STRANGE DISTRIBUTIONALLY CHAOTIC TRIANGULAR MAPS III

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Abstract

In the class $\mathcal{T}$ of triangular maps of the square we consider the strongest notion of distributional chaos, *DC1*, originally introduced by Schweizer and Smítal [Trans. Amer. Math. Soc. **344** (1994), 737 - 854] for continuous maps of the interval. We show that a map $F \in \mathcal{T}$ is *DC1* if F has a periodic orbit with period $\neq 2^n$, for any $n \geq 0$. Consequently, a map in $\mathcal{T}$ is *DC1* if it has a homoclinic trajectory. This result is important since in general systems like $\mathcal{T}$, positive topological entropy itself does not imply *DC1*. It contributes to the solution of a long-standing open problem of A. N. Sharkovsky concerning classification of triangular maps of the square.

Key words: Compact metric spaces, discrete dynamical systems, distributional chaos, homoclinic trajectory, topological entropy, triangular maps.

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1. Introduction and main results

The present work aims to clarify relations between the strongest version of distributional chaos, *DC1*, as introduced in [9], and topological entropy. There are *DC1* continuous maps of a compact metric space with zero toplogical entropy (cf., e.g., [6]). There are also maps with positive topological entropy which are not *DC1* [11]; it should be noted that these maps are distributionally chaotic in a weaker sense. Such examples do exist even in a special class $\mathcal{T}$ of triangular, or skew-product maps of the square which has been intensively studied because its dynamics is sufficiently complicated but still they have some essential similarities with one-dimensional maps.

We are able to provide a condition sufficient for *DC1* in $\mathcal{T}$: existence of a periodic orbit whose period is not a power of 2. In $\mathcal{T}$, this condition is stronger than positive topological entropy but, for maps in $\mathcal{T}$, there is no weaker condition implying positive topological entropy (except for trivial condition that a factor of the map has positive topological entropy). See, e.g., [5], [7] or [8] for details and other references. Our result also contributes to the solution of a problem by A. N. Sharkovsky from the eighties concerning classification of triangular maps. We believe that our result will be useful also for those who are interested in applications of distributional chaos (in quantum physics, for example).

Let φ be a map from a compact metric space (M, ρ) into itself. For a pair (x, y) of points in M and a positive integer n, define a distribution function $\Phi_{xy}^{(n)} : \mathbb{R} \to [0, 1]$ by

$$\Phi_{xy}^{(n)}(t) = \frac{1}{n}\#\{0 \le i < n;\ \rho(\varphi^i(x), \varphi^i(y)) < t\}. \tag{1}$$

Then $\Phi_{xy}^{(n)}$ is a non-decreasing function, $\Phi_{xy}^{(n)}(t) = 0$ for $t \le 0$, and $\Phi_{xy}^{(n)}(t) = 1$ for $t > \operatorname{diam}(M)$. Define the *lower* and *upper distribution function* generated by φ, x and y as

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t), \quad \text{and} \quad \Phi_{xy}^*(t) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t),$$

respectively. Obviously, $\Phi_{xy} \le \Phi_{xy}^*$. If there are points $x, y \in M$ such that

$$\Phi_{xy}^* \equiv 1 \quad \text{and} \quad \Phi_{xy}(t) = 0, \quad \text{for some } t > 0, \tag{2}$$

then φ *exhibits distributional chaos of type 1*, briefly, *DC1*. Recall that *DC1* was originally introduced in [9] for the class $\mathcal{C}$ of continuous maps of the interval, for weaker notions *DC2 and DC3* see, e.g., [11] or [2].

Throughout this paper we consider distributional chaos on the class $\mathcal{T}$ of triangular maps of the square. Recall that $F \in \mathcal{T}$ if $F : I^2 \to I^2$ is a continuous map of the form $F(x, y) = (f(x), g_x(y))$, for any (x, y) in I^2. In I^2 we use the metric assigning to a pair of points (x_i, y_i), $i = 1, 2$, the distance $\max\{|x_1 - x_2|, |y_1 - y_2|\}$.

In [11] there is an example of a map $F \in \mathcal{T}$ with positive topological entropy which is not *DC1*. This is because in $\mathcal{T}$, but not in $\mathcal{C}$, there are maps of type 2^∞ with positive topological entropy. Neverthless, the next theorem shows that, contrary to this, the maps in $\mathcal{T}$ still have similar behavior as the continuous maps of the interval.

Theorem 1. *If $F \in \mathcal{T}$ has a periodic point whose period is different from 2^n, for any $n \geq 0$, then F is DC1.*

In $\mathcal{T}$, existence of a homoclinic trajectory implies existence of a periodic point of period $\neq 2^n$, $n \geq 0$, cf. [5]. Hence, we have the following:

Corollary. *If $F \in \mathcal{T}$ has a homoclinic trajectory then it is DC1.*

It should be emphasized that, for maps in $\mathcal{T}$, Theorem 1 and Corollary exhibite all known nontrivial implications between *DC1* and other properties of dynamical systems which are in $\mathcal{C}$ equivalent to positive topological entropy, cf. [4], [7], [5] and [8] for more details. Recall also that a weaker form of Theorem 1, for triangular maps which are nondecreasing on the *fibres* $I_x = \{x\} \times I$, is proved in [8]. Its proof is relatively simple, and cannot be addapted to the general case of our Theorem 1.

The paper is organized as follows. In the next section we recall some and prove some other results concerning continuous maps of the interval with zero topological entropy. They will be used in Section 3 which contains auxillary results. In particular, Lemma 3 (and its corollary, Lemma 5, concerning nonautonomous dynamical systems) which is essential for our proof of the main result is interesting in itself. Proof of Theorem 1 can be found in Section 4. In the sequel we use standard terminology and notation. On places, some notions and basic facts are recalled. For other notions and related results, see, e.g., [1], [5], [7] or [8].

2. Maps of the interval with zero topological entropy.

We recall some basic properties of maps $f \in \mathcal{C}$ with $h(f) = 0$, i.e., with zero topological entropy (cf., e.g., [1] or [10]). Any periodic point of such map has period 2^n, for some $n \in \mathbb{N}$. Moreover, any ω-limit set $\tilde{\omega}$ of f is *simple*, i.e., either it is a singleton – a fixed point, or it splits into two compact periodic portions $\tilde{\omega}_0$ and $\tilde{\omega}_1$

of period 2; thus, $f^2(\tilde{\omega}_i) = \tilde{\omega}_i$, $i = 0, 1$. In particular, for any infinite $\tilde{\omega} = \omega_f(x)$, there is a decreasing sequence $\{V_n(\tilde{\omega})\}_{n=0}^{\infty}$ of minimal compact periodic intervals such that, for any n, $V_n(\tilde{\omega})$ has period 2^n and its orbit contains $\tilde{\omega}$. Thus,

$$\tilde{\omega} \subseteq \bigcap_{n=0}^{\infty} \bigcup_{j=0}^{2^n-1} f^j(V_n(\tilde{\omega})) =: S(\tilde{\omega}). \tag{3}$$

The set $S(\tilde{\omega})$ is the *solenoid* containing $\tilde{\omega}$. For simplicity, denote $V_n^j(\tilde{\omega}) := f^j(V_n(\tilde{\omega}))$. Then, for any infinite ω-limit sets $\tilde{\omega}$ and $\tilde{\omega}'$ of f, not necessarily distinct, and for any $i, j, m, n \in \mathbb{N}$,

$$V_m^i(\tilde{\omega}) \cap V_n^j(\tilde{\omega}') \neq \emptyset \;\Rightarrow\; V_m^i(\tilde{\omega}) \subseteq V_n^j(\tilde{\omega}') \text{ or } V_m^i(\tilde{\omega}) \supseteq V_n^j(\tilde{\omega}'). \tag{4}$$

If, for an infinite ω-limit set $\tilde{\omega}$, $V_1^{j_1}(\tilde{\omega}) \supset V_2^{j_2}(\tilde{\omega}) \supset \cdots$ is a nested sequence then $M = \bigcap_{n=1}^{\infty} V_n^{j_n}$ either is a singleton hence, a point of $\tilde{\omega}$, or is a wandering interval. Then M belongs to the set $CR(f)$ of chain recurrent points of f. In particular, if $\omega(f)$ denotes the set of ω-limit points of f then, for any $x \in CR(f) \setminus \omega(f)$,

$$\text{for any } n \in \mathbb{N} \text{ there is } j \in \mathbb{N} \text{ with } x \in V_n^j(\omega_f(x)) \tag{5}$$

The following lemmas are used to prove our theorem.

Lemma 1. (Cf. [4].) *Let $f \in \mathcal{C}$, $h(f) = 0$, and let $\omega_f(x)$ be infinite. Let $U = [u, v]$ be the convex hull of $\omega_f(x)$, and $V = [a, b]$ the minimal compact invariant interval containing U. Then*

(i) $V \setminus U$ contains no fixed point of f;

(ii) there is an interval J relatively open in I such that $U \subseteq J$, $J \setminus U$ contains no fixed point of f, and $f(\overline{J}) \subseteq J$.

Lemma 2. *Let $f \in \mathcal{C}$, $h(f) = 0$, and let $L \subseteq \omega(f)$ be an interval. Then $L \subseteq Per(f)$ and there is an $n \in \mathbb{N}$ such that all points in L have periods less or equal to 2^{n+1}.*

Proof. This result must be known but we are not able to give a reference. So let $h(f) = 0$ and $L \subseteq \omega(f)$. Since $\omega(f)$ is closed we may assume $L = [a, b]$. Then

$$L \subseteq \overline{Per(f)}. \tag{6}$$

To see this assume that $x \in (a, b) \setminus Per(f)$. Then x belongs to an infinite ω-limit set $\tilde{\omega}$. If $S(\tilde{\omega})$ is nowhere dense then, by (3), $x \in \overline{Per(f)}$. If $S(\tilde{\omega})$ fails to be nowhere dense then, by (3), $S(\tilde{\omega})$ contains an interval. This interval is wandering, and has a preimage in any $V_n^j(\tilde{\omega})$. Consequently, L would contain a wandering interval J which is impossible since $\omega(f) \cap J = \emptyset$.

Thus, for some $n \in \mathbb{N}$, there is a periodic point $p \in L$ of period 2^n. Then, if we replace f by $g = f^{2^n}$, we have that $K := \bigcup_{i=0}^{\infty} g^i(L)$ is an interval with $g(K) \subseteq K$. Since $K = L \cup g(K)$, if $K \setminus g(K)$ contains an interval J, then, for any $i > 0$, $g^i(J) \cap J = \emptyset$, contrary to (6). So we have $\overline{K} = g(\overline{K})$ and, by [3], this implies that $g|\overline{K}$ is a homeomorphism, $\overline{K} \subseteq Per(g)$, and no point in $\overline{K}$ has period other than 1 or 2. $\square$

Lemma 3. *Let $f \in \mathcal{C}$ have zero topological entropy. Then, for any $\varepsilon > 0$, there is an $N_\varepsilon \in \mathbb{N}$ with the following properties: For any $n \geq N_\varepsilon$ in $\mathbb{N}$ there are disjoint compact intervals $H_1, H_2, \cdots, H_k$ such that*

(i) $H = H_1 \cup H_2 \cup \cdots \cup H_k$ is a neighborhood of $CR(f)$;

(ii) for any i, $\lambda(H_i) < \varepsilon$, or H_i contains an interval K_i such that $\lambda(H_i \setminus K_i) < \varepsilon$, and either K_i is a wandering interval contained in $CR(f)$, or K_i consists of periodic points of f;

(iii) for any i, $f^{2^n}(H_i \cap CR(f)) = H_i \cap CR(f)$.

Proof. Assume $h(\varphi) = 0$ and let $\varepsilon > 0$. By Lemma 2, there is an $N_1 \in \mathbb{N}$ such that any interval $L \subseteq \omega(f)$ either consists of periodic points of period not greater than 2^{N_1}, or has diameter less than $\varepsilon/3$. Moreover, there is an $N_\varepsilon \in \mathbb{N}$ such that, for any infinite ω-limit set $\tilde{\omega}$ of f and any $n, j \in \mathbb{N}$ with $n \geq N_\varepsilon$, $V_n^j(\tilde{\omega})$ either has diameter less than $\varepsilon/3$, or contains a wandering interval $K_n^j(\tilde{\omega})$ such that $V_n^j(\tilde{\omega}) \setminus K_n^j(\tilde{\omega})$ has measure less than $\varepsilon/3$. Moreover, since the periodic orbits of f are simple we may assume that any periodic orbit of $f^{2^{N_\varepsilon}}$ has diameter less than $\varepsilon/3$. Finally, assume that $N_\varepsilon \geq N_1$.

Fix an $n \in \mathbb{N}$, $n \geq N_\varepsilon$, and let X be the union of the sets $V_n^j(\tilde{\omega})$, for all $j \in \mathbb{N}$ and all infinite ω-limit sets $\tilde{\omega}$. By (4), X is the union $\bigcup_{i=1}^{\infty} V_i$ of a countable family of disjoint intervals V_i. The family $\mathcal{V}$ of their closures $\overline{V_i}$ also is disjoint. To see this note that if V_i is not of the form $V_n^j(\tilde{\omega})$ then it is the union of a strictly increasing sequence $\{V_n^{j_i}(\tilde{\omega}_i)\}_{i=1}^{\infty}$. Hence, $f^{2^n}(V_i) = V_i$. Let $a_i < b_i$ be the minimum and maximum of $V_n^{j_i}(\tilde{\omega}_i) \cap \tilde{\omega}_i$, respectively. Since $V_n^{j_1}(\tilde{\omega}_1)$ is periodic it is disjoint from any $\tilde{\omega}_i$, $i > 1$.

Then, $a_i < \min V_n^{j_1}(\tilde{\omega}_1)$ and $b_i > \max V_n^{j_1}(\tilde{\omega}_1)$ since if, e.g., $b_i < \max V_n^{j_1}(\tilde{\omega}_1)$ then the intervals $V_n^{j_1}(\tilde{\omega}_1)$ and $V_n^{j_i}(\tilde{\omega}_i)$ being minimal would be disjoint. Thus, for any $i > 0$,

$$f^{2^n}(a_i) - a_i > \lambda(V_n^{j_1}(\tilde{\omega}_1)), \quad b_i - f^{2^n}(b_i) > \lambda(V_n^{j_1}(\tilde{\omega}_1)). \tag{7}$$

By the continuity, (7) implies that no two distinct intervals in $\mathcal{V}$ can have a point in common, and that the interval between any two intervals from $\mathcal{V}$ cannot be a subset of $Per(f)$. Consequently, there is a family of disjoint open intervals $\mathcal{A} = \{A_i\}_{i=1}^{\infty}$ such that, for any i,

$$\text{the endpoints of } A_i \text{ are in } I \setminus \omega(f), \tag{8}$$

$$\overline{V_i} \subseteq A_i \subseteq B_{\varepsilon/3}(V_i), \tag{9}$$

where $B_\delta(M)$ denotes the open δ-neighborhood of a set M, and

$$f^{2^n}(A_i \cap CR(f)) = A_i \cap CR(f). \tag{10}$$

Property (8) follows by Lemma 2 since the intervals between neighbor sets from $\mathcal{A}$ cannot be subsets of $Per(f)$, (9) follows since $n > N_1$, and (10) by (5). Then $A = \bigcup \mathcal{A}$ is open and hence, $Y := \omega(f) \setminus A$ is a closed subset of $Per(f)$.

Now we need a system $\mathcal{B} = \{B_i\}_{i=1}^{\infty}$ of open intervals covering Y, satisfying conditions (8) and (10), with A_i replaced by B_i, and such that any two intervals in $\mathcal{A} \cup \mathcal{B}$ either are disjoint or one is contained in the other one. More precisely, the intervals from $\mathcal{B}$ are disjoint but may contain infinitely many intervals from $\mathcal{A}$. Finally, we need that any interval B_i either has length less than ε, or contains an interval $K_i \subseteq Per(f)$ such that $B_i \setminus K_i$ has measure less than ε. It is easy to construct such a family $\mathcal{B}$ using (8) – (10) and Lemma 2, e.g., by transfinite induction.

To finish the argument note that $\mathcal{A} \cup \mathcal{B}$ is an open cover of $\omega(f)$, hence there is a finite subcover $G_1, \cdots, G_k$. Take $H_i = \overline{G_i}$, for $1 \leq i \leq k$. The property (i) follows since, by (5), $\mathcal{A} \cup \mathcal{B}$ covers $CR(f)$, (ii) and (iii) since $G_i \in \mathcal{A} \cup \mathcal{B}$ and any interval in $\mathcal{A} \cup \mathcal{B}$ has these properties. The intervals H_i are obviously disjoint. $\square$

We conclude this section by the following simple result.

Lemma 4. *Let p be a fixed point of $f \in \mathcal{C}$ and $\varepsilon > 0$. Assume that, for any* *neighborhood W of p, there are a point x and positive integers $m < n$ such that $|f^m(x) - p| > \varepsilon$ and $x, f^n(x) \in W$. Then $h(f) > 0$.*

Proof. This result must be known, and it is easy to prove it since the hypothesis implies existence of a horseshoe or, equivalently, existence of a trajectory homoclinic to p. It is well-known that in either case, $h(f) > 0$. Cf. also [1]. □

3. Preliminary results.

Before stating the next lemma we introduce some terminology. If $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence of continuous maps from a compact metric space X to itself, then $\mathcal{S} = (X, \{\varphi_n\}_{n=1}^{\infty})$ is a *nonautonomous dynamical system.* The $\mathcal{S}$-*trajectory* $\{y_n\}_{n=0}^{\infty}$ of a point $y \in X$ is defined by $y_0 = y$, and $y_n = \varphi_n(y_{n-1})$, for $n > 0$.

Lemma 5. *Let $\varphi \in \mathcal{C}$ and $h(\varphi) = 0$. Then, for any $\varepsilon > 0$, there are $N, T \in \mathbb{N}$, $\delta > 0$, and disjoint compact sets $M_1, \cdots, M_k$ with the following properties.*

(i) $M = M_1 \cup \cdots \cup M_k$ is a neighborhood of $CR(\varphi)$ and each M_k is a finite union of disjoint intervals;

(ii) for any i, $\lambda(M_i) < \varepsilon$, or M_i contains an interval K_i such that $\lambda(M_i \setminus K_i) < \varepsilon$, and either K_i is a wandering interval contained in $CR(\varphi)$, or K_i consists of periodic points of φ;

(iii) for any i, $\varphi^{2^N}(M_i \cap CR(\varphi)) = M_i \cap CR(\varphi)$.

Moreover, if $\mathcal{S} = (I, \{\varphi_i\}_{i=1}^{\infty})$ is a nonautonomous dynamical system, with any $\varphi_i \in \mathcal{C}$ satisfying

$$||\varphi_i - \varphi^{2^N}|| < \delta, \text{ for any } i \geq 1, \tag{11}$$

then, for any trajectory $\{y_n\}_{n=0}^{\infty}$ of $\mathcal{S}$,

(iv) the number of n's for which $y_n \notin M$, is less than T;

(v) if, for some n, $y_n, y_{n+1} \in M$ then there is an i such that $y_n, y_{n+1} \in M_i$.

Proof. Fix an $\varepsilon > 0$, and let H, H_i and N_ε be as in Lemma 3, and let $N \geq N_\varepsilon$.

Let $\delta_0 = \min_{i \neq j} \operatorname{dist}(H_i, H_j)$. Put

$$D_i = \left\{ x \in H_i; \varphi^{2^N}(x) \in B_{\delta_0/2}(H \setminus H_i) \right\}, \; 1 \leq i \leq k, \tag{12}$$

and let $D = D_1 \cup \cdots \cup D_k$. For any interval J complementary to $CR(\varphi)$, let $D(J) \subseteq J$ be the minimal interval such that $D \cap J = D \cap D(J)$. By Lemma 3(iii), $D \cap CR(\varphi) = \emptyset$, hence, by the continuity of φ, the distance between each D_i and $CR(\varphi)$ is positive. It follows that there are finitely many J such that $D(J) \neq \emptyset$. Let W be the union of all sets $D(J)$ and, for any i, $1 \leq i \leq k$, let $M_i = H_i \setminus W$. Then W is the union of finitely many open intervals, hence $M := M_1 \cup \cdots \cup M_k$ is a finite union of compact intervals. Since $W \cap CR(\varphi) = \emptyset$, Lemma 3 implies that M satisfies the first three conditions (i) – (iii).

It remains to prove (iv) and (v). It is well-known that, for any $x \in I \setminus \omega(\varphi)$, there is an interval U_x such that any trajectory of φ visits U_x no more than two times [1]. Since, by (i), $X = \overline{I \setminus M}$ is disjoint from $CR(\varphi) \supseteq \omega(\varphi)$, there is a finite cover of X by sets $U_{x_1}, U_{x_2}, \cdots, U_{x_s}$. Put $T = 2s$. Then, by (12), any trajectory $\{y_i\}_{i=0}^{\infty}$ of φ^{2^N} satisfies (iv) and (v). To finish the argument note that $f \mapsto CR(f)$ is upper semicontinuous [1]. Hence if $\delta \leq \delta_0/2$ is sufficiently small than, for any $\varphi_i \in \mathcal{C}$ with $||\varphi_i - \varphi^{2^N}|| < \delta$, M is a neighborhood of $CR(\varphi_i)$ and, by (12), $\varphi_i(M_k) \cap M_l = \emptyset$ for $k \neq l$. □

Lemma 6. (Cf. [8].) *Let $g \in \mathcal{C}$ have positive topological entropy. Then, for an integer $N \geq 1$, the map $f = g^N$ has a basic set $\tilde{\omega} \subset I$(i.e., infinite maximal ω-limit set containing periodic points) with the following properties:*
(i) $\tilde{\omega}$ contains fixed points $p < q$ of f.
(ii) For any interval $J \subset I$ having an infinite intersection with $\tilde{\omega}$ there is an integer $n \geq 0$ such that $\{p, q\} \subset \operatorname{Int}(f^n(J))$.

Lemma 7. (See [8].) *Let k, s be positive integers, and let $a_0 > 0$ and $a_i \geq 0$, if $1 \leq i \leq s$. If*

$$a_1 + a_2 + \cdots + a_s \geq a_0(1 + k)^s \tag{13}$$

then, for some i, $0 \leq i < s$,

$$\frac{a_{i+1}}{a_0 + a_1 + \cdots + a_i} > k. \tag{14}$$

4. PROOF OF THEOREM 1.

Assume that F has a periodic point $w = (p, q)$ of period $k = mn$, where m is an odd integer. If n is a multiple of the period s of p, then w is a periodic point of $\varphi := F^s|I_p$ of period k/s and so φ is (isomorphic to) a map from $\mathcal{C}$ which is not of type 2^∞. Therefore φ is *DC1* [9] and consequently, also F is *DC1*.

So we may assume that the base map f has positive topological entropy and that, for each periodic point p of f with period s, the function $F^s|I_p$ is of type 2^∞.

Since F^N is *DC1* iff F is *DC1*, we may assume that f has the properties indicated in Lemma 6 (i) and (ii). Hence there are compact interval neighborhoods U_0, V_0 of p and q, respectively, such that

$$\operatorname{dist}(U_0, V_0) = \varepsilon > 0 \tag{15}$$

and, for any interval $J \subset I$ containing infinitely many points of $\tilde{\omega}$,

$$f^n(J) \supset U_0 \cup V_0, \text{ for some } n \geq 1. \tag{16}$$

We prove the theorem by showing that there are points $\alpha = (p, z)$, $\beta = (x, y)$ in I^2 such that

$$\Phi_{\alpha\beta}(\varepsilon) = 0, \quad \text{and} \quad \Phi^*_{\alpha\beta} \equiv 1. \tag{17}$$

Our proof depends on an increasing sequence

$$\nu = \{n(i)\}_{i=0}^{\infty}, \tag{18}$$

of positive integers which will be specified later, see (31) and (34) – (36). Now we proceed with three stages.

STAGE 1. Our first aim is to find $x \in I$ such that

$$\Phi_{px}(\varepsilon) = 0 \quad \text{and} \quad \Phi^*_{px} \equiv 1. \tag{19}$$

Notice that the distribution functions in (17) are generated by the map F while these in (19) by f. Clearly, the first condition in (19) implies the first condition in (17), for any choice of z and y. Since any basic set is perfect, any neighborhood J of p or q has an uncountable intersection with Q and hence, by Lemma 6, (16) is satisfied for such J. Consequently, by the induction argument, there are decreasing sequences $\{U_i\}_{i=0}^{\infty}$ and $\{V_i\}_{i=0}^{\infty}$ of compact interval neighborhoods of p and q, respectively, such that

$$f(U_{i+1}) = U_i \quad \text{and} \quad f(V_{i+1}) = V_i, \quad \text{for every } i \geq 0, \tag{20}$$

and hence,

$$\bigcap_{i=0}^{\infty} U_i = \{p\} \quad \text{and} \quad \bigcap_{i=0}^{\infty} V_i = \{q\}. \tag{21}$$

By (16) there are minimal integers u_i and v_i such that

$$f^{u_i}(U_i) \supseteq V_0, \quad \text{and} \quad f^{v_i}(V_i) \supseteq U_0, \quad \text{for any } i \geq 0. \tag{22}$$

Now we proceed as follows. We pick up a point $x \in U_{n(0)}$ with the itinerary

$$U_{n(0)} \xrightarrow{m_1} U_0 \xrightarrow{m_2} V_{n(1)} \xrightarrow{m_3} V_1 \xrightarrow{m_4} U_{n(2)} \xrightarrow{m_5} U_2 \xrightarrow{m_6} V_{n(3)} \xrightarrow{m_7} V_3 \xrightarrow{m_8} U_{n(4)} \cdots \tag{23}$$

where $A \xrightarrow{m} B$ means that $f^m(A) \supseteq B$. Existence of x with itinerary (23) follows by (20), (22) and the Itinerary Lemma. To prove (17) it suffices to estimate the corresponding lower and upper distribution functions generated by the pair p and x, and choose (18) properly.

Thus, by (20) and (22), we always can satisfy (23) by taking, for any $k \geq 0$,

$$m_{4k+1} = n(2k) - 2k, \quad m_{4k+3} = n(2k+1) - (2k+1). \tag{24}$$

$$m_{4k+2} \geq u_{2k}, \quad m_{4k+4} \geq v_{2k+1}. \tag{25}$$

For simplicity, denote

$$t_0 = 0, \quad \text{and} \quad t_k = \sum_{i=1}^{k} m_i, \quad \text{for any } k \geq 1. \tag{26}$$

Note that the estimation of m_i with even i in (25) is independent of the choice of (18). Therefore we may take m_{4k+2} and m_{4k+2} arbitrarily big in order to assure that, given an $N_k \in \mathbb{N}$, t_{4k} is divisible by 2^{N_k}, cf. (34).

Then, by (23),

$$f^i(x) \in U_{2k}, \quad \text{if } t_{4k} \leq i \leq t_{4k} + m_{4k+1}, \tag{27}$$

and

$$f^i(x) \in V_{2k+1}, \quad \text{if } t_{4k+2} \leq i \leq t_{4k+2} + m_{4k+3}. \tag{28}$$

Thus, by (21), (26) and (27)

$$\Phi^*_{px}(\delta) \geq \limsup_{k\to\infty} \frac{m_{4k+1}}{t_{4k} + m_{4k+1}}, \quad \text{for any } \delta > 0, \tag{29}$$

and similarly, by (15), (26) and (28),

$$\Phi_{px}(\varepsilon) \leq \liminf_{k\to\infty} \frac{t_{4k+2}}{t_{4k+2} + m_{4k+3}}. \tag{30}$$

Now if the sequence ν in (18) satisfies

$$\frac{n(k)}{t_{2k}} > k, \text{ for any } k > 0, \tag{31}$$

then

$$\frac{m_{4k+1}}{t_{4k} + m_{4k+1}} > 1 - \frac{1}{k} \text{ , and } \frac{t_{4k+2}}{t_{4k+2} + m_{4k+3}} < \frac{1}{k}, \text{ for any } k > 0$$

and, by (29) and (30), (19) is satisfied.

STAGE 2. Now we specify (18). In Lemma 5, put $\varphi = F|I_p$ and, for any $k \geq 1$, let $\varepsilon = \frac{1}{k}$ to obtain compact sets $M_1^k, M_2^k, \cdots, M_{m(k)}^k$, integers $N = N_k > 0$, $T = T_k > 0$, and $\delta = \delta_k > 0$. Assume that

$$\text{for any } k,\ N_k < N_{k+1},\ T_k < T_{k+1}, \text{ and } \delta_k > \delta_{k+1} \to 0. \tag{32}$$

Denote

$$M^k = M_1^k \cup M_2^k \cup \cdots \cup M_{m(k)}^k. \tag{33}$$

By induction define (18) satisfying (31) and such that, for any $k > 0$,

$$t_{4k} \text{ is divisible by } 2^{N_k}, \tag{34}$$

$$n(2k) > Q_k + 2k, \text{ where } Q_k = (1+k)^{T_k}(t_{4k} + T_k), \tag{35}$$

and

$$u \in U_{n(2k)} \text{ implies } \|\, F^i|I_p - F^i|I_u \,\| < \delta_k,\ 0 \le i \le Q_k. \tag{36}$$

Condition (36) can be satisfied by (20) and (21).

STAGE 3. Let $\{y_n\}_{n=0}^\infty$ be a trajectory of the nonautonomous system $\mathcal{F} = (I, \{F|I_{x_n}\}_{n=0}^\infty)$ where $\{x_n\}$ is the trajectory of x chosen in Stage 1. This trajectory depends on the choice of ν in (18) which has been fixed in Stage 2. We show that, for any $k > 0$, there is a positive integer S_k satisfying $t_{4k} < S_k \le t_{4k+1}$ and such that

$$\lim_{k\to\infty} \Phi_{\alpha\beta_0}^{S_k}(\eta) = 1, \quad \text{for any } \eta > 0, \tag{37}$$

where $\beta_0 = (x_0, y_0)$, and $\alpha = (p, z)$ is a suitable point in I_p. Then (17) follows by (19). Here the distribution function $\Phi_{\alpha\beta_0}^n(t)$ for the nonautonomous system $\mathcal{F}$ is defined similarly as in (1).

By (34) – (36), and by (iv) and (v) of Lemma 5, the set

$$\{t_{4k} \le n < t_{4k} + Q_k; y_n \in M^k \text{ and } n \equiv 0 \pmod{2^{N_k}}\} \tag{38}$$

consists of less than T_k blocks A_i of consecutive integers. Apply Lemma 7, with $a_0 = t_{4k} + T_k$, $s = T_k$, and $a_i \ge 0$ the length of A_i, to find $j(k), H_k, S_k \in \mathbb{N}$ such that

$$t_{4k} \le H_k < S_k \le Q_k, \quad \frac{S_k}{H_k} > k, \tag{39}$$

and

$$y_n \in M^k_{j(k)}, \text{ if } n \equiv 0 \pmod{2^{N_k}} \text{ and } H_k \le n < S_k. \tag{40}$$

Since $\varepsilon = 1/k \to 0$ the family $\{M^k_{j(k)}\}_{k=1}^\infty$ contains an infinite decreasing subsequence $\{M^{k_i}_{j(k_i)}\}_{i=1}^\infty$. Denote by M_∞ its intersection. By (i) and (ii) of Lemma 5, $M_\infty \subseteq CR(F)$. By (40) there is a subsequence of $\{y_{H_k}\}_{k=1}^\infty$ converging to a point $z \in M_\infty$. Without loss of generality assume $\lim_{k\to\infty} y_{H_k} = z$.

Denote $G = F|I_p$. Since, for any $\varphi \in \mathcal{C}$ with zero topological entropy the set $CR(\varphi)\setminus \omega(\varphi)$ consists of intervals contained in solenoids [1] there are two possible cases, either $\tilde{\omega} := \omega_G(z)$ is infinite or $z \in Per(G)$.

Fix an $\eta > 0$ and assume first that $\tilde{\omega}$ is infinite. Then it is contained in a solenoid, hence there is a decreasing sequence $\{L_k\}_{k=1}^\infty$ of compact periodic intervals generating this solenoid such that $L_k \supseteq M_\infty$ has period 2^{N_k}. Fix a $k_0 > 0$ such that $\delta_{k_0} < \eta/2$. If z is an interior point of L_{k_0} then $y_{H_k} \in L_{k_0}$, for any sufficiently large k. Hence, by (36),

$$|G^n(z) - y_{H_k+n}| \le |G^n(L_{k_0})| + |G^n(z) - G^n(y_{H_k})| < \eta,$$

for any n between 0 and $S_k - H_k$, and any $k \ge k_0$, with no more than $2/\eta$ exceptions for n since the intervals in the orbit of any L_k are disjoint. This, by (39), proves (37) provided z is an interior point of L_{k_0}. In the other case note that $z \notin Per(G)$ so, for $i = 2^{N_{k_0}}$ or $i = 2^{N_{k_0}+1}$, z is an interior point of L_{k_0}. Then, by the continuity, $y_{H_k+i} \in L_{k_0}$, for any large k, and similarly as before we obtain (37).

It remains to consider the other case when $z \in Per(F)$. Without loss of generality assume p is a fixed point of G. If M_∞ is an interval then, by Lemma 5 (ii), it consists of fixed points of G. If z is an interior point then, for any sufficiently large k, y_{H_k} is also an interior point hence, fixed a point of G. Thus, by (36) and (40), for any sufficiently lagre k,

$$|G^n(z) - y_{H_k+n}| \le |z - y_{H_k}| + |G^n(y_{H_k}) - y_{H_k+n}| < |z - y_{H_k}| + \delta_k < \eta,$$

whenever $0 \le n < S_k - H_k$. This gives (37). Finally, if M_∞ is a singleton or an interval of fixed points with p as is its endpoint then, by Lemma 4, there is a neighborhood W of z such that if $y_{H_k} \in W$ then $|G^n(y_{H_k}) - z| < \eta/2$, for $0 \le n < S_k - H_k$.

Consequently, for any sufficiently large k, by (36) and (40), $|z - y_n| < \eta$ whenever $0 \leq n < S_k - H_k$, and (37) follows. $\square$

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