GENERALIZED CONIC FUNCTIONS OF HV-CONVEX PLANAR SETS: CONTINUITY PROPERTIES AND RELATIONS TO X-RAYS

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ABSTRACT. In the paper we investigate the continuity properties of the mapping Φ which sends any non-empty compact connected hv-convex planar set K to the associated generalized conic function f_K. The function f_K measures the average taxicab distance of the points in the plane from the focal set K by integration. The main area of the applications is the geometric tomography because f_K involves the coordinate X-rays' information as second order partial derivatives [8]. We prove that the Hausdorff-convergence implies the convergence of the conic functions with respect to both the supremum-norm and the L_1-norm provided that we restrict the domain to the collection of non-empty compact connected hv-convex planar sets contained in a fixed box (reference set) with parallel sides to the coordinate axes. We also have that Φ^{-1} is upper semi-continuous as a set-valued mapping. The upper semi-continuity establishes an approximating process in the sense that if f_L is close to f_K then L must be close to an element K' such that $f_K = f_{K'}$. Therefore K and K' have the same coordinate X-rays almost everywhere. Lower semi-continuity is usually related to the existence of continuous selections. If a set-valued mapping is both upper and lower semi-continuous at a point of its domain it is called continuous. The last section of the paper is devoted to the case of non-empty compact convex planar sets. We show that the class of convex bodies that are determined by their coordinate X-rays coincides with the family of convex bodies K for which f_K is a point of lower semi-continuity for Φ^{-1}.

1. INTRODUCTION

The idea motivating our investigations is the application of generalized conics' theory [6], [7] and [8] in geometric tomography. Let K be a compact planar set in the Euclidean plane $\mathbb{R}^2$, $p \geq 1$ and consider the distance function

$$d_p\big((x,y),(\alpha,\beta)\big) = \sqrt[p]{|x - \alpha|^p + |y - \beta|^p}$$

induced by the p - norm. Pairs of the form (x,y), (α,β) and (c_1,c_2) denote elements of $\mathbb{R}^2$. Let us define the sets

$$x <_1 K := \{(\alpha,\beta) \in K \mid x < \alpha\}, \quad K <_1 x := \{(\alpha,\beta) \in K \mid \alpha < x\},$$

$$y <_2 K := \{(\alpha,\beta) \in K \mid y < \beta\}, \quad K <_2 y := \{(\alpha,\beta) \in K \mid \beta < y\},$$

$$x =_1 K := \{(\alpha,\beta) \in K \mid \alpha = x\}, \quad y =_2 K := \{(\alpha,\beta) \in K \mid \beta = y\},$$

where the index refers to the usual ordering of the coordinates. A planar set K is said to be hv-convex if the sections $x =_1 K$ and $y =_2 K$ are convex sets for all $x, y \in \mathbb{R}$.

Definition 1. *The* X-ray functions into the coordinate directions *are*

$$Y_K(x) := \lambda_1(x =_1 K) \quad and \quad X_K(y) := \lambda_1(y =_2 K),$$

where λ_1 denotes the one-dimensional Lebesgue measure.

X-ray functions (especially coordinate X-rays) are typical objects in geometric tomography [2]. We start from a compact set K in the plane to construct a convex function carrying the information of coordinate X-rays as second order derivatives.

Definition 2. [8] *The generalized conic function f_K associated to K is defined by the formula*

$$f_K(x, y) := \int_K d_1\big((x, y), (\alpha, \beta)\big) \, d\alpha d\beta.$$

The levels of this function are called generalized conics *with K as the* focal set.

Using that the 1 - norm is decomposable the generalized conic function f_K can be expressed in terms of the coordinate X-rays as follows:

$$(1) \qquad f_K(x, y) = \int_{-\infty}^{\infty} |x - \alpha| Y_K(\alpha) \, d\alpha + \int_{-\infty}^{\infty} |y - \beta| X_K(\beta) \, d\beta;$$

moreover

$$\frac{\partial f_K}{\partial x}(x, y) = \lambda_2(K <_1 x) - \lambda_2(x <_1 K)$$

and, in a similar way,

$$\frac{\partial f_K}{\partial y}(x, y) = \lambda_2(K <_2 y) - \lambda_2(y <_2 K),$$

where λ_2 denotes the Lebesgue measure on $\mathbb{R}^2$. By the Cavalieri's principle

$$\lambda_2(K <_1 x) = \int_{-\infty}^{x} Y_K(s) \, ds \quad and \quad \lambda_2(x <_1 K) = \int_{x}^{\infty} Y_K(s) \, ds,$$

$$\lambda_2(K <_2 y) = \int_{-\infty}^{y} X_K(t) \, dt \quad and \quad \lambda_2(y <_2 K) = \int_{y}^{\infty} X_K(t) \, dt.$$

Lebesgue differentiation theorem leads us to

$$(2) \qquad \frac{\partial^2 f_K}{\partial x \partial x}(x, y) = 2Y_K(x) \quad and \quad \frac{\partial^2 f_K}{\partial y \partial y}(x, y) = 2X_K(y)$$

except on a set of measure zero. Equations (1) and (2) show that $f_K = f_L$ if and only if K and L have the same coordinate X-rays almost everywhere.

Remark 1. *If $\lambda_2(K) > 0$ then the weighted function*

$$F_K := \frac{1}{\lambda_2(K)} f_K$$

can be also introduced. We have that $F_K = F_L$ if and only if the coordinate X-rays are proportional to each other [8].

Our main result is that the Hausdorff-convergence implies the convergence of the conic functions with respect to both the supremum-norm and the L_1-norm provided that we restrict the domain to the collection of non-empty compact connected hv-convex planar sets contained in a fixed box (reference set) with parallel sides to the coordinate axes. We also have that Φ^{-1} is upper semi-continuous as a set-valued mapping. The upper semi-continuity establishes an approximating process in the sense that if f_L is close to f_K then L must be close to an element K' such that $f_K = f_{K'}$. Therefore K and K' have the same coordinate X-rays almost everywhere. R. Gardner and M. Kiderlen [3] presented an algorithm for reconstructing convex bodies from noisy X-ray measurements with a full proof of convergence in 2007. In this sense the reconstruction means to give the unknown set as a limit of a convergent sequence. The algorithm uses four directions which is related to the minimal number of directions for all convex bodies to be determined by their X-rays in these directions. Our approach means an alternative way for the reconstruction/approximation. The method uses only two directions (coordinate X-rays) and we can apply the theory to the wider class of non-empty compact connected hv-convex planar sets: let f_K be the input data and consider the optimization problem

$$\text{minimize } \|f_L - f_K\| \text{ subject to } L \in \mathcal{H},$$

where the set $\mathcal{H}$ is the subcollection of non-empty compact connected hv-convex sets which are constituted by the subrectangles belonging to a partition of the reference set [9]. It is typically a rectangle B with parallel sides to the coordinate axes such that $K \subset B$.

2. General observations

In what follows $\mathcal{F}(\mathbb{R}^2)$ denotes the metric space of non-empty bounded and closed (i.e. compact) subsets in the plane equipped with the Hausdorff metric [1]. The outer parallel body K^ε is the union of all closed Euclidean balls centered at the points of $K \in \mathcal{F}(\mathbb{R}^2)$ with radius $\varepsilon > 0$. The Hausdorff distance between K and $L \in \mathcal{F}(\mathbb{R}^2)$ is given by the formula

$$H(K, L) := \inf\{\varepsilon > 0 \mid K \subset L^\varepsilon \text{ and } L \subset K^\varepsilon\}.$$

Let pr_1 and pr_2 be the orthogonal projections onto the coordinate axes and consider a rectangle B with parallel sides to the coordinate axes. The level set of $\text{pr}_1 \times \text{pr}_2$ belonging to B is defined as

$$\mathcal{L}_B = \{L \in \mathcal{F}(\mathbb{R}^2) \mid \text{pr}_1(L) \times \text{pr}_2(L) = B\}.$$

We also introduce a kind of sublevel set

$$\mathcal{M}_B = \bigcup_{B^* \subset B} \mathcal{L}_{B^*}$$

with respect to the partial ordering induced by the inclusion, where the union is taken with respect to all rectangles with parallel sides to the coordinate axes and contained in B.

Proposition 1. *Both $\mathcal{L}_B$ and $\mathcal{M}_B$ are compact with respect to the Hausdorff metric.*

Proof. Since $\mathbb{R}^2$ is complete, the space $\mathcal{F}(\mathbb{R}^2)$ equipped with the Hausdorff metric is also complete, see [1]. By Hausdorff's theorem any closed and totally bounded subset in a complete metric space is compact. That $\mathcal{L}_B$ (or $\mathcal{M}_B$) is totally bounded follows from the well-known version of Blaschke's selection theorem for compact sets [5] [Theorem 1.8.4]. The closedness can be concluded from the continuity of the mapping

$$L \mapsto \mathrm{pr}_1(L) \times \mathrm{pr}_2(L),$$

where pr_1 and pr_2 denote the orthogonal projections onto the coordinate axes. The orthogonal projections are obviously continuous mappings with respect to the Hausdorff metric because the projection of any closed ball is a lower dimensional closed ball. In other words the projected parallel body is just the parallel body (with the same radius) of the projected set. □

Proposition 2. *Both $\mathcal{L}_B$ and $\mathcal{M}_B$ are convex in the sense that $L_1 \in \mathcal{L}_B/\mathcal{M}_B$ and $L_2 \in \mathcal{L}_B/\mathcal{M}_B$ implies that $tL_1 + (1-t)L_2 \in \mathcal{L}_B/\mathcal{M}_B$ for all $0 \le t \le 1$.*

Proof. Since the projections preserve the convex combination of the elements the statement follows directly from the definition of $\mathcal{L}_B/\mathcal{M}_B$. □

Theorem 1. *The mapping $\Phi\colon L \in \mathcal{L}_B \to f_L$ is concave in the sense that for any $(x,y) \in \mathbb{R}^2$*

$$f_{tL_1+(1-t)L_2}(x,y) \ge tf_{L_1}(x,y) + (1-t)f_{L_2}(x,y),$$

where L_1, $L_2 \in \mathcal{L}_B$ and $0 \le t \le 1$.

Proof. Let $(\alpha, \beta) \in B$ be a fixed point. Then

$$\big(\alpha =_1 (tL_1 + (1-t)L_2)\big) \supset \big(t\alpha =_1 tL_1\big) + \big((1-t)\alpha =_1 (1-t)L_2\big),$$

$$\big(\beta =_2 (tL_1 + (1-t)L_2)\big) \supset \big(t\beta =_2 tL_1\big) + \big((1-t)\beta =_2 (1-t)L_2\big).$$

Since none of the sets $\alpha =_1 L_i$ $(i=1,2)$ are empty the numbers

$$c_1 := \inf\{y \mid (t\alpha, y) \in tL_1\} = t\inf\{y \mid (\alpha, y) \in L_1\},$$

$$d_1 := \sup\{y \mid (t\alpha, y) \in tL_1\} = t\sup\{y \mid (\alpha, y) \in L_1\}$$

and

$$c_2 := \inf\{y \mid ((1-t)\alpha, y) \in (1-t)L_2\} = (1-t)\inf\{y \mid (\alpha, y) \in L_2\},$$

$$d_2 := \sup\{y \mid ((1-t)\alpha, y) \in (1-t)L_2\} = (1-t)\sup\{y \mid (\alpha, y) \in L_2\}$$

are well-defined. Consider the sets

$$P := (t\alpha, c_1) + (1-t)(\alpha =_1 L_2) \subset \mathrm{conv}\big\{(\alpha, c_1 + c_2), (\alpha, c_1 + d_2)\big\}$$

and

$$Q := ((1-t)\alpha, d_2) + t(\alpha =_1 L_1) \subset \mathrm{conv}\big\{(\alpha, d_2 + c_1), (\alpha, d_2 + d_1)\big\}.$$

Since they are contained in adjacent segments and

$$P \cup Q \subset \left(\alpha =_1 tL_1 + (1-t)L_2 \right)$$

it follows that

$$Y_{tL_1+(1-t)L_2}(\alpha) \geq \lambda(P \cup Q) = \lambda_1(P) + \lambda_1(Q).$$

Therefore

$$(3) \qquad Y_{tL_1+(1-t)L_2}(\alpha) \geq tY_{L_1}(\alpha) + (1-t)Y_{L_2}(\alpha)$$

and, in a similar way,

$$(4) \qquad X_{tL_1+(1-t)L_2}(\beta) \geq tX_{L_1}(\beta) + (1-t)X_{L_2}(\beta).$$

According to equation (1)

$$(5) \qquad f_{tL_1+(1-t)L_2}(x,y) \geq tf_{L_1}(x,y) + (1-t)f_{L_2}(x,y) \quad (0 \leq t \leq 1)$$

for any $(x,y) \in \mathbb{R}^2$. $\square$

Remark 2. Integrating both sides of inequality (3) (or (4)) it follows that

$$\lambda_2(tL_1 + (1-t)L_2) \geq t\lambda_2(L_1) + (1-t)\lambda_2(L_2).$$

If we omit the condition of the common axis parallel bounding box then the statement is false as the following example shows:

$$L_1 := [-3,3] \times [-3,3], \quad L_2 = [-1,1] \times [-1,1],$$

$$\frac{1}{2}L_1 + \frac{1}{2}L_2 = [-2,2] \times [-2,2]$$

and

$$\lambda_2\left(\frac{1}{2}L_1 + \frac{1}{2}L_2\right) = 16, \quad \frac{1}{2}\lambda_2(L_1) + \frac{1}{2}\lambda_2(L_2) = \frac{1}{2}36 + \frac{1}{2}4 = 20.$$

The condition of the common axis parallel bounding box is used as *none of the sets $\alpha =_1 L_i$ $(i = 1, 2)$ are empty* because L_1 and $L_2 \in \mathcal{L}_B$.

Since the generalized conic function is defined by the integral over the focal set Φ obviously preserves the ordering with respect to the inclusion and the pointwise upper semi-continuity

$$\limsup_{n \to \infty} f_{L_n}(x,y) \leq f_L(x,y)$$

follows immediately for any $(x,y) \in \mathbb{R}^2$, where the sequence L_n tends to $L \in \mathcal{F}(\mathbb{R}^2)$ with respect to the Hausdorff metric. In the forthcoming section we are going to investigate the continuity properties of the mapping Φ restricted to the class of compact connected hv-convex planar sets.

3. The case of compact connected hv-convex sets

In what follows $\mathcal{F}_{hv}(\mathbb{R}^2)$ denotes the metric space of non-empty compact hv-convex sets in the plane equipped with the Hausdorff metric. Let the rectangle B with parallel sides to the coordinate axes be given as the Cartesian product $[a, b] \times [c, d]$. Recall that $\mathcal{L}_B$ is just the collection of non-empty compact planar sets L for which

(6) $$\mathrm{pr}_1(L) \times \mathrm{pr}_2(L) = B.$$

The axis parallel bounding box of L is the intersection of all axis parallel boxes that contain L. It can be easily seen that any non-empty compact connected hv-convex set L is between the upper and the lower bound functions

(7) $$h_L(x) := \sup\{y \mid (x, y) \in L\} \quad \text{and} \quad g_L(x) := \inf\{y \mid (x, y) \in L\}$$

in the sense that

$$L = \{(x, y) \mid a \leq x \leq b \text{ and } g_L(x) \leq y \leq h_L(x)\}.$$

On the other hand

$$Y_L(x) = h_L(x) - g_L(x)$$

and for any sequence $x_n \to x$

$$\limsup_{n\to\infty} Y_L(x_n) \leq \limsup_{n\to\infty} h_L(x_n) - \liminf_{n\to\infty} g_L(x_n) \leq h_L(x) - g_L(x) = Y_L(x)$$

which means that the coordinate X-ray function Y_L is upper semi-continuous on the interval $[a, b]$. A similar statement can be formulated in terms of X_L. For the general theory of parallel X-rays see [2].

Lemma 1. *The set*

(8) $$\mathcal{L}_B^{hv} = \mathcal{L}_B \cap \mathcal{F}_{hv}(\mathbb{R}^2)$$

consists of all non-empty compact connected hv-convex sets with axis parallel bounding box B. The set

(9) $$\mathcal{M}_B^{hv} = \mathcal{M}_B \cap \mathcal{F}_{hv}(\mathbb{R}^2)$$

consists of all non-empty compact connected hv-convex sets with axis parallel bounding box contained in B.

Proof. For the non-trivial direction let L be a non-empty compact hv-convex set satisfying condition (6) and suppose that L is not connected. This means that there is a non-constant, continuous function $f \colon L \to \{0, 1\}$. Since L is hv-convex f must be constant along each horizontal or vertical segments running in L. Therefore we can construct a continuous function $\tilde{f} \colon \mathrm{pr}_1(L) \to \{0, 1\}$ such that $\tilde{f}$ makes the diagram

$$L \xmapsto{f} \{0, 1\}$$
$$\mathrm{pr}_1 \downarrow \quad \nearrow$$
$$\mathrm{pr}_1(L)$$

commutative. This contradicts to the connectedness of $\mathrm{pr}_1(L) = [a, b]$. $\square$

Lemma 2. *If L is a non-empty compact connected hv-convex set then the outer parallel body L^ε is connected and hv-convex for any $\varepsilon > 0$.*

Proof. To prove that L^ε is hv-convex suppose, in contrary, that it is not true. Without loss of generality we can suppose that the points $Q_1(0,0)$ and $Q_2(0,m)$ belong to the parallel body but the segment joining Q_1 and Q_2 contains a point $Q(0,y) \notin L^\varepsilon$ and $0 < y < m$. Therefore we can choose points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ from the intersections of L with the closed disks centered at Q_1 and Q_2 with radius ε but L must be disjoint from the closed disk D centered at Q with radius ε. We have that $y_1 < y$ because P_1 must be under the perpendicular bisector of the segment Q_1Q (otherwise P_1 would be closer to Q than to Q_1 which is a contradiction). In a similar way $y < y_2$. Therefore $y_1 < y < y_2$. On the other hand $x_1 \neq x_2$ because the convexity into the vertical direction says that if $x_1 = x_2$ then the segment joining P_1 and P_2 belong to L. The parallel body of the segment P_1P_2 is a convex set and the points Q_1 and Q_2 belong to $(P_1P_2)^\varepsilon \subset L^\varepsilon$. So does the point Q which is a contradiction. Using reflection about the vertical coordinate line if necessary suppose that $x_1 < x_2$. We claim that $h_L(x_1) < y$ (for the definition of the upper bound function see (7)). In opposite case we have

$$y_1 < y \leq h_L(x_1)$$

and the vertical line segment joining $P_1(x_1,y_1)$ and $(x_1, h_L(x_1))$ intersects D which is a contradiction. Obviously $h_L(x_2) \geq y_2 > y$. Let us define the number

$$0 \leq s := \sup\{t \geq 0 \mid h_L(x) < y \ \text{ for all } \ x \in [x_1, x_1 + t]\} \leq x_2 - x_1.$$

Then we can choose a sequence $s_n^+ \to s$ such that $h_L(s_n^+) \geq y$ and, by the upper semi-continuity of the upper bound function h_L, it follows that

$$y \leq \limsup_{n \to \infty} h_L(s_n^+) \leq h_L(s).$$

We can also choose a sequence $s_n^- \to s$ such that $g_L(s_n^-) \leq h_L(s_n^-) < y$ and, by the lower semi-continuity of the lower bound function g_L, it follows that

$$y \geq \liminf_{n \to \infty} g_L(s_n^-) \geq g_L(s).$$

The convexity into the vertical direction gives a contradiction because the (vertical) segment joining $(s, h_L(s))$ with $(s, g_L(s))$ intersects D. Since L is connected and hv-convex Lemma 1 implies that

$$\mathrm{pr}_1(L) \times \mathrm{pr}_2(L) = B,$$

where $B = [a,b] \times [c,d]$ is the axis parallel bounding box of L. Therefore

$$\mathrm{pr}_1(L^\varepsilon) \times \mathrm{pr}_2(L^\varepsilon) = [a - \varepsilon, b + \varepsilon] \times [c - \varepsilon, d + \varepsilon]$$

and the connectedness follows by using Lemma 1 again. $\square$

Lemma 3. *The limit $L \in \mathcal{F}(\mathbb{R}^2)$ of the sequence L_n of non-empty compact connected hv-convex sets is a connected hv-convex set.*

Proof. We are going to discuss the convexity only into the horizontal direction (the discussion of the vertical direction is similar). Suppose, in contrary, that there exist points $P_1(x_1, y)$, $P_2(x_2, y) \in L$ such that the segment joining P_1 with P_2 contains a point $Q(x, y) \notin L$. Since the Euclidean distance of Q from L is strictly positive we can choose a positive real number ε in such a way that $L^{2\varepsilon}$ is disjoint from the closed disk D centered at Q with
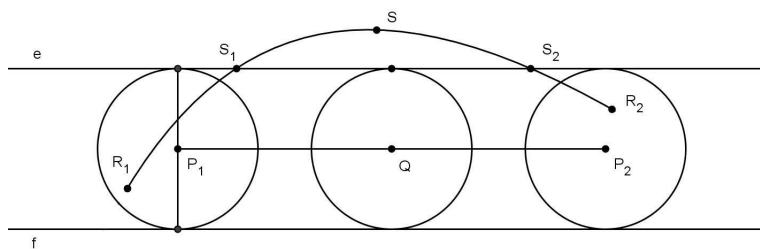
FIGURE 1.

radius ε. The Hausdorff convergence $L_n \to L$ implies the existence of $n \in \mathbb{N}$ such that $L_n \subset L^\varepsilon$ and $L \subset L_n^\varepsilon$. Therefore there exist points $R_1, R_2 \in L_n$ in the closed disks D_1 and D_1 centered at P_1 and P_2 with radius ε, respectively. Since $L_n^\varepsilon \subset (L^\varepsilon)^\varepsilon \subset L^{2\varepsilon}$ we have that L_n^ε is disjoint from D.

Consider the (common) tangent lines e and f of the disks D_1 and D_2. They are tangent to D at the same time. Let H_e be the closed half plane bounded by e containing the line f and, in a similar way, H_f denotes the closed half plane bounded by f containing the line e. In view of Lemma 2, the parallel body L_n^ε is connected together with its interior containing L_n. Therefore int L_n^ε is arcwise connected as a connected open subset of the Euclidean plane. There exists a continuous arc $s \subset$ int L_n^ε joining R_1 and R_2 but L_n^ε together with s is disjoint from D (the closed disk centered at Q with radius ε). Taking a point $S \in s$ such that $S \notin H_e \cap H_f$ (suppose, for example, that $S \notin H_e$) we can divide s into the union of continuous arcs s_1 and s_2 intersecting the line e at the points S_1 and S_2, respectively. The horizontal segment between S_1 and S_2 intersects D. Since L_n^ε is hv-convex it follows that D and L_n^ε has a common point which is a contradiction. The connectedness follows easily from Lemma 1 because

$$B = \mathrm{pr}_1(L) \times \mathrm{pr}_2(L)$$

is just the limit of the sequence of rectangles $B_n = \mathrm{pr}_1(L_n) \times \mathrm{pr}_2(L_n)$ with parallel sides to the coordinate axes. $\square$

Corollary 1. *Both $\mathcal{L}_B^{hv}$ and $\mathcal{M}_B^{hv}$ are compact with respect to the Hausdorff metric.*

Lemma 4. *Let $\varepsilon > 0$ be an arbitrary positive real number and consider a finite simple polygonal chain $\mathcal{P}$ in the plane. Then*

(10) $$\lambda_2\left(\mathcal{P}^\varepsilon\right) \leq 2l\varepsilon + \varepsilon^2\pi,$$

where l is the length of $\mathcal{P}$. Especially, if $\mathcal{P}$ is closed then

(11) $$\lambda_2\left(\mathcal{P}^\varepsilon\right) \leq 2l\varepsilon.$$

Proof. The proof is an induction for $m \in \mathbb{N}$, where m denotes the number of segments constituting the chain. If the polygonal chain consists of only one segment then estimation (10) is obviously true (especially we have equality). Suppose that estimation (10) is true for all polygonal chains consisting
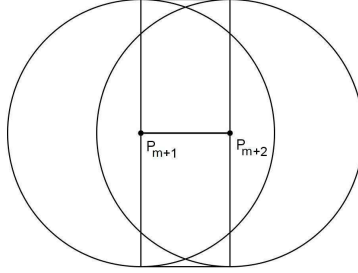
FIGURE 2.

of m segments and consider a chain $\mathcal{P}_{m+1}$ with vertices $P_1, \ldots, P_{m+1}, P_{m+2}$. Taking

$$\mathcal{P}_{m+1} = \mathcal{P}_m \cup P_{m+1}P_{m+2}$$

we have $\mathcal{P}_{m+1}^\varepsilon$ as the union of $\mathcal{P}_m^\varepsilon$ and the parallel body of the segment joining P_{m+1} with P_{m+2} (see Figure 2). It is clear that the disk around P_{m+1} with radius ε has a twofold covering. Therefore

$$\lambda_2 \left(\mathcal{P}_{m+1}^\varepsilon \right) \leq \lambda_2 \left(\mathcal{P}_m^\varepsilon \right) + \lambda_2((P_{m+1}P_{m+2})^\varepsilon) - \varepsilon^2 \pi,$$

where the area of the parallel body of the segment $P_{m+1}P_{m+2}$ is

$$\lambda_2((P_{m+1}P_{m+2})^\varepsilon) = 2\varepsilon \cdot d_2(P_{m+1}, P_{m+2}) + \varepsilon^2 \pi.$$

Therefore

$$\lambda_2 \left(\mathcal{P}_{m+1}^\varepsilon \right) \leq \lambda_2 \left(\mathcal{P}_m^\varepsilon \right) + 2\varepsilon \cdot d_2(P_{m+1}, P_{m+2})$$

and the inductive hypothesis gives the estimation

$$\lambda_2 \left(\mathcal{P}_{m+1}^\varepsilon \right) \leq 2l\varepsilon + \varepsilon^2 \pi$$

as was to be stated. Estimation (11) follows from (10) by subtracting the areas coming from the twofold coverings of the disks around P_{m+1} and $P_{m+2} = P_1$. $\square$

Theorem 2. *Suppose that $L \subset \mathbb{R}^2$ is a non-empty compact connected hv-convex set with axis parallel bounding box $B = [a,b] \times [c,d]$. Then*

$$\lambda_2 \left(L^\varepsilon \right) - \lambda_2(L) \leq 2k\varepsilon,$$

where k denotes the perimeter of B.

Proof. Let $r_n \to 0^+$ be an arbitrary sequence and consider partitions $x_0 = a < x_1 < \ldots < x_m = b$, $y_0 = c < y_1 < \ldots < y_m = d$ such that $\operatorname{diam} B_{ij}^n < r_n$, where

$$B_{ij}^n := [x_{i-1}, x_i] \times [y_{j-1}, y_j] \quad (i, j = 1, 2, \ldots, m).$$

For the minimal covering

$$L_n := \bigcup_{B_{ij}^n \cap L \neq \emptyset} B_{ij}^n$$

of L as the union of subrectangles having a non-empty intersection with L we have that $H(L_n, L) \leq r_n$, i.e. the sequence L_n tends to L with respect

to the Hausdorff metric. On the other hand $\lambda_2(L_n) \to \lambda_2(L)$ because of $L \subset L_n \subset L^{r_n}$. Therefore for any positive real number $\varepsilon > 0$

$$\lambda_2\left(L^\varepsilon\right) - \lambda_2(L) = \lambda_2\left(L^\varepsilon\right) - \lim_{n\to\infty} \lambda_2(L_n) = \lim_{n\to\infty}\left(\lambda_2\left(L^\varepsilon\right) - \lambda_2(L_n)\right)$$

and thus

$$\lambda_2\left(L^\varepsilon\right) - \lambda_2(L) \leq \lim_{n\to\infty}\left(\lambda_2\left(L_n^\varepsilon\right) - \lambda_2(L_n)\right)$$

using that $L \subset L_n$. We claim that L_n is hv-convex. The discussion will be restricted to the convexity into the horizontal direction (the discussion of the vertical direction is similar). Suppose, in contrary, that there exist points $P_1(x_1, y)$, $P_2(x_2, y) \in L_n$ such that the segment joining P_1 with P_2 contains a point $Q(x, y) \notin L_n$. This means that P_1 and P_2 are in disjoint subrectangles $B_{i_1 j_1}^n$ and $B_{i_2 j_2}^n$, respectively. According to the definition of L_n these subrectangles contain points R_1 and $R_2 \in L$ but the subrectangle $B_{i_3 j_3}^n$ containing Q must be disjoint from the set L. Since L is compact we can choose a positive real number $0 < \delta$ such that $L^\delta \cap B_{i_3 j_3}^n = \emptyset$. Lemma 2 implies that L^δ is connected together with its interior. Therefore int L^δ is arcwise connected as a connected open subset of the Euclidean plane and the points R_1, R_2 can be joined by a continuous arc in the interior of L^δ. The argumentation can be finished in the same way as in the proof of Lemma 3 (see Figure 1 with rectangular domains instead of disks). Since

$$\mathrm{pr}_1(L_n) \times \mathrm{pr}_2(L_n) = B$$

it follows by Lemma 2 that it is a connected set: L_n is actually a special kind of parallel body constructed from L by adding rectangular domains instead of disks. Therefore the boundary of L_n is a finite simple closed polygonal chain $\mathcal{P}_n$ in the plane. The lenght of $\mathcal{P}_n$ is the perimeter of the box B. Since $L_n^\varepsilon \setminus L_n \subset \mathcal{P}_n^\varepsilon$ we have by Lemma 4 that

$$\lambda_2\left(L_n^\varepsilon\right) - \lambda_2(L_n) = \lambda_2(L_n^\varepsilon \setminus L_n) \leq \lambda_2(\mathcal{P}_n^\varepsilon) \leq 2k\varepsilon$$

and, consequently,

$$\lambda_2\left(L^\varepsilon\right) - \lambda_2(L) \leq 2k\varepsilon$$

as was to be stated. $\square$

Theorem 3. *The mapping $\Phi\colon L \in \mathcal{M}_B^{hv} \to f_L$ is continuous between $\mathcal{M}_B^{hv}$ equipped with the Hausdorff metric and the function space equipped with the norm*

$$\|f_L\|_{\infty, B} := \sup_{(x,y)\in B} |f_L(x,y)|.$$

Proof. Suppose that $L_n \to L$ with respect to the Hausdorff metric, where L_n and L are non-empty compact connected hv-convex sets with axis parallel bounding box contained in B. From the definition of the generalized conic function we have that

$$f_{L_n}(x,y) \leq f_{L^{r_n}}(x,y) = f_L(x,y) + \int\limits_{L^{r_n}\setminus L} d_1\big((x,y),(\alpha,\beta)\big)\,d\alpha d\beta$$

where $r_n := H(L_n, L)$. The integrand is obviously bounded from above by the following way: since $L^{r_n} \subset B^{r_n}$ and $L \subset B$ we have that

$$(12) \qquad\qquad d_1\big((x,y),(\alpha,\beta)\big) \leq \frac{k}{2} + 2r_n$$

for any $(x, y) \in B$, where k is the perimeter of B: $k/2$ is just the diameter of B with respect to the taxicab norm but the parallel body L^{r_n} allows us two additional steps of lenght r_n into the vertical or the horizontal directions. Therefore

$$f_{L_n}(x, y) \leq f_L(x, y) + \left(\frac{k}{2} + 2r_n \right) \left(\lambda_2(L^{r_n}) - \lambda_2(L) \right) \leq$$

$$f_L(x, y) + \left(\frac{k}{2} + 2r_n \right) 2kr_n$$

because of Theorem 2. Conversely

$$f_L(x, y) \leq f_{L_n^{r_n}}(x, y) = f_{L_n}(x, y) + \int\limits_{L_n^{r_n} \setminus L_n} d_1\big((x, y), (\alpha, \beta)\big) \, d\alpha d\beta$$

where $r_n := H(L_n, L)$. The integrand is obviously bounded from above by the same way as in (12):

(13) $$d_1\big((x, y), (\alpha, \beta)\big) \leq \frac{k}{2} + 2r_n.$$

Therefore

$$f_L(x, y) \leq f_{L_n}(x, y) + \left(\frac{k}{2} + 2r_n \right) \left(\lambda_2(L_n^{r_n}) - \lambda_2(L_n) \right) \leq$$

$$f_{L_n}(x, y) + \left(\frac{k}{2} + 2r_n \right) 2kr_n$$

because of Theorem 2. These inequalities imply that for any $(x, y) \in B$

(14) $$|f_L(x, y) - f_{L_n}(x, y)| \leq \left(\frac{k}{2} + 2r_n \right) 2kr_n,$$

where the right hand side is a quadratic polynomial expression of the Hausdorff distance which is independent of the choice of $(x, y) \in B$. In other words the convergence $f_{L_n} \to f_L$ is uniform over the reference set B and the statement of Theorem 3 follows immediately. $\square$

Corollary 2. *The mapping* $\Phi \colon L \in \mathcal{M}_B^{hv} \to f_L$ *is continuous between* $\mathcal{M}_B^{hv}$ *equipped with the Hausdorff metric and the function space equipped with the norm*

$$\|f_L\|_{1,B} := \int\limits_B |f_L(x, y)| \, dxdy.$$

Corollary 3. *If* L_n *is a sequence of non-empty compact connected hv-convex sets tending to the limit* $L \in \mathcal{F}(\mathbb{R}^2)$ *then for any* $(x, y) \in \mathbb{R}^2$

(15) $$\lim_{n \to \infty} f_{L_n}(x, y) = f_L(x, y).$$

Proof. Lemma 3 says that L is a non-empty compact connected hv-convex set and we can use Theorem 3 under the choice of a sufficiently large reference set B. $\square$

Corollary 4. *Let L_n be a sequence of non-empty compact connected hv-convex sets contained in B. If $f_{L_n} \to f_K$ with respect to the L_1-norm or the supremum norm then any convergent subsequence of L_n tends to a set K' having the same coordinate X-rays as K almost everywhere. If K is uniquely detemined by the coordinate X-rays then K' is equal to K modulo a set of measure zero.*

Proof. Consider the case of the supremum norm. If K' is the limit of a subsequence L_{n_k} then

$$\|f_{K'} - f_K\|_{\infty,B} \le \|f_{K'} - f_{L_{n_k}}\|_{\infty,B} + \|f_{L_{n_k}} - f_K\|_{\infty,B},$$

where the first term tends to zero in view of Theorem 3. So does the second term because of the condition $f_{L_n} \to f_K$ with respect to the supremum norm. Taking the limit as $k \to \infty$

$$\|f_{K'} - f_K\|_{\infty,B} = 0$$

which means that $f_K(x,y) = f_{K'}(x,y)$ for any $(x,y) \in B$ because of the continuity of the generalized conic functions. Therefore K and K' have the same coordinate X-rays almost everywhere. $\square$

As a sequence-free version we can formulate the following theorem of approximation.

Theorem 4. *Suppose that $K \in \mathcal{M}_B^{hv}$. For any $\varepsilon > 0$ there exists $\delta_\infty > 0$ or $\delta_1 > 0$ such that for any $L \in \mathcal{M}_B^{hv}$*

$$\|f_L - f_K\|_{\infty,B} < \delta_\infty \quad or \quad \|f_L - f_K\|_{1,B} < \delta_1$$

implies that $H(L, K') < \varepsilon$ for some $K' \in \mathcal{M}_B^{hv}$, where K' has the same coordinate X-rays as K almost everywhere.

The theorem says that if $f_L \approx f_K$ with respect to the supremum norm or the L_1-norm then L approximates at least one of the sets in $\Phi^{-1}(f_K)$.

4. THE CASE OF COMPACT CONVEX PLANAR BODIES: GARDNER'S PROBLEM

In what follows $\mathcal{F}_c(\mathbb{R}^2)$ denotes the metric space of non-empty compact convex sets in the plane equipped with the Hausdorff metric. Let the rectangle B with parallel sides to the coordinate axes be given as the Cartesian product $[a,b] \times [c,d]$. Recall that $\mathcal{L}_B$ is just the collection of non-empty compact planar sets L for which

$$(16) \qquad \qquad \mathrm{pr}_1(L) \times \mathrm{pr}_2(L) = B.$$

Rådström's embedding theorem [4] says that the collection of non-empty compact convex sets (equipped with the Hausdorff metric) can be isometrically embedded into a normed vector space V as a cone. It is a continuous embedding because of the distance preserving property. Therefore

$$(17) \qquad \qquad \mathcal{L}_B^c = \mathcal{L}_B \cap \mathcal{F}_c(\mathbb{R}^2)$$

can be interpreted as a compact convex subset in V (Proposition 1 and Proposition 2) and $\Phi \colon \mathcal{L}_B^c \subset V \to W$ is a bounded order-preserving[1] concave

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[1]Lattice properties [1] were added to the theory by A. G. Pinsker in 1966.

(Theorem 1) and continuous mapping (Theorem 3) into the normed vector space W of continuous functions on B equipped with the supremum or the L_1-norm (over B).

Definition 3. *Let X and Y be Hausdorff topological spaces and consider the mapping $F\colon X \to 2^Y$. It is upper semi-continuous at $x_0 \in X$ if for any open neighbourhood $\mathcal{V}$ of $F(x_0)$ there exists an open neighbourhood $\mathcal{U}$ of x_0 such that $F(x) \subset \mathcal{V}$ for any $x \in \mathcal{U}$. The mapping F is lower semi-continuous at x_0 if for any open set $\mathcal{V}$ which intersects $F(x_0)$ there exists an open neighbourhood $\mathcal{U}$ of x_0 such that $F(x) \cap \mathcal{V} \neq \emptyset$ for any $x \in \mathcal{U}$. A set-valued mapping which is both upper- and lower semi-continuous is called continuous.*

Since $\Phi\colon L \in \mathcal{L}_B^c \to f_L$ is a continuous mapping defined on a compact metric space its inverse (as a set-valued mapping) is upper semi-continuous at any element f_K of the range: in view of Theorem 4 for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f_L - f_K\| < \delta$ implies that $L \in \left(\Phi^{-1}(f_K)\right)^\varepsilon \cap \mathcal{L}_B^c$, where

$$\Phi^{-1}(f_K) := \{K' \in \mathcal{L}_B^c \mid f_K(x,y) = f_{K'}(x,y) \text{ for all } (x,y) \in B\}.$$

Under the notation of Definition 3

$$\mathcal{V} := \left(\Phi^{-1}(f_K)\right)^\varepsilon \cap \mathcal{L}_B^c$$

and $\mathcal{U}$ is just the ball with radius δ around $x_0 := f_K$. The upper semi-continuity establishes an approximating process. In view of Michael's selection theorem the lower semi-continuity is related to the existence of continuous selections. In what follows we prove that the lower semi-continuity of Φ^{-1} at f_K is equivalent to the determination of K by the coordinate X-rays in the class $\mathcal{F}_c^\circ(\mathbb{R}^2)$ of non-empty compact convex bodies. A compact convex set is called a body if it has a non-empty interior. The determination of the compact convex body K by the coordinate X-rays means that for any compact convex body K' the relation $f_K = f_{K'}$ implies that $K = K'$. To *characterize those convex bodies that can be determined by two X-rays* is an open problem due to R. J. Gardner [2], Problem 1.1, p. 51. According to the affine nature of the problem we can suppose that the X-ray directions correspond to the coordinate axes without loss of generality.

Theorem 5. *The body $K \in \mathcal{F}_c(\mathbb{R}^2)$ is determined by the coordinate X-rays if and only if the mapping Φ^{-1} is lower semi-continuous at f_K.*

Proof. If K is uniquely determined by the coordinate X-rays then Corollary 4 implies immediately the continuity (especially the lower semi-continuity) of the inverse mapping at f_K. Conversely, suppose that the inverse mapping is lower semi-continuous. Together with the upper semi-continuity we can conclude that Φ^{-1} is continuous at f_K. Since the set of bodies that can be determined by their coordinate X-rays is dense in $\mathcal{F}_c^\circ(\mathbb{R}^2)$ [2] [Theorem 1.2.17] we can choose a sequence $f_{L_n} \to f_K$ such that $\Phi^{-1}(f_{L_n})$ is a singleton. So is $\Phi^{-1}(f_K)$. $\square$

Remark 3. The theorem is a way to rephrase the problem of determination in terms of the function Φ. Although it is probably hard to check the continuity of Φ^{-1} at f_K continuity properties may give answers in terms of

algorithms by finding the possible alternatives: let f_K be the input data and consider the optimization problem

$$\text{minimize } \|f_L - f_K\| \text{ subject to } L \in \mathcal{H},$$

where the set $\mathcal{H}$ is the subcollection of non-empty compact connected hv-convex sets which are constituted by the subrectangles belonging to a partition of the reference set [9]. It is typically a rectangle B with parallel sides to the coordinate axes such that $K \subset B$.

Acknowledgements

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