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REMARKS ON MULTISYMPLECTIC REDUCTION

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Abstract

The problem of reduction of multisymplectic manifolds by the action of Lie groups is stated and discussed, as a previous step to give a fully covariant scheme of reduction for classical field theories with symmetries.

Key words: *Multisymplectic manifolds, actions of Lie groups, reduction, momentum maps, classical field theories.*

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1 Introduction

Multisymplectic manifolds constitute one of the most used and generic geometric frameworks for classical field theories. Then, the covariant reduction of field theories by symmetries requires, as a previous step, to study the reduction of multisymplectic manifolds.

This procedure should be based on the pioneering Marsden-Weinstein technique of reduction of symplectic manifolds [32] which was applied later to many different situations such as, for instance, the reduction of autonomous and non-autonomous Hamiltonian and Lagrangian systems (regular or singular) [1, 3, 10, 17, 18, 22, 24, 37], non-holonomic systems [4, 7, 11, 27], control systems [6, 21, 33, 35, 39, 40], and in other cases (Poisson, Dirac, Euler-Poincaré, Routh and implicit reduction) [5, 16, 29, 30, 31].

In relation to the problem of reduction of classical field theories, only partial results have been achieved in the context of Lagrangian, Poisson and Euler-Poincaré reduction, and other particular situations in multisymplectic field theories [8, 13, 14, 15, 23, 26, 38, 41]. Nevertheless, the problem of establishing a complete scheme of reduction for the multisymplectic case (in the line of the Marsden-Weinstein theorem), which should give a fully covariant reduction of the theory, is still unsolved.

The aim of this letter is to review the statement of the problem and give some insights in this way. In particular, a brief discussion about Noether invariants in Lagrangian field theory illustrates these considerations.

In this paper, manifolds are real, paracompact, connected and C^∞ , and maps are C^∞ .

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2 Multisymplectic manifolds. Actions of Lie groups

Let \mathcal{M} be a m -dimensional differentiable manifold, and $\Omega \in \Omega^{k+1}(\mathcal{M})$ a differentiable form in \mathcal{M} ($k+1 \leq m$). For every $x \in \mathcal{M}$, the form Ω_x establish a correspondence $\hat{\Omega}_r(x)$ between the set of r -vectors, $\Lambda^r T_x \mathcal{M}$, and the set of $(k+1-r)$ -forms, $\Lambda^{k+1-r} T_x^* \mathcal{M}$, as

$$\hat{\Omega}_r(x): \Lambda^r T_x \mathcal{M} \longrightarrow \Lambda^{k+1-r} T_x^* \mathcal{M} \quad ; \quad v \mapsto i(v)\Omega_x .$$

If v is homogeneous, $v = v_1 \wedge \dots \wedge v_r$, then $i(v)\Omega_x = i(v_1 \wedge \dots \wedge v_r)\Omega_x = i(v_1) \dots i(v_r)\Omega_x$. Thus, an r -vector field $X \in \mathfrak{X}^r(\mathcal{M})$ (that is, a section of $\Lambda^r T\mathcal{M}$) defines a contraction $i(X)$ of degree r of the algebra of differential forms in \mathcal{M} . The $(k+1)$ -form Ω is 1-*nondegenerate* if $\ker \Omega := \{X \in \mathfrak{X}(\mathcal{M}) \mid \hat{\Omega}_1(x)(X) = 0; x \in \mathcal{M}\} = \{0\}$.

A couple (\mathcal{M}, Ω) is a *multisymplectic manifold* if Ω is closed and 1-*nondegenerate*. The degree $k+1$ of the form Ω will be called *the degree of the multisymplectic manifold*. $X \in \mathfrak{X}(\mathcal{M})$ is a *Hamiltonian vector field* if $i(X)\Omega$ is an exact k -form; that is, there exists $\zeta \in \Omega^{k-1}(\mathcal{M})$ such that

$$i(X)\Omega = d\zeta . \tag{1}$$

ζ is defined modulo closed $(k-1)$ -forms. The class $\bar{\zeta} \in \Omega^{k-1}(\mathcal{M})/Z^{k-1}(\mathcal{M})$ defined by ζ is called the *Hamiltonian* for X , and every element in this class is a *Hamiltonian form* for X . Furthermore, $X \in \mathfrak{X}(\mathcal{M})$ is a *locally Hamiltonian vector field* if $i(X)\Omega$ is a closed k -form. Then, for every $x \in \mathcal{M}$, there is an open neighbourhood $W \subset \mathcal{M}$ and $\zeta \in \Omega^{k-1}(W)$ such that (1) holds on W . As above, changing \mathcal{M} by W , we obtain the *Hamiltonian* for X , $\bar{\zeta} \in \Omega^{k-1}(W)/Z^{k-1}(W)$, and the local Hamiltonian forms for X .

Conversely, $\zeta \in \Omega^{k-1}(\mathcal{M})$ (resp. $\zeta \in \Omega^{k-1}(W)$) is a *Hamiltonian form* (resp. a *local Hamiltonian form*) if there exist a vector field $X_\zeta \in \mathfrak{X}(\mathcal{M})$ (resp. $X_\zeta \in \mathfrak{X}(\mathcal{M})$) such that (1) holds (resp. on W). Of course, a vector field $X \in \mathfrak{X}(\mathcal{M})$ is a locally Hamiltonian vector field if, and only if, the Lie derivative $L(X)\Omega = 0$. If X, Y are locally Hamiltonian vector fields, then $[X, Y]$ is a Hamiltonian vector field with Hamiltonian form $i(X \wedge Y)\Omega$. We denote by $\mathcal{H}^{k-1}(\mathcal{M})$ the \mathbb{R} -vector space of Hamiltonian $(k-1)$ -forms. There is a natural Lie algebra structure on $\mathcal{H}^{k-1}(\mathcal{M})$ given by (see [12, 19])

$$\{\xi, \zeta\} := -i(X_\xi)i(X_\zeta)\Omega .$$

Definition 1 Let $\Phi: G \times \mathcal{M} \rightarrow \mathcal{M}$ be an action of a Lie group G on a multisymplectic manifold (\mathcal{M}, Ω) . We say that Φ is a multisymplectic action (or also that G acts multisymplectically on \mathcal{M} by Φ) if, for every $g \in G$, Φ_g is a multisymplectomorphism, that is, $\Phi_g^* \Omega = \Omega$. Then \mathcal{M} is a multisymplectic G -space, or also that G is a symmetry group of (\mathcal{M}, Ω) .

We denote by $\tilde{\mathfrak{g}} \subset \mathfrak{X}(\mathcal{M})$ the real Lie algebra of fundamental vector fields. As a consequence of the definition, the fundamental vector field $\tilde{\xi} \in \tilde{\mathfrak{g}}$ associated with every $\xi \in \mathfrak{g}$ by Φ is a locally Hamiltonian vector field, $\tilde{\xi} \in \tilde{\mathfrak{X}}_{lh}(\mathcal{M})$ (conversely, if for every $\tilde{\xi} \in \tilde{\mathfrak{g}}$, we have that $\tilde{\xi} \in \tilde{\mathfrak{X}}_{lh}(\mathcal{M})$, then Φ is a multisymplectic action of G on \mathcal{M}). So we have that, for every $\xi \in \mathfrak{g}$, $L(\tilde{\xi})\Omega = 0$ or, what is equivalent, $i(\tilde{\xi})\Omega \in Z^k(\mathcal{M})$ (it is a closed k -form). Then, following the same terminology as for actions of Lie groups on symplectic manifolds [2], [25], [42], we define:

Definition 2 Let $\Phi: G \times \mathcal{M} \rightarrow \mathcal{M}$ be a multisymplectic action of G on \mathcal{M} . Φ is a strongly multisymplectic or Hamiltonian action if $\tilde{\mathfrak{g}} \subseteq \tilde{\mathfrak{X}}_h(\mathcal{M})$ or, what is equivalent, for every $\xi \in \mathfrak{g}$, $i(\tilde{\xi})\Omega$ is an exact form. Otherwise, it is called a locally Hamiltonian action.

In particular, if (\mathcal{M}, Ω) is an exact multisymplectic manifold (that is, there exists $\Theta \in \Omega^k(\mathcal{M})$ such that $d\Theta = \Omega$), and Φ an exact action; that is, $\Phi_g^* \Theta = \Theta$, for every $g \in G$, then $\tilde{\mathfrak{g}} \subset \tilde{\mathfrak{X}}_h(\mathcal{M})$ and, for every $\tilde{\xi} \in \tilde{\mathfrak{g}}$, its Hamiltonian form is $\zeta_\xi = -i(\tilde{\xi})\Theta$. Hence, Φ is strongly multisymplectic.

3 Momentum map for multisymplectic actions

From now on (\mathcal{M}, Ω) will be a m -dimensional multisymplectic manifold of degree $k + 1$, and $\Phi: G \times \mathcal{M} \rightarrow \mathcal{M}$ a strongly multisymplectic action of a Lie group G on \mathcal{M} , with $\dim G = n$.

Definition 3 A comomentum map associated with Φ is a map $\mathcal{J}^*: \mathfrak{g} \rightarrow \mathcal{H}^{k-1}(\mathcal{M})$, such that

$$i(\tilde{\xi})\Omega = d\mathcal{J}^*(\xi) \quad ; \quad \xi \in \mathfrak{g} .$$

A momentum map associated with Φ is a map $\mathcal{J}: \mathcal{M} \rightarrow \mathfrak{g}^* \otimes_{\mathcal{M}} \Lambda^{k-1}T^*\mathcal{M}$ such that

$$\mathcal{J}(x)(\xi) := \mathcal{J}^*(\xi)(x) \in \Lambda^{k-1}T_x^*\mathcal{M} \quad ; \quad x \in \mathcal{M}, \xi \in \mathfrak{g} .$$

The comomentum maps are parametrized by $L_{\mathbb{R}}(\mathfrak{g}, Z^{k-1}(\mathcal{M}))$; that is, the real space of the linear maps from \mathfrak{g} to $Z^{k-1}(\mathcal{M})$. In fact, if $\mathcal{F}: \mathfrak{g} \rightarrow Z^{k-1}(\mathcal{M})$ is a continuous linear map, and \mathcal{J}^* is a comomentum map, so is $\mathcal{J}'^* = \mathcal{J}^* + \mathcal{F}$. Furthermore, for all $\xi_1, \xi_2 \in \mathfrak{g}$, with $i(\tilde{\xi}_i)\Omega = d\zeta_{\xi_i}$, ($i = 1, 2$), then $i([\tilde{\xi}_1, \tilde{\xi}_2])\Omega = d\zeta_{[\xi_2, \xi_1]}$, and we have that

$$d\{\zeta_{\xi_1}, \zeta_{\xi_2}\} = i([\tilde{\xi}_1, \tilde{\xi}_2])\Omega = d\zeta_{[\xi_2, \xi_1]} ,$$

therefore $d\{\zeta_{\xi_1}, \zeta_{\xi_2}\} = d\zeta_{[\xi_2, \xi_1]}$, and $\{\zeta_{\xi_1}, \zeta_{\xi_2}\} = \zeta_{[\xi_2, \xi_1]} + \gamma(\tilde{\xi}_1, \tilde{\xi}_2)$, where $\gamma: \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \rightarrow Z^{k-1}(\mathcal{M})$ is a skewsymmetric bilinear map. Then the comomentum map is a Lie algebra homomorphism if, and only if, $\gamma = 0$.

Definition 4 Φ is a Poissonian or strongly Hamiltonian action if there exists a comomentum map which is a Lie algebra homomorphism; that is, $\{\mathcal{J}^*(\xi_1), \mathcal{J}^*(\xi_2)\} = \mathcal{J}^*([\xi_1, \xi_2])$, for every $\xi_1, \xi_2 \in \mathfrak{g}$.

Φ is a Coad-equivariant action if there exists a momentum map which is Ad^* -equivariant; that is, for every $g \in G$, we have the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{J} & \\ \Phi_g & \begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathfrak{g}^* \otimes \Lambda^{k-1}T^*\mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathfrak{g}^* \otimes \Lambda^{k-1}T^*\mathcal{M} \\ & \mathcal{J} & \end{array} & \text{Ad}_g^* \otimes \Lambda^{k-1}T^*\Phi_{g^{-1}} \end{array}$$

Using the same reasoning than for actions of Lie groups on symplectic manifolds [2], one can prove that every Coad-equivariant action is Poissonian. As a particular case we have:

Proposition 1 If (\mathcal{M}, Ω) is an exact multisymplectic manifold with $\Omega = d\Theta$, and the action Φ is exact, then a comomentum map exists which is given by $\mathcal{J}^*(\xi) = -i(\tilde{\xi})\Theta$, $\xi \in \mathfrak{g}$, and the action is Coad-equivariant and Poissonian.

(Proof) In fact, for every $x \in \mathcal{M}$, $\xi \in \mathfrak{g}$, and $X_1, \dots, X_{k-1} \in T_x\mathcal{M}$, we have that

$$\begin{aligned} (\mathcal{J} \circ \Phi_g)(x)(\xi; X_1, \dots, X_{k-1}) &= \mathcal{J}(\Phi_g(x))(\xi; X_1, \dots, X_{k-1}) = \mathcal{J}^*(\xi)(\Phi_g(x); X_1, \dots, X_{k-1}) \\ &= -[i(\tilde{\xi})\Theta](\Phi_g(x); X_1, \dots, X_{k-1}) \\ &= -[\Phi_g^* i(\tilde{\xi})\Theta](x; T_{\Phi_g(x)}\Phi_{g^{-1}}X_1, \dots, T_{\Phi_g(x)}\Phi_{g^{-1}}X_{k-1}) ; \end{aligned}$$

but, bearing in mind that $(\widetilde{Ad_g\xi}) = \Phi_{g^{-1}*}\tilde{\xi}$, we have

$$\Phi_g^* i(\tilde{\xi})\Theta = i(\Phi_{g^{-1}*}\tilde{\xi})\Phi_g^*\Theta = i(\Phi_{g^{-1}*}\tilde{\xi})\Theta = i(\widetilde{Ad_g\xi})\Theta \implies \mathcal{J}^*(Ad_g\xi) = \Phi_g^* i(\tilde{\xi})\Theta = -\Phi_g^*(\mathcal{J}^*(\xi)) ,$$

and hence

$$\mathcal{J}(x)(\text{Ad}_g \xi; X_1, \dots, X_{k-1}) = \mathcal{J}^*(\text{Ad}_g \xi)(x; X_1, \dots, X_{k-1}) = -[\Phi_g^* i(\tilde{\xi})\Theta](x; X_1, \dots, X_{k-1}).$$

Therefore

$$\begin{aligned} (\mathcal{J} \circ \Phi_g)(x)(\xi; X_1, \dots, X_{k-1}) &= \mathcal{J}(x)(\text{Ad}_g \xi; T_{\Phi_g(x)} \Phi_{g^{-1}} X_1, \dots, T_{\Phi_g(x)} \Phi_{g^{-1}} X_{k-1}) \\ &= \mathcal{J}(x)[(\text{Ad}_g \otimes \Lambda^{k-1} T \Phi_{g^{-1}})(\xi; X_1, \dots, X_{k-1})] \\ &= [(\text{Ad}_g \otimes \Lambda^{k-1} T \Phi_{g^{-1}})^t(\mathcal{J}(x))](\xi; X_1, \dots, X_{k-1}) \\ &= [(\text{Ad}_g^* \otimes \Lambda^{k-1} T \Phi_{g^{-1}}) \circ \mathcal{J}](x)(\xi; X_1, \dots, X_{k-1}); \end{aligned}$$

thus the action is Coad-equivariant. Hence it is Poissonian too, and

$$\zeta_{[\xi_1, \xi_2]} = -\Theta([\tilde{\xi}_1, \tilde{\xi}_2]) = -i([\tilde{\xi}_1, \tilde{\xi}_2])\Theta = -L(\tilde{\xi}_1) i(\tilde{\xi}_2)\Theta = \{\zeta_{\xi_1}, \zeta_{\xi_2}\}.$$

■

4 Momentum-type submanifolds and multisymplectic reduction

Definition 5 *A submanifold S of \mathcal{M} , with natural embedding $j_S: S \hookrightarrow \mathcal{M}$, is a momentum-type submanifold if:*

1. S is a closed submanifold of \mathcal{M} .
2. $j_S^* i(\tilde{\xi})\Omega = 0$, for every $\xi \in \mathfrak{g}$; that is, S is an integral submanifold of the exterior differential system $\{i(\tilde{\xi})\Omega; \xi \in \mathfrak{g}\}$.
3. S is maximal, in the order established by the inclusion, among all the submanifolds verifying the above conditions.

Let $G_S \subset G$ be the maximal subgroup of G (with respect to the inclusion) that leaves S invariant (the isotropy group of S).

Consider the submodule $\mathcal{G} \subset \mathfrak{X}(\mathcal{M})$ defined by $\mathcal{G} := C^\infty(\mathcal{M}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}$. This module generates the distribution tangent to the orbits of the action of G on \mathcal{M} ; that is, if G_p denotes the orbit passing through $p \in \mathcal{M}$, and $\langle \xi_1, \dots, \xi_n \rangle$ is a basis of \mathfrak{g} , then $T_p G_p = \langle (\tilde{\xi}_1)_p, \dots, (\tilde{\xi}_n)_p \rangle \subset T_p \mathcal{M}$.

If S is a closed submanifold of \mathcal{M} , and $\mathfrak{g}_S \subset \mathfrak{g}$ is the Lie subalgebra associated with G_S , we define $\mathcal{G}_S := C^\infty(\mathcal{M}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}_S$. Then, for every momentum-type submanifold S we have that \mathcal{G}_S is a submodule closed under the Lie bracket. Observe that, for every $\tilde{\xi} \in \mathcal{G}_S$, as $\tilde{\xi}$ is tangent to S , there exists $\tilde{\xi}_S \in \mathfrak{X}(S)$ such that $j_{S*} \tilde{\xi}_S = \tilde{\xi}|_S$, and we have that

$$j_S^* [i(\tilde{\xi})\Omega] = i(\tilde{\xi}_S)(j_S^* \Omega) = 0. \quad (2)$$

Consider the ideal $Nul(S) := \{f \in C^\infty(\mathcal{M}) \mid j_S^* f = 0\}$, and let $\mathfrak{X}(S)$ be the set of vector fields in \mathcal{M} which are tangent to S , that is, $\mathfrak{X}(S) := \{X \in \mathfrak{X}(\mathcal{M}) \mid L(X)(\overline{Nul(S)}) \subset Nul(S)\}$. Then we have the following obvious result:

Proposition 2 *Let S be a closed submanifold of \mathcal{M} with $\dim S \geq k$, and let $\mathfrak{X}^k(S)$ be the set of k -vector fields in \mathcal{M} which are tangent to S . Then S is an integral submanifold of the exterior differential system $\{i(\tilde{\xi})\Omega; \xi \in \mathfrak{g}\}$ if, and only if,*

$$C^\infty(\mathcal{M}) \otimes_{\mathbb{R}} \{i(\tilde{\xi})\Omega; \xi \in \mathfrak{g}\} \subset (\mathfrak{X}^k(S))',$$

where $(\mathfrak{X}^k(S))' := \{\alpha \in \Omega^k(\mathcal{M}) \mid i(X)\alpha = 0, X \in \mathfrak{X}^k(S)\}$ is the annihilator of $\mathfrak{X}^k(S)$.

If $\dim S = s \geq k$, as $\dim \mathfrak{X}^k(S) = \binom{s}{k}$, and $\dim (\mathfrak{X}^k(S))' = \binom{m}{k} - \binom{s}{k}$, then, the condition for S to be an integral submanifold of the above exterior differential system is that

$$n \leq \binom{m}{k} - \binom{s}{k} \quad ; \quad \text{that is, } \binom{s}{k} \leq \binom{m}{k} - n, \quad (3)$$

(which is a condition on $s = \dim S$). In particular,

$$\binom{s}{k} = \binom{m}{k} - n \iff C^\infty(\mathcal{M}) \otimes_{\mathbb{R}} \{i(\tilde{\xi})\Omega; \xi \in \mathfrak{g}\} = (\mathfrak{X}^k(S))'. \quad (4)$$

Definition 6 *Let S be a momentum-type submanifold. S will be called an optimal momentum-type submanifold if*

$$s \equiv \dim S = \sup \left\{ q \in \mathbb{N} \mid \binom{q}{k} \leq \binom{m}{k} - n \right\}.$$

In particular, S will be called a maximal momentum-type submanifold if (4) holds.

Let $j_S: S \hookrightarrow \mathcal{M}$ be a momentum-type submanifold, with $\dim S \geq k + 1$. Denote $\omega = j_S^*\Omega \in \Omega^{k+1}(S)$, which is a closed but, in general, 1-degenerate form, and let \mathcal{K}_ω be the distribution associated with $\ker \omega$ (the characteristic distribution of ω). We have that \mathcal{G}_S is an involutive distribution (also denoted by \mathcal{G}_S) which, as a consequence of (2), is a subbundle of \mathcal{K}_ω . Therefore:

Theorem 1 *If the action of G_S on S is free and proper then:*

1. S/G_S is a differentiable manifold and the projection $\pi: S \rightarrow S/G_S$ is a surjective submersion.
2. There exists an unique closed form $\tilde{\omega} \in \Omega^{k+1}(S/G_S)$ such that $\pi^*\tilde{\omega} = \omega$.
(Observe that, as $\mathcal{G}_S \subseteq \mathcal{K}_\omega$, then $\tilde{\omega}$ is 1-degenerate, in general).

(*Proof*) The proof of the item 1 follows the same pattern as in the classical theorems of reduction (see [2], [25], [32]). For the second item, from (2), it is easy to prove that ω is a π -basic form. Thus the existence of $\tilde{\omega}$ is assured, and it is obviously a closed form. The uniqueness is a consequence of the fact that π is a surjective submersion, and hence π^* is injective. \blacksquare

Remarks:

- If $\dim S < k + 1$ then $\omega = 0$ and the result is trivial. Moreover $\dim S/G_S \geq k + 1$, as $\mathcal{G}_S \subset \mathcal{K}_\omega$.
- Observe that if we make the quotient $S/\ker \omega$ we obtain a reduced form $\tilde{\omega} \in \Omega^{k+1}(S/\ker \omega)$ which is nondegenerate (that is, multisymplectic), but then we have removed more degrees of freedom than the corresponding to the symmetries introduced by the group G .
- In the case of reduction of symplectic and presymplectic manifolds, if the action is *Poissonian* (that is, there exist a *momentum map* $J: \mathcal{M} \rightarrow \mathfrak{g}^*$, which is Ad^* -equivariant), there is a natural way of obtaining momentum-type submanifolds: they are the level sets of the momentum map, $J^{-1}(\mu)$, for every weakly regular value $\mu \in \mathfrak{g}^*$. These level sets are the maximal integral submanifolds of the Pfaff system $\{i(\tilde{\xi})\Omega = 0, \forall \xi \in \mathfrak{g}\}$ [22], and all of them are optimal momentum-type submanifolds, since in these cases $k = 1$ and the equality in (3) holds since it reduces to $s = n - m$.
- In the multisymplectic case, the momentum map \mathcal{J} associated with a Poissonian action allows us to define an exterior differential system which is generated by the forms $\{i(\tilde{\xi})\Omega = d\mathcal{J}^*(\xi), \xi \in \mathfrak{g}\}$, and whose maximal integral submanifolds are momentum-type submanifolds, which are called the *integral submanifolds of the momentum map* \mathcal{J} . Then, theorem 1 holds in this context.

5 Noether invariants in Lagrangian field theory

A Lagrangian field theory is characterized giving a *configuration bundle* $\pi: E \rightarrow M$, where M is a k -dimensional oriented manifold with volume form $\omega \in \Omega^k(M)$, and a *Lagrangian density* which is a $\bar{\pi}^1$ -semibasic k -form on $J^1\pi$, where $\pi^1: J^1\pi \rightarrow E$ is the jet bundle of local sections of π , and $\bar{\pi}^1 = \pi \circ \pi^1: J^1\pi \rightarrow M$ gives another fiber bundle structure. We have that $\mathcal{L} = \mathcal{L}\bar{\pi}^{1*}\omega$, where $\mathcal{L} \in C^\infty(J^1\pi)$ is the *Lagrangian function* associated with \mathcal{L} and ω . The *Poincaré-Cartan forms* associated with \mathcal{L} , denoted $\Theta_{\mathcal{L}} \in \Omega^k(J^1\pi)$ and $\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}} \in \Omega^{k+1}(J^1\pi)$, are constructed using the canonical jet bundle elements. $(J^1\pi, \Omega_{\mathcal{L}})$ is a *Lagrangian system*, which is *regular* if $\Omega_{\mathcal{L}}$ is 1-nondegenerate. The Lagrangian problem consists in finding sections $\phi \in \Gamma(\pi)$ (the set of sections of π), such that, if $j^1\phi$ denotes the canonical lifting of ϕ to $J^1\pi$, then $(j^1\phi)^*i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$. These are the Euler-Lagrange field equations. (See [20] for details).

Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a regular Lagrangian system, and $\Phi: G \times J^1\pi \rightarrow J^1\pi$ be an action of a Lie group G on $J^1\pi$. If Φ is an exact action (hence strongly multisymplectic), and $\Phi_g^*\mathcal{L} = \mathcal{L}$, for every $g \in G$, then G is a *symmetry group* of $(J^1\pi, \Omega_{\mathcal{L}})$. Therefore, as described above, a comomentum map exists for which the action is Coad-equivariant and Poissonian, and it is given by

$$\mathcal{J}^*(\xi) = i(\tilde{\xi})\Theta_{\mathcal{L}} \text{ , for every } \xi \in \mathfrak{g} \text{ .} \quad (5)$$

But $i(\tilde{\xi})\Theta_{\mathcal{L}}$ are just the *Noether invariants* associated with this symmetry group. In fact, *Noether's theorem* states that, for every $\phi \in \Gamma(\pi)$ solution to the field equations, from $L(\tilde{\xi})\Theta_{\mathcal{L}} = d i(\tilde{\xi})\Theta_{\mathcal{L}} + i(\tilde{\xi})d\Theta_{\mathcal{L}} = 0$, for every $\xi \in \mathfrak{g}$, we have

$$0 = (j^1\phi)^*d i(\tilde{\xi})\Theta_{\mathcal{L}} + (j^1\phi)^*i(\tilde{\xi})\Omega_{\mathcal{L}} = (j^1\phi)^*d i(\tilde{\xi})\Theta_{\mathcal{L}} \text{ .}$$

Observe that $(j^1\phi)^*i(\tilde{\xi})\Omega_{\mathcal{L}} = 0$ implies that $\{d i(\tilde{\xi})\Theta_{\mathcal{L}} = i(\tilde{\xi})\Omega_{\mathcal{L}} \text{ , } \xi \in \mathfrak{g}\} = [\mathfrak{X}^k(\text{Im } j^1\phi)]'$, and then, for every $\phi \in \Gamma(\pi)$ solution, $\text{Im } j^1\phi$ are momentum-type submanifolds of $J^1\pi$. So, we call them *G-Noether-type submanifolds*. All of this allows us to state:

Proposition 3 *If N_G is a G-Noether-type submanifolds which contains a Cauchy data submanifold S , and ϕ is a solution to the field equations on it, then $\text{Im } j^1\phi \subset N_G$.*

The remaining question is under what conditions the reduction procedure described above can be applied to reduce the field equations. Results in this way have been already obtained [23, 38].

Observe also that, if \mathcal{J}'^* is another comomentum map, then

$$d\mathcal{J}'^*(\xi) = i(\tilde{\xi})\Omega_{\mathcal{L}} = d i(\tilde{\xi})\Theta_{\mathcal{L}} \implies d(\mathcal{J}'^*(\xi) - i(\tilde{\xi})\Theta_{\mathcal{L}}) = 0 \text{ .}$$

Thus, it is obvious that the construction of *G-Noether-type submanifolds* depends only on the group action, and not on the choice of a comomentum map. The relevant fact is the existence of the comomentum map given by (5).

It is interesting to point out that, in the realm of first-order Lagrangian field theories, the existence of momentum-type submanifolds is assured (the images of the sections solution to the field equations). Nevertheless, we cannot assure the existence of those manifolds verifying the condition of maximal dimensionality (i.e., for being maximal momentum-type submanifolds).

6 Discussion and outlook

Recently, a very generic scheme of reduction in the ambient of the so-called k -symplectic or polysymplectic formulations of classical field theories has been completed [28], eestablishing sufficient conditions in order to do this reduction possible. In a previous paper [36], the relation between the

k -symplectic (polysymplectic) and the multisymplectic formalisms (for certain kinds of multisymplectic manifolds) was studied. Bearing in mind this relationship and the results in [28], we could study also the equivalence between the k -symplectic reduction and a suitable reduction programme for the multisymplectic case. This line of work will be the object of a future research.

Finally, another way of approaching the problem of the multisymplectic reduction could be using the so-called *higher-Dirac structures*, since the reduction of these types of structures would generalize the multisymplectic and polysymplectic reduction, in the same way that the reduction of *Dirac structures* encompasses other reduction procedures such as the symplectic (Marsden-Weinstein), presymplectic and Poisson cases [9].

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References

- [1] C. Albert, “Le théorème de réduction de Marsden-Weinstein en géométrie cosymplectique et de contact”, *J. Geom. Phys.* **6**(4) (1989) 627-649. (doi.org/10.1016/0393-0440(89)90029-6).
- [2] R. Abraham, J.E. Marsden, *Foundations of Mechanics* (2nd ed.), Addison-Wesley, Reading, 1978.
- [3] J.M. Arms, R.H. Cushman, M.J. Gotay, “A Universal Reduction Procedure for Hamiltonian Group Actions”, T. Ratiu Ed., *M.S.R.I. Series* **22**, Springer-Verlag (1991) 33-52.
- [4] L. Bates, J. Sniatycki, “Non-holonomic Reduction”, *Rep. Math. Phys.* **32**(1) (1993) 99-115. (doi.org/10.1016/0034-4877(93)90073-N).
- [5] G. Blankenstein, A.J. Van der Schaft, “Optimal Control and Implicit Hamiltonian systems”, *Nonlinear Control in the Year 2000*, Eds. A. Isidori, F. Lamnabhi-Lagarrigue, W. Respondek. Springer-Verlag, LNCIS **258** (2000) 185-206.
- [6] A.M. Bloch, P. Crouch, “Optimal control, optimization and analytical mechanics”, in *Mathematical Control Theory*, L. Bailieul and J.C. Willems Eds., Springer Verlag (1999) 265-321.
- [7] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, R.M. Murray, “Nonholonomic mechanical systems with symmetry”, *Arch. Rational Mech. Anal.* **136**(1) (1996) 21-99. (doi.org/10.1007/BF02199365).
- [8] H. Bursztyn, A. Cabrera, D. Iglesias, “Multisymplectic geometry and Lie groupoids”, In Chang D., Holm D., Patrick G., Ratiu T. (eds) *Geometry, Mechanics, and Dynamics*. Fields Institute Communications **73**. Springer, New York (2015).
- [9] H. Bursztyn, N. Martínez-Alba, A.R. Rubio, “On Higher Dirac Structures”, *Int. Math. Research Notices*, rnx163 (2017). (doi.org/10.1093/imrn/rnx163).
- [10] F. Cantrijn, J.F. Cariñena, M. Crampin, L.A. Ibort, “Reduction of degenerate Lagrangian systems”, *J. Geom. Phys.* **3**(3) (1986) 353-400. (doi.org/10.1016/0393-0440(86)90014-8)
- [11] F. Cantrijn, M. de León, J.C. Marrero, D. Martín de Diego, “Reduction of nonholonomic mechanical systems with symmetries”, *Rep. Math. Phys.* **42**(1-2) (1998) 25-45. (doi.org/10.1016/S0034-4877(98)80003-7).
- [12] F. Cantrijn, A. Ibort, M. de León, “Hamiltonian Structures on Multisymplectic manifolds”, *Rend. Sem. Mat. Univ. Pol. Torino* **54** (1996) 225-236.
- [13] M. Castrillón-López, P.L. García Pérez, C. Rodrigo, “Euler-Poincaré reduction in principal fibre bundles and the problem of Lagrange”, *Diff. Geom. Appl.* **25**(6) (2007) 585-593. (doi.org/10.1016/j.difgeo.2007.06.007).

- [14] M. Castrillón-López, J.E. Marsden, “Some remarks on Lagrangian and Poisson reduction for field theories”, *J. Geom. Phys.* **48**(1) (2003) 52-83. (doi.org/10.1016/S0393-0440(03)00025-1).
- [15] M. Castrillón-López, T. Ratiu, “Reduction in principal bundles: covariant Lagrange-Poincaré equations”, *Comm. Math. Phys.* **236**(2) (2003) 223–250. (doi.org/10.1007/s00220-003-0797-5).
- [16] M. de León, D. Martín de Diego, “Almost Product Structures and Poisson Reduction of presymplectic Systems”, *Extracta Math.* **10**(1) (1995) 37-45.
- [17] M. de León, M.H. Mello, P.R. Rodrigues, “Reduction of degenerate non-autonomous Lagrangian systems” *Mathematical Aspects of Classical Field Theory*, Contemp. Math. **132**, 275-305, Eds. M.J. Gotay, J.E. Marsden, V. Moncrief, (Seattle, Washington, American Math. Society). 1992.
- [18] M. de León, M. Saralegi, “Cosymplectic reduction for singular momentum maps”, *J. Phys. A: Math. Gen.* **26**(19) (1993) 5033. (doi.org/10.1088/0305-4470/26/19/032).
- [19] A. Echeverría-Enríquez, A. Ibort, M.C. Muñoz-Lecanda, N. Román-Roy, “Invariant Forms and Automorphisms of Locally Homogeneous Multisymplectic Manifolds” *J. Geom. Mech.* **4**(4) (2012) 397-419. (doi.org/10.3934/jgm.2012.4.397).
- [20] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Geometry of Lagrangian first-order classical field theories”, *Fortschr. Phys.* **44**(3) (1996) 235–280. (doi.org/10.1002/prop.2190440304).
- [21] A. Echeverría-Enríquez, J. Marín-Solano, M.C. Muñoz-Lecanda, N. Román-Roy, “Geometric reduction in optimal control theory with symmetries”. *Rep. Math. Phys.* **52**(1) (2003) 89-113. (doi.org/10.1016/S0034-4877(03)90006-1).
- [22] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Reduction of presymplectic manifolds with symmetry”, *Rev. Math. Phys.* **11**(10) (1999) 1209-1247. (doi.org/10.1142/S0129055X99000386).
- [23] A.P. Fernández, P.L. García, C. Rodrigo, “Lagrangian reduction and constrained variational calculus”, *Proc. IX Fall Workshop on Geometry and Physics* (Vilanova i la Geltrú, 2000), Eds. X. Gràcia *et al.*. Pubs. R.S.M.E. **3** (2001) 53-64.
- [24] L.A. Ibort, J. Marín-Solano, “A geometric classification of Lagrangian functions and the reduction of the evolution space”, *J. Phys. A: Math. Gen.* **25**(11) (1992) 3353-3367. (doi.org/10.1088/0305-4470/25/11/036).
- [25] P. Libermann, C.M. Marle, *Symplectic geometry and analytical dynamics*, D. Reidel Publishing Company, Dordrecht, 1987.
- [26] T.B. Madsen, A. Swann, “Multi-moment maps”, *Adv. Math.* **229**(4) (2012) 2287-2309. (doi.org/10.1016/j.aim.2012.01.002).
- [27] C.L. Marle, “Reduction of Constrained Mechanical Systems and Stability of Relative Equilibria”, *Comm. Math. Phys.* **174**(2) (1995) 295-318. (doi.org/10.1007/BF02099604).
- [28] J.C. Marrero, N. Román-Roy, M. Salgado, S. Vilarino, “Reduction of polysymplectic manifolds”, *J. Phys. A: Math. Theor.* **48**(5) (2015) 055206 (43pp) (doi:10.1088/1751-8113/48/5/055206).
- [29] J.E. Marsden, T.S. Ratiu, “Reduction of Poisson manifolds”, *Lett. Math. Phys.* **11**(2) (1986) 161-170. (doi.org/10.1007/BF00398428).
- [30] J.E. Marsden, T.S. Ratiu, A. Weinstein, “Semi-direct Products and Reduction in Mechanics”, *Trans. Am. Math. Soc.* **281** (1984) 147-177. (doi.org/10.1090/S0002-9947-1984-0719663-1).
- [31] J.E. Marsden, J. Scheurle, “The Reduced Euler-Lagrange Equations”, *Fields Inst. Comm.* **1** (1993) 139-164.
- [32] J.E. Marsden, A. Weinstein, “Reduction of symplectic manifolds with symmetry”, *Rep. Math. Phys.* **5**(1) (1974) 121-130. (doi.org/10.1016/0034-4877(74)90021-4).
- [33] E. Martínez, “Reduction in optimal control theory”, *Rep. Math. Phys.* **53**(1) (2004) 79-90. (doi.org/10.1016/S0034-4877(04)90005-5).
- [34] N. Martínez-Alba, *On higher Poisson and higher Dirac structures*, Ph D Thesis, IMPA, Rio de Janeiro (Brazil) 2015.
- [35] S. Martínez, J. Cortés, M. de León, “Symmetries in vakonomic dynamics. Applications to optimal control”, *J. Geom. Phys.* **38**(3-4) (2001) 343-365. (doi.org/10.1016/S0393-0440(00)00069-3).

- [36] N. Román-Roy, A.M. Rey, M. Salgado, S. Vilariño, “On the k -symplectic, k -cosymplectic and multisymplectic formalisms of classical field theories”, *J. Geom. Mech.* **3**(1) (2011) 113-137. (doi.org/10.3934/jgm.2011.3.113).
- [37] R. Sjamaar, E. Lerman, “Stratified symplectic spaces and reduction”, *Ann. Math.* **134**(2) (1991) 375-422. (doi.org/10.2307/2944350).
- [38] J. Sniatycki, “Multisymplectic reduction for proper actions”, *Canad. J. Math.* **56**(3) (2004) 638-654. (doi.org/10.4153/CJM-2004-029-8).
- [39] H.J. Sussmann, “Symmetries and integrals of motion in Optimal Control”, in *Geometry in Nonlinear Control and Differential Inclusions*, A. Fryszkowski, B. Jakubczyk, W. Respondek, T. Rzezuchowski Eds., Banach Center Pubs. **32**. Math. Inst. Polish Acad. Sci., Warsaw, Poland (1995) 379-393.
- [40] A.J. Van der Schaft, “Symmetries in optimal control”, *SIAM J. Control and Optimization* **25**(2) (1987) 245-259. (doi.org/10.1137/032501).
- [41] J. Vankerschaver, “Euler-Poincaré reduction for discrete field theories”, *J. Math. Phys.* **48**(3) (2007) 032902, 17 pp. (doi.org/10.1063/1.2712419).
- [42] F.W. Warner, *Foundations on differentiable manifolds and Lie groups*, Scott, Foresman and Co., Glenview, 1971.