular tax system is selected which seems to be overall inequality reducing for any reference family type, with some restrictions on the tax schedule and on the income domain.

2 AN AFFINE EQUIVALENT INCOME FUNCTION AND ITS PROPERTIES

As stated in the last section, no relation is required between the inequality concept and the form of the equivalent income function, for a tax system to be overall inequality reducing,.

[24] Then $v_r(y)$ is θ progressive too.

This independence no longer holds if the tax system has to be both overall and within type inequality reducing. In this case, the equivalent income function and the adopted inequality concept cannot be separated. Proposition 4 maintains that two equivalent income functions have to be considered: the former when absolute inequality is evaluated, the latter when relative or an intermediate inequality concept is taken into account. In this section, we introduce an income transformation function which is a particular case of the affine function defined at (8b).

We discuss this function in § 2.1. In section 2.2, supposing θ=0 and θ=1, we investigate how the adopted function can yield a tax system either overall inequality reducing and reference independent.

2.1 The adopted equivalent income function

Following Ebert (2000), it is assumed that each household with the same characteristics has identical preferences and that each household member enjoys the same level of utility. The equivalent income function is then generated by a utility function that, for the representative individual of the *i*-type family, takes the form:

$$U_i\left(y\right)=h\left(\frac{y-\gamma_i}{m_i}\right) \tag{26}$$

where $h(.)$ is a suitable function. The parameter γ_i represents the subsistence income of the *i* type family, while parameter m_i takes into account the needs of household *i* over the subsistence level. Both γ_i and m_i depends on the characteristics of the family, which are here restricted to its composition; the parameters are considered as exogenous[25]. The values of the parameters m_i are supposed to be positive, those of the parameters γ_i are assumed to be non negative for all *i*; we stress that they do not depend on income, but only on family characteristics. Both γ_i and m_i are assumed to be non-decreasing with respect to the family dimension expressed by *i*; at least one m_i is different from 1 and one γ_i is different from zero. In the pursue, we will consider the tax liabilities for families with different incomes and sizes. We assume that no tax areas may differ for each type of households and that parameters γ_i represent no tax area thresholds. In this framework we will consider $U_i\left(y\right)=h\left[(y-\gamma_i)/m_i\right]\leq 0$ for all $y\leq\gamma_i$. This means that we assume *i* type families with income $y\leq\gamma_i$ have no tax capability.
Summarizing:

(1) m_i>0 for all *i*;
(2) $\gamma_i\geq 0$ for all *i*;
(3) $m_i\geq m_h$, $\gamma_i\geq\gamma_h$, if *i*>*h*.

From (26), it follows that a generic *i*-type family enjoys a given utility level $\bar{u}$, if its income y_i is such that:

$$y_i=\gamma_i+m_i\cdot g\left(\bar{u}\right) \tag{27}$$

[25] The value of the parameters is assigned by the decision maker to pursue political goals; they can be also estimated by proper methods (e.g. econometric methods).

where $g(.)$ is the inverse function of $h(.)$.
From (27), it immediately follows that we can express $g(\bar{u})$ as a function of income for a reference family of type r; in doing so we obtain:

$$g(\bar{u}) = \frac{y_r - \gamma_r}{m_r} \tag{27'}$$

Obtaining $g(\bar{u})$ from the above expression, if we substitute (27') into (27), we get that the income level y_i, which is necessary to families of type i to attain the utility level $\bar{u}$, can be expressed as a function of the income level $y_{r,i}$, that allows the reference family r to enjoy the same utility level $\bar{u}$ as family of type i does.
Then:

$$y_i = \gamma_i - \frac{m_i}{m_r} \cdot \gamma_r + \frac{m_i}{m_r} \cdot y_{r,i}$$

$y_{r,i}$ is defined as the *equivalent income* (that is the income which gives to r type an utility level equal to the utility level enjoyed by i type). Given the income y_i for the i type family, $y_{r,i}$ can be obtained from the relation[26]:

$$y_{r,i} = S_{r,i}(y_i) = \frac{m_r}{m_i}\left(y_i - \gamma_i + \gamma_r \frac{m_i}{m_r} \right) = \gamma_r + \frac{m_r}{m_i}(y_i - \gamma_i) \tag{28}$$

We now verify that expression (28) satisfies the properties listed in §1.1 for an equivalent income function.

PROPERTY (a): $S_{r,r}(y) = S_{r,r}(y) = U_r^{-1} \circ U_r(y) = y$.
Considering (28), it is easy to see that

$$S_{r,r}(y) = \gamma_r + m_r \cdot g(\bar{u}) = \gamma_r + m_r \cdot \frac{y - \gamma_r}{m_r} = y$$

PROPERTY (b): $S_{r,i}(y)$ is continuous and increasing in y for all $r,i \in H$.
$S_{r,i}(y)$ is actually continuous in y. The derivative of $S_{r,i}(y)$ w.r.t. y is

$$\frac{dS_{r,i}}{dy} = \frac{m_r}{m_i} \text{ for all } y \in D.$$

By assumption m_i>0, for all $i \in H$, this derivative is positive.

PROPERTY (c): $S_{r,i}(y) = S_{r,h}(y) \circ S_{h,i}(y)$.
To test the property (c), that is path independence, we must verify that

$$S_{r,i}(y) = S_{r,h}\left[\left(S_{h,i}(y) \right) \right]$$

[26] Observe that if all γ_i where zero, (28) would collapse to $S_{r,i}(y) = y/sd_i$ (see expression (2a) in §1.1), whilst if all m_i where 1, it would simplify into $S_{r,i}(y) = y - sa_i$ (see expression (3a) in §1.1).

Now, from expression (28), taking type h as the reference family, one has

$$S_{h,i}(y) = \gamma_h + \frac{m_h}{m_i}(y - \gamma_i)$$

then

$$S_{r,h}\left[\left(S_{h,i}(y)\right)\right] = \gamma_r + \frac{m_r}{m_h}\left[S_{h,i}(y) - \gamma_h\right] =$$

$$= \gamma_r + \frac{m_r}{m_h}\left[\left(\gamma_h + m_h \frac{y - \gamma_i}{m_i}\right) - \gamma_h\right] = \gamma_r + \frac{m_r}{m_i}(y - \gamma_i) =$$

$$= S_{r,i}(y)$$

and so the property is verified.

From property (c) it follows immediately that $S_{r,i}\left[\left(S_{i,r}(y)\right)\right] = y$, as it can be easily verified substituting h with i and i with r in the above expression

$$S_{r,i}\left[S_{i,r}(y)\right] = \gamma_r + \frac{m_r}{m_r}(y - \gamma_r) = y$$

PROPERTY (d): $S_{r,i}(y)$ is non increasing in i for all $i \in H$ and $y \in D$.

Property (d) implies $S_{r,h}(y) \geq S_{r,i}(y)$ for $h<i$. Considering expression (28), property (d) is verified if it holds:

$$\frac{y - \gamma_h + m_h\gamma_r}{m_h} \geq \frac{y - \gamma_i + m_i\gamma_r}{m_i} \quad \text{with } m_i \geq m_h \text{ and } \gamma_i \geq \gamma_h \tag{29}$$

When $m_i = m_h$ inequality (29) is satisfied if $\gamma_i \geq \gamma_h$. Then, being, by assumption, γ_i non decreasing in i, in this case property (d) is always satisfied.

If $m_i > m_h$, solving (29) for y one obtains:

$$y \geq \frac{\gamma_h m_i - \gamma_i m_h}{m_i - m_h} \tag{29'}$$

The value of the ratio at the r.h.s. of the above inequality is always smaller than γ_h: it is exactly γ_h when $\gamma_h = \gamma_i$. We remember that for h type families the taxable income domain is $y_h \geq \gamma_h$: then inequality (29') is always satisfied in the taxable income domain. Nevertheless, inequality (29') imposes severe restrictions on values of parameters. Indeed, the sign of the ratio may be positive or negative. Being

$$\frac{\gamma_h m_i - \gamma_i m_h}{m_i - m_h} > 0$$

it would imply that only a particular set of positive y satisfies inequality (29). This case should to be avoided.

The sign of the ratio is not positive $\left(\frac{\gamma_h m_i - \gamma_i m_h}{m_i - m_h} \le 0\right)$ if

$$\frac{m_i}{m_h} \le \frac{\gamma_i}{\gamma_h} \quad \text{for } \gamma_h \ne \gamma_i \tag{30}$$

So, if one considers only non negative income, inequality (30) should be sufficient[27] to ensure that $S_{r,i}(y)$ is decreasing[28] in *i* for all $y \in R_+$.

Summing up, PROPERTY (d) is satisfied if one considers R_+ as income domain and fixing the scale at the subsistence level, expressed by the ratio between γ_i and γ_h, higher than the scale for income levels over subsistence expressed by m_i and m_h. When these restrictions hold, expression (28) may be considered an equivalent income function.

2.2 Properties of the adopted equivalent income function

Without any loss of generality, one can put $m_r = 1$ for the reference family and, consequently, rescale all the other coefficients m_i. According with expression (9) and (9a) in § 1.1, which are respectively, $sa_i(y_i) = \left[(R(i)-1)y_i + A(i)\right]/R(i)$ and $sd(y_i) = R(i)y/\left[y - A(i)\right]$

the equivalent income function $S_{r,i}(y_i) = \gamma_r + (y_i - \gamma_i)/m_i$, defined in expression (28), leads to the following absolute scale:

$$sa_i(y_i) = \frac{(m_i - 1)\cdot y_i + \gamma_i - m_i \cdot \gamma_r}{m_i} \tag{31}$$

or to the relative scale:

$$sd_i(y_i) = \frac{m_i \cdot y_i}{y_i - \gamma_i + m_i \cdot \gamma_r} \tag{32}$$

Both scales (31) and (32) depend on income.

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[27] Of course if $i<h$ it must hold

$S_{r,i} \ge S_{r,h} \quad (m_i < m_h;\ \gamma_i < \gamma_h)$

and condition (30) becomes $\frac{m_h}{m_i} \ge \frac{\gamma_h}{\gamma_i}$

[28] Remember that it is common practice to normalize *m* coefficients imposing m_r=1. In this case inequality (29') becomes

if $i>r$ $(m_i > m_r = 1;\ \gamma_i > \gamma_r)$ $\quad y \ge \frac{\gamma_r m_i - \gamma_i}{m_i - 1}$

as $(m_i - 1)$>0, condition (30) requires $(\gamma_r m_i - \gamma_i)$<0 that is $m_i \le \frac{\gamma_i}{\gamma_r}$

if $i<r$

$(m_i < m_r = 1;\ \gamma_i < \gamma_r)$ $\quad y \ge \frac{\gamma_i - \gamma_r m_i}{1 - m_i}$

as $(1 - m_i)$>0, condition (30) requires $(\gamma_i - \gamma_r m_i)$<0 that is $m_i \ge \frac{\gamma_i}{\gamma_r}$

Let us recall horizontal equity conditions when scale are income dependent. The H.E. condition on tax function is (see expression 16):

$$t_i(y_i) = t_r(y_i - sa_i(y_i)) + \left[sa_i(y_i) - sa_i(v_i(y_i))\right]$$

when an absolute scale is used, while
the H.E. condition on tax function is (see expression 14):

$$t_i(y_i) = sd_i(y_i)t_r\left(\frac{y_i}{sd_i(y_i)}\right) + \left[sd_i(y_i) - sd_i(v_i(y_i))\right]v_r\left(\frac{y_i}{sd_i(y_i)}\right)$$

when the relative scale is used.

Substituting $sa_i(y)$ given in (31) into expression (16), or $sd_i(y)$ given in (32) into expression (14), after some passages, one obtains the tax function for *i* type families

$$t_i(y_i) = m_i \cdot t_r\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right) \tag{33}$$

The tax function in (33) ensures that the horizontal equity condition on post tax incomes $v_i(y) = S_{r,i}^{-1} \circ v_r \circ S_{r,i}(y)$ (see expression 12 and note 10) is respected[29].
Then, our first result is that the tax system defined by $S_{r,i}(y_i) = \gamma_r + (y_i - \gamma_i)/m_i$ and $t_i(y_i) = m_i \cdot t_r\left[(y_i - \gamma_i + m_i \cdot \gamma_r)/m_i\right]$ is horizontally equitable.
As said above (see proposition 2 in § 1.4), a tax system is overall θ inequality reducing if horizontal equity condition on net income is respected and if the tax function is θ progressive [30] for the reference type *r*. Then, supposing that there is accord on *r* as reference type and provided that the tax function $t(y)$ is θ progressive for *r* type, the above specified tax system is overall θ inequality reducing. This is our second result.
It remains to verify if a tax function as in expression (33) originates a tax system which can be also within type inequality reducing and then reference independent.
As observed in § 1.4 reference independence and within type inequality reduction are equivalent, provided that the tax system is overall inequality reducing. Remembering that, if a tax system is overall inequality reducing, within type inequality reduction requires that the tax system is θ progressive for all $i \in H$ (see proposition 3), we verify WIR and, in so doing, we test also RI.
One has to remember that proposition 4 of § 1.4 lists two equivalent income functions: the first has to be considered when θ=0, the second when $0 < \theta \le 1$; if and only if these functions are used *any* OIR tax system is reference independent. Concentrating our attention on the two main cases: θ=1 (average rate progression) and θ=0 (minimal progression) we propose a single equivalent income function: $S_{r,i}(y_i) = \gamma_r + (y_i - \gamma_i)/m_i$ and we try to seek out if there exists particular conditions under which the tax function $t_i(y_i) = m_i \cdot t_r\left[(y_i - \gamma_i + m_i \cdot \gamma_r)/m_i\right]$ guarantees reference independence.

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[29] In fact: $v_r \circ S_{r,i}(y) = \gamma_r + (y_i - \gamma_i)/m_i - t_r(\gamma_r + (y_i - \gamma_i)/m_i)$
and $S_{r,i}^{-1} \circ v_r \circ S_{r,i}(y) = \gamma_i + m_i \cdot \left[(y_i - \gamma_i)/m_i - t_r(\gamma_r + (y_i - \gamma_i)/m_i)\right] = y_i - m_i \cdot t_r(\gamma_r + (y_i - \gamma_i)/m_i)$.

[30] We remember that the definition of θ progression requires that $t(y)/(\theta y + 1 - \theta)$ (expression 24) is non decreasing in y.

For what above reported, checking within type inequality reduction is equivalent to verify that

$$\frac{t_i(y_i)}{\theta \cdot y_i + 1 - \theta} = \frac{m_i t_r\left(\dfrac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right)}{\theta \cdot y_i + 1 - \theta} \tag{34}$$

is not decreasing in y_i for all i.

It is important to note that, differently than Ebert and Moyes, we analyze the behavior of expression (34), restricting the income domain to R_+ and that, for each family type, we consider $y_i \geq \gamma_i$ as taxable incomes, being $t_i(y) = 0$ for $y \leq \gamma_i$

(*a*) When θ=0, (34) is non decreasing in y if

$$t_r'\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right) \geq 0 \tag{35}$$

We remember that in homogeneous environment tax function is non decreasing w.r.t. income (see §1.4). Then, on considering the set of household with size r, inequality (35) holds and, consequently, (34) is non decreasing in y for all i. In the case of minimal progression, unlike Ebert and Moyes, we are not constrained to impose $m_i = 1 \ \forall \ i \in H$ because we have restricted the income domain to R_+.

(*b*) When θ=1, the first derivative of expression (34) w.r.t. y_i is not negative, that is (34) is a non decreasing function in y_i, if the following inequality is verified:

$$t_r'\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right) \geq \frac{t_r\left(\dfrac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right)}{\dfrac{y_i}{m_i}} \tag{36}$$

Observe that for the reference family, the definition of average rate progression implies:

$$t_r'(y) \geq \frac{t_r(y)}{y} \tag{37}$$

and consequently that:

$$t_r'(y/m_i) \geq \frac{t_r(y/m_i)}{\dfrac{y}{m_i}} \tag{37'}$$

Observe now that if $\gamma_i > m_i\gamma_r$, then $(y_i - \gamma_i + m_i \cdot \gamma_r)/m_i < y_i/m_i$. If in (37') we substitute y in the argument of $t(\cdot)$ and of its derivative $t'(\cdot)$ with a smaller quantity, a fortiori inequality (36) is verified.

For i>r, being the r.h.s. in (29') negative, γ_i is greater than $m_i\gamma_r$

Otherwise, if $(y_i - \gamma_i + m_i \cdot \gamma_r)/m_i$ is greater [31] than y_i/m_i, it is the restriction adopted on taxable incomes that is sufficient for (36) to be verified. This restriction works through the following specifications:

(*i*) the tax system admits an exemption $e_r = \gamma_r$ [32] and
(*ii*) the tax function can be expressed as:

$$t\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right) = t_*\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i} - e_r\right) \tag{38}$$

where the function $t_*(.)$ has a first derivative positive and not decreasing w.r.t. *y*.
Under these assumptions, condition (36) for average rate progression becomes:

$$t'_{*r}\left(\frac{y_i - \gamma_i}{m_i}\right) \geq \frac{t_{*r}\left(\frac{y_i - \gamma_i}{m_i}\right)}{\frac{y_i}{m_i}} \tag{39}$$

which a fortiori holds if (37') holds, being $y_i \geq y_i - \gamma_i$.

We conclude our analysis with the further consideration on the contradiction lying inside a tax credit, when it is preferred to a more proper exemption.

Consider the case when the tax schedule admits for the reference family a tax credit c_r. The tax function can now be represented as

$$t\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right) = t_\circ\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right) - c_r \tag{40}$$

where the function $t_\circ(.)$ coincides with $t(.)$ but for the presence of c_r. Expression (36) specializes now into the following expression

$$t'_\circ\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right) \geq \frac{t_\circ\left(\frac{y_i - \gamma_i + m_i \cdot \gamma_r}{m_i}\right) - c_r}{\frac{y_i}{m_i}} \tag{41}$$

In piecewise linear a tax system, similar to the one described in Table A4 in the Appendix, as long as everything lies in the firs brackets and $(c_r/\alpha_1) > \gamma_r - (\gamma_i/m_i)$, we can apply the same considerations as when $e_r > \gamma_r - (\gamma_i/m_i)$. Conversely, for $y \to \infty$, (41) becomes less than 1.

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[31] γ_i/γ_r is smaller than m_i when the subsistence scale is greater than the scale over the subsistence but the *i* type family is "smaller" than the reference one.

[32] It would be enough that $e_r \geq \gamma_r - \gamma_i/m_i$, but in order to avoid unpleasant effects it is convenient that $e_r = \gamma_r$: for instance if $e_r < \gamma_r$ it might happen that if the coefficient m_i for the *i* type family is augmented, the tax increases instead of diminishing.

It follows that in a piecewise linear tax scheduling where (*i*) marginal bracket tax rates are not decreasing w.r.t. income and (*ii*) for the reference family is admitted a basic exemption $e_r = \gamma_r$, (33) is overall income inequality reducing, whatever is the reference family chosen and, consequently, is within type inequality reducing. Otherwise, with the same system, if the basic exemption is substituted by a tax credit $c_r = \alpha_1 \cdot e_r$, (25) may not be satisfied.

In what it concerns tax progression, it is worth to stress that the elasticity of (33) w.r.t. income is the same for equivalent incomes, only when the parameter $\gamma_i = \gamma_r$, are the same $\forall i$; otherwise it is higher for families with $\gamma_i > \gamma_r$.

3 LINES FOR A MORE FAMILY ORIENTED TAX SYSTEM IN POLAND

In the present Polish personal tax system no family allowances are scheduled: the only distinction is made between singles and couples, being indifferent for the latter having or not having children.

We try to introduce some corrections intended to design a more fair system: the corrections consists in children exemptions. In order to limit the loss in tax revenue we design a tax schedule with intermediate income brackets and a higher maximum marginal rate. We consider also the reduction in spouse coefficient.

Together with the present tax system, we analyze two further systems: both the new proposed systems deal with the same tax schedule, present the same children exemptions but differ in what it concerns the spouse coefficient.

The proposed systems are, first of all, compared with the present one on the basis of nominal incomes and on the average rates of family type actual taxes with respect to family incomes.

The changes in distribution of taxes and, on the other side, the loss in tax revenue induced by the reforms are evaluated by simulations on the data collected in 2001 by Polish Central Statistical Office in Household Budget Survey.

After the analysis based on nominal incomes, results are evaluated by the application of an equivalent income function which is a mixture of the minimum survival incomes and the OECD scale.

Results concerning tax burdens in both original and theoretical systems are obtained by application of appropriate tax schedule to the incomes of the taxpayers. Such treatment enabled comparisons of systems, but results obtained should be treated as significant approximation. Firstly, the distribution of personal income tax seems not to be representative for the whole population of Poland. Secondly, data collected in HBS is not directly intended to enable analysis of tax burdens and – as a result – appropriate fiscal categories could not be unambiguously identified.

From the whole sample, consisting of almost 32000 households, to the simulations only sub-sample – 20430 families – has been used. Because of the character of the analysis only single households and married couples with no more than 3 children were taken into account[33].

3.1 Two suggestions

At present, the personal income tax adopts single households as reference family type. The marginal tax rates are reported in Table 1. It is worth observing that the same final tax would be obtained if the marginal rate 19%, were applied to all incomes from zero to 37024 PLN, allowing for a tax credit equal to 0.19·2790 PLN.

[33] In the next of the paper the term "family" will be used for all analyzed types of households.

Couples with or without children may add up their incomes and then apply the tax schedule reported in Table 1 to each separate half of total income: the resulting tax is doubled. Such taxation treats couple as if there were two singles with income equal to half of the couple's income.

Table 1. Present income tax schedule

Income bracket [PLN]		Tax rate
0	2790	0%
2790	37024	19%
37024	74048	30%
74048		40%

Source: Polish Ministry of Finance (http://www.mf.gov.pl).

The spouse coefficient, being equal to 2, generates two effects. A first effect consists in doubling the tax credit for couples (with or without children): this effect acts at any income level. A second effect is produced only when family income falls into a bracket superior to the lowest: if this is the case, the tax saving[34] is generated because on taxing twice half income, and then doubling the tax for each half, a smaller proportion of family income (if not any at all) is subjected to higher the marginal rate(s) that would correspond family income y, and a greater (if not the whole) to lower rate(s). Of course, this mechanism generates tax savings – in comparison to the individual taxation of spouses – only in case of asymmetry in within-household income distribution: the splitting is ineffective when each spouse enjoys the same income as the other and it is maximum when one of the two has income zero.

If we formalize the present tax system (labeled P) according to the notation introduced in the previous sections, the γ coefficients equals to zero for each family type, while the coefficient m for the single is m_s=1 and for couples with or without children it is indifferently m_i=2 (see table 2).

The present system results to be absolutely unfair for family with children because it does not keep into account the costs that parents have to face for their presence. In this context the question arises, how to get some hints to keep into account the presence of children?

One possible solution is to look at the minimum survival income levels for families of a given composition. For Poland, such income levels are published yearly by the Institute of Labor and Social Matters (IPiSS) for selected family types. Analyzing published data – see IPiSS (2001) – subsistence minimum is on average equal to 3300 PLN for each child per year– no matter, if it is first, second or consecutive one. This result might justify the introduction of an exemption of 3300 PLN for each child; this exemption, if associated to the present couple splitting, would result in generating a tax system similar to the German one: we label this hypothesis of reform as *Modification A* (*M-a*).

According to the Polish personal tax system, we take as reference type family the single. Then, if $t_r(\cdot)$ is the tax function conceived for the single, remembering the form of the equivalent income function (28) and that of the tax function (33), for couples with children we would have, respectively

$$y_{r,i} = S_{r,i}(y_i) = \gamma_r + \frac{1}{m_i}(y_i - \gamma_i) = \frac{1}{2}(y_i - nch_i \cdot 3300)$$

[34] Let $t_r(y_i)$ be the tax paid by the reference household at income y_i and $t_i(y_i)$ the tax for the taxpayer with family of type i, then the tax saving is simply given by: $ac_i(y_i) = t_r(y_i) - t_i(y_i)$.

$$t_i(y_i) = 2 \cdot t_r\left(\frac{y_i - nch_i \cdot 3300}{2}\right) \tag{42}$$

where nch_i is the number of children and 3300 is the exemption evaluated on the basis of minimum survival income. In such system parameters γ_i and m_i take values reported in the second part of Table 2.

Table 2. Family coefficients and exemptions

		Single	Couple	Couple with 1 child	Couple with 2 children	Couple with 3 children	Couple with *n* children
Present system (P)	m_i	1	2	2	2	2	2
	γ_i	0	0	0	0	0	0
Modification A (M-a)	m_i	1	2	2	2	2	2
	γ_i	0	0	3300	6600	9900	*n*·3300
Modification B (M-b)	m_i	1	1.6	1.6	1.6	1.6	1.6
	γ_i	0	1116	1116+3300	1116+6600	1116+9900	1116+*n*·3300

Source: own proposals based on IPiSS (2001).

The new tax function (42), applied to the present income brackets and associated marginal rates (Table 1), induces a 15.24% tax loss for the Government. To decrease loss in tax revenue spouse coefficient could be lowered – keeping account of some scale economies – to 1.6. In order to maintain the same tax saving, induced by the present tax credit in favor of spouses, there will be introduced an exemption equal to 40% of 2790. We label this further hypothesis of reform *Modification B* (*M-b*). The new values of γ and m coefficients are reported in third part of Table 2. The equivalent income function (28) and the tax function (33) for couples with children, now become:

$$y_{r,i} = S_{r,i}(y_i) = \gamma_r + \frac{1}{m_i}(y_i - \gamma_i) = \frac{1}{1.6}(y_i - 1116 - nch_i \cdot 3300)$$

$$t_i(y_i) = 1.6 \cdot t_r\left(\frac{y_i - 1116 - nch_i \cdot 3300}{1.6}\right) \tag{43}$$

In a tax schedule which allows a tax credit equal to $0.19 \cdot 2790$=530.10 for the single, (42) doubles the tax credit for families with children; (43) gives couples with children 160% of 530.10, but recognizes an exemption equal to 1116 to the spouse. This means that, as long as the equivalent income falls in the first income bracket, the spouse exemption compensates the loss in tax credit due to diminishing of the splitting coefficient. It smoothes the loss of tax saving in higher income brackets.

For mentioned tax schedules it could be observed that

$$(1/1.6)(y_i - 1116 - nch_i \cdot 3300) \geq (1/2)(y_i - nch_i \cdot 3300)$$

when $y_i \geq 2 \cdot 2790$. If the tax is calculated on the same reference schedule (the one conceived for the single) and this schedule presents tax-free amount equal to 2790, it results that (43) gives tax values which are not lower than those implied by (42). More precisely, if equivalent incomes, being an argument of tax function in (42) and (43), fall in the first income bracket, the taxes calculated for the two schedules result to be exactly the same. Tax calculated for (43) is higher than the one obtained for (42), if $(y_i - 1116 - nch_i \cdot 3300)/1.6$ falls in an income bracket higher than the first one.

The percentage loss of tax revenue when *M-b* is applied together with the present tax schedule equals to 14.59%. The gain obtained through lowering the spouse coefficient is a bit more than a half (0.65) per cent point. In order to restrict the tax loss for the Government we draft a more progressive tax schedule for the single, that is the reference type. The proposed tax schedule is reported in Table 3.

Table 3. Modified tax schedule

Income bracket [PLN]		Tax rate
0	2790	0%
2790	15000	19%
15000	20000	22%
20000	30000	26%
30000	45000	30%
45000	60000	33%
60000	75000	36%
75000	100000	40%
100000		45%

Source: own proposal.

Applications of *M-a* to the new tax schedule leads to a tax loss of 11.59%, whilst for *M-b* the loss is equal to 9.60%. The difference between the two systems is a bit higher than before – nearly 2 per cent point. This happens because a modification in a splitting coefficient is the more effective the higher is the progression in marginal rates.

In the next of the paragraph we shall use the following abbreviations for the mentioned tax schedules: *ps* for the present tax schedule (Table 1) and *ms* for the modified tax schedule (Table 3).

The average[35] tax rates induced by *M-a/ms* and *M-b/ms*, together with those associated to *P/ps*, are plotted in Figure 1. With the proposed reforms, in spite of more progressive tax schedule, families with children face a reduction in average tax rates, at least up to a certain level of income and this level increases with the number of children. For couples with one child the level is 40000 PLN (*M-b/ms*) and 45000 PLN (*M-a/ms*). It increases to 55000 PLN (*M-b/ms*) and to 60000 PLN (*M-a/ms*) when there are two children in the family and reaches 70000 PLN (*M-b/ms*) and 75000 PLN (*M-a/ms*) for couples with three children.

Above these limits, for the family with one offspring the maximum difference in average tax rate between *M-b/ms* and *P/ps* is 5 per cent points and it is reached at 140000 PLN; it is contained at 3 per cent points for *M-a/ms*. These differences are on decrease with increase in number of children: 4 per cent points over 140000 PLN for *M-b/ms* for families with two children. For families with three children the difference reaches 3 per cent points (*M-b/ms*) for incomes greater than 120000 PLN and is restricted to 1 per cent point (over 75000 PLN) for *M-a/ms*. Of course, as it is conceived, the reforms do not bring any tax-cuts to couples without children and to singles. Tax amounts are indifferent to the choice of tax schedule up to 30000 PLN for childless couples, and up to 15000 PLN for singles. Over these limits, for singles the average rates increases up to a maximum of 4 per cent points, which is reached over 70000 PLN, for either modified systems. For the couple without children, with *M-b/ms* the rise reaches a maximum of 6 per cent points over 160000 PLN (4 per cent points over 55000 PLN), while stops at a maximum increase of 4 per cent points, over 140000 PLN, when *M-a/ms* reform is chosen.

Table 4 reports average tax rates for all families within each family type for the analyzed sample.

[35] Some information on the distribution of income within analyzed sample are presented in Table 6.

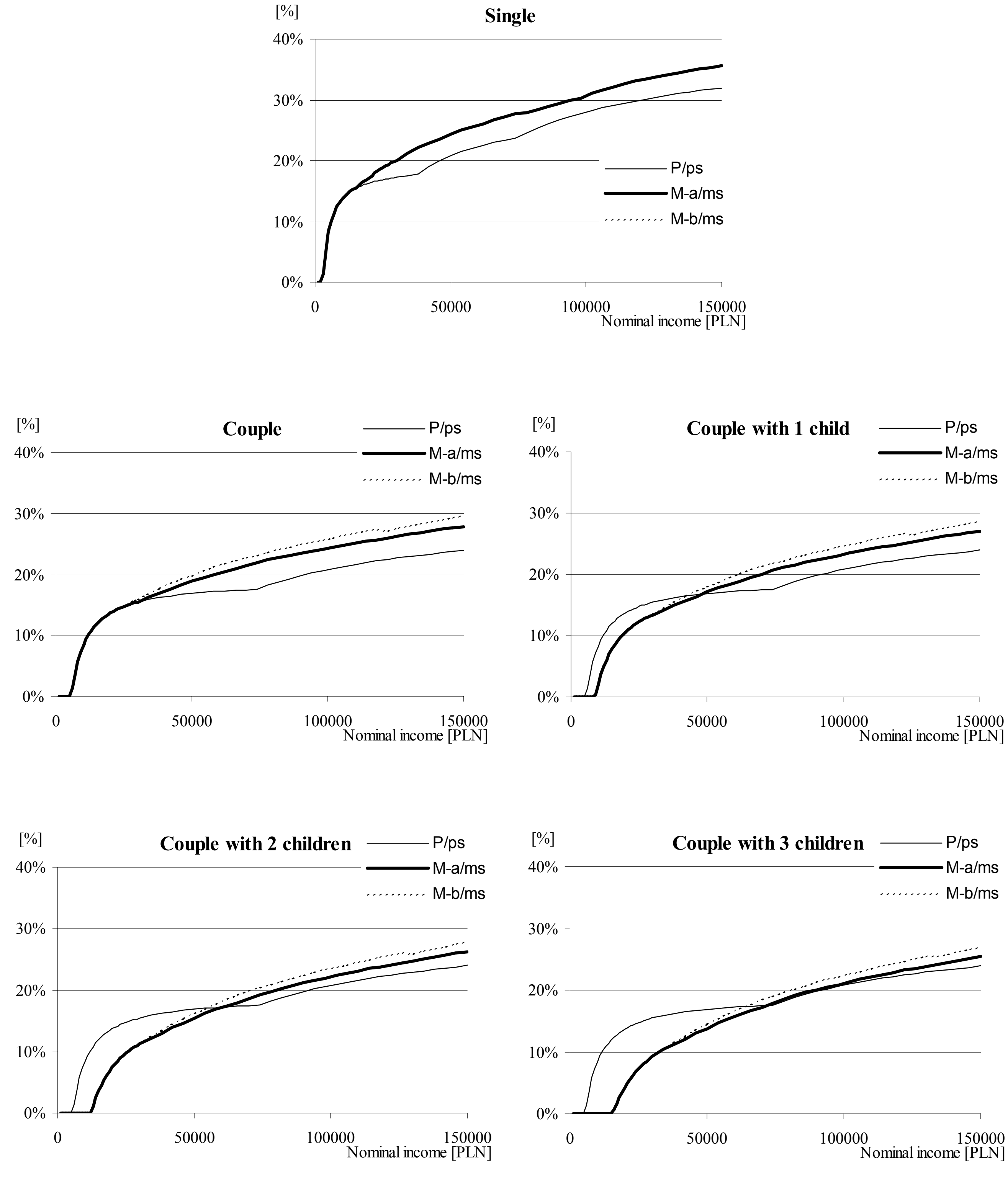

Figure 1. Average tax rates for *M-a/ms*, *M-b/ms* and *P/ps* for nominal incomes.
Source: own calculations.

On average, the reform cost would equal to 0.65 per cent points for singles and 0.4 per cent points (*M-a/ms*) or 0.78 per cent points (*M-b/ms*) for couples. Tax-cuts would be yielded by families with children – the highest for a family with three children: 6.31 per cent points (*M-b/ms*) or 6.52 per cent points (*M-a/ms*) on average. Quite high reduction could be observed also for families with two children – about 3.5 per cent points (*M-b/ms*) and 4 per cent points (*M-a/ms*). For families with just one offspring tax-cut would equal to 1.3 per cent points and 1.7 per cent points, respectively.

Table 4. Average tax rates for nominal incomes

	Single	Couple	Couple with 1 child	Couple with 2 children	Couple with 3 children
P/ps	14.92%	14.32%	15.03%	15.04%	14.55%
M-a/ms	15.57%	14.77%	13.33%	11.00%	8.03%
M-b/ms	15.57%	15.10%	13.74%	11.33%	8.24%

P/ps: present system with present tax schedule; *M-a/ms*: modified (a) system with the more progressive tax schedule; *M-b/ms*: modified (b) system with the more progressive tax schedule.

Source: own calculations.

3.2 Evaluations of the proposed reforms

In the previous section we based our considerations on nominal incomes. In this section we analyze the performance of the present tax systems and the ones of the proposed reforms, on the basis of equivalent incomes.

The transformation of nominal incomes into equivalent incomes cannot be carried on unambiguously, as it depends on the choice of equivalence function that is judged to be the "true" one. Nominal incomes themselves may be considered as equivalent, if one accepts a transformation function which assumes everywhere the constant value 1.

Because of the type of the analysis performed, we decided to employ officially used statistics to construct function of equivalent incomes. Legal character of tax system requires defining possibly general rules and stable sources of data.

In Polish official statistics there are used two equivalent scales – the OECD scale and the modified OECD scale. The former one assigns 1 for first adult, 0.7 for consecutive adults and 0.5 for each child. In the latter these values are equal to 1, 0.5 and 0.3 respectively. The modified scale, widely accepted in European official statistics, is said to become more appropriate with increase of the wealth and is going to completely replace the original scale in Polish official statistics. Additionally, tax systems obtained using the modified scale are less costly for the Government. For these reasons in following analyses modified OECD scale will be employed.

On the side of exemptions, mentioned earlier subsistence minimum levels for Poland – see IPiSS (2001) – will be applied. Values of parameters m_i and γ_i based on this sources are presented in Table 5. These coefficients and exemptions, together with the more progressive tax schedule (Table 3) will constitute our benchmark tax system.

Table 5. Family coefficients and exemptions for the benchmark transformation function

		Single	Couple	Couple with 1 child	Couple with 2 children	Couple with 3 children	Couple with *n* children
OECD scale coefficients	m_i	1	1.5	1.8	2.1	2.4	$1.5 + n\cdot 0.3$
Minimum survival incomes	γ_i	4000	6700	10000	13300	16600	$6700 + n\cdot 3300$

Source: own proposal based on IPiSS (2001) and information from Polish Central Statistical Office.

Values adopted in the present tax system in Poland are lower than values resulting from published data on subsistence minimum. Minimum income for the single amounts to 4000 PLN, which is much greater than the present no tax limit of 2790. Analogously, for the couple a value of 6700 PLN is also greater than twice 2790. So a tax credit of 0.19·2790 for a single, and of 2·0.19·2790, and even more if it is 1.6·0.19·2790 appears as inadequate if compared with an exemption of 4000 PLN for the single, and of 6700 PLN for a couple.

If we adopt the coefficients reported in Table 5 with the present tax schedule, substituting the present tax credit with the new exemptions for the adults, that is 4000 PLN for each single and 6700 PLN for each couple, the loss for the tax revenue would amount to the 21.57%; if applied to the more progressive tax schedule, given in Table 3, the loss would be 18.67%.

Of course: demographic or social reasons, together with budget restrictions, can interact with the economic theory hints and produce a final scale which may differ substantially from theoretical one. So we do not pretend that here adopted benchmark generates an actual scale for taxation. However, it constitutes a useful device that can help in analyzing the fairness of a tax system.

General characteristics of the sample that was used in evaluations of proposed tax reforms is presented in Table 6. Analyzing average income of given family types it could be observed that in terms of nominal values, income of couples with or without children is quite similar. This situation significantly changes, when equivalent values are taken into account: income of couples with three children is almost three times lower in comparison with childless couples.

Table 6. Basic statistics for nominal and equivalent income

	Single	Couple	Couple with 1 child	Couple with 2 children	Couple with 3 children
Nominal income					
Mean	11232	21745	24552	24111	20752
Standard deviation	9063	12833	16895	17119	16185
Minimum	0	0	0	0	0
Maximum	240000	222600	216000	240000	168670
Percentage of positives	95	99	97	96	92
Percentage of zeros	5	1	3	4	8
Percentage of negatives	0	0	0	0	0
Equivalent income					
Mean	11232	14030	12085	9148	5730
Standard deviation	9063	7884	9386	8152	6744
Minimum	0	-467	-1556	-2333	-2917
Maximum	240000	147933	118444	111952	67362
Percentage of positives	95	99	97	94	86
Percentage of zeros	5	0	0	0	0
Percentage of negatives	0	1	3	6	14

Source: own calculations.

Relatively the worst situation of couples with two and three children is confirmed by percentages of families gaining income equal to zero (in terms of nominal values) or lower than zero (in terms of equivalent income). Both statistics take the highest values for couples with 3 children: 8% of such families gains no income (subjecting to personal income tax) and income of further 6% of these families is so low that its equivalent value is lower than zero.

Presented results, together with moderate values of standard deviation, suggest underprivileged position of families with more than one child – in comparison to others – in the context of income taxation.

This conclusion is confirmed by the average tax rates. In Table 7 average tax rates with respect to equivalent incomes are given. In terms of equivalent incomes, on average, the cost for the couple is roughly the same as in terms of nominal incomes (for the single it is the same by definition of equivalent income) – see Table 4. Tax savings resulting from modified systems are easily seen for families with children. In terms of equivalent incomes they are even higher than in terms of nominal ones. From the table it also results that with the present system families with children are taxed with an average rate which reaches almost 22%.

Table 7. Average tax rates for equivalent incomes

	Single	Couple	Couple with 1 child	Couple with 2 children	Couple with 3 children
P/ps	14.92%	14.80%	16.97%	18.87%	21.95%
M-a/ms	15.57%	15.26%	15.05%	13.81%	12.12%
M-b/ms	15.57%	15.60%	15.51%	14.22%	12.44%
Benchmark	13.37%	14.41%	14.05%	12.64%	10.86%

P/ps: present system with present tax schedule; *M-a/ms*: modified (a) system with the more severe tax schedule; *M-b/ms*: modified (b) system with the more progressive tax schedule. *Benchmark* is the system adopting coefficients and exemptions given in Table 5, with the more progressive tax schedule.

Source: own calculations.

In the last row of the table there are given average rates, in equivalent terms, that would yield the benchmark system. It appears that, on average, the reforms present average rates still above the benchmark values. For families with children differences from the benchmark system range from 1 per cent point to 1.60 per cent points, according to the family type and the hypothesis of reform. Even more contained are differences for the couple. Slightly bigger is the difference for the single – 2 per cent points; this is due to the fact that the present tax credit is lower than the minimum survival income, reported in Table 5.

Figure 2 presents differences – in terms of nominal values – between the average tax rates for the benchmark system, the present one and the two hypotheses of reform. The present systems, shows the highest distances from the benchmark especially at lower incomes and the unfairness is greater the more numerous children are: the maximum difference is 3 per cent points for the couple without children and 5 per cent points for a single, it jumps above 8 per cent points when there is a child and yields 12 per cent points with three children. As incomes grow, the unfairness decrease and the present systems may become more advantageous than the *benchmark*. For the single this happens at 25000 PLN, for the couple without children at 35000 PLN, for the couple with one child at 55000 PLN, at 70000 PLN for the couple with two children and never – in the considered interval – for the couple with three children, even if the benchmark has a more progressive tax schedule.

It is then evident how the reforms correct an inequity which is very strong at lower income levels. Actually, if the reform were applied, all considered family types should be taxed more than at present, at higher income levels; the point where the present situation switches to be better than the proposed reforms, is higher the more numerous a family is: 15000 PLN for the single, 30000 PLN for the couple, nearly at 50000 PLN with one child, about at 65000 PLN with two children and 75000 PLN for the couple with three children.

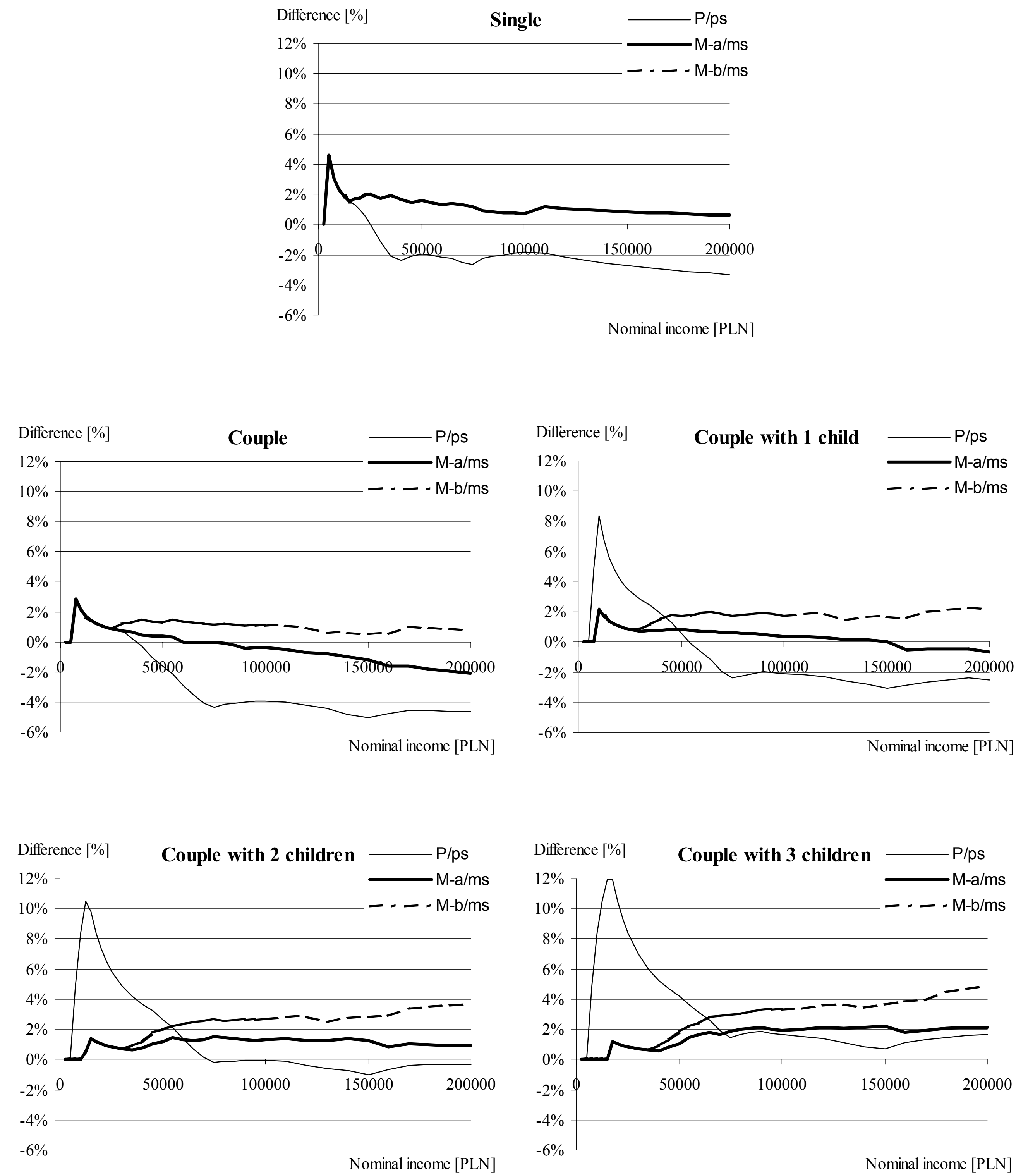

Figure 2. Differences in tax rates between *M-a/ms*, *M-b/ms* and *P/ps* and the benchmark system (for nominal incomes).
Source: own calculations.

In what it concerns the relationship of the reforms with the benchmark, we see that *M-b/ms* has average tax rates always greater than the latter. *M-a/ms* has always greater average tax rates than the benchmark for all family types[36] except for the couple without children and the couple with one

[36] Actually for the single *M-a/ms* and *M-b/ms* coincide.

child: in the former case the line of *M-a/ms* crosses the income axis at 11000 PLN and in the latter at 150000 PLN, in any case much after than the present system does.

It is due to the lack of a suitable no tax area that makes the present tax rates so high – in relation to the benchmark – at lower incomes.

Summarizing, starting from the present tax system, it would be desired to improve situation of those, whose actual situations is relatively the worst, that is families with children. However, reformed systems proposed in the paper, do not make worse the present situation – at low level of income – either for the single or for the couple without children.

Table 8 presents Gini concentration, redistribution and progressivity indexes within each family type; analogous indexes for the whole sample can be found in Table 9. Indexes have been calculated on equivalent incomes:

(i) adding to each equivalent income the maximum value of absolute negative value which resulted to be 2916.67, in order to get rid of negative values and then

(ii) weighting each income by the m_i coefficient, reported in Table 5.

The shift introduced in the procedure leads to the same effect that would be generated if nominal incomes were transformed into equivalent ones, taking as reference type the couple with three children instead of the single[37].

Due to the higher progression of the modified schedule, the modified systems are more inequality reducing than the original one. Moreover, *M-b/ms* reduces inequality stronger than *M-a/ms* due to a lower coefficient for spouses, which is decreased from 2 to 1.6.

Table 8. Within family types Gini indexes for equivalent incomes before and after taxation. Redistribution and progressivity indexes

		Single	Couple	Couple with 1 child	Couple with 2 children	Couple with 3 children
Equivalent incomes before taxation						
1		0.2576	0.2378	0.3123	0.3331	0.3835
Equivalent incomes after taxation						
2	*P/ps*	0.2355	0.2192	0.2926	0.3148	0.3659
3	*M-a/ms*	0.2314	0.2161	0.2833	0.3028	0.3545
4	*M-b/ms*	0.2314	0.2140	0.2809	0.3008	0.3531
Equivalent taxes						
5	*P/ps*	0.4221	0.3706	0.4362	0.4424	0.4869
6	*M-a/ms*	0.4435	0.3879	0.5221	0.5915	0.7155
7	*M-b/ms*	0.4435	0.3983	0.5318	0.6000	0.7214
*Redistribution indexes (Reynolds-Smolensky)**						
8	(1)-(2)	0.0221	0.0186	0.0197	0.0183	0.0176
9	(1)-(3)	0.0262	0.0217	0.0290	0.0303	0.0290
10	(1)-(4)	0.0262	0.0238	0.0314	0.0323	0.0304
*Progressivity indexes (Kakwani)**						
11	(5)-(1)	0.1645	0.1328	0.1239	0.1093	0.1035
12	(6)-(1)	0.1859	0.1501	0.2098	0.2584	0.3320
13	(7)-(1)	0.1859	0.1605	0.2195	0.2669	0.3379

* For definition of both indexes and detailed discussion see for example Lambert (2001).

Source: own calculations.

[37] Amount 2916.67 is exactly equal to 4000-(1/2.4)·16600. For an alternative procedures see Chau-Nan Chen et al. (1982).

Table 9. Overall concentration indexes for equivalent incomes and taxes, before and after taxation order. Redistribution and progressivity indexes

		A	B	C	D	E	F
		Incomes *(proper order)* *Overall*	*Incomes* *(before tax order)* *Overall*	*Taxes* *(proper order)* *Overall*	*Taxes* *(before tax order)* *Overall*	*Overall redistribution (Reynolds-Smolensky) index*	*Overall progressivity (Kakwani) index*
Before tax							
1		0.3120	0.3120				
After tax							
2	*P/ps*	0.2956	0.2954	0.4299	0.4219	0.0166 *	0.1099 °
3	*M-a/ms*	0.2843	0.2843	0.5230	0.5226	0.0277 **	0.2106 °°
4	*M-b/ms*	0.2825	0.2825	0.5308	0.5305	0.0295 ***	0.2185 °°°

* B1-B2. ** B1-B3. *** B1-B4.
° D2-B1. °° D3-B1. °°° D4-B1.
Source: own calculations.

In what it concerns the within analysis, the income order before and after taxation is the same as it is expected due to the properties of (42) and (43), applied to progressive tax schedule. Thus, all concentration indexes presented in Table 8 are Gini coefficients.

Tax systems (42) and (43) preserve original ranking (ranking of incomes before taxation) in after tax incomes – in case of analysis made within given family type. The same, however, may not be true in case of between group analysis, where the equivalent income transformation – used for overall evaluation – is not the same as those used to obtain the tax function.

Results shown in Table 9 indicate that indexes calculated on the proper order (appearing after taxation) are different from those calculated keeping the order existing before taxation. In fact, concentration of taxes, measured by Gini coefficients (with proper order, column D) is systematically greater than measured by correspondent concentration indexes (ordered as if it was income before taxation, column C; these concentration indexes are – despite the same formula – no longer Gini coefficients). These differences are relatively stronger for the present system then for the reformed ones – they indicate level of vertical inequity.

Analyzing columns A and B, we see that for the proposed reformed systems, indexes are practically the same – there are very small differences, which disappear in approximating. For the present systems we observe a difference, which indicate extent of re-ranking – different values of mentioned indexes prove that tax system violates axiom of maximal progression (see Kakwani, Lambert, 1998). It means that vertical inequality (indicated by differences between values in columns C and D) is so significant that caused changes in the ranking of incomes before and after taxation.

This would indicate that present system is unfair not only because it does not keep into account the presence of children, but even generates somewhere an unfair ordering in what it concerns disposable incomes after taxation. This unfair results are present also in the reformed systems, but their extent is even much more contained, especially for the case of exact splitting between spouses.

Both proposed reforms result in significant increase in progressivity. Such change – observed for all analyzed family types – is a result of modifications of both tax rates and the structure of the tax system. Also redistributive properties of proposed reforms are improved (in comparison to the present system). In case of bigger families percentage of income, distributed from higher- to lower-income taxpayers (measured by Reynolds-Smolensky's indexes) is almost doubled.

Additional information on the structure of changes induced by proposed tax reforms could be get through decomposition of obtained Gini indexes. It is well known that the Gini index cannot generally be decomposed into a between-family-types part and a within-family-types part. However, there are presented in literature some proposals of decomposition that generally come down to the following form (Dagum, 1997; cf. Aronson, Johnson, Lambert, 1994):

$$G = G_B + G_W + G_t \qquad (44)$$

where
G_B is the index that would result if each family within a specific type were given the average income of its own type (between-family-type part);
$G_W = \sum_i p_i q_i G_i$, where G_i is the Gini index for the specific family type i (i=1,2,3,4,5; see Table 8), p_i is the share all equivalent people in family type i (i=1,2,3,4,5) with respect to all equivalent people in the sample, q_i is the share of overall equivalent income possessed by all people in family type i (i=1,2,3,4,5);
G_t is the transvariation coefficient. It is the difference between the Gini index in the left hand side of equation (44) and the concentration index C that results when ranking all incomes according to the ranking of family type average incomes[38]. The decompositions for Gini indexes are reported in Table 10. In the same table, below the values of indexes, there is the ratio of each of the three components of the index, with respect to the global index.

Table 10. Gini indexes decomposition

	G	G_B	G_W	G_t
Before tax	0.3121	0.1021	0.0649	0.1450
Before tax (relative)	1	0.3272	0.2080	0.4648
After P/ps tax	0.2956	0.1066	0.0605	0.1286
After P/ps (relative)	1	0.3604	0.2047	0.4349
After M-a/ms tax	0.2843	0.0951	0.0590	0.1302
After M-a/ms (relative)	1	0.3346	0.2076	0.4578
After M-b/ms tax	0.2825	0.0950	0.0586	0.1290
After M-b/ms (relative)	1	0.3362	0.2073	0.4565

Source: own calculations.

From Table 10 it appears that the greatest changes in the relative composition of Gini index is shown by the present tax system, much more contained are the relative changes induced by the two proposed tax reforms. For the present tax system G_B/G rises from 32.72% to 36.04%, while for *M-a/ms* and *M-b/ms* the ratio shifts, respectively, to 33.46% and 33.62%, that is less than one per cent point. Rather stable is the ratio for G_W/G. The ratio G_t/G is in some way complementary to G_B/G. Given that the marginal tax rates progression for *M-a/ms* and *M-b/ms* is much stronger than for

[38] Firsty, incomes are ordered by family type, taking into account average income for every family type. Secondly they are ordered – within each type – by amounts of income.
In the case here considered, we see from the second part of Table 6 that the ranking for average incomes, from the lowest to the highest, is: couples with 3 children, couples with 2 children, singles, couples with 1 child and couples without children. In the particular case when all the incomes of each specific family type do not overlap with incomes of other type – if the richest equivalent person in the group of couples with 3 children were less rich then the poorest in the group of couples with 2 children and so on – G_t would be equal to zero.

P/ps, the sensible alteration for G_B/G induced by the present tax system, indicates how this system biases horizontal equity, given the adopted benchmark scale.

Deeper insight into structure of the relative changes, resulting from the changes in tax system, could be given by application of further decomposition of Gini indexes. According to results obtained by Dagum (1997)[39], if we order the H=5 family types such that their average incomes μ_j (i=1,2,3,4,5) are in a decreasing order, that is $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq \mu_5$, we can express the term G_B, which appears in formula (44), as

$$G_B = \sum_{j}^{H} \sum_{h=j+1}^{H} \left[\sum_{l}^{H_j} \sum_{m}^{H_h} n_{l,j} n_{m,h} \left| y_{l,j} - y_{m,h} \right| \cdot \left(\frac{1}{\sum_{g}^{H} \mu_g \cdot \sum_{g}^{H} n_g \cdot \mu_g} \right) \cdot D_{jh} \right] \tag{45}$$

and the term G_w as

$$G_t = \sum_{j}^{H} \sum_{h=j+1}^{H} \left[\sum_{l}^{H_j} \sum_{m}^{H_h} n_{l,j} n_{m,h} \left| y_{l,j} - y_{m,h} \right| \cdot \left(\frac{1}{\sum_{g}^{H} n_g \cdot \sum_{g}^{H} n_g \cdot \mu_g} \right) \cdot \left(1 - D_{jh}\right) \right] \tag{45'}$$

in (45) and (45') H_j represents the number of families in the j^{th} group (j=1,2,3,4,5) and the term D_{jh} is defined as

$$D_{jh} = \frac{n_j \cdot n_h \cdot \left(\mu_j - \mu_h\right)}{\sum_{l}^{H_j} \sum_{m}^{H_h} n_{l,j} n_{m,h} \left| y_{l,j} - y_{m,h} \right|} = $$

$$= \frac{\sum_{l}^{H_j} \sum_{m}^{H_h} n_{l,j} n_{m,h} \left| y_{l,j} - y_{m,h} \right|_{y_{l,j} \geq y_{m,h}} - \sum_{l}^{H_j} \sum_{m}^{H_h} n_{l,j} n_{m,h} \left| y_{l,j} - y_{m,h} \right|_{y_{l,j} < y_{m,h}}}{\sum_{l}^{H_j} \sum_{m}^{H_h} n_{l,j} n_{m,h} \left| y_{l,j} - y_{m,h} \right|_{y_{l,j} \geq y_{m,h}} + \sum_{l}^{H_j} \sum_{m}^{H_h} n_{l,j} n_{m,h} \left| y_{l,j} - y_{m,h} \right|_{y_{l,j} < y_{m,h}}} \tag{46}$$

In expressions (45), (45') and (46) indexes j and h refers to two different family types (j=1,2,3,4,5 and h=j+1,..5): $\mu_j \geq \mu_h$. $n_{l,j}$ is the equivalent number of components of family l which is of type j, and, conversely, $n_{m,h}$ is the equivalent number of components of family m which is of type h. In the denominator of (46) there are added all the weighted absolute differences between equivalent incomes of the two different types, taken into considerations. The numerator reports the same differences with their relative signs: provided that the average of type j is greater then the average of type h, the numerator is positive. If all incomes in j are greater than incomes in h, D_{jh} is 1; D_{jh} becomes zero if the two averages are equal.

Dagum observes that D_{jh} can be interpreted as a *directional economic distance* ratio or a *relative economic affluence* (*REA*) measure.

We stress that D_{jh} (that is *REA*s between the j and h family types) is a factor either of G_B or of G_t.

Table 11 reports *REA*s for the considered family types. The tables ranks the family types according to the average (equivalent) incomes (in decreasing order). Each line in the table indicates

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[39] Formulae (45), (45') and (46) are a bit different from those presented by Dagum (1997): there the ranking for average incomes is reversed w.r.t. the one adopted in the present article, that is $\mu_1 \leq \mu_2 \leq \mu_3 \leq ...$, moreover in (45), (45') and (46) the differences are weighted by $n_{i,j}$ (i=1,2,..,H_j; j=1,2,3,4,5) and n_g (i=1,2,3,4,5) that may be not integer and, last, (45) and (45') are obtained from Dagum's (35) and (36) after some simplifications of redundant terms.

how affluent is the group of families to which the line refers, with respect to the group marked at the head of a column. For instance, before taxation, the Relative Economic Affluence of the childless couples ranges from a relative advantage of 21.69% (with respect to the couples with 1 child), to 80.39% (when compared to couples with 3 children). Before taxation the *REA* of singles is 26.28% w.r.t. couples with 2 children and it is 65.63% when compared to couples with 3 children.

Table 11 shows that all tax systems preserve the average equivalent incomes ranking which existed before taxation[40].

The present tax system is the one which mostly changes *REA* values existing before taxation; it results to be strongly favourable first of all for couples and singles. With just one exception, all positive variations are greater than 2.3 per cent points, with a maximum of 6.46, and favour families with less children with respect to families with more children[41].

Table 11. Relative economic affluence (*REA*) effects and *REA* variations due to taxation

		C+1 REA	*C+1 Tax ΔREA · 100*	*S REA*	*S Tax ΔREA · 100*	*C+2 REA*	*C+2 Tax ΔREA · 100*	*C+3 REA*	*C+3 Tax ΔREA · 100*
Before tax	*C*	0.2169		0.3469		0.5359		0.8039	
P/ps		0.2619	4.50	0.3665	1.97	0.5943	5.83	0.8414	3.74
M-a/ms tax		0.2274	1.05	0.3750	2.81	0.5467	1.08	0.8111	0.75
M-b/ms tax		0.2314	1.45	0.3702	2.33	0.5508	1.49	0.8137	0.97
Before tax	*C+1*			0.1011		0.3263		0.6668	
P/ps				0.0703	-3.08	0.3560	2.97	0.7005	3.37
M-a/ms tax				0.1169	1.58	0.3286	0.23	0.6697	0.29
M-b/ms tax				0.1093	0.82	0.3298	0.35	0.6707	0.39
Before tax	*S*					0.2628		0.6532	
P/ps						0.3274	6.46	0.7083	5.51
M-a/ms tax						0.2523	-1.05	0.6503	-0.29
M-b/ms tax						0.2596	-0.32	0.6545	0.13
Before tax	*C+2*							0.4359	
P/ps								0.4591	2.32
M-a/ms tax								0.4375	0.16
M-b/ms tax								0.4380	0.21

Source: own calculations.

Table 12. Head count ratio

	Single	Couple	Couple with 1 child	Couple with 2 children	Couple with 3 children
Before Taxation					
	6,68%	4,24%	13,46%	23,99%	42,89%
After taxation					
P/ps	7,09%	4,52%	16,14%	29,34%	51,90%
M-a/ms	7,09%	4,52%	14,26%	24,64%	44,37%
M-b/ms	7,09%	4,52%	14,26%	24,64%	44,37%

Source: own calculations.

[40] The REAs ranking and the average incomes ranking must be necessarily the same.

[41] Generally, it significantly increases differences in relative affluence between family types. The only exception is relative affluence of couples with 1 child and singles, which is diminished by this system by 3.08 per cent points (but this reduction means improvement in situation of singles).

Definitely the two proposed reforms are much more family oriented: the only variations which are greater than 2 per cent points are registered between couples and singles. Singles shows a negative variation when compared with the couple with 2 children, either for *M-a/ms* or *M-b/ms*, and with couples with 3 children, for *M-a/ms*. Even if differences are quite contained, if we look at variations in *REA*s, *M-a/ms* shows to be a bit more family friendly than *M-b/ms.*The role of exemptions is additionally enlightened by values of head count ratio, presented in Table 12. This ratios describe percentage of families of a given type that get income lower than the minimum survival one (see Table 5).

Despite the more progressive tax schedule, applied in both proposed reforms, introducing exemptions for children enabled to significantly reduce impact of taxation on the percentage of families with children, for whom after tax income lies below the minimum survival line. As a result, in case of both reform proposals, percentage of such families is only slightly higher than for income before taxation. It seems to be more coherent with assumption that such low incomes should not be taxed at all.

CONCLUSION

In this paper we analyzed some properties and the behavior of a family of affine income equivalent function in what it concerns inequality reduction and the pursuing of horizontal equitable tax systems. We chose this function because it allows to introduce the specification – independently for each family type – both of a lump sum and of family coefficients which transforms taxable incomes into equivalent incomes, with respect to a reference family.

In section 2 we found out the restrictions on the income domain and on the tax schedule that allow the adopted transformation function to be overall inequality reducing whatever the reference family is.

To the paper, there is attached an appendix intended to throw more lights on the relations existing among the traditional instruments for family charges: tax credit, exemption and family quotient.

In the empirical part we considered the problem of reforming the Polish personal income tax system, which, at present, does not allow for any exemption, quotient or tax credit for the presence of children in general. The core of our suggested reform hypotheses laid in two specifications of the equivalent transformation function analyzed in the second paragraph: these specifications adopted a spouse coefficient – different for two proposals – and children exemptions. Both proposed functions were based on benchmark equivalent income function which belonged to the same family. This benchmark function, specified on the basis of the minimum survival income and the OECD scale was further used for assessment of an equivalent income and changes in its distribution.

We verified that the proposed functions would yield a less unequal tax system, both in vertical and in horizontal sense. The results obtained through the analysis of the average tax rates curves and simulations conducted on a sample encompassing families of selected type – singles, childless couples and couples with no more than three children – showed that the proposed reforms would sensibly improve the situation of families with children, both in sense of nominal and equivalent incomes.

Pragmatic reasons, together with results obtained from analysis of average tax rates and simulations performed, would suggest to maintain the present spouse coefficient (splitting into to equal parts) at least when children are present. A spouse coefficient lower than 2 might be taken into consideration for spouses without children. In any case spouses should be allowed to decide either for a joint or a separate personal income tax, but impact of possibility of such a decision was not analyzed in the present version of the paper.

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APPENDIX

SOME CONSIDERATIONS ON THE ANALYTICAL RELATIONS AMONG QUOTIENT, EXEMPTION AND TAX CREDIT

A tax system can be evaluated in what it concerns its capability to keep into account family burdens just looking at the amount of tax saving or to the number of explicit or implicit shares into which it splits family incomes or to income amounts which are (explicitly or implicitly) deduced from taxable income, due to the presence of family burdens. In this appendix we take into consideration tax credit c_i, exemption e_i and family quotient q_i, when invariant w.r.t income and depending only on family characteristics. In particular we analyze the effects that a tax credit c_i, an exemption e_i and a family quotient q_i, have on the taxation for a family type i, under the aspect of *tax saving*, *not taxable income* and virtual *number of family shares* into which the tax payer income y_i is divided.

(a) The *tax saving* can be immediately calculated: it is the difference between the tax due by a reference tax payer r, generally without family burdens, and the tax due by another tax payer i with the same income y_i as r, but with family burdens. Let $t_r(y_i)$ be r's tax and $t_i(y_i)$ the tax for the one with family burdens of type i: then the tax saving is simply given by:

$$ac_i(y_i) = t_r(y_i) - t_i(y_i) \tag{A1}$$

(b) In order to evaluate the *not taxable income*, we have to determine the income ${}_t y_r$ for which the reference subject r must pay a tax $t_r({}_t y_r)$ equal to the same tax $t_i(y_i)$ which is due by the subject with family burdens and income y_i. Once ${}_t y_r$ has been detected, the income amount not taxable for the presence of family burdens is:

$$ae_i(y_i) = y_i - {}_t y_r \quad \text{where } {}_t y_r \text{ s.t. } t_i(y_i) = t_r({}_t y_r) \tag{A2}$$

(c) For the virtual *number of family shares* we have to find ${}_n y_r$ at which r faces the same average tax rate as the tax payer with family burdens. The virtual number of shares into which is divided the income y_i of the latter is given by the ratio:

$$aq_i(y_i) = y_i / {}_n y_r \quad \text{s.t. } \left[t_i(y_i)/y_i\right] = \left[t_r({}_n y_r)/{}_n y_r\right] \tag{A3}$$

On substituting ${}_n y_r$ by ${}_n y_r = y_i/aq_i(y_i)$ into the equation which imposes the two average tax rates to be equal, we get $\left[t_i(y_i)/y_i\right] = \left[t_r\left(y_i/aq_i(y_i)\right)/\left(y_i/aq_i(y_i)\right)\right]$ and then $t_i(y_i) = aq_i(y_i) \cdot t_r\left(y_i/aq_i(y_i)\right)$, from which we get that the tax due by the tax payer with family burdens can be interpreted as the result of a taxation applied to $aq_i(y_i)$ independent subjects each having an income equal to ${}_n y_r = y_i/aq_i(y_i)$.

Notice that ${}_n y_r < {}_t y_r$: for, in correspondence of ${}_t y_r$, generally we have:

$$\frac{t_r\left({}_n y_r\right)}{{}_n y_r} = \frac{t_i\left(y_i\right)}{y_i} < \frac{t_r\left({}_t y_r\right)}{{}_t y_r} \tag{A4}$$

because, according to (A2), $t_i\left(y_i\right) = t_r\left({}_t y_r\right)$, while, if the fiscal law adopts any horizontal equity instrument, $y_i > {}_t y_r$. If the average tax rate is increasing w.r.t. income[42], when ${}_n y_r > {}_t y_r$ inequality (A4) can but increase: then if a solution exists this must happen for

$${}_n y_r < {}_t y_r \tag{A5}$$

We shall now take into consideration either the case when the marginal tax rate is constant or the case when it is increasing w.r.t income; we moreover assume that there is a positive income under which no tax is due or, in case, the is a negative income tax. In the former case the tax for the reference family *i*, is given by equation $t_r\left(y_r\right) = \alpha y_r - \beta$ (β positive). Parameter β determines the level of no tax area and causes the average tax rate to increase w.r.t. income, even if the marginal tax rate is constant. In the latter case, for the sake of simplicity, we assume the marginal tax is continuously first differentiable. We assume there is one income earner in the family.

Table A1 reports no tax area levels when applying each of the three instruments and for each of the two tax functions; c_i, e_i and q_i are determined in such a way the no tax area level $y_{0,i}$ is always the same, given family typology *i*, independently from the equity instrument adopted[43]: this is a necessary assumption in order to make meaningful comparison among instruments.

Table A1. Levels for the no tax area

	No tax area limit			
Tax function	Reference subject	Family *i* tax credit $t_i(y_i) = \alpha y_i - \beta - c_i$	Family *i* exemption $t_i(y_i) = \alpha\left(y_i - e_i\right) - \beta$	Family *i* quotient $t_i(y_i) = q_i\left[\alpha\left(y_i/q_i\right) - \beta\right]$
$t_r(y_r) = \alpha y_r - \beta$	β/α	$(\beta+c_i)/\alpha$	$(\beta/\alpha)+e_i$	$q_i \cdot(\beta/\alpha)$
Marginal tax rate increasing w.r.t. income	y_0	$t^{-1}(c_i)$	y_0+e_i	$q_i \cdot y_0$

Source: own calculations.

Table A2 summarizes the effects of the three instruments according to the three evaluation principles described in the previous paragraph when tax is $t_r(y_r) = \alpha y_r - \beta$. It is easily seen that with this taxation function the result never depends on income, whatever the instruments is. Moreover, as it is easily seen, the effect is always the same provided that c_i, e_i and q_i are chosen in

[42] If for the reference tax payer, the one without family burdens, the average tax rate is constant, if any horizontal equity instrument is adopted, (A3) has no solution either for the exemption or the tax credit, moreover it is undetermined with the quotient.
[43] If β were equal 0, there would not be any exempt income with the quotient.

order to generate the same level for the no tax area: once one out of these three parameters has been given a value, the remaining two can be obtained univocally from the one previously fixed[44].

When the marginal tax rate is increasing w.r.t. income, the situation is more complicated. The effects given by the three instruments are summarized in Table A3.

Table A2. $t_r(y_r)=\alpha\cdot y_r-\beta$: effects generated by tax credit, exemption and quotient

	Tax credit $t_i(y_i)=\alpha\cdot y-\beta-c_i$	Exemption $t_i(y_i)=\alpha\cdot(y-e_i)-\beta$	Quotient $t_i(y_i)= q_i\cdot[\alpha\cdot(y_i/q_i)-\beta]$
(*i*) $ac_i(y_i)= t_r(y_i)-t_i(y_i)$	$ac_i =c_i$	$ac_i =\alpha\cdot e_i$	$ac_i = (q_i -1)\cdot\beta$
(*ii*) $ae_i(y_i) = y_i - {}_t\, y_r$ with $t_i (y_i) = t_r({}_t\, y_r)$	$ae_i = c_i /\alpha$	$ae_i = e_i$	$ae_i =[(q_i -1)\cdot\beta]/\alpha$
(*iii*) $aq_i(y_i)= y_i / {}_n\, y_r$ with $t_i(y_i)/y_i = t_r({}_n y_r)/{}_n y_r$	$aq_i=1+(c_i /\beta)$	$aq_i=1+(\alpha\cdot e_i /\beta)$	$aq_i = q_i$

Source: own calculations.

Table A3. Marginal tax rate increasing w.r.t. income: effects generated by tax credit, exemption and quotient (income levels above the no tax area)

	Tax credit $t_i(y_i)= t_r(y_i)- c_i$	Exemption $t_i(y_i)= t_r(y_i-e_i)$	Quotient $t_i(y_i)= q_i\cdot t_r(y_i/q_i)$
(*i*) $ac_i(y_i)= t_r(y_i)-t_i(y_i)$	$ac_i = c_i$ constant	$ac_i(y_i) = t_r(y_i)- t_r(y_i -e_i)$ increasing w.r.t. y_i	$ac_i(y_i) = t_r(y_i)- q_i\cdot t_r(y_i/q_i)$ increasing w.r.t. y_i $ac_i(y_i)_{quotient}> ac_i(y_i)_{exemption}$
(*ii*) $ae_i(y_i) = y_i - {}_t\, y_r$ with $t_i (y_i) = t_r ({}_t\, y_r)$	$ae_i(y_i)= y_i -t_r^{-1}[t_r(y_i)- c_i]< e_i$ decreasing w.r.t. y_i	$ae_i =e_i$ constant	$ae_i(y_i) = y_i - t_r^{-1}[q_i\cdot t_r(y_i/q_i)]> e_i$ increasing w.r.t. y_i
(*iii*) $aq_i(y_i)= y_i / {}_n\, y_r$ with $t_i(y_i)/y_i = t_r ({}_n y_r)/{}_n y_r$	$aq_i(y_i) \cdot t_r(y_i/ aq_i(y_i))= = t_r(y_i)- c_i$ decreasing w.r.t y_i	$aq_i(y_i) \cdot t_r(y_i/ aq_i(y_i))= =t_r(y_i -e_i)$ decreasing w.r.t y_i $aq_i(y_i)_{exemption} > aq_i(y_i)_{tax\ credit}$	$aq_i =q_i$ constant

Source: own calculations.

According to their definitions the exemption gives constant tax savings, the tax credit has constant no table income and the quotient presents always the same number of shares, once is given

[44] E.g., given q_i, we get the same values in each of the three rows in the table if $c_i=(q_i-1)\cdot\beta$ and $e_i =[(q_i-1)\cdot\beta]/\alpha$. It should be stressed that when tax is exactly proportional to y, and $\beta = 0$, tax credit and exemption are still each other univocally linked, but for quotient tax saving (and the no tax area as well) is zero.

the family typology. Outside these specific criteria, the three instruments show effects depending on income.

Marginal tax rate increasing w.r.t. income:(i) tax saving

Tax saving is constant by construction with tax credit, is increasing w.r.t. income when applying either exemption or quotient; moreover, for y above the no tax area, tax saving induced by the quotient is higher than that generated by the exemption, which, in turn, is greater than c_i.

Tax saving with exemption is $ac_i(y_i) = t_r(y_i) - t_r(y_i - e_i)$: the first derivative of $ac_i(y_i)$ w.r.t. y is positive, because the marginal tax rate at y_i is higher than at $(y_i - e_i)$.

With the quotient $ac_i(y_i) = t_r(y_i) - q_i \cdot t(y_i/q_i)$, and the first derivative of $ac_i(y_i)$ w.r.t. y is $ac_i^{'}(y_i) = t_r^{'}(y_i) - t_r^{'}(y_i/q_i) > 0$, because the marginal tax rate is increasing w.r.t. y.

Consider

$$ac_i(y_i)_{quotient} - ac_i(y_i)_{exemption} = t_r(y_i - e_i) - q_i \cdot t_r(y_i/q_i)$$

if we define y_i from the no tax area level for the specific family typology, we have $y_i = (y_0 + e_i) + \Delta y_i = q_i \cdot y_0 + \Delta y_i$, so the difference between the above tax savings can be written as

$$ac_i(y_i)_{quotient} - ac_i(y_i)_{exemption} = t_r(y_0 + \Delta y_i) - q_i \cdot t_r(y_0 + \Delta y_i/q_i) \qquad \text{(A6)}$$

the first derivative of (A6) w.r.t Δy, $t_r'(y_0 + \Delta y) - t_r'(y_0 + \Delta y/m_h)$, is positive, because the marginal tax rate is increasing; moreover (A6) is zero when $\Delta y_i = 0$, because at y_0 tax is zero whatever the equity instrument is, then, for $\Delta y_i > 0$ (A6) can be but positive.

In what it concerns elasticity w.r.t. income of tax credit effect of exemption and quotient, we can observe that the former is greater than 1, when

$$t_r^{'}(y_i) \cdot y_i - t_r^{'}(y_i - e_i) \cdot y_i > t_r(y_i) - t_r(y_i - e_i) \qquad \text{(A7)}$$

that is

$$t_r^{'}(y_i) - \frac{t_r(y_i)}{y_i} > t_r^{'}(y_i - e_i) - \frac{t_r(y_i - e_i)}{y_i} \qquad \text{(A7')}$$

According to (A7'), the elasticity of exemption tax credit effect is grater than 1, when the difference between the marginal and the average tax rate is greater before the application of the exemption than that which results after the application of the exemption.

For the elasticity of quotient tax credit effect to be grater than 1, it must be verified that

$$t_r^{'}(y_i) \cdot y_i - t_r^{'}\left(\frac{y_i}{q_i}\right) \cdot y_i > t_r(y_i) - q_i \cdot t_r\left(\frac{y_i}{q_i}\right) \qquad \text{(A8)}$$

or, which is the same, comparing differences between marginal tax rates and average tax rates

$$t_r^{'}(y_i) - t_r^{'}\left(\frac{y_i}{q_i}\right) > \frac{t_r(y_i)}{y_i} - \frac{t_r(y_i/q_i)}{y_i/q_i} \qquad \text{(A8')}$$

which means that the elasticity of quotient tax credit effect is greater than 1, when the difference between the marginal tax rate at y_i and that at y_i/q_i is greater than the difference between the average tax rates calculated at the same income levels.

Marginal tax rate increasing w.r.t. income: (ii) non taxable income

Non taxable income, for family burdens, decreases with constant tax credit, is constant with exemption and increases with quotient.

On indicating by $w=t^{-1}(g)$ the inverse function of $g=t(w)$, with tax credit first derivative of non taxable income w.r.t y, is negative

$$\frac{\partial ae_i(y_i)}{\partial y_i}=1-\frac{\partial t^{-1}\left[t_r(y_i)-c_i\right]}{\partial\left[t_r(y_i)-c_i\right]}\cdot t_r^{'}(y_i)=1-\frac{1}{t_r^{'}\left({}_t y_r\right)}\cdot t_r^{'}(y_i)<0$$

because[45] $t_r^{'}\left({}_t y_r\right)<t_r^{'}(y_i)$, as the former is the marginal tax rate at a value lower than y_i.

In what it concerns the quotient, remembering its definition, non taxable income is $ae_i = y_i - {}_t y_r = y_i - t_r^{-1}\left[q_i\cdot t_r\left(y_i/q_i\right)\right]$. Partial differentiation w.r.t. y gives

$$\frac{\partial ae_i}{\partial y_i}=1-\frac{\partial t_r^{-1}\left[q_i\cdot t_r\left(y_i/q_i\right)\right]}{\partial y_i}=1-\frac{\partial t_r^{-1}\left[q_i\cdot t_r\left(y_i/q_i\right)\right]}{\partial\left[q_i\cdot t_r\left(y_i/q_i\right)\right]}\cdot t_r^{'}\left(y_i/q_i\right)=1-\frac{1}{t_r^{'}\left({}_t y_r\right)}\cdot t_r^{'}\left({}_n y_r\right)>0$$

as, by definition $(y_i/q_i)={}_n y_r$, $t_r({}_n y_r)<t_r({}_t y_r)$ and the marginal tax rate is increasing w.r.t. income.

As, by construction, the three methods at no tax area level must have the same no taxable income, we have that the quotient effect is stronger both than that of exemption and of tax credit, the income is higher; in the meanwhile the tax credit effect gets worse than either the quotient or the exemption one.

In what it concerns the elasticity w.r.t. income of the non taxable income generated by the quotient, by definition it is:

$$\frac{\dfrac{\partial\left[y_i-t_r^{-1}\left(q_i t_r\left(\dfrac{y_i}{q_i}\right)\right)\right]}{\partial y_i}\cdot y_i}{y_i-t_r^{-1}\left(q_i t_r\left(\dfrac{y_i}{q_i}\right)\right)} \qquad \text{(A9)}$$

On defining ${}_{ae,q}y_i = t_r^{-1}\left(q_i t_r\left(y_i/q_i\right)\right)$, we can write (A9) as:

$$\frac{y_i-\left[1/t_r^{'}\left({}_{ae,q}y_i\right)\right]\cdot t_r^{'}\left(\dfrac{y_i}{q_i}\right)\cdot y_i}{y_i-{}_{ae,q}y_i} \qquad \text{(A9')}$$

In order (A9') is greater than 1, we must have:

$$t_r^{'}\left(\frac{y_i}{q_i}\right)\cdot y_i < t_r^{'}\left({}_{ae,q}y_i\right)\cdot {}_{ae,q}y_i$$

[45] As $g=t(w)$ is monotone, $\left[\partial t^{-1}(g)/\partial g\right]=\left[1/t'(w)\right]$.

Then on dividing both sides of the above expression by $q_i t_r\left(y_i/q_i\right)$, and remembering that $q_i t_r\left(y_i/q_i\right)=t_i\left(y_i\right)=t_r\left({}_{ae,q}y_i\right)$, we have immediately:

$$\frac{t_r'\left(\frac{y_i}{q_i}\right)\cdot\frac{y_i}{q_i}}{t_r\left(\frac{y_i}{q_i}\right)}<\frac{t_r'\left({}_{ae,q}y_i\right)\cdot{}_{ae,q}y_i}{t_r\left({}_{ae,q}y_i\right)}$$

Or, defining with $\eta_r\left(y_i\right)$ the elasticity of tax at income y_i for the reference family (the single), the elasticity of non taxable income effect generated by the quotient is greater than 1, when:

$$\eta_r\left(\frac{y_i}{q_i}\right)<\eta_r\left({}_{ae,q}y_i\right) \qquad \text{(A10)}$$

Then when (A10) is verified, an increase of income generates a more than proportional increase of non taxable income; the contrary holds when the inequality sign in (A10) is reversed.

Marginal tax rate increasing w.r.t. income: (iii) the number of shares into which the tax payer income y is divided

Either for exemption or tax credit the number shares, into which the tax payer income y is divided, decreases w.r.t. y. In what it concerns the tax credit, from the definition of number of shares $aq_i(y_i)$ we have that $t_i\left(y_i\right)-aq_i\left(y_i\right)\cdot t_r\left(y_i/aq_i\left(y_i\right)\right)=0$. Let us now differentiate this expression w.r.t. y, imposing this function to remain constant (that is, equal to 0). Remembering that for the tax credit, $t_i(y_i)= t_r(y_i)-c_i$, we get:

$$t_r'\left(y_i\right)-aq_i'\left(y_i\right)\cdot t_r\left[\frac{y_i}{aq_i\left(y_i\right)}\right]-aq_i\left(y_i\right)\cdot t_r'\left[\frac{y_i}{aq_i\left(y_i\right)}\right]\cdot\left\{\frac{aq_i\left(y_i\right)-aq_i'\left(y_i\right)\cdot y_i}{\left[aq_i\left(y_i\right)\right]^2}\right\}=0$$

from which it follows:

$$aq_i'\left(y_i\right)=-\frac{t_r'\left(y_i\right)-t_r'\left[\frac{y_i}{aq_i\left(y_i\right)}\right]}{\frac{y_i}{aq_i\left(y_i\right)}\cdot t_r'\left[\frac{y_i}{aq_i\left(y_i\right)}\right]-t\left[\frac{y_i}{aq_i\left(y_i\right)}\right]}<0 \qquad \text{(A11)}$$

In (A11) both the numerator and the denominator are positive: the former because $t'(w)$ increases w.r.t. its argument and the latter because by multiplying w times the marginal tax rate $t'(w)$ we get something greater than the tax $t(w)$.

Analogously for the exemption, we differentiate $t_r\left(y_i-e_i\right)-aq_i\left(y_i\right)\cdot t_r\left[y/aq_i\left(y_i\right)\right]=0$, w.r.t. y, keeping the equation to be constant, that is equal to zero, and we get:

$$t_r^{'}\left(y_i-e_i\right)-aq_i^{'}\left(y_i\right)\cdot t_r\left[\frac{y_i}{aq_i\left(y_i\right)}\right]-aq_i\left(y_i\right)\cdot t_r^{'}\left[\frac{y_i}{aq_i\left(y_i\right)}\right]\cdot\left\{\frac{aq_i\left(y_i\right)-aq_i^{'}\left(y_i\right)\cdot y_i}{\left[aq_i\left(y_i\right)\right]^2}\right\}=0$$

from which:

$$aq_i^{'}\left(y_i\right)=-\frac{t_r^{'}\left(y_i-e_i\right)-t_r^{'}\left[\dfrac{y_i}{aq_i\left(y_i\right)}\right]}{\dfrac{y_i}{aq_i\left(y_i\right)}\cdot t_r^{'}\left[\dfrac{y_i}{aq_i\left(y_i\right)}\right]-t\left[\dfrac{y_i}{aq\left(y_i\right)}\right]}<0 \qquad \text{(A12)}$$

In what it concerns the denominator in (A12) we note that still holds what we observed for the denominator in(A7); in what it concerns the numerator, if $t_r\left(y_i-e_i\right)=aq_i\left(y_i\right)\cdot t_r\left(y_i/aq_i\left(y_i\right)\right)$, then $t_r\left(y_i-e_i\right)>t_r\left(y_i/aq_i\left(y_i\right)\right)$ and consequently, when the marginal tax rate is increasing, $t_r^{'}\left(y_i-e_i\right)>t_r^{'}\left(y_i/aq_i\left(y_i\right)\right)$.

Moreover $aq_i(y_i)_{exemption}> aq_i(y_i)_{tax\ credit}$. Equating average tax rate, requests that for the tax credit:

$$t_r\left(y_i\right)-c_i=aq_i\left(y_i\right)_{tax\,credit}\cdot t_r\left(\frac{y_i}{aq_i\left(y_i\right)_{tax\,credit}}\right)$$

and analogously for the exemption:

$$t_r\left(y_i-e_i\right)=aq_i\left(y_i\right)_{exemption}\cdot t_r\left(\frac{y_i}{aq_i\left(y_i\right)_{exemption}}\right)$$

remember now that if, above the no tax area, tax saving for the exemption is greater that that for the tax credit, as $t_r\left(y_i-e_i\right)<t_r\left(y_i\right)-c_i$, it must be true that:

$$aq_i\left(y_i\right)_{exemption}\cdot t_r\left(\frac{y_i}{aq_i\left(y_i\right)_{exemption}}\right)<q_i\left(y_i\right)_{tax\,credit}\cdot t_r\left(\frac{y_i}{aq_i\left(y_i\right)_{tax\,credit}}\right)$$

Now, with increasing marginal tax rate, when y_i constant, the expression $aq_i\left(y_i\right)\cdot t_r\left(y_i/aq\left(y_i\right)\right)$ decreases as $aq_i(y_i)$ increases[46], from which it follows immediately that $aq_i(y_i)_{exemption}> aq_i(y_i)_{tax\ credit}$.

Summarizing:

- we have compared three instruments in order to decrease tax liability due to the presence of family burdens: *family quotient*, *exemption* and *tax credit*, all depending only on family characteristics and invariant with respect to income;

[46] Note $\partial\{q\cdot t(y/q)\}/\partial q=t(y/q)-(y/q)\cdot t(y/q)<0$, because when the marginal tax rate is increasing w.r.t. income the tax for y/q is less than y/q times the marginal tax rate at y/q itself.

- for a given tax scheduling and established no tax area limits for each family typology, whatever instrument is us adopted for family charges:

  (1) with a constant marginal tax rate the three methods present the same effect;
  (2) when the marginal tax rate is increasing, the quotient is the most effective one in decreasing tax liability and the tax credit is the less effective one.

Applications to actual income tax systems

Actual income tax systems present marginal tax rates which are constant for a specific income interval, as described in the following Table A4.

Table A4. Example of a tax system piecewise linear with increasing bracket tax rates

Income interval	Marginal tax rate	Total tax at the upper interval limit	Total tax
$y_0 < y \leq L_1$	α_1	T_1	$\alpha_1 \cdot (y - y_0) = \alpha_1 \cdot y - \beta$
$L_1 < y \leq L_2$	α_2	T_2	$T_1 + \alpha_2 \cdot (y - L_1)$
$L_2 < y \leq L_3$	α_3	T_3	$T_2 + \alpha_3 \cdot (y - L_2)$
$L_2 < y \leq L_3$	α_4	T_4	$T_3 + \alpha_4 \cdot (y - L_3)$
……….	……….	……….	……….
$L_K < y$	α_{K+1}		$T_K + \alpha_{K-1} \cdot (y - L_K)$

Source: own calculations.

For the sake of simplicity let's suppose that c_i and the parameter of the first bracket are such that $c_i < \alpha_1 \cdot y - \beta$, for $\forall\, i$. If we express the exemption e_i and the quotient q_i in terms of the tax credit c_i, restricting the no tax area limit to be in any case the same for type i family, whatever the instrument adopted is, on the basis of Table A1, we have $e_i = c_i / \alpha_1$ and $q_i = (c_i + \beta)/\beta$, with $\beta = \alpha_1 \cdot y_0$.
As long as y_i lies in the first bracket, the situation is exactly the one tackled in Table A2, so the results there reported can be directly applied.
Suppose now that y_i lies in the second bracket, and that, always for the sake of simplicity, either $(y_i - e_i)$ or y_i / q_i still lies in the second bracket[47]. It is quite simple to verify that the application of the family exemption $e_i = c_i / \alpha_1$ implies a tax credit effect $c_i \cdot (\alpha_2 / \alpha_1)$, greater than c_i and it is immediate to check that either for exemption or for quotient the tax credit effect becomes larger as y_i lifts into a higher bracket. Moreover the latter is greater than the former.
In fact, starting from the tax generated by the quotient:

$$\frac{c_i + \beta}{\beta} \cdot \left(\alpha_1 \cdot L_1 - \beta + \alpha_2 \cdot y_i \cdot \frac{\beta}{c_i + \beta} - \alpha_2 \cdot L_1 \right) \tag{A13}$$

after some simplifications, the tax credit effect of the quotient becomes $(\alpha_2 - \alpha_1) \cdot L_1 \cdot (c_i / \beta) + c_i$; the inequality:

$$(\alpha_2 - \alpha_1) \cdot L_1 \cdot \frac{c_i}{\beta} + c_i > c_i \cdot \frac{\alpha_2}{\alpha_1}$$

[47] If either $(y_i - e_i)$ or y_i / q_i should shift into a lower bracket, the reported results hold *a fortiori*.

is verified, because

$$\left(\alpha_2-\alpha_1\right)\cdot L_1>\left(\alpha_2-\alpha_1\right)\cdot\frac{\beta}{\alpha_1}$$

as $L_1>\beta/\alpha_1$.

The exemption effect of the quotient is greater e_i than and depends on income.
Still in the case that either y_i and y_i/q_i lies in the second bracket, we must find $ae_i(y_i)$ s.t. the tax expressed as:

$$\alpha_1\cdot L_1-\beta+\alpha_2\cdot\left[y_i-ae_i\left(y_i\right)-L_1\right]$$

is equal to the tax expressed as in (A11). The solution for $ae_i(y_i)$ is:

$$ae_i\left(y_i\right)=\left(\alpha_2-\alpha_1\right)\cdot L_1\frac{1}{\alpha_2}+\frac{c_i}{\alpha_2} \tag{A14}$$

(A14) is directly related to income, via the bracket in which the income itself lies, and is greater than $e_i=c_i/\alpha_1$; in fact the inequality:

$$ae_i\left(y_i\right)=\left(\alpha_2-\alpha_1\right)\cdot L_1\frac{1}{\alpha_2}+\frac{c_i}{\alpha_2}>\frac{c_i}{\alpha_1}$$

is verified because $L_1>c_i/\alpha_1$.

In what it concerns the elasticity of tax, we only observe that in the system outlined in Table A4, we just observe that in an actual tax system it is not possible for the tax elasticity to have a monotonic behavior. In fact, let us consider the elasticity at L_i and that at $y=L_i+\Delta$, with Δ s.t. $L_i+\Delta<L_{i+1}$. Defining:

$$\eta(L_i)=\frac{\alpha_i\cdot L_i}{T_i}\qquad\qquad \eta(L_i+\Delta)=\frac{\alpha_{i+1}\cdot(L_i+\Delta)}{T_i+\alpha_{i+1}\cdot\Delta}$$

Then $\eta(L_i)>\eta(L_i+\Delta)$ when $\frac{\alpha_i\cdot L_i}{T_i}>\frac{\alpha_{i+1}\cdot(L_i+\Delta)}{T_i+\alpha_{i+1}\cdot\Delta}$ which, solved w.r.t. Δ, gives:

$$\Delta>\frac{T_i\cdot L_i\cdot\left(\alpha_{i+1}-\alpha_i\right)}{\alpha_{i+1}\cdot\left(\alpha_i\cdot L_i-T_i\right)} \tag{A15}$$

Observe that in the r.h.s. of (A15) the denominator is positive.
It follows that in an actual tax system it is not possible for the tax elasticity to be monotonically decreasing: $\eta(L_i)>\eta(L_i+\Delta)$ when Δ is large enough to satisfy (A15), which depends on the difference between marginal tax rates for contiguous income intervals and on the length of the previous interval. Conversely, in order that $\eta\left(L_i\right)<\eta\left(L_i+\Delta\right)$, Δ should be small enough to satisfy:

$$\Delta<\frac{T_i\cdot L_i\cdot\left(\alpha_{i+1}-\alpha_i\right)}{\alpha_{i+1}\cdot\left(\alpha_i\cdot L_i-T_i\right)} \tag{A15'}$$