

A GENERALIZATION OF THE CONCEPT OF DISTANCE BASED ON THE SIMPLEX INEQUALITY

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ABSTRACT. We introduce and discuss the concept of n -distance, a generalization to n elements of the classical notion of distance obtained by replacing the triangle inequality with the so-called simplex inequality

$$d(x_1, \dots, x_n) \leq K \sum_{i=1}^n d(x_1, \dots, x_n)_i^z, \quad x_1, \dots, x_n, z \in X,$$

where $K = 1$. Here $d(x_1, \dots, x_n)_i^z$ is obtained from the function $d(x_1, \dots, x_n)$ by setting its i th variable to z . We provide several examples of n -distances, and for each of them we investigate the infimum of the set of real numbers $K \in]0, 1]$ for which the inequality above holds. We also introduce a generalization of the concept of n -distance obtained by replacing in the simplex inequality the sum function with an arbitrary symmetric function.

1. INTRODUCTION

The notion of metric space, as first introduced by Fréchet [13] and later developed by Hausdorff [14], is one of the key ingredients in many areas of pure and applied mathematics, particularly in analysis, topology, geometry, statistics, and data analysis.

Denote the half-line $[0, +\infty[$ by \mathbb{R}_+ . Recall that a *metric space* is a pair (X, d) , where X is a nonempty set and d is a distance on X , that is, a function $d: X^2 \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- $d(x_1, x_2) \leq d(x_1, z) + d(z, x_2)$ for all $x_1, x_2, z \in X$ (triangle inequality),
- $d(x_1, x_2) = d(x_2, x_1)$ for all $x_1, x_2 \in X$ (symmetry),
- $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$ (identity of indiscernibles).

Generalizations of the concept of distance in which $n \geq 3$ elements are considered have been investigated by several authors (see, e.g., [5, Chapter 3] and the references therein). The three conditions above may be generalized to n -variable functions $d: X^n \rightarrow \mathbb{R}_+$ in the following ways. For any integer $n \geq 1$, we set $[n] = \{1, \dots, n\}$. For any $i \in [n]$ and any $z \in X$, we denote by $d(x_1, \dots, x_n)_i^z$ the function obtained from $d(x_1, \dots, x_n)$ by setting its i th variable to z . Let also denote by S_n the set of all permutations on $[n]$. A function $d: X^n \rightarrow \mathbb{R}_+$ is said to be an $(n-1)$ -*semimetric* [7] if it satisfies

- (i) $d(x_1, \dots, x_n) \leq \sum_{i=1}^n d(x_1, \dots, x_n)_i^z$ for all $x_1, \dots, x_n, z \in X$,
- (ii) $d(x_1, \dots, x_n) = d(x_{\pi(1)}, \dots, x_{\pi(n)})$ for all $x_1, \dots, x_n \in X$ and all $\pi \in S_n$,

and it is said to be an $(n-1)$ -*hemimetric* [5, 6] if additionally it satisfies

- (iii') $d(x_1, \dots, x_n) = 0$ if and only if x_1, \dots, x_n are not pairwise distinct.

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Condition (i) is referred to as the *simplex inequality* [5, 7]. For $n = 3$, this inequality can be interpreted as follows: the area of a triangle face of a tetrahedron does not exceed the sum of the areas of the remaining three faces.

The following variant of condition (iii') can also be naturally considered:

(iii) $d(x_1, \dots, x_n) = 0$ if and only if $x_1 = \dots = x_n$.

For $n = 3$, functions satisfying conditions (i), (ii), and (iii) were introduced by Dhage [8] and called *D-distances*. Their topological properties were investigated subsequently [9–11], but unfortunately most of the claimed results are incorrect, see [23]. Moreover, it turned out that a stronger version of *D-distance* is needed for a sound topological use of these functions [16, 23, 24].

In this paper we introduce and discuss the following simultaneous generalization of the concepts of distance and *D-distance* by considering functions with $n \geq 2$ arguments.

Definition 1.1 (see [17]). Let $n \geq 2$ be an integer. We say that (X, d) is an *n-metric space* if X is a nonempty set and d is an *n-distance* on X , that is, a function $d: X^n \rightarrow \mathbb{R}_+$ satisfying conditions (i), (ii), and (iii).

We observe that for any *n-distance* $d: X^n \rightarrow \mathbb{R}_+$, the set of real numbers $K \in]0, 1]$ for which the condition

$$(1) \quad d(x_1, \dots, x_n) \leq K \sum_{i=1}^n d(x_1, \dots, x_n)_i^z, \quad x_1, \dots, x_n, z \in X,$$

holds has an infimum K^* . We call it the *best constant* associated with the *n-distance* d . Determining the value of K^* for a given *n-distance* is an interesting problem that might be mathematically challenging. It is the purpose of this paper to provide natural examples of *n-distances* and to show how elegant the investigation of the values of the best constants might be.

It is worth noting that determining the best constant K^* is not relevant for nonconstant $(n-1)$ -hemimetrics because we always have $K^* = 1$ for those functions. Indeed, we have

$$0 < d(x_1, \dots, x_n) = \sum_{i=1}^n d(x_1, \dots, x_n)_i^{x_n}$$

for any pairwise distinct elements x_1, \dots, x_n of X .

The paper is organized as follows. In Section 2 we provide some basic properties of *n-metric spaces* as well as some examples of *n-distances* together with their corresponding best constants. In Section 3 we investigate the values of the best constants for Fermat point based *n-distances* and discuss the particular case of median graphs. In Section 4 we consider some geometric constructions (smallest enclosing sphere and number of directions) to define *n-distances* and study their corresponding best constants. In Section 5 we introduce a generalization of the concept of *n-distance* by replacing in condition (i) the sum function with an arbitrary symmetric *n-variable* function. Finally, in Section 6 we conclude the paper by proposing topics for further research.

Remark 1. A *multidistance* on X , as introduced by Martín and Mayor [19], is a function $d: \bigcup_{n \geq 1} X^n \rightarrow \mathbb{R}_+$ such that, for every integer $n \geq 1$, the restriction of d to X^n satisfies conditions (ii), (iii), and

(i') $d(x_1, \dots, x_n) \leq \sum_{i=1}^n d(x_i, z)$ for all $x_1, \dots, x_n, z \in X$.

Properties of multidistances as well as instances including the Fermat multidistance and smallest enclosing ball multidistances have been investigated for example in [2, 18–20]. Note that multidistances have an indefinite number of arguments whereas *n-distances* have

a fixed number of arguments. In particular, an n -distance can be defined without referring to any given 2-distance. Interestingly, some of the n -distances we present in this paper cannot be constructed from the concept of multidistance (see Section 6).

2. BASIC EXAMPLES AND GENERAL PROPERTIES OF n -DISTANCES

Let us illustrate the concept of n -distance by giving a few elementary examples. Other classes of n -distances will be investigated in the next sections. We denote by $|E|$ the cardinality of any set E .

Example 2.1 (Drastic n -distance). For every integer $n \geq 2$, the map $d: X^n \rightarrow \mathbb{R}_+$ defined by $d(x_1, \dots, x_n) = 0$, if $x_1 = \dots = x_n$, and $d(x_1, \dots, x_n) = 1$, otherwise, is an n -distance on X for which the best constant is $K_n^* = \frac{1}{n-1}$. Indeed, let $x_1, \dots, x_n, z \in X$ and assume that $d(x_1, \dots, x_n) = 1$. If there exists $k \in [n]$ such that $x_i = x_j \neq x_k$ for all $i, j \in [n] \setminus \{k\}$, then we have

$$\sum_{i=1}^n d(x_1, \dots, x_n)_i^z = \begin{cases} n-1, & \text{if } z \in \{x_1, \dots, x_n\} \setminus \{x_k\}, \\ n, & \text{otherwise.} \end{cases}$$

In all other cases we have $\sum_{i=1}^n d(x_1, \dots, x_n)_i^z = n$. \square

Example 2.2 (Cardinality based n -distance). For every integer $n \geq 2$, the map $d: X^n \rightarrow \mathbb{R}_+$ defined by

$$d(x_1, \dots, x_n) = |\{x_1, \dots, x_n\}| - 1$$

is an n -distance on X for which the best constant is $K_n^* = \frac{1}{n-1}$. Indeed, let $x_1, \dots, x_n, z \in X$ and assume that $d(x_1, \dots, x_n) \geq 1$. The case $n = 2$ is trivial. So let us further assume that $n \geq 3$. For every $i \in [n]$, set $m_i = |\{j \in [n] \mid x_j = x_i\}|$. If $|\{x_1, \dots, x_n\}| < n$ (which means that there exists $j \in [n]$ such that $m_j \geq 2$), then it is straightforward to see that

$$\begin{aligned} \sum_{i=1}^n d(x_1, \dots, x_n)_i^z &\geq n d(x_1, \dots, x_n) - |\{i \in [n] \mid m_i = 1\}| \\ &\geq (n-1) d(x_1, \dots, x_n), \end{aligned}$$

where the first inequality is an equality if and only if $z = x_j$ for some $j \in [n]$ such that $m_j \geq 2$, and the second inequality is an equality if and only if there is exactly one $j \in [n]$ such that $m_j \geq 2$. If $|\{x_1, \dots, x_n\}| = n$, then

$$\sum_{i=1}^n d(x_1, \dots, x_n)_i^z \geq (n-1) d(x_1, \dots, x_n),$$

with equality if and only if $z \in \{x_1, \dots, x_n\}$. \square

Example 2.3 (Diameter). Given a metric space (X, d) and an integer $n \geq 2$, the map $d_{\max}: X^n \rightarrow \mathbb{R}_+$ defined by

$$d_{\max}(x_1, \dots, x_n) = \max_{\{i, j\} \subseteq [n]} d(x_i, x_j)$$

is an n -distance on X for which we have $K_n^* = \frac{1}{n-1}$. Indeed, let $x_1, \dots, x_n, z \in X$ and assume without loss of generality that $d_{\max}(x_1, \dots, x_n) = d(x_1, x_2)$. For every $i \in [n]$ we have

$$d_{\max}(x_1, \dots, x_n)_i^z \geq \begin{cases} d(x_2, z), & \text{if } i = 1, \\ d(x_1, z), & \text{if } i = 2, \\ d(x_1, x_2), & \text{otherwise.} \end{cases}$$

Using the triangle inequality, we then obtain

$$\begin{aligned} \sum_{i=1}^n d_{\max}(x_1, \dots, x_n)_i^z &\geq (n-2)d(x_1, x_2) + d(x_1, z) + d(x_2, z) \\ &\geq (n-1)d(x_1, x_2) = (n-1)d_{\max}(x_1, \dots, x_n), \end{aligned}$$

which proves that $K_n^* \leq \frac{1}{n-1}$. To prove that $K_n^* = \frac{1}{n-1}$, note that if $x_1 = \dots = x_{n-1} = z$ and $x_n \neq z$, then $\sum_{i=1}^n d_{\max}(x_1, \dots, x_n)_i^z = (n-1)d_{\max}(x_1, \dots, x_n)$. \square

Example 2.4 (Sum based n -distance). Given a metric space (X, d) and an integer $n \geq 2$, the map $d_\Sigma: X^n \rightarrow \mathbb{R}_+$ defined by

$$d_\Sigma(x_1, \dots, x_n) = \sum_{\{i,j\} \subseteq [n]} d(x_i, x_j)$$

is an n -distance on X for which we have $K_n^* = \frac{1}{n-1}$. Indeed, for fixed $x_1, \dots, x_n, z \in X$, we have

$$\sum_{i=1}^n d_\Sigma(x_1, \dots, x_n)_i^z = (n-2) \sum_{\{i,j\} \subseteq [n]} d(x_i, x_j) + (n-1) \sum_{i=1}^n d(x_i, z).$$

Using the triangle inequality we obtain

$$(n-1) \sum_{i=1}^n d(x_i, z) = \sum_{\{i,j\} \subseteq [n]} (d(x_i, z) + d(x_j, z)) \geq \sum_{\{i,j\} \subseteq [n]} d(x_i, x_j).$$

Therefore, we finally obtain

$$\sum_{i=1}^n d_\Sigma(x_1, \dots, x_n)_i^z \geq (n-1) \sum_{\{i,j\} \subseteq [n]} d(x_i, x_j) = (n-1)d_\Sigma(x_1, \dots, x_n),$$

which proves that $K_n^* \leq \frac{1}{n-1}$. To prove that $K_n^* = \frac{1}{n-1}$, note that if $x_1 = \dots = x_{n-1} = z$ and $x_n \neq z$, then $\sum_{i=1}^n d_\Sigma(x_1, \dots, x_n)_i^z = (n-1)d_\Sigma(x_1, \dots, x_n)$. \square

Example 2.5 (Arithmetic mean based n -distance). For any integer $n \geq 2$, the map $d: \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by

$$d(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i - x_{(1)} = \frac{1}{n} \sum_{i=1}^n (x_i - x_{(1)}),$$

where $x_{(1)} = \min\{x_1, \dots, x_n\}$, is an n -distance on \mathbb{R} for which $K_n^* = \frac{1}{n-1}$. Indeed, let $x_1, \dots, x_n, z \in \mathbb{R}$. By symmetry of d we may assume that $x_1 \leq \dots \leq x_n$. We then obtain

$$d(x_1, \dots, x_n) = \frac{1}{n} \left(\sum_{i=1}^n x_i \right) - x_1$$

and

$$\sum_{i=1}^n d(x_1, \dots, x_n)_i^z = \left(1 - \frac{1}{n}\right) \left(\sum_{i=1}^n x_i \right) + z - (n-1) \min\{x_1, z\} - \min\{x_2, z\}.$$

It follows that condition (1) holds for $K_n = \frac{1}{n-1}$ if and only if

$$(n-1)(x_1 - \min\{x_1, z\}) + (z - \min\{x_2, z\}) \geq 0.$$

We then observe that this inequality is trivially satisfied, which proves that $K_n^* \leq \frac{1}{n-1}$. To prove that $K_n^* = \frac{1}{n-1}$, just take $x_1, \dots, x_n, z \in \mathbb{R}$ so that $x_1 < z < x_2 = \dots = x_n$. \square

In the next result, we show how to construct an $(n-1)$ -hemimetric from an n -distance.

Proposition 2.6. *Let (X, d) be an n -metric space for some integer $n \geq 2$. The function $d': X^n \rightarrow \mathbb{R}_+$ defined as*

$$d'(x_1, \dots, x_n) = \begin{cases} 0, & \text{if } x_1, \dots, x_n \text{ are not pairwise distinct,} \\ d(x_1, \dots, x_n), & \text{otherwise,} \end{cases}$$

is an $(n-1)$ -hemimetric.

Proof. It is easy to see that d' satisfies conditions (ii) and (iii'). To see that condition (i) holds, let $x_1, \dots, x_n, z \in X$ and assume that $d'(x_1, \dots, x_n) > 0$. If $d'(x_1, \dots, x_n)_i^z = d(x_1, \dots, x_n)_i^z$ for every $i \in [n]$, then the simplex inequality holds for d' . Otherwise, we must have $z \in \{x_1, \dots, x_n\}$ and then $\sum_{i=1}^n d'(x_1, \dots, x_n)_i^z = d'(x_1, \dots, x_n)$. This shows that condition (i) holds. \square

The next proposition shows that two of the standard constructions of distances from existing ones are still valid for n -distances. The proof uses the following lemma.

Lemma 2.7. *For any $a_1, \dots, a_n, a \in \mathbb{R}_+$ such that $a \leq \sum_{i=1}^n a_i$, we have*

$$\frac{a}{1+a} \leq \sum_{i=1}^n \frac{a_i}{1+a_i}.$$

Proof. We proceed by induction on $n \geq 1$. The result is easily obtained for $n \in \{1, 2\}$. Assume that the result holds for $k \in \{1, \dots, n-1\}$ for some $n \geq 3$, and that $a \leq \sum_{i=1}^n a_i$ for some $a, a_1, \dots, a_n \in \mathbb{R}_+$. Letting $b = \max\{0, a - a_n\}$, we obtain

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{a_n}{1+a_n} \leq \sum_{i=1}^n \frac{a_i}{1+a_i},$$

where the first inequality is obtained by the induction hypothesis applied to $a \leq b + a_n$, and the second to $b \leq \sum_{i=1}^{n-1} a_i$. \square

Proposition 2.8. *Let d and d' be n -distances on X and let $\lambda > 0$. The following assertions hold.*

- (a) $d + d'$ and λd are n -distances on X .
- (b) $\frac{d}{1+d}$ is an n -distance on X , with values in $[0, 1]$.

Proof. (a) is a simple verification. For (b) we note that condition (i) holds for $\frac{d}{1+d}$ by Lemma 2.7. \square

Remark 2. In the same spirit as Proposition 2.8 we observe that if $d: X \rightarrow \mathbb{R}_+$ is an n -distance and $d_0: X \rightarrow \mathbb{R}_+$ is an $(n-1)$ -hemimetric, then $d + d_0$ is an n -distance.

3. FERMAT POINT BASED n -DISTANCES

Recall that, given a metric space (X, d) and an integer $n \geq 2$, the *Fermat set* F_Y of any n -element subset $Y = \{x_1, \dots, x_n\}$ of X is defined as

$$F_Y = \left\{ x \in X \mid \sum_{i=1}^n d(x_i, x) \leq \sum_{i=1}^n d(x_i, z) \text{ for all } z \in X \right\}.$$

Elements of F_Y are the *Fermat points* of Y . The problem of finding the Fermat point of a triangle in the Euclidean plane was formulated by Fermat in the early 17th century, and was first solved by Torricelli around 1640. The general problem stated for $n \geq 2$ in any metric space was considered by many authors, and applications were found for instance in geometry, combinatorial optimization, and facility location. We refer to [3, Chapter II]

and [12] for an account of the history of this problem. Also, in [15], the location problem is extended in various directions and studied also for very general metrics – more general than those of normed spaces.

We observe that F_Y need not be nonempty in a general metric space. However, it follows from the continuity of the function $h: X \rightarrow \mathbb{R}_+$ defined by $h(x) = \sum_{i=1}^n d(x_i, x)$ that F_Y is nonempty whenever (X, d) is a proper metric space. (Recall that a metric space is proper if every closed ball is compact.) In this section we will therefore assume that (X, d) is a proper metric space.

Proposition 3.1. *For any proper metric space (X, d) and any integer $n \geq 2$, the map $d_F: X^n \rightarrow \mathbb{R}_+$ defined as*

$$d_F(x_1, \dots, x_n) = \min_{x \in X} \sum_{i=1}^n d(x_i, x),$$

is an n -distance on X and we call it the Fermat n -distance.

Proof. The map d_F clearly satisfies conditions (ii) and (iii). Let us show that it satisfies condition (i). Assume first that $n = 2$ and let $y_1, y_2 \in X$ be such that

$$d_F(z, x_2) = d(z, y_1) + d(x_2, y_1) \quad \text{and} \quad d_F(x_1, z) = d(x_1, y_2) + d(z, y_2).$$

By applying the triangle inequality, we obtain

$$\begin{aligned} d_F(z, x_2) + d_F(x_1, z) &= (d(x_1, y_2) + d(z, y_2)) + (d(z, y_1) + d(x_2, y_1)) \\ &\geq d(x_1, x_2) = d(x_1, x_1) + d(x_1, x_2) \geq d_F(x_1, x_2). \end{aligned}$$

Assume now that $n \geq 3$ and let $y_1, \dots, y_n \in X$ be such that

$$d_F(x_1, \dots, x_n)_i^z = \sum_{j \neq i} d(x_j, y_i) + d(z, y_i), \quad i = 1, \dots, n.$$

It follows that

$$\begin{aligned} \sum_{i=1}^n d_F(x_1, \dots, x_n)_i^z &\geq \sum_{i=1}^n \sum_{j \neq i} d(x_j, y_i) \\ &\geq (d(x_1, y_n) + d(x_2, y_n)) + \sum_{i=2}^{n-1} (d(x_1, y_i) + d(x_{i+1}, y_i)), \end{aligned}$$

that is, by applying the triangle inequality,

$$\sum_{i=1}^n d_F(x_1, \dots, x_n)_i^z \geq \sum_{i=2}^n d(x_1, x_i) = \sum_{i=1}^n d(x_1, x_i) \geq d_F(x_1, \dots, x_n),$$

where the last inequality follows from the definition of d_F . \square

In the next proposition we use rough counting arguments to obtain bounds for the best constant K_n^* associated with the Fermat n -distance.

Proposition 3.2. *For every $n \geq 2$, the best constant K_n^* associated with the Fermat n -distance satisfies the inequalities $\frac{1}{n-1} \leq K_n^* \leq \frac{1}{\lfloor n/2 \rfloor}$.*

Proof. Let $x_1, \dots, x_n \in X$ and let z be a Fermat point of $\{x_1, \dots, x_n\}$. For every $i \in [n]$, denote by y_i a Fermat point of $\{z\} \cup \{x_1, \dots, x_n\} \setminus \{x_i\}$. We then have

$$\begin{aligned} (2) \quad d_F(x_1, \dots, x_n)_i^z &= \sum_{j \neq i} d(x_j, y_i) + d(z, y_i) \\ &\leq \sum_{j \neq i} d(x_j, z) + d(z, z) = \sum_{j \neq i} d(x_j, z). \end{aligned}$$

By summing over $i = 1, \dots, n$, we obtain

$$\sum_{i=1}^n d_F(x_1, \dots, x_n)_i^z \leq (n-1) \sum_{i=1}^n d(x_i, z) = (n-1) d_F(x_1, \dots, x_n),$$

which shows that $K_n^* \geq 1/(n-1)$.

Now, if z denotes any element of X and if y_1, \dots, y_n are defined as in the first part of the proof, the identity (2) holds for every $i \in [n]$. Then, for $i = 1, \dots, n-1$, we have

$$d_F(x_1, \dots, x_n)_i^z + d_F(x_1, \dots, x_n)_{i+1}^z \geq d(z, y_i) + d(z, y_{i+1}) + d(x_i, y_{i+1}) + \sum_{j \neq i} d(x_j, y_i)$$

$$(3) \quad \geq d(x_i, y_i) + \sum_{j \neq i} d(x_j, y_i)$$

$$(4) \quad \geq d_F(x_1, \dots, x_n),$$

where (3) is obtained by a double application of the triangle inequality and (4) is obtained by definition of d_F .

It follows from (4) that $\sum_{i=1}^n d_F(x_1, \dots, x_n)_i^z \geq \lfloor n/2 \rfloor d_F(x_1, \dots, x_n)$, which proves that $K_n^* \leq \lfloor n/2 \rfloor^{-1}$. \square

The next proposition uses a more refined counting argument to provide an improvement of the upper bound obtained for K_n^* in Proposition 3.2. Let us first state an immediate generalization of the hand-shaking lemma, which is folklore in graph theory.

Lemma 3.3. *Let $G = (V, E, w)$ be a weighted simple graph, where $w: E \rightarrow \mathbb{R}_+$ is the weighting function. If $f: V \rightarrow \mathbb{R}_+$ is such that $f(x) + f(y) \geq w(e)$ for every $e = \{x, y\} \in E$, then*

$$\sum_{x \in V} f(x) \deg_G(x) \geq \sum_{e \in E} w(e),$$

where $\deg_G(x)$ is the degree of x in G .

Proposition 3.4. *For every $n \geq 2$, the best constant K_n^* associated with the Fermat n -distance satisfies $K_n^* \leq (4n-4)/(3n^2-4n)$.*

Proof. Let $z, x_1, \dots, x_n, y, y_1, \dots, y_n \in X$ be such that y is a Fermat point of $\{x_1, \dots, x_n\}$ and such that equation (2) holds for every $i \in [n]$. For any distinct $i, j \in [n]$, by the triangle inequality we have

$$(5) \quad d(z, y_i) + d(z, y_j) + d(x_i, y_j) \geq d(x_i, y_i).$$

By summing (5) over all $j \in [n] \setminus \{i\}$ we obtain

$$(6) \quad (n-1) d(z, y_i) + \sum_{j \neq i} (d(z, y_j) + d(x_i, y_j)) \geq (n-1) d(x_i, y_i).$$

By summing (6) over all $i \in [n]$ we then obtain

$$(7) \quad 2(n-1) \sum_{i=1}^n d(z, y_i) + \sum_{i=1}^n \sum_{j \neq i} d(x_j, y_i) \geq (n-1) \sum_{i=1}^n d(x_i, y_i).$$

Let us set $S = \sum_{i=1}^n \sum_{j \neq i} d(x_j, y_i)$. We then have

$$(8) \quad 2(n-1) \sum_{i=1}^n d_F(x_1, \dots, x_n)_i^z = (2n-3)S + S + 2(n-1) \sum_{i=1}^n d(z, y_i)$$

$$(9) \quad \geq (2n-3)S + (n-1) \sum_{i=1}^n d(x_i, y_i)$$

$$(10) \quad = (n-2)S + (n-1) \sum_{i=1}^n \sum_{j=1}^n d(x_j, y_i),$$

where (8) follows by the definitions of S and d_F , (9) follows by (7), and (10) by the definition of S .

Now, on the one hand, by the definition of d_F we have

$$(11) \quad (n-1) \sum_{i=1}^n \sum_{j=1}^n d(x_j, y_i) \geq n(n-1) d_F(x_1, \dots, x_n).$$

On the other hand, let us fix $i \in [n]$ and set $V = \{x_1, \dots, x_n\} \setminus \{x_i\}$. Define the function $f: V \rightarrow \mathbb{R}_+$ by $f(x_j) = d(x_j, y_i)$ for any $j \neq i$, and consider the complete weighted graph $G = (V, \binom{V}{2}, w)$ defined by $w(\{x_\ell, x_j\}) = d(x_\ell, x_j)$ for any distinct $x_\ell, x_k \in V$. It follows from Lemma 3.3 that

$$(12) \quad (n-2) \sum_{j \neq i} d(x_j, y_i) \geq \sum_{\{x_k, x_\ell\} \in \binom{V}{2}} d(x_k, x_\ell).$$

By summing (12) over all $i \in [n]$, we get

$$(13) \quad \begin{aligned} (n-2)S &\geq (n-2) \sum_{\{k, \ell\} \in \binom{[n]}{2}} d(x_k, x_\ell) = \frac{n-2}{2} \sum_{k=1}^n \sum_{\ell=1}^n d(x_k, x_\ell) \\ &\geq n \frac{n-2}{2} d_F(x_1, \dots, x_n), \end{aligned}$$

where (13) is obtained by definition of d_F . By substituting (11) and (13) into (10), we finally obtain

$$\sum_{i=1}^n d_F(x_1, \dots, x_n)_i^z \geq \frac{n(3n-4)}{4(n-1)} d_F(x_1, \dots, x_n),$$

which proves that $K_n^* \leq (4n-4)/(3n^2-4n)$. \square

We observe that Proposition 3.4 provides a better upper bound than Proposition 3.2 for every $n \geq 2$, but the difference between these bounds converges to zero as n tends to infinity. The high number of inequalities involved in the proof of Proposition 3.4 suggests that it is in general very difficult to obtain the exact value of K_n^* (we have to find $x_1, \dots, x_n, z \in X$ that turn these inequalities into equalities). However, we will now show that we can determine the value of K_n^* when d_F is the Fermat n -distance associated with the distance function in median graphs.

Recall that a *median graph* is a connected undirected simple graph in which, for any triplet of vertices u, v, w , there is one and only one vertex $\mathbf{m}(u, v, w)$ that is at the intersection of shortest paths between any two elements among u, v, w . Cubes and trees are instances of median graphs. In a median graph $G = (V, E)$, the Fermat 3-distance is the function $d_{\mathbf{m}}: V^3 \rightarrow \mathbb{R}_+$ defined by

$$(14) \quad d_{\mathbf{m}}(u, v, w) = \min_{y \in V} (d(u, y) + d(v, y) + d(w, y)),$$

where d denotes the usual distance function between vertices in a connected graph.

Proposition 3.5. *If $G = (V, E)$ is a median graph, then the best constant K^* associated with its Fermat 3-distance $d_{\mathbf{m}}$ is equal to $\frac{1}{2}$. Moreover, the only Fermat point of $\{u, v, w\}$ is $\mathbf{m}(u, v, w)$.*

Proof. The minimum in (14) is realized by any $y_0 \in V$ that realizes the minimum of the values

$$(15) \quad (d(u, y) + d(v, y)) + (d(w, y) + d(u, y)) + (d(v, y) + d(w, y))$$

for $y \in V$. By definition, the vertex $y_0 = \mathbf{m}(u, v, w)$ is on shortest paths between any two elements among u, v, w , which shows that it realizes the minimum of each of the three terms in (15), and hence the minimum in (14).

It follows that

$$\begin{aligned} d_{\mathbf{m}}(u, v, z) &= d(u, y_0) + d(v, y_0) + d(z, y_0) \\ &= \frac{1}{2}(d(u, y_0) + d(v, y_0) + d(z, y_0) + d(u, y_0) + d(v, y_0) + d(z, y_0)) \\ &= \frac{1}{2}(d(u, v) + d(u, z) + d(v, z)), \end{aligned}$$

which shows that $\min_{z \in V} d_{\mathbf{m}}(u, v, z)$ is equal to $d(u, v)$, and is realized by any element z_0 on a shortest path between u and v . We conclude that the minimum of

$$d_{\mathbf{m}}(z, v, w) + d_{\mathbf{m}}(u, z, w) + d_{\mathbf{m}}(u, v, z)$$

for $z \in V$ is realized by $z_0 = \mathbf{m}(u, v, w)$, and is equal to $d(v, w) + d(u, w) + d(u, v) = 2d_{\mathbf{m}}(u, v, w)$. We have proved that the best constant K^* associated with $d_{\mathbf{m}}$ is $\frac{1}{2}$. \square

4. EXAMPLES OF n -DISTANCES BASED ON GEOMETRIC CONSTRUCTIONS

In this section we introduce n -distances defined from certain geometric constructions and investigate their corresponding best constants. In what follows, we denote by d the Euclidean distance on \mathbb{R}^k for some integer $k \geq 2$.

The first n -distances we investigate are based on the following construction.

Definition 4.1. For any $n \geq 2$ and any $x_1, \dots, x_n \in \mathbb{R}^k$, we denote by $S(x_1, \dots, x_n)$ the smallest $(k-1)$ -dimensional sphere enclosing $\{x_1, \dots, x_n\}$. For any $i \in [n]$ and any $z \in \mathbb{R}^k$, we denote by $S(x_1, \dots, x_n)_i^z$ the smallest $(k-1)$ -dimensional sphere enclosing $\{x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n\}$.

The sphere introduced in Definition 4.1 always exists and is unique. Moreover, it can be computed in linear time [21, 22] or expected linear time [26].

When $k = 2$, we have the following fact.

Fact 4.2. *Let A, B, C be the vertices of a triangle in \mathbb{R}^2 .*

- (a) *If ABC forms an acute triangle with angles α, β and γ , respectively, then $S(A, B, C)$ is the circumcircle \mathcal{C} of ABC whose radius R satisfies*

$$(16) \quad R = \frac{a}{2 \sin \alpha} = \frac{b}{2 \sin \beta} = \frac{c}{2 \sin \gamma},$$

where $a = d(B, C)$, $b = d(A, C)$, and $c = d(A, B)$. Let A^* be one of the two points of the circle \mathcal{C} that is on the bisector of BC . Then the perimeter of the triangle ABC strictly decreases as A moves along \mathcal{C} from A^* to B (or from A^* to C).

- (b) If ABC is obtuse in A , then $S(A, B, C)$ contains B and C , and its diameter is equal to a .
- (c) It follows from (a) and (b) that the radius R of $S(A, B, C)$ satisfies

$$(17) \quad R \geq \max\left\{\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right\}.$$

Proposition 4.3 (Radius of $S(x_1, \dots, x_n)$ in \mathbb{R}^2). *For any $n \geq 2$, the map $d_r: (\mathbb{R}^2)^n \rightarrow \mathbb{R}_+$ that associates with any $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ the radius of $S(x_1, \dots, x_n)$ is an n -distance for which we have $K_n^* = \frac{1}{n-1}$.*

Proof. Let us show that the map d_r satisfies the simplex inequality for $K_n = \frac{1}{n-1}$. Since d_r is a continuous function, we can assume that its arguments are pairwise distinct.

Consider first the case where $n = 2$. For any distinct $A, B \in \mathbb{R}^2$, we have $d_r(A, B) = \frac{1}{2} d(A, B)$, which proves that the simplex inequality holds for $n = 2$.

Suppose now that $n = 3$ and let us show that, for any $A, B, C, Z \in \mathbb{R}^2$, with A, B, C pairwise distinct, we have

$$(18) \quad 2 d_r(A, B, C) \leq d_r(Z, B, C) + d_r(A, Z, C) + d_r(A, B, Z).$$

Set $a = d(B, C)$, $b = d(A, C)$, and $c = d(A, B)$. By (17) we have

$$(19) \quad d_r(Z, B, C) \geq \frac{a}{2}, \quad d_r(A, Z, C) \geq \frac{b}{2}, \quad d_r(A, B, Z) \geq \frac{c}{2},$$

and hence

$$(20) \quad d_r(Z, B, C) + d_r(A, Z, C) + d_r(A, B, Z) \geq \frac{a+b+c}{2} \geq \max\{a, b, c\}.$$

Suppose first that ABC is not acute, assuming for instance that $\beta \geq \frac{\pi}{2}$. Then $2 d_r(A, B, C) = b$, and then (18) immediately follows from (20). Suppose now that ABC is acute, with circumcircle \mathcal{C} , and consider the triangle $A'BC$, with sides a, b', c' , such that $A' \in \mathcal{C}$ and $\sphericalangle A'BC = \frac{\pi}{2}$. By Fact 4.2 (a) we have

$$\frac{a+b+c}{2} \geq \frac{a+b'+c'}{2} \geq b' = 2 d_r(A', B, C) = 2 d_r(A, B, C),$$

and then again (18) follows from (20). Finally, the equality is obtained in (18) by taking $A \neq B = C = Z$.

We now prove the general case where $n \geq 3$. Let $A_1, \dots, A_n, Z \in \mathbb{R}^2$, with A_1, \dots, A_n pairwise distinct. It is a known fact [4] that either there are $j, k \in [n]$ such that A_j and A_k are distinct and

$$S(A_1, \dots, A_n) = S(A_j, A_k)$$

or there are $j, k, \ell \in [n]$ such that A_j, A_k , and A_ℓ are distinct and

$$S(A_1, \dots, A_n) = S(A_j, A_k, A_\ell).$$

Let us consider the latter case (the proof in the former case can be dealt with similarly). On the one hand, using (17) it is easy to see that

$$(21) \quad d_r(A_1, \dots, A_n)_i^Z \geq d_r(A_1, \dots, A_n), \quad i \notin \{j, k, \ell\}.$$

On the other hand, the following inequalities hold:

$$\begin{aligned} d_r(A_1, \dots, A_n)_j^Z &\geq d_r(Z, A_k, A_\ell), \\ d_r(A_1, \dots, A_n)_k^Z &\geq d_r(A_j, Z, A_\ell), \\ d_r(A_1, \dots, A_n)_\ell^Z &\geq d_r(A_j, A_k, Z). \end{aligned}$$

Indeed, $S(A_1, \dots, A_n)_j^Z$ encloses the points Z , A_k , and A_ℓ and hence cannot have a radius strictly smaller than that of $S(Z, A_k, A_\ell)$.

Adding up these inequalities and then using (18), we obtain

$$(22) \quad \begin{aligned} & d_r(A_1, \dots, A_n)_j^Z + d_r(A_1, \dots, A_n)_k^Z + d_r(A_1, \dots, A_n)_\ell^Z \\ & \geq d_r(Z, A_k, A_\ell) + d_r(A_j, Z, A_\ell) + d_r(A_j, A_k, Z) \\ & \geq 2 d_r(A_j, A_k, A_\ell) = 2 d_r(A_1, \dots, A_n). \end{aligned}$$

Combining (21) with (22), we finally obtain

$$\sum_{i=1}^n d_r(A_1, \dots, A_n)_i^Z \geq (n-1) d_r(A_1, \dots, A_n),$$

which proves that $K_n^* \leq \frac{1}{n-1}$. To prove that $K_n^* = \frac{1}{n-1}$, just consider $A_2 = \dots = A_n = Z$ and $A_1 \neq A_2$. \square

Proposition 4.4 (Area bounded by $S(x_1, \dots, x_n)$ in \mathbb{R}^2). *For any $n \geq 3$, the map $d_s: (\mathbb{R}^2)^n \rightarrow \mathbb{R}_+$ that associates with any $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ the surface area bounded by $S(x_1, \dots, x_n)$ is an n -distance for which we have $K_n^* = (n - \frac{3}{2})^{-1}$.*

Proof. Let us show that the map $d_s = \pi d_r^2$ satisfies the simplex inequality with constant $K_n = (n - \frac{3}{2})^{-1}$. Since d_r is continuous, we can assume that its arguments are pairwise distinct.

Consider first the case where $n = 3$ and let us show that, for any $A, B, C, Z \in \mathbb{R}^2$, with A, B, C pairwise distinct, we have

$$(23) \quad d_r(A, B, C)^2 \leq \frac{2}{3} (d_r(Z, B, C)^2 + d_r(A, Z, C)^2 + d_r(A, B, Z)^2).$$

If the triangle ABC is acute, then we may assume for instance that $\frac{\pi}{3} \leq \alpha \leq \frac{\pi}{2}$, which implies $\frac{\sqrt{3}}{2} \leq \sin \alpha \leq 1$. Using (16), we then have

$$(24) \quad d_r(A, B, C)^2 \leq \frac{a^2}{3} \leq \frac{2}{3} \left(\frac{a^2}{4} + \frac{a^2}{4} \right) \leq \frac{2}{3} \left(\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4} \right),$$

where the latter inequality holds by the law of cosines. We then obtain (23) by combining (19) with (24).

If ABC is obtuse in C , then $d_r(A, B, C) = \frac{c}{2}$. Using the triangle inequality and the square and arithmetic mean inequality, we also have

$$\frac{a^2 + b^2}{2} \geq \left(\frac{a+b}{2} \right)^2 \geq \frac{c^2}{4}.$$

Combining these observations with (19), we obtain

$$\begin{aligned} & \frac{2}{3} (d_r(Z, B, C)^2 + d_r(A, Z, C)^2 + d_r(A, B, Z)^2) \\ & \geq \frac{2}{3} \left(\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4} \right) \geq \frac{2}{3} \frac{3}{8} c^2 = \left(\frac{c}{2} \right)^2 = d_r(A, B, C)^2. \end{aligned}$$

To see that the general case where $n \geq 3$ also holds, it suffices to proceed as in the proof of Proposition 4.3. This shows that $K_n^* \leq (n - \frac{3}{2})^{-1}$. To prove that $K_n^* = (n - \frac{3}{2})^{-1}$, just consider $A_1 \neq A_2$ and $A_3 = \dots = A_n = Z = (A_1 + A_2)/2$, where $(A_1 + A_2)/2$ is the midpoint of A_1 and A_2 . \square

Remark 3. The map d_s defined in Proposition 4.4 can be naturally extended to the case where $n = 2$. However, in this case d_s no longer satisfies condition (i) and hence is not a 2-distance. Indeed, for any $A, B, Z \in \mathbb{R}^2$, with A, B distinct, we have

$$d_s(A, B) \leq 2(d_s(A, Z) + d_s(Z, B)),$$

or equivalently,

$$d(A, B)^2 \leq 2d(A, Z)^2 + 2d(Z, B)^2,$$

where the constant 2 is optimal (take A and B distinct and $Z = (A+B)/2$). To see that this inequality holds, set $A = (0, 0)$, $B = (b, 0)$, and $Z = (x, y)$. Then, the inequality becomes

$$b^2 \leq 2(x^2 + y^2) + 2(x - b)^2 + 2y^2,$$

which always holds because it is algebraically equivalent to

$$(2x - b)^2 + 4y^2 \geq 0.$$

Remark 4. In an attempt to generalize the previous two propositions to \mathbb{R}^k ($k \geq 2$), we may consider the following open questions:

- (a) Prove (or disprove) that Proposition 4.3 still holds in \mathbb{R}^k .
- (b) Prove (or disprove) that, for any $n \geq 3$, the map $d_v: (\mathbb{R}^k)^n \rightarrow \mathbb{R}_+$ that associates with any $(x_1, \dots, x_n) \in (\mathbb{R}^k)^n$ the k -dimensional volume bounded by $S(x_1, \dots, x_n)$ is an n -distance for which we have $K_n^* = (n - 2 + 2^{1-k})^{-1}$.

Note that the problem in (b) above is motivated by the fact that the corresponding simplex inequality with $K_n = (n - 2 + 2^{1-k})^{-1}$ holds when x_1 and x_2 are distinct and $x_3 = \dots = x_n = z$ is the midpoint of x_1 and x_2 .

We now show that counting the number of different directions defined by pairs of distinct elements among n points in the plane defines an n -distance.

For any distinct $x, y \in \mathbb{R}^2$, we denote by \overline{xy} the direction $\pm(x - y)/\|x - y\|$. Here we assume that \overline{xy} and \overline{yx} represent the same direction.

Proposition 4.5 (Number of directions in \mathbb{R}^2). *For any $n \geq 3$, the map $d_n: (\mathbb{R}^2)^n \rightarrow \mathbb{R}_+$ that associates with any $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ the cardinality $|\Delta|$ of the set*

$$\Delta = \{\overline{x_i x_j} \mid i, j \in [n] \text{ and } x_i \neq x_j\}$$

is an n -distance for which we have $\frac{1}{n-2+\frac{2}{n}} \leq K_n^ < \frac{1}{n-2}$.*

Proof. Let $x_1, \dots, x_n, z \in \mathbb{R}^2$. For any $i \in [n]$, let

$$\Delta_i = \{\overline{x_j x_k} \mid j, k \in [n] \setminus \{i\} \text{ and } x_j \neq x_k\}.$$

On the one hand, we clearly have $|\Delta_i| \leq d_n(x_1, \dots, x_n)_i^z$ for every $i \in [n]$. On the other hand, it is easy to see that each direction in Δ is counted at least $(n - 2)$ times in the sum $\sum_{i=1}^n |\Delta_i|$. From these observations it follows that

$$(25) \quad (n - 2) d_n(x_1, \dots, x_n) = (n - 2) |\Delta| \leq \sum_{i=1}^n |\Delta_i| \leq \sum_{i=1}^n d_n(x_1, \dots, x_n)_i^z,$$

which proves that $K_n^* \leq \frac{1}{n-2}$.

We now show by contradiction that the latter inequality is strict. Assume that there exist $x_1, \dots, x_n, z \in \mathbb{R}^2$ such that

$$(n - 2) d_n(x_1, \dots, x_n) = \sum_{i=1}^n d_n(x_1, \dots, x_n)_i^z.$$

It follows that for these points we can replace both inequalities in (25) with equalities. The first equality then means that each direction in Δ is counted exactly $(n - 2)$ times in the sum $\sum_{i=1}^n |\Delta_i|$. It is easy to see that this condition also means that no three of the points x_1, \dots, x_n are collinear. Let us now consider the second inequality. Since $|\Delta_i| \leq d_n(x_1, \dots, x_n)_i^z$ for every $i \in [n]$, we must have $|\Delta_i| = d_n(x_1, \dots, x_n)_i^z$ for every $i \in [n]$. Suppose first that $n \geq 4$. It follows from the latter condition that both sets $\{x_2, \dots, x_n\}$ and $\{z, x_2, \dots, x_n\}$ generate the same number of directions. Since no three of the points x_2, \dots, x_n are collinear, we should have $z = x_\ell$ for some $\ell \in \{2, \dots, n\}$. But then we have $|\Delta_\ell| < d_n(x_1, \dots, x_n)_\ell^z$, a contradiction. A similar contradiction can be easily reached when $n = 3$.

Let us now establish the lower bound for K_n^* . Let x_1, \dots, x_n be pairwise distinct and placed clockwise on the unit circle. Let also $z = x_1$. Then we have

$$d_n(x_1, \dots, x_n) = \binom{n}{2} \quad \text{and} \quad d_n(x_1, \dots, x_n)_i^z = \begin{cases} \binom{n}{2} & \text{if } i = 1, \\ \binom{n-1}{2} & \text{if } i \neq 1, \end{cases}$$

and hence

$$\begin{aligned} \sum_{i=1}^n d_n(x_1, \dots, x_n)_i^z &= \binom{n}{2} + (n-1) \binom{n-1}{2} = \left(n - 2 + \frac{2}{n}\right) \binom{n}{2} \\ &= \left(n - 2 + \frac{2}{n}\right) d_n(x_1, \dots, x_n), \end{aligned}$$

which completes the proof. \square

Remark 5. An n -distance $d: (\mathbb{R}^k)^n \rightarrow \mathbb{R}_+$ is said to be *homogeneous of degree* $q \geq 0$ if, for any $t > 0$, we have

$$d(tx_1, \dots, tx_n) = t^q d(x_1, \dots, x_n), \quad x_1, \dots, x_n \in \mathbb{R}^k.$$

This means that under any dilation $x \mapsto tx$, the n -distance d is magnified by the factor t^q . Since a distance on \mathbb{R}^k usually represents a linear dimension, we could expect any n -distance on \mathbb{R}^k to be homogeneous of degree 1. This is for instance the case for the n -distance defined in Proposition 4.3. Surprisingly enough, the n -distances defined in Examples 2.1, 2.2, and Proposition 4.5 are homogeneous of degree 0, that is, invariant under any dilation. Also, the n -distance defined in Proposition 4.4 is homogeneous of degree 2.

5. A GENERALIZATION OF THE CONCEPT OF n -DISTANCE

The concept of n -distance as defined in Definition 1.1 can naturally be generalized by relaxing condition (i) as follows.

Definition 5.1. Let $g: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ be a symmetric function, i.e., invariant under any permutation of its arguments. We say that a function $d: X^n \rightarrow \mathbb{R}_+$ is a *g -distance* if it satisfies conditions (ii), (iii), and

$$d(x_1, \dots, x_n) \leq g(d(x_1, \dots, x_n)_1^z, \dots, d(x_1, \dots, x_n)_n^z)$$

for all $x_1, \dots, x_n, z \in X$.

In view of Proposition 2.8, it is natural to require $d + d'$, λd , and $\frac{d}{1+d}$ to be g -distances whenever so are d and d' . The following proposition provides sufficient conditions on g for these properties to hold. Recall that a function $g: \mathbb{R}_+^n \rightarrow \mathbb{R}$ is *positively homogeneous* if $g(\lambda \mathbf{r}) = \lambda g(\mathbf{r})$ for all $\mathbf{r} \in \mathbb{R}_+^n$ and all $\lambda > 0$. It is said to be *superadditive* if $g(\mathbf{r} + \mathbf{s}) \geq$

$g(\mathbf{r}) + g(\mathbf{s})$ for every $\mathbf{r}, \mathbf{s} \in \mathbb{R}_+^n$. Also, it is *additive* if $g(\mathbf{r} + \mathbf{s}) = g(\mathbf{r}) + g(\mathbf{s})$ for every $\mathbf{r}, \mathbf{s} \in \mathbb{R}_+^n$.

Proposition 5.2. *Let $g: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ be a symmetric function, and let $d, d': X^n \rightarrow \mathbb{R}_+$ be g -distances. The following assertions hold.*

- (a) *If g is positively homogeneous, then λd is a g -distance for every $\lambda > 0$.*
- (b) *If g is superadditive, then $d + d'$ is a g -distance.*
- (c) *If g is both positively homogeneous and superadditive, then it is concave.*
- (d) *The function g is additive if and only if there exists $\lambda \geq 0$ such that*

$$(26) \quad g(\mathbf{r}) = \lambda \sum_{i=1}^n r_i, \quad \mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}_+^n.$$

- (e) *If g satisfies (26) for some $\lambda \geq 1$, then $\frac{d}{1+d}$ is a g -distance.*

Proof. (a) and (b) follow from the definitions.

(c) For any $\lambda \in [0, 1]$, we have

$$g(\lambda \mathbf{r} + (1 - \lambda)\mathbf{s}) \leq g(\lambda \mathbf{r}) + g((1 - \lambda)\mathbf{s}) = \lambda g(\mathbf{r}) + (1 - \lambda)g(\mathbf{s}),$$

where the inequality follows from superadditivity and the equality from positive homogeneity.

(d) The sufficiency is trivial. To see that the necessity holds, note that g is additive and bounded from below (since it ranges in \mathbb{R}_+) and hence it is continuous and there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $g(\mathbf{r}) = \sum_{i=1}^n \lambda_i r_i$; see [1, Cor. 2, p. 35]. The result then follows from the symmetry of g .

(e) Let $x_1, \dots, x_n, z \in X$ and set $d = d(x_1, \dots, x_n)$ and $d_i = d(x_1, \dots, x_n)_i^z$ for every $i \in [n]$. Since $\lambda \geq 1$, we have $\lambda r / (1 + \lambda r) \leq \lambda r / (1 + r)$ for every $r \geq 0$. It then follows that

$$\frac{1}{1+d} \leq \sum_{i=1}^n \frac{\lambda d_i}{1 + \lambda d_i} \leq \sum_{i=1}^n \frac{\lambda d_i}{1 + d_i},$$

where the first inequality follows from Lemma 2.7 and the fact that d is a g -distance. \square

6. CONCLUSION AND FURTHER RESEARCH

In this paper we have introduced and discussed the concept of n -distance as a natural generalization of the concept of distance to functions of $n \geq 2$ variables. There are two key features in this generalization: one is an n -ary version of the identity of indiscernibles, and the other is the simplex inequality, which is a natural generalization of the triangle inequality. We have observed that any n -distance d has an associated best constant $K_n^* \in]0, 1]$ satisfying inequality (1). Also, we have provided many natural examples of n -distances, and have shown that searching for their associated best constant may be mathematically challenging and may sometimes require subtle arguments. The examples we have discussed might suggest that we have $K_n^* < 1$ for any n -distance. The following example, which was communicated to us by Roberto Ghiselli Ricci [25], shows that this is not the case.

Example 6.1. Let $n \geq 3$ and $a \in \mathbb{R}$. Let also $\mathcal{A}(a, n)$ be the set of n -tuples whose components are consecutive elements of arithmetic progressions with common difference a . Consider the map $d_n: \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined as

$$d_n(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_1 = \dots = x_n, \\ 1 & \text{if } (x_1, \dots, x_n) \in \mathcal{A}(a, n) \text{ for some } a \neq 0, \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

We prove that d_n is an n -distance for which we have $K_n^* = 1$. Conditions (ii) and (iii) are easily verified. To see that condition (i) holds, consider $x_1, \dots, x_n, z \in \mathbb{R}$. First assume that $d_n(x_1, \dots, x_n) = \frac{1}{n}$. There is at most one $i \in [n]$ such that $d_n(x_1, \dots, x_n)_i^z = 0$. Thus, we obtain

$$\sum_{i=1}^n d_n(x_1, \dots, x_n)_i^z \geq \frac{n-1}{n} \geq d_n(x_1, \dots, x_n).$$

Assume now that $d(x_1, \dots, x_n) = 1$. It follows that $d_n(x_1, \dots, x_n)_i^z \geq \frac{1}{n}$ for all $i \in [n]$, which shows that the simplex inequality holds in that case as well. To prove that $K_n^* = 1$, just consider $x_1 = 1, x_2 = 2, \dots, x_n = n$, and $z = -1$. \square

We also observe that certain n -distances cannot be constructed from the concept of multidistance as defined by Martín and Mayor [19] (see Remark 1). Instances of such n -distances are given, e.g., in Propositions 4.4 and 4.5.

We conclude this paper by proposing a few topics for further research.

- (a) Improve the bounds for the best constant associated with the Fermat n -distance (at least in some given proper metric spaces).
- (b) Consider and solve the problems stated in Remark 4.
- (c) Investigate properties of topological spaces based on n -metric spaces. On this issue we observe that in [24] the authors introduced a stronger version of 3-metric space called *G-metric space* (see also [16]). It is shown that there is a natural metric space associated with any G -metric space. Finding an appropriate generalization of the notion of G -metric space as a stronger version of n -metric space and investigating its topological properties seems to be an interesting topic of research.

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