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ENTIRE SOLUTIONS OF SINGULAR ELLIPTIC INEQUALITIES ON COMPLETE MANIFOLDS

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Abstract. We present some qualitative properties for solutions of singular quasilinear elliptic differential inequalities on complete Riemannian manifolds, such as the validity of the weak maximum principle at infinity, and non–existence results.

1. **Introduction.**

In this paper we are interested in the qualitative study of solutions of quasilinear elliptic differential inequalities on complete Riemannian manifolds. In particular, we establish a weak maximum principle at "*infinity*" under generally mild assumptions on the quasilinear operators and on the manifolds themselves. In this introduction, in order to clarify the presentation, the results are given for the canonical divergence structure differential inequalities

$$\operatorname{div}\{A(|\nabla u|)\nabla u\} - f(u) \geq 0, \tag{1.1}$$

on a connected, complete, non–compact Riemannian manifold, $(\mathcal{M}, \langle\cdot,\cdot\rangle)$, of dimension $m \geq 2$. We fix an origin O and denote by $r = r(x)$ the distance function from O to x. Clearly r is of class $\mathrm{Lip}(\mathcal{M})$. Then $B_R = \{x \in \mathcal{M} \ : \ r(x) < R\}$ indicates the geodesic ball of radius $R > 0$ centered at O. Here ∇u denotes the gradient of the given function $u = u(x)$, $x \in \mathcal{M}$. The main assumptions on $A = A(\rho)$, $\Phi := \rho A(\rho)$ and $f = f(u)$ are:

(A1) $A \in C^1(\mathbb{R}^+)$;
(A2) $\Phi'(\rho) > 0$ *for* $\rho > 0$ *and* $\Phi(\rho) \to 0$ *as* $\rho \to 0^+$;
(A3) $\Phi(\rho) \leq c\rho^\sigma$ *on* $[0, \varpi)$ *for some* c, ϖ, $\sigma > 0$.

Condition (A2) is a requirement for ellipticity of (1.1) and allows singular and degenerate behavior of the operator A at $\rho = 0$, that is, at critical points of u. We emphasize that f is assumed only continuous in $\mathbb{R}$, unless otherwise stated.

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By a *semi–classical* (*classical*) *solution* of (1.1) on $\mathcal{M}$ we mean a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{M})$ ($u \in C^1(\mathcal{M})$) which satisfies (1.1) in the distribution sense, that is, for all $\varphi \in C_0^\infty(\mathcal{M})$, $\varphi \geq 0$,

$$\int_{\mathcal{M}} \{\langle A(|\nabla u|)\nabla u, \nabla\varphi\rangle + f(u)\varphi\} d\mathcal{M} \leq 0.$$

With the aid of (A2), we extend Φ by continuity on $\mathbb{R}_0^+$ by setting $\Phi(0) = 0$ and complete the definition of Φ on the entire real line putting $\Phi(\rho) = -\Phi(-\rho)$ if $\rho < 0$. Introduce

$$H(\rho) = \rho\Phi(\rho) - \int_0^\rho \Phi(s)ds, \qquad \rho \geq 0. \tag{1.2}$$

The function H is easily seen to be strictly increasing in $\mathbb{R}_0^+$.

For the Laplace operator, that is when (1.1) takes the classical form

$$\Delta v - f(v) \geq 0, \qquad v \geq 0,$$

the results are $A(\rho) \equiv 1$ and $H(\rho) = \frac{1}{2}\rho^2$. Similarly, for the degenerate p–Laplace operator, $p > 1$, we have $A(\rho) = \rho^{p-2}$ and $H(\rho) = (p-1)\rho^p/p$, while for the mean curvature operator, one has $A(\rho) = 1/\sqrt{1+\rho^2}$ and $H(\rho) = 1 - 1/\sqrt{1+\rho^2}$. In the last example, note the anomalous behavior $\Phi(\infty) = H(\infty) = 1$, a possibility which occasionally requires extra care in the statement and treatment of the results.

It is also worth observing that (1.1), when equality holds, is precisely the Euler–Lagrange equation for the variational integral

$$I[v] = \int_{\mathcal{M}} \{\mathcal{G}(|\nabla v|) + F(v)\} d\mathcal{M}, \qquad F(v) = \int_0^v f(s)ds,$$

where $\mathcal{G}$ and A are related by $\mathcal{G}'(\rho) = \rho A(\rho) = \Phi(\rho)$, $\rho > 0$. In this case $H(\rho) = \rho\mathcal{G}'(\rho) - \mathcal{G}(\rho)$, is the pre–Legendre transform of $\mathcal{G}$. Further comments and other examples of operators satisfying (A1)–(A3) are given in [16] and [18].

As a further remark we observe that, while globally the distance function on $\mathcal{M}$ is in general only Lipschitz, we can always find at any point $x \in \mathcal{M}$ a small geodesic ball $B_R(x)$ such that the distance from x, that is, $\mathrm{dist}(x, \cdot)$ is a smooth function on $B_R(x) \setminus \{x\}$. We shall always call such a ball a *regular ball* without any further mentioning.

We shall usually work by comparing the manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ with a model manifold in the sense of Greene and Wu [8]. This latter can be briefly described as follows. A *model* $\mathcal{N} = \mathcal{N}(g)$ is a smooth Riemannian manifold of dimension $m \geq 2$ such that:

i) $\mathcal{N}$ has a pole O, that is the exponential map is a diffeomorphism of $T_O(\mathcal{N})$ onto $\mathcal{N}$;
ii) every linear isometry $\gamma : T_O(\mathcal{N}) \to T_O(\mathcal{N})$ is realized as the differential of an isometry $\Gamma : \mathcal{N} \to \mathcal{N}$, that is, $\Gamma(O) = O$ and $\Gamma_{*O} = \gamma$, where Γ_{*O} is the differential of Γ at O.

Clearly, $\mathcal{N}$ is complete and it may be identified with $T_O(\mathcal{N})$ via the exponential map. In geodesic polar coordinates $(r, \vartheta) \in \mathbb{R}^+ \times S^{m-1} \backsimeq \mathcal{N} \setminus \{O\}$, the Riemannian metric can be expressed in the form

$$\langle\cdot,\cdot\rangle = dr^2 + g(r)^2 d\vartheta^2, \tag{1.3}$$

where $d\vartheta^2$ is the standard metric on S^{m-1}, and g satisfies the following natural analytic assumptions:

(g1) $g \in C^\infty(\mathbb{R}_0^+)$, $g^{(2k)}(0) = 0$ for all $k = 0, 1, \dots$, $g'(0) = 1$;
(g2) $g(r) > 0$ for $r > 0$,

which in particular guarantee that the metric defined in (1.3) can be extended smoothly on all of $\mathcal{N}$. Thus, for instance, the Euclidean space $\mathbb{R}^m$ and the hyperbolic space $\mathbb{H}^m$ of constant sectional curvature -1 are realized by the choices respectively $g(r) = r$ and $g(r) = \sinh r$.

The model $r(x) = \operatorname{dist}(x, O)$ is smooth outside O and satisfies

$$\Delta r = (m-1)\frac{g'(r)}{g(r)}, \qquad \operatorname{Hess}(r) = \frac{g'(r)}{g(r)}[\langle\cdot,\cdot\rangle - dr \otimes dr] \qquad \text{in } \mathcal{M} \setminus \{O\}. \tag{1.4}$$

The classical Laplacian and Hessian comparison theorems allow us to estimate from above and below (in general only in the weak sense) the Laplacian and the Hessian of the distance function on a generic manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ via (1.4) of an appropriate model $\mathcal{N} = \mathcal{N}(g)$ constructed through curvature conditions on the original manifold $\mathcal{M}$. By way of example observe that any complete manifold verifies condition

($\mathcal{M}$1) $\operatorname{Ricc}_{(\mathcal{M},\langle\cdot,\cdot\rangle)}(\nabla r, \nabla r) \ge -(m-1)G(r)$ *in* $\mathcal{M}$, *for some positive non–decreasing function* $G \in C^1(\mathbb{R}_0^+)$.

Hence by Lemma 2.1 of [16] the function g defined by

$$g(r) = \frac{\exp\left\{D\int_0^r \sqrt{G(s)}ds\right\} - 1}{D\sqrt{G(0)}}, \tag{1.5}$$

where $D > 0$ is sufficiently large, is such that

$$\Delta r(x) \le (m-1)\frac{g'(r(x))}{g(r(x))} \qquad \text{on} \quad \mathcal{M} \setminus [\{O\} \cup \operatorname{cut}(O)], \tag{1.6}$$

where $\operatorname{cut}(O)$ is the cut locus of the origin O, and (1.6) holds weakly on all of $\mathcal{M}$.

When $\mathcal{N}$ is a model $\mathcal{N} = \mathcal{N}(g)$ the function defined by (1.5) does not coincide (in general) with the original function g associated to the model itself. This is certainly clear when we observe that the left hand side of the inequality in ($\mathcal{M}$1) is simply $-(m-1)g''/g$, so that G must only bound g''/g from above. However, we adopt this abuse of notation since in the main proofs the function g in (1.5) will play the role of the function g of a model manifold $\mathcal{N} = \mathcal{N}(g)$.

This comparison technique will be repeatedly used in the sequel. Furthermore, on stating and commenting some of our results we shall often explicitly consider the special case of models with a twofold purpose: namely, through them we easily compare with the more familiar Euclidean setting and, when relevant, we may underline the influence of geometry.

To grasp the global structure of the manifold, we resort to a type of maximum principle, which has its roots in the work of Omori [13], on immersions of minimal submanifolds into cones of $\mathbb{R}^n$ and which relies on the following simple observation: *if* $u : \mathbb{R} \to \mathbb{R}$ *is a* $C^2(\mathbb{R})$ *function, with* $u^* = \sup u < \infty$*, then there exists a sequence* $\{x_k\}_k \subset \mathbb{R}$ *such that*

$$u(x_k) > u^* - 1/k, \qquad |u'(x_k)| < 1/k \quad \textit{and} \quad u''(x_k) < 1/k \quad \textit{for all } k \in \mathbb{N}. \tag{1.7}$$

Omori established a version of this principle on a complete Riemannian manifold, with sectional curvature bounded from below, and he also provided examples for which his global form of the maximum principle fails. This new idea was taken up by Yau who refined the principle for the Laplace–Beltrami operator in a series of papers [21], some of which in collaboration with Cheng [1], and applied it to find, in an elegant way, solutions to several geometrical problems. Some refinements and extensions have been recently given in [16].

Theorem 1.1. (The Strong Maximum Principle at Infinity.) *Suppose that the Ricci radial curvature of* $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ *satisfies* $(\mathcal{M}1)$, *with*

$$G(t) \leq z(t)^{2\sigma} \qquad \textit{for } t >> 1, \tag{1.8}$$

where $z \in C^1(\mathbb{R}_0^+)$ *is a positive non–decreasing function such that* $1/z \notin L^1([1,\infty))$. *Let* $u \in C^1(\mathcal{M})$ *be such that* $u^* = \sup_{\mathcal{M}} u < \infty$. *Let* $\wp > 0$ *and*

$$E_\wp = \{x \in \mathcal{M} \;:\; u(x) > u^* - \wp,\ |\nabla u| < \wp\}.$$

Then for every $\varepsilon > 0$ *the function* u *is not a classical solution of the differential inequality*

$$\operatorname{div}\{A(|\nabla u|)\nabla u\} \geq \varepsilon \qquad \textit{in } E_\wp. \tag{1.9}$$

Clearly when $u \in C^2(\mathcal{M})$ and the vector field $A(|\nabla u|)\nabla u$ is of class $C^1(\mathcal{M}, T\mathcal{M})$, the above conclusion can be restated in a form similar to (1.7), in other words, there exists a sequence $\{x_k\}_k$ with the properties

$$u(x_k) > u^* - 1/k, \quad |\nabla u(x_k)| < 1/k, \quad \operatorname{div}\{A(|\nabla u|)h(\nabla u,\cdot)^\sharp\}(x_k) < 1/k \tag{1.10}$$

for all $k \in \mathbb{N}$.

Theorem 1.1 continues to hold if $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{M})$ is a semi–classical solution of (1.9) provided that the set $E_\wp$ is replaced by the larger open set

$$\tilde{E}_\wp = \{x \in \mathcal{M} \;:\; u(x) > u^* - \wp\}.$$

To illustrate the possible use of Theorem 1.1, we consider the following geometrical example. An old result of Heinz [10] (originally stated for surfaces and then generalized by Chern [2] and Flanders [7] to any dimension) implies that a constant mean curvature graph on $\mathbb{R}^m$ is necessarily minimal. We recall that a graph on a manifold $\mathcal{M}$ is the immersion $\Gamma_u : \mathcal{M} \to \mathcal{M} \times \mathbb{R}$ defined by $\Gamma_u(x) = (x, u(x))$ for some $u \in C^\infty(\mathcal{M})$. As well known, a graph is of constant mean curvature when u satisfies

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = c \qquad \text{on } \mathcal{M}, \tag{1.11}$$

for some constant c, which can be assumed non–negative. Minimality is equivalent to $c = 0$. It is not hard to see that if $h_C(\mathcal{M})$, the Cheeger constant of $\mathcal{M}$, is null, as in the Euclidean case, then $c = 0$. This extends Heinz' result. However, the conclusion is in general false if $h_C(\mathcal{M}) > 0$. For instance, on $\mathbb{H}^m$ with metric $\langle\cdot,\cdot\rangle = dr^2 + \sinh^2 r d\vartheta^2$ outside the pole, $h_C(\mathbb{H}^m) = m - 1 > 0$, and for any $c \in (0, m-1]$ the smooth function

$$u(x) = \int_0^{r(x)} \frac{\varphi(t)}{\sqrt{1-|\varphi(t)|^2}}\,dt, \qquad \varphi(t) = c(\sinh t)^{1-m}\int_0^t (\sinh s)^{m-1} ds,$$

satisfies (1.11). Hence u produces a graph of constant mean curvature c/m. Note that in this case $u^* = \infty$. Recalling that a graph Γ_u on $\mathcal{M}$ is said to be bounded if $|u|^* < \infty$, as a consequence of Theorem 1.1, we have that any bounded constant mean curvature graph on $\mathcal{M}$ is minimal if

$$\mathrm{Ricc}_{(\mathcal{M},\langle\cdot,\cdot\rangle)}(\nabla r, \nabla r) \geq -(m-1)C(1+r^2), \tag{1.12}$$

for some $C > 0$. See the more general Corollary 1.2 below.

Furthermore, we emphasize that the geometrical assumptions in Theorem 1.1 are sharp. This can be indirectly seen with the following reasoning. Recall that *stochastic completeness* is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold this means that the total probability of the particle to be found in the state space is constantly equal to 1 (for an introduction to the subject see the excellent book by Emery [4]). In [16] it has been proved that stochastic completeness of the Riemannian manifold $\mathcal{M} = (\mathcal{M}, \langle\cdot,\cdot\rangle)$ is equivalent to the following analytical property: *for each $f \in C(\mathbb{R})$ and for each $u \in C^2(\mathcal{M})$, with $u^* < \infty$, satisfying $\Delta u \geq f(u)$, we have $f(u^*) \leq 0$.*

Then the conclusion of Theorem 1.1 for the Laplace–Beltrami operator implies that the complete Riemannian manifold $\mathcal{M}$ is stochastically complete. To prove that the geometrical assumption $(\mathcal{M}1)$ together with (1.8) are sharp is therefore enough to show that if we relax them, then $\mathcal{M}$ is no longer stochastically complete and hence the conclusion of Theorem 1.1 is false.

Towards this aim we consider an m–dimensional model $\mathcal{M} = \mathcal{M}(g)$ such that

$$\int^{\infty} g(r)^{1-m} \int_0^r g(t)^{m-1} dt dr < \infty. \tag{1.13}$$

It is well known, see for instance [16] or Grigor'yan [9], that in this case $\mathcal{M} = \mathcal{M}(g)$ is not stochastically complete. Let us choose

$$g(r) = \exp\left(\frac{r^2}{m-1}(\log r)^{1+\varepsilon}\right) \qquad \text{for } r \geq 2 \tag{1.14}$$

and some $\varepsilon > 0$. Clearly

$$g(r)^{1-m} \int_0^r g(t)^{m-1} dt \sim \frac{1}{2r(\log r)^{1+\varepsilon}} \quad \text{as } r \to \infty,$$

so that (1.13) is satisfied. On the other hand,

$$\mathrm{Ricc}_{(\mathcal{M},\langle\cdot,\cdot\rangle)}(\nabla r, \nabla r) = -(m-1)\frac{g''(r)}{g(r)} \sim -\frac{4r^2(\log r)^{2(1+\varepsilon)}}{m-1} \quad \text{as } r \to \infty.$$

Thus (1.8) with $\sigma = 1$, that is for the Laplace–Beltrami operator, barely fails to be met.

Let us now introduce the main assumption on the nonlinearity f we consider, that is,

(F1) *$f \in C(\mathbb{R}_0^+)$, with $f(0) = 0$ and f positive on $\mathbb{R}^+$.*

As an immediate consequence of Theorem 1.1 and of the previous remark, we have

Corollary 1.2. *Let f satisfy* (F1). *Under the assumption of Theorem 1.1 there are no positive bounded above semi–classical entire solutions of the differential inequality*

$$\mathrm{div}\{A(|\nabla u|)\nabla u\} \geq f(u) \quad \textit{on } \mathcal{M}. \tag{1.15}$$

Note that in the assumptions of Corollary 1.2 positive unbounded semi–classical or even classical solutions of (1.15) may exist. In fact, they do exist under the additional assumptions $f(0) = 0$, (F2) given below, (A3) with $\varpi = \infty$, and

$$\int^{\infty} \frac{ds}{H^{-1}(F(s))} = \infty. \tag{1.16}$$

See Proposition 4.2 in the Appendix.

Confining ourselves to classical solutions, the conclusion of Corollary 1.2 continues to hold if we substitute the right hand side of (1.15) with a more general nonlinearity B satisfying

(B) *the function* $B : \mathcal{M} \times \mathbb{R}_0^+ \times T\mathcal{M} \to \mathbb{R}$ *is continuous and verifies*

$$B(x, u, \boldsymbol{\xi}) \geq -\kappa T(|\boldsymbol{\xi}|) + f(u) \qquad \text{for } x \in \mathcal{M},\ u \geq 0 \text{ and } |\boldsymbol{\xi}| \leq 1,$$

for some $\kappa \geq 0$*, where* f *satisfies* (F1)*, and* T *is a continuous function on* $\mathbb{R}_0^+$*, with* $T(0) = 0$*,*

see Corollary 2.5. In both Corollaries 1.2 and 2.5 the condition $f(0) = 0$ in (F1) is not needed.

An important prototype is the inequality

$$\Delta_p u + \kappa|\nabla u|^q - f(u) \geq 0, \qquad p > 1, \quad \kappa \geq 0, \quad q > 0,$$

where $T(\varrho) = \kappa\varrho^q$. There are a number of important papers concerning this example of great interest in applications; the reader is referred to [18] and the references thereby.

We also observe that under the additional assumption

(F2) $\quad$ *f is non–decreasing on some interval* $\mathbb{R}_0^+$;

the apriori request in Corollary 1.2, that the entire solutions are bounded above, can be removed. However the result is no more applicable to the mean curvature operator.

Theorem 1.3. *Let the assumptions of Theorem 1.1 be satisfied, with* $\Phi(\infty) = \infty$*. Let* f *verify* (F1)*,* (F2) *and*

$$\liminf_{u\to\infty} \frac{f(u)}{u^\tau} > 0 \quad \text{for some } \tau > \max\{1,\ \sigma\}. \tag{1.17}$$

Assume that ($\mathcal{M}1$) *is verified with*

$$G(t) = O(t^{2\sigma}) \quad \text{as } t \to \infty. \tag{1.18}$$

Then (1.15) *has no positive semi–classical entire solutions.*

As noted for Corollary 1.2, Theorem 1.3 can also be extended to nonlinearity of type B, see Theorem 3.3 below.

It is a simple matter to realize that assumption (1.17) cannot be relaxed too much. Indeed, a result of Fisher–Colbrie and Schoen in [6] implies that on any $\mathcal{M}$ we can find a positive solution of $\Delta u = u$. Nevertheless, observe that (1.17) implies a condition somewhat *dual* to (1.16), that is

$$\int^\infty \frac{ds}{H^{-1}(F(s))} < \infty. \tag{1.19}$$

When $A \equiv 1$ then (1.3) is also sufficient to get the conclusion of Theorem 1.3, see [16, Theorem 1.31], that is when (1.15) reduces to the Laplace–Beltrami inequality

$$\Delta u \geq f(u) \qquad \text{on } \mathcal{M}.$$

In this case (1.19) becomes

$$\int^\infty \frac{ds}{\sqrt{F(s)}} < \infty,$$

which is the well known Keller–Osserman condition when $\mathcal{M} = \mathbb{R}^m$. Actually Theorem 1.3 is the extension to the Riemannian setting of [12, Theorem 2] of Naito and Usami. For further extensions in Euclidean space of the Naito and Usami results to the vectorial case as well as to divergence operators with diffusion terms $\varphi(u)$, possibly singular or degenerate, we refer to the recent paper [5] of Filippucci.

The generalization of Theorem 1.3, when (1.17) is replaced by (1.19), for (1.15) on a general manifold $\mathcal{M}$ is still an open problem.

For the classical space forms, as $\mathbb{R}^m$ and $\mathbb{H}^m$, condition (1.18) of Theorem 1.3 holds when $G(t) =$Const.> 0 for all $\sigma > 0$. Clearly, for the p–Laplace–Beltrami operators, $p > 1$, Theorem 1.3 applies for all exponents $\tau > \max\{1, p-1\}$ in (1.17).

In this paper we extend the above results to a larger class of elliptic differential inequalities by replacing $f = f(u)$ with a term of the type $B = B(x, u, \nabla u)$, and the differential operator $\operatorname{div}\{A(|\nabla u|)\nabla u\}$ by the more general $\operatorname{div}\{A(|\nabla u|)h(\nabla u, \cdot)^\sharp\}$, where h is a symmetric positive definite 2–covariant tensor field on $\mathcal{M}$ and $^\sharp$ denotes the musical isomorphism. In particular Theorems 1.1, 1.3 and Corollary 1.2 will be consequences respectively of Theorems 2.3, 3.3 and Corollary 3.1.

2. **Maximum principle at infinity.**

In this section we suppose the validity of (A1), (A2), (A3), and in order to introduce the more general operator alluded to in the introduction we consider condition

(H1) *h is a positive definite, symmetric, 2–covariant tensor field on $\mathcal{M}$ for which there exist functions α, λ, $\Lambda \in C(\mathbb{R}^+_0)$ such that for all $r \in \mathbb{R}^+$, $x \in \partial B_r$, $\boldsymbol{X} \in T_x\mathcal{M}$, $|\boldsymbol{X}| = 1$,*

$$(i) \quad 0 < \lambda(r) \le h(\boldsymbol{X}, \boldsymbol{X}) \le \Lambda(r), \qquad (ii) \quad |(\operatorname{div} h)(\boldsymbol{X})| \le \alpha(r).$$

We also require, without loss of generality, that

$$\liminf_{r\to\infty} \Lambda(r) > 0. \tag{2.1}$$

The operator we shall be concerned with is then defined by

$$\operatorname{div}\{A(|\nabla v|)h(\nabla v, \cdot)^\sharp\}, \tag{2.2}$$

where the vector field $h(\nabla v, \cdot)^\sharp$ is characterized by the property $\langle h(\nabla v, \cdot)^\sharp, \boldsymbol{Y}\rangle = h(\nabla v, \boldsymbol{Y})$ for each vector field $\boldsymbol{Y} \in \mathcal{X}(\mathcal{M})$. Note that the above definition makes sense for every function $v \in Lip_{\text{loc}}(\mathcal{M})$. Observe that in the special case when h is the metric $\langle \cdot, \cdot\rangle$ on $\mathcal{M}$, the operator in (2.2) reduces to $\operatorname{div}\{A(|\nabla u|)\nabla u\}$, with (H1) satisfied by the choices

$$\lambda(r) = \Lambda(r) \equiv 1, \qquad \alpha(r) \equiv 0.$$

We shall be interested also in cases in which $v(x) = u(r(x))$ is a radial function, with $u \in C^1(\mathbb{R}^+_0)$. Then an easy calculation yields

$$\operatorname{div}\{A(|\nabla v|)h(\nabla v, \cdot)^\sharp\} = h(\nabla r, \nabla r)\{A(|u'|)u'\}' + A(|u'|)u' \operatorname{div} h(\nabla r, \cdot)^\sharp,$$

in the weak sense in $\mathcal{M} \setminus \operatorname{cut}(O)$. Therefore, since

$$\operatorname{div} h(\boldsymbol{X}, \cdot)^\sharp = (\operatorname{div} h)(\boldsymbol{X}) + \langle \nabla \boldsymbol{X}^\flat, h\rangle, \tag{2.3}$$

with $\nabla \boldsymbol{X}^\flat$ determined by $(\nabla \boldsymbol{X}^\flat)(\boldsymbol{Y}, \boldsymbol{Z}) = \langle \nabla_{\boldsymbol{Y}} \boldsymbol{X}, \boldsymbol{Z}\rangle$, see [17], we get

$$\begin{aligned} \operatorname{div}\{A(|\nabla v|)h(\nabla v, \cdot)^\sharp\} = h(\nabla r, \nabla r)\{A(|u'|)u'\}' \\ + A(|u'|)u'[(\operatorname{div} h)(\nabla r) + \langle \operatorname{Hess} r, h\rangle]. \end{aligned} \tag{2.4}$$

In matrix notation

$$\langle \operatorname{Hess} r, h\rangle = \operatorname{tr}(\operatorname{Hess} r \cdot h) \quad \text{in } \mathcal{M} \setminus [\{O\} \cup \operatorname{cut}(O)]. \tag{2.5}$$

Furthermore, recalling the definition of Φ in $\mathbb{R}$, we rewrite (2.4) as

$$\begin{aligned}\operatorname{div}\{A(|\nabla v|)h(\nabla v,\cdot)^\sharp\} = h(\nabla r,\nabla r)\{\Phi(u')\}' \\ + [(\operatorname{div} h)(\nabla r) + \langle \operatorname{Hess} r, h\rangle]\,\Phi(u').\end{aligned} \tag{2.6}$$

When h is the metric $\langle\cdot,\cdot\rangle$ on $\mathcal{M}$, then (2.6) reduces to

$$\operatorname{div}\{A(|\nabla u|)\nabla u\} = [\Phi(u')]' + \Delta r\,\Phi(u').$$

It is also worth to observe that if we are on a model manifold $\mathcal{N} = \mathcal{N}(g)$ and $h = a(r)\langle\cdot,\cdot\rangle$, with $a \in C^1(\mathbb{R}_0^+)$, then (2.6) becomes

$$\operatorname{div}\{a(r)A(|\nabla u|)\nabla u\} = a(r)\left\{[\Phi(u')]' + \left[(m-1)\frac{g'}{g} + \frac{a'}{a}\right]\Phi(u')\right\}.$$

This operator often appears in geometrical problems. For instance, for $\Phi(\varrho) = \varrho$, it is used in the study of equivariant harmonic maps associated to large group actions. See, e.g., [11].

Next, let ${}^{\mathcal{M}}K_r$ denote the radial sectional curvatures of $\mathcal{M}$, that is the sectional curvatures evaluated over the 2–planes containing ∇r. From now on we also assume the validity of

($\mathcal{M}2$) *there exists a positive non–decreasing function $G \in C^1(\mathbb{R}_0^+)$ such that*

$$^{\mathcal{M}}K_r \geq -G(r);$$

($\mathcal{M}3$) *there exist a neighborhood U of ∞, a function ζ and a number $\eta > 0$, such that*

$$\zeta \in C^1(U), \quad 1/\zeta \notin L^1(\infty), \quad \zeta > 0, \quad \zeta' \geq 0 \quad \textit{in } U,$$
$$\zeta^\sigma \geq \eta\frac{\Lambda}{\lambda}[\alpha + \Lambda\sqrt{G}] \quad \textit{in } U,$$

where α, λ and Λ are the functions appearing in (H1).

Every complete manifold verifies ($\mathcal{M}2$) and it results

$$\operatorname{Hess} r \leq \frac{g'(r)}{g(r)}\,[\langle\cdot,\cdot\rangle - dr\otimes dr] \qquad \text{on } \mathcal{M}\setminus[\{O\}\cup\operatorname{cut}(O)], \tag{2.7}$$

where g is defined as in (1.5) and $D > 0$ is sufficiently large. Condition ($\mathcal{M}2$) implies ($\mathcal{M}1$) with the same G. Moreover, if $\mathcal{M} = \mathcal{M}(g)$ is a model, see (1.3), then (2.7) is valid with equality sign and with $\operatorname{cut}(O) = \emptyset$ by (1.4).

Fix $x \in \mathcal{M}$ and a local orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^m$ which diagonalizes h at x. Let $\lambda_k(x) > 0$, $k = 1,\ldots,m$, be the eigenvalues of h at x. Set $\nabla r = \sum_{i=1}^m r_i\boldsymbol{e}_i$, by Gauss' lemma

$$\sum_{i=1}^m r_i^2 = 1, \tag{2.8}$$

and by (2.5), (2.7) and (2.8), for $x \in \mathcal{M}\setminus[\{O\}\cup\operatorname{cut}(O)]$, we have

$$\begin{aligned}\langle\operatorname{Hess} r, h\rangle &= \sum_{i=1}^m(\operatorname{Hess} r)(\boldsymbol{e}_i,\boldsymbol{e}_i)h(\boldsymbol{e}_i,\boldsymbol{e}_i) \leq \sum_{i=1}^m \frac{g'(r)}{g(r)}(1-r_i^2)\lambda_i(x) \\ &\leq \frac{g'(r)}{g(r)}\left(m - \sum_{i=1}^m r_i^2\right)\max_k \lambda_k(x) = (m-1)\frac{g'(r)}{g(r)}\max_k\lambda_k(x),\end{aligned} \tag{2.9}$$

where the first inequality is an equality when $\mathcal{M} = \mathcal{M}(g)$ is a model.

We denote by k the function defined by

$$k(r) = \exp\left(\frac{1}{m-1}\int_R^r \frac{\mathcal{H}(s)}{\lambda(s)}\,ds\right), \tag{2.10}$$

with $\mathcal{H}$ given by

$$\mathcal{H}(r) = \alpha(r) + (m-1)\Lambda(r)g'(r)/g(r). \tag{2.11}$$

For $\varepsilon > 0$, $R > 0$, $w_R \in \mathbb{R}$ set

$$w_\varepsilon(r) = w_R + \int_R^r \Phi^{-1}\left(\varepsilon k^{1-m}(s)\int_R^s \frac{k^{m-1}(t)}{\Lambda(t)}dt\right) ds.$$

We have

Lemma 2.1. *For $\varepsilon > 0$ sufficiently small, there exists w_ε non–decreasing in $[R, \infty)$ and satisfying*

$$\begin{cases} \Lambda[k^{m-1}\Phi(w'_\varepsilon)]' = \varepsilon k^{m-1} & \text{in } (R,\infty), \\ w_\varepsilon(R) = w_R, \qquad w'_\varepsilon(R) = 0. \end{cases} \tag{2.12}$$

Furthermore $w'_\varepsilon > 0$ on (R, ∞) and

$$\lim_{r\to\infty} w_\varepsilon(r) = \infty. \tag{2.13}$$

Moreover the following hold

$$w_\varepsilon(r) \to w_R \quad \text{as } \varepsilon \to 0^+ \text{ uniformly on compact subsets of } [R,\infty), \tag{2.14}$$

$$w'_\varepsilon(r) \to 0 \quad \text{as } \varepsilon \to 0^+ \text{ uniformly on } [R,\infty). \tag{2.15}$$

Proof. For $r \geq R$ we set

$$S(r) = k^{1-m}(r)\int_R^r \frac{k^{m-1}(t)}{\Lambda(t)}dt. \tag{2.16}$$

Because of (H1) (i) and (2.1) we have

$$0 \leq S(r) \leq Ck^{1-m}(r)\int_R^r k^{m-1}(t)dt \tag{2.17}$$

for some $C > 0$. Set

$$\beta(r) = \frac{\alpha(r) + (m-1)\Lambda(r)(g'/g)(r)}{\lambda(r)} = \frac{\mathcal{H}(r)}{\lambda(r)}. \tag{2.18}$$

Since $g'/g \sim D\sqrt{G}$ as $r \to \infty$ by (1.5) for D sufficiently large, it is easy to check that

$$\beta(r) \geq N\sqrt{G(r)}$$

for an appropriate constant $N > 0$. Hence

$$\liminf_{r\to\infty} \beta(r) > 0.$$

Applying de l'Hôpital's rule to the right hand side of (2.17), we see that

$$S^* = \sup_{[R,\infty)} S(r) < \infty.$$

We can now choose $\bar{\varepsilon} > 0$ so that for each $\varepsilon \in (0, \bar{\varepsilon})$

$$\varepsilon S(r) \in [0, \omega) = \Phi(\mathbb{R}_0^+) \quad \text{for all } r \geq R.$$

This shows that for such values of ε the function w_ε is well defined on $[R,\infty)$. It is now a simple checking to verify the validity of (2.12). To prove (2.13), according to (2.16), it is enough to show that

$$\Phi^{-1}(\varepsilon S(r)) \geq \frac{C}{\zeta(r)} \quad \text{for } r >> 1 \tag{2.19}$$

and for some constant $C>0$, where ζ is as in $(\mathcal{M}3)$. This follows from

$$\frac{S(r)}{\Phi(C/\zeta(r))} \geq \frac{1}{\varepsilon} \quad \text{for } r >> 1. \tag{2.20}$$

Now we have

$$\frac{S(r)}{\Phi(C/\zeta(r))} = \frac{\displaystyle\int_R^r \frac{k^{m-1}(t)}{\Lambda(t)}dt}{k^{m-1}(r)\Phi(C/\zeta(r))}. \tag{2.21}$$

We assume, without loss of generality, that $\zeta(r) \to \infty$ as $r \to \infty$, so that, using (A3), we obtain for $r >> 1$

$$\frac{S(r)}{\Phi(C/\zeta(r))} \geq \frac{\zeta(r)^\sigma \displaystyle\int_R^r \frac{k^{m-1}(t)}{\Lambda(t)}dt}{c\, C^\sigma k^{m-1}(r)} := \frac{\Xi(r)}{\Upsilon(r)}. \tag{2.22}$$

Since $\Xi(r) \to \infty$ as $r \to \infty$, then (2.19) is trivially satisfied if $\Upsilon(r) = O(1)$ as $r \to \infty$. Otherwise,

$$\liminf_{r\to\infty} \frac{\Xi(r)}{\Upsilon(r)} \geq \liminf_{r\to\infty} \frac{\Xi'(r)}{\Upsilon'(r)}. \tag{2.23}$$

Using $(\mathcal{M}3)$, we have

$$\Xi'(r) \geq \zeta(r)^\sigma \frac{k^{m-1}(r)}{\Lambda(r)}, \qquad \Upsilon'(r) = (m-1)c\, C^\sigma k^{m-2}(r)k'(r).$$

Therefore by $(\mathcal{M}3)$ again for $r >> 1$

$$\frac{\Xi'(r)}{\Upsilon'(r)} \geq \frac{1}{(m-1)cC^\sigma D} \cdot \frac{\zeta(r)^\sigma}{\Lambda(r)} \cdot \frac{k(r)}{k'(r)} \geq \frac{\eta}{cC^\sigma} \cdot \frac{\beta(r)}{m-1} \cdot \frac{k(r)}{k'(r)} \geq \frac{\eta}{cC^\sigma}. \tag{2.24}$$

Using (2.22)–(2.24), we can thus choose $C>0$ sufficiently small (and depending on ε) so that (2.20) is satisfied, and in turn also (2.19). Therefore (2.13) holds.

Now we prove (2.15). Since Φ^{-1} is monotone increasing, we have

$$w'_\varepsilon(r) = \Phi^{-1}(\varepsilon S(r)) \leq \Phi^{-1}(\varepsilon S^*).$$

Since $\Phi^{-1}(0^+) = 0$, it follows that

$$\max_{[R,\infty)} w'_\varepsilon(r) \leq \sup_{[R,\infty)} \Phi^{-1}(\varepsilon S^*) \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Of course (2.14) now follows from (2.15). ☐

In what follows we shall need the following comparison result which is a special case of Theorem 5.3 of [17]. Since in this case the proof is very simple, for the sake of completeness, we repeat it here. For further applications via the comparison principle see [19].

From now on we also assume

(H2) *for all $x \in \mathcal{M}$ and for all $\boldsymbol{\xi} \in T_x\mathcal{M}$, $\boldsymbol{\xi} \neq \mathbf{0}$, the bilinear form*

$$\frac{A'(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\langle \boldsymbol{\xi}, \cdot\rangle \odot h(\boldsymbol{\xi}, \cdot) + A(|\boldsymbol{\xi}|)h(\cdot, \cdot)$$

is symmetric and positive definite.

With $\odot$ we shall indicate the symmetric tensor product. Thus if ω_1, ω_2 are 1–forms on $\mathcal{M}$,

$$\omega_1 \odot \omega_2 = \tfrac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1).$$

The symmetry of the expression in (H2) is equivalent to the symmetry of h.

Note that if $h = a(x)\langle \cdot, \cdot\rangle$, $a > 0$, by assumption (A2)

$$\varrho A'(\varrho) > -A(\varrho) \qquad \text{on } \mathbb{R}^+$$

and therefore (H2) is satisfied if

$$a(x)A(|\boldsymbol{\xi}|)\left(|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2 - |\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle|^2\right) \geq 0 \tag{2.25}$$

for all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in T_x\mathcal{M} \setminus \{\mathbf{0}\}$, which is of course valid thanks to the Schwarz inequality. In particular, (H2) is automatic when h is the metric on $\mathcal{M}$.

For a wide discussion on the validity of (H2) – that is for its positive definiteness – when $\mathcal{M}$ reduces to $\mathbb{R}^m$ and the divergence part $\operatorname{div}\{A(|\nabla v|)h(\nabla v, \cdot)^\sharp\}$ is of the form $\partial_i[A(|\nabla u|)a_{ij}(x, u)\partial_j u]$, we refer to [3].

Lemma 2.2. *Let Ω be a relatively compact domain in $\mathcal{M}$. Let u, $v \in Lip_{\mathrm{loc}}(\Omega)$ satisfy*

$$\begin{cases} \operatorname{div}\{A(|\nabla u|)h(\nabla u, \cdot)^\sharp\} \geq \operatorname{div}\{A(|\nabla v|)h(\nabla v, \cdot)^\sharp\} & \text{in } \Omega, \\ u \leq v \quad \text{on } \partial\Omega. \end{cases} \tag{2.26}$$

Then $u \leq v$ in Ω.

Proof. First, we observe that, because of (H2) and compactness of $\overline{\Omega}$, there exists $\lambda > 0$ such that in Ω

$$L(\nabla v, \nabla u) := \langle A(|\nabla v|)h(\nabla v, \cdot)^\sharp - A(|\nabla u|)h(\nabla u, \cdot)^\sharp, \nabla v - \nabla u\rangle \geq \lambda|\nabla v - \nabla u|^2.$$

This is easily seen, with the aid of (H2) once we realize that

$$L(\nabla v, \nabla u) = \int_0^1 \bigg\{ h(\boldsymbol{X}_t, \nabla v - \nabla u)\langle \boldsymbol{X}_t, \nabla v - \nabla u\rangle \frac{A'(|\boldsymbol{X}_t|)}{|\boldsymbol{X}_t|} + A(|\boldsymbol{X}_t|)h(\nabla v - \nabla u, \nabla v - \nabla u)\bigg\} dt,$$

where $\boldsymbol{X}_t = t\nabla v + (1-t)\nabla u \neq \mathbf{0}$, $t \in [0, 1]$.

Let $w = v - u$ *and assume for contradiction the existence of* $x_0 \in \Omega$ such that $w(x_0) < 0$. Fix $\varepsilon > 0$ so small that $w(x_0) + \varepsilon < 0$ and set $w_\varepsilon = \min\{w + \varepsilon, 0\}$. Clearly $w_\varepsilon = 0$ on $\partial\Omega$ since $u \leq v$ on $\partial\Omega$. Thus $-w_\varepsilon$ is a non–negative compactly supported Lipschitz continuous function which we can use as a test function for (2.26). Let Ω_{x_0} be the connected component of $\{x \in \Omega : w(x) + \varepsilon < 0\}$ containing x_0. Then

$$\begin{aligned} \int_{\Omega_{x_0}} \lambda|\nabla v - \nabla u|^2 &\leq \int_{\Omega_{x_0}} L(\nabla v, \nabla u) \\ &= \int_{\Omega_{x_0}} \langle A(|\nabla v|)h(\nabla v, \cdot)^\sharp - A(|\nabla u|)h(\nabla u, \cdot)^\sharp, \nabla w_\varepsilon\rangle \leq 0, \end{aligned}$$

so that $\nabla u = \nabla v$ a.e. in Ω_{x_0}. Therefore, by a connectedness argument, $\Omega_{x_0} = \Omega$ and $v = u + w(x_0)$ in Ω, with $w(x_0) < 0$. This contradicts the fact that $u \le v$ on $\partial\Omega$. □

The next requirement relates h with the geometry of $\mathrm{cut}(O)$ and it will be assumed throughout the rest of the paper.

(H3) *Let either* $\mathrm{cut}(O) = \emptyset$ *or suppose the existence of a telescoping sequence of smooth domains* $\{\Omega_n\}_n$ *exhausting* $\mathcal{M} \setminus \mathrm{cut}(O)$ *such that, denoting with* ν_n *the exterior unit normal to* $\partial\Omega_n$*, one has* $h(\nabla r, \nu_n) \ge 0$.

By Yau [20] property (H3) is automatically satisfied whenever $h = a(x)\langle\cdot,\cdot\rangle$ for some positive function a.

With this preparation we are now able to establish the following

Theorem 2.3. (Maximum principle at infinity). *Assume that* $u \in C^1(\mathcal{M})$ *is such that* $u^* = \sup_{\mathcal{M}} u < \infty$. *Then for each* $\wp > 0$

$$\inf_{E_\wp} \mathrm{div}\{A(|\nabla u|)h(\nabla u, \cdot)^\sharp\} \le 0 \tag{2.27}$$

holds in the weak sense, where

$$E_\wp = \{x \in \mathcal{M} \ : \ u(x) > u^* - \wp, \ \ |\nabla u(x)| < \wp\}. \tag{2.28}$$

Proof. We argue by contradiction. Suppose that for some $\wp > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varphi \in C_0^\infty(E_\wp)$, $\varphi \ge 0$,

$$\int_{E_\wp} \{A(|\nabla u|)h(\nabla u, \nabla\varphi) + \varepsilon_0\varphi\} \le 0. \tag{2.29}$$

First note that u^* cannot be achieved at any point $x_0 \in \mathcal{M}$, for otherwise $x_0 \in E_\wp$ and on the open set $E_\wp$ it holds weakly

$$\mathrm{div}\{A(|\nabla u|)h(\nabla u, \cdot)^\sharp\} \ge 0.$$

Thus u is constant in every connected component of $E_\wp$ by Lemma 2.2 (see also the comparison Theorem 5.3 of [17]), applied to $v = u - u^*$. This contradicts (2.29) because $\varepsilon_0 > 0$.

Since u^* is not attained in $\mathcal{M}$, there is a divergent sequence $\{r_j\}_j$ such that

$$\sup_{\partial B_{r_j}} u \to u^* \qquad \text{as } j \to \infty. \tag{2.30}$$

Choose $R > 0$ in such a way that

$$u_R^* := \sup_{B_R} u > u^* - \wp.$$

To simplify the reasoning we first assume that O is a pole so that r is smooth on $\mathcal{M} \setminus \{O\}$.

Next fix $w_R \in (u_R^*, u^*)$ and choose $\varepsilon \in (0, \varepsilon_0)$ sufficiently small to apply Lemma 2.1. Define

$$v_\varepsilon(x) = w_\varepsilon(r(x)) \qquad \text{on } \mathcal{M}_R = \mathcal{M} \setminus \overline{B_R},$$

where w_ε is given in Lemma 2.1. Then, according to (2.13), we have

$$v_\varepsilon(x) \to \infty \qquad \text{as } r(x) \to \infty. \tag{2.31}$$

Now, since w_ε satisfies (2.12), from the definition of k in (2.10), we see that w_ε satisfies the following problem

$$\begin{cases} [\Phi(w_\varepsilon')]' + \beta(r)\Phi(w_\varepsilon') - \varepsilon/\Lambda(r) \le 0 & \text{in } (R,\infty),\\ w_\varepsilon(R) = w_R \ge 0, \qquad w_\varepsilon'(R) = 0, \qquad w_\varepsilon' \ge 0 & \text{in } [R,\infty), \end{cases}$$

where β is the function given in (2.18). Thus, using (2.6), (2.9) and (H1), similarly to what has been shown in the proof of Lemma 4.1 of [17], we have that v_ε is a classical solution of

$$\begin{cases} \operatorname{div}\{A(|\nabla v_\varepsilon|)h(\nabla v_\varepsilon, \cdot)^\sharp\} \le \varepsilon & \text{in } \mathcal{M}_R,\\ v_\varepsilon \ge 0 \quad \text{in } \mathcal{M}_R, \qquad v_\varepsilon(x) = w_R & \text{for } x \in \partial B_R. \end{cases} \tag{2.32}$$

We claim that if ε is sufficiently small, then $u - v_\varepsilon$ attains a positive maximum M_ε in $\mathcal{M}_R$. Indeed, by (2.30) we can choose N sufficiently large so that, having set $R_1 = r_N$, we obtain

$$R_1 > R \qquad \text{and} \qquad \sup_{\partial B_{R_1}} u > w_R.$$

Select $\bar\wp > 0$ so small that $w_R + \bar\wp < \sup_{\partial B_{R_1}} u$. Finally, according to Lemma 2.1 and (2.14), we choose $\varepsilon = \varepsilon(R_1, \bar\wp) \in (0, \varepsilon_0)$ so small that

$$w_R \le w_\varepsilon(r) \le w_R + \bar\wp \quad \text{in } [R, R_1].$$

For every such ε we have

$$v_\varepsilon(x) = w_\varepsilon(R) = w_R > \sup_{\overline{B_R}} u \ge \sup_{\partial B_R} u \ge u(x) \qquad \text{for all } x \in \partial B_R,$$

so that

$$u - v_\varepsilon < 0 \qquad \text{on } \partial B_R. \tag{2.33}$$

Furthermore, if $\bar x \in \partial B_{R_1}$ is such that $\sup_{\partial B_{R_1}} u = u(\bar x)$, then

$$u(\bar x) - v_\varepsilon(\bar x) = \sup_{\partial B_{R_1}} u - w_\varepsilon(R_1) \ge \sup_{\partial B_{R_1}} u - w_\varepsilon(R) - \bar\wp > 0.$$

Finally (2.31) and the assumption that $u^* < \infty$ imply that

$$(u - v_\varepsilon)(x) < 0 \qquad \text{for } r(x) >> 1. \tag{2.34}$$

Thus, $u - v_\varepsilon$ achieves its absolute, positive maximum M_ε in $\mathcal{M}_R$, proving the claim.

Moreover the set

$$\Gamma_\varepsilon = \{x \in \mathcal{M}_R \ : \ (u - v_\varepsilon)(x) = M_\varepsilon\}$$

is compact by (2.31) and (2.33).

Our next goal is to show that, up to choosing $\varepsilon > 0$ small enough,

$$\Gamma_\varepsilon \subset E_\wp. \tag{2.35}$$

Towards this end, we first observe that for every $\tau > 0$ there exists $\varepsilon_1 = \varepsilon_1(\tau) > 0$ such that whenever $0 < \varepsilon < \varepsilon_1$

$$v_\varepsilon(x) < \tau + w_R \qquad \text{for all } x \in \Gamma_\varepsilon.$$

Indeed, $v_\varepsilon(x) = w_\varepsilon(r(x))$ and $r(\Gamma_\varepsilon) \subset (R, \infty)$ is compact. Therefore we can use property (2.14) of Lemma 2.1. Next, from (2.15) and Gauss' lemma for each $\tau > 0$ there exists $\varepsilon_2 = \varepsilon_2(\tau) > 0$ such that if $0 < \varepsilon < \varepsilon_2$

$$|\nabla v_\varepsilon(x)| = w_\varepsilon'(r(x)) < \tau \qquad \text{for all } x \in \Gamma_\varepsilon.$$

We may therefore choose $\varepsilon > 0$ so small that

$$u(x) > u^* - \wp/2 \quad \text{and} \quad |\nabla v_\varepsilon(x)| < \wp/2 \quad \text{in } \Gamma_\varepsilon.$$

Since $|\nabla u(x)| = |\nabla v_\varepsilon(x)|$ for each $x \in \Gamma_\varepsilon$, by definition of Γ_ε, inclusion (2.35) is valid.

In particular, since $E_\wp$ is open and Γ_ε is compact, there is a small neighborhood of Γ_ε contained in $E_\wp$. Now, pick a point $y \in \Gamma_\varepsilon$, fix $\tau \in (0, M_\varepsilon)$ and call $\Omega_{\tau,y}$ the connected component containing y of the set

$$\{x \in \mathcal{M}_R \ : \ (u - v_\varepsilon)(x) > \tau\}.$$

Clearly $\Omega_{\tau,y}$ is bounded by (2.34), $y \in \Omega_{\tau,y}$ and $\overline{\Omega_{\tau,y}} \subset \mathcal{M}_R$, since $u - v_\varepsilon < 0$ on ∂B_R. Furthermore, $u = v_\varepsilon + \tau$ on $\partial\Omega_{\tau,y}$ and

$$u(x) > v_\varepsilon(x) + \tau \geq w_R > \sup_{B_R} u > u^* - \wp \qquad \text{in } \Omega_{\tau,y}.$$

Therefore, by (2.35) we can choose $\tau > 0$ sufficiently near M_ε so that $\overline{\Omega_{\tau,y}} \subset E_\wp$. But, according to (2.32) and (2.29) in $\Omega_{\tau,y}$, we have for all $\varphi \in C_0^\infty(\Omega_{\tau,y})$, $\varphi \geq 0$,

$$\int_{\Omega_{\tau,y}} A(|\nabla u|)h(\nabla u, \nabla\varphi) \leq -\varepsilon_0 \int_{\Omega_{\tau,y}} \varphi < -\varepsilon \int_{\Omega_{\tau,y}} \varphi \leq \int_{\Omega_{\tau,y}} A(|\nabla v_\varepsilon|)h(\nabla v_\varepsilon, \nabla\varphi).$$

This contradicts Proposition 6.1, Remark 6.1, of [16], and completes the proof in the case in which O is a pole.

It remains to consider the case in which O is not a pole, but this can be dealt exactly as in the proof of Theorem 6.4 of [16]. ◻

Remarks. Theorem 2.3 continues to hold if $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{M})$ provided that the set $E_\wp$ is replaced by the larger open set

$$\tilde{E}_\wp = \{x \in \mathcal{M} \ : \ u(x) > u^* - \wp\}. \tag{2.36}$$

This can be easily recognized by a careful inspection of the proof above.

Theorem 1.1 is a direct consequence of Theorem 2.3 in case h is the metric $\langle \cdot, \cdot \rangle$, since conditions (H1) and (H2) are automatic. Indeed, in this case div $h = 0$, with $\lambda = \Lambda = 1$ and $\alpha = 0$, while (H2) follows from (A2) and Schwarz' inequality, see (2.25).

Corollary 2.4. *In the assumptions of Theorem 2.3 let u be a semi–classical solution with $u^* < \infty$ of*

$$\mathrm{div}\{A(|\nabla u|)h(\nabla u, \cdot)^\sharp\} \geq f(u) \qquad \text{in } E_\wp, \tag{2.37}$$

for some $f \in C(\mathbb{R})$. Then $f(u^*) \leq 0$.

Proof. By contradiction assume that $f(u^*) \geq 2\varepsilon > 0$. Fix $\wp > 0$ sufficiently small that on the open set $\tilde{E}_\wp$ in (2.36), we have $f(u(x)) \geq \varepsilon$. Let $\tilde{\varphi} \in C_0^\infty(\tilde{E}_\wp)$, $\tilde{\varphi} \geq 0$, then by definition of semi–classical solution of (2.37) we have

$$\int_{\tilde{E}_\wp} A(|\nabla u|)h(\nabla u, \nabla\tilde{\varphi}) \leq -\varepsilon \int_{\tilde{E}_\wp} \tilde{\varphi}.$$

This contradicts (2.27) of Theorem 2.3 and the above remark. ◻

Corollary 2.5. *In the assumptions of Theorem 2.3 let u be a classical solution with $u^* < \infty$ of*

$$\operatorname{div}\{A(|\nabla u|)h(\nabla u, \cdot)^\sharp\} \geq B(x, u, \nabla u) \qquad \text{in } E_\wp, \tag{2.38}$$

where B satisfies (B) *for some f assumed only continuous in $\mathbb{R}$. Then $f(u^*) \leq 0$.*

Proof. By contradiction assume that $f(u^*) \geq 2\varepsilon > 0$. Fix $\wp > 0$ sufficiently small that on the open set $E_\wp$ in (2.28), we have $f(u(x)) \geq \varepsilon$. Let $\varphi \in C_0^\infty(E_\wp)$, $\varphi \geq 0$, then by definition of classical solution of (2.38) we have

$$\int_{E_\wp} A(|\nabla u|)h(\nabla u, \nabla\varphi) \leq -\int_{E_\wp} B(x, u, \nabla u)\varphi \leq \kappa \int_{E_\wp} T(|\nabla u|)\varphi - \varepsilon \int_{E_\wp} \tilde{\varphi}.$$

By the properties of T in (B), this easily contradicts (2.27) of Theorem 2.3. □

3. **Non–existence theorems.** The conclusions of Corollaries 2.4 and 2.5 can be used to prove the following

Corollary 3.1. (Non–existence result.) *In the assumptions of Theorem 2.3 let f satisfy* (F1). *Then* (2.37) *has no positive bounded semi–classical entire solutions. Furthermore, under assumption* (B), *inequality* (2.38) *admits no positive bounded classical entire solutions.*

Note that in both cases of Corollary 3.1 it is no longer necessary to require that $f(0) = 0$.

From now on we assume (A1), (A2), $\Phi(\infty) = \infty$, and (F1). We shall now remove the assumption $u^* < \infty$. This will be achieved via a comparison principle given in Theorem 5.3 of [17], with the careful construction of the comparison function contained in the next

Lemma 3.2. *Assume* (1.17) *and*

$$\text{(A3)}' \qquad \Phi(\rho) \leq c\rho^\sigma \qquad \text{in } \mathbb{R}_0^+$$

for some c, $\sigma > 0$. Let $b \in C(\mathbb{R}_0^+)$ be positive, with

$$\sup_{t\in\mathbb{R}_0^+} b(t) < \infty; \qquad b(t) \geq dt^{-\mu} \quad \text{for } t >> 1, \tag{3.1}$$

for some $d > 0$ and for some $\mu \in [0, 1+\sigma)$. Let $\tilde{g} \in C^\infty(\mathbb{R}_0^+)$ satisfy (g1), (g2) *and*

$$\liminf_{t\to\infty} t^{\mu-\sigma-1} \log \int_0^t \tilde{g}^{n-1}(s)ds < \infty \tag{3.2}$$

for some $n \in \mathbb{N}$, with $n \geq 2$.

Let $w = w(t)$ be a C^1 solution defined on its maximal interval of definition $[0, T)$ of the problem

$$\begin{cases} [\tilde{g}^{n-1}\Phi(w')]' = \tilde{g}^{n-1}b(t)f(w), \\ w(0) = w_0 > 0, \qquad w'(0) = 0. \end{cases} \tag{3.3}$$

Then $T < \infty$, $w' > 0$ in $(0, T)$, and

$$\lim_{t\to T^-} w(t) = \infty. \tag{3.4}$$

Proof. We divide the argument into several steps following some lines in [16].

Step 1. Integrating (3.3) over $[0,t]$, $t \in (0,T)$ and using the fact that $\tilde{g}(0) = 0$, (A2) and $w'(0) = 0$, we obtain

$$\Phi(w'(t)) = \tilde{g}^{1-n}(t) \int_0^t \tilde{g}^{n-1}(s) b(s) f(w(s)) ds.$$

Since $b(t) > 0$ for $t > 0$, $f(u) > 0$ for $u > 0$ and since Φ is invertible from $\mathbb{R}_0^+$ onto $\mathbb{R}_0^+$, we get

$$w'(t) = \Phi^{-1}\left(\tilde{g}^{1-n}(t) \int_0^t \tilde{g}^{n-1}(s) b(s) f(w(s)) ds\right), \tag{3.5}$$

whence the positivity of w' on $(0,T)$. From the initial data in (3.3) it follows that $w(t) > 0$ on $[0,T)$ and there exists

$$\lim_{t \to T^-} w(t). \tag{3.6}$$

Step 2. Now reason by contradiction and suppose $T = \infty$. We shall show in this case that

$$w^* = \sup_{t \in \mathbb{R}_0^+} w(t) < \infty.$$

Indeed, assume the contrary and let $\gamma_0 \geq 0$ be so large that by (1.17) there is $a > 0$ with

$$f(w(t)) \geq a w(t)^\tau \quad \text{for } t \in \Omega_\gamma = \{t \in \mathbb{R}_0^+ \ : \ w(t) > \gamma\} \neq \emptyset$$

for each $\gamma \geq \gamma_0$. Note that for any fixed $\gamma \geq \gamma_0$ in Ω_γ the function w solves the differential inequality

$$[\tilde{g}^{n-1}\Phi(w')]' \geq a\tilde{g}^{n-1} b w^\tau. \tag{3.7}$$

We choose $R > 0$ sufficiently large so that

$$[r_\gamma, R) = [0,R) \cap \Omega_\gamma \neq \emptyset, \qquad w(r_\gamma) = \gamma.$$

Since $\tau > \max\{1,\ \sigma\}$ it is possible to find $\zeta > 1$ such that

$$(i) \quad 2 + \frac{1+\sigma}{\tau - 1}\left(\frac{1}{\zeta} - 1\right) \geq 0, \qquad\qquad (ii) \quad 2 + \frac{1+\sigma}{\tau-\sigma}\left(\frac{1}{\zeta} - 1\right) \geq 0. \tag{3.8}$$

Having fixed such a ζ, we let $r \geq R$ and choose $\psi : \mathbb{R}_0^+ \to [0,1]$, a smooth cut–off function, with the properties

$$(i) \quad \psi \equiv 1 \quad \text{on } [0,r], \qquad (ii) \quad \psi \equiv 0 \quad \text{on } [2r, \infty), \qquad (iii) \quad |\psi'| \leq \frac{C}{r}\psi^{1/\zeta} \tag{3.9}$$

for some constant $C = C(\zeta) > 0$. We note that this is possible since $\zeta > 1$. Let $\phi : \mathbb{R} \to \mathbb{R}_0^+$ be a C^1 non–decreasing function such that

$$\phi(u) = 0 \quad \text{for } u \leq \gamma \quad \text{and} \quad \phi(u) = 1 \quad \text{for } u \geq \gamma + 1. \tag{3.10}$$

Fix $\alpha > 1$ and multiply the right hand side of (3.7) by $\psi^{2(\alpha+\tau-1)}\phi(w)w^{\alpha-1}$. We integrate by parts on $[0, 2r]$ and using (3.7), (A3)$'$ and the fact that $\phi' \geq 0$

$$\begin{aligned}\int_0^{2r} a\psi^{2(\alpha+\tau-1)}\phi(w)w^{\alpha+\tau-1}b\tilde{g}^{n-1}dt \\ \leq \int_0^{2r} 2(\alpha+\tau-1)\psi^{2(\alpha+\tau-1)-1}\phi(w)w^{\alpha-1}\Phi(w')|\psi'|\tilde{g}^{n-1}dt \\ - \int_0^{2r} \frac{\alpha-1}{c^{1/\sigma}}\psi^{2(\alpha+\tau-1)}\phi(w)w^{\alpha-2}\Phi(w')^{1+1/\sigma}\tilde{g}^{n-1}dt.\end{aligned} \tag{3.11}$$

Since $\alpha > 1$, we set

$$p = 1 + \frac{1}{\sigma}, \qquad p' = 1 + \sigma, \qquad \varepsilon = \left(\frac{\alpha-1}{c^{1/\sigma}}\right)^{\sigma/(1+\sigma)} > 0,$$

and use the inequality

$$\xi\eta \leq \frac{\varepsilon^p \xi^p}{p} + \frac{\eta^{p'}}{p'\varepsilon^{p'}},$$

valid for all ξ, $\eta \geq 0$, to obtain

$$\begin{aligned}\int_0^{2r} 2(\alpha+\tau-1)\psi^{2(\alpha+\tau-1)-1}\phi(w)w^{\alpha-1}\Phi(w')|\psi'|\tilde{g}^{n-1}dt \\ \leq \int_0^{2r} \frac{\alpha-1}{c^{1/\sigma}}\psi^{2(\alpha+\tau-1)}\phi(w)w^{\alpha-2}\Phi(w')^p\tilde{g}^{n-1}dt \\ + C\frac{(\alpha+\tau-1)^{1+\sigma}}{(\alpha-1)^\sigma}\int_0^{2r}\psi^{2(\alpha+\tau-1)}\phi(w)w^{\alpha+\sigma-1}(\psi^{-1}|\psi'|)^{p'}\tilde{g}^{n-1}dt\end{aligned}$$

for some constant $C = C(c, \sigma) > 0$. Hence, inserting into (3.11) and using the fact that $\phi \geq 0$, we get

$$\begin{aligned}\int_0^{2r} \psi^{2(\alpha+\tau-1)}\phi(w)w^{\alpha+\tau-1}b\tilde{g}^{n-1}dt \\ \leq C\frac{(\alpha+\tau-1)^{1+\sigma}}{(\alpha-1)^\sigma}\int_0^{2r}\psi^{2(\alpha+\tau-1)+(-1+1/\zeta)p'} \\ \cdot \phi(w)w^{\alpha+\sigma-1}(\psi^{-1/\zeta}|\psi'|)^{p'}\tilde{g}^{n-1}dt,\end{aligned} \tag{3.12}$$

with $C = C(c, \sigma, a) > 0$.

Let $\tilde{p}$ and $\tilde{p}'$ be conjugate exponents to be chosen later. Since $b > 0$ in $\mathbb{R}_0^+$, by (3.9) (iii) we obtain

$$\begin{aligned}W = \int_0^{2r}\psi^{2(\alpha+\tau-1)+(-1+1/\zeta)p'}\phi(w)w^{\alpha+\sigma-1}(\psi^{-1/\zeta}|\psi'|)^{p'}\tilde{g}^{n-1}dt \\ \leq \left\{\int_0^{2r}\psi^{2(\alpha+\tau-1)}\phi(w)w^{(\alpha+\sigma-1)\tilde{p}}b\,\tilde{g}^{n-1}dt\right\}^{1/\tilde{p}} \\ \cdot \left\{\int_0^{2r}\psi^{2(\alpha+\tau-1)+\tilde{p}'p'(-1+1/\zeta)}\phi(w)b^{1-\tilde{p}'}\tilde{g}^{n-1}dt\right\}^{1/\tilde{p}'}\frac{C}{r^{p'}}\end{aligned} \tag{3.13}$$

for some constant $C = C(\zeta) > 0$.

Next we need to consider two cases separately.

Case $\sigma \le 1$. Since $\tau > 1$, we can choose

$$\tilde{p} = p_1 = \frac{\alpha + \tau - 1}{\alpha} > 1, \qquad \tilde{p}' = p_1' = \frac{\alpha + \tau - 1}{\tau - 1}.$$

Note that $(\sigma - 1)/\alpha \le 0$ and therefore

$$w^{(\alpha+\sigma-1)/\alpha} \le \gamma^{(\sigma-1)/\alpha} w \qquad \text{on } \Omega_\gamma.$$

Using this latter, (3.13), (3.8) (i), (3.1) and the fact that $\psi \le 1$ yields

$$W \le \frac{C\gamma^{\sigma-1}}{r^{p'-\mu}} \left\{ \int_0^{2r} \psi^{2(\alpha+\tau-1)} \phi(w) b w^{\alpha+\tau-1} \tilde{g}^{n-1} dt \right\}^{1/p_1} \cdot \left\{ \int_0^{2r} \phi(w) b \tilde{g}^{n-1} dt \right\}^{1/p_1'},$$

and inserting into (3.12) gives

$$\begin{aligned} &\int_0^{2r} \psi^{2(\alpha+\tau-1)} \phi(w) w^{\alpha+\tau-1} b \tilde{g}^{n-1} dt \\ &\qquad \le \left\{ \frac{C(\alpha+\tau-1)^{p'} \gamma^{\sigma-1}}{(\alpha-1)^{\sigma} r^{p'-\mu}} \right\}^{p_1'} \left\{ \int_0^{2r} \phi(w) b \tilde{g}^{n-1} dt \right\}, \end{aligned} \tag{3.14}$$

with $C = C(c, \sigma, a, \zeta) > 0$. Therefore, since contributions to the integral are obtained only for $w \ge \gamma$, putting $\tilde{\tau} = \tau - \max\{1,\ \sigma\} = \tau - 1$, we have

$$\int_0^{r} \phi(w) b \tilde{g}^{n-1} dt \le \left\{ \frac{C(\alpha+\tau-1)^{p'} \gamma^{\sigma-\tau}}{(\alpha-1)^{\sigma} r^{p'-\mu}} \right\}^{(\alpha+\tau-1)/\tilde{\tau}} \int_0^{2r} \phi(w) b \tilde{g}^{n-1} dt. \tag{3.15}$$

Case $\sigma > 1$. This case is similar to the above and one proceeds from (3.13) with the choices

$$\tilde{p} = p_2 = \frac{\alpha + \tau - 1}{\alpha + \sigma - 1} > 1, \qquad \tilde{p}' = p_2' = \frac{\alpha + \tau - 1}{\tau - \sigma},$$

observing that

$$w^{\alpha/(\alpha+\sigma-1)} \le \gamma^{(1-\sigma)/(\alpha+\sigma-1)} w \quad \text{on } \Omega_\gamma$$

to get (3.15) also in this case, with $\tilde{\tau} = \tau - \max\{1,\ \sigma\} = \tau - \sigma$. Next we set

$$\tilde{G}(r) = \int_0^r \phi(w) b \tilde{g}^{n-1} dt,$$

so that (3.15) becomes

$$\tilde{G}(r) \le \left\{ \frac{C(\alpha+\tau-1)^{p'}}{\gamma^{\tau-\sigma}(\alpha-1)^{\sigma} r^{p'-\mu}} \right\}^{(\alpha+\tau-1)/\tilde{\tau}} \tilde{G}(2r). \tag{3.16}$$

We choose

$$\alpha = \alpha(r) = \frac{\gamma^{\tau-\sigma}}{8C} r^{p'-\mu},$$

so that, up to have chosen $R > 0$ sufficiently large,

$$\frac{(\alpha+\tau-1)^{p'}}{(\alpha-1)^{\sigma}} \le 4\alpha \quad \text{for each } r \ge R.$$

It follows that, for some appropriate constant $c_1 > 0$, independent of γ, and $r \ge R$, from (3.16) we deduce that

$$\tilde{G}(r) \le 2^{-c_1 r^{p'-\mu} \gamma^{\tau-\sigma}} \tilde{G}(2r), \qquad r \ge R, \tag{3.17}$$

with $\tilde{G}:[R,\infty)\to\mathbb{R}_0^+$ non–decreasing and $p'-\mu>0$. Applying Lemma 4.7 of [16], we obtain the existence of a constant $c_2=c_2(\sigma,\mu)>0$ such that, for each $r\geq 2R$,

$$r^{\mu-p'}\log\int_0^r\phi(w)b\tilde{g}^{n-1}dt\geq r^{\mu-p'}\log\int_0^R\phi(w)b\tilde{g}^{n-1}dt+c_1c_2\gamma^{\tau-\sigma}\log 2. \tag{3.18}$$

Next, since $\sup b<\infty$, $\phi\equiv 1$ for $t\geq\gamma+1$, we have

$$r^{\mu-p'}\log\int_0^r\tilde{g}^{n-1}dt\geq r^{\mu-p'}\log\int_0^R\phi(w)b\tilde{g}^{n-1}dt+c_1c_2\gamma^{\tau-\sigma}\log 2,$$

for some $C>0$. Then, observing that $\mu-p'=\mu-1-\sigma$ and choosing r and γ sufficiently large, we contradict (3.2). It follows that $w^*<\infty$.

Step 3. Having proved that $w^*<\infty$, we now contradict $T=\infty$. Towards this end we consider the model manifold $\mathcal{N}=\mathbb{R}_0^+\times S^{n-1}$, with metric

$$\langle\cdot,\cdot\rangle=dr^2+\tilde{g}^2d\vartheta^2\qquad\text{on }\mathcal{N}\setminus\{0\}.$$

By (g1) and (g2) this metric can be smoothly extended to all of $\mathcal{N}$. Since w is a positive C^1 solution of (3.3) on $\mathbb{R}_0^+$ and $w'\geq 0$, the function $v(x)=w(r(x))$ is a positive C^1 classical solution of

$$\operatorname{div}\{A(|\nabla v|)\nabla v\}=b(r(x))f(v)\qquad\text{in }\mathcal{N},$$

with $v^*=\sup_{\mathcal{N}}v<\infty$. Moreover $b(r)\geq Cr^{-\mu}$, $r>>1$, $b(r(x))>0$ on $\mathcal{N}$ and

$$\liminf_{r\to\infty}r^{\mu-p'}\log\operatorname{vol}(B_r)=\liminf_{r\to\infty}r^{\mu-p'}\log\int_0^r\tilde{g}^{n-1}dt<\infty$$

because of (3.2), where vol denotes the Riemannian measure in $\mathcal{N}$. It follows from Theorem A of [15] that $f(v^*)\leq 0$, so that $v^*=0$ by (F1) since $\delta=\infty$, contradicting the initial data in (3.3). Hence $T<\infty$.

Step 4. We claim that (3.4) holds. By the previous steps T is finite. Arguing by contradiction and using (3.6), we have

$$\lim_{t\to T^-}w(t)=w_T\in\mathbb{R}.$$

From (3.5)

$$\lim_{t\to T^-}w'(t)=w'_T\in\mathbb{R}.$$

We claim that the problem

$$\begin{cases}[\tilde{g}^{n-1}\Phi(\tilde{w}')]'=\tilde{g}^{n-1}b(t)f(\tilde{w}),\\ \tilde{w}(T)=w_T>0,\qquad \tilde{w}'(T)=w'_T\end{cases} \tag{3.19}$$

admits a C^1 solution on $[T,T+\varepsilon)$, for some $\varepsilon>0$. Indeed, by the change of variable $z=\Phi(\tilde{w}')$, the initial value problem can be written in the equivalent form

$$\begin{cases}\tilde{w}'=\Phi^{-1}(z)\\ z'=-(n-1)\dfrac{\tilde{g}'}{\tilde{g}}\,z+b(t)f(\tilde{w}),\\ \tilde{w}(T)=w_T>0,\quad z(T)=w'_T\end{cases}\qquad\begin{cases}\boldsymbol{x}'=\boldsymbol{\Psi}(t,\boldsymbol{x}),\quad \boldsymbol{x}=(\tilde{w},z)\in\mathbb{R}^2,\\ \boldsymbol{x}(T)=(w_T,w'_T),\end{cases} \tag{3.20}$$

where $\boldsymbol{\Psi}$ is a continuous function from $[T,\infty)\times\mathbb{R}^2$. By standard theory (3.20) has at least a C^1 solution $\boldsymbol{x}=(\tilde{w},z)$ defined in some interval $[T,T+\varepsilon)$, $\varepsilon>0$, and therefore $\tilde{w}$ is a C^1 solution of (3.19) in $[T,T+\varepsilon)$, proving the claim.

Now the C^1 function

$$\hat{w}(t) = \begin{cases} w(t), & t \in [0, T), \\ \tilde{w}(t), & t \in [T, T + \varepsilon), \end{cases}$$

is a solution of (3.3), contradicting the maximality of T. $\square$

Actually problem (3.3) admits the required solution at least when $\tilde{g}$ is non–decreasing, see Proposition 4.1 in the Appendix. In the next result we recall that under assumption $(\mathcal{M}2)$ we have $g'/g \sim D\sqrt{G}$ as $t \to \infty$ by (1.5). We are now ready to prove

Theorem 3.3. *Assume also* (A3)$'$, (H2), (H3), (F2) *and* (1.17), *where* $\sigma > 0$ *is the number given in* (A3)$'$, *and* (H1) *with*

$$\Lambda(r) = O(r^{\mu}) \qquad \text{as } r \to \infty, \tag{3.21}$$

for some $\mu \in [0, \sigma)$. *Let* B *satisfy* (B), *with* $T = \Phi$ *and suppose also that*

$$\tilde{\beta}(r) = \frac{\mathcal{H}(r) + \kappa}{\lambda(r)} = O(r^{\sigma-\mu}) \qquad \text{as } r \to \infty. \tag{3.22}$$

Then the differential inequality

$$\operatorname{div}\{A(|\nabla u|)h(\nabla u, \cdot)^{\sharp}\} \geq B(x, u, \nabla u) \tag{3.23}$$

admits no positive semi–classical entire solutions.

Proof. We first observe that without loss of generality we can assume

$$\Lambda(r) \geq 1 \qquad \text{on } \mathbb{R}_0^+. \tag{3.24}$$

Next, we argue by contradiction and assume the existence of a positive solution u of (3.23) on $\mathcal{M}$. Since $f(u) > 0$ for $u > 0$, the solution u cannot be constant, since $\Phi(0) = 0$. Hence $\nabla u(O') \neq \mathbf{0}$ for some $O' \in \mathcal{M}$. With no loss of generality, we can suppose $O' = O$. Using Proposition 4.1 of the Appendix there is a solution w of (3.3) on $[0, T)$, $T \leq \infty$, with

$$0 < w(0) = w_0 < u(0), \qquad b(t) = 1/\Lambda(t), \qquad n \geq 1 + (m-1)/\lambda(0), \tag{3.25}$$

and with $\tilde{g}$ being the solution of the problem

$$\begin{cases} (n-1)\tilde{g}' - \gamma(t)\tilde{g} = 0 \\ \tilde{g}(0) = 0, \quad \tilde{g}'(0) = 1 \end{cases} \tag{3.26}$$

where γ is a fixed function of class $C^{\infty}(\mathbb{R}^+)$ such that

$$\gamma(t) = \begin{cases} \dfrac{n-1}{t}, & 0 < t << 1, \\ Ct^{\sigma-\mu}, & t >> 1, \end{cases} \tag{3.27}$$

for some $C > 0$, and also such that

$$\gamma(t) \geq \tilde{\beta}(t) \qquad \text{in } \mathbb{R}_0^+, \tag{3.28}$$

where explicitly

$$\tilde{\beta}(t) = \frac{\alpha(t) + \kappa + (m-1)\Lambda(t)g'(t)/g(t)}{\lambda(t)},$$

and g is given in (1.5) under assumption $(\mathcal{M}2)$. Inequality (3.28) is clearly possible by (3.25) and (3.27) for some appropriate $C > 0$. Note that the solution $\tilde{g}$ of (3.26) is of class $C^{\infty}(\mathbb{R}_0^+)$ since γ is smooth. Moreover $\tilde{g}(t) > 0$ for $t > 0$ and $\tilde{g}(t) = t$

near 0. Thus $\tilde g'(0) = 1$ and $\tilde g^{(2k)}(0) = 0$ for each $k = 0, 1, 2, \ldots$. Furthermore $\tilde g$ is strictly increasing in $\mathbb{R}^+_0$.

An easy calculation shows that (3.2) is satisfied. Furthermore, (3.21), (3.24) and (3.25) yield (3.1), which will be needed to apply Proposition 4.1 below. Moreover Lemma 3.2 can be applied so that $T < \infty$, $w'(t) > 0$ for $t \in (0,T)$ and (3.4) holds. We also observe that by (3.26) and (3.28) the solution w of (3.3) and (3.25) satisfies also

$$\begin{cases} [\Phi(w')]' + \tilde\beta(r)\Phi(w') \le f(w)/\Lambda & \text{in } [0,T), \\ w(0) > 0, \quad w'(0) = 0, \quad w' > 0 & \text{in } (0,T). \end{cases} \tag{3.29}$$

It follows, similarly to what has been shown in the proof of Lemma 4.1 of [17], that $v(x) = w(r(x))$ is a semi–classical solution of

$$\begin{cases} \operatorname{div}\{A(|\nabla v|)h(\nabla v, \cdot)^\sharp\} \le f(v) - \kappa\Phi(|\nabla v|) & \text{in } B_T, \\ v(O) < u(O), \quad v > 0 \text{ in } B_T, \quad v(x) \to \infty \text{ as } x \to \partial B_T. \end{cases} \tag{3.30}$$

Put $\Omega = \{x \in \mathcal{M} \ : \ u(x) > v(x)\}$. Then $O \in \Omega \ne \emptyset$ and $\overline{\Omega} \subset B_T$ since $v(x) \to \infty$ as $x \to \partial B_T$. Moreover, $v \equiv u$ on $\partial\Omega$. By the comparison Theorem 5.3 of [17], applied with $B(x, z, \boldsymbol{\xi}) = f(z) - \kappa\Phi(|\boldsymbol{\xi}|)$, we deduce that $v \ge u$ in Ω. This contradicts the fact that $O \in \Omega$. □

Of course Theorem 1.3 is an immediate consequence of Theorem 3.3.

4. **Appendix.** The next existence result has been used in the proof of Theorem 3.3, with a weaker version of the structural assumptions (A1), (A2), that is

(a1) $A \in C(\mathbb{R}^+)$,
(a2) $\Phi(\rho)$ *is strictly increasing in* $\mathbb{R}^+$ *and* $\Phi(\rho) \to 0$ *as* $\rho \to 0^+$.

Proposition 4.1. *Assume* (a1), (a2) *and* (F1). *Problem* (3.3) *admits a* C^1 *solution* w *in* $[0,T)$, $T \le \infty$, *for all* $\tilde g$ *that are continuous, monotone non–decreasing in* $\mathbb{R}^+_0$, *with* $\tilde g(0) = 0$, *and for all* $b \in C(\mathbb{R}^+_0)$ *satisfying* $(3.1)_1$, *with* b *positive.*

Proof. First, without loss of generality, we suppose that $\sup_{\mathbb{R}^+_0} b \le 1$. Now any possible local classical solution of (3.3), for small $t > 0$, must be a fixed point of the operator

$$\mathcal{T}[w](t) = w_0 + \int_0^t \Phi^{-1}\left(\int_0^s \left[\frac{\tilde g(\tau)}{\tilde g(s)}\right]^{n-1} b(\tau) f(v(\tau))d\tau\right) ds. \tag{4.1}$$

We denote by $C[0,t_0]$, $t_0 > 0$, the usual Banach space of continuous real functions on $[0,t_0]$, endowed with the uniform norm $\|\cdot\|_\infty$.

Fix $\varepsilon > 0$ so small that $[w_0 - \varepsilon, w_0 + \varepsilon] \subset (0,\delta)$, and put

$$C = \{w \in C[0,t_0] \ : \ \|w - w_0\|_\infty \le \varepsilon\}.$$

By (F1)

$$0 < \min_{[w_0-\varepsilon, w_0+\varepsilon]} f(u) \ \le \max_{[w_0-\varepsilon, w_0+\varepsilon]} f(u) = M < \infty.$$

If $w \in C$ then $w([0,t_0]) \subset [w_0 - \varepsilon, w_0 + \varepsilon]$, and in turn $0 < f(w(t)) \le M$. Now by $(3.1)_1$

$$0 \le \int_0^s \left[\frac{\tilde g(\tau)}{\tilde g(s)}\right]^{n-1} b(\tau) f(v(\tau))d\tau \le \int_0^s f(w(\tau))d\tau, \qquad 0 < s \le t_0,$$

and the last integral approaches 0 as $s \to 0$ by $(F1)$. Thus the operator $\mathcal{T}$ in (4.1) is well defined.

We shall show that $\mathcal{T} : C \to C$ and is compact provided t_0 is so small that $Mt_0 < \Phi(\infty)$ and $t_0\Phi^{-1}(Mt_0) \le \varepsilon$. Indeed, by $(3.1)_1$ for $w \in C$ we have

$$\|\mathcal{T}[w] - w_0\|_\infty \le \int_0^{t_0} \Phi^{-1}\left(\int_0^s \left[\frac{\tilde{g}(\tau)}{\tilde{g}(s)}\right]^{n-1} b(\tau)f(w(\tau))d\tau\right) ds \le t_0\Phi^{-1}(Mt_0) \le \varepsilon$$

and in turn $\mathcal{T}[w] \in C$. Hence $\mathcal{T}(C) \subset C$. Let $\{w_k\}_k$ be a sequence in C and let s, t be two points in $[0, t_0]$. Then

$$|\mathcal{T}[w_k](t) - \mathcal{T}[w_k](s)| \le \Phi^{-1}(M)\,|t - s|.$$

By the Ascoli-Arzelà theorem this means that $\mathcal{T}$ maps bounded sequences into relatively compact sequences with limit points in C, since C is closed.

Finally $\mathcal{T}$ is continuous, because if $w \in C$ and $\{w_k\}_k \subset C$ are such that the sequence $\|w_k - w\|_\infty$ tends to 0 as $k \to \infty$, then by Lebesgue's dominated convergence theorem, we can pass under the sign of integrals twice in (4.1), and so $\mathcal{T}[w_k]$ tends to $\mathcal{T}[w]$ pointwise in $[0, t_0]$ as $k \to \infty$. By the above argument, it is obvious that $\|\mathcal{T}[w_k] - \mathcal{T}[w]\|_\infty \to 0$ as $k \to \infty$ as claimed.

By the Schauder Fixed Point theorem, $\mathcal{T}$ possesses a fixed point w in C. Clearly, $w \in C[0, t_0] \cap C^1[0, t_0)$ by the representation formula (4.1), that is

$$w(t) = w_0 - \int_0^t \Phi^{-1}\left(\int_0^s \left[\frac{\tilde{g}(\tau)}{\tilde{g}(s)}\right]^{n-1} b(\tau)f(v(\tau))d\tau\right) ds, \tag{4.2}$$

so that the fixed point w is a C^1 solution of (3.3). □

Once it is known that a solution of (3.3) exists in $[0, T)$, then it necessarily obeys (4.2) in the entire $[0, T)$.

The next existence result provides the counterexample mentioned after Corollary 1.2.

Proposition 4.2. *Assume* (a1), (a2), (F1), (F2) *and that* $\Phi(\infty) = \infty$. *Let* g *be continuous, monotone non–decreasing in* $\mathbb{R}_0^+$, *with* $g(0) = 0$, *and suppose* $f \not\equiv 0$. *Then initial value problem*

$$\begin{cases} [\operatorname{sign} w'(t)] \cdot [g(t)^{m-1}\Phi(w'(t))]' - Cg(t)^{m-1}f(w(t)) = 0, & t > 0, \\ w(0) = w_0 > 0, \quad w'(0) = 0, \end{cases}$$

admits a non–decreasing C^1 *solution defined in the entire* $\mathbb{R}_0^+$, *whenever* (1.16) *holds and* $C > 0$ *is a suitable constant.*

If furthermore also (A3)′, (F1) *hold and* $g(\infty) = \infty$, *then* $w(t) \to \infty$ *as* $t \to \infty$.

Proof. The proof of the first part is essentially given in Lemma 3.1 of [14] with the use of Lemma 3.1 (ii) of [18].

For the second part of the proof we can proceed as in the proof of Theorem A of [14]. □

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