DETERMINANTS OF BLOCK TRIDIAGONAL MATRICES

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Abstract

An identity is proven that evaluates the determinant of a block tridiagonal matrix with (or without) corners as the determinant of the associated transfer matrix (or a submatrix of it).

Key words: Block tridiagonal matrix, transfer matrix, determinant
1991 MSC: 15A15, 15A18, 15A90

1 Introduction

A tridiagonal matrix with entries given by square matrices is a block tridiagonal matrix; the matrix is banded if off-diagonal blocks are upper or lower triangular. Such matrices are of great importance in numerical analysis and physics, and to obtain general properties is of great utility. The blocks of the inverse matrix of a block tridiagonal matrix can be factored in terms of two sets of matrices[10], and decay rates of their matrix elements have been investigated[14]. While the spectral properties of tridiagonal matrices have been under study for a long time, those of tridiagonal block matrices are at a very initial stage[1,2].

What about determinants? A paper by El-Mikkawy[4] on determinants of tridiagonal matrices triggered two interesting generalizations for the evaluation of determinants of block-tridiagonal and general complex block matrices, respectively by Salkuyeh[15] and Sogabe[17]. These results encouraged me to re-examine a nice identity that I derived in the context of transport[11], and

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extend it as a mathematical result for general block-tridiagonal complex matrices.

For ordinary tridiagonal matrices, determinants can be evaluated via multiplication of 2×2 matrices:

$$\det\begin{bmatrix} a_1 & b_1 & & c_0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ b_n & & c_{n-1} & a_n \end{bmatrix} = (-1)^{n+1}(b_n\cdots b_1 + c_{n-1}\cdots c_0)$$
$$+\,\mathrm{tr}\left[\begin{pmatrix} a_n & -b_{n-1}c_{n-1} \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_2 & -b_1c_1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & -b_nc_0 \\ 1 & 0 \end{pmatrix}\right] \tag{1}$$

$$\det\begin{bmatrix} a_1 & b_1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & c_{n-1} & a_n \end{bmatrix} = \left[\begin{pmatrix} a_n & -b_{n-1}c_{n-1} \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_2 & -b_1c_1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 1 & 0 \end{pmatrix}\right]_{11} \tag{2}$$

Do these procedures generalize to block-tridiagonal matrices? The answer is affirmative. If the matrix has corner blocks, the determinant is proportional to that of an associated transfer matrix, in general of much smaller size. The proof is simple and is given in section 2. A simple modification yields a formula for the determinant when corner blocks are absent, and is given in section 3. The relation with Salkuyeh's recursion formula is then shown.

2 The Duality Relation

Consider the following block-tridiagonal matrix $\mathrm{M}(z)$ with blocks A_i, B_i and C_{i-1} ($i = 1,\dots,n$) that are complex $m\times m$ matrices. It is very useful to introduce also a complex parameter z in the corner blocks:

$$\mathrm{M}(z) = \begin{bmatrix} A_1 & B_1 & & \frac{1}{z}C_0 \\ C_1 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ zB_n & & C_{n-1} & A_n \end{bmatrix} \tag{3}$$

It is required that off-diagonal blocks are nonsingular: $\det B_i \neq 0$ and $\det C_{i-1} \neq 0$ for all i. As it will be explained, the matrix is naturally associated with a *transfer matrix*, built as the product of n matrices of size $2m \times 2m$:

$$\mathrm{T} = \begin{bmatrix} -B_n^{-1}A_n & -B_n^{-1}C_{n-1} \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} -B_1^{-1}A_1 & -B_1^{-1}C_0 \\ I_m & 0 \end{bmatrix} \tag{4}$$

where I_m is the $m \times m$ unit matrix. The transfer matrix is nonsingular, since

$$\det \mathrm{T} = \prod_{i=1}^{n} \det[B_i^{-1}C_{i-1}] \tag{5}$$

The main result, the duality relation, relies on the following lemma:

Lemma 1 $\qquad \det \mathrm{M}(z) \;=\; \dfrac{(-1)^{nm}}{(-z)^m} \det[\,\mathrm{T} - z\, I_{2m}]\, \det[\, B_1 \dots B_n\,]$

Proof: The equation $\mathrm{M}(z)\Psi = 0$ has a nontrivial solution provided that $\det \mathrm{M}(z) = 0$, and corresponds to the following linear system in terms of the blocks of the matrix and the components $\psi_k \in \mathbb{C}^m$ of the null vector Ψ:

$$A_1\,\psi_1 + B_1\,\psi_2 + z^{-1}C_0\,\psi_n \;=\; 0 \tag{6}$$
$$B_k\psi_{k+1} + A_k\psi_k + C_{k-1}\psi_{k-1} = 0 \qquad (k = 2, \dots, n-1) \tag{7}$$
$$z\,B_n\psi_1 + A_n\,\psi_n + C_{n-1}\,\psi_{n-1} \;=\; 0 \tag{8}$$

The equations (7) are recursive and can be put in the form

$$\begin{bmatrix} \psi_{k+1} \\ \psi_k \end{bmatrix} = \begin{bmatrix} -B_k^{-1}A_k & -B_k^{-1}C_{k-1} \\ I_m & 0 \end{bmatrix} \begin{bmatrix} \psi_k \\ \psi_{k-1} \end{bmatrix}$$

and iterated. Inclusion of the boundary equations (6) and (8) produces an eigenvalue equation for the full transfer matrix (4) that involves only the end vector-components:

$$\mathrm{T} \begin{bmatrix} \psi_1 \\ \frac{1}{z}\psi_n \end{bmatrix} \;=\; z\, \begin{bmatrix} \psi_1 \\ \frac{1}{z}\psi_n \end{bmatrix} \tag{9}$$

Equation (9) has a nontrivial solution if and only if $\det[\mathrm{T} - zI_{2m}] = 0$, which is dual to the condition $\det \mathrm{M}(z) = 0$. Both $z^m \det \mathrm{M}(z)$ and $\det[\mathrm{T} - zI_{2m}]$

are polynomials in z of degree $2m$ and share the same roots, which cannot be zero by (5). Therefore, the polynomials coincide up to a constant of proportionality, which is found by considering the limit case of large z: $\det \mathrm{M}(z) \approx (-1)^{nm}(-z)^m \det[B_1 \cdots B_n]$. ■

Before proceeding, let us show that in the special case of tridiagonal matrices with corners ($m = 1$), Lemma 1 with $z = 1$ yields (1).

The factorization

$$\begin{pmatrix} -\frac{a_{k-1}}{b_{k-1}} & -\frac{c_{k-2}}{b_{k-1}} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{b_{k-1}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{k-1} & -c_{k-2}b_{k-2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{b_{k-2}} \end{pmatrix} \tag{10}$$

is introduced for all factors in the transfer matrix T and produces intermediate factors $\frac{1}{b_k} I_2$ that commute, and allow us to simplify the determinant of the lemma:

$$\begin{aligned}
&\det \left[\begin{pmatrix} -\frac{a_n}{b_n} & -\frac{c_{n-1}}{b_n} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -\frac{a_1}{b_1} & -\frac{c_0}{b_1} \\ 1 & 0 \end{pmatrix} - I_2 \right] \\
&= \det \left[\frac{(-1)^{n-1}}{b_1 \cdots b_{n-1}} \begin{pmatrix} -\frac{1}{b_n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_n & -b_{n-1}c_{n-1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & -b_n c_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{b_n} \end{pmatrix} - I_2 \right] \\
&= \frac{1}{b_1^2 \cdots b_n^2} \det \left[\begin{pmatrix} a_n & -b_{n-1}c_{n-1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & -b_n c_0 \\ 1 & 0 \end{pmatrix} - (-1)^n b_1 \cdots b_n I_2 \right] \\
&= \frac{z_1 z_2}{b_1^2 \cdots b_n^2} - (-1)^n \frac{z_1 + z_2}{b_1 \cdots b_n} + 1 \\
&= -\frac{(-1)^n}{b_1 \cdots b_n} [(z_1 + z_2) - (-1)^n (b_1 \cdots b_n + c_0 \cdots c_{n-1})]
\end{aligned}$$

z_1 and z_2 are the eigenvalues of the transfer matrix in (1), whose trace is $z_1 + z_2$ and whose determinant is $z_1 z_2 = (b_1 \cdots b_n)(c_0 \cdots c_{n-1})$. ■

Multiplication of Lemma 1 by $\det \mathrm{T}^{-1}$ gives a variant of it:

$$\det \mathrm{M}(z) = (-1)^{nm}(-z)^m \det(\mathrm{T}^{-1} - \frac{1}{z}) \det[C_0 \ldots C_{n-1}]$$

Multiplication of Lemma 1 by the previous equation, with parameter $1/z$, gives another variant:

$$\det \mathrm{M}(z) \det \mathrm{M}(1/z) \;=\; \det \left[\mathrm{T} + \mathrm{T}^{-1} - \left(z + \frac{1}{z}\right)\right] \det[B_1 C_0 \dots B_n C_{n-1}]$$

Instead of $\mathrm{M}(z)$, consider the matrix $\mathrm{M}(z) - \lambda I_{nm}$ and the corresponding transfer matrix $\mathrm{T}(\lambda)$ obtained by replacing the entries A_i with $A_i - \lambda I_m$. Then Lemma 1 has a symmetric form, where the roles of eigenvalue and parameter exchange between the matrices. For this reason it is called a *duality relation*.

Theorem 1 (The Duality Relation)

$$\det[\lambda I_{nm} - \mathrm{M}(z)] \;=\; (-z)^{-m} \det[\,\mathrm{T}(\lambda) - z I_{2m}\,] \det[B_1 \cdots B_n]$$

It shows that the parameter z, which enters in $\mathrm{M}(z)$ as a boundary term, is related to eigenvalues of the matrix $\mathrm{T}(\lambda)$ that connects the eigenvector of $\mathrm{M}(z)$ at the boundaries.

The duality relation was initially obtained and discussed for Hermitian block matrices[11–13]. For $n = 2$ it is due to Lee and Ioannopoulos[9]. Here I have shown that it holds for generic block-tridiagonal matrices, and the proof given is even simpler. The introduction of corner values z and $1/z$ in Hermitian tridiagonal matrices ($c_k = b_k^*$) was proposed by Hatano and Nelson [7] in a model for vortex depinning in superconductors, as a tool to link the decay of eigenvectors to the permanence of corresponding eigenvalues on the real axis. It has been a subject of intensive research[16,5,6,18]. The generalization to block matrices is interesting for the study of transport in discrete structures such as nanotubes or molecules[8,3,19].

3 Block tridiagonal matrix with no corners

By a modification of the proof of the lemma, one obtains an identity for the determinant of block-tridiagonal matrices $\mathrm{M}^{(0)}$ with no corners ($B_n = C_0 = 0$ in the matrix (3)):

Theorem 2 $\quad \det \mathrm{M}^{(0)} = (-1)^{nm} \det[\mathrm{T}_{11}^{(0)}] \det[B_1 \cdots B_{n-1}]$

where $\mathrm{T}_{11}^{(0)}$ *is the upper left block of size* $m \times m$ *of the transfer matrix*

$$\mathrm{T}^{(0)} = \begin{bmatrix} -A_n & -C_{n-1} \\ I_m & 0 \end{bmatrix} \begin{bmatrix} -B_{n-1}^{-1} A_{n-1} & -B_{n-1}^{-1} C_{n-2} \\ I_m & 0 \end{bmatrix} \cdots \begin{bmatrix} -B_1^{-1} A_1 & -B_1^{-1} \\ I_m & 0 \end{bmatrix}$$

Proof: The linear system $\mathrm{M}^{(0)}\Psi = 0$ can be translated into the following equation, via the transfer matrix technique:

$$\begin{bmatrix} \psi_n \\ -C_{n-1}^{-1} A_n \psi_n \end{bmatrix} = \begin{bmatrix} -B_{n-1}^{-1} A_{n-1} & -B_{n-1}^{-1} C_{n-2} \\ I_m & 0 \end{bmatrix} \times \dots \\ \times \begin{bmatrix} -B_2^{-1} A_2 & -B_2^{-1} C_1 \\ I_m & 0 \end{bmatrix} \begin{bmatrix} -B_1^{-1} A_1 \psi_1 \\ \psi_1 \end{bmatrix} \tag{11}$$

Right multiplication by the nonsingular matrix

$$\begin{bmatrix} -A_n & -C_{n-1} \\ I_m & 0 \end{bmatrix}$$

and rewriting the right-hand vector as the product

$$\begin{bmatrix} -B_1^{-1} A_1 & -B_1^{-1} \\ I_m & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix}$$

transform (11) into an equation for the transfer matrix $\mathrm{T}^{(0)}$, that connects the boundary components with $\psi_{n+1} = 0$ and $\psi_0 = 0$:

$$\begin{bmatrix} 0 \\ \psi_n \end{bmatrix} = \mathrm{T}^{(0)} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \tag{12}$$

Equation (12) implies that $\det \mathrm{T}_{11}^{(0)} = 0$, which is dual to $\det \mathrm{M}^{(0)} = 0$. The implication translates into an identity by introducing the parameter λ and comparing the polynomials $\det[\lambda I_{nm} - \mathrm{M}^{(0)}]$ and $\det \mathrm{T}^{(0)}(\lambda)$ (obtained by replacing blocks A_i with $A_i - \lambda I_m$). Since both are polynomials in λ of degree nm and with the same roots, they must be proportional. Their behaviour for large λ fixes the constant. ■

For tridiagonal matrices ($m = 1$) blocks are just scalars and, by means of (10), one shows Theorem 2 simplifies to (2).

The formula for the evaluation of $\det \mathrm{M}^{(0)}$ requires $n-1$ inversions B_k^{-1}, multiplication of n matrices of size $2m \times 2m$, and the final evaluation of a determinant. Salkuyeh[15] proposed a different procedure for the evaluation of the same determinant:

$$\det \mathrm{M}^{(0)} = \prod_{k=1}^{n} \det \Lambda_k$$

$$\Lambda_k = A_k - C_{k-1}\Lambda_{k-1}^{-1}B_{k-1}, \qquad \Lambda_1 = A_1$$

It requires $n-1$ inversions of matrices of size $m \times m$, and the evaluation of their determinants. I show that the two procedures are related.

The transfer matrix $\mathrm{T}^{(0)} = \mathrm{T}(n)$ is the product of n matrices. Let $T(k)$ be the partial product of k matrices. Then:

$$\mathrm{T}(k) = \begin{bmatrix} -B_k^{-1}A_k & -B_k^{-1}C_{k-1} \\ I_m & 0 \end{bmatrix} \mathrm{T}(k-1)$$

This produces a two-term recurrence relation for blocks

$$\mathrm{T}(k)_{11} = -B_k^{-1}A_k\mathrm{T}(k-1)_{11} - B_k^{-1}C_{k-1}\mathrm{T}(k-2)_{11}$$

with $T(1)_{11} = -B_1^{-1}A_1$ and $T(0)_{11} = I_m$. The equations by Salkuyeh result for $\Lambda_k = -B_k\mathrm{T}(k)_{11}[\mathrm{T}(k-1)_{11}]^{-1}$.

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