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Elliptic Systems with Nonlinearities of Arbitrary Growth

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Abstract. In this paper we study the existence of nontrivial solutions for the following system of coupled semilinear Poisson equations:

$$\begin{cases} -\Delta u = v^p\,, & \text{in } \Omega, \\ -\Delta v = f(u)\,, & \text{in } \Omega, \\ u = 0 \text{ and } v = 0\,, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$. We assume that $0 < p < \frac{2}{N-2}$, and the function f is superlinear and with no growth restriction (for example $f(s) = s\,e^s$); then the system has a nontrivial (strong) solution.

1. Introduction

We consider the system of equations

$$\begin{cases} -\Delta u = g(v)\,, \text{ in } \Omega \\ -\Delta v = f(u)\,, \text{ in } \Omega \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \end{cases}, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$. It is known, see [5], [11], [15], that for the "model case"

$$f(s) = s^q\,,\ q > 1\,, \quad \text{and} \quad g(s) = s^p\,,\ p > 1\,,$$

(here and in what follows, $s^\alpha := \text{sgn(s)}|s|^\alpha$) the system (1) has a nontrivial solution provided that

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N} \tag{2}$$

For $N = 2$ this condition is satisfied for any $p > 1$ and $q > 1$.
For $N \geq 3$, the curve of $(p, q) \in \mathbb{R}^2$ satisfying $\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}$ is the so-called "critical hyperbola": for points (p, q) on this curve one finds the typical problems of non-compactness, and non-existence of solutions, as it was proved in [23], [18], using Pohozaev type arguments.

The case N=2
As mentioned above, for $N = 2$ any pair of powers $(p, q) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfies the inequality (2). Actually, even a higher growth than polynomial is admitted: by the inequality of Trudinger-Moser, see [22], [19], [20], *subcritical growth* for a single equation is given by the condition (see [10])

$$\lim_{|t|\to\infty} \frac{g(t)}{e^{\alpha t^2}} = 0\,,\ \forall\, \alpha > 0$$

It follows from a result in de Figueiredo-do Ó-Ruf [8] that system (1) has a non-trivial solution for nonlinearities f and g with such subcritical growth (and satisfying an Ambrosetti-Rabinowitz condition, see [2]). Also existence results for certain nonlinearities with critical growth are given in [8]. In this paper we consider a different type of extension of the known results: We will show that if one nonlinearity, say g, has polynomial growth (of any order), then, to prove existence of solutions, *no growth restriction* is required on the other nonlinearity f (other than the Ambrosetti-Rabinowitz condition).

The case N=3
Note that for $N = 3$ the critical hyperbola has the asymptotes $p_\infty = 2$ and $q_\infty = 2$. In particular, if $g(s) = s^p$ with $1 < p < 2$, then the cited existence results say that there exists a solution (u, v) for system (1) with $f(s) = s^q$, for any $q > 1$. Also in this case we show that existence of solutions can be proved requiring *no growth restriction* whatsoever on the nonlinearity f (other than the Ambrosetti-Rabinowitz condition).

The case N $\geq$ 4
For $N \geq 4$ the asymptotes of the critical hyperbola are in the values $p_\infty = \frac{2}{N-2} \leq 1$ and $q_\infty = \frac{2}{N-2} \leq 1$. Note that for an exponent $p < 1$, the corresponding equation in the system is *sublinear*. i.e. we have a system with one sublinear and one superlinear equation. In this situation, the proposed approach is no longer applicable. However, in this case a reduction of the system to a single equation is possible (see Clément-Felmer-Mitidieri [6] and Felmer - Martínez [12]), which allows to prove again a result of the same form; moreover this approach also allows to extend to the whole range the cases $N = 2$ and $N = 3$, that is for $N = 2\ :\ 0 < p < +\infty$, and for $N = 3\ :\ 0 < p < 2$.

The main result of the paper is stated in the following theorem:

Theorem 1.1. *Suppose that*

1) $g(s) = s^p$, *with* $\begin{cases} 0 < p\ , & \text{if } N = 2 \\ 0 < p < \frac{2}{N-2}\ , & \text{if } N \geq 3 \end{cases}$

2) $f \in C(\mathbb{R})$, *and set* $F(s) = \int_0^s f(t)dt$;

- there exist constants $\theta > \begin{cases} 2\ ,\ if\ p \geq 1 \\ 1 + \frac{1}{p}\ ,\ if\ p < 1 \end{cases}$ *and* $s_0 \geq 0$ *such that*

$\theta F(s) \leq f(s)s\ ,\ \forall\ |s| \geq s_0$

- and for s near 0: $f(s) = \begin{cases} o(s)\ , & if\ p \geq 1 \\ o(s^{1/p})\ , & if\ p < 1 \end{cases}$

Then the system

$$\begin{cases} -\Delta u = v^p & in \quad \Omega, \\ -\Delta v = f(u) & in \quad \Omega, \\ u = 0\ ,\quad v = 0 & on \quad \partial\Omega, \end{cases} \tag{3}$$

has a nontrivial (strong) solution.

Remarks
1) It is somewhat surprising that *no growth restriction* needs to be imposed on f, since for the single equation $-\Delta u = f(u)$ growth restrictions are, in general, necessary to prove the existence of solutions; we refer to the non-existence result in [9] for $N = 2$, and to [20] for $N \geq 3$.
2) In the cases with $p > 1$, the nonlinearity $g(s) = s^p$ may be replaced by more general functions, satisfying an Ambrosetti-Prodi type condition like $f(s)$, and the growth restriction

$$|g(s)| \leq c|s|^p + d\ , \quad \text{for some constants} \quad c, d > 0, \quad \text{and} \begin{cases} 1 < p\ , & N = 2 \\ 1 < p < 2\ , & N = 3 \end{cases}$$

For the sake of simplicity, we restrict here to the case $g(s) = s^p$.

For completeness we also state the following theorem:

Theorem 1.2. *Suppose that*

1) (p, q) *satisfy* $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}$, *and* $\frac{2}{N-2} \leq p \leq 1$.

2) $f \in C(\mathbb{R})$, *and there exist constants* $\theta > \frac{p+1}{p}$ *and* $s_0 \geq 0$ *such that*

$$\theta F(s) := \theta \int_0^s f(t)dt \leq f(s)s \ , \quad \forall \ |s| \geq s_0 \ ,$$

and

$$|f(s)| \leq c|s|^q + d \ , \quad \textit{for some constants} \quad c, d > 0 \ .$$

Then the system

$$\begin{cases} -\Delta u & = & v^p & in & \Omega, \\ -\Delta v & = & f(u) & in & \Omega, \\ u = 0 & , & v = 0 & on & \partial\Omega, \end{cases} \tag{4}$$

has a nontrivial (strong) solution.

In the literature we have only found the cases of (p, q) below the critical hyperbola, and with the restriction that $p > 1$ and $q > 1$ (see [5], [15], [11]) and the case $0 < p \cdot q < 1$ (see Felmer-Martínez [12]). This does not cover the whole region below the critical hyperbola. The above theorem covers also the remaining cases below the critical hyperbola, namely

$$0 < p \leq 1 \quad \text{and} \quad p \cdot q \geq 1 \ ;$$

note that we need to make the restriction that the sublinear function v^p is in the form of a power, while the superlinear function $f(u)$ may be of more general form.

2. Proof: the case $p > 1$

In this section we consider the case $1 < p < \frac{2}{N-2}$, i.e. $N = 2, 3$.

2.1. The setting

A natural functional associated to system (1) is

$$J(u, v) = \int_\Omega \nabla u \nabla v dx - \int_\Omega (F(u) + G(v))dx \ , \tag{5}$$

with $F(s) = \int_0^s f(t)dt$ and $G(s) = \int_0^s g(t)dt$. The natural space to consider this functional is the Sobolev space $H_0^1(\Omega) \times H_0^1(\Omega)$; however, in order to have a well-defined C^1-functional on this space, one has to impose certain *growth restrictions*:

in $N = 2$: F and G subcritical in the sense of Trudinger-Moser (see above)

in $N = 3$: $|F(s)| \leq c|s|^6 + d \ , \quad |G(s)| \leq c|s|^6 + d$

These conditions are on the one hand too loose for $G(s) = \frac{1}{p+1}s^{p+1}$, where a more restrictive growth is given, and too strong on $F(s)$, where we do not want any growth limitation.

We therefore follow an idea of de Figueiredo-Felmer [11] and Hulshoff-vanderVorst [15], defining a related functional on suitable *fractional* Sobolev spaces.

Consider the Laplacian as the operator

$$-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega) \ ,$$

and $\{e_i\}_{i=1}^\infty$ a corresponding system of orthogonal and L^2-normalized eigenfunctions, with eigenvalues $\{\lambda_i\}$. Then, writing

$$u = \sum_{n=1}^\infty a_n e_n \ , \quad \text{with} \quad a_n = \int_\Omega u e_n dx \ ,$$

we set

$$E^s = \{u \in L^2(\Omega) \ : \ \sum_{n=1}^{\infty} \lambda_n^s |a_n|^2 < \infty\}$$

and define a linear operator on $L^2(\Omega)$ by

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^{s/2} a_n e_n \ , \ \forall \ u \in D(A^s) := E^s \ .$$

The spaces E^s are *fractional* Sobolev spaces with the inner product

$$(u, v)_s = \int_\Omega A^s u A^s v dx \ ,$$

see Lions-Magenes [16], and we have

$$\begin{aligned} &E^s = H^s(\Omega) \ \text{ if } \ 0 \le s < \tfrac{1}{2} \ , \qquad E^{1/2} \subset H^{1/2}(\Omega) \ , \\ &E^s = \{u \in H^s(\Omega) \mid u|_{\partial\Omega} = 0\} \ \text{ if } \ \tfrac{1}{2} < s \le 2 \ , \ s \ne \tfrac{3}{2} \ , \ \text{and} \\ &E^{3/2} \subset \{u \in H^{3/2}(\Omega) \mid u|_{\partial\Omega} = 0\} \end{aligned}$$

By the Sobolev imbedding theorem we therefore have continuous imbeddings

$$E^s \subset L^p(\Omega) \ , \ \text{ if } \ \frac{1}{p} \ge \frac{1}{2} - \frac{s}{N} \ ,$$

and these imebbedings are compact if $\frac{1}{p} > \frac{1}{2} - \frac{s}{N}$.

2.2. The functional

With these definitions, we now define the Hilbert space $E := E^t \times E^s$, endowed with the norm

$$\|(u, v)\|_E = (\|u\|_{E^t}^2 + \|v\|_{E^s}^2)^{\frac{1}{2}}$$

On the space E we consider the functional

$$\begin{aligned} &I : E \to \mathbb{R} \ , \\ &I(u, v) = \int_\Omega A^t u A^s v - \int_\Omega (\frac{1}{p+1}|v|^{p+1} + F(u)) dx \end{aligned} \tag{6}$$

with s and t such that $s + t = 2$; loosely speaking, this means that we distribute the two derivatives given in the first term of the functional J, see (5), differently on the variables u and v. Of course, it is crucial to recuperate from critical points (u, v) of this functional solutions of system (3). We state this in the following

Proposition 2.1. *Suppose that $(u, v) \in E^t \times E^s$ is a critical point of the functional I, i.e. u and v are weak solutions of the system*

$$\left\{ \begin{aligned} &\int_\Omega A^t u A^s \phi = \int_\Omega v^p \phi \ , \ \forall \ \phi \in E^s \\ &\int_\Omega A^t \psi A^s v = \int_\Omega f(u) \psi \ , \ \forall \ \psi \in E^t \ . \end{aligned} \right. \tag{7}$$

Then $v \in W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$ and $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \forall \ q \ge 1$, and hence u and v are "strong" solutions of (3), i.e.

$$\left\{ \begin{aligned} &\int_\Omega (-\Delta u) \phi = \int_\Omega v^p \phi \ , \ \forall \ \phi \in C_0^\infty(\Omega) \\ &\int_\Omega (-\Delta v) \psi = \int_\Omega f(u) \psi \ , \ \forall \ \psi \in C_0^\infty(\Omega). \end{aligned} \right. \tag{8}$$

From this proposition follows by standard bootstrap arguments that u and v are classical solutions of (3) if f and Ω are smooth.

The proof of this proposition follows ideas of de Figueiredo - Felmer [11], and will be given in subsection 2.5.

In the following subsection we prove that there exist values s and t with $s+t=2$ such that the functional I is a well-defined C^1 functional, and that it has a non-trivial critical level.

2.3. The choice of the spaces E^s and E^t

We begin by proving the following Lemma:

Lemma 2.2.
Let $1 < p$ ($N = 2$), or $1 < p < 2$ ($N = 3$). Then there exist parameters $s > 0$ and $t > 0$ with $s + t = 2$ such that the following embeddings are continuous and compact:

$$E^s(\Omega) \subset L^{p+1}(\Omega) \quad , \qquad E^t(\Omega) \subset C^0(\Omega)$$

Proof. Note that $H^s(\Omega) \subset L^{p+1}(\Omega)$ compactly, iff $\frac{1}{p+1} > \frac{1}{2} - \frac{s}{N}$.
For $N = 2$, we get thus the condition

$$s > 1 - \frac{2}{p+1}$$

Choose $s < 1$ satisfying the previous condition, and set $t = 2 - s > 1$. We have a compact embedding $E^t(\Omega) \subset C^0(\Omega)$ for

$$\frac{t}{N} > \frac{1}{2} \ , \quad \text{i.e. for} \ \ t > 1 \ ;$$

and hence the Lemma holds for $N = 2$.

For $N = 3$, we get the condition

$$s > \frac{3}{2} - \frac{3}{p+1} \ .$$

Since

$$\sup\{\frac{3}{2} - \frac{3}{p+1} \mid 1 < p < 2\} = \frac{1}{2} \ ,$$

we can choose $s < \frac{1}{2}$, and then $t > \frac{3}{2}$, and hence $E^t(\Omega) \subset C^0(\Omega)$ compactly.

□

Thus, we now fix s and t as in Lemma 2.2, and define the functional $I(u, v)$ given by (6) on the space $E^t \times E^s =: E$.

In the next Lemma we collect a few properties of the operators A^s and the spaces E^s.

Lemma 2.3. *Let $s > 0$ and $t > 0$.*

1) $z \in E^s$ iff $A^s z \in L^2$, and $\|z\|_{E^s} = \|A^s z\|_{L^2}$

2) Let $z \in E^{s+t} = E^2 = H^2$; then $A^{s+t}z = A^s A^t z = A^t A^s z$.

Proof. 1) follows immediately from the definitions.

2) we have

$$A^{s+t}z = \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{(s+t)/2} e_i = \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{s/2} \lambda_i^{t/2} e_i = A^s \sum_{i\in\mathbb{N}} \alpha_i \lambda_i^{t/2} e_i = A^s A^t z$$

□

2.4. Existence of a non-trivial critical point

The functional $I(u,v) : E = E^t \times E^s$ is strongly indefinite near zero, in the sense that there exist infinite dimensional subspaces E^+ and E^- with $E^+ \oplus E^- = E$ such that the functional is (near zero) positive definite on E^+ and negative definite on E^-. Li-Willem [17] prove the following general existence theorem for such situations, which can be applied in our case:

Theorem 2.4 (Li-Willem, 1995).
Let $\Phi : E \to \mathbb{R}$ be a strongly indefinite C^1-functional satisfying
A1) Φ has a local linking at the origin, i.e. for some $r > 0$:

$$\Phi(z) \geq 0 \;\; for \;\; z \in E^+ \; , \; \|z\|_E \leq r \quad , \quad \Phi(z) \leq 0 \; , \; for \;\; z \in E^- \; , \; \|z\|_E \leq r \; .$$

A2) Φ maps bounded sets into bounded sets.
A3) Let E_n^+ be any n-dimensional subspace of E^+; then $\phi(z) \to -\infty$ as $\|z\| \to \infty$, $z \in E_n^+ \oplus E^-$.
A4) Φ satisfies the Palais-Smale condition (PS) (Li-Willem [17] require a weaker "(PS*)-condition", however, in our case the classical (PS) condition will be satisfied).
Then Φ has a nontrivial critical point.

We now verify that our functional satisfies the assumptions of this theorem.

First, it is clear, with the choices of s and t made above, that $I(u,v)$ is a C^1-functional on $E^s \times E^t$.

A1) Following de Figueiredo-Felmer [11] we can define the spaces

$$E^+ = \{(u, A^{t-s}u) \mid u \in E^t\} \; , \text{ and } E^- = \{(u, -A^{t-s}u) \mid u \in E^t\}$$

which give a natural splitting $E^+ \oplus E^- = E$. It is easy to see that $I(u,v)$ has a local linking with respect to E^+ and E^- at the origin.

A2) Let $B \subset E^t \times E^s$ be a bounded set, i.e. $\|u\|_{E^t} \leq c$, $\|v\|_{E^s} \leq c$, for all $(u,v) \in B$. Then

$$\begin{aligned} |I(u,v)| &\leq \|A^t u\|_{L^2} \|A^s v\|_{L^2} + \int_\Omega |v|^{p+1} + \int_\Omega |f(u)| \\ &\leq \|u\|_{E^t} \|v\|_{E^s} + c\|v\|_{E^s}^{p+1} + \sup_{x\in\Omega} |f(u(x))| \cdot |\Omega| \leq C \end{aligned}$$

A3) Let $z_k = z_k^+ + z_k^- \in E = E_n^+ \oplus E^-$ denote a sequence with $\|z_k\|_E \to \infty$. By the above, z_k may be written as

$$z_k = (u_k, A^{t-s}u_k) + (w_k, -A^{t-s}w_k) \; , \quad \text{with} \;\; u_k \in E_n^t, \;\; w_k \in E^t \; ,$$

where E_n^t denotes an $n-$dimensional subspace of E^t. Thus, the functional $I(z_k)$ takes the form

$$\begin{aligned} I(z_k) &= \int_\Omega A^t u_k A^s A^{t-s} u_k - \int_\Omega A^t w_k A^s A^{t-s} w_k - \\ &\qquad -\frac{1}{p+1} \int_\Omega |A^{t-s}(u_k - w_k)|^{p+1} - \int_\Omega F(u_k + w_k) \\ &= \int_\Omega |A^t u_k|^2 - \int_\Omega |A^t w_k|^2 - \frac{1}{p+1} \int_\Omega |A^{t-s}(u_k - w_k)|^{p+1} - \int_\Omega F(u_k + w_k) \end{aligned}$$

Note that $\|z_k\| \to \infty \Longleftrightarrow \int |A^t u_k|^2 + \int |A^t w_k|^2 = \|u_k\|_{E^t}^2 + \|w_k\|_{E^t}^2 \to \infty$.

Now, if

1) $\|u_k\|_{E^t} \leq c$, then $\|w_k\|_{E^t} \to \infty$, and then $I(z_k) \to -\infty$

2) $\|u_k\|_{E^t} \to \infty$, then we estimate (c, c_1 and c_2 are positive constants) using the fact that $t - s > 0$ and $p > 1$

$$\int_\Omega |A^{t-s}(u_k - w_k)|^{p+1} \geq c \left(\int_\Omega |A^{t-s}(u_k - w_k)|^2 \right)^{\frac{p+1}{2}} \geq c_1 \|u_k - w_k\|_{L^2}^{p+1}$$

and

$$\int_\Omega F(u_k + w_k) \geq c_2 \int_\Omega |u_k + w_k|^{p+1} - d \geq c_1 \|u_k + w_k\|_{L^2}^{p+1} - d$$

and hence we obtain the estimate

$$I(z_k) \leq \frac{1}{2}\|u_k\|_{E^t}^2 - c_1(\|u_k - w_k\|_{L^2}^{p+1} + \|u_k + w_k\|_{L^2}^{p+1}) + d$$

Since $\phi(t) = t^{p+1}$ is convex, we have $\frac{1}{2}(\phi(t) + \phi(s)) \geq \phi(\frac{1}{2}(s+t))$, and hence

$$\begin{aligned} I(z_k) &\leq \frac{1}{2}\|u_k\|_{E^t}^2 - c_1 \frac{1}{2^p}(\|u_k - w_k\|_{L^2} + \|u_k + w_k\|_{L^2})^{p+1} + d \\ &\leq \frac{1}{2}\|u_k\|_{E^t}^2 - c_1 \frac{1}{2^p}\|u_k\|_{L^2}^{p+1} + d \end{aligned}$$

Since on E_n^t the norms $\|u_k\|_{E^t}$ and $\|u_k\|_{L^2}$ are equivalent, we conclude that also in this case $J(z_k) \to -\infty$.

A4) Let $\{z_n\} \subset E$ denote a (PS)-sequence, i.e. such that

$$|I(z_n)| \to c \ , \quad \text{and} \quad |(\Phi'(z_n), \eta)| \leq \epsilon_n \|\eta\|_E \ , \ \forall\, \eta \in E, \quad \text{and} \quad \epsilon_n \to 0 \tag{9}$$

We first show:

Lemma 2.5. *The (PS)-sequence $\{z_n\}$ is bounded in E.*

Proof. By (9) we have for $z_n = (u_n, v_n)$

$$I(u_n, v_n) = \int_\Omega A^t u_n A^s v_n - \frac{1}{p+1}\int_\Omega v_n^{p+1} - \int_\Omega F(u_n) \to c \tag{10}$$

$$I'(u_n, v_n)(\phi, \psi) = \int_\Omega A^t u_n A^s \psi + \int_\Omega A^s v_n A^t \phi - \int_\Omega v_n^p \psi - \int_\Omega f(u_n)\phi = \epsilon_n \|(\phi, \psi)\|_E \tag{11}$$

Choosing $(\phi, \psi) = (u_n, v_n) \in E^t \times E^s$ we get by (11)

$$2\int_\Omega A^t u_n A^s v_n - \int v_n^{p+1} - \int_\Omega f(u_n) u_n = \epsilon_n (\|u_n\|_{E^t} + \|v_n\|_{E^s}) \tag{12}$$

and subtracting this from $2\, I(u_n, v_n)$ we obtain, using assumption 2) of Theorem 1.1

$$(1 - \frac{2}{p+1})\int_\Omega v_n^{p+1} + (1 - \frac{2}{\theta})\int_\Omega f(u_n) u_n \leq C + \epsilon_n (\|u_n\|_{E^t} + \|v_n\|_{E^s}) \tag{13}$$

and thus

$$\int_\Omega v_n^{p+1} \leq C + \epsilon_n (\|u_n\|_{E^t} + \|v_n\|_{E^s}) \tag{14}$$

$$\int_\Omega f(u_n) u_n \leq C + \epsilon_n (\|u_n\|_{E^t} + \|v_n\|_{E^s}) \tag{15}$$

Choosing $(\phi, \psi) = (0, A^{t-s} u_n) \in E^t \times E^s$ in (11) we get

$$\int_\Omega |A^t u_n|^2 = \int_\Omega v_n^p A^{t-s} u_n + \epsilon_n \|A^{t-s} u_n\|_{E^s}$$

and hence by Hölder

$$\|u_n\|_{E^t}^2 = \|A^t u_n\|_{L^2}^2 \leq (\int_\Omega |v_n|^{p+1})^{\frac{p}{p+1}} (\int_\Omega |A^{t-s} u_n|^{p+1})^{\frac{1}{p+1}} + \epsilon_n \|u_n\|_{E^t}$$

Noting that

$$(\int_\Omega |A^{t-s} u_n|^{p+1})^{\frac{1}{p+1}} \leq c\|A^{t-s} u_n\|_{E^s} = c\|A^t u_n\|_{L^2} = c\|u_n\|_{E^t}$$

we obtain, using (14)

$$\|u_n\|_{E^t}^2 \leq [C + \epsilon_n (\|u_n\|_{E^t} + \|v_n\|_{E^s})]^{p/(p+1)} \cdot c\|u_n\|_{E^t} + \epsilon_n \|u_n\|_{E^t}$$

and thus

$$\|u_n\|_{E^t} \leq C + \epsilon_n (\|u_n\|_{E^t} + \|v_n\|_{E^s})^{p/(p+1)} \tag{16}$$

Similarly as above we note that $A^{s-t}v_n \in E^t$, and thus, choosing $(\phi, \psi) = (A^{s-t}v_n, 0) \in E^t \times E^s$ in (11) we get

$$\int_\Omega |A^s v_n|^2 = \int_\Omega f(u_n) A^{s-t} v_n + \epsilon_n \|A^{s-t} v_n\|_{E^t} \leq \|A^{s-t} v_n\|_\infty \int_\Omega |f(u_n)| + \epsilon_n \|v_n\|_{E^s}$$

Using that $\|A^{s-t}v_n\|_{E^t} = \|A^s v_n\|_{L^2} = \|v_n\|_{E^s}$, and the fact that $E^t \subset C^0$ we then obtain, using (15)

$$\begin{aligned} \|v_n\|_{E^s} &\leq c \textstyle\int_\Omega |f(u_n)| + \epsilon_n = \int_{[|u_n| \leq s_0]} \max_{|t| \leq s_0} |f(t)| + \int_{[|u_n| > s_0]} f(u_n) u_n + \epsilon_n \\ &\leq C + \epsilon_n (\|u_n\|_{E^t} + \|v_n\|_{E^s}) \end{aligned} \tag{17}$$

Joining (16) and (17) we finally get

$$\|u_n\|_{E^t} + \|v_n\|_{E^s} \leq C + 2\epsilon_n (\|u_n\|_{E^t} + \|v_n\|_{E^s})$$

Thus, $\|u_n\|_{E^t} + \|v_n\|_{E^s}$ is bounded. □

With this it is now possible to complete the proof of the (PS)-condition: since $\|u_n\|_{E^t}$ is bounded, we find a weakly convergent subsequence $u_n \rightharpoonup u$ in E^t. Since the mappings $A^t : E^t \to L^2$ and $A^{-s} : L^2 \to E^s$ are continuous isomorphisms, we get $A^t(u_n - u) \rightharpoonup 0$ in L^2 and $A^{t-s}(u_n - u) \rightharpoonup 0$ in E^s. Since $E^s \subset L^{p+1}$ compactly, we conclude that $A^{t-s}(u_n - u) \to 0$ strongly in L^{p+1}.

Similarly, we find a subsequence of $\{v_n\}$ which is weakly convergent in E^s and such that v_n^p is strongly convergent in $L^{\frac{p+1}{p}}$

Choosing $(\phi, \psi) = (0, A^{t-s}(u_n - u) \in E^t \times E^s$ in (11) we thus conclude

$$\int_\Omega A^t u_n A^t (u_n - u) = \int_\Omega v_n^p A^{t-s}(u_n - u) + \epsilon_n \|A^{t-s}(u_n - u)\|_{E^s} \tag{18}$$

By the above considerations, the righthand-side converges to 0, and thus

$$\int_\Omega |A^t u_n|^2 \to \int_\Omega |A^t u|^2$$

Thus, $u_n \to u$ strongly in E^t.

To obtain the strong convergence of $\{v_n\}$ in E^s, one proceeds similarly: as above, one finds a subsequence $\{v_n\}$ converging weakly in E^s to v, and then $A^{s-t}v_n \rightharpoonup A^{s-t}v$ weakly in A^t and $A^{s-t}v_n \to A^{s-t}v$ strongly in C^0. Choosing in (9) $(\phi, \psi) = (A^{s-t}(v_n - v), 0)$, we get

$$\int_\Omega A^s(v_n - v) A^s v_n = \int f(u_n) A^{s-t}(v_n - v) + \epsilon_n (\|A^{s-t}(v_n - v)\|) \tag{19}$$

The first term on the right is estimated by $\|A^{s-t}(v_n - v)\|_{C^0} \int_\Omega |f(u_n)| \to 0$, and thus one concludes again that

$$\int_\Omega |A^s v_n|^2 \to \int_\Omega |A^s v|^2$$

and hence also $v_n \to v$ strongly in E^s.

Thus, the conditions of Theorem 2.4 are satisfied; hence, we find a positive critical point (u, v) for the functional I, which yields a weak solution to system (3).

2.5. Strong solutions

In this section we prove Proposition 2.1.

Consider the first equation in the system (7). We can follow the arguments of [11]: If $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$\int_\Omega A^t u A^s \phi = \int_\Omega u A^2 \phi = \int_\Omega u(-\Delta\phi) \tag{20}$$

On the other hand, $v^p \in L^{\frac{p+1}{p}}(\Omega)$, and hence (see [13]) there exists a unique solution

$$y \in W^{2,\frac{p+1}{p}}(\Omega) \quad \text{of} \quad -\Delta y = v^p \ .$$

By the choice of s we have $\frac{1}{p+1} > \frac{1}{2} - \frac{s}{N}$, which is equivalent to $\frac{1}{2} > \frac{p}{p+1} - \frac{s}{N}$, which in turn implies that $W^{2,\frac{p+1}{p}}(\Omega) \subset L^2(\Omega)$. Thus, we conclude that

$$\int_\Omega v^p \phi = \int_\Omega (-\Delta y)\phi = \int_\Omega y(-\Delta\phi) \ , \ \forall \ \phi \in H^2(\Omega) \cap H_0^1(\Omega) \tag{21}$$

Comparing (20) and (21) yields

$$\int_\Omega (y-u)(-\Delta\phi) = 0 \ , \ \forall \ \phi \in H^2(\Omega) \cap H_0^1(\Omega)$$

and hence $u = y$; thus $u \in W^{2,\frac{p+1}{p}}(\Omega)$.

Consider now the second equation in system (7). Again, for $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ we have

$$\int_\Omega (-\Delta\psi)v = \int_\Omega A^t\psi A^s v = \int_\Omega f(u)\psi \ , \ \forall \ \psi \in E^t \ .$$

On the other hand, $E^t \subset \{u \in H^t(\Omega) \mid u|_{\partial\Omega} = 0\} \subset C^\lambda(\Omega)$, with $\lambda = t - \frac{N}{2}$.

By our choices of s and t we have

$$\begin{cases} 1 < t < 2 \ , & N = 2 \\ \frac{3}{2} < t < 2 \ , & N = 3 \end{cases}$$

and hence in both cases $u \in C^\lambda(\Omega)$ with $\lambda > 0$. This implies that $f(u) \in L^\infty(\Omega)$, and hence there exists a unique solution

$$w \in W^{2,q}(\Omega) \ , \ \forall \ q \geq 1 \ , \quad \text{of} \quad -\Delta w = f(u)$$

Note that if $f \in C^\lambda$ and $\partial\Omega$ is sufficiently smooth, then $w \in C^{2,\lambda}(\Omega)$.
We finish by concluding as above that $w = v$, and that therefore $v \in W^{2,q}, \forall \, q \geq 1$, respectively $v \in C^{2,\lambda}(\Omega)$.

3. Proof: the case $p \leq 1$

In this section we consider the cases $0 < p \leq 1$ ($N = 2, 3$) and $0 < p < \frac{2}{N-2}$ ($N \geq 4$), i.e. we consider the situation where one equation has a sublinear nonlinearity in the form of a power, and the other equation has a superlinear nonlinearity.

3.1. The functional

We consider now the system

$$\begin{cases} -\Delta u = v^p \ , & \text{with } 0 < p \leq 1 \\ -\Delta v = f(u) \end{cases} \tag{22}$$

System (22) can be written as

$$\begin{cases} (-\Delta u)^{1/p} = v \ , & \text{with } 0 < p \leq 1 \\ -\Delta v = f(u) \end{cases} \tag{23}$$

and thus we have the equivalent equation

$$\begin{cases} -\Delta(-\Delta u)^{1/p} = -\Delta v = f(u) \\ u = \Delta u = 0 \qquad \partial\Omega \end{cases} \tag{24}$$

To equation (24) we may associate the following functional

$$I(u) = \frac{p}{p+1} \int_\Omega |\Delta u|^{\frac{p+1}{p}} - \int_\Omega F(u) \ . \tag{25}$$

Indeed, the derivative of $I(u)$ in direction v yields

$$I'(u)\, v = \int_\Omega (-\Delta u)^{1/p}(-\Delta v) - \int_\Omega f(u)v \ ,$$

and thus critical points of I correspond to weak solutions of equation (23) and thus of system (22).

3.2. Existence of critical points

Note that the first term of the functional I is defined on the space $E = W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$. Since by assumption $p < \frac{2}{N-2}$ we have $\frac{p+1}{p} > 1 + \frac{N-2}{2} > \frac{N}{2}$, and thus

$$W^{2,\frac{p+1}{p}}(\Omega) \ \subset\subset \ C(\Omega)$$

Thus, the second term of the functional I is defined if F is continuous, and no growth restriction on F is necessary. Since F is differentiable, the functional I is a well-defined C^1-functional on the space E.

We now show that the classical mountain-pass theorem of Ambrosetti-Rabinowitz may be applied to the functional I. Indeed, I has a local minimum in the origin:

$$I(u) = \frac{p}{p+1} \int_\Omega |\Delta u|^{\frac{p+1}{p}} - \int_\Omega F(u) \geq c\, \frac{p}{p+1}\, \|u\|_C^{\frac{p+1}{p}} - o(\|u\|_C^{\frac{p+1}{p}})$$

Next, let u_1 be any fixed element of E. Then

$$I(su_1) \leq \frac{p}{p+1}\, s^{\frac{p+1}{p}} \int_\Omega |\Delta u_1|^{\frac{p+1}{p}} - s^\theta \|u\|_C^\theta + d$$

with $\theta > \frac{p+1}{p}$ (by assumption), and thus $I(su_1) \to -\infty$ as $s \to \infty$.

Finally, we show that I satisfies the Palais-Smale condition (PS). Let $(u_n) \subset E$ be a (PS)-sequence, i.e.

$$|I(u_n)| \leq c \quad , \quad \text{and} \quad |I'(u_n)v| \leq \epsilon_n \|v\|_E \ , \ \epsilon_n \to 0 \ , \ \forall\, v \in E \ .$$

We have

$$\begin{aligned} c + \epsilon_n \|u_n\|_E \quad & \geq |\theta I(u_n) - I'(u_n)u_n| \\ & \geq (\theta \tfrac{p}{p+1} - 1) \textstyle\int_\Omega |\Delta u_n|^{\frac{p+1}{p}} - \theta \int_\Omega F(u_n) + \int_\Omega f(u_n)u_n \\ & \geq (\theta \tfrac{p}{p+1} - 1) \textstyle\int_\Omega |\Delta u_n|^{\frac{p+1}{p}} - c \\ & \geq \delta \|u\|_E^{\frac{p+1}{p}} - c \ , \end{aligned}$$

and thus (u_n) is bounded in E. Since E is compactly imbedded in $C(\Omega)$, we find a convergent subsequence in $C(\Omega)$, and then it is standard to conclude that u_n converges strongly also in E.

Thus, by the Mountain-Pass theorem we obtain a (non-trivial) critical point u, which gives rise to a solution to system (3).

3.3. Proof of Theorem 1.2

The proof follows the same lines as in section 3.2. We just observe that for $\frac{2}{N-2} \leq p \leq 1$

$$W^{2,\frac{p+1}{p}}(\Omega) \subset L^{\frac{N(p+1)}{Np-2(p+1)}}(\Omega) \ .$$

The exponent $\frac{N(p+1)}{Np-2(p+1)}$ satisfies

$$\frac{1}{p+1} + \frac{1}{\frac{N(p+1)}{Np-2(p+1)}} = 1 - \frac{2}{N} \ ,$$

i.e. we are on the critical hyperbola. Hence, for $q+1 < \frac{N(p+1)}{Np-2(p+1)}$ we are below the hyperbola, and we have $E \subset\subset L^{q+1}(\Omega)$ compactly. We can then proceed exactly as above, to obtain a critical point via the Mountain-Pass theorem.

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