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Parametric quantum electrodynamics

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von

M. Sc. Marcel Golz

Präsidentin der Humboldt-Universität zu Berlin:

Prof. Dr.-Ing. Dr. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät:

Prof. Dr. Elmar Kulke

Gutachter:

1. Prof. Dr. Dirk Kreimer
2. Prof. Dr. Francis Brown
3. Prof. Dr. Jan Plefka

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Berlin,
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Marcel Golz

献给我的妻子王潇睿

Abstract

This thesis is concerned with the study of Schwinger parametric Feynman integrals in quantum electrodynamics. Using a variety of tools from combinatorics and graph theory, significant simplification of the integrand is achieved.

After a largely self-contained introduction to Feynman graphs and integrals, the derivation of the Schwinger parametric representation from the standard momentum space integrals is reviewed in full detail for both scalar theories and quantum electrodynamics. The derivatives needed to express Feynman integrals in quantum electrodynamics in their parametric version are found to contain new types of graph polynomials based on cycle and bond subgraphs.

Then the tensor structure of quantum electrodynamics, products of Dirac matrices and their traces, is reduced to integer factors with a diagrammatic interpretation of their contraction. Specifically, chord diagrams with a particular colouring are used. This results in a parametric integrand that contains sums of products of cycle and bond polynomials over certain subsets of such chord diagrams.

Further study of the polynomials occurring in the integrand reveals connections to other well-known graph polynomials, the Dodgson and spanning forest polynomials. This is used to prove an identity that expresses some of the very large sums over chord diagrams in a very concise form. In particular, this leads to cancellations that massively simplify the integrand.

Zusammenfassung

In dieser Dissertation geht es um Schwinger-parametrische Feynmanintegrale in der Quantenelektrodynamik. Mittels einer Vielzahl von Methoden aus der Kombinatorik und Graphentheorie wird eine signifikante Vereinfachung des Integranden erreicht.

Nach einer größtenteils in sich geschlossenen Einführung zu Feynmangraphen und -integralen wird die Herleitung der Schwinger-parametrischen Darstellung aus den klassischen Impulsraumintegralen ausführlich erläutert, sowohl für skalare Theorien als auch Quantenelektrodynamik. Es stellt sich heraus, dass die Ableitungen, die benötigt werden um Integrale aus der Quantenelektrodynamik in ihrer parametrischen Version zu formulieren, neue Graphpolynome enthalten, die auf Zykeln und minimalen Schnitten (engl. “bonds”) basieren.

Danach wird die Tensorstruktur der Quantenelektrodynamik, bestehend aus Dirac-Matrizen und ihren Spuren, durch eine diagrammatische Interpretation ihrer Kontraktion zu ganzzahligen Faktoren reduziert. Dabei werden insbesondere gefärbte Sehnendiagramme benutzt. Dies liefert einen parametrischen Integranden, der über bestimmte Teilmengen solcher Diagramme summierte Produkte von Zykel- und Bondpolynomen enthält.

Weitere Untersuchungen der im Integranden auftauchenden Polynome decken Verbindungen zu Dodgson- und Spannwaldpolynomen auf. Dies wird benutzt um eine Identität zu beweisen, mit der sehr große Summen von Sehnendiagrammen in einer kurzen Form ausgedrückt werden können. Insbesondere führt dies zu Aufhebungen, die den Integranden massiv vereinfachen.

Contents

1 Introduction

Dimidium facti qui coepit habet; sapere aude; incipe!
He who has begun has half done. Dare to be wise; begin!

Quintus Horatius Flaccus, *Epistularum liber primus*, 20 BCE

1.1 Motivation

The purpose of theoretical physics is to explain observed physical phenomena and then predict further observations for experimentalists to look for. Nowadays, the latter part often amounts to the computation of *Feynman integrals* in perturbative quantum field theory. Enormous amounts of ever more complicated integrals need to be dealt with in order to keep up with the extremely high measurement accuracies achieved at modern particle colliders like the Large Hadron Collider (LHC).

Often these integrals cannot be computed with existing mathematical tools and require a collaboration of mathematicians and physicists in order to introduce more advanced concepts that are not commonly in a physicists toolbox, and push the boundaries towards new mathematics. The Schwinger parametric representation of Feynman integrals is particularly interesting in this regard, since it unveils the deep connections of Feynman integrals to algebraic geometry.

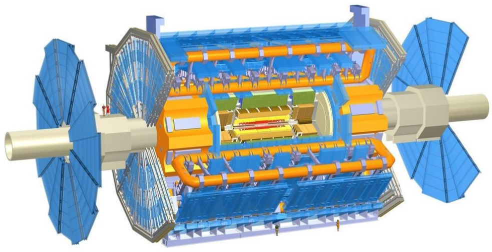


Figure 1.1: Drawing of the ATLAS detector at the LHC. [1]

[1] By Argonne National Laboratory licensed under CC-BY-SA-2.0

It has been used to great effect in so-called scalar quantum field theories. However, these are only a small subset of the physically relevant theories like quantum chromodynamics or Yang-Mills theory. In principle Schwinger parameters can be used for these cases as well, but in praxis the resulting integrals are so large and complicated that alternative methods (like the traditional momentum space representation) are much more efficient. Even for quantum electrodynamics the Schwinger parametric representation is already prohibitively complex such that it has barely been used in decades.

However, in light of the impressive progress the scalar Schwinger parametric representation has allowed for over the last decade, it is now the time to return to the gauge theory cases, starting with the simplest, and ask the central question of this thesis:

> Can the Schwinger parametric Feynman integral in QED be simplified, such that it becomes feasible to study and compute it?

By the end of this thesis we will be able to emphatically answer this question with yes. Building on this we can then ask the logical follow up:

> What new insights can be gained with this simplified parametric Feynman integral?

The obvious answer is that a simpler integral is also generally easier to compute and may yield previously unreachable results. Beyond that the main appeal of parametric Feynman integrals is that they offer an alternative point of view from which it might be possible to tackle certain longstanding questions about the structure of QED. The latter point in particular includes the puzzling cancellation of zeta values in the beta function of QED that originally motivated this work [26, 76, 114]. Unsurprisingly these unsolved problems are in general very hard and their solutions lie beyond this thesis, but with the simplified integral we will be able to lay out a plan for future work that may make them accessible in the foreseeable future.

1.2 Background

1.2.1 A brief history of quantum field theory

Quantum field theory (QFT) is somewhat of a curiosity in the sense that – unlike other physical theories – it does not seem to have a single canonical definition. One rather vague statement that most texts about QFT seem to agree on with slight variations in phrasing is:

Quantum field theory is the theoretical and mathematical framework for particle physics.

Other similarly common definitions characterise it as the generalisation of single particle quantum mechanics to fields, i.e. infinite degrees of freedom, or the combination of special relativity and quantum mechanics. However, while they are intuitive, historically motivated and very popular in introductory courses and texts on QFT, these marginally less vague definitions are already conceptually and philosophically problematic [99]. Consequently it is not surprising that there is a wide range of literature on QFT, from accessible introductions and reviews [139, 142] to older standard textbooks [12] and comprehensive tomes [137] to more mathematically rigorous treatments [126] and philosophical studies [34].

Historically quantum field theory was developed in the late 1920s in order to include the photon – the quintessential relativistic particle, moving at the speed of light – into the newly developed quantum mechanics. After early attempts to quantise the electromagnetic field [22] it is usually Dirac's article *"The quantum theory of the emission and absorption of radiation"* [53] that is viewed as the birth of quantum field theory. Among other notable things it contains the first mention of "quantum electrodynamics".

Early QFT produced some notable results, like Dirac's prediction of the existence of positrons, but physicists noted very quickly that it is plagued by infinities in the form of divergent integrals. It took until 1948 for Feynman [62–64], Schwinger [117], and Tomonaga [127] to develop methods to systematically control these infinities and Dyson [55] showed that all three are in fact equivalent.

From there, research continued apace with theoretical predictions of a variety of new particles, followed by experimental confirmation. In 1956 the neutrinos hypothesised by Pauli 23 years earlier are detected [49]. After the invention of Yang-Mills theory [140] prompted research into non-abelian gauge theories the ideas of quantum electrodynamics were extended to develop electro-weak theory, quantum chromodynamics and spontaneous symmetry breaking [58, 67–71, 79, 83, 111, 115, 136]. The predicted quarks [59, 60], gluons [43], vector bosons [44, 45], and most recently the Higgs boson [41, 42] were then sooner or later discovered in experiments, confirming the validity of the standard model of elementary particle physics. Of course, the standard model does not give a complete description of the universe. Most notably, all attempts to include gravity have failed and while there are promising extensions like string theory, experimentally verifiable predictions remain out of reach. Hence, we take 't Hooft's attitude toward QFT [124],

"[We] *know where its limits are, and these limits are far away."*

and continue studying. There is still much to learn, even about the simpler quantum field theories like QED, and one should hope that a deeper understanding of QFT will eventually bring about a breakthrough.

1.2.2 Perturbative quantum field theory

Since QFT poses such complex problems, concrete computations usually boil down to approximations via perturbation theory. The idea of perturbation theory is to model a complicated problem that cannot be solved exactly as a small modification ("perturbation") of an easier problem with a known solution. In quantum mechanics one typically does this by considering a Hamiltonian $H = H_0 + \epsilon V$, where H_0 is the Hamiltonian of a well understood system (e.g. the quantum harmonic oscillator or the hydrogen atom), $\epsilon > 0$ is a sufficiently small parameter and V describes the perturbation. The solution of the full system, i.e. the energy or its eigenstates, can then be expressed as a series $E^{(0)} + \epsilon E^{(1)} + \cdots$.

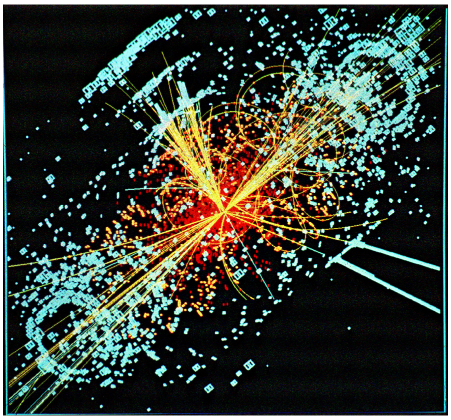


Figure 1.2: CMS: Simulated Higgs to two jets and two electrons [2]

In quantum field theory the idea is the same. The easy part is the description of a free particle moving through space. The full system we want to study then involves interactions between different particles. Following the usual formalisms found in any quantum field theory text book one finds the (probability, scattering, transition or Feynman) *amplitude*

$$\mathcal{A} = \mathcal{A}^{(0)} + \lambda \mathcal{A}^{(1)} + \lambda^2 \mathcal{A}^{(2)} + \cdots$$

in terms of such a series. Its modulus squared corresponds to a probability and can be used to compute observables like cross sections or decay rates that experimentalists can measure in particle colliders. There are a number of conceptual problems with such series (e.g. the fact

[2] By Lucas Taylor (CERN) licensed under CC-BY-SA-4.0

that they most likely diverge, according to a famous argument by Dyson [56]). In this thesis we will not be bothered by any of this, since, in Dyson's own words [56]:

"The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, [...]"

In fact, the computation of the anomalous magnetic dipole moment of the electron, sometimes called just "$g-2$" since it is the deviation of g from the value of 2 predicted by the Dirac equation, is the most accurate prediction of a physical quantity in history and agrees with experiment to 10 significant figures [6, 7, 100, 116], [81].

1.2.3 Feynman integrals

The amplitude contributions $\mathcal{A}^{(i)}$ are given in terms of sums of the aforementioned Feynman integrals [65]. An example for such an integral in a scalar theory is[3]

$$\int_{\mathbb{R}^D} \mathrm{d}^D k_1 \int_{\mathbb{R}^D} \mathrm{d}^D k_2 \frac{1}{((k_1+q)^2+m_1^2)(k_2^2+m_2^2)((k_1+k_2)^2+m_3^2)}, \tag{1.1}$$

where $q \in \mathbb{R}^D$ is the "external momentum" of some particle and the k_i are "internal momenta" corresponding to the degrees of freedom of virtual particles facilitating the interaction. $\mathcal{A}$ is a sum over all such integrals satisfying certain constraints given by the particular theory and the number and types of particles involved in the process that is studied. The i-th order contribution contains all integrals with i integrations of momentum vectors over D-dimensional space-time.

The integral is simple enough to write down. However, this simplicity is deceptive. In this case it already results in highly complicated elliptical polylogarithms, if one assumes the masses to be generic, and has kept physicists busy for decades [3, 4, 8, 25, 78, 101]. Hence, we will mostly restrict ourselves to the massless case, but occasionally comment on the massive case.

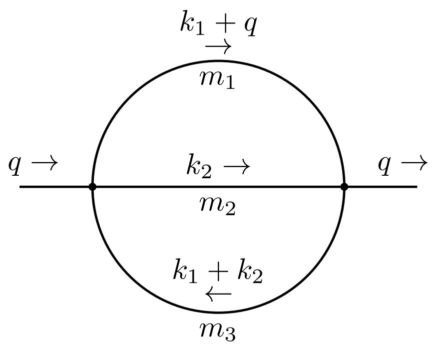


Figure 1.3: The two-loop banana/ sunrise/sunset graph.

These integrals can be visualised as graphs by associating parts of the integral to each edge, vertex and loop of a graph ("Feynman rules"). Specifically, each edge corresponds to a *propagator*, i.e. a function involving the inverse squares

[3]Note that this is the so-called Euclidean version. See section 2.1.3 for a more in-depth discussion of this choice.

of the momenta and masses associated to the edge, each independent loop corresponds to an integration of a momentum vector and momentum conservation has to be enforced at each vertex. The graph corresponding to the integral above is depicted in fig. 1.3 and has been given many different names over the years. Beyond their use as mere mnemonics these graphs have become interesting objects of study in their own right and one can gain a surprising amount of knowledge about the integrals just from their combinatorics [20, 31, 32, 105, 141].

1.2.4 Schwinger parameters

When studying the combinatorics of Feynman integrals it is useful to express the integral in a different form that is more closely connected to the underlying graph. For example, up to trivial factors and masses the example from eq. (1.1) can be rewritten as

$$\int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2 \int_0^\infty \mathrm{d}\alpha_3 \frac{\exp(-\frac{q^2\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3})}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{D/2}}. \tag{1.2}$$

The new dimensionless variables α_i that we associate to each edge are called *Schwinger parameters*[4] and the polynomials $q^2\alpha_1\alpha_2\alpha_3$ and $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$ are defined through the properties of the Feynman graph from fig. 1.3 in a way we discuss in detail in section 2.1.1. This alternative form of Feynman integrals has been known for a long time and was very useful in the early days of quantum field theory [9, 10, 51, 87, 103, 104, 106, 123]. Even back then it already piqued the interest of mathematicians who may otherwise not be particularly invested in physics [119].

Later the parametric integral fell somewhat out of use since other methods became much more efficient for higher order calculations. It reemerged when its deep connections to algebraic geometry were discovered [13]. It has since been used in a variety of applications and promises to reveal deeper structures behind Feynman amplitudes [5, 27–32, 92, 96, 108].

1.2.5 Quantum electrodynamics

In this thesis we are specifically concerned with quantum electrodynamics. Compared to what we introduced above the Feynman integrals in QED are slightly different. Not only do the propagators now also have momenta in the numerator, but we also get some new objects: Dirac matrices.

[4] *Julian Seymour Schwinger* (1918-1994). Schwinger's name is curiously absent from the literature about this topic. Since he notoriously disliked Feynman diagrams and integrals [118] and supposedly even banned them from his quantum field theory courses it seems even more bizarre that these types of integrals are now often named after him.

The Dirac gamma matrices are a set of four complex 4×4 matrices that satisfy the anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{1}_{4\times 4} \qquad \mu, \nu = 0, 1, 2, 3$$

and hence generate a representation of a Clifford algebra. Since there are different types of particles in QED we have to use different lines in the Feynman graphs, namely $\longrightarrow$ for electrons and $\sim\!\sim\!\sim$ for photons. Moreover, we have the restriction that each vertex should have exactly one photon and two electrons with opposite direction incident to it. Some graphs satisfying these constraints are depicted in fig. 1.4.

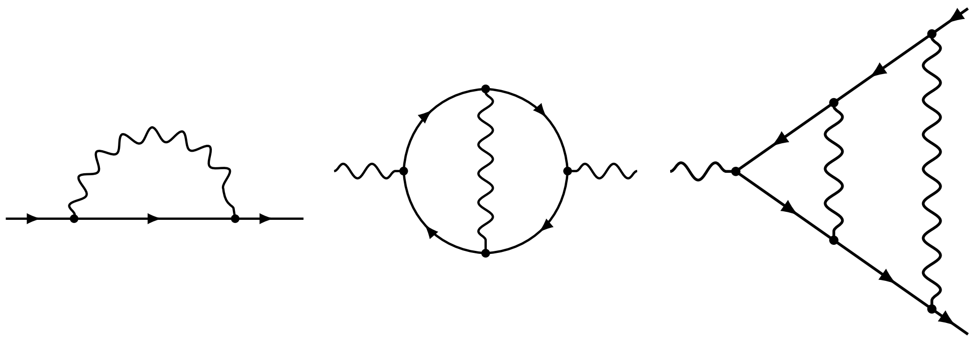


Figure 1.4: Three QED Feynman graphs.

The classical method to deal with these additional objects in the Feynman integral is to compute the contractions and possibly traces of the Dirac matrices, which is usually not difficult given modern computer algebra. This results in a set of effectively scalar Feynman integrals as shown above, except that the numerator may contain products of momenta like $k_i \cdot k_j$, $q \cdot k_i$ etc. These integrals can then be computed with the usual methods, which may or may not include translation into the Schwinger parametric version. However, this ignores a significant amount of combinatorial structure that we will be able to exploit below in order to achieve much simpler integrals.

1.3 Overview

Chapter 2 begins with a review of Feynman integrals, the main subject of interest for us, and some graph theory that will be needed throughout. Since we are particularly interested in the Schwinger parametric version of Feynman integrals we show in detail how to derive it from the classical momentum space integrals for scalar theories and then generalise to the case of quantum electrodynamics.

In section 2.3 we then come to our first new results. We analyse the combinatorics of the numerous derivatives in Schwinger parametric Feynman integral for QED and introduce a polynomial with the help of which the integrand can be expressed explicitly, free of derivatives. These results were previously published in [73].

Even with these new expressions for the integrands we still have the problem that they yield ill-defined divergent integrals. In **Chapter 3** we therefore very briefly introduce the general theory and Hopf algebraic structure of renormalisation and then explicitly work out the renormalisation of parametric QED integrands for superficial divergences and simple subdivergences.

Having renormalised the integrals we can turn again to the combinatorics of the integrand. Even expressed with the polynomials of chapter 2 it is still quite large and unfit for practical integration. In **chapter 4** we remove the entire tensor structure – products of Dirac matrices and traces thereof – from the integral. This is achieved by abstracting the algorithmic contraction of these objects to combinatorics on words and then interpreting it diagrammatically via chord diagrams. We end up with a purely scalar integrand given as a sum of such chord diagrams and the integer factors that resulted from the contraction of Dirac matrices are directly given by the properties of these diagrams. These results are published in [74].

The integrand has yet more structure that we can exploit to simplify it and make it easier to integrate. As we will discover in **chapter 5**, the polynomials introduced in chapter 2 can be viewed as special cases of *Dodgson polynomials*. This in particular means that they satisfy a family of identities which allow us to rewrite the large sums over chord diagrams that we found in the last chapter in a much briefer form. Most notably, this includes quite wondrous factorisations and cancellations that reduce the size of the integrand. These results will be published separately in [75].

Finally, in **chapter 6** we explicitly compute examples and highlight the achieved simplifications compared to the previous status quo. We also discuss observations made and what future work should be next.

2 Parametric Feynman integrals

世上无难事，只怕有心人.

Nothing in this world is difficult,
but thinking makes it seem so.

吴承恩 (Wu Cheng'en), 西游记 *(Journey to the West)*, 1592

2.1 Feynman graphs, rules and integrals

2.1.1 Preliminaries on graph theory

Graphs are the central combinatorial object of study for us, so we introduce all the necessary basics that will be needed. For more extensive reviews of graph theory that also keep the connection to physics in mind, the reader is referred to [16] and [105].

Graphs

A *graph* G is an ordered pair (V_G, E_G) of the set of *vertices* $V_G = \{v_1, \ldots, v_{|V_G|}\}$ and the set of *edges* $E_G = \{e_1, \ldots, e_{|E_G|}\}$, together with a map $\partial : E_G \to V_G \times V_G$. In this thesis we will always assume that G is connected, unless it is explicitly defined as a disjoint union $G = \sqcup_i G_i$ with connected components G_i.

Often we will need directed graphs, in which a direction is assigned to each edge $e \in E_G$ by specifying an ordered pair $\partial(e) = (\partial_-(e), \partial_+(e))$, where the vertex $\partial_-(e) \in V_G$ is called start or initial vertex while $\partial_+(e) \in V_G$ is called target or final vertex. The particular choice of direction for each edge will usually have no influence on the end results that we are interested in but in some cases we will fix a particular physically motivated choice to simplify notation.

When considering general graphs we make no restrictions on the edges. In particular, multiple edges between the same vertices ($\partial(e_i) = \partial(e_j)$ for $e_i \neq e_j$) as well as edges with identical start and target ($\partial_-(e) = \partial_+(e)$, “self-cycles” or “tadpoles”) are allowed. Only when we explicitly work with Feynman graphs we have the usual physical constraints that may exclude such types of graphs.

Subgraphs $g \subseteq G$ are usually identified with their edge set $E_g \subseteq E_G$ and it is implicitly assumed that g does not contain isolated vertices, i.e. $V_g = \partial_+(E_g) \cup \partial_-(E_g)$. The notable exception to this are forests, which are one of a number of types of graph that we will be interested in:

- A *tree* T is a graph that is connected and simply connected.
- A disjoint union of trees $F = \sqcup_{i=1}^{k} T_i$ is called a *k-forest*, such that a tree is a 1-forest.
- A subgraph $g \subset G$ that contains all vertices of G, i.e. $V_g = V_G$, is called *spanning*.
- A *bond* $B \subset G$ is a minimal subgraph G such that $G \setminus B$ has exactly two connected components.
- A simple *cycle* $C \subset G$ is a subgraph of G that is 2-regular, i.e. all vertices have exactly two edges incident to it, and has only one connected component.

For a given graph G we denote the sets of all spanning k-forests, bonds, and simple cycles with $\mathcal{T}_G^{[k]}$, $\mathcal{B}_G$, and $\mathcal{C}_G^{[1]}$ respectively. The number of independent cycles (loops, in physics nomenclature) is denoted $h_1(G)$, the first Betti number of the graph. If the graph is unambiguously clear from context we often just write $h_1 \equiv h_1(G)$.

Example 2.1.1. *Let G be the banana graph with three edges depicted in fig. 2.1. It has three spanning trees*

$$T_1 = \{e_1\} \qquad T_2 = \{e_2\} \qquad T_3 = \{e_3\},$$

each consisting of a single edge. The only spanning 2-forest has no edges but only the two isolated vertices. There is also only one bond, $B_1 = \{e_1, e_2, e_3\}$, since all edges have to be removed to separate the graph into two components.

Remark 2.1.2. *Consider the vector space of edge subsets of a graph G over $\mathbb{Z}_2$, where addition is given by the symmetric difference*

$$E_1 \triangle E_2 := (E_1 \setminus E_2) \cup (E_2 \setminus E_1) = (E_1 \cup E_2) \setminus (E_1 \cap E_2) \tag{2.1}$$

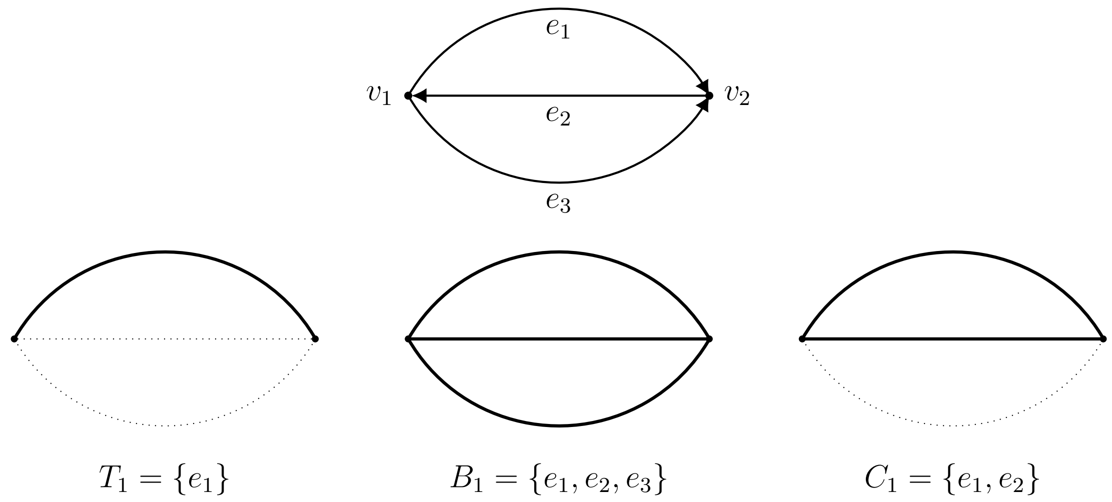


Figure 2.1: The banana graph with 3 edges and examples for a spanning tree, a bond and a cycle subgraph of it.

and the (degenerate) inner product is

$$\langle E_1, E_2 \rangle := \begin{cases} 1 & \text{if } |E_1 \cap E_2| \text{ odd,} \\ 0 & \text{if } |E_1 \cap E_2| \text{ even.} \end{cases} \tag{2.2}$$

The set of all cycles (not just simple cycles) $\mathcal{C}_G = \bigcup_i \mathcal{C}_G^{[i]}$ *and the set of bonds* $\mathcal{B}_G$ *are each others orthogonal complement in this vector space and thus span it. While we do not really explicitly use this anywhere, this duality between bonds and cycles in a graph underlies many of the combinatoric results of this thesis.*

Two operations on graphs that we make extensive use of are the *deletion* and *contraction* of an edge. Deletion is rather self-explanatory – the edge is simply removed from the edge set. The resulting graph is denoted $G \setminus e := (V_G, E_G \setminus \{e\})$. If the removal of an edge disconnects the graph then e is called a *bridge*, and if none of its edges are bridges then G is called bridgeless, bridgefree, 2-edge-connected[1] or, in physics literature, one-particle irreducible (1PI). Contraction additionally identifies the two end points of a deleted edge, i.e. the resulting graph is $G /\!\!/ e := (V_G|_{\partial_+(e)=\partial_-(e)}, E_G \setminus \{e\})$.

For edge subsets $E \subset E_G$ with more than one element the operations also apply and the order of contraction or deletion does not matter. As long as $E \cap E' = \emptyset$

[1]Note that these notions vary slightly for disconnected graphs. A disconnected graph is bridgeless if each connected component is 2-edge-connected.

they can also be combined to yield $(G \setminus E) /\!\!/ E' = (G /\!\!/ E') \setminus E$.

Note that we use the double slash in the contraction to differentiate between two slightly different notions. Since we usually identify graphs with their edge sets we often have notation like $G /\!\!/ g = G /\!\!/ E_g$ for the (edge) contraction of some subgraph $g \subset G$. On the other hand, we use G/g to denote the quotient graph in the algebraic sense (see section 3.1.1). These notions often coincide, but differ slightly for propagator Feynman graphs, as we will discuss in chapter 3.

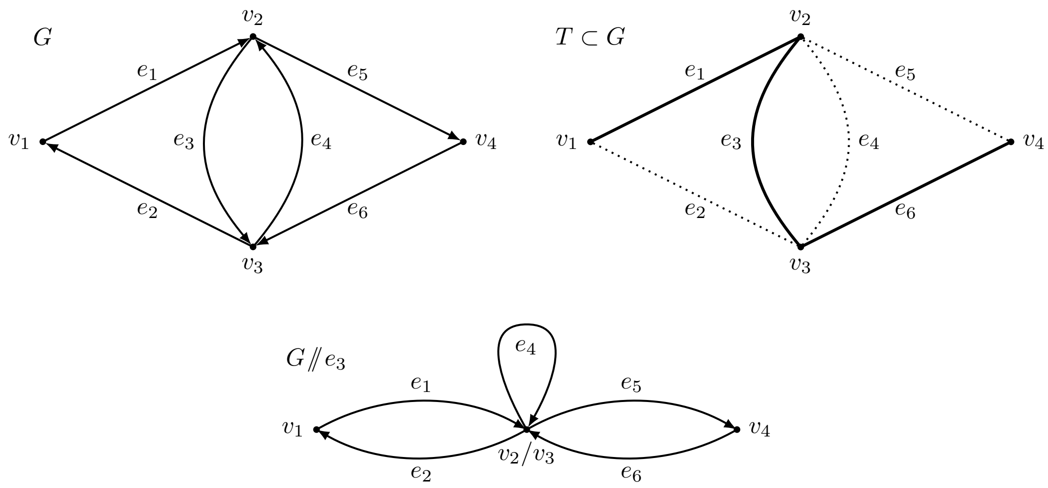


Figure 2.2: A graph G, one of its spanning trees, and the graph that results from contraction of one of its edges.

The Kirchhoff polynomial

Graphs have many invariants that happen to be polynomials. Most famously there is the Tutte polynomial [128, 129] and its various specialisations like the chromatic polynomial [11, 138], the Jones polynomial in knot theory [85] or the partition function of the Potts model in statistical physics [112]. The one that we are interested in differs from these in that it is a polynomial in variables $\alpha = (\alpha_e)_{e \in E_G}$ assigned to the edges of a graph, whereas the others are usually univariate or bivariate[2].

The *Kirchhoff polynomial*, which is especially in the physics literature also often

[2]However, the Kirchhoff polynomial can in fact be seen as a limiting case of the multivariate generalisation of the Tutte polynomial, as described in [16, section 6], [93].

called the first Symanzik polynomial, is defined as

$$\Psi_G(\alpha) := \sum_{T \in \mathcal{T}_G^{[1]}} \prod_{e \notin T} \alpha_e. \tag{2.3}$$

It has been known for a very long time and was first introduced by Kirchhoff in his study of electrical circuits [88]. We will often make use of the abbreviation

$$\alpha_S := \prod_{e \in S} \alpha_e \tag{2.4}$$

for any edge subset $S \subset E_G$, such that $\Psi_G(\alpha) = \sum_{T \in \mathcal{T}_G^{[1]}} \alpha_{E_G \setminus T}$.

Two important properties are obvious directly from the definition: Ψ_G is

- homogeneous of degree $h_1(G)$ in α, and
- linear in each α_e.

Moreover, it also satisfies the famous *contraction-deletion relation*,

$$\Psi_G = \Psi_{G/\!\!/e} + \alpha_e \Psi_{G\setminus e} \tag{2.5}$$

which means that the polynomials belonging to graphs that are related via contraction or deletion of edges can be recovered easily from the original graph polynomial:

$$\Psi_{G/\!\!/e} = \Psi_G\big|_{\alpha_e=0} \tag{2.6}$$

$$\Psi_{G\setminus e} = \frac{\partial}{\partial \alpha_e} \Psi_G \tag{2.7}$$

Note that two cases have to be excluded: One is the contraction of a tadpole edge, which is the same as just deleting the edge since its endpoints are already identified. The other is deletion of a bridge, which would disconnect the graph. Since a bridge is necessarily contained in all spanning trees Ψ_G is independent of the corresponding edge variable such that the derivative vanishes. While this is not in itself inconsistent it would conflict with the common and useful definition

$$\Psi_G := \prod_i \Psi_{G_i}, \tag{2.8}$$

for disjoint unions $G = \bigsqcup_i G_i$. This product is the Kirchhoff polynomial of a vertex-1-connected graph that consists of the components G_i arranged in a chain, each component overlapping with the next in only one vertex[3]. It is sensible to define the polynomials for disconnected graphs like this since there is clearly a one-to-one correspondence between spanning trees of such a vertex-1-connected graph and tuples of spanning trees, containing one tree from each component.

[3]See also [29] and the "circular joins" used therein.

Example 2.1.3. *We have already seen one example in the introductory chapter. The polynomial* $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$ *in the integral in eq. (1.2) is the Kirchhoff polynomial of the graph in fig. 2.1 and also the Feynman graph in fig. 1.3. For a more elaborate example let* G *be the graph from fig. 2.2. The spanning tree depicted in that figure corresponds to the monomial* $\alpha_2\alpha_4\alpha_5$*. The full polynomial is a sum over 12 spanning trees:*

$$\begin{aligned}\Psi_G = \quad & \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_6 + \alpha_1\alpha_4\alpha_5 + \alpha_1\alpha_4\alpha_6 + \alpha_2\alpha_3\alpha_4 \\ & + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_4\alpha_5 + \alpha_2\alpha_4\alpha_6 + \alpha_3\alpha_4\alpha_5 + \alpha_3\alpha_4\alpha_6 \\ =& \alpha_3\alpha_4(\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6) + (\alpha_3 + \alpha_4)(\alpha_1\alpha_5 + \alpha_1\alpha_6 + \alpha_2\alpha_5 + \alpha_2\alpha_6) \end{aligned} \tag{2.9}$$

Matrices

Many properties of graphs can be captured by matrices[4], and we discuss here some of the well known relations between graphs, matrices and the Kirchhoff polynomial.

The *incidence matrix* I is an $|E_G| \times |V_G|$ matrix

$$I_{ev} := \begin{cases} \pm 1 & \text{if } v = \partial_\pm(e) \\ 0 & \text{if } e \text{ is not incident to } v. \end{cases} \tag{2.10}$$

The second matrix we need is the *Laplacian matrix* L. It is defined as the difference of the degree and adjacency matrices of the graph. Since we will not need either of those two going forward we instead use a well known identity to define the Laplacian as the product of incidence matrix and its transpose

$$L := I^T I. \tag{2.11}$$

This is a standard result discussed in many graph theory courses [18]. A detailed proof can also be found in the author's master thesis [72, Lemma 1.2.14].

Instead of the full matrices we will actually always need the smaller matrices in which one column (of I) or one column and one row (of L) corresponding to an arbitrarily chosen vertex of G are deleted. From now on we use I' and L' for these $|E_G| \times |V_G| - 1$ and $|V_G| - 1 \times |V_G| - 1$ matrices, called reduced incidence and reduced Laplacian matrix.

Finally, let A be the diagonal $|E_G| \times |E_G|$ matrix with entries $A_{ij} := \delta_{ij}\alpha_{e_i}$. With this setup the well known *Matrix-Tree-Theorem* [35] tells us that

$$\Psi_G = \alpha_{E_G} \det({I'}^T A^{-1} I'). \tag{2.12}$$

[4]Or, more generally, by matroids [107], [97], [16, section 8]. While we do not use them in this thesis, matroids seem like a useful tool that is currently woefully underused in physics and should be kept in mind for future work.

The original theorem states that the determinant of $L' = I'^T I'$ counts the spanning trees of the corresponding graph. Replacing L' by the weighted Laplacian $I'^T A^{-1} I'$ yields the sum over spanning trees with monomials α_T^{-1} for each spanning tree, so multiplying with all edge parameters turns it into the Kirchhoff polynomial.

Remark 2.1.4. *The polynomial $\Psi_G^* = \det(I'^T A I')$ is sometimes called dual Kirchhoff polynomial. If G is planar then it is the Kirchhoff polynomial of its planar dual graph G^*.*

Often I' and A are arranged in a block matrix

$$M := \begin{pmatrix} A & I' \\ -I'^T & 0 \end{pmatrix}. \tag{2.13}$$

This is called the *graph matrix* of G [13,28], and with the block matrix determinant identity

$$\det \begin{pmatrix} S & T \\ U & V \end{pmatrix} = \det(S) \det(V - US^{-1}T) \tag{2.14}$$

one sees that indeed

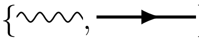
$$\det(M) = \det(A) \det(I'^T A^{-1} I') = \Psi_G. \tag{2.15}$$

2.1.2 Feynman graphs

Feynman graphs are graphs with some extra information that can be used to encode the Feynman integrals we are interested in. In order to distinguish them from the usual graphs G we denote them with Γ. There are three major significant differences compared to usual graphs. Firstly, there can be different types of edges that represent different types of particles. Which particles are present depends on the theory. For example, in scalar ϕ^k-theories there is only a single particle

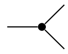
$\mathcal{R}_E^{\phi^k} = \{$  $\}, \qquad (2.16)$

whereas quantum electrodynamics has photons and electrons

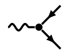
$\mathcal{R}_E^{QED} = \{$  $\}, \qquad (2.17)$

and many more appear in other theories. In QED we will denote the edge subsets consisting only of photons and fermions with $E_\Gamma^{(p)}$ and $E_\Gamma^{(f)}$ respectively. In the case of fermions we will always assume that the edge orientation is aligned with the fermion flow, indicated by the arrow on the edge.

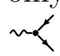
Secondly, there are restrictions on how these edges can be combined. They are implemented by specifying a (usually finite) set of allowed *corollas* – vertices together with incident half-edges. In the two most common scalar theories that would be

$$\mathcal{R}_V^{\phi^3} = \{$$  $$\} \quad \text{and} \quad \mathcal{R}_V^{\phi^4} = \{$$  $$\}, \tag{2.18}$$

and QED also only has one,

$$\mathcal{R}_V^{QED} = \{$$  $$\}. \tag{2.19}$$

Again, other theories like quantum chromodynamics have more and more complicated types of corollas but these are the cases we will mostly discuss in this thesis.

Finally, Feynman graphs contain so called *external edges*, which are half-edges only incident to one vertex. The set of the corresponding external vertices is denoted $V_\Gamma^{ext} \subseteq V_\Gamma$, and V_Γ^{int} is its complement. The external edges carry information about physical data like momenta of the incoming and outgoing particles they represent and are not counted as members of a Feynman graph's edge set E_Γ. The type and number of external edges is encoded in the *residue* $\text{res}(\Gamma)$ of a Feynman graph. If there are two external edges of the same type then it is a propagator $\text{res}(\Gamma) \in \mathcal{R}_E$, whereas it is a corolla if there are more than two. In this thesis we only focus on QED Feynman graphs whose residues are in $\mathcal{R}^{QED} = \{$, ,  $\}$.

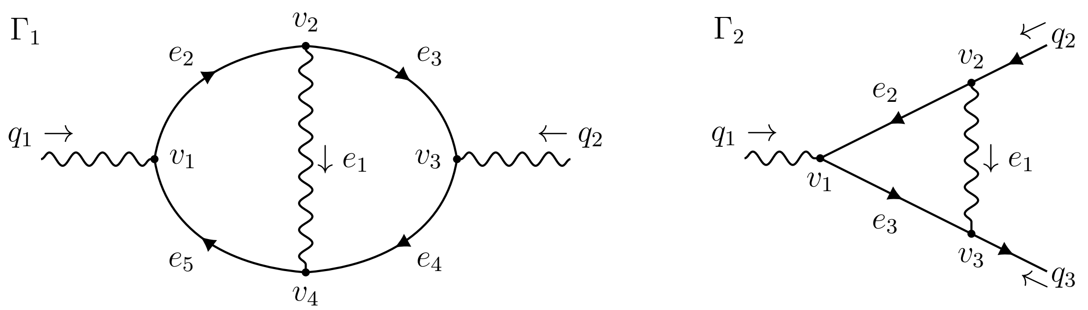


Figure 2.3: Two labelled QED Feynman graphs.

The second Symanzik polynomial

With some of the additional information contained in Feynman graphs, namely external particle momenta, we can define a second graph polynomial that appears all throughout this thesis. The *second Symanzik polynomial* (with the first Symanzik being an alternative name for the Kirchhoff polynomial) is traditionally defined similarly to the Kirchhoff polynomial, by summing over spanning 2-forests instead of trees [123]:

$$\Phi_\Gamma(\alpha, q) := \sum_{(T_1,T_2)\in\mathcal{T}_\Gamma^{[2]}} s(q, T_1, T_2) \prod_{e\notin T_1\cup T_2} \alpha_e \tag{2.20}$$

The function s is the square of a linear combination of external momenta flowing from one component of the spanning forest to the other. The examples below should elucidate what that means and we will make it a bit more precise when we give a different interpretation of the second Symanzik polynomial in section 2.3. Analogously to the Kirchhoff polynomial we can generalise the definition to disconnected graphs $\Gamma = \bigsqcup_i \Gamma_i$:

$$\Phi_\Gamma := \sum_i \Phi_{\Gamma_i} \prod_{j\neq i} \Psi_{\Gamma_j} \tag{2.21}$$

If there are only two external momenta (such that $q_1 = -q_2 \equiv q$ by momentum conservation), then the second Symanzik polynomial factorises and we write

$$\Phi_\Gamma = q^2 \varphi_\Gamma. \tag{2.22}$$

Example 2.1.5. *The other polynomial in eq. (1.2),*

$$q^2\alpha_1\alpha_2\alpha_3, \tag{2.23}$$

is the second Symanzik polynomial of the graph in fig. 1.3. There is only a single spanning 2-forest that consists of the two vertices and no edges. The momentum between those two edges is simply q, which enters the graph via one external edge and exits through the other.

Let Γ_1, Γ_2 be the two Feynman graphs from fig. 2.3. Their Kirchhoff polynomials are

$$\Psi_{\Gamma_1} = (\alpha_2 + \alpha_5)(\alpha_3 + \alpha_4) + \alpha_1(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5), \tag{2.24}$$

$$\Psi_{\Gamma_2} = \alpha_1 + \alpha_2 + \alpha_3. \tag{2.25}$$

For Γ_1 there are a total of 10 spanning 2-forests, but not all of them contribute to the second Symanzik polynomial. Consider the spanning 2-forest with $T_1 = \{e_2, e_5\}$ and T_2 just the isolated vertex v_3 without edges. The external momentum q_1 enters T_1 in the vertex v_1 and q_2 (which has to be $-q_1$ due to momentum conservation) enters T_2 in v_3. Hence, the momentum transfer between these two components of *the forest is* $\pm q_1$ *and the corresponding monomial is* $q_1^2\alpha_1\alpha_3\alpha_4$. *An example of a forest that does not contribute is* $T_1 = \{e_2, e_3\}$ *and* T_2 *just the vertex* v_4. *The external momentum enters* T_1 *in* v_1 *and exits in* v_3 *whereas* T_2 *is not connected to any external edges at all. Hence* $s(q_1, q_2, T_1, T_2) = 0$ *in this case. Overall, 8 of the 10 forests contribute to yield the second Symanzik polynomial*

$$\Phi_{\Gamma_1} = q^2\Big(\alpha_2\alpha_5(\alpha_1 + \alpha_3 + \alpha_4) + \alpha_3\alpha_4(\alpha_1 + \alpha_2 + \alpha_5) + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_5\Big). \tag{2.26}$$

If there are more than two external edges the situation becomes more complicated, so we choose a one-loop graph example. By momentum conservation one of the three external momenta of Γ_2 *is the negative sum of the other two, say* $q_1 = -q_2 - q_3$. *There are three spanning 2-forests, each consisting of one edge in one tree and the remaining isolated vertex in the other. Hence, we have a momentum transfer* $\pm q_i$ *where* v_i *is the isolated vertex in each of those tree pairs and the full polynomial is*

$$\begin{aligned}\Phi_{\Gamma_2} &= q_2^2\alpha_1\alpha_2 + q_3^2\alpha_1\alpha_3 + q_1^2\alpha_2\alpha_3 \\ &= q_2^2\alpha_1\alpha_2 + q_3^2\alpha_1\alpha_3 + (q_2 + q_3)^2\alpha_2\alpha_3.\end{aligned} \tag{2.27}$$

Remark 2.1.6. *In addition to all this structure, Feynman graphs also happen to form Hopf algebras [47,48,94,131]. However, except for implicitly using this when discussing renormalisation in chapter 3 we will not really make use of this fact in this thesis.*

2.1.3 Feynman integrals in momentum space

Now that we know what Feynman graphs are, we can go back to the integrals they were invented to visualise. For a given Feynman graph Γ the *Feynman rules* are a map that assigns to it a complex multivalued function $\phi_\Gamma(q, m)$. The arguments contain all kinematic information[5] in the form of external momenta $q = (q_1, \ldots, q_{|V_\Gamma^{ext}|}) \in \mathbb{C}^{D\times|V_\Gamma^{ext}|}$ and particle masses $m = (m_1, \ldots, m_{|E_\Gamma|}) \in \mathbb{C}^{|E_\Gamma|}$.

In its *momentum space representation* ϕ_Γ is given as an integral

$$\phi_\Gamma(q, m) = \frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d} \tilde{I}_\Gamma(q, m, k)\mathrm{d}^d k \tag{2.28}$$

with $d = |E_\Gamma|D$. The specific form of the integrand depends on the theory at hand but always follows these steps:

[5]These physical objects should of course be real valued but it often is useful to consider analytic continuations. For our purposes they will mostly just be formal parameters and it will not play a big role if they are complex or real.

- All edges $e \in E_\Gamma$ are assigned the corresponding propagator terms, which are then multiplied together. In the scalar case this is

$$= \frac{1}{k_e^2 + m_e^2}, \tag{2.29}$$

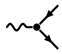
which we have seen in eq. (1.1) before. For QED the two types of edges are

$$= \frac{\gamma_{\mu_e} k_e^{\mu_e} + m_e}{k_e^2 + m_e^2} \tag{2.30}$$

$$= \frac{g^{\nu_u \nu_v} + \varepsilon \frac{k_e^{\nu_u} k_e^{\nu_v}}{k_e^2}}{k_e^2}, \qquad (u, v) = \partial(e). \tag{2.31}$$

Here we use ε as the gauge parameter which is more commonly written as $-(1-\zeta)$ or $-(1-\chi)$. For our purposes it makes more sense to use this notation, where $\varepsilon = 0$ corresponds to Feynman gauge.

- All vertices $v \in V_\Gamma$ are assigned a delta function (enforcing momentum conservation in each vertex) plus assorted factors like the Dirac matrix in QED:



$$= \gamma_{\nu_v} \pi^{D/2} \delta^{(D)}\Big(q_v + \sum_{e \in E_\Gamma} I_{ev} k_e\Big) \tag{2.32}$$

I_{ev} is an entry of the incidence matrix of Γ, as defined in eq. (2.10) and q_v is the external momentum entering the graph in v (so $q_v = 0$ if $v \in V_\Gamma^{int}$). Here we adopt the convention of [109] and include a factor of $\pi^{D/2}$ for each vertex. This ensures that the parametric integrals we are interested in below will be free of powers of π. Sometimes the coupling constant is also included as a factor for each vertex, but since we view the integrals as coefficients of powers of the coupling in the perturbation expansion we do not do so here.

- Other theory dependent factors or modifications: The Dirac matrices do not generally commute, so they have to be multiplied in a specific order determined by the graph. If a product of Dirac matrices is associated to

  - an open fermion line (going from external fermion to external fermion), then it is ordered opposite to fermion flow, starting from the external vertex with the outgoing external fermion.
  - a closed fermion loop (a cycle $C \in \mathcal{C}_\Gamma^{[1]}$ that is also a subset of $E_\Gamma^{(f)}$), then one takes the trace of the product and multiplies with an overall factor of -1.

For the scalar case we have seen the integral eq. (1.1) in the introduction. Note that one of the two delta functions was integrated, leaving only two integrations and modifying the denominators. The other delta function enforces the overall momentum conservation and is not written explicitly. This is commonly done in momentum space, but to move to parametric space we will need the delta functions explicitly in the integrand. For QED examples we return to the two graphs from fig. 2.3.

Example 2.1.7. *Let Γ_1 and Γ_2 be the graphs from fig. 2.3. With $\not{k}_i = \gamma_{\mu_i} k_i^{\mu_i}$ the integrand of the first is*

$$\begin{aligned}
&\tilde{I}_{\Gamma_1}(q,m,k) \\
&=(-1)\pi^8 \frac{g^{\nu_2\nu_4} + \varepsilon \frac{k_1^{\nu_2} k_1^{\nu_4}}{k_1^2}}{k_1^2} \frac{\operatorname{tr}\left(\gamma^{\nu_1}(\not{k}_2+m)\gamma^{\nu_2}(\not{k}_3+m)\gamma^{\nu_3}(\not{k}_4+m)\gamma^{\nu_4}(\not{k}_5+m)\right)}{(k_2^2+m^2)(k_3^2+m^2)(k_4^2+m^2)(k_5^2+m^2)} \\
&\quad \delta^{(4)}\Big(q_1+k_5-k_2\Big)\delta^{(4)}\Big(k_2-k_1-k_3\Big)\delta^{(4)}\Big(q_2+k_3-k_4\Big)\delta^{(4)}\Big(k_1+k_4-k_5\Big).
\end{aligned} \tag{2.33}$$

Here we used that all electrons have the same mass and set the dimension to $D = 4$.

Ignoring masses, which is what we mostly do from now on, the trace reduces to

$$\bar{\gamma}_{\Gamma_1} = (-1)\operatorname{tr}(\gamma^{\nu_1}\gamma^{\mu_2}\gamma^{\nu_2}\gamma^{\mu_3}\gamma^{\nu_3}\gamma^{\mu_4}\gamma^{\nu_4}\gamma^{\mu_5}). \tag{2.34}$$

We denote with μ_i the index of the matrix associated to the edge e_i and with ν_i the matrix of a vertex v_i. The coefficient of m^2 contains six traces of the form

$$\operatorname{tr}(\gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\nu_3}\gamma^{\mu_4}\gamma^{\nu_4}\gamma^{\mu_5}). \tag{2.35}$$

Since this would also be contracted with only two instead of four momenta $k_e^{\mu_e}$ it is essentially just a simpler version of the mass free term that we will always consider below.

For Γ_2 the situations is similar, except that there are fewer edges and vertices and no trace. Instead, one gets the product

$$\bar{\gamma}_{\Gamma_2} = \gamma^{\nu_3}\gamma^{\mu_3}\gamma^{\nu_1}\gamma^{\mu_2}\gamma^{\nu_2}, \tag{2.36}$$

again ignoring the mixed terms with masses.

Divergences

Given an integral $\phi_\Gamma(q,m)$ it is natural to ask if it is even well-defined. As it turns out, most interesting cases are actually divergent. For an extensive discussion of divergence and convergence, regularisation, analytic continuation etc. we refer to [109, Section 2.2] and the references therein, in particular the work of Speer [120,121] and Weinberg [135]. In essence the situation is thus: For Euclidean kinematics (momenta q and masses m) the behaviour of the integrals is fully understood. If one regularises the integral by modifying the powers of the denominators, the dimension, or both, then there always exist values of these parameters such that $\phi_\Gamma(q,m)$ is absolutely convergent. The most important consequence for us is that the Schwinger trick, which requires an exchange of the integrations over the momenta and Schwinger parameters, is applicable. The regularised integral then extends to a meromorphic function of the regularisation parameters. It can be renormalised to remove the divergences, which allows the return to the original values of the regularisation parameters, e.g. $D = 4$. The dependence on q and m is much more complicated, namely multi-valued with highly non-trivial monodromies and far from understood [14,15]. This discussion also applies, with some modifications like the infrared variant of Weinstein's ultraviolet power counting theorem [102], to infrared divergences, e.g. resulting from $k_e \to 0$ in the photon propagator, or in any other propagator if we omit their masses. The parametric integral is especially interesting in that regard since in projective space UV ($\alpha_e \to 0$) and IR ($\alpha_e \to \infty$) divergences correspond to the same subset of the integration domain [109, Remark 2.2.11].

The Minkowski space version is much less understood, so this is the main reason – besides the notational convenience of not having to carry factors of i – why we choose to work in the Euclidean setting.

Alternatively, instead of first applying the Schwinger trick and then renormalising in parametric space as we do in chapter 3, one could take the fully renormalised momentum space integral [46, 82, 143] and then apply the Schwinger trick to the full integral, i.e. all terms in Zimmermann's forest formula. The integrals we are working with are then convergent at all times and one never needs to think about regularisation. However, this would unnecessarily complicate the exposition of the Schwinger trick and the introduction to the parametric integral.

2.2 The Schwinger parametric representation

We give a step-by-step derivation of the Schwinger parametric integral, based on the Schwinger trick and starting from the momentum space integral introduced above. Other texts doing so with varying degrees of detail and rigour are for example the original article [106], the review article [16], or the textbook [84, Sec. 6-2-3]. As an alternative approach one could follow the more abstract geometric derivation in [13, Sec. 6].

2.2.1 Scalar theories

In section 2.1.3 we learned how to construct Feynman integrals in momentum space. For a generic scalar theory the integrand simply takes the form

$$\tilde{I}_\Gamma(q,m,k) = \Big(\prod_{e\in E_\Gamma} \frac{1}{k_e^2+m_e^2}\Big)\Big(\prod_{v\in V_\Gamma} \pi^{D/2}\delta^{(D)}\Big(q_v + \sum_{e\in E_\Gamma} I_{ev}k_e\Big)\Big). \tag{2.37}$$

Now the *Schwinger trick*[6]

$$\frac{1}{X} = \int_0^\infty e^{-\alpha X}\mathrm{d}\alpha \qquad \forall X>0 \tag{2.38}$$

and the Fourier transform

$$\delta^{(D)}(p) = \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} e^{ipx}\mathrm{d}^D x \tag{2.39}$$

are applied to rewrite each propagator and delta function:

$$\begin{aligned}\tilde{I}_\Gamma(q,m,k) &= \frac{1}{(4\pi)^{\frac{D}{2}|V_\Gamma|}}\int_{\mathbb{R}_+^{|E_\Gamma|}} \mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_{E_\Gamma}\int_{\mathbb{R}^{D|V_\Gamma|}}\mathrm{d}^D x_1\cdots\mathrm{d}^D x_{|V_\Gamma|}\\ &\quad\times\exp\Big(-\sum_{e\in E_\Gamma}\alpha_e(k_e^2+m_e^2) + i\sum_{v\in V_\Gamma} x_v\Big(q_v+\sum_{e\in E_\Gamma} I_{ev}k_e\Big)\Big)\\ &= \frac{1}{(4\pi)^{\frac{D}{2}|V_\Gamma|}}\int_{\mathbb{R}_+^{|E_\Gamma|}} \mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_{E_\Gamma}e^{-\sum\alpha_e m_e^2}\int_{\mathbb{R}^{D|V_\Gamma|}}\mathrm{d}^D x_1\cdots\mathrm{d}^D x_{|V_\Gamma|}e^{i\sum x_v q_v}\\ &\quad\times\exp\Big(-\sum_{e\in E_\Gamma}\alpha_e\Big(k_e^2-\frac{i}{\alpha_e}k_e\sum_{v\in V_\Gamma}x_v I_{ev}\Big)\Big).\end{aligned} \tag{2.40}$$

In the last exponent one completes the square

$$k_e^2 - \frac{i}{\alpha_e}k_e\sum_{v\in V_\Gamma}x_v I_{ev} = \Big(k_e - \frac{i}{2\alpha_e}\sum_{v\in V_\Gamma}x_v I_{ev}\Big)^2 + \frac{1}{4\alpha_e^2}\Big(\sum_{v\in V_\Gamma}x_v I_{ev}\Big)^2 \tag{2.41}$$

[6] More generally one one could use $X^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty \alpha^{\nu-1}e^{-\alpha X}\mathrm{d}\alpha$, with $\mathrm{Re}(\nu)>0$, which changes nothing except for the additional factor for each edge.

such that the k integration in eq. (2.28) becomes a Gaußian integral that gives

$$\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d} \mathrm{d}^d k \exp\Bigg(-\sum_{e\in E_\Gamma}\alpha_e\Big(k_e-\frac{i}{2\alpha_e}\sum_{v\in V_\Gamma}x_v I_{ev}\Big)^2\Bigg)=\prod_{e\in E_\Gamma}\alpha_e^{-\frac{D}{2}}. \tag{2.42}$$

Note that $d=|E_\Gamma|D$, so there are $|E_\Gamma|$ Gaußian integrals, each yielding a factor $\alpha_e^{-\frac{D}{2}}$ for a different edge. The remaining part of eq. (2.41) is a squared sum over vertices. Choose an arbitrary vertex $v_0\in V_\Gamma$, transform $x_v\to x_{v_0}+x_v$ for all $v\neq v_0$ and define $V'_\Gamma=V_\Gamma\setminus\{v_0\}$. Then

$$\begin{aligned}&\frac{1}{4\alpha_e}\Big(\sum_{v\in V_\Gamma}x_v I_{ev}\Big)^2\\ &\quad\to\frac{1}{4\alpha_e}\Big(x_{v_0}\underbrace{\sum_{v\in V_\Gamma}I_{ev}}_{=0}+\frac{1}{4\alpha_e}\sum_{v\in V'_\Gamma}x_v I_{ev}\Big)^2=\frac{1}{4\alpha_e}\Big(\sum_{v\in V'_\Gamma}x_v I_{ev}\Big)^2.\end{aligned} \tag{2.43}$$

Summing this over all edges yields

$$\sum_{e\in E_\Gamma}\alpha_e^{-1}\Big(\sum_{v\in V'_\Gamma}x_v I_{ev}\Big)^2=\sum_{v_1,v_2\in V'_\Gamma}x_{v_1}x_{v_2}\sum_{e\in E_\Gamma}\alpha_e^{-1}I_{ev_1}I_{ev_2}=x^T(I'^TA^{-1}I')x \tag{2.44}$$

where $I'^TA^{-1}I'=:\tilde{L}'$ is the (weighted and reduced) Laplacian matrix of Γ with both column and row v_0 removed as introduced in section 2.1.1.

The term linear in x_v in eq. (2.40) gives

$$\sum_{v\in V_\Gamma}x_v q_v\to x_{v_0}\sum_{v\in V_\Gamma}q_v+\sum_{v\in V'_\Gamma}x_v q_v. \tag{2.45}$$

The first term allows for integration of x_{v_0} which returns a delta function $\delta^{(D)}(\sum q_v)$ that enforces overall (external) momentum conservation. The Feynman integral has now taken the form

$$\begin{aligned}\phi_\Gamma(q,m)=&\frac{\pi^{\frac{D}{2}}\delta^{(D)}(\sum q_v)}{(4\pi)^{\frac{D}{2}|V'_\Gamma|}}\int_{\mathbb{R}_+^{|E_\Gamma|}}\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_{E_\Gamma}\prod_{e\in E_\Gamma}\frac{e^{-\alpha_e m_e^2}}{\alpha_e^{D/2}}\\ &\times\int_{\mathbb{R}^{D|V'_\Gamma|}}\mathrm{d}^Dx_1\cdots\mathrm{d}^Dx_{V'_\Gamma}\exp\Big(-\frac{1}{4}x^T\tilde{L}'x+iQ'x\Big)\end{aligned} \tag{2.46}$$

where Q' is the vector $(q_v)_{v\in V'_\Gamma}$.

This is another Gaußian integration such that

$$\phi_\Gamma(q,m) = \pi^{\frac{D}{2}} \delta^{(D)}(\sum q_v) \int_{\mathbb{R}_+^{|E_\Gamma|}} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{E_\Gamma} \left(\prod_{e \in E_\Gamma} \frac{e^{-\alpha_e m_e^2}}{\alpha_e^{D/2}} \right) \frac{\exp\left(- Q'^T \tilde{L}'^{-1} Q' \right)}{\det(\tilde{L}')^{D/2}}.$$

Now we use the Matrix-Tree theorem (eq. (2.12)) to identify the Kirchhoff polynomial in the denominator. What remains to be worked out is

$$\begin{aligned} Q'^T \tilde{L}'^{-1} Q' &= \sum_{v_1, v_2 \in V'_\Gamma} q_{v_1} \cdot q_{v_2} (\tilde{L}'^{-1})_{v_1 v_2} \\ &= \frac{1}{\det(\tilde{L}')} \sum_{v_1, v_2 \in V'_\Gamma} q_{v_1} \cdot q_{v_2} (-1)^{v_1 + v_2} \det(\tilde{L}'^{\{v_1\}}_{\{v_2\}}) \\ &= \frac{\alpha_{E_\Gamma}}{\Psi_\Gamma} \sum_{v_1, v_2 \in V'_\Gamma} q_{v_1} \cdot q_{v_2} (-1)^{v_1 + v_2} \det(\tilde{L}'^{\{v_1\}}_{\{v_2\}}). \end{aligned} \tag{2.47}$$

where we used the cofactor inversion for $\tilde{L}'^{-1}$, i.e. the subscript and superscript $\{v_i\}$ indicates a deleted row and column of $\tilde{L}'$. Analogous to the Matrix-Tree theorem one then shows that the minor $\det(\tilde{L}'^{\{v_1\}}_{\{v_2\}})$ with the usual cofactor sign $(-1)^{v_1+v_2}$ and a product α_{E_Γ} of all edges is indeed the correct coefficient of $q_{v_1} \cdot q_{v_2}$ in the second Symanzik polynomial (see e.g. [109, theorem 2.1.3] for details). However, this will also become much more obvious below after we reinterpret the second Symanzik polynomial in section 2.3.2 and then again in section 5.3.

To finish the computation and find the parametric representation of a Feynman integral we now just need to collect all terms:

$$\phi_\Gamma(q,m) = \pi^{\frac{D}{2}} \delta^{(D)}(\sum q_v) \int_{\mathbb{R}_+^{|E_\Gamma|}} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{E_\Gamma} \frac{\exp\left(-\frac{\Phi_\Gamma}{\Psi_\Gamma} - \sum \alpha_e m_e^2 \right)}{\Psi_\Gamma^{D/2}} \tag{2.48}$$

The delta function is usually omitted in favour of manually imposing momentum conservation as discussed in example 2.1.5 and eq. (2.27). Note that sometimes one finds the mass terms incorporated into the second Symanzik polynomial. Whenever we do this we will explicitly use the notation

$$\bar{\Phi}_\Gamma \equiv \bar{\Phi}_\Gamma(\alpha, q, m) := \Phi_\Gamma(\alpha, q) + \Psi_\Gamma \sum_{e \in E_\Gamma} \alpha_e m_e^2. \tag{2.49}$$

With this last bit of notation, we have now reached the (unrenormalised) *scalar parametric integrand*

$$I_\Gamma(\alpha, q, m) = \frac{\exp\left(-\frac{\bar{\Phi}_\Gamma}{\Psi_\Gamma} \right)}{\Psi_\Gamma^{D/2}}. \tag{2.50}$$

2.2.2 Quantum electrodynamics

Using again the Feynman rules given in section 2.1.3, the massless momentum space integrand for quantum electrodynamics is

$$\begin{aligned}\tilde{I}_\Gamma(q,k) &= \bar{\gamma}_\Gamma \Big(\prod_{e\in E_\Gamma^{(f)}} \frac{k_e^{\mu_e}}{k_e^2} \Big) \Big(\prod_{e\in E_\Gamma^{(p)}} \frac{g^{\nu_u\nu_v} + \varepsilon \frac{k_e^{\nu_u} k_e^{\nu_v}}{k_e^2}}{k_e^2} \Big) \Big(\prod_{v\in V_\Gamma} \delta^{(D)}\Big(q_v + \sum_{e\in E_\Gamma} I_{ev} k_e\Big)\Big) \\ &= \bar{\gamma}_\Gamma \Big(\prod_{e\in E_\Gamma^{(f)}} k_e^{\mu_e} \Big) \Big(\prod_{e\in E_\Gamma^{(p)}} g^{\nu_u\nu_v} + \varepsilon \frac{k_e^{\nu_u} k_e^{\nu_v}}{k_e^2} \Big) \tilde{S}_\Gamma(q,k) \end{aligned} \tag{2.51}$$

where $\tilde{S}_\Gamma(q,k)$ is the scalar momentum space integrand of Γ, i.e. the scalar integrand $\tilde{I}_\Gamma(q,k)$ from eq. (2.37) for a scalar Feynman graph that one would get by replacing the fermion and photon edges of Γ with scalar edges. We relabel it $\tilde{S}_\Gamma(q,k)$ here, since it is only a part of the full QED integrand $\tilde{I}_\Gamma(q,k)$.

The Dirac matrices have been factored into the term $\bar{\gamma}_\Gamma$ to be treated separately in chapter 4. The numerator terms in eq. (2.51) contain the momenta k_e which prohibits the direct Gaußian integration used in section 2.2.1. Instead, we first generalise the Schwinger trick by introducing certain auxiliary variables and derivatives. For fermions it is quite simple,

$$\frac{k_e^{\mu_e}}{k_e^2} = k_e^{\mu_e} \int_0^\infty e^{-\alpha_e k_e^2} \mathrm{d}\alpha_e = -\int_0^\infty \lim_{\xi_e\to 0} \frac{1}{2\alpha_e} \frac{\partial}{\partial \xi_{e,\mu_e}} e^{-\alpha_e(k_e+\xi_e)^2} \mathrm{d}\alpha_e, \tag{2.52}$$

while the photon is slightly more complicated,

$$\begin{aligned}\frac{g^{\nu_u\nu_v} + \varepsilon \frac{k_e^{\nu_u} k_e^{\nu_v}}{k_e^2}}{k_e^2} &= \int_0^\infty \left(g^{\nu_u\nu_v} + \varepsilon \alpha_e k_e^{\nu_u} k_e^{\nu_v}\right) e^{-\alpha_e k_e^2} \mathrm{d}\alpha_e \\ &= \int_0^\infty \lim_{\xi_e\to 0} \left(\frac{2+\varepsilon}{2} g^{\mu_u\mu_v} + \frac{\varepsilon}{4\alpha_e} \frac{\partial^2}{\partial \xi_{e,\mu_u} \partial \xi_{e,\mu_v}} \right) e^{-\alpha_e(k_e+\xi_e)^2} \mathrm{d}\alpha_e, \end{aligned} \tag{2.53}$$

but the idea is the same in both cases. We also introduce the auxiliary momentum into the delta functions by replacing

$$\delta^{(D)}\Big(q_v + \sum_{e\in E_\Gamma} I_{ev} k_e\Big)\Big) \to \delta^{(D)}\Big(q_v + \sum_{e\in E_\Gamma} I_{ev} (k_e + \xi_e)\Big)\Big). \tag{2.54}$$

Now the derivation of the scalar parametric integrand mostly goes through as above. The momenta k_e are replaced everywhere by $k_e + \xi_e$, but the completion of the square and first Gaußian integration work without change. The external momenta are substituted by $q_v \to q_v + \sum_e I_{ev}\xi_e$, which leaves eq. (2.45) unchanged since $\sum_v I_{ev} = 0$. Hence, the second Gaußian integration also goes through as before and eq. (2.47) again yields the second Symanzik polynomial – but now all spanning 2-forests contribute, since all entries of Q' are non-zero due to $\sum_e I_{ev}\xi_e$, even if the corresponding q_v is zero (cf. examples 2.1.5 and 2.3.4). We then have found the scalar parametric integrand from eq. (2.50), except without masses and with additional auxiliary momenta. Moreover, from now on we will drop the external momenta q_v and only carry the ξ_e. This is more convenient in the notation and we will be able to recover them by sending ξ_e to q_i or 0 in a certain way (see section 2.3.2), rather than just sending them all to zero.

Overall, the parametric integrand for Γ is therefore

$$I_\Gamma(\alpha,\xi) = \bar{\gamma}_\Gamma \partial_\Gamma S_\Gamma(\alpha,\xi), \tag{2.55}$$

where $S_\Gamma(\alpha,\xi)$ is the scalar parametric integrand, and

$$\partial_\Gamma = \prod_{e\in E_\Gamma^{(f)}} \left(-\frac{1}{2\alpha_e}\frac{\partial}{\partial \xi_{e,\mu_e}}\right) \prod_{\substack{e\in E_\Gamma^{(p)}\\ \partial(e)=(u,v)}} \left(\frac{2+\varepsilon}{2} g^{\mu_u\mu_v} + \frac{\varepsilon}{4\alpha_e}\frac{\partial^2}{\partial \xi_{e,\mu_u}\partial \xi_{e,\mu_v}}\right). \tag{2.56}$$

2.2.3 General gauge theories: The corolla polynomial

The entirety of section 2.3 will be devoted to the question of how I_Γ in eq. (2.55) can be expressed explicitly, without any derivatives. Before we begin with that we briefly discuss what this generalised Schwinger trick and the differential ∂_Γ would look like in more complicated gauge theories other than QED. The two main references for everything mentioned here are [98] for the corolla polynomial and [96] for the application in gauge theories.

Let G be a 3-regular graph (like the QED Feynman graphs we discussed here so far). Instead of its edges consider half-edges, and assign to each a variable $a_{e,v}$ where e is the edge and v the vertex that half of the edge is incident to. Then define for each vertex

$$D_v := \sum_{e\in\bar{\partial}^{-1}(v)} a_{e,v} \tag{2.57}$$

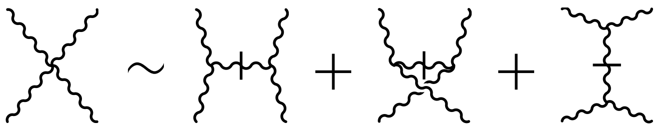
where $\bar{\partial}^{-1}(v) \subset E_G$ is the set of all edges incident to v, i.e. all $e \in E_G$ with $\partial(e) = (v,\bullet)$ or $(\bullet, v)$. Let $\mathcal{C}_G^{[i]}$ be the set of cycles with i edge-connected components, i.e. pairwise disjoint unions of i simple cycles $C \in \mathcal{C}_G^{[1]}$ as defined in section 2.1.1, and $\mathcal{C}_G = \bigcup_i \mathcal{C}_G^{[i]}$. Since G is 3-regular each vertex in a cycle has a unique edge incident to it that is not in that cycle. Call this edge $\tilde{e}(v,C)$. Then the *Corolla polynomial* is

$$C(G) := \sum_{C\in\mathcal{C}_G} (-1)^{h_1(C)} \Big(\prod_{v\in V_C} a_{\tilde{e}(v,C),v} \Big) \Big(\prod_{v\in V_G\setminus V_C} D_v \Big). \tag{2.58}$$

The half-edge variables are then replaced by certain differential operators w.r.t. auxiliary momenta ξ_e, analogous to the generalised Schwinger trick we used above. For example, for a fermion edge e_1, incident to a vertex v, together with two other edges e_2 and e_3, one has

$$a_{e_1,v} \to \gamma_{\mu_2}\gamma_{\nu_v} \frac{1}{\alpha_2} \frac{\partial}{\partial \xi_{2,\mu_1}} - \gamma_{\nu_v}\gamma_{\mu_3} \frac{1}{\alpha_3} \frac{\partial}{\partial \xi_{3,\mu_1}}. \tag{2.59}$$

Note that a fixed cyclic ordering of the edges e_1, e_2, e_3 around the vertex v needs to be observed to get the signs correct. For the case of QED these terms indeed reduce to the simpler derivatives in eq. (2.56) that we use here [96, example 6.15], but the corolla polynomial with this type of differential operator replacing its variables can be used to express the integrands of Feynman integrals from any gauge theory, and even the full standard model [113]. This includes theories whose Feynman graphs are not 3-regular, like QCD with its four-gluon vertex. However, these types of graphs can be decomposed into sums of 3-regular graphs, by IHX-type relations like [96, section 4, eq. (8)]:



Moreover, since the corolla polynomial, which is defined as a sum over cycles, is quite similar to the polynomial we will use in the next section to simplify the QED integrand it seems reasonable to believe that the approach of this thesis will be readily generalisable to other theories.

2.3 The QED integrand without derivatives

In this section we will rewrite eq. (2.55) by computing all derivatives in $\partial_\Gamma S_\Gamma$ and analysing the structure of the resulting polynomials. The following computations rely heavily on the properties of graph polynomials discussed in section 2.1.1. Moreover, we need two additional graph polynomials that are based on cycle and bond subgraphs (as opposed to spanning trees and forests in the cases of the Kirchhoff and Symanzik polynomials) and it will be our first task to define and study them. It then turns out that the rewriting of $I_\Gamma(\alpha, \xi)$ in theorem 2.3.9 follows almost immediately from application of the lemmata 2.3.6, 2.3.7 and 2.3.8.

2.3.1 Cycle and bond polynomials: Definition

First let us define two more notions of edge orientation, relative to a cycle and a bond. Each edge already has a fixed orientation, given by the order of the pair of vertices $\partial(e) = (\partial_-(e), \partial_+(e))$. Now we also fix an arbitrary orientation for all simple cycles $C \in \mathcal{C}_G^{[1]}$ by specifying a direction in which to traverse it. Then the relative orientation of an edge e with respect to a simple cycle C is

$$\mathrm{o}_C(e) \coloneqq \begin{cases} +1 & \text{if } e \text{ is traversed along its direction,} \\ -1 & \text{if } e \text{ is traversed opposite its direction,} \\ \ \ 0 & \text{if } e \notin C. \end{cases} \tag{2.60}$$

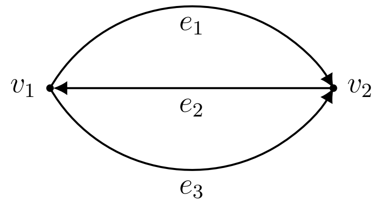
While $o_C(e)$ clearly depends on the initial choice of orientation for C, a product of two orientations $\mathrm{o}_C(e_1) o_C(e_2) \eqqcolon \mathrm{o}_C(e_1, e_2)$ does not. Only such products will appear in the integrands.

Similarly one can define the orientation of a bond by designating the two connected components in its complement as positive and negative, $G \setminus B = G_+ \sqcup G_-$. Then

$$\mathrm{o}_B(e) \coloneqq \begin{cases} +1 & \text{if } \partial_\pm(e) \in G_\pm, \\ -1 & \text{if } \partial_\pm(e) \in G_\mp, \\ \ \ 0 & \text{if } e \notin B. \end{cases} \tag{2.61}$$

As in the cycle case, we abbreviate the product $\mathrm{o}_B(e_1, e_2) \coloneqq \mathrm{o}_B(e_1) o_C(e_2)$, which is independent of the initial choice of G_+ and G_-.

Example 2.3.1. *Consider the three edge banana graph*



from fig. 2.1, previously discussed in example 2.1.1. Choose the bond orientation such that $G_- = (\emptyset, \{v_1\})$ *and* $G_+ = (\emptyset, \{v_2\})$*. Then* $\mathrm{o}_{B_1}(e_1) = +1 = \mathrm{o}_{B_1}(e_3)$*, while* $\mathrm{o}_{B_1}(e_2) = -1$*. There are three cycles in G, each given by a pair of edges:*

$$C_1 = \{e_1, e_2\} \qquad C_2 = \{e_1, e_3\} \qquad C_3 = \{e_2, e_3\}$$

Choose an orientation for the cycles, say clockwise. Then the relative orientations of each edge with respect to each cycle are

$$\begin{array}{lll} \mathrm{o}_{C_1}(e_1) = +1 & \mathrm{o}_{C_1}(e_2) = +1 & \mathrm{o}_{C_1}(e_3) = 0 \\ \mathrm{o}_{C_2}(e_1) = +1 & \mathrm{o}_{C_2}(e_2) = 0 & \mathrm{o}_{C_2}(e_3) = -1 \\ \mathrm{o}_{C_3}(e_1) = 0 & \mathrm{o}_{C_3}(e_2) = -1 & \mathrm{o}_{C_3}(e_3) = -1 \end{array}$$

Definition 2.3.2. (Cycle polynomials)
Let G be a connected graph, $\mathcal{C}_G^{[1]}$ *its set of simple cycles and* $\xi = (\xi_1, \ldots, \xi_{|E_\Gamma|})$ *a tuple of formal variables assigned to each edge. Then the cycle polynomial of G is*

$$\chi_G(\alpha, \xi) := \sum_{C \in \mathcal{C}_G^{[1]}} \left(\sum_{e \in C} \mathrm{o}_C(e) \xi_e \right)^2 \Psi_{G/\!/C}(\alpha). \tag{2.62}$$

Furthermore, for any two (not necessarily distinct) edges $e_i, e_j \in E_G$ *define*

$$\chi_G^{(e_i|e_j)}(\alpha) := \sum_{C \in \mathcal{C}_G^{[1]}} \mathrm{o}_C(e_i, e_j) \Psi_{G/\!/C}(\alpha). \tag{2.63}$$

Definition 2.3.3. (Bond polynomials)
Let G be a connected graph, $\mathcal{B}_G$ *its set of bonds and* $\xi = (\xi_1, \ldots, \xi_{|E_\Gamma|})$ *a tuple of formal variables assigned to each edge. Then the bond polynomial of G is*

$$\beta_G(\alpha, \xi) := \sum_{B \in \mathcal{B}_G} \left(\sum_{e \in B} \mathrm{o}_B(e) \xi_e \right)^2 \alpha_B \Psi_{G \setminus B}(\alpha). \tag{2.64}$$

Furthermore, for any two (not necessarily distinct) edges $e_i, e_j \in E_G$ *define*

$$\beta_G^{(e_i|e_j)}(\alpha) := \sum_{B \in \mathcal{B}_G} \mathrm{o}_B(e_i, e_j) \alpha_B \Psi_{G \setminus B}(\alpha). \tag{2.65}$$

The definitions extend to disconnected graphs $G = \sqcup_i G_i$ similarly to the Kirchhoff and Symanzik polynomials in equations (2.8) and (2.21). The bond and cycle sets are simply the union of the sets for each component and for a cycle or bond in the i-th component one has

$$\Psi_{G_i /\!/ C}(\alpha) \prod_{j \neq i} \Psi_{G_j} \qquad \text{or} \qquad \alpha_B \Psi_{G_i \setminus B}(\alpha) \prod_{j \neq i} \Psi_{G_j}$$

in the summand.

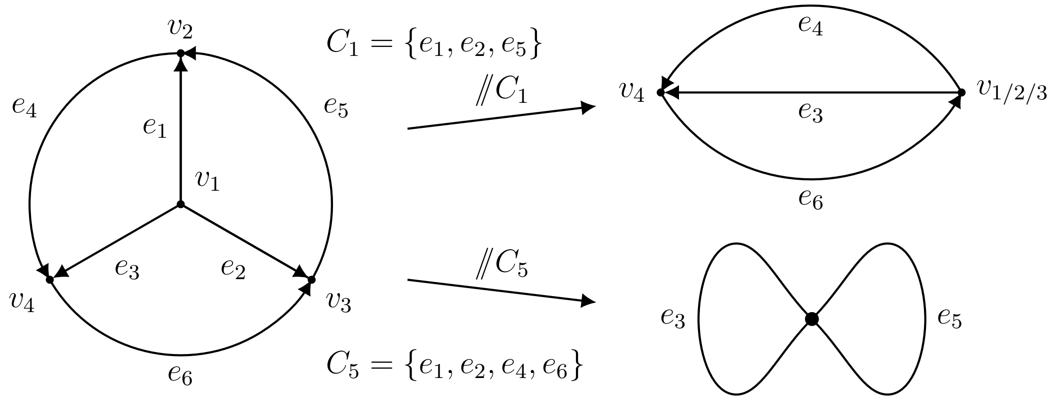


Figure 2.4: The *wheel with 3 spokes* graph WS_3 and the contraction of two of its cycles.

Examples

Let $G = WS_3$ be the wheel with three spokes as depicted in fig. 2.4. It contains seven cycles, all of which are simple:

$$C_1 = \{e_1, e_2, e_5\} \qquad C_2 = \{e_1, e_3, e_4\} \qquad C_3 = \{e_2, e_3, e_6\} \qquad C_4 = \{e_4, e_5, e_6\}$$

$$C_5 = \{e_1, e_2, e_4, e_6\} \qquad C_6 = \{e_1, e_3, e_5, e_6\} \qquad C_7 = \{e_2, e_3, e_4, e_5\}$$

Contracting the 3-edge cycles C_1, C_2, C_3, C_4 results in a 3-edge banana graph, while contraction of the 4-edge cycles returns a rose with the two remaining edges. Hence,

$$\Psi_{G/\!/C_1}(\alpha) = \alpha_3\alpha_4 + \alpha_3\alpha_6 + \alpha_4\alpha_6 \qquad \Psi_{G/\!/C_2}(\alpha) = \alpha_2\alpha_5 + \alpha_2\alpha_6 + \alpha_5\alpha_6$$

$$\Psi_{G/\!/C_3}(\alpha) = \alpha_1\alpha_4 + \alpha_1\alpha_5 + \alpha_4\alpha_5 \qquad \Psi_{G/\!/C_4}(\alpha) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$

$$\Psi_{G/\!/C_5}(\alpha) = \alpha_3\alpha_5 \qquad \Psi_{G/\!/C_6}(\alpha) = \alpha_2\alpha_4 \qquad \Psi_{G/\!/C_7}(\alpha) = \alpha_1\alpha_6$$

and the full cycle polynomial is

$$\begin{aligned}\chi_G(\alpha,\xi) = \\ &(\xi_1-\xi_2-\xi_5)^2(\alpha_3\alpha_4+\alpha_3\alpha_6+\alpha_4\alpha_6)+(\xi_1-\xi_3+\xi_4)^2(\alpha_2\alpha_5+\alpha_2\alpha_6+\alpha_5\alpha_6)\\ &+(\xi_2-\xi_3-\xi_6)^2(\alpha_1\alpha_4+\alpha_1\alpha_5+\alpha_4\alpha_5)+(\xi_4+\xi_5+\xi_6)^2(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\\ &+(\xi_1-\xi_2+\xi_4+\xi_6)^2\alpha_3\alpha_5+(\xi_1-\xi_3-\xi_5-\xi_6)^2\alpha_2\alpha_4+(\xi_2-\xi_3+\xi_4+\xi_5)^2\alpha_1\alpha_6.\end{aligned} \tag{2.66}$$

The restricted version, say for $e_i = e_j = e_1$ contains only the terms corresponding to cycles that contain e_1, so C_1, C_2, C_5 and C_6. Hence,

$$\chi_G^{(e_1|e_1)}(\alpha) = (\alpha_2+\alpha_3)(\alpha_4+\alpha_5)+\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5). \tag{2.67}$$

Finally, consider two edges, say e_1, e_2. The only cycles that contain both are C_1 and C_5, and e_1, e_2 have opposing directions in both those cycles such that $\mathrm{o}_{C_1}(e_1,e_2) = -1 = \mathrm{o}_{C_5}(e_1,e_2)$. Hence

$$\chi_G^{(e_1|e_2)}(\alpha) = -\alpha_3(\alpha_4+\alpha_5+\alpha_6)-\alpha_4\alpha_6. \tag{2.68}$$

For a different example consider e_1, e_6. There are again two cycles containing this pair, C_5 and C_6. However, this time $\mathrm{o}_{C_5}(e_1,e_6) = +1$ and $\mathrm{o}_{C_6}(e_1,e_6) = -1$, so

$$\chi_G^{(e_1|e_6)}(\alpha) = \alpha_3\alpha_5-\alpha_2\alpha_4. \tag{2.69}$$

Consider the bonds of the same graph. They are

$$B_1 = \{e_1,e_2,e_3\} \qquad B_2 = \{e_1,e_4,e_5\} \qquad B_3 = \{e_2,e_5,e_6\} \qquad B_4 = \{e_3,e_4,e_6\}$$

$$B_5 = \{e_1,e_2,e_4,e_6\} \qquad B_6 = \{e_1,e_3,e_5,e_6\} \qquad B_7 = \{e_2,e_3,e_4,e_5\}.$$

Note that the bonds consisting of three edges differ from the cycles but B_5, B_6 and B_7 are simultaneously cycles and bonds. Removing any of the first four bonds leaves an isolated vertex (with $\Psi_{G_1} = 1$) and a cycle on three edges (with $\Psi_{G_2} = \alpha_i+\alpha_j+\alpha_k$) in the two components. The other three bonds leave the graph with two single edges (i.e. trees with $\Psi_{G_1} = \Psi_{G_2} = 1$) in both components. The full bond polynomial is therefore

$$\begin{aligned}\beta_G(\alpha,\xi) = \\ &(\xi_1+\xi_2+\xi_3)^2\alpha_1\alpha_2\alpha_3(\alpha_4+\alpha_5+\alpha_6)+(\xi_1-\xi_4+\xi_5)^2\alpha_1\alpha_4\alpha_5(\alpha_2+\alpha_3+\alpha_6)\\ &+(\xi_2-\xi_5+\xi_6)^2\alpha_2\alpha_5\alpha_6(\alpha_1+\alpha_3+\alpha_4)+(\xi_3+\xi_4-\xi_6)^2\alpha_3\alpha_4\alpha_6(\alpha_1+\alpha_2+\alpha_5)\\ &+(\xi_1+\xi_2-\xi_4+\xi_6)^2\alpha_1\alpha_2\alpha_4\alpha_6+(\xi_1+\xi_3+\xi_5-\xi_6)^2\alpha_1\alpha_3\alpha_5\alpha_6\\ &+(\xi_2+\xi_3+\xi_4-\xi_5)^2\alpha_2\alpha_3\alpha_4\alpha_5.\end{aligned} \tag{2.70}$$

Here it should be noted that possibly $o_B(e) \neq o_C(e)$, even if B and C happen to be the same edge subset.

2.3.2 Bond polynomials and the second Symanzik

By noting that the pairs of spanning trees in the two connected components of $G \setminus B$ precisely correspond to the two components of spanning 2-forests of G one finds that the bond polynomial and the second Symanzik polynomial are closely related. In fact, if we take the second Symanzik polynomial $\Phi_\Gamma(\alpha, \xi)$ contained in the scalar part $S_\Gamma(\alpha, \xi)$ of the integrand in eq. (2.55), then that is exactly the bond polynomial β_Γ.

The second Symanzik polynomial is traditionally only defined for Feynman graphs with external momenta, but the bond polynomial can be defined for any graph, so we should try to clearly differentiate between the two cases of $\Phi_\Gamma(\alpha, q)$ and $\beta_\Gamma \equiv \Phi_\Gamma(\alpha, \xi)$. In this sense, the second Symanzik is the evaluation of the bond polynomial with the following evaluation map:

1. Choose $n + 1$ vertices to be external and call one of them v_0.
2. For each of the n other external vertices choose an arbitrary[7] directed path $P_{v,v_0} \subset E_\Gamma$ from from v to v_0.
3. Then evaluate all auxiliary momenta:

$$\xi_e \to \sum_{v \in V_\Gamma^{ext} \setminus \{v_0\}} o_{P_{v,v_0}}(e) q_v \tag{2.71}$$

where the sign $o_{P_v}(e) = \pm 1$ if the edge orientation of e is aligned with or opposite that of the path, and vanishes if $e \notin P_{v,v_0}$.

Example 2.3.4. *The 3-edge banana from fig. 2.1 has the bond polynomial*

$$\beta_G(\alpha, \xi) = (\xi_1 - \xi_2 + \xi_3)^2 \alpha_1 \alpha_2 \alpha_3. \tag{2.72}$$

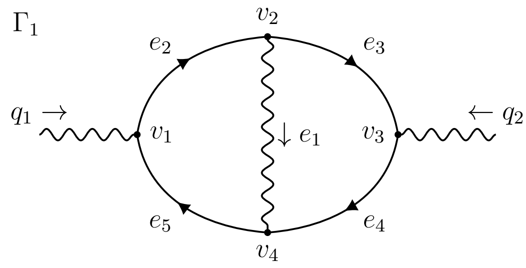
One can choose any single edge as the path between the two external vertices. Replacing the corresponding ξ_e by q and setting the other two to 0 one gets

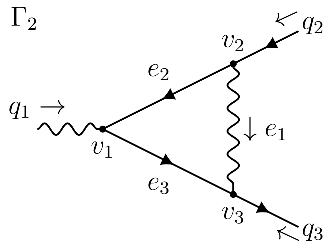
$$\Phi_G(\alpha, q) = q^2 \alpha_1 \alpha_2 \alpha_3, \tag{2.73}$$

which we are very familiar with by now.

For more complicated examples we go again to the Feynman graphs Γ_1 and Γ_2 from fig. 2.3, whose second Symanziks we computed in example 2.1.5.

[7] The path independence of this is essentially Kirchhoff's voltage law, with momenta replacing voltages [88].





Their bond polynomials are:

$$\begin{aligned}\Phi_{\Gamma_1}(\alpha,\xi) = {} & (\xi_2-\xi_5)^2\alpha_2\alpha_5(\alpha_1+\alpha_3+\alpha_4)+(\xi_3-\xi_4)^2\alpha_3\alpha_4(\alpha_1+\alpha_2+\alpha_5)\\ & +(\xi_1-\xi_2+\xi_4)^2\alpha_1\alpha_2\alpha_4+(\xi_1+\xi_3-\xi_5)^2\alpha_1\alpha_3\alpha_5\\ & +(\xi_1-\xi_2+\xi_3)^2\alpha_1\alpha_2\alpha_3+(\xi_1+\xi_4-\xi_5)^2\alpha_1\alpha_4\alpha_5\end{aligned} \tag{2.74}$$

$$\Phi_{\Gamma_2}(\alpha,\xi) = (\xi_1+\xi_2)^2\alpha_1\alpha_2+(\xi_1+\xi_3)^2\alpha_1\alpha_3+(\xi_2-\xi_3)^2\alpha_2\alpha_3 \tag{2.75}$$

For Γ_1 we need to choose any path between the vertices v_1 and v_3, say $\{e_2, e_3\}$, directed from v_1 to v_3. The external momenta entering these two vertices are q_1 and q_2, and by momentum conservation we know $-q_2 = q_1 \equiv q$. Consequently we have to evaluate $\xi_2, \xi_3 = q$ and $\xi_1, \xi_4, \xi_5 = 0$ and find

$$\begin{aligned}\Phi_{\Gamma_1}(\alpha,q) = {} & (q-0)^2\alpha_2\alpha_5(\alpha_1+\alpha_3+\alpha_4)+(q-0)^2\alpha_3\alpha_4(\alpha_1+\alpha_2+\alpha_5)\\ & +(0-q+0)^2\alpha_1\alpha_2\alpha_4+(0+q-0)^2\alpha_1\alpha_3\alpha_5\\ & +(0-q+q)^2\alpha_1\alpha_2\alpha_3+(0+0-0)^2\alpha_1\alpha_4\alpha_5\\ = {} & q^2\Big(\alpha_2\alpha_5(\alpha_1+\alpha_3+\alpha_4)+\alpha_3\alpha_4(\alpha_1+\alpha_2+\alpha_5)+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_5\Big),\end{aligned} \tag{2.76}$$

which is exactly the second Symanzik as computed in eq. (2.26).

For Γ_2 one has to consider two different paths. Choose v_1 as the vertex towards which we direct the paths starting in v_2 and v_3. To keep things simple choose the single edges e_2 and e_3 for these paths. Then we have to evaluate $\xi_1 \to 0$, $\xi_2 \to q_2$ and $\xi_3 \to -q_3$. We find

$$\begin{aligned}\Phi_{\Gamma_2}(\alpha,q_1,q_2) &= (0+q_2)^2\alpha_1\alpha_2+(0-q_3)^2\alpha_1\alpha_3+(q_2+q_3)^2\alpha_2\alpha_3\\ &= q_2^2\alpha_1\alpha_2+q_3^2\alpha_1\alpha_3+(q_2+q_3)^2\alpha_2\alpha_3.\end{aligned} \tag{2.77}$$

which is the same polynomial we found in eq. (2.27).

2.3.3 Cycle and bond polynomials: Properties

Being essentially the second Symanzik polynomial, $\beta_G(\alpha,\xi)$ has all its well known properties. Similarly, the cycle polynomials $\chi_G(\alpha,\xi)$ inherit linearity in each α_e as well as homogeneity of degree $h_1(G)-1$ from the Kirchhoff polynomial. It also satisfies the usual contraction-deletion relations, which we prove in

Proposition 2.3.5. *Let G be a connected graph. Then for every $e \in E_G$*

$$\chi_{G\backslash e}(\alpha,\xi) = \frac{\partial}{\partial \alpha_e}\,\chi_G(\alpha,\xi)\big|_{\xi_e=0}\,, \tag{2.78}$$

and for every $e \in E_G$ except self-cycles

$$\chi_{G/\!\!/e}(\alpha,\xi) = \chi_G(\alpha,\xi)\big|_{\alpha_e,\xi_e=0}\,. \tag{2.79}$$

Proof. Firstly, the cycle set of G can be separated into two disjoint subsets, depending on whether or not the cycles contain a given edge e and $\mathcal{C}^{[1]}_{G\backslash e} = \{C \in \mathcal{C}^{[1]}_G \mid e \notin C\}$. Secondly, if a cycle C contains an edge e, then $\Psi_{G/\!\!/C}$ is independent of α_e and the derivative w.r.t. α_e vanishes. For the cycles not containing C we can employ the contraction-deletion relation for the Kirchhoff polynomial and commutativity of contracting and deleting to find

$$\begin{aligned}
\frac{\partial}{\partial \alpha_e}\,\chi_G(\alpha,\xi)\big|_{\xi_e=0} &= \sum_{C\in\mathcal{C}^{[1]}_G}\left(\sum_{\substack{e'\in C\\ e'\neq e}} \mathrm{o}_C(e')\xi'_e\right)^2 \frac{\partial}{\partial\alpha_e}\Psi_{G/\!\!/C}(\alpha)\\
&= \sum_{\substack{C\in\mathcal{C}^{[1]}_G\\ e\notin C}}\left(\sum_{\substack{e'\in C\\ e'\neq e}} \mathrm{o}_C(e')\xi'_e\right)^2 \Psi_{(G/\!\!/C)\backslash e}(\alpha)\\
&= \sum_{C\in\mathcal{C}^{[1]}_{G\backslash e}}\left(\sum_{e'\in C} \mathrm{o}_C(e')\xi'_e\right)^2 \Psi_{(G\backslash e)/\!\!/C}(\alpha) = \chi_{G\backslash e}(\alpha,\xi).
\end{aligned} \tag{2.80}$$

For the contraction we need to consider three different cases. let $C \in \mathcal{C}^{[1]}_G$ and $e \in E_G$ any edge that is not a self-cycle, $\partial_+(e) \neq \partial_-(e)$. If $e \in C$ then one observes that after contraction $C/\!\!/e$ is still a cycle of $G/\!\!/e$ and their corresponding Kirchhoff polynomials are unchanged since they are independent of α_e. If $e \notin C$, but both endpoints of e lie in C, then two things happen. On the one hand, contracting e joins four edges of C in one vertex such that C is not a simple cycle of $G /\!\!/ e$ anymore. On the other hand, contracting C turns e into a self-cycle, such that $\Psi_{G/\!\!/C}(\alpha)$ vanishes when evaluating at $\alpha_e = 0$. Therefore, these terms vanish on both sides of the contraction relation. Finally, when $e \notin C$ and at least one of its endpoints is not incident to a vertex of C, then C remains a cycle in $G /\!\!/ e$ and

$$\Psi_{G/\!\!/C}(\alpha)\Big|_{\alpha_e=0} = \Psi_{(G/\!\!/C)/\!\!/e}(\alpha) = \Psi_{(G/\!\!/e)/\!\!/C}(\alpha). \tag{2.81}$$

□

One finds that the same relations hold for $\chi_G^{(e_1|e_2)}(\alpha)$, as long as $e \neq e_1, e_2$, by simply restricting the discussion in the proof to the subsets of cycles that contain e_1 and e_2.

Identities

Additionally, there are several identities relating cycle, bond and Kirchhoff polynomials to each other. Specifically, in the following we will prove:

lemma 2.3.6: $\chi_G^{(e|e)} = \Psi_{G\setminus e} = \frac{\partial}{\partial\alpha_e}\Psi_G$

lemma 2.3.7: $\beta_G^{(e|e)} = \alpha_e \Psi_{G/\!\!/e} = \alpha_e \; \Psi_G\big|_{\alpha_e=0}$

lemma 2.3.8: $\beta_G^{(e|e')} = -\alpha_e \alpha_{e'} \chi_G^{(e|e')}$ for $e \neq e'$

Not only do these lemmata already contain most of the work needed to prove theorem 2.3.9 in the following section, they also yield two interesting expressions for the Kirchhoff polynomial:

$$\Psi_G = \alpha_e^{-1} \beta_G^{(e|e)} + \alpha_e \chi_G^{(e|e)} \tag{2.82}$$

$$\Psi_G = \frac{1}{h_1(G)} \sum_{C \in \mathcal{C}_G^{[1]}} \Psi_C \Psi_{G/\!\!/C} \tag{2.83}$$

For the former one simply uses the first two lemmata to replace the Kirchhoff polynomials in eq. (2.5) while one invokes Euler's homogenous function theorem and the first lemma to write

$$\Psi_G = \frac{1}{h_1(G)} \sum_{e \in E_G} \alpha_e \frac{\partial}{\partial \alpha_e} \Psi_G = \frac{1}{h_1(G)} \sum_{e \in E_G} \alpha_e \chi_G^{(e|e)}, \tag{2.84}$$

such that one only needs to exchange summations and note that $\Psi_C = \sum_{e \in C} \alpha_e$ to prove the latter.

Lemma 2.3.6. *Let G be a connected graph and $e \in E_G$ any of its edges that is not a bridge*[8]*. Then*

$$\chi_G^{(e|e)} = \Psi_{G \setminus e}. \tag{2.85}$$

Proof. The left hand side written out in full is

$$\chi_G^{(e|e)} = \sum_{\substack{C \in \mathcal{C}_G^{[1]} \\ C \ni e}} \Psi_{G /\!\!/ C} = \sum_{\substack{C \in \mathcal{C}_G^{[1]} \\ C \ni e}} \sum_{T \in \mathcal{T}_{G /\!\!/ C}^{[1]}} \alpha_{E_G \setminus (C \cup T)}. \tag{2.86}$$

By definition all cycles summed over here contain e, and contracting a cycle C is the same as contracting all but one of its edges and deleting the last edge (which is only a self-cycle after contraction of all the others and thus not contained in any spanning trees). Hence, we can replace the sum over cycles by a sum over paths P between vertices v, w in $G \setminus e$, where $(v, w) = \partial(e)$, i.e.

$$\chi_G^{(e|e)} = \sum_{P \in \mathcal{P}_{G \setminus e}^{v,w}} \Psi_{(G \setminus e) /\!\!/ P}. \tag{2.87}$$

Two observations suffice to finish the proof. Firstly, a spanning tree remains a spanning tree after contraction of any number of its edges, i.e. for a connected graph H one of its spanning trees T and any edge subset $T' \subseteq T$ it follows that $T /\!\!/ T'$ is also a spanning tree of $H /\!\!/ T'$. Secondly, every spanning tree contains a unique path between any two vertices of the graph it spans. Therefore

$$\mathcal{T}_{(G \setminus e) /\!\!/ P}^{[1]} \simeq \{T \in \mathcal{T}_{G \setminus e}^{[1]} \mid T \supset P\} \tag{2.88}$$

and

$$\bigcup_{P \in \mathcal{P}_{G \setminus e}^{v,w}} \mathcal{T}_{(G \setminus e) /\!\!/ P}^{[1]} \simeq \mathcal{T}_{G \setminus e}^{[1]}, \qquad \bigcap_{P \in \mathcal{P}_{G \setminus e}^{v,w}} \mathcal{T}_{(G \setminus e) /\!\!/ P}^{[1]} = \emptyset. \tag{2.89}$$

Finally, for the monomial it is irrelevant whether an edge is contained in the spanning tree or was contracted – each monomial contains variables associated to edges that are still in the graph and not in the spanning tree. In other words, for spanning trees $T_1 \in \mathcal{T}_{(G \setminus e) /\!\!/ P}^{[1]}$ and $T_2 \in \mathcal{T}_{G \setminus e}^{[1]}$ with $P \subseteq T_2$ one has $\alpha_{E_G \setminus (e \cup P \cup T_1)} = \alpha_{E_G \setminus (e \cup T_2)}$ such that

$$\chi_G^{(e|e)} = \sum_{P \in \mathcal{P}_{G \setminus e}^{v,w}} \sum_{T \in \mathcal{T}_{(G \setminus e) /\!\!/ P}^{[1]}} \alpha_{E_G \setminus (e \cup P \cup T)} = \sum_{T \in \mathcal{T}_{G \setminus e}^{[1]}} \alpha_{E_G \setminus (e \cup T)} = \Psi_{G \setminus e}. \tag{2.90}$$

□

[8]If it were a bridge then no cycle could contain it such that $\chi_G^{(e|e)}(\alpha) = 0 = \frac{\partial}{\partial \alpha_e} \Psi_G(\alpha)$. But since we explicitly defined the Kirchhoff polynomial for disconnected graphs as a product over the connected components, such that $\Psi_{G \setminus e}(\alpha) \neq 0$ even for bridges e, this case has to be excluded here.

Lemma 2.3.7. *Let G be a connected graph and $e \in E_G$ any of its edges that is not a self-cycle. Then*

$$\beta_G^{(e|e)} = \alpha_e \Psi_{G/\!/e}. \tag{2.91}$$

Proof. Let $B \in \mathcal{B}_G$ be a bond that contains e and $G'(B)$ the graph obtained from $G \setminus B = G_1 \sqcup G_2$ by identifying the vertices that e is incident to, i.e.

$$G'(B) = (G \setminus (B \setminus e)) /\!/ e. \tag{2.92}$$

Then $\mathcal{T}^{[1]}_{G\setminus B} = \mathcal{T}^{[1]}_{G_1} \times \mathcal{T}^{[1]}_{G_2} \simeq \mathcal{T}^{[1]}_{G'(B)}$ and $\Psi_{G\setminus B} = \Psi_{G'(B)}$. Moreover, $\mathcal{T}^{[1]}_{G'(B)} \subset \mathcal{T}^{[1]}_{G/\!/e}$. Consider any spanning tree $T \in \mathcal{T}^{[1]}_{G/\!/e}$. The edges of T form a spanning 2-forest in G and their complement in G contains exactly one of the bonds $B \in \mathcal{B}_G$, given by the edges that are bridges between the two connected components. Therefore,

$$\bigcup_{\substack{B\in\mathcal{B}_G\\ B\ni e}} \mathcal{T}^{[1]}_{G'(B)} = \mathcal{T}^{[1]}_{G/\!/e} \quad \text{and} \quad \bigcap_{\substack{B\in\mathcal{B}_G\\ B\ni e}} \mathcal{T}^{[1]}_{G'(B)} = \emptyset. \tag{2.93}$$

Moreover, for each monomial one sees that

$$\alpha_B \alpha_{E_G\setminus(T\cup B)} = \alpha_{E_G\setminus T} = \alpha_e \alpha_{E_G\setminus(e\cup T)} \tag{2.94}$$

from which we conclude that

$$\beta_G^{(e|e)} = \sum_{\substack{B\in\mathcal{B}_G\\ B\ni e}} \alpha_B \sum_{T\in\mathcal{T}^{[1]}_{G'(B)}} \alpha_{E_G\setminus(T\cup B)} = \sum_{T\in\mathcal{T}^{[1]}_{G/\!/e}} \alpha_{E_G\setminus T} = \alpha_e \Psi_{G/\!/e}. \tag{2.95}$$

□

Lemma 2.3.8. *Let G be a connected graph and $e, e' \in E_G$ two distinct edges. Then*

$$\beta_G^{(e|e')} = -\alpha_e \alpha_{e'} \chi_G^{(e|e')}. \tag{2.96}$$

Proof. First, consider some special cases. If e or e' is a bridge then there is no cycle that contains it and any bond that contains it cannot contain the other edge, since bonds are minimal subgraphs and removing a bridge already disconnects the graph. Hence, both sides of eq. (2.96) vanish. Analogously, if either edge is a self-cycle then there is no bond that contains it and the cycle that contains it cannot contain the other edge such that again both sides vanish. For the remainder of the proof assume that both e and e' are neither a bridge nor a self-cycle.

By definition

$$\beta_G^{(e|e')} = \sum_{B\in\mathcal{B}_G} \mathrm{o}_B(e,e')\alpha_B \Psi_{G\setminus B}. \tag{2.97}$$

Consider the two connected components G_1, G_2 of $G \setminus B$ and remember that $\Psi_{G\setminus B}(\alpha) = \Psi_{G_1}(\alpha)\Psi_{G_2}(\alpha)$. In each component let $v_i, v_i' \in V_{G_i}$ be the two (not necessarily distinct) vertices that e and e' are incident to respectively. Following similar arguments as in the proof of lemma 2.3.6 above we can decompose the Kirchhoff polynomial as a sum

$$\Psi_{G_i} = \sum_{P_i} \Psi_{G_i/\!\!/P_i}. \tag{2.98}$$

over all paths $P_i \in \mathcal{P}_{G_i}^{v_i,v_i'}$ between these two vertices. After contraction of any two paths P_1, P_2 the edges e and e' start and end in the same two vertices, such that we get another isomorphism on spanning trees.

$$\mathcal{T}^{[1]}_{G_1/\!\!/P_1} \times \mathcal{T}^{[1]}_{G_2/\!\!/P_2} \simeq \mathcal{T}^{[1]}_{(G\setminus B')/\!\!/C_0} \tag{2.99}$$

where $B' = B \setminus \{e,e'\}$ and C_0 is the simple cycle formed by the two paths and e, e'. Every simple cycle of $G \setminus B'$ that contains e and e' can be constructed by combining the two edges with any pair of paths P_1, P_2, i.e.

$$\{P_1\} \oplus \{P_2\} \oplus e \oplus e' \simeq \{C \in \mathcal{C}^{[1]}_{G\setminus B'} \mid C \supseteq \{e,e'\}\} = \{C \in \mathcal{C}^{[1]}_G \,|C \cap B = \{e,e'\}\}.$$

On the level of the Kirchhoff polynomial we can therefore write

$$\Psi_{G\setminus B} = \sum_{P_1}\sum_{P_2} \Psi_{G_1/\!\!/P_1}\Psi_{G_2/\!\!/P_2} = \sum_{\substack{C\in\mathcal{C}^{[1]}_G \\ C\cap B=\{e,e'\}}} \Psi_{(G\setminus B')/\!\!/C}. \tag{2.100}$$

We also see from this construction that for any bond B and simple cycle C that intersect exactly in these two edges the relative orientations of e, e' are related via $\mathrm{o}_B(e,e') = -\mathrm{o}_C(e,e')$. Furthermore, contraction and deletion of edge subsets commute as long as the contracted and deleted set do not intersect. We can therefore exchange summation over bonds and cycles as follows:

$$\begin{aligned}\sum_{B\in\mathcal{B}_G} \mathrm{o}_B(e,e')\alpha_B\Psi_{G\setminus B}(\alpha) &= \sum_{B\in\mathcal{B}_G} \mathrm{o}_B(e,e')\alpha_B \sum_{\substack{C\in\mathcal{C}^{[1]}_G \\ C\cap B=\{e,e'\}}} \Psi_{(G\setminus B')/\!\!/C}(\alpha) \\ &= -\sum_{C\in\mathcal{C}^{[1]}_G} \mathrm{o}_C(e,e') \sum_{\substack{B\in\mathcal{B}_G \\ B\ni e,e'}} \alpha_B\Psi_{(G/\!\!/C)\setminus B'}\end{aligned} \tag{2.101}$$

Note also that in the sum over bonds we can weaken the requirement $C \cap B = \{e, e'\}$ to $e, e' \in B$ since removing edges that have previously been contracted would yield a vanishing Kirchhoff polynomial and thus give no contribution anyway. Let now G' be the graph G in which the edges $C \setminus \{e, e'\}$ have been contracted and e' deleted. Then $G /\!\!/ C = G' /\!\!/ e$ and we can apply lemma 2.3.7 to G' to get

$$\sum_{B \in \mathcal{B}_G} \mathrm{o}_B(e, e') \alpha_B \Psi_{G \setminus B} = -\alpha_e \alpha_{e'} \sum_{C \in \mathcal{C}_G^{[1]}} \mathrm{o}_C(e, e') \Psi_{G /\!\!/ C} = -\alpha_e \alpha_{e'} \chi_G^{(e,e')}. \tag{2.102}$$

□

2.3.4 Rewriting the integrand

Notation

Before stating and proving the derivative free form of the integrand $I_\Gamma(\alpha, \xi)$ in theorem 2.3.9, we need to introduce some more notation to be able cover all technicalities.

Firstly, the formal variables ξ_e in the cycle and bond polynomials are now specified to be vectors $\xi_e \in \mathbb{R}^D$ and are again interpreted as auxiliary momenta.

Secondly, there are two particular combinations of cycle polynomials that occur. The first is

$$\bar{\chi}_\Gamma^{(e_i|e_j)} := \chi_\Gamma^{(e_i|e_j)} + 2\delta_{ij} \frac{\Psi_\Gamma}{\varepsilon \alpha_{e_i}} \tag{2.103}$$

where ε is the gauge parameter (with $\varepsilon \to 0$ corresponding to Feynman gauge) and δ_{ij} is the Kronecker delta. The other is

$$\mathrm{X}_\Gamma^{e,\mu}(\alpha, \xi) := \frac{1}{2\alpha_e} \frac{\partial}{\partial \xi_{e,\mu}} \beta_\Gamma(\alpha, \xi) = \alpha_e^{-1} \sum_{e' \in E_\Gamma} \xi_{e'}^\mu \beta_\Gamma^{(e|e')}(\alpha). \tag{2.104}$$

Note that, using lemma 2.3.7 for the term with $e' = e$ and lemma 2.3.8 for the others, one finds

$$\begin{aligned} \mathrm{X}_\Gamma^{e,\mu} &= \xi_e^\mu \Psi_{\Gamma /\!\!/ e} - \sum_{\substack{e' \in E_\Gamma \\ e' \neq e}} \xi_{e'}^\mu \alpha_{e'} \chi_\Gamma^{(e|e')} \\ &= \xi_e^\mu \Psi_\Gamma - \sum_{e' \in E_\Gamma} \xi_{e'}^\mu \alpha_{e'} \chi_\Gamma^{(e|e')}. \end{aligned} \tag{2.105}$$

Furthermore, the result involves a sum over all pairings of fermion edges and a subset of the vertices. A pairing of a set S with $2n$ elements is a partition into n sets with 2 elements each and the set of all such pairings is denoted $\mathcal{P}_2(S)$. Additionally, use the notation analogous to the binomial coefficient to denote the set of all possible choices of $2k$ elements out of a set S:

$$\binom{S}{2k} \tag{2.106}$$

Note that here $|S|$ does not necessarily need to be even.

Some notational complication arises due to the mixing of vertices and edges in these pairings. We address the problem by defining a map $\bar{e} : E_\Gamma^{(f)} \cup V_\Gamma \to E_\Gamma$ that is the identity for fermion edges but assigns to a vertex the unique photon edge incident to it:

$$\bar{e}(x) = \begin{cases} e & \text{if } x = e \in E_\Gamma^{(f)} \\ e' \in \left(\partial^{-1}(v \times V_\Gamma) \cap E_\Gamma^{(p)}\right) & \text{if } x = v \in V_\Gamma \end{cases} \tag{2.107}$$

Finally, in the following we only need those vertices that have an internal photon edge incident to it, not those whose photon edge is an external half-edge. Hence we define the vertex subset

$$V_\Gamma^{(p)} := \{v \in V_\Gamma \mid \bar{e}(v) \neq \emptyset\} = V_\Gamma \setminus \ker \bar{e}. \tag{2.108}$$

Note that this is not the same as the set of external vertices V_Γ^{ext} if Γ has at least one external fermion edge.

Statement and proof

Theorem 2.3.9. *Let Γ be a Feynman graph in quantum electrodynamics, ∂_Γ the differential operator given in eq. (2.56) and*

$$S_\Gamma(\alpha, \xi) = \frac{\exp\left(-\frac{\Phi_\Gamma(\alpha,\xi)}{\Psi_\Gamma}\right)}{\Psi_\Gamma^2} \tag{2.109}$$

the scalar part of its parametric integrand as derived in section 2.2.2.
Then $\partial_\Gamma S_\Gamma = N_\Gamma S_\Gamma$ where

$$N_\Gamma(\alpha, \xi) = \varepsilon^{|E_\Gamma^{(p)}|} \alpha_{E_\Gamma^{(p)}} \sum_{k=0}^{m} \sum_{A \in \binom{M}{2(m-k)}} \left(\prod_{l \notin A} \frac{X_\Gamma^{\bar{e}(l),\mu_l}}{\Psi_\Gamma} \right) \sum_{P \in \mathcal{P}_2(A)} \prod_{(i,j) \in P} \frac{g^{\mu_i \mu_j}}{2} \frac{\bar{\chi}_\Gamma^{(\bar{e}(i)|\bar{e}(j))}}{\Psi_\Gamma} \tag{2.110}$$

with $M = V_\Gamma^{(p)} \cup E_\Gamma^{(f)}$ and $m = \lfloor |M|/2 \rfloor$. Note that these numbers are chosen such that $k = 0$ corresponds to the case in which as many elements of M as possible are paired.

In Feynman gauge the derivatives from photon propagators drop out, $m = \lfloor |E_\Gamma^{(f)}|/2 \rfloor$, and the result simplifies to

$$N_\Gamma(\alpha,\xi) = \Big(\prod_{e \in E_\Gamma^{(p)}} g^{\mu_{\partial_+(e)}\mu_{\partial_-(e)}} \Big) \sum_{k=0}^{m} \sum_{A \in \binom{E_\Gamma^{(f)}}{2(m-k)}} \Big(\prod_{l \notin A} \frac{\mathrm{X}_\Gamma^{l,\mu_l}}{\Psi_\Gamma} \Big) \sum_{P \in \mathcal{P}_2(A)} \prod_{(i,j) \in P} \frac{g^{\mu_i \mu_j}}{2} \frac{\chi_\Gamma^{(i|j)}}{\Psi_\Gamma}. \tag{2.111}$$

Proof. Consider the derivative of the bond polynomial, as in eq. (2.104) and (2.105):

$$\mathrm{X}_\Gamma^{e,\mu}(\alpha,\xi) = \frac{1}{2\alpha_e} \frac{\partial}{\partial \xi_{e,\mu}} \Phi_\Gamma(\alpha,\xi) = \xi_e^\mu \Psi_{\Gamma/\!\!/e}(\alpha) - \sum_{\substack{e' \in E_\Gamma \\ e' \neq e}} \xi_{e'}^\mu \alpha_{e'} \chi_\Gamma^{(e|e)}(\alpha) \tag{2.112}$$

Using this we can quickly compute the derivatives of $\exp(-\Phi_\Gamma/\Psi_\Gamma)$ that appear in $\partial_\Gamma S_\Gamma$. For a single photon $e \in E_\Gamma^{(p)}$ between vertices $u, v \in V_\Gamma$ one finds

$$\begin{aligned}
&\Big(\frac{2+\varepsilon}{2} g^{\mu_u \mu_v} + \frac{\varepsilon}{4\alpha_e} \frac{\partial^2}{\partial \xi_{e,\mu_u} \partial \xi_{e,\mu_v}} \Big) \exp\Big(-\frac{\Phi_\Gamma}{\Psi_\Gamma} \Big) \\
&\quad = \Big(\frac{2+\varepsilon}{2} g^{\mu_u \mu_v} + \varepsilon \alpha_e \Big(-\frac{g^{\mu_u\mu_v} \Psi_{\Gamma/\!\!/e}}{2\alpha_e \Psi_\Gamma} + \frac{\mathrm{X}_\Gamma^{e,\mu_u} \mathrm{X}_\Gamma^{e,\mu_v}}{\Psi_\Gamma^2} \Big) \Big) \exp\Big(-\frac{\Phi_\Gamma}{\Psi_\Gamma} \Big) \\
&\quad = \Big(g^{\mu_u\mu_v} \frac{2\Psi_\Gamma + \varepsilon(\Psi_\Gamma - \Psi_{\Gamma/\!\!/e})}{2\Psi_\Gamma} + \varepsilon \alpha_e \frac{\mathrm{X}_\Gamma^{e,\mu_u} \mathrm{X}_\Gamma^{e,\mu_v}}{\Psi_\Gamma^2} \Big) \exp\Big(-\frac{\Phi_\Gamma}{\Psi_\Gamma} \Big) \\
&\quad = \varepsilon \alpha_e \Big(g^{\mu_u\mu_v} \frac{\frac{2\Psi_\Gamma}{\varepsilon \alpha_e} + \Psi_{\Gamma \setminus e}}{2\Psi_\Gamma} + \frac{\mathrm{X}_\Gamma^{e,\mu_u} \mathrm{X}_\Gamma^{e,\mu_v}}{\Psi_\Gamma^2} \Big) \exp\Big(-\frac{\Phi_\Gamma}{\Psi_\Gamma} \Big) \\
&\quad = \varepsilon \alpha_e \Big(\frac{g^{\mu_u\mu_v} \bar{\chi}_\Gamma^{(e|e)}}{2\Psi_\Gamma} + \frac{\mathrm{X}_\Gamma^{e,\mu_u} \mathrm{X}_\Gamma^{e,\mu_v}}{\Psi_\Gamma^2} \Big) \exp\Big(-\frac{\Phi_\Gamma}{\Psi_\Gamma} \Big).
\end{aligned} \tag{2.113}$$

Similarly the combined derivative for two fermion edges $e_1, e_2 \in E_\Gamma^{(f)}$ is

$$\frac{1}{4\alpha_{e_1}\alpha_{e_2}}\frac{\partial^2}{\partial\xi_{e_1,\mu_1}\partial\xi_{e_2,\mu_2}}\exp\Big(-\frac{\Phi_\Gamma}{\Psi_\Gamma}\Big) = -\frac{1}{2\alpha_{e_1}}\frac{\partial}{\partial\xi_{e_1,\mu_1}}\left(\frac{X_\Gamma^{e_2,\mu_2}}{\Psi_\Gamma}\exp\Big(-\frac{\Phi_\Gamma}{\Psi_\Gamma}\Big)\right)$$
$$= \left(\frac{g^{\mu_1\mu_2}\chi_\Gamma^{(e_1|e_2)}}{2\Psi_\Gamma} + \frac{X_\Gamma^{e_1,\mu_1}X_\Gamma^{e_2,\mu_2}}{\Psi_\Gamma^2}\right)\exp\Big(-\frac{\Phi_\Gamma}{\Psi_\Gamma}\Big). \tag{2.114}$$

By the Leibniz rule the desired result is then just a sum over certain combinations of these basic results, exactly given by the pairings introduced above, proving eq. (2.110). For $\varepsilon \to 0$ the photon contribution simplifies to a single metric tensor, such that only pairings of fermions remain and one finds eq. (2.111). □

Examples

Consider again the graphs Γ_1 and Γ_2 from fig. 2.3 for which we computed the Kirchhoff and second Symanzik polynomials in example 2.1.5. We will use Feynman gauge for Γ_1 and general gauge for the smaller Γ_2. For the sake of simpler notation we will in this section use ν_i as space-time index corresponding to a vertex v_i and μ_i for those corresponding to edges e_i.

Example 2.3.10. Γ_1 *has three simple cycles*

$$C_1 = \{e_1, e_2, e_5\} \quad C_2 = \{e_1, e_3, e_4\} \quad C_3 = \{e_2, e_3, e_4, e_5\}$$

which give the polynomials

$$\Psi_{\Gamma_1/\!\!/C_1} = \alpha_3 + \alpha_4, \qquad \Psi_{\Gamma_1/\!\!/C_2} = \alpha_2 + \alpha_5, \qquad \Psi_{\Gamma_1/\!\!/C_3} = \alpha_1.$$

In Feynman gauge we only need pairings of fermion edges $\{e_2, e_3, e_4, e_5\}$. *There are* $\binom{4}{2} = 6$ *with one pair,* $3!! = 3$ *with two pairs and the empty pairing. The relevant cycle polynomials are*

$$\begin{aligned}
\chi_{\Gamma_1}^{(e_2|e_3)} &= \Psi_{\Gamma_1/\!\!/C_3} = \alpha_1 \qquad \chi_{\Gamma_1}^{(e_2|e_4)} = \Psi_{\Gamma_1/\!\!/C_3} = \alpha_1\\
\chi_{\Gamma_1}^{(e_3|e_5)} &= \Psi_{\Gamma_1/\!\!/C_3} = \alpha_1 \qquad \chi_{\Gamma_1}^{(e_4|e_5)} = \Psi_{\Gamma_1/\!\!/C_3} = \alpha_1\\
\chi_{\Gamma_1}^{(e_2|e_5)} &= \Psi_{\Gamma_1/\!\!/C_1} + \Psi_{\Gamma_1/\!\!/C_3} = \alpha_1 + \alpha_3 + \alpha_4\\
\chi_{\Gamma_1}^{(e_3|e_4)} &= \Psi_{\Gamma_1/\!\!/C_2} + \Psi_{\Gamma_1/\!\!/C_3} = \alpha_1 + \alpha_2 + \alpha_5
\end{aligned}$$

and one finds $N_{\Gamma_1} = N_{\Gamma_1}^{(0)} + N_{\Gamma_1}^{(1)} + N_{\Gamma_1}^{(2)}$, *where*

$$N_{\Gamma_1}^{(0)} = \frac{g^{\nu_2\nu_4}}{4\Psi_{\Gamma_1}^2}\Big((g^{\mu_2\mu_3}g^{\mu_4\mu_5} + g^{\mu_2\mu_4}g^{\mu_3\mu_5})\alpha_1^2$$
$$+ g^{\mu_2\mu_5}g^{\mu_3\mu_4}(\alpha_1+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_5)\Big), \tag{2.115}$$

$N_{\Gamma_1}^{(1)}$ consists of six terms of the form

$$\frac{g^{\nu_2\nu_4}}{2\Psi_{\Gamma_1}^3} g^{\mu_2\mu_3} \chi_{\Gamma_1}^{(e_2|e_3)} \mathrm{X}_{\Gamma_1}^{e_4,\mu_4} \mathrm{X}_{\Gamma_1}^{e_5,\mu_5}, \tag{2.116}$$

where e.g.

$$\begin{aligned} \mathrm{X}_{\Gamma_1}^{e_5,\mu_5} &= \xi_5^{\mu_5}\Big((\alpha_1+\alpha_2)(\alpha_3+\alpha_4)+\alpha_1\alpha_2\Big) \\ &\quad - \xi_2^{\mu_5}\alpha_2(\alpha_1+\alpha_3+\alpha_4) - \xi_3^{\mu_5}\alpha_1\alpha_3 - \xi_4^{\mu_5}\alpha_1\alpha_4, \end{aligned} \tag{2.117}$$

and

$$N_{\Gamma_1}^{(2)} = \frac{g^{\nu_2\nu_4}}{\Psi_{\Gamma_1}^4} \mathrm{X}_{\Gamma_1}^{e_2,\mu_2} \mathrm{X}_{\Gamma_1}^{e_3,\mu_3} \mathrm{X}_{\Gamma_1}^{e_4,\mu_4} \mathrm{X}_{\Gamma_1}^{e_5,\mu_5}. \tag{2.118}$$

Evaluating ξ to get physical momenta simplifies matters. Choose $\{e_2, e_3\}$ as the momentum path. Then

$$\begin{aligned} \mathrm{X}_{\Gamma_1}^{e_2,\mu_2} &= q^{\mu_2}\Big((\alpha_1+\alpha_5)(\alpha_3+\alpha_4)+\alpha_1\alpha_5-\alpha_1\alpha_3\Big), \\ \mathrm{X}_{\Gamma_1}^{e_3,\mu_3} &= q^{\mu_3}\Big((\alpha_1+\alpha_4)(\alpha_2+\alpha_5)+\alpha_1\alpha_4-\alpha_1\alpha_2\Big), \\ \mathrm{X}_{\Gamma_1}^{e_4,\mu_4} &= -q^{\mu_4}\Big(\alpha_1\alpha_2+\alpha_3(\alpha_1+\alpha_2+\alpha_5)\Big), \\ \mathrm{X}_{\Gamma_1}^{e_5,\mu_5} &= -q^{\mu_5}\Big(\alpha_2(\alpha_1+\alpha_3+\alpha_4)+\alpha_1\alpha_3\Big). \end{aligned}$$

Example 2.3.11. *Γ_2 contains only a single simple cycle that encompasses the entire graph, except for the external half-edges. Hence $\Psi_{\Gamma_2/\!\!/C} = 1$ and the three cycle polynomials $\chi_{\Gamma_2}^{(e_1,e_2)}$, $\chi_{\Gamma_2}^{(e_1,e_3)}$ and $\chi_{\Gamma_2}^{(e_2,e_3)}$ are also all just ± 1. Beyond that we have*

$$\begin{aligned} \mathrm{X}_{\Gamma_2}^{e_1,\nu_i} &= \xi_1^{\nu_i}(\alpha_2+\alpha_3) - \xi_2^{\nu_i}\alpha_2 - \xi_3^{\nu_i}\alpha_3 &&\to\quad q_2^{\nu_i}(\alpha_2+\alpha_3) - q_1^{\nu_i}\alpha_3 \qquad i=2,3 \\ \mathrm{X}_{\Gamma_2}^{e_2,\mu_2} &= \xi_2^{\mu_2}(\alpha_1+\alpha_3) - \xi_1^{\mu_2}\alpha_1 - \xi_3^{\mu_2}\alpha_3 &&\to\quad -q_2^{\mu_2}\alpha_1 - q_1^{\mu_2}\alpha_3 \\ \mathrm{X}_{\Gamma_2}^{e_3,\mu_3} &= \xi_3^{\mu_3}(\alpha_1+\alpha_2) - \xi_1^{\mu_3}\alpha_1 - \xi_2^{\mu_3}\alpha_2 &&\to\quad q_1^{\mu_3}(\alpha_1+\alpha_2) - q_2^{\mu_3}\alpha_1 \end{aligned}$$

$$\bar{\chi}_{\Gamma_2}^{(e_1|e_1)} = 1 + \frac{2\Psi_{\Gamma_2}}{\varepsilon\alpha_1}$$

In this example we again have to consider pairings of a set of four objects, but this time we include vertices and have to remember $\bar{e}(v_2) = e_1 = \bar{e}(v_3)$. We again have $N_{\Gamma_2} = N_{\Gamma_2}^{(0)} + N_{\Gamma_2}^{(1)} + N_{\Gamma_2}^{(2)}$, where the summands are constructed from the *polynomials above in the same way as in the last example. In order to illustrate the difference between Feynman gauge and general gauge consider the first term:*

$$\begin{aligned} N_{\Gamma_2}^{(0)} &= \frac{\varepsilon\alpha_1}{4\Psi_{\Gamma_2}^2}\left(g^{\mu_2\mu_3}g^{\nu_2\nu_3}\left(1+\frac{2\Psi_{\Gamma_2}}{\varepsilon\alpha_1}\right)+g^{\mu_2\nu_2}g^{\mu_3\nu_3}+g^{\mu_2\nu_3}g^{\mu_3\nu_2}\right) \\ &= \frac{g^{\mu_2\mu_3}g^{\nu_2\nu_3}}{2\Psi_{\Gamma_2}}+\varepsilon\alpha_1\frac{g^{\mu_2\mu_3}g^{\nu_2\nu_3}+g^{\mu_2\nu_2}g^{\mu_3\nu_3}+g^{\mu_2\nu_3}g^{\mu_3\nu_2}}{4\Psi_{\Gamma_2}^2} \end{aligned} \tag{2.119}$$

3 Renormalisation

Personne ne se rend compte que certaines personnes dépensent une force herculéenne pour être seulement normales.

Albert Camus, *Notebooks: 1942-1951*

3.1 Preparation

3.1.1 BPHZ and Hopf-algebraic renormalisation

For a Feynman graph Γ define the *superficial degree of divergence* to be the number

$$\omega_\Gamma^D = Dh_1(\Gamma) - \sum_i \omega_E^i |E_\Gamma^{(i)}| - \sum_j \omega_V^j |V_\Gamma^{(j)}| \tag{3.1}$$

where the sums are over the different types of edges and vertices in the particular theory under consideration, and $\omega_E^i, \omega_V^j \geq 0$ are weights associated to them. For example, in a scalar theory all vertices have weight zero and there is only a single type of edge with weight $\omega_E = 2$, stemming from the power of the momentum in a propagator $1/(k^2+m^2)$. This *power counting* is also the reason behind the term $Dh_1(\Gamma)$, since each independent loop corresponds to a D-fold integration $\int \mathrm{d}^D k$.

In QED there are two types of edges, and the fermion has another momentum in the numerator countering the square in the denominator, such that

$$\omega_\Gamma^4 = 4h_1(\Gamma) - |E_\Gamma^{(f)}| - 2|E_\Gamma^{(p)}| \tag{3.2}$$

in four dimensions.

A graph with $\omega_\Gamma^D \geq 0$ is called *superficially divergent* and any 1PI subgraph $\gamma \subset \Gamma$ with $\omega_\gamma^D \geq 0$ is called a *subdivergence* of Γ. The fact that a Feynman integral converges if and only if $\omega^D_\gamma < 0$ for all 1PI subgraphs $\gamma \subseteq \Gamma$ is Weinberg's famous powercounting theorem [135].

With divergences characterised like that, the next step is to find a mathematically consistent method to remove them while keeping certain properties – causality, Lorentz invariance, etc. – intact. The BPHZ renormalisation scheme, named for Bogoliubov and Parasiuk [17], Hepp, [82], and Zimmermann [143], is such a method. It provides a systematic procedure to construct counterterms T_Γ whose subtraction from a Feynman integral renders it finite. For a graph without subdivergences this is as simple as $\phi^R_\Gamma = \phi_\Gamma - T_\Gamma$. In general, Zimmermann's famous forest formula[1]

$$\phi^R_\Gamma = \sum_{f \in \mathcal{F}_\Gamma} (-1)^f \phi_{\Gamma/f} T_f \tag{3.3}$$

removes all divergences from a given Feynman integral. The book [46] is recommended for a more in-depth exposition of these topics.

Hopf algebras

Hopf algebras $\mathcal{H}$ are bialgebras, together with a map called antipode. In the 1990s it was discovered that Feynman graphs have the structure of a Hopf algebra and that this can be use to give an algebraic interpretation to renormalisation. Aside from the seminal articles [47, 48, 94] we recommend the reviews [95, 110] as an introduction to the topic.

A Hopf algebra of Feynman graphs consists of the divergent one-particle irreducible graphs in a given theory. The unit element $\mathbb{1} \in \mathcal{H}$ is the empty graph and the product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is the disjoint union of two graphs. The coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is given by

$$\Delta\Gamma = \sum_{\substack{\gamma \subseteq \Gamma \\ \omega_D(\gamma) \geq 0 \\ \gamma \text{ 1PI}}} \gamma \otimes \Gamma/\gamma. \tag{3.4}$$

The unit $u : \mathbb{R} \to \mathcal{H}$ with $u(1) = \mathbb{1}$ and counit $\varepsilon : \mathcal{H} \to \mathbb{R}$ with $\varepsilon(\mathbb{1}) = 1$ and zero otherwise turn $(\mathcal{H}, m, \Delta, u, \varepsilon)$ into a bialgebra. The antipode $S : \mathcal{H} \to \mathcal{H}$ is then recursively defined as

$$S(\Gamma) = -\sum_{\substack{\gamma \subseteq \Gamma \\ \omega_D(\gamma) \geq 0 \\ \gamma \text{ 1PI}}} S(\gamma)\Gamma/\gamma, \tag{3.5}$$

[1] Note that these forests $f \in \mathcal{F}_\Gamma$ are collections of divergent 1PI subgraphs of Γ, which is completely unrelated to the notion of forests previously defined in section 2.1.1. See section 3.3.4 for more details.

starting with $S(\mathbb{1}) = \mathbb{1}$. This recursion can be solved by what is essentially the forest formula eq. (3.3) on the level of graphs, rather than Feynman integrals.

Feynman rules ϕ in this setting are seen as a character on $\mathcal{H}$, i.e. as a homomorphism with certain nice properties from $\mathcal{H}$ to an algebra. Then a Birkhoff decomposition of this map provides the renormalised Feynman rules and counterterms, i.e. what we denoted above with ϕ_Γ^R and T_Γ for a specific graph.

Note that the quotient graph Γ/γ differs from $\Gamma /\!\!/ \gamma$ if $\mathrm{res}(\gamma)$ is a propagator.

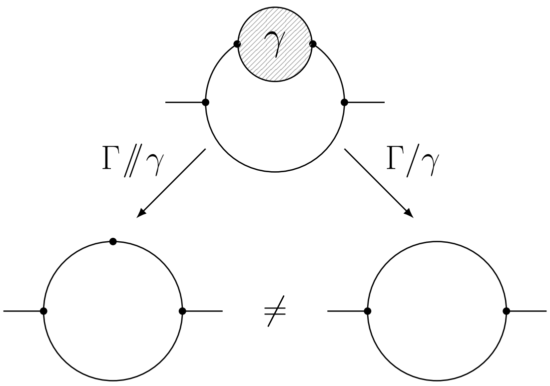


Figure 3.1: A graph with a propagator subgraph γ and the two different contractions.

Contracting only the edge set of γ leaves a graph with a 2-valent vertex that is generally not permitted in usual Feynman rules. Moreover, the additional edge messes with the powercounting, and in order to get a proper element of the Hopf algebra of Feynman graphs one needs to also contract one of the two connecting edges of the subgraph.

Specifics of QED

It is a simple combinatorial exercise to show that the superficial degree of divergence for QED Feynman graphs only depends on their residue. Hence, the only divergent types of graphs are the two propagators and , the electron-photon-vertex , the 3-photon vertex, and the 4-photon vertex. However, the amplitude with three external photons vanishes due to Furry's theorem [66] and the 4-photon vertex turns out to be finite even though it has $\omega_4(\Gamma) = 0$. It should be noted that the latter statement does not appear to be proved anywhere in the literature but is rather a folklore theorem whose veracity is doubted by nobody. In fact, it appears that a potential rigorous proof of this statement is another possible application of the results of this thesis in future work.

Taking the finiteness of the 4-photon vertex for granted we are left with three relevant types of QED graphs. For the upcoming combinatorics heavy chapters it will be useful to collect some information about them in one place. We will make extensive use of these numbers, so the reader is advised to keep this table in mind.

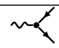
$\mathrm{res}(\Gamma)$			
$\lvert V_\Gamma \rvert$	$2h_1$	$2h_1$	$2h_1+1$
$\lvert E_\Gamma \rvert$	$3h_1-1$	$3h_1-1$	$3h_1$
$\lvert E_\Gamma^{(f)} \rvert$	$2h_1$	$2h_1-1$	$2h_1$
$\lvert E_\Gamma^{(p)} \rvert$	h_1-1	h_1	h_1

Table 3.1: The number of vertices and edges of QED Feynman graphs in terms of their first Betti number.

QED Feynman graphs also generate a Hopf algebra [89,131]. As a gauge theory it has more structure than the scalar case, which manifests itself for example in the form of the Ward identity [134]. One of its notable consequences is the transversality of the photon amplitude. This means that, while a single Feynman integral of a graph with $\mathrm{res}(\Gamma) =$ will have the form $g^{\mu\nu}q^2A + q^\mu q^\nu B$, the full amplitude is transversal, i.e. of the form $(g^{\mu\nu}q^2 - q^\mu q^\nu)C$, such that it suffices to compute only A or B in each integral.

Parametric renormalisation

The BPHZ scheme, forest formula, etc. were originally worked out for momentum space Feynman integrals, but soon parametric adaptations followed [9, 10]. In the modern Hopf algebraic context, parametric renormalisation was studied in [29]. The conclusions of that article hold quite generally, but only one scalar theory was discussed specifically. Most of the rest of this chapter will be concerned with working out the complications arising in QED. As a trade-off, to keep notation somewhat within bounds, we will continue to only study the massless case and also omit the angles. Concretely, the latter statement only means that we will set one of the external momenta of the vertex function to zero, such that we have a single external momentum q in all three cases (the two propagators and the vertex).

3.1.2 Degrees and momenta

The Feynman integral we want to renormalise is

$$\phi_\Gamma(q) = \int_{\mathbb{R}_+^{|E_\Gamma|}} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{E_\Gamma} \bar{\gamma}_\Gamma N_\Gamma S_\Gamma \tag{3.6}$$

where $N_\Gamma(\alpha, q)$ and $S_\Gamma(\alpha, q)$ are the physical momentum evaluations (following section 2.3.2) of the corresponding expressions in theorem 2.3.9. We will investigate what the contraction of $\bar{\gamma}_\Gamma$ with N_Γ looks like in detail in the following chapter. For now, it suffices to know the powers of q and external tensor structure, and the degrees in the variables α_e of each summand. Define $I_\Gamma := \bar{\gamma}_\Gamma N_\Gamma S_\Gamma$ and rewrite it as

$$I_\Gamma = \frac{e^{-\frac{\Phi_\Gamma}{\Psi_\Gamma}}}{\Psi_\Gamma^{2+n}} \sum_{k=0}^{m} \frac{1}{\Psi_\Gamma^k} I_\Gamma^{(k)}, \tag{3.7}$$

where m, n are positive integers determined by the combinatorics of Γ. Each $I_\Gamma^{(k)}$ splits into (possibly a sum of) of terms of the form $f(q) J_\Gamma^{(k)}(\alpha)$, where each factor depends only on momenta or only the Schwinger parameters respectively. To get more specific, we need to look at the three graph types separately.

For $\operatorname{res}(\Gamma) =$ $\longrightarrow$ we have

$$m = \lfloor (|E_\Gamma^{(f)}| + 2|E_\Gamma^{(p)}|)/2 \rfloor = \lfloor 2h_1 - \frac{1}{2} \rfloor = 2h_1 - 1. \tag{3.8}$$

Each pairing corresponds to a factor with one Kirchhoff polynomial in the denominator, and each pair that is replaced by two unpaired elements raises the power of Ψ_Γ in the denominator by one. Finally, all possible contractions[2] of the Dirac matrices – whether they involve metric tensors $g^{\mu_i \mu_j}$ from the pairs or momenta q^{μ_i} from unpaired elements – result in $\not{q}$ times a power of q^2, such that

$$I_\Gamma = \frac{e^{-\frac{\Phi_\Gamma}{\Psi_\Gamma}}}{\Psi_\Gamma^{2h_1+2}} \sum_{k=0}^{2h_1-1} \frac{I_\Gamma^{(k)}}{\Psi_\Gamma^k} = \not{q} \frac{e^{-\frac{\Phi_\Gamma}{\Psi_\Gamma}}}{\Psi_\Gamma^{2h_1+2}} \sum_{k=0}^{2h_1-1} \frac{(q^2)^k}{\Psi_\Gamma^k} J_\Gamma^{(k)}. \tag{3.9}$$

For $\operatorname{res}(\Gamma) =$ $\sim\!\sim\!\sim$ the situation is more complicated. The powers of Ψ_Γ are analogous, and the coefficient of $J_\Gamma^{(0)}$ can only be $g^{\mu\nu}$, where μ, ν are the spacetime indices of the photon's external vertices. However, for the higher orders the situation becomes problematic, since the contraction can yield three different results.

[2] Readers completely unfamiliar with Dirac matrix manipulations may want to skip ahead and read the introduction to chapter 4, or just take these claims for granted, for now.

The coefficient of a summand in $I_\Gamma^{(k)}$ is $(q^2)^{k-1}$ times either $q^2 g^{\mu\nu}$, or $q^\mu q^\nu$, or the combination $2q^\mu q^\nu - q^2 g^{\mu\nu}$. Hence, the integrand splits into

$$I_\Gamma = \frac{e^{-\frac{\Phi_\Gamma}{\Psi_\Gamma}}}{\Psi_\Gamma^{2h_1+1}} \Big(g^{\mu\nu} J_\Gamma^{(0)} + \sum_{k=1}^{2h_1-1} \frac{(q^2)^{k-1}}{\Psi_\Gamma^k} \Big(q^2 g^{\mu\nu} J_{\Gamma,+}^{(k)} + (2q^\mu q^\nu - q^2 g^{\mu\nu}) J_{\Gamma,0}^{(k)} + q^\mu q^\nu J_{\Gamma,-}^{(k)} \Big)\Big). \tag{3.10}$$

While one could of course sort $J_{\Gamma,0}^{(k)}$ into the other two parts it makes sense to keep them separate, since they arise from different situations (corresponding to the indices $+$, 0, $-$) that will be discussed in chapter 4.

Vertices (res(Γ) = ) are again a bit more involved. The coefficient of $J_\Gamma^{(0)}$ is just one matrix γ^μ, where μ is the spacetime index of the vertex with the external photon line. Analogous to the photon case the other terms can appear with three different coefficients, $q^2\gamma^\mu$, $q^\mu \not{q}$, and the combination $2q^\mu \not{q} - q^2\gamma^\mu$:

$$I_\Gamma = \frac{e^{-\frac{\Phi_\Gamma}{\Psi_\Gamma}}}{\Psi_\Gamma^{2h_1+2}} \Big(\gamma^\mu J_\Gamma^{(0)} + \sum_{k=1}^{2h_1} \frac{(q^2)^{k-1}}{\Psi_\Gamma^k} \Big(q^2 \gamma^\mu J_{\Gamma,+}^{(k)} + (2q^\mu \not{q} - q^2 \gamma^\mu) J_{\Gamma,0}^{(k)} + q^\mu \not{q} J_{\Gamma,-}^{(k)} \Big)\Big) \tag{3.11}$$

Note that the power of Ψ_Γ in the first denominator, $2h_1 + 2$, and the upper limit of the summation, $2h_1$, differ from the photon case due to the different numbers of edges (see table 3.1).

Now we can analyse the degrees of the numerator and denominator polynomials in each case. The cycle polynomials are of degree $h_1 - 1$ and the $\mathrm{X}_\Gamma^{e,\mu_e}$ are of degree h_1, just like Ψ_Γ. Additionally there are $|E_\Gamma^{(p)}|$ Schwinger parameters. Counting the occurrences of these polynomials in each term one finds that the k-th summand of I_Γ has degree $k - 3h_1$ for the photon and vertex and $k + 1 - 3h_1$ for the fermion.

Feynman gauge

In the second half of this thesis we will mostly constrain ourselves to Feynman gauge, since this massively reduces the size of the integrand while still exhibiting the same general structure. Comparing eq. (2.111) and eq. (2.110) one sees that pairings of fermion edges and vertices are simply reduced to pairings of only fermions. Explicitly, the definition of $\bar{\chi}_\Gamma^{(e_i|e_j)}$ in eq. (2.103) ensures that pairings of vertices that do not correspond to endpoints of the same photon get a higher power in the gauge parameter, such that

$$J_\Gamma^{(k)} = \Psi_\Gamma^{|E_\Gamma^{(p)}|} J_\Gamma^{(k,0)} + \varepsilon \Psi_\Gamma^{|E_\Gamma^{(p)}|-1} J_\Gamma^{(k,1)} + \ldots + \varepsilon^{|E_\Gamma^{(p)}|} J_\Gamma^{(k,|E_\Gamma^{(p)}|)}. \tag{3.12}$$

Hence, in order to reduce the results we work out below to Feynman gauge, we only need to change the exponents of the Kirchhoff polynomials by $|E_\Gamma^{(p)}|$ and replace $J_\Gamma^{(k)}$ with $J_\Gamma^{(k,0)}$ (or the respective cograph and subgraph quantities) everywhere.

3.1.3 Parametrising the divergence

We now consider a general integral of the same form as our Feynman integral, namely

$$\int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \sum_{k=-1}^{m} e^{-X} I_k, \tag{3.13}$$

where X and all I_k are rational functions in α_i with overall degree (degree of numerator minus degree of denominator) 1 and $k-n$ respectively.

We can introduce an auxiliary variable t by inserting $1 = \int_0^\infty \delta(t - \sum_i \lambda_i \alpha_i)\mathrm{d}t$, where each $\lambda_i \in \{0,1\}$ and at least one of them non-zero. Then scaling all Schwinger parameters by $\alpha_i \mapsto t\alpha_i$ turns eq. (3.13) into

$$\int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \; \delta(1 - \sum_i \lambda_i \alpha_i) \sum_{k=-1}^{m} I_k T_k \tag{3.14}$$

with

$$T_k = \int_0^\infty \mathrm{d}t \; t^{k-1} e^{-tX} = X^{-k}\Gamma(k). \tag{3.15}$$

The Gamma function has poles at negative integers and zero, corresponding here to quadratic and logarithmic divergences for $k = -1$ and 0. They can be parametrised for further study by regularising the t-integration with an $\epsilon > 0$:

$$T_0 \overset{\epsilon \to 0}{=} \int_\epsilon^\infty t^{-1} e^{-tX} \mathrm{d}t = -\log\epsilon - \log X - \gamma_E + \mathcal{O}(\epsilon) \tag{3.16}$$

$$T_{-1} \overset{\epsilon \to 0}{=} \int_\epsilon^\infty t^{-2} e^{-tX} \mathrm{d}t = \frac{e^{-\epsilon X}}{\epsilon} - X \underbrace{\int_\epsilon^\infty t^{-1} e^{-tX} \mathrm{d}t}_{=T_0} \tag{3.17}$$

We see that the divergent terms are isolated and a simple subtraction like

$$T_0 - T_0' = -\log \frac{X}{X'}, \tag{3.18}$$

is already enough to cancel a logarithmic divergence. However, for the Feynman integral this is complicated somewhat by the momentum parameters that also appear in each summand as well as the exponent. We will need to slightly modify this subtraction to adapt it to the graph at hand and make sure the subtraction scheme is compatible with the Ward identities.

Before we finally begin with this discussion we remark upon some more useful notation. Firstly, instead of using a delta function, the integral in eq. (3.14) can equivalently be written projectively[3]

$$\int_{\mathbb{R}^n} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \, \delta(1 - \sum_i \lambda_i \alpha_i) \sum_{k=-1}^{m} I_k T_k = \int_{\sigma_\Gamma} \Omega_\Gamma \sum_{k=-1}^{m} I_k T_k \tag{3.19}$$

where $\Omega_\Gamma = \sum_{i=1}^n (-1)^{i-1} \alpha_i \mathrm{d}\alpha_1 \wedge \cdots \wedge \widehat{\mathrm{d}\alpha_i} \wedge \cdots \wedge \mathrm{d}\alpha_n$ and one integrates over the subset of real projective space in which all parameters are positive

$$\sigma_\Gamma = \{[\alpha_1 : \ldots : \alpha_n] \mid \alpha_i > 0 \; \forall \; i = 1, \ldots, n\}. \tag{3.20}$$

In addition to this, abbreviations for certain combinations of graph polynomials will allow us to express the renormalisation procedure a bit more concisely. First, let $\Phi_\Gamma^0 \equiv \Phi_\Gamma(\alpha, \mu)$ for some reference momentum $\mu \neq q$, and $\varphi_\Gamma \coloneqq \Phi_\Gamma / q^2$. Then define for a Feynman graph Γ with a (possibly empty) subgraph γ:

$$\mathrm{x}_\gamma^\Gamma \coloneqq \frac{\Phi_{\Gamma/\gamma}}{\Psi_{\Gamma/\gamma}} + \frac{\Phi_\gamma^0}{\Psi_\gamma} \to q^2 \frac{\varphi_{\Gamma/\gamma}}{\Psi_{\Gamma/\gamma}} + \mu^2 \frac{\varphi_\gamma}{\Psi_\gamma} \tag{3.21}$$

$$\mathrm{x}_\gamma^{\Gamma,0} \coloneqq \frac{\Phi_{\Gamma/\gamma}^0}{\Psi_{\Gamma/\gamma}} + \frac{\Phi_\gamma^0}{\Psi_\gamma} \tag{3.22}$$

$$\Delta_\gamma^\Gamma \coloneqq \frac{\mathrm{x}_\gamma^\Gamma}{\mathrm{x}_\gamma^{\Gamma,0}} = \frac{\Phi_{\Gamma/\gamma}\Psi_\gamma + \Psi_{\Gamma/\gamma}\Phi_\gamma^0}{\Phi_{\Gamma/\gamma}^0\Psi_\gamma + \Psi_{\Gamma/\gamma}\Phi_\gamma^0} \to \frac{\frac{q^2}{\mu^2}\varphi_{\Gamma/\gamma}\Psi_\gamma + \Psi_{\Gamma/\gamma}\varphi_\gamma}{\varphi_{\Gamma/\gamma}\Psi_\gamma + \Psi_{\Gamma/\gamma}\varphi_\gamma} \tag{3.23}$$

The arrow indicates the simplification one finds when assuming a single scale, as we always do here. In that case, we also define $s \coloneqq q^2/\mu^2$ and

$$\Upsilon_\gamma^\Gamma(s) \coloneqq s\varphi_{\Gamma/\gamma}\Psi_\gamma + \Psi_{\Gamma/\gamma}\varphi_\gamma \tag{3.24}$$

Readers should beware that this notation slightly differs from that used in [29]. x_γ^Γ is not the same as x_γ^Γ there, instead $\Delta_\gamma^\Gamma = 1 + x_\gamma^\Gamma$. On the other hand, $\Upsilon_\gamma^\Gamma(s)$ corresponds exactly to $\Upsilon_{\gamma;\Gamma}(s)$ in [29]. Note that this notation in particular includes

$$\mathrm{x}_\emptyset^\Gamma = \frac{\Phi_\Gamma}{\Psi_\Gamma} \qquad \text{and} \qquad \Delta_\emptyset^\Gamma = \frac{\Phi_\Gamma}{\Phi_\Gamma^0} \to \frac{q^2}{\mu^2}.$$

[3] For a more thorough discussion of the bijection between $\mathbb{R}^n$ and (a certain subset of) projective space induced by the introduction of the delta function see [109, sec. 2.1.3]. Of note in particular is the fact that it is completely independent of the choice of the parameters λ_i, which is sometimes called "Cheng-Wu theorem".

3.2 Superficial Renormalisation

In this section we begin with the assumption that Γ does not have any subdivergences, i.e. $\omega_4(\Gamma) \geq 0$ but $\omega_4(\gamma) < 0$ for all proper 1PI subgraphs $\gamma \subsetneq \Gamma$. The three types of divergent graphs need to be treated separately.

3.2.1 Fermions

We want to subtract the coefficient of $\not{q}$ at $q^2 = \mu^2$, i.e. when writing $I_\Gamma = \not{q} I'_\Gamma(q^2)$ the correct subtraction evaluates $I'_\Gamma(\mu^2)$ but leaves $\not{q}$ as is. We have $|E_\Gamma| = 3h_1 - 1$, putting the overall degree – including the measure – of the k-th summand at k. Hence, we start with the logarithmically divergent term at $k = 0$, which we can subtract as in the simple example in eq. (3.18) to find

$$\not{q} \frac{J_\Gamma^{(0)}}{\Psi_\Gamma^{2h_1+2}} \int_0^\infty \mathrm{d}t\ t^{-1}\left(e^{-t\mathrm{x}_\emptyset^\Gamma} - e^{-t\mathrm{x}_\emptyset^{\Gamma,0}}\right) = -\not{q} \log \frac{q^2}{\mu^2} \frac{J_\Gamma^{(0)}}{\Psi_\Gamma^{2h_1+2}}. \tag{3.25}$$

For all other terms with $k \geq 1$ one has

$$\begin{aligned}
&\not{q} \frac{J_\Gamma^{(k)}}{\Psi_\Gamma^{2h_1+2+k}} \int_0^\infty \mathrm{d}t\ t^{k-1}\Big((q^2)^k e^{-t\mathrm{x}_\emptyset^\Gamma} - (\mu^2)^k e^{-t\mathrm{x}_\emptyset^{\Gamma,0}}\Big) \\
&\qquad = \not{q} \frac{J_\Gamma^{(k)}}{\Psi_\Gamma^{2h_1+2+k}} \left(\left(\frac{q^2}{\mathrm{x}_\emptyset^\Gamma}\right)^k - \left(\frac{\mu^2}{\mathrm{x}_\emptyset^{\Gamma,0}}\right)^k\right) \\
&\qquad = 0.
\end{aligned} \tag{3.26}$$

The renormalised integral is then

$$\phi_\Gamma^R \coloneqq -\not{q} L \int_{\sigma_\Gamma} \Omega_\Gamma \frac{J_\Gamma^{(0)}}{\Psi_\Gamma^{2h_1+2}} \tag{3.27}$$

where we introduced the abbreviation $L \coloneqq \log \frac{q^2}{\mu^2}$.

3.2.2 Photons

We have again $|E_\Gamma| = 3h_1 - 1$, but this time the quadratic divergence needs to be included. Using partial integration as in eq. (3.17) we find

$$\begin{aligned}
&\frac{g^{\mu\nu}}{\Psi_\Gamma^{2h_1+1}} J_\Gamma^{(0)} \int_0^\infty \mathrm{d}t\ t^{-2} e^{-\mathrm{x}_\emptyset^\Gamma} \\
&\qquad \to \frac{g^{\mu\nu} q^2}{\Psi_\Gamma^{2h_1+1}} J_\Gamma^{(0)} \left((\epsilon q^2)^{-1} - \frac{\varphi_\Gamma}{\Psi_\Gamma} - \frac{\varphi_\Gamma}{\Psi_\Gamma} \int_\epsilon^\infty \mathrm{d}t\ t^{-1} e^{-\mathrm{x}_\emptyset^\Gamma}\right) + \mathcal{O}(\epsilon).
\end{aligned} \tag{3.28}$$

The boundary term is a bit problematic since it does not vanish after the usual subtraction at $q=\mu$. To fix this we first subtract on-shell, i.e. at $q^2=0$ for the photon propagator. The only thing this changes is that it removes the leading 1 from the exponential in the boundary term such that the term $\sim \epsilon^{-1}$ does not appear after partial integration. The boundary term consists then simply of the constant $\varphi_\Gamma/\Psi_\Gamma$ and higher order terms $\mathcal{O}(\epsilon)$ such that everything vanishes after the usual subtraction and taking $\epsilon \to 0$. Only the logarithmic term remains, to give

$$q^2 g^{\mu\nu} L \frac{\varphi_\Gamma J_\Gamma^{(0)}}{\Psi_\Gamma^{2h_1+2}}. \tag{3.29}$$

For $k=1$ we get another logarithmic divergence which results in

$$-\frac{L}{\Psi_\Gamma^{2h_1+2}}\Big(q^2 g^{\mu\nu}\Big(J_{\Gamma,+}^{(1)} - J_{\Gamma,0}^{(1)}\Big) - q^\mu q^\nu\Big(J_{\Gamma,-}^{(1)} + 2J_{\Gamma,0}^{(1)}\Big)\Big) \tag{3.30}$$

and the convergent terms vanish as in the fermion case. Overall the integrand now contains

$$q^2 g^{\mu\nu}\Big(\varphi_\Gamma J_\Gamma^{(0)} + J_{\Gamma,0}^{(1)} - J_{\Gamma,+}^{(1)}\Big) - q^\mu q^\nu\Big(2J_{\Gamma,0}^{(1)} + J_{\Gamma,-}^{(1)}\Big), \tag{3.31}$$

and since the amplitude has to be transversal it makes sense to impose this condition on each individual graph, in order to simplify integration. This yields the superficially renormalised integral

$$\phi_\Gamma^R := (q^2 g^{\mu\nu} - q^\mu q^\nu) L \int_{\sigma_\Gamma} \Omega_\Gamma \frac{\varphi_\Gamma J_\Gamma^{(0)} + J_{\Gamma,0}^{(1)} - J_{\Gamma,+}^{(1)}}{\Psi_\Gamma^{2h_1+2}}. \tag{3.32}$$

3.2.3 Vertices

The vertex renormalisation is slightly different. There is no quadratic divergence. Instead, because $|E_\Gamma| = 3h_1$ the $k=0$ term is the logarithmic divergence:

$$-\gamma^\mu L \frac{J_\Gamma^{(0)}}{\Psi_\Gamma^{2h_1+2}} \tag{3.33}$$

The renormalisation prescription for the vertex only subtract the coefficient of γ^μ, which removes the corresponding convergent parts of the integrand, but leaves the contributions with $\not{q} q^\mu$. Using eq. (3.15) they are

$$q^\mu \not{q} \frac{2J_{\Gamma,0}^{(k)} + J_{\Gamma,-}^{(k)}}{\Psi_\Gamma^{2h_1+2+k}} (q^2)^{k-1} \left(\frac{q^2 \varphi_\Gamma}{\Psi_\Gamma}\right)^{-k} = \frac{q^\mu \not{q}}{q^2} \frac{2J_{\Gamma,0}^{(k)} + J_{\Gamma,-}^{(k)}}{\Psi_\Gamma^{2h_1+2} \varphi_\Gamma^k}, \tag{3.34}$$

and the integral is

$$\phi_\Gamma^R := -\int_{\sigma_\Gamma} \Omega_\Gamma \frac{1}{\Psi_\Gamma^{2h_1+2}} \left(\gamma^\mu L J_\Gamma^{(0)} + \frac{\not{q} q^\mu}{q^2} \sum_{k=1}^{2h_1} \frac{2J_{\Gamma,0}^{(k)} + J_{\Gamma,-}^{(k)}}{\varphi_\Gamma^k} \right). \tag{3.35}$$

3.3 Subdivergences

Very few general Feynman graphs and indeed no photon propagator graphs beyond one loop are primitive[4]. Hence, we need to consider further subtractions of divergences resulting from subgraphs $\gamma \subsetneq \Gamma$, i.e. from integration of a certain subsets of edge parameters $E_\gamma \subsetneq E_\Gamma$. For the sake of brevity we only explicitly work out the case of photon propagator graphs with simple, i.e. non-nested subdivergences with a single connected component. The other two graph types work analogously and are left to the reader. Likewise, more general "forests" of divergent subgraphs can be incorporated via minor changes in notation and iteration of the simple case. This generalisation will be discussed briefly in the next section but for details on the Hopf algebraic structure of such forests as well as rigorous proofs of convergence we refer to [29].

In order to isolate the subdivergent part of the Feynman integral let us go back to the momentum space representation in eq. (2.51). One can simply separate the factors belonging to γ – including the Dirac matrices, which are either a sequence of consecutive matrices within the trace belonging to the fermion cycle of the overall photon propagator, or an entirely separate trace. That expression can then be parametrised with the Schwinger trick to get an I_γ, which we renormalise as shown above. Hence, in order to produce a term that subtracts the subdivergence out of the already superficially renormalised Feynman integral, we replace I_γ by its counterterms and then proceed to apply the Schwinger trick to the remaining terms belonging to Γ/γ. Those terms are then also superficially renormalised so as not to introduce a new divergence while trying to remove the subdivergence. The details vary slightly in each case, so we now have to differentiate between the types of subgraphs.

[4]See e.g. [21] for asymptotic enumeration of various types of graphs – primitive, 1PI, connected etc. – in a number of different theories.

3.3.1 Vertex subgraphs

Since propagator subgraphs lead to some further complications we begin with vertices. The unrenormalised integrand was found to be

$$I_\Gamma = \frac{e^{-\frac{\Phi_\Gamma}{\Psi_\Gamma}}}{\Psi_\Gamma^{2h_1+2}}\Big(\gamma^\mu J_\Gamma^{(0)} + \sum_{k=1}^{2h_1} \frac{(q^2)^k}{\Psi_\Gamma^k}\Big(\gamma^\mu (J_{\Gamma,+}^{(k)} - J_{\Gamma,0}^{(k)}) + \frac{q^\mu \not{q}}{q^2}(2J_{\Gamma,0}^{(k)} + J_{\Gamma,-}^{(k)})\Big)\Big)$$

in eq. (3.11). The vertex subgraph inherits a single-scale structure from the full graph. It depends on the choice of path along which one routes the external momentum through Γ but the choice does not influence the end result as long as it is kept consistent over all subdivergence subtractions. The counterterms are

$$T_\gamma = \gamma^\mu \frac{e^{-\mu^2 \frac{\varphi_\gamma}{\Psi_\gamma}}}{\Psi_\gamma^{2h_1^\gamma+2}}\Big(J_\gamma^{(0)} + \sum_{k=1}^{2h_1^\gamma}\Big(\frac{\mu^2}{\Psi_\gamma}\Big)^k \Big(J_{\gamma,+}^{(k)} - J_{\gamma,0}^{(k)}\Big)\Big). \tag{3.36}$$

The one Dirac matrix we have left represents the vertex of the cograph Γ/γ that γ is contracted to. Moreover, looking back at the derivation of the Schwinger parametric representation in section 2.2 we remember that this parametrisation also yields a delta function for the "external" vertices of γ, i.e. the momenta entering or leaving this vertex in Γ/γ. Hence, everything we need to proceed with the Schwinger trick for Γ/γ is present.

This yields the unrenormalised photon propagator integrand for the cograph, as in eq. (3.10),

$$I_{\Gamma/\gamma} = (q^2 g^{\mu\nu} - q^{\mu\nu})\frac{e^{-q^2 \frac{\varphi_{\Gamma/\gamma}}{\Psi_{\Gamma/\gamma}}}}{\Psi_{\Gamma/\gamma}^{2h_1^{\Gamma/\gamma}+2}}\Big(- \varphi_{\Gamma/\gamma} J_{\Gamma/\gamma}^{(0)} + \sum_{l=1}^{2h_1^{\Gamma/\gamma}-1} \frac{(q^2)^{l-1}}{\Psi_{\Gamma/\gamma}^{l-1}}\Big(J_{\Gamma/\gamma,+}^{(l)} - J_{\Gamma/\gamma,0}^{(l)}\Big)\Big), \tag{3.37}$$

slightly rewritten here with transversality and the partial integration of the quadratic divergence already included. The combined degree of the product of the l-th summand from this and the k-th summand from the counterterm is $3h_1^{\Gamma/\gamma} - l + 3h_1^\gamma - k = 3h_1 - l - k$. There are $3h_1 - 1$ edges in total, so we have logarithmic divergences for $k = 0$ with $l = 1$, and $l = 0$ after partial integration. After renormalisation these terms leave a contribution of

$$\frac{(\varphi_{\Gamma/\gamma} J_{\Gamma/\gamma}^{(0)} + \tilde{J}_{\Gamma/\gamma}^{(1)}) J_\gamma^{(0)}}{\Psi_{\Gamma/\gamma}^{2h_1^{\Gamma/\gamma}+2} \Psi_\gamma^{2h_1^\gamma+2}} \log \Delta_\gamma^\Gamma, \tag{3.38}$$

where we use the abbreviation

$$\tilde{J}_\gamma^{(k)} = J_{\gamma,0}^{(k)} - J_{\gamma,+}^{(k)}, \qquad 1 \le k \le 2h_1^\gamma, \tag{3.39}$$

$$\tilde{J}_{\Gamma/\gamma}^{(l)} = J_{\Gamma/\gamma,0}^{(l)} - J_{\Gamma/\gamma,+}^{(l)}, \quad 1 \le l \le 2h_1^{\Gamma/\gamma} - 1. \tag{3.40}$$

For $k, l > 0$ we encounter an unfortunate side effect of subdivergences: The convergent terms do not vanish in the subtraction anymore. Each summand now contains the integral

$$\begin{aligned}
&\int_0^\infty \mathrm{d}t\ t^{(k+l-1)-1}\Big(e^{-t\mathrm{x}_\gamma^\Gamma} - e^{-t\mathrm{x}_\gamma^{\Gamma,0}}\Big) \\
&\quad = \Big(\frac{q^2\varphi_{\Gamma/\gamma}\Psi_\gamma + \mu^2\varphi_\gamma\Psi_{\Gamma/\gamma}}{\Psi_{\Gamma/\gamma}\Psi_\gamma}\Big)^{-(k+l-1)} - \Big(\frac{\mu^2\varphi_{\Gamma/\gamma}\Psi_\gamma + \mu^2\varphi_\gamma\Psi_{\Gamma/\gamma}}{\Psi_{\Gamma/\gamma}\Psi_\gamma}\Big)^{-(k+l-1)} \\
&\quad = \Big(\frac{\mu^2}{\Psi_{\Gamma/\gamma}\Psi_\gamma}\Big)^{-(k+l-1)}\Big(\Upsilon_\gamma^\Gamma(s)^{-(k+l-1)} - \Upsilon_\gamma^\Gamma(1)^{-(k+l-1)}\Big). \qquad (3.41)
\end{aligned}$$

Altogether we have the expression

$$\begin{aligned}
M_\gamma^\Gamma := &\frac{\Omega_\Gamma}{\Psi_{\Gamma/\gamma}^{2h_1^{\Gamma/\gamma}+2}\Psi_\gamma^{2h_1^\gamma+2}}\Big((\varphi_{\Gamma/\gamma}J_{\Gamma/\gamma}^{(0)} + \tilde{J}_{\Gamma/\gamma}^{(1)})J_\gamma^{(0)}\log\Delta_\gamma^\Gamma \\
&+ \sum_{k,l\geq 1}\Big(s\Psi_\gamma\Big)^{l-1}\Big(\Psi_{\Gamma/\gamma}\Big)^k\tilde{J}_{\Gamma/\gamma}^{(l)}\tilde{J}_\gamma^{(k)}\Big(\Upsilon_\gamma^\Gamma(s)^{-(k+l-1)} - \Upsilon_\gamma^\Gamma(1)^{-(k+l-1)}\Big)\Big) \qquad (3.42)
\end{aligned}$$

to subtract from our already superficially renormalised integrand. Using the same notation for the superficially renormalised part one has

$$M_\emptyset^\Gamma := \frac{L(\varphi_\Gamma J_\Gamma^{(0)} + \tilde{J}_\Gamma^{(1)})}{\Psi_\Gamma^{2h_1+2}}\Omega_\Gamma, \qquad (3.43)$$

and the fully renormalised integral for a single vertex subdivergence is then

$$\phi_\Gamma^R := (g^{\mu\nu}q^2 - q^\mu q^\nu)\int_{\sigma_\Gamma}(M_\emptyset^\Gamma - M_\gamma^\Gamma). \qquad (3.44)$$

3.3.2 Fermion subgraphs

For propagator subgraphs one encounters a new problem: The cograph $\Gamma/\!\!/\gamma \neq \Gamma/\gamma$ is not itself a valid Feynman graph since the contraction leaves a 2-valent vertex. This is repaired by choosing one of the two connectors, i.e. edges of $\Gamma/\!\!/\gamma$ incident to γ, and contracting one of them. That choice is called *squashing*. In a general setting with multiple propagator subgraphs one needs to ensure that no connector is chosen twice (e.g. if it is the fermion edge between two sequential fermion subdivergences), but it is evident that such a choice always exists.

Let now $e_1^\gamma, e_2^\gamma \in E_{\Gamma/\!\!/\gamma}$ be the two connectors for the subgraph γ. After parametrising the subdivergence one finds a delta function $\delta^{(4)}(k_{e_1^\gamma} - k_{e_2^\gamma})$ that allows us to identify the two connector momenta and removes one of their two integrations. The counterterms are (cf. eq. (3.9))

$$T_\gamma = \not{k}_{e_1^\gamma} \frac{e^{-\mu^2 \frac{\varphi_\gamma}{\Psi_\gamma}}}{\Psi_\gamma^{2h_1^\gamma+2}} \sum_{l=0}^{2h_1^\gamma-1} \frac{(\mu^2)^l}{\Psi_\gamma^l} J_\gamma^{(l)} \tag{3.45}$$

where the Dirac matrix in $\not{k}_{e_1^\gamma} = \gamma_\nu k^\nu_{e_1^\gamma}$ represents the 2-valent vertex of $\Gamma /\!\!/ \gamma$. Including the connector propagators one now has

$$\frac{\not{k}_{e_1^\gamma} \not{k}_{e_1^\gamma} \not{k}_{e_1^\gamma}}{k^4_{e_1^\gamma}} = \frac{\not{k}_{e_1^\gamma}}{k^2_{e_1^\gamma}}, \tag{3.46}$$

such that one of the two connector propagators, including its Dirac matrix, cleanly cancels and only terms of the correct cograph Γ/γ remain. Now the cograph can be parametrised as usual, which brings us again to eq. (3.37). We again look at the degrees and collect terms accordingly, where we remember that the measure now only contributes $|E_{\Gamma/\gamma}| + |E_\gamma| = 3h_1^{\Gamma/\gamma} - 1 + 3h_1^\gamma - 1 = 3h_1 - 2$ instead of $|E_\Gamma| = 3h_1 - 1$. The result is analogous to the vertex subdivergence:

$$\begin{aligned} M_\gamma^\Gamma := \frac{\Omega_{\bar\Gamma}}{\Psi_{\Gamma/\gamma}^{2h_1^{\Gamma/\gamma}+2} \Psi_\gamma^{2h_1^\gamma+2}} \Big(& (\varphi_{\Gamma/\gamma} J^{(0)}_{\Gamma/\gamma} + \tilde{J}^{(1)}_{\Gamma/\gamma}) J_\gamma^{(0)} \log \Delta_\gamma^\Gamma \\ & + \sum_{k,l\geq 1} \left(s\Psi_\gamma\right)^{l-1} \left(\Psi_{\Gamma/\gamma}\right)^k \tilde{J}^{(l)}_{\Gamma/\gamma} J_\gamma^{(k)} \left(\Upsilon_\gamma^\Gamma(s)^{-(k+l-1)} - \Upsilon_\gamma^\Gamma(1)^{-(k+l-1)}\right)\Big) \end{aligned} \tag{3.47}$$

Squashing in $M_\emptyset^\Gamma$

We have analysed the subdivergence of a vertex and a fermion and happily notice that the resulting term M_γ^Γ in the forest formula can be expressed more or less identically in both cases. However, for the fermion there is now a different problem: Since an edge is missing we now also have a different form $\Omega_{\bar\Gamma}$, where

$$\bar\Gamma := \Gamma /\!\!/ e_2^\gamma \tag{3.48}$$

is the *squashed graph*, in which one of the two connectors is contracted. We have two integrals, $\int_{\sigma_\Gamma} M_\emptyset^\Gamma$ and $\int_{\sigma_{\bar\Gamma}} M_\gamma^\Gamma$, neither of which is well-defined on its own. They cannot be combined since they involve different differential forms and chains of integration. Hence, we need to modify one of the two. It turns out, we can exploit the additional information about connectors of γ to also express $M_\emptyset^\Gamma$ just in terms of the variables of $\bar\Gamma$.

Consider again the subgraph parametrisation as in eq. (3.45), but do not introduce the reference momentum μ. Instead, we exploit the cancellation of one connector propagator and continue parametrising the cograph. Since now $k_{e_1^\gamma}$ is still contained in the exponential and each summand, another slightly modified Schwinger trick needs to be employed:

$$\begin{aligned}(k_{e_1^\gamma}^2)^{l-1}\exp\Big(-k_{e_1^\gamma}^2\frac{\varphi_\gamma}{\Psi_\gamma}\Big) &= \int_0^\infty \mathrm{d}\alpha_{e_1^\gamma}\ (k_{e_1^\gamma}^2)^l \exp\Big(-k_{e_1^\gamma}^2\Big(\alpha_{e_1^\gamma}+\frac{\varphi_\gamma}{\Psi_\gamma}\Big)\Big)\\ &= \int_0^\infty \mathrm{d}\alpha_{e_1^\gamma}\ (-\partial_{e_1^\gamma})^l \exp\Big(-k_{e_1^\gamma}^2\Big(\alpha_{e_1^\gamma}+\frac{\varphi_\gamma}{\Psi_\gamma}\Big)\Big).\end{aligned} \tag{3.49}$$

The other edges are parametrised as before. Usually the l-th summand corresponds to degree l in the integrand, but the l-fold derivative reduces it by l, so the overall degree of the full integrand depends only on the summands of the cograph. Hence, only the first summand of the cograph (including the partially integrated term with quadratic divergence) contributes, but it is combined with all summands of the subgraph, i.e.

$$M_\emptyset^\Gamma := L\frac{\Omega_{\bar\Gamma}}{\Psi_\gamma^{2h_1^\gamma+2}}\sum_{l=0}^{2h_1^\gamma-1}\frac{J_\gamma^{(l)}}{\Psi_\gamma^l}(-\partial_{e_1^\gamma})^l\left(\frac{\varphi_{\Gamma/\gamma}J_{\Gamma/\gamma}^{(0)}+\tilde J_{\Gamma/\gamma}^{(1)}}{\Psi_{\Gamma/\gamma}^{2h_1^{\Gamma/\gamma}+2}}\right)\Bigg|_{\alpha_{e_1^\gamma}\to\alpha_{e_1^\gamma}+\frac{\varphi_\gamma}{\Psi_\gamma}} \tag{3.50}$$

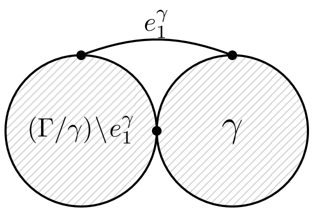
Since the graph polynomials are linear in each variable the replacement of $\alpha_{e_1^\gamma}$ simply yields

$$\Psi_{\Gamma/\gamma}\to\Psi_{\Gamma/\gamma}+\frac{\varphi_\gamma}{\Psi_\gamma}\partial_{e_1^\gamma}\Psi_{\Gamma/\gamma} \qquad \text{and} \qquad \varphi_{\Gamma/\gamma}\to\varphi_{\Gamma/\gamma}+\frac{\varphi_\gamma}{\Psi_\gamma}\partial_{e_1^\gamma}\varphi_{\Gamma/\gamma}.$$

In the case of propagator type subgraphs one finds a convenient relation that identifies these terms as the corresponding quantities of the full squashed graph $\bar\Gamma$:

$$\begin{aligned}\Psi_{\bar\Gamma} &= \Psi_{\Gamma/\gamma}\Psi_\gamma+\varphi_\gamma\partial_{e_1^\gamma}\Psi_{\Gamma/\gamma}\\ \varphi_{\bar\Gamma} &= \varphi_{\Gamma/\gamma}\Psi_\gamma+\varphi_\gamma\partial_{e_1^\gamma}\varphi_{\Gamma/\gamma}.\end{aligned}$$

These identities become rather evident when we visualise the squashed graph as

$\bar\Gamma =$ 

and remember the usual contraction-deletion relations. In the scalar case this is sufficient to express $M_\emptyset^\Gamma$ entirely with graph polynomials of $\bar\Gamma$, but here we still have $J_{\Gamma/\gamma}^{(0)}$ and $\tilde J_{\Gamma/\gamma}^{(1)}$ in the integrand. While similar identities also hold for each individual cycle polynomial $\chi^{(i|j)}_{\Gamma/\gamma}$, the trace and sum over pairings in $\tilde{J}^{(1)}_{\Gamma/\gamma}$ keep it explicitly dependent on the cograph. For now this good enough for us, but it might be interesting to investigate this more in future work. Since $M^\Gamma_\emptyset$ and M^Γ_γ are now compatible we can write the renormalised integral as

$$\phi^R_\Gamma := (g^{\mu\nu}q^2 - q^\mu q^\nu)\int_{\sigma_{\bar{\Gamma}}}(M^\Gamma_\emptyset - M^\Gamma_\gamma) \tag{3.51}$$

3.3.3 Photon subgraphs

Analogous to the previous cases one finds

$$\begin{aligned} M^\Gamma_\gamma := \frac{\Omega_{\bar{\Gamma}}}{\Psi_{\Gamma/\gamma}^{2h_1^{\Gamma/\gamma}+2}\Psi_\gamma^{2h_1^\gamma+2}} \Bigg(&(\varphi_\Gamma J^{(0)}_{\Gamma/\gamma} + \tilde{J}^{(1)}_{\Gamma/\gamma})\tilde{J}^{(1)}_\gamma \log\Delta^\Gamma_\gamma \\ &+ \sum_{\substack{k,l\geq 1\\ k+l>1}} \Big(s\Psi_\gamma\Big)^{l-1}\Big(\Psi_{\Gamma/\gamma}\Big)^k \tilde{J}^{(l)}_{\Gamma/\gamma} J^{(k)}_\gamma \Big(\Upsilon^\Gamma_\gamma(s)^{-(k+l-1)} - \Upsilon^\Gamma_\gamma(1)^{-(k+l-1)}\Big)\Bigg) \end{aligned} \tag{3.52}$$

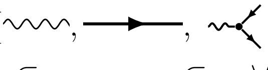
with some minor modifications. We also need to modify $M^\Gamma_\emptyset$ again, similarly to the fermion case. The cancellation of one of the connectors (cf. eq. (3.46)) takes the form

$$\begin{aligned} &\frac{1}{k^4_{e^\gamma_1}}\Bigg(g^{\nu_a\nu_b} + \varepsilon\frac{k^{\nu_a}_{e^\gamma_1}k^{\nu_b}_{e^\gamma_1}}{k^2_{e^\gamma_1}}\Bigg)\Big(g^{\nu_b\nu_c}k^2_{e^\gamma_1} - k^{\nu_b}_{e^\gamma_1}k^{\nu_c}_{e^\gamma_1}\Big)\Bigg(g^{\nu_c\nu_d} + \varepsilon\frac{k^{\nu_c}_{e^\gamma_1}k^{\nu_d}_{e^\gamma_1}}{k^2_{e^\gamma_1}}\Bigg) \\ &\qquad = \frac{1}{k^4_{e^\gamma_1}}\Big(g^{\nu_a\nu_d}k^2_{e^\gamma_1} - k^{\nu_a}_{e^\gamma_1}k^{\nu_d}_{e^\gamma_1}\Big). \end{aligned} \tag{3.53}$$

Note that the connector photon edge is projected onto the transversal component, independent of the chosen gauge parameter. One can then proceed further, following the steps from the previous section to find the adapted form of $M^\Gamma_\emptyset$.

3.3.4 Forests of subdivergences

A forest f of a Feynman graph Γ is a collection of superficially divergent 1PI proper subgraphs $\gamma_i \subsetneq \Gamma$ (including the empty graph) such that any two distinct subgraphs in a forest are either disjoint or nested (i.e. one is a proper subgraph of the other). The set of all forests f of Γ is then denoted $\mathcal{F}_\Gamma$:

$\mathcal{F}_\Gamma := \{f = \sqcup_{i\in I}\,\gamma_i \mid \gamma_i \subsetneq \Gamma$ s.t. $\operatorname{res}(\gamma_i) \in \{$  $\}$ and

$$\gamma_i \cap \gamma_j = \emptyset \ \text{ or } \ \gamma_i \subsetneq \gamma_j \ \text{ or } \ \gamma_j \subsetneq \gamma_i \ \forall i,j\in I,\ i\neq j\}. \tag{3.54}$$

For disjoint graphs we have seen in eqs. (2.8) and (2.21) how to generalise the graph polynomials and that can be extended to nested graphs. Let $f = \sqcup_i \gamma_i$ be a forest of n nested graphs (sometimes called *flag*) such that $\gamma_1 \subsetneq \gamma_2 \subsetneq \ldots \subsetneq \gamma_n$. Then

$$\Psi_f := \Psi_{\gamma_1} \prod_{i=2}^{n} \Psi_{\gamma_i/\gamma_{i-1}}, \tag{3.55}$$

and

$$\Phi_f := \Phi_{\gamma_n/\gamma_{n-1}} \Psi_{\gamma_1} \prod_{i=2}^{n-1} \Psi_{\gamma_i/\gamma_{i-1}}. \tag{3.56}$$

The subdivergences in a forest are partially ordered by their nesting. Following this ordering we can parametrise and subtract each subdivergence analogous to the case of the single subdivergence. This yields a general forest formula integrand M_f^Γ which is the obvious but rather unwieldy generalisation of the terms found in the previous sections. Note that for each propagator subgraph a connector has to be chosen for the squashing $E_\Gamma^{\text{sq}} \subset E_\Gamma$, $\bar{\Gamma} = \Gamma /\!\!/ E_\Gamma^{\text{sq}}$, and the corresponding edge parameters are suppressed in all terms similarly to $\alpha_{e_2^\gamma}$ in $M_\emptyset^\Gamma$ in section 3.3.2.

The fully renormalised integral is then

$$\phi_\Gamma^R := (g^{\mu\nu} q^2 - q^\mu q^\nu) \int_{\sigma_{\bar{\Gamma}}} \sum_{f \in \mathcal{F}_\Gamma} (-1)^{|f|} M_f^\Gamma. \tag{3.57}$$

4

Structure of the integrand I: Contraction of Dirac matrices

O caos é uma ordem por decifrar.
Chaos is order yet undeciphered.

José Saramago, *O hohem duplicado*, 2002

4.1 Dirac matrices

Now that we have made the integrals well-defined we can apply ourselves to the study of the structure of their integrands, which was so far conveniently isolated into the $J_\Gamma^{(k)}$. The aim of this chapter is to eliminate the Dirac matrices. To do so, we study the combinatorics of the contraction procedure outlined below, which will allow us to completely circumvent this computation and directly write down the resulting integer factors just from certain combinatorial properties of the Feynman graph.

We begin by recapitulating the basics and usual techniques that are used to deal with the Dirac matrices in Feynman integrals. As was already mentioned in the introduction, Dirac gamma matrices are a set of four complex 4×4 matrices that satisfy the anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \mathbb{1}_{4\times 4} \qquad \mu, \nu = 0, 1, 2, 3. \tag{4.1}$$

They generate a Clifford algebra, and there are many different possible choices of concrete representations for these matrices that are useful in different circumstances. However, one of the main points of this chapter is that we focus entirely on the combinatorics, which are independent of the representation, so we will not delve into this topic her.

On a related note, since we chose to write our integrals in Euclidean rather than Minkowski space we should actually work with yet another set of Euclidean Dirac matrices γ_E^μ and have a Kronecker delta $\delta^{\mu\nu}$, rather than the metric tensor. But since the combinatorics are the same either way and we never use an explicit representation we refrain from blowing up the notation with yet another subscript.

Contracting Dirac matrices the old-fashioned way

Traditionally the contraction is computed by iteratively applying the Clifford algebra relation eq. (4.1), or rather, an identity that can be derived from it:

$$\gamma_\mu \gamma_{\nu_1} \dots \gamma_{\nu_n} \gamma^\mu = \begin{cases} -2\gamma_{\nu_n} \dots \gamma_{\nu_1} & \text{if } n \text{ odd} \\ 2(\gamma_{\nu_n}\gamma_{\nu_1} \dots \gamma_{\nu_{n-1}} + \gamma_{\nu_{n-1}} \dots \gamma_{\nu_1}\gamma_{\nu_n}) & \text{if } n \text{ even} \end{cases} \tag{4.2}$$

It was first proved (independently and with different methods) by Caianello and Fubini [33] and Chisholm [37]. After all duplicate indices within one product of Dirac matrices are contracted one can continue by combining traces with the *Chisholm identity*[1] [38]

$$\gamma_\mu \operatorname{tr}(\gamma^\mu S) = 2(S + \tilde{S}) \tag{4.3}$$

where S is a product containing an odd number of Dirac matrices and $\tilde{S}$ is the same product reversed. When that identity cannot be applied anymore the remaining traces are expressed in terms of metric tensors with the recursion formula

$$\operatorname{tr}(\gamma^{\mu_1} \dots \gamma^{\mu_n}) = \sum_{i=2}^{n} (-1)^i g^{\mu_1\mu_i} \operatorname{tr}(\gamma^{\mu_2} \dots \widehat{\gamma^{\mu_i}} \dots \gamma^{\mu_n}). \tag{4.4}$$

Remark 4.1.1. *Note that the even case of the contraction relation can alternatively be expressed in the form*

$$\gamma_\mu \gamma_{\nu_1} \dots \gamma_{\nu_n} \gamma^\mu = 2(\gamma_{\nu_{k+1}} \dots \gamma_{\nu_n}\gamma_{\nu_1} \dots \gamma_{\nu_k} + \gamma_{\nu_k} \dots \gamma_{\nu_1}\gamma_{\nu_n} \dots \gamma_{\nu_{k+1}}) \tag{4.5}$$

for any odd $k < n$. This is discussed in more detail in section 4.2. To summarise: By exploiting the equivalence of these different choices we can reduce the recursive trace formula eq. (4.4) to a much shorter, non-recursive formula from which – among other things – the Chisholm identity eq. (4.3) follows as a trivial special case. This simplification in turn allows for the diagrammatic and combinatorial interpretation of contraction in section 4.3.

[1] Sometimes the previous eq. (4.2) is also called Chisholm identity, but here we will always use the name to refer to eq. (4.3)

Example 4.1.2. *We return to one of our examples from chapter 2. Consider the graph Γ_1 from fig. 2.3. Its Dirac matrix structure is given in eq. (2.34):*

$$\bar{\gamma}_{\Gamma_1} = (-1)\,\mathrm{tr}(\gamma^{\nu_1}\gamma^{\mu_2}\gamma^{\nu_2}\gamma^{\mu_3}\gamma^{\nu_3}\gamma^{\mu_4}\gamma^{\nu_4}\gamma^{\mu_5})$$

Its contraction with two metric tensors looks as follows:

$$\begin{aligned} g_{\nu_2\nu_4}g_{\mu_2\mu_4}\gamma_{\Gamma_1} &= -\,\mathrm{tr}(\gamma^{\nu_1}\gamma^{\mu_2}\underbrace{\gamma^{\nu_2}\gamma^{\mu_3}\gamma^{\nu_3}\gamma^{\mu_2}\gamma^{\nu_2}}_{=-2\gamma^{\mu_2}\gamma^{\nu_3}\gamma^{\mu_3}}\gamma^{\mu_5}) \\ &= 2\,\mathrm{tr}(\gamma^{\nu_1}\underbrace{\gamma^{\mu_2}\gamma^{\mu_2}}_{=4}\gamma^{\nu_3}\gamma^{\mu_3}\gamma^{\mu_5}) \\ &= 32(g_{\nu_1\nu_3}g_{\mu_3\mu_5} - g_{\nu_1\mu_3}g_{\nu_3\mu_5} + g_{\nu_1\mu_5}g_{\mu_3\nu_3}) \end{aligned} \tag{4.6}$$

Products of traces can be combined as follows:

$$\begin{aligned} &\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\nu_1}\gamma^{\nu_2})\,\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\nu_3}\gamma^{\nu_4}) \\ &\quad= \mathrm{tr}(\underbrace{\gamma^{\mu_1}\,\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\nu_1}\gamma^{\nu_2})}_{=2(\gamma^{\mu_2}\gamma^{\nu_1}\gamma^{\nu_2}+\gamma^{\nu_2}\gamma^{\nu_1}\gamma^{\mu_2})}\gamma^{\mu_2}\gamma^{\nu_3}\gamma^{\nu_4}) \\ &\quad= 2\Big(\mathrm{tr}(\underbrace{\gamma^{\mu_2}\gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\mu_2}}_{\substack{=2(\gamma^{\nu_2}\gamma^{\nu_1}+\gamma^{\nu_1}\gamma^{\nu_2})\\ =4g^{\nu_1\nu_2}}}\gamma^{\nu_3}\gamma^{\nu_4}) + \mathrm{tr}(\gamma^{\nu_2}\gamma^{\nu_1}\underbrace{\gamma^{\mu_2}\gamma^{\mu_2}}_{=4}\gamma^{\nu_3}\gamma^{\nu_4})\Big) \\ &\quad= 8\Big(g^{\nu_1\nu_2}\,\mathrm{tr}(\gamma^{\nu_3}\gamma^{\nu_4}) + \mathrm{tr}(\gamma^{\nu_2}\gamma^{\nu_1}\gamma^{\nu_3}\gamma^{\nu_4})\Big) \\ &\quad= 32\Big(2g^{\nu_1\nu_2}g^{\nu_3\nu_4} - g^{\nu_1\nu_4}g^{\nu_2\nu_3} + g^{\nu_1\nu_3}g^{\nu_2\nu_4}\Big) \end{aligned} \tag{4.7}$$

Computer algorithms for contraction (e.g. implemented as `trace4` in `FORM` [132]) typically try to successively apply the three equations (4.2), (4.3) and (4.4) until full contraction is achieved. However, as far back as the 1960s there have been attempts to find alternative contraction methods that bear some similarities to our approach [86]. Kahane developed an algorithm which involves instructions on how to first draw a diagram based on a given sequence of Dirac matrices. Following that the algorithm describes how to parse the diagram, simultaneously multiplying the result with certain factors depending on what one encounters. In our approach we use chord diagrams – a very well understood type of graph – together with a colouring to carry all the necessary information. Moreover, we isolate the relevant combinatorial property of the chord diagrams – the number of cycle subgraphs with a certain colouring – such that our result is a closed formula instead of an algorithm. Finally, Kahane's proofs are based on using a certain basis for the Clifford algebra generated by the Dirac matrices, while our results are entirely concluded from the contraction relation eq. (4.2). In fact, in section

4.2 we completely abstract the process of contraction from Dirac matrices to combinatorial sequences of letters representing the different space-time indices.

Kahane's algorithm was later generalised to products of traces by Chisholm [39], using his identity eq. (4.3). Working with Kahane's diagrams, the computations with this generalised algorithm become quite cumbersome[2]. Following our approach the general case follows very directly and with only marginally more complicated notation as corollary 4.3.13 from our single trace result theorem 4.3.9.

There are a multitude of modern methods that have been developed to deal with the problem of overly complicated contractions (e.g. spin-helicity, BCFW recursion [24, 57, 61]) and the reader may not yet be convinced that studying the combinatorics of the "traditional" contraction process is a worthwhile enterprise. However, especially outside of supersymmetric theories, such on-shell methods are not immune to becoming complicated and tedious either, and the standard contraction of Dirac matrices is still very much used today (e.g. in [36,77]). Instead of circumventing the contraction process, like these methods, we completely work it out, in a way that does not depend on any particular choice of representation for the gamma matrices or spinor basis, and give its end result for any QED graph, at any loop-order, in terms of simple chord diagrams. Moreover, our focus here lies of course on Feynman integrals in the parametric context, in which the above methods are plainly not applicable.

[2]In the words of J.S.R. Chisholm himself [39] : *"The proof of our final result is long and tedious, and even the statement of it is fraught with notational difficulties. We therefore explain it by an example, [...]"*

4.2 Combinatorics on words

4.2.1 Modelling Dirac matrices

The algebra of Dirac words

In this section we define an algebra that will serve as an abstraction of products of Dirac matrices and allow us to study their contraction and traces without any of the unnecessary ballast they carry.

Let $\mathsf{A} := \{\mathsf{a}_i \mid i \in \mathbb{N}\}$ be an alphabet. Then A^* with * denoting the Kleene star [90] is the set of words w ("noncommutative monomials") over A. The length, i.e. the number of letters, of a word w is denoted $|\mathsf{w}|$. We say a word is even (odd) if its length is even (odd) and we only consider words of finite length. $\tilde{\mathsf{w}}$ is the reversed word. Evidently, A^* is a free monoid. Moreover, A generates a free algebra $\mathbb{Z}\langle\mathsf{A}\rangle$ and we also use the nomenclature "word" for elements $\mathsf{w} = \sum c_j \mathsf{w}_j$ of this algebra. Unless explicitly stated otherwise we consider homogeneous words in which all "monomial words" have the same coefficient and are just rearrangements of the same letters. By linearity the discussion below holds in general, but we will see that we are only really interested in this kind of word.

In order to model Dirac matrices we have to satisfy three additional conditions:

- Each space-time index (i.e. each letter $\mathsf{a}_i \in \mathsf{A}$) appears at most twice.
- An analogon of the contraction relation eq. (4.2) holds.
- The word $\mathsf{g}_{ij} := \frac{1}{2}(\mathsf{a}_i\mathsf{a}_j + \mathsf{a}_j\mathsf{a}_i) \in \mathbb{Z}\langle\mathsf{A}\rangle$ has the right properties to serve as an analogon for the metric tensor.

We implement the first condition in our definition of Dirac words.

Definition 4.2.1. (Dirac words)
Let A *be the alphabet introduced above and* $I_k := \langle \mathsf{a}_i^k \mid i \in \mathbb{N}\rangle$ *the ideal generated by k-th powers of its letters. Then we define* Dirac words *as elements of the free algebra* $\mathbb{Z}\langle\mathsf{A}\rangle$ *divided by all third powers*

$$\mathsf{D} := \mathbb{Z}\langle\mathsf{A}\rangle/I_3. \tag{4.8}$$

Moreover, we define fully contracted Dirac words *as those Dirac words in which each letter appears at most once, i.e.*

$$\bar{\mathsf{D}} := \mathbb{Z}\langle\mathsf{A}\rangle/I_2. \tag{4.9}$$

The contraction relation eq. (4.2) is translated to letters and words in the obvious way as

$$\mathsf{a}_i\mathsf{u}\mathsf{a}_i = -2\tilde{\mathsf{u}} \qquad\qquad \mathsf{a}_i\mathsf{a}_j\mathsf{u}\mathsf{a}_i = 2(\mathsf{u}\mathsf{a}_j + \mathsf{a}_j\tilde{\mathsf{u}}) \tag{4.10}$$

for any odd $\mathsf{u} \in \mathsf{D}$. In remark 4.1.1 we discussed that the even case can be expressed in different but equivalent ways. We extend this discussion in section 4.2.1 which will allow us to formulate the contraction relation more elegantly in eq. (4.20), but for now this version suffices. Note that the even case also includes length 0, i.e. the empty word, as $\mathsf{a}_i^2 = 2(1+1) = 4$. Hence, each letter is up to an integer factor its own multiplicative inverse. This generalises to (monomial) words as $\mathsf{w}^{-1} = 2^{-2|\mathsf{w}|}\tilde{\mathsf{w}}$.

Finally, we can also introduce an analogue to the metric tensor by simply defining it as an abbreviation for a certain element of D that turns out to have exactly the desired properties.

Proposition 4.2.2. *Let* $\mathsf{g}_{ij} = \frac{1}{2}(\mathsf{a}_i\mathsf{a}_j + \mathsf{a}_j\mathsf{a}_i) \in \mathsf{D}$. *Then:*

(*i*) $\mathsf{g}_{ii} = 4$ (*ii*) $\mathsf{g}_{ij}\mathsf{a}_j = \mathsf{a}_i$ (*iii*) $\mathsf{g}_{ij}\mathsf{w} = \mathsf{w}\mathsf{g}_{ij}\ \forall \mathsf{w} \in \mathsf{D}$

Proof. The first equation follows directly from $\mathsf{a}_i^2 = 4$. For (*ii*) we employ the contraction relations (4.10) to find

$$\mathsf{g}_{ij}\mathsf{a}_j = \frac{1}{2}(\mathsf{a}_i\mathsf{a}_j\mathsf{a}_j + \mathsf{a}_j\mathsf{a}_i\mathsf{a}_j) = \frac{1}{2}(4\mathsf{a}_i - 2\mathsf{a}_i) = \mathsf{a}_i. \tag{4.11}$$

In order to prove (iii) note first that the exchange of a letter that we just proved also works if there is a word between g_{ij} and a_j, i.e. for $\mathsf{u} \in \mathsf{D}$ with $\mathsf{a}_j \notin \mathsf{u}$

$$\mathsf{g}_{ij}\mathsf{u}\mathsf{a}_j = \frac{1}{2}(\mathsf{a}_i\mathsf{a}_j\mathsf{u}\mathsf{a}_j + \mathsf{a}_j\mathsf{a}_i\mathsf{u}\mathsf{a}_j) = -\mathsf{a}_i\tilde{\mathsf{u}} + (\mathsf{u}\mathsf{a}_i + \mathsf{a}_i\tilde{\mathsf{u}}) = \mathsf{u}\mathsf{a}_i \tag{4.12}$$

if $|\mathsf{u}|$ odd, and

$$\mathsf{g}_{ij}\mathsf{u}\mathsf{a}_j = \frac{1}{4}\mathsf{a}_k\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{u}\mathsf{a}_j = -\frac{1}{2}\mathsf{a}_k\mathsf{a}_i\tilde{\mathsf{u}}\mathsf{a}_k = \mathsf{u}\mathsf{a}_i, \tag{4.13}$$

if $|\mathsf{u}|$ even. In the latter case we used eq. (4.10) to rewrite g_{ij} as

$$\mathsf{g}_{ij} = \frac{1}{2}(\mathsf{a}_i\mathsf{a}_j + \mathsf{a}_j\mathsf{a}_i) = \frac{1}{4}\mathsf{a}_k\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k \tag{4.14}$$

for some $k \neq i, j$. This is now used to show commutativity with a single letter, which suffices since a word can be commuted by sequentially commuting its letters:

$$\mathsf{g}_{ij}\mathsf{a}_l = \frac{1}{4}\mathsf{a}_k\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{a}_l = -\frac{1}{4}\mathsf{a}_k\mathsf{a}_i\mathsf{a}_j\mathsf{a}_l\mathsf{a}_k + \frac{1}{2}\mathsf{a}_k\mathsf{a}_i\mathsf{a}_j\mathsf{g}_{kl} = \frac{1}{2}\mathsf{a}_l\mathsf{a}_j\mathsf{a}_i + \frac{1}{2}\mathsf{a}_l\mathsf{a}_i\mathsf{a}_j = \mathsf{a}_l\mathsf{g}_{ij} \tag{4.15}$$

□

Remark 4.2.3. *The reader might be wondering why we did not simply use* $\mathsf{g}_{ij} = \frac{1}{2}(\mathsf{a}_i\mathsf{a}_j + \mathsf{a}_j\mathsf{a}_i)$ *as a Clifford algebra equation and derive the contraction relations from there as one does with Dirac matrices. However, that is not possible in this setting. In the Dirac matrix setting one can only derive eq. (4.2) from eq. (4.1) with the help of the additional information that there are only four Dirac matrices* $\gamma_0, \gamma_1, \gamma_2, \gamma_3$*. Therefore we include eq. (4.10) by definition and derive everything we need from there.*

Symmetry equivalence and commutativity

Define a symmetrisation/antisymmetrisation map sym : $\mathsf{D} \to \mathsf{D}$ with

$$\mathrm{sym}(\mathsf{w}) = \frac{1}{2}(\mathsf{w} + (-1)^{|\mathsf{w}|}\tilde{\mathsf{w}}). \tag{4.16}$$

such that $\mathrm{sym}(\mathsf{D}) \subset \mathsf{D}$ is the subset of even symmetric and odd antisymmetric Dirac words. Let furthermore $s_k : \mathsf{D} \to \mathsf{D}$ be the k-fold cyclic shift, i.e. for a (monomial) word $\mathsf{a}_i\mathsf{a}_j\mathsf{v}$ one has $s_1(\mathsf{a}_i\mathsf{a}_j\mathsf{v}) = \mathsf{a}_j\mathsf{v}\mathsf{a}_i$, $s_2(\mathsf{a}_i\mathsf{a}_j\mathsf{v}) = \mathsf{v}\mathsf{a}_i\mathsf{a}_j$ and so on. Using this new notation, reconsider the contraction relation eq. (4.10). The even case is

$$\mathsf{a}_i\mathsf{a}_j\mathsf{u}\mathsf{a}_i = 2(\mathsf{u}\mathsf{a}_j + \mathsf{a}_j\tilde{\mathsf{u}}) = 4\,\mathrm{sym}(s_1(\mathsf{a}_j\mathsf{u})). \tag{4.17}$$

We mentioned above that different decompositions are possible. Using the odd case of the contraction relation we find for an even word $\mathsf{w} = \mathsf{v}\mathsf{u}$ with $|\mathsf{v}|, |\mathsf{u}|$ odd that

$$\begin{aligned} \mathsf{a}_i\mathsf{v}\mathsf{u}\mathsf{a}_i &= -\frac{1}{2}\mathsf{a}_i\mathsf{v}\mathsf{a}_k\tilde{\mathsf{u}}\mathsf{a}_k\mathsf{a}_i \\ &= +\frac{1}{2}\mathsf{a}_i\mathsf{v}\mathsf{a}_k\tilde{\mathsf{u}}\mathsf{a}_i\mathsf{a}_k - \mathsf{g}_{ik}\mathsf{a}_i\mathsf{v}\mathsf{a}_k\tilde{\mathsf{u}} \\ &= -\mathsf{u}\mathsf{a}_k\tilde{\mathsf{v}}\mathsf{a}_k - \mathsf{a}_k\mathsf{v}\mathsf{a}_k\tilde{\mathsf{u}} = 2(\mathsf{u}\mathsf{v} + \tilde{\mathsf{v}}\tilde{\mathsf{u}}) = 4\,\mathrm{sym}(s_{|\mathsf{v}|}(\mathsf{w})) \end{aligned} \tag{4.18}$$

We see that – as far as the symmetrisation map is concerned – all the odd cyclic shifts of even words are the same. In other words:

Proposition 4.2.4. *Let* $\mathsf{u} \in \mathsf{D}$ *be a Dirac word with* $|\mathsf{u}|$ *even. Then*

$$\mathrm{sym}(\mathsf{u}) = \mathrm{sym}(s_{2k}(\mathsf{u})) \qquad \forall k \in \mathbb{N}. \tag{4.19}$$

The symmetrisation map induces an equivalence relation on D given by $\mathsf{u} \sim_{\mathrm{sym}} \mathsf{v}$ if and only if $\mathrm{sym}(\mathsf{u}) = \mathrm{sym}(\mathsf{v})$. For a given even word w there are two equivalence classes related by odd cyclic shifts: $[\mathsf{w}] \coloneqq \{s_{2k}(\mathsf{w}) \mid k \in \mathbb{N}\}$ and $[\mathsf{w}^*] \coloneqq \{s_{2k+1}(\mathsf{w}) \mid k \in \mathbb{N}\}$. Whenever no confusion can arise, we simply write w, w^*

for (an arbitrary representative of) the equivalence classes, such that odd cyclic shifts become maps $s_{2k+1}(\mathsf{w}) = \mathsf{w}^*$ and vice versa. The contraction relation in this notation becomes

$$\mathsf{a}_i \mathsf{u} \mathsf{a}_i = \begin{cases} -2\tilde{\mathsf{u}} & \text{if } |\mathsf{u}| \text{ odd}, \\ 4\,\mathrm{sym}(\mathsf{u}^*) & \text{if } |\mathsf{u}| \text{ even}. \end{cases} \tag{4.20}$$

Above we observed that $\mathsf{g}_{ij} = \frac{1}{2}(\mathsf{a}_i\mathsf{a}_j + \mathsf{a}_j\mathsf{a}_i) = \mathrm{sym}(\mathsf{a}_i\mathsf{a}_j)$ commutes with all other words. We can generalise this commutation property to longer words as follows.

Proposition 4.2.5. *Let* $\mathsf{v}, \mathsf{w} \in \mathsf{D}$ *with* $|\mathsf{v}|$ *even. Then*

$$\mathsf{w}\,\mathrm{sym}(\mathsf{v}) = \begin{cases} \mathrm{sym}(\mathsf{v})\mathsf{w} & \text{if } |\mathsf{w}| \text{ even}, \\ \mathrm{sym}(\mathsf{v}^*)\mathsf{w} & \text{if } |\mathsf{w}| \text{ odd}, \end{cases} \tag{4.21}$$

for all $\mathsf{w} \in \mathsf{D}$. *Moreover, a word* $\mathsf{u} \in \mathsf{D}$ *is a central element of* D, *i.e.* $\mathsf{uw} = \mathsf{wu}$ *for all* $\mathsf{w} \in \mathsf{D}$, *if and only if there exists an even* $\mathsf{v} \in \mathsf{D}$ *such that*

$$\mathsf{u} = \mathrm{sym}(\mathsf{v} + \mathsf{v}^*). \tag{4.22}$$

Proof. Consider commutation of a letter,

$$\mathsf{a}_i\,\mathrm{sym}(\mathsf{v}) = \frac{1}{4}\mathsf{a}_i\mathsf{a}_j\mathsf{v}^*\mathsf{a}_j = \frac{1}{4}(-\mathsf{a}_j\mathsf{a}_i + 2\mathsf{g}_{ij})\mathsf{v}^*\mathsf{a}_j = \mathrm{sym}(\mathsf{v}^*)\mathsf{a}_i.$$

Hence, successively commuting an odd or even number of letters in a word produces the first claim eq. (4.21) and commutativity of any $\mathsf{u} = \mathrm{sym}(\mathsf{v} + \mathsf{v}^*) = \mathrm{sym}(\mathsf{v}) + \mathrm{sym}(\mathsf{v}^*)$ is an immediate consequence. To see that all central elements have to be of this form consider the following two conditions. If u is central and even (i.e. $|\mathsf{u}|$ us even) then on the one hand

$$\mathsf{a}_i\mathsf{u} = \mathsf{u}\mathsf{a}_i = -\frac{1}{2}\mathsf{a}_j\mathsf{a}_i\tilde{\mathsf{u}}\mathsf{a}_j = -\frac{1}{2}\mathsf{a}_j\mathsf{a}_i\mathsf{a}_j\tilde{\mathsf{u}} = \mathsf{a}_i\tilde{\mathsf{u}}, \tag{4.23}$$

i.e. $\mathsf{u} \overset{!}{=} \tilde{\mathsf{u}}$. On the other hand one also has

$$\mathsf{a}_i\,\mathrm{sym}(\mathsf{u}) = \mathrm{sym}(\mathsf{u})\mathsf{a}_i = \mathsf{a}_i\,\mathrm{sym}(\mathsf{u}^*) \tag{4.24}$$

by commutativity and eq. (4.21), so $\mathsf{u} \overset{!}{=} \mathsf{u}^*$. Finally, there can be no odd commutative word since that would directly contradict the odd case of eq. (4.21). □

4.2.2 Traces and contraction

We have seen in the beginning that after contraction of all duplicate indices the trace of a product of Dirac matrices is computed with a recursion formula that decomposes it into metric tensors. We can translate that formula to our algebra to define the trace of Dirac words as a linear automorphism

$$\mathrm{tr}: \ \mathsf{a}_{i_1} \dots \mathsf{a}_{i_n} \mapsto \sum_{j=2}^{n} (-1)^j \mathsf{g}_{i_1 i_j} \, \mathrm{tr}(\mathsf{a}_{i_2} \dots \hat{\mathsf{a}}_{i_j} \dots \mathsf{a}_{i_n}), \qquad \forall n \geq 2 \tag{4.25}$$

on D, with the trace of the empty word $\mathrm{tr}(1) := 4$ corresponding to the trace of the 4×4 unit matrix in the Dirac matrix case. The trace $\mathrm{tr}(\mathsf{w}) \in \mathsf{D}$ is clearly central for every $\mathsf{w} \in \mathsf{D}$, so by proposition 4.2.5 there exists a word $\mathsf{w}' \in \mathsf{D}$ such that

$$\mathrm{tr}(\mathsf{w}) = \mathrm{sym}(\mathsf{w}' + \mathsf{w}'^*) \tag{4.26}$$

and w' differs from w at most by a constant factor, which we discuss in the following

Theorem 4.2.6. *For all* $\mathsf{w} \in \mathsf{D}$ *with* $|\mathsf{w}|$ *even*

$$\mathrm{tr}(\mathsf{w}) = 2\,\mathrm{sym}(\mathsf{w} + \mathsf{w}^*). \tag{4.27}$$

Proof. For $|\mathsf{w}| \in \{0, 2\}$ we can check explicitly that the claim holds:

$$2\,\mathrm{sym}(1 + 1) = 4 = \mathrm{tr}(1) \tag{4.28}$$

$$2(\mathrm{sym}(\mathsf{a}_i \mathsf{a}_j) + \mathrm{sym}(\mathsf{a}_j \mathsf{a}_i)) = 4\mathsf{g}_{ij} = \mathrm{tr}(\mathsf{a}_i \mathsf{a}_j) \tag{4.29}$$

Exploiting the recursive trace formula we then show the general case. Consider the word $\mathsf{a}_1 \mathsf{a}_2 \cdots \mathsf{a}_n$ and commute the first letter all the way to the end,

$$\begin{aligned} \mathsf{a}_1 \mathsf{a}_2 \cdots \mathsf{a}_n &= -\mathsf{a}_2 \mathsf{a}_1 \mathsf{a}_3 \cdots \mathsf{a}_n + 2\mathsf{g}_{12} \mathsf{a}_3 \cdots \mathsf{a}_n \\ &\vdots \\ &= (-1)^{n-1} \underbrace{\mathsf{a}_2 \cdots \mathsf{a}_n \mathsf{a}_1}_{= s_1(\mathsf{a}_1 \cdots \mathsf{a}_n)} + 2 \sum_{i=2}^{n} (-1)^i \mathsf{g}_{1i} \mathsf{a}_2 \cdots \hat{\mathsf{a}}_i \cdots \mathsf{a}_n. \end{aligned} \tag{4.30}$$

Using w_{ij} for the word w after removal of the i-th and j-th letter we can therefore write

$$2 \sum_{i=2}^{|\mathsf{w}|} (-1)^i \mathsf{g}_{1i} \mathsf{w}_{1i} = \mathsf{w} + (-1)^{|\mathsf{w}|} s_1(\mathsf{w}). \tag{4.31}$$

which is $\mathsf{w}+\mathsf{w}^*$ for even words. When trying to do the same for a sum $\sum(-1)^i \mathsf{g}_{1i}(\mathsf{w}_{1i})^*$ one encounters problems since $(\mathsf{w}_{ij})^* \neq (\mathsf{w}^*)_{ij}$. However, exploiting the symmetrisation and proposition 4.2.4 one quickly shows that for an even word w

$$2\sum_{i=2}^{|\mathsf{w}|}(-1)^i \mathsf{g}_{1i}\,\mathrm{sym}((\mathsf{w}_{1i})^*) = \mathrm{sym}(\mathsf{w}+\mathsf{w}^*) = 2\sum_{i=2}^{|\mathsf{w}|}(-1)^i \mathsf{g}_{1i}\,\mathrm{sym}(\mathsf{w}_{1i}). \tag{4.32}$$

The trick is to move each $\mathsf{g}_{1i} = \frac{1}{2}(\mathsf{a}_1\mathsf{a}_i + \mathsf{a}_i\mathsf{a}_1)$ into the i-th slot of w_{1i}, i.e. the place where the i-the letter has been removed. In the sum on the rhs this leads to a telescopic sum in which only half of the first and last terms remain. Due to the symmetrisation and proposition 4.2.4 the same trick can be applied to the sum with $(\mathsf{w}_{1i})^*$ albeit with slightly less obvious cancellations. Hence, one recursively finds

$$\mathrm{tr}(\mathsf{w}) = \sum_{i=2}^{|\mathsf{w}|}(-1)^i \mathsf{g}_{1i}\,\mathrm{tr}(\mathsf{w}_{1i}) = 2\sum_{i=2}^{|\mathsf{w}|}(-1)^i \mathsf{g}_{1i}\,\mathrm{sym}(\mathsf{w}_{1i}+(\mathsf{w}_{1i})^*) = 2\,\mathrm{sym}(\mathsf{w}+\mathsf{w}^*). \tag{4.33}$$

□

Remark 4.2.7. *With the above expression for traces one immediately sees Chisholm's identity eq. (4.3) as a special case:*

$$\mathsf{a}_i\,\mathrm{tr}(\mathsf{a}_i\mathsf{w}) = 2\mathsf{a}_i\,\mathrm{sym}(\mathsf{a}_i\mathsf{w}+\mathsf{w}\mathsf{a}_i) = \mathsf{a}_i^2\mathsf{w} + \mathsf{a}_i\tilde{\mathsf{w}}\mathsf{a}_i + \mathsf{a}_i\mathsf{w}\mathsf{a}_i + \mathsf{a}_i^2\tilde{\mathsf{w}} = 2(\mathsf{w}+\tilde{\mathsf{w}}) \tag{4.34}$$

Example 4.2.8. *Consider the trace of a word of length* 6 *which gives 15 terms in its usual expansion:*

$$\begin{aligned}\frac{1}{4}\,\mathrm{tr}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6) = {} & \mathsf{g}_{12}\mathsf{g}_{34}\mathsf{g}_{56} - \mathsf{g}_{12}\mathsf{g}_{35}\mathsf{g}_{46} + \mathsf{g}_{12}\mathsf{g}_{36}\mathsf{g}_{45} - \mathsf{g}_{13}\mathsf{g}_{24}\mathsf{g}_{56} + \mathsf{g}_{13}\mathsf{g}_{25}\mathsf{g}_{46} \\ & - \mathsf{g}_{13}\mathsf{g}_{26}\mathsf{g}_{45} + \mathsf{g}_{14}\mathsf{g}_{23}\mathsf{g}_{56} - \mathsf{g}_{14}\mathsf{g}_{25}\mathsf{g}_{36} + \mathsf{g}_{14}\mathsf{g}_{26}\mathsf{g}_{35} - \mathsf{g}_{15}\mathsf{g}_{23}\mathsf{g}_{46} \\ & + \mathsf{g}_{15}\mathsf{g}_{24}\mathsf{g}_{36} - \mathsf{g}_{15}\mathsf{g}_{26}\mathsf{g}_{34} + \mathsf{g}_{16}\mathsf{g}_{23}\mathsf{g}_{45} - \mathsf{g}_{16}\mathsf{g}_{24}\mathsf{g}_{35} + \mathsf{g}_{16}\mathsf{g}_{25}\mathsf{g}_{34}\end{aligned} \tag{4.35}$$

On the contrary, our new expression for the trace has only four terms:

$$\begin{aligned}\mathrm{tr}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6) &= 2\,\mathrm{sym}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6 + \mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6\mathsf{a}_1) \\ &= \mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6 + \mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6\mathsf{a}_1 + \mathsf{a}_6\mathsf{a}_5\mathsf{a}_4\mathsf{a}_3\mathsf{a}_2\mathsf{a}_1 + \mathsf{a}_1\mathsf{a}_6\mathsf{a}_5\mathsf{a}_4\mathsf{a}_3\mathsf{a}_2\end{aligned} \tag{4.36}$$

Moreover, this version of the trace has four terms for any length of word, while the number of terms in the recursive expansion grows factorially as $(|\mathsf{w}|-1)!!$.

4.3 Chord diagrams

Definition 4.3.1. (Chord diagram)
A chord diagram *D of order n is a graph, consisting of a cycle on $2n$ vertices (the* base*) and $k \leq n$ more edges that pairwise connect $2k$ of the vertices of that cycle (the* chords*). We denote with $\mathcal{D}$ the set of all chord diagrams and with $\mathcal{D}_k^n$ the restriction to those of order n with k chords.*

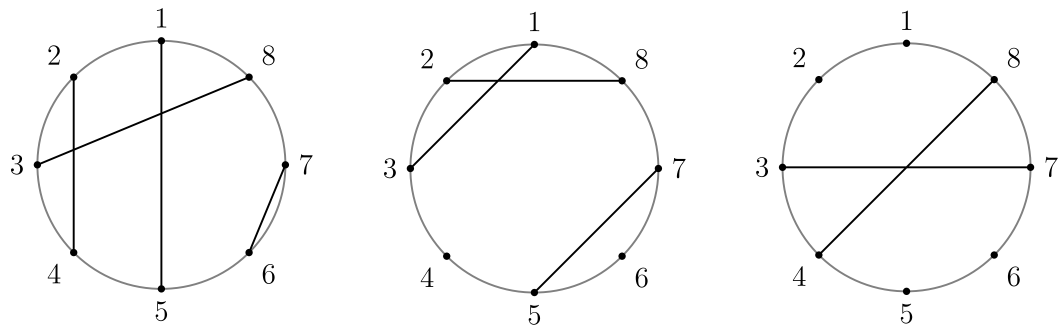


Figure 4.1: Three chord diagrams of order $n = 4$ with one base cycle and four, three and two chords respectively.

There is an obvious bijection between traces of (monomial) Dirac words w and chord diagrams that assigns to each vertex a letter (respecting the relative ordering) and represents duplicate letters by chords. The cyclicity of $\operatorname{tr}(\mathsf{w}) = 2\operatorname{sym}(\mathsf{w} + \mathsf{w}^*)$ is manifest in the base cycle of the chord diagram and since it is also symmetric it does not make a difference whether we choose to label the vertices clockwise or anti-clockwise. In somewhat of an abuse of notation we use D to denote this map $\mathsf{D} \to \mathcal{D}$, i.e. we write $D(\mathsf{w})$ for the chord diagram associated to the word w, but also continue to use the letter D generically for chord diagrams.

In order to include products of traces, in particular those that contain contractions of matrices in different traces, the usual definition of chord diagrams is not enough, so we generalise as follows:

Definition 4.3.2. (Generalised chord diagram)
A generalised chord diagram *of order $n = (n_1, \ldots, n_\ell)$ is a graph that consists of ℓ chord diagrams $D_i \in \mathcal{D}_{k_i}^{n_i}$, that may additionally contain edges between vertices in different base cycles – which we will also call chords – while still keeping each vertex at most 3-valent. We write $N = \sum n_i$ for the total order of the diagram and denote with $\mathcal{D}_k^{(n_1,\ldots,n_\ell)}$ the set of chord diagrams with the respective number and size of base cycles and $k \leq N$ chords.*

In the following we will always just write chord diagram for the general version. The sets of the two types of vertices of a chord diagram – 2-valent and 3-valent – are denoted $V_D^{(2)}$ and $V_D^{(3)}$. Furthermore, in addition to the distinction between base edges and chords we will need to introduce more properties to differentiate between certain types of edge subsets. This is achieved via colouring.

4.3.1 Colours and cycles

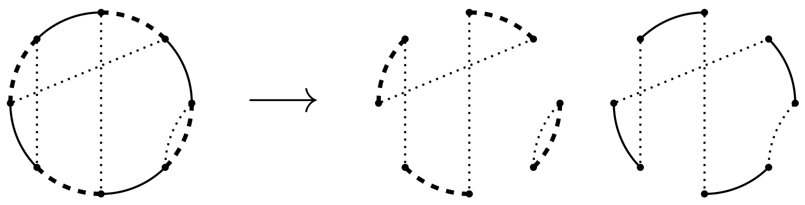
Definition 4.3.3. (Edge k-colouring)
Let G be a graph and K a finite set consisting of k colours. *Then a map $\kappa : E_G \to K$ is called a k-*edge-coloring *if for every vertex v of G all edges incident to it are assigned different colours, i.e. if κ is injective on $\partial^{-1}(\{v\} \times V_G) \subset E_G$ for all $v \in V_G$.*

The number of colours needed to colour a given graph is given by Vizing's theorem to be either the maximal degree Δ of the graph or $\Delta + 1$ [133]. Clearly, each chord diagram D admits an edge 3-coloring $\kappa : E_D \to \{0, 1, 2\}$ - sometimes called *Tait colouring* [125] - where two alternating colours 1 and 2 are assigned to the edges of the base cycles and the third colour 0 to all chords. Fix one of the 2^ℓ possibilities of such a colouring. This edge colouring induces a unique (up to permutations of colours) double cover $\{E_D^{01}, E_D^{02}, E_D^{12}\}$ of the chord diagram in which the components $E_D^{ij} = \kappa^{-1}(\{i, j\})$ are given by edge subsets that have exactly two different colours. Analogously we write E_D^0, E_D^1 and E_D^2 for the respective single colour edge subsets. Furthermore, each two-coloured edge subset can be decomposed into collections $\mathcal{C}_D^{ij}, \mathcal{P}_D^{ij}$ of cycles and paths with $\mathcal{P}_D^{12} = \emptyset$ and $|\mathcal{C}_D^{12}| = \ell(D)$ since the bases are the only cycles with these two colours. The bicoloured paths between the 2-valent vertices of D can always be combined to form tricoloured cycles by joining all paths in their shared initial or final vertices. Contracting each path in $\mathcal{P}_D^{01}$ and $\mathcal{P}_D^{02}$ to a single edge of colour 1 or 2 projects the tricoloured cycles onto a generalised chord diagram D' that consists of a disjoint union of base cycles without any chords. Specifically, it defines a map

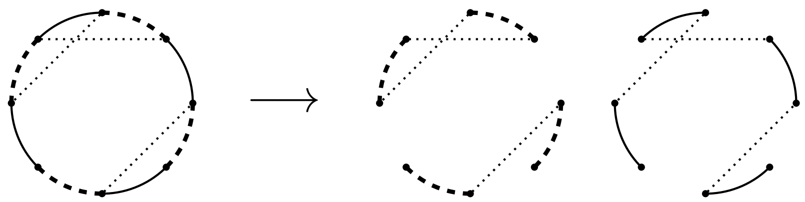
$$\pi_0 : \mathcal{D}_k^n \to \mathcal{D}_0^{n'} \tag{4.37}$$

with $n' = (n'_1, \dots, n'_{\ell'})$, $N' = N - k$ and $\max\{0, \ell - k\} \leq \ell' \leq \ell + k$. The number of bicoloured (excluding bases) and tricoloured cycles is the central combinatorial property that we will need later, so we introduce a separate notation for it:

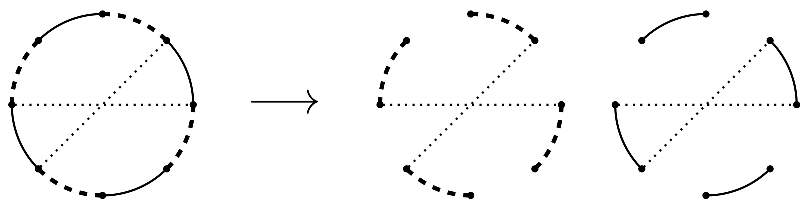
$$c_2(D) := |\mathcal{C}_D^{01}| + |\mathcal{C}_D^{02}| \qquad\qquad c_3(D) := \ell(\pi_0(D)) \tag{4.38}$$



(a) There are no free vertices, so there are no paths but a total of 3 bicoloured cycles.



(b) The 01-component consists of a single path while the 02-component contains a path and a bicoloured cycle. Note that the two paths share initial and final vertices, such that they combine to a tricoloured cycle.



(c) There are four paths in total, but they all combine to form a single tricoloured cycle.

Figure 4.2: Colour decompositions of the chord diagrams from fig. 4.1.

Example 4.3.4. *In drawings we use different line types to represent the colours:*

$0 \sim$ ·············· $1 \sim$ ——— $2 \sim$ - - - - - - -

Let D_1, D_2, D_3 *be the three chord diagrams from fig. 4.2 (a)-(c). For* D_1 *there are no free vertices, so all bicoloured components are cycles and*

$$c_2(D_1) = 2 + 1 = 3 \qquad c_3(D_1) = 0.$$

D_2 *contains two free vertices - 4 and 6 with the labelling from fig. 4.1 - with two* *different bicoloured paths between them, forming a tricoloured cycle. Overall*

$$c_2(D_2) = 1 \qquad c_3(D_2) = 1.$$

Finally, D_3 has four free vertices. There is one cycle bicoloured with $\{0, 1\}$. The four paths form a single tricoloured cycle, so

$$c_2(D_3) = 1 \qquad c_3(D_3) = 1.$$

The action of the projection map, contracting this tricoloured cycle to a base cycle without chords is depicted in fig. 4.3 below.

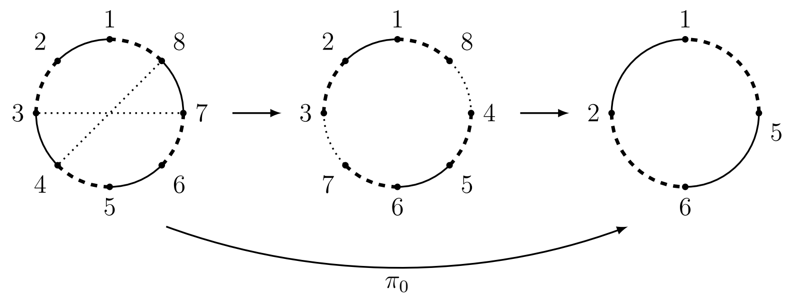


Figure 4.3: Visualisation of the projection map π_0 for the case of diagram D_3 from fig. 4.2c.

Finally, it is convenient to have a notion of distance between two free vertices, $u, v \in V_D^{(2)}$. If both are contained within the same tricoloured cycle then the two vertices split it into two segments and we define $d_+(u, v)$ and $d_-(u, v)$ to be the number of paths in the longer and shorter of these two segments. As a convention we say $d_\pm(u, v) = -1$ if u and v are in different cycles. Moreover, we also define $\operatorname{sgn}(u, v) := d_\pm(u, v) \mod 2$.

In the following proposition we show how the cycle numbers c_2 and c_3 change upon addition of another chord to a diagram. This will provide us with the foundation we need to prove the main results of the following section.

Proposition 4.3.5. *Let $D_0 \in \mathcal{D}_{k-1}^n$ with $1 \leq k \leq N = \sum n_i$ and $D \in \mathcal{D}_k^n$ result from D_0 by adding a chord between two vertices $u, v \in V_{D_0}^{(2)}$. In other words, if $D_0 = D(\mathsf{w})$ for some Dirac word w, then $D = D(\mathsf{g}_{uv}\mathsf{w})$. Then there are the following possibilities.*

1. *If $d_\pm(u, v) > 0$ and*

(a) $d_+(u,v) = 1 = d_-(u,v)$*, then*

$$c_2(D) = c_2(D_0) + 2 \qquad c_3(D) = c_3(D_0) - 1.$$

(b) $d_\pm(u,v) = 1$ *and* $d_\mp(u,v) > 1$ *(necessarily odd), then*

$$c_2(D) = c_2(D_0) + 1 \qquad c_3(D) = c_3(D_0).$$

(c) $d_+(u,v)$ *and* $d_-(u,v)$ *are both even, then*

$$c_2(D) = c_2(D_0) \qquad c_3(D) = c_3(D_0).$$

(d) $d_\pm(u,v) > 1$ *and both are odd, then*

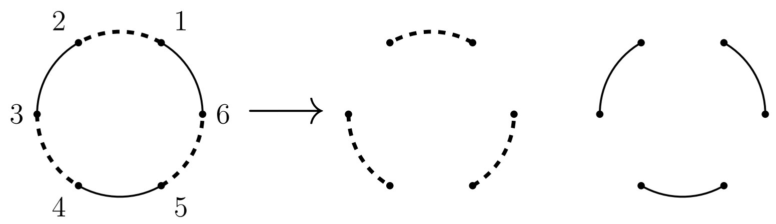
$$c_2(D) = c_2(D_0) \qquad c_3(D) = c_3(D_0) + 1.$$

2. If $d_\pm(u,v) = -1$*, then*

$$c_2(D) = c_2(D_0) \qquad c_3(D) = c_3(D_0) - 1.$$

Proof. The cases *1.(a)* and *1.(b)* are apparent since any single path is completed by a chord to form a new bicoloured cycle, while the other segment remains a tricoloured cycle with the chord in place of the former path. In *1.(c)* both segments have two differently coloured edges on their ends and their oppositely coloured ends are incident to each other in u and v. Hence the new chord bridges the equally coloured endings which results in a new tricoloured cycle. Visually, a plane tricoloured cycle is twisted into an ∞-shape, or alternatively, one segment is cut out, flipped and glued back into the cycle with chords as glue. In *1.(d)* both ends of either segment have the same colour, such that the new chord cleanly separates the tricoloured cycle into two new cycles. Finally, in the second case the edges of either colour incident to u are connected by the chord to the equally coloured edge incident to v in the other cycle such that a single new cycle results. □

Example 4.3.6. *Adding a chord between the two free vertices of* D_2 *from example 4.3.4 (cf. fig. 4.2b) falls into case 1.(a). All six possible ways to add a chord between any two vertices of* D_3 *(fig. 4.2c) are examples of case 1.(b). To illustrate the other cases one needs either larger very complicated diagrams or almost trivial cases, so for simplicity consider* $D_0 \in \mathcal{D}_0^3$ *to be the empty chord diagram of order* 3*, in which every single base edge is itself a path:*



New chords between any pair of vertices separated by one other vertex (say $(1,3)$*) correspond to case 1.(c). Adding a chord between any of the pairs* $(1,4)$*,* $(2,5)$ *or* $(3,6)$ *corresponds to case 1.(d), where the base cycle is split into two new three-coloured cycles. Say the chord* $(1,4)$ *is added. Then additionally connecting* $(2,5)$ *or* $(3,6)$ *would be examples for case 2.*

4.3.2 Chord diagrams and words

Cycle words and diagram contraction

For this section we only consider the single base cycle case $\ell = 1$. The results are then generalised in the following section. Above we already mentioned the relation between traces of monomial Dirac words and chord diagrams. Let $\mathsf{w} \in \mathsf{D}$ be a Dirac word such that $D(\mathsf{w}) \in \mathcal{D}^n_k$ for $k < n$. Then $D(\mathsf{w})$ contains at least one tricoloured cycle and $|V^{(2)}_D| = 2(n-k)$ free vertices, corresponding to the non-duplicate letters of w. The structure of D then tells us how to arrange these letters into new words in $\bar{\mathsf{w}}(D) \in \bar{\mathsf{D}}$ which will allow us to compute the contractions of duplicate letters easily.

Definition 4.3.7. (Cycle words)
Let $D \in \mathcal{D}^n_k$ *be a chord diagram with the canonical edge 3-colouring introduced above and* $D' = \sqcup_{i=1}^{\ell'} D'_i = \pi_0(D)$*. Then for each* D'_i *consider the words* $\mathsf{u}_i \in \bar{\mathsf{D}}$ *that satisfy* $D(\mathsf{u}_i) = D'_i$*. Up to cyclic shifts there are four such words for each* D'_i *and they are related to each other as* $\mathsf{u}_i, \tilde{\mathsf{u}}_i, \mathsf{u}^*_i$ *and* $\widetilde{\mathsf{u}^*_i}$*. Using these words we define the cycle word associated to* D *as*

$$\bar{\mathsf{w}}(D) \coloneqq \prod_{i=1}^{\ell'} \mathrm{sym}(\mathsf{u}_i) + \prod_{i=1}^{\ell'} \mathrm{sym}(\mathsf{u}^*_i). \tag{4.39}$$

Example 4.3.8. *Consider the chord diagram* D_3 *from fig. 4.3 and fig. 4.2c, previously discussed in examples 4.3.4 and 4.3.6. It has the four free vertices* $1, 2, 5$ *and* 6*, with four paths* $1-2$*,* $2-3-7-6$*,* $6-5$ *and* $5-4-8-1$ *combining to one tricoloured cycle. Note that after projection to a base cycle the free vertices are not in the original order anymore. Choose for example* $\mathsf{u} = \mathsf{a}_1\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5$*. Then*

$$\bar{\mathsf{w}}(D_3) = \frac{1}{2}\Big(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5 + \mathsf{a}_5\mathsf{a}_6\mathsf{a}_2\mathsf{a}_1 + \mathsf{a}_2\mathsf{a}_6\mathsf{a}_5\mathsf{a}_1 + \mathsf{a}_1\mathsf{a}_5\mathsf{a}_6\mathsf{a}_2\Big) \tag{4.40}$$

For an example with multiple cycles consider the empty order 3 diagram also discussed in example 4.3.6 together with a single chord between $(3,6)$*. One cycle consists of the two paths* $2-3-6-1$ *and* $1-2$ *which gives the word* $\mathsf{u}_1 = \mathsf{a}_2\mathsf{a}_1$

while the other in analogous fashion gives $\mathsf{u}_2 = \mathsf{a}_4\mathsf{a}_5$. *The cycle word is then*

$$\frac{1}{4}\Big((\mathsf{a}_2\mathsf{a}_1 + \mathsf{a}_1\mathsf{a}_2)(\mathsf{a}_4\mathsf{a}_5 + \mathsf{a}_5\mathsf{a}_4) + (\mathsf{a}_1\mathsf{a}_2{+}\mathsf{a}_2\mathsf{a}_1)(\mathsf{a}_5\mathsf{a}_4 + \mathsf{a}_4\mathsf{a}_5)\Big)$$
$$= \frac{1}{2}\Big((\mathsf{a}_2\mathsf{a}_1 + \mathsf{a}_1\mathsf{a}_2)(\mathsf{a}_4\mathsf{a}_5 + \mathsf{a}_5\mathsf{a}_4)\Big). \quad (4.41)$$

In the case $k = 0$ one has $\pi_0(D(\mathsf{w})) = D(\mathsf{w}) \in \mathcal{D}_0^n$ with $\ell' = \ell = 1$ and therefore

$$\bar{\mathsf{w}}(D(\mathsf{w})) = \mathrm{sym}(\mathsf{w}) + \mathrm{sym}(\mathsf{w}^*) = \frac{1}{2}\,\mathrm{tr}(\mathsf{w}) \quad (4.42)$$

by theorem 4.2.6. This is quite sensible since we can interpret the "contraction" of a word without duplicate letters to contract as the expansion into g_{ij} via the trace recursion formula, divided by $4 = \mathrm{tr}(1)$. On the other hand one sees that if $k = n$ then there are no more tricoloured cycles in $D(\mathsf{w})$ and $\bar{\mathsf{w}}(D(\mathsf{w})) = 2$. The overall factor of 2 in both cases is due to the fact that in the case of a single base cycle the two terms $\mathrm{sym}(\mathsf{w})$ and $\mathrm{sym}(\mathsf{w}^*)$ correspond to the two equivalent alternating colourings of base cycle edges with the two possible colours. We could absorb this by including a 1/2 in the definition of $\bar{\mathsf{w}}(D(\mathsf{w}))$ but it will be more useful, especially when generalising to multiple base cycles where different colourings yield completely different cycle words, to keep all integer factors isolated outside of the cycle word.

In general we find the following relation between w and $\bar{\mathsf{w}}$.

Theorem 4.3.9. *Let* $\mathsf{w} \in \mathsf{D}$ *be a monomial Dirac word such that the associated chord diagram* $D(\mathsf{w}) \in \mathcal{D}_k^n$, $0 \leq k \leq n$. *Then*

$$\mathrm{tr}(\mathsf{w}) = -(-2)^{k+c_2(D(\mathsf{w}))+c_3(D(\mathsf{w}))}\ \bar{\mathsf{w}}(D(\mathsf{w})). \quad (4.43)$$

Proof. As discussed above the case $k = 0$ gives $\mathrm{tr}(\mathsf{w}) = 2\bar{\mathsf{w}}(D(\mathsf{w})) = -(-2)\bar{\mathsf{w}}(D(\mathsf{w}))$ where $c_3 = 1$ and $c_2 = 0$, such that the claim holds for all n. For $0 < k \leq n$ we prove by induction over the number of chords. Let $\mathsf{w}' \in \mathsf{D}$ such that $\mathsf{w} = \mathsf{g}_{ij}\mathsf{w}'$. We abbreviate $c_2 \equiv c_2(D(\mathsf{w}))$, $c_3 \equiv c_3(D(\mathsf{w}))$ and $\bar{\mathsf{w}} \equiv \bar{\mathsf{w}}(D(\mathsf{w}))$, and write c_2', c_3' and $\bar{\mathsf{w}}'$ for the corresponding objects resulting from w'. $D(\mathsf{w})$ is $D(\mathsf{w}')$ together with a chord between vertices i and j and we need to consider the same five cases as in proposition 4.3.5. The idea is the same for all of them: Use the contraction relation eq. (4.20) to compute $\mathsf{g}_{ij}\bar{\mathsf{w}}' = N\bar{\mathsf{w}}$ where N is some integer factor. Then confirm that both the change in term structure of the cycle word and the new integer factor is in accordance with change in cycle structure and cycle numbers c_2 and c_3 as discussed in proposition 4.3.5.

1. (a) A single path in both segments yields the simple two letter words $\mathrm{sym}(\mathsf{u}_{ij}) = \frac{1}{2}(\mathsf{a}_i\mathsf{a}_j + \mathsf{a}_j\mathsf{a}_i) = \mathrm{sym}(\mathsf{u}^*_{ij})$ such that $\mathsf{g}_{ij}\bar{\mathsf{w}}' = 4\bar{\mathsf{w}}$, where $\bar{\mathsf{w}}$ is the same as $\bar{\mathsf{w}}'$ except that the entire base cycle that contained i and j – and no other vertices – has been removed from both products. Hence, in accordance with 4.3.5 we have $c_3 = c'_3 - 1$. Furthermore we have $c_2 = c'_2 + 2$ and one more chord $(k-1 \to k)$ such that we can identify the integer factor 4 as $(-2)^{1+2-1}$ and find

$$\begin{aligned} \mathrm{tr}(\mathsf{w}) = \mathsf{g}_{ij}\,\mathrm{tr}(\mathsf{w}') &= -(-2)^{k-1+c'_2+c'_3}\mathsf{g}_{ij}\bar{\mathsf{w}}' \\ &= -(-2)^{k-1+c'_2+c'_3}4\bar{\mathsf{w}} \\ &= -(-2)^{k+c_2+c_3}\bar{\mathsf{w}}. \end{aligned} \tag{4.44}$$

(b) Now one segment is longer but the other still contains only one path, such that the two letters associated to i and j are still neighbours in the cycle word. That is, there are words $\mathsf{v}_1, \mathsf{v}_2 \in \bar{\mathsf{D}}$ such that $\mathsf{u}_{ij} = \mathsf{v}_1\mathsf{a}_i\mathsf{a}_j\mathsf{v}_2$. Multiplying with g_{ij} extracts the factor $4 = \mathsf{a}_i^2$ but otherwise leaves the product structure of $\bar{\mathsf{w}}'$ intact (and in particular $c_3 = c'_3$). There is one additional chord and one new bicoloured cycle, absorbing the factor $4 = (-2)^{1+1+0}$:

$$\mathrm{tr}(\mathsf{w}) = -(-2)^{k-1+c'_2+c'_3}4\bar{\mathsf{w}} = -(-2)^{k+c_2+c_3}\bar{\mathsf{w}} \tag{4.45}$$

(c) With multiple paths in both segments a_i and a_j are not neighbours anymore. Instead, $\mathsf{u}_{ij} = \mathsf{v}_1\mathsf{a}_i\mathsf{v}_2\mathsf{a}_j\mathsf{v}_3$ for words $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3 \in \bar{\mathsf{D}}$ and in this case $|\mathsf{v}_2|$ odd. One can use

$$\mathsf{g}_{ij}\mathsf{a}_i\mathsf{v}_2\mathsf{a}_j = \mathsf{a}_i\mathsf{v}_2\mathsf{a}_i = -2\tilde{\mathsf{v}}_2 \tag{4.46}$$

and as before one finds (with $c_2 = c'_2$ and $c_3 = c'_3$)

$$\mathrm{tr}(\mathsf{w}) = -(-2)^{k-1+c'_2+c'_3}(-2)\bar{\mathsf{w}} = -(-2)^{k+c_2+c_3}\bar{\mathsf{w}}. \tag{4.47}$$

The reversal of v_2 clearly mirrors the twisting of the tricoloured cycle in the diagram.

(d) Here we have $\mathsf{u}_{ij} = \mathsf{v}_1\mathsf{a}_i\mathsf{v}_2\mathsf{a}_j\mathsf{v}_3$ analogously to the previous case, but now $|\mathsf{v}_2|$ is even. One can use proposition 4.2.5 to find

$$\begin{aligned} \mathsf{g}_{ij}\mathsf{u}_{ij} = \mathsf{v}_1\mathsf{a}_i\mathsf{v}_2\mathsf{a}_i\mathsf{v}_3 &= 4\mathsf{v}_1\,\mathrm{sym}(\mathsf{v}_2^*)\mathsf{v}_3 \\ &= 4\mathsf{v}_1\mathsf{v}_3\begin{cases}\mathrm{sym}(\mathsf{v}_2^*) & \text{if } |\mathsf{v}_1|, |\mathsf{v}_3| \text{ even,} \\ \mathrm{sym}(\mathsf{v}_2) & \text{if } |\mathsf{v}_1|, |\mathsf{v}_3| \text{ odd,}\end{cases} \end{aligned} \tag{4.48}$$

and write

$$\operatorname{sym}(\mathsf{g}_{ij}\mathsf{u}_{ij}) = 2\begin{cases} \mathsf{v}_1\mathsf{v}_3 \operatorname{sym}(\mathsf{v}_2^*) + \operatorname{sym}(\mathsf{v}_2^*)\tilde{\mathsf{v}}_3\tilde{\mathsf{v}}_1 & \text{if } |\mathsf{v}_1|, |\mathsf{v}_3| \text{ even} \\ \mathsf{v}_1\mathsf{v}_3 \operatorname{sym}(\mathsf{v}_2) + \operatorname{sym}(\mathsf{v}_2)\tilde{\mathsf{v}}_3\tilde{\mathsf{v}}_1 & \text{if } |\mathsf{v}_1|, |\mathsf{v}_3| \text{ odd} \end{cases}$$

$$= 4\operatorname{sym}(\mathsf{v}_1\mathsf{v}_3)\begin{cases} \operatorname{sym}(\mathsf{v}_2^*) & \text{if } |\mathsf{v}_1|, |\mathsf{v}_3| \text{ even} \\ \operatorname{sym}(\mathsf{v}_2) & \text{if } |\mathsf{v}_1|, |\mathsf{v}_3| \text{ odd.} \end{cases} \tag{4.49}$$

If the even (odd) case applies to u_{ij} then the odd (even) case can be used to find the analogous result (with $(\mathsf{v}_1\mathsf{v}_3)^*$) for u_{ij}^*. We see the expected splitting into two tricoloured cycles realised in the products. Altogether one finds

$$\operatorname{tr}(\mathsf{w}) = -(-2)^{k-1+c_2'+c_3'} 4\bar{\mathsf{w}} = -(-2)^{k+c_2+c_3}\bar{\mathsf{w}}. \tag{4.50}$$

2. For this final case we work the other way around. We begin with a product $\operatorname{sym}(\mathsf{u}_i)\operatorname{sym}(\mathsf{u}_j)$. One can always choose representatives u_i and u_j such that a_i and a_j are either their first or last letter respectively. Hence, there exist words $\mathsf{v}_1, \mathsf{v}_2 \in \bar{\mathsf{D}}$ such that

$$\mathsf{g}_{ij}\operatorname{sym}(\mathsf{u}_i)\operatorname{sym}(\mathsf{u}_j) = \frac{1}{4}(\mathsf{a}_i\mathsf{v}_1 + \tilde{\mathsf{v}}_1\mathsf{a}_i)(\mathsf{a}_i\mathsf{v}_2 + \tilde{\mathsf{v}}_2\mathsf{a}_i) = \operatorname{sym}(\tilde{\mathsf{v}}_2\mathsf{v}_1). \tag{4.51}$$

The factor here is $1 = (-2)^{1+0-1}$ where we have one more chord but lost one tricoloured cycle, so here, too, everything works out as claimed.

□

Remark 4.3.10. *Above we only discussed contraction of traces of even words. In practice one would also like to contract odd words, which are associated to "open" fermion lines in a Feynman graph. For contraction of such a word* $\mathsf{w}' \in \mathsf{D}$ *with* $|\mathsf{w}'|$ *odd consider the word* $\mathsf{w} = \mathsf{w}_1\mathsf{a}_i\mathsf{w}_2$ *where* $\mathsf{a}_i \in \mathsf{A}$ *is a dummy letter that does not occur in* w', $\mathsf{w}_1\mathsf{w}_2 = \mathsf{w}'$ *and* w_2 *starts with the first letter that occurs only once in the* w'*, i.e. is not contracted. The trace and its contraction of* w *can be computed as above. The contraction of the odd word is then simply obtained by dividing by 4 and "unsymmetrising" the factor corresponding to the tricoloured cycle that contains the dummy vertex in the two products in the cycle word, i.e.*

$$\operatorname{sym}(\mathsf{u}_i) \to \mathsf{u}_i \text{ or } \tilde{\mathsf{u}}_i \qquad \text{and} \qquad \operatorname{sym}(\mathsf{u}_i^*) \to \mathsf{u}_i^* \text{ or } \tilde{\mathsf{u}}_i^*$$

where the choice is fixed by demanding that upon evaluation $\mathsf{a}_i \to 1$ *the first letter of the unsymmetrised word is the first letter of* w_2*. See also example 4.3.11 below.*

Example 4.3.11. *We return again to the contraction of* $g_{\nu_2\nu_4} g_{\mu_2\mu_4} \gamma_{\Gamma_1}$ *from example 4.1.2, corresponding to the Dirac word*

$$\mathsf{w} = \mathsf{g}_{37}\mathsf{g}_{48}\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6\mathsf{a}_7\mathsf{a}_8 \tag{4.52}$$

whose chord diagram is again $D(\mathsf{w}) = D_3$*, the rightmost diagram in fig. 4.1 which we already discussed in all the previous examples. In example 4.3.4 we found* $c_2(D_3) = 1$ *and* $c_3(D_3) = 1$*, and in example 4.3.8 we saw*

$$\bar{\mathsf{w}}(D_3) = \mathrm{sym}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5) + \mathrm{sym}(\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5\mathsf{a}_1) = \frac{1}{2}\,\mathrm{tr}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5). \tag{4.53}$$

Therefore $\mathrm{tr}(\mathsf{w}) = -(-2)^{2+1+1}\frac{1}{2}\,\mathrm{tr}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5) = -8\,\mathrm{tr}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5)$*, which is the same result as in the previous manual computation.*

Let $\mathsf{v} = \mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6\mathsf{a}_3\mathsf{a}_4$ *be the odd word such that* $\mathsf{w} = \mathsf{a}_1\mathsf{v}$*. We compute its contraction following remark 4.3.10. There is only one factor in the cycle word to be unsymmetrised and the choice is such that* a_2 *is the first letter after removal of* a_1*. One finds*

$$\mathsf{v} = -(-2)^4\frac{1}{2}\frac{1}{2}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5 + \mathsf{a}_2\mathsf{a}_6\mathsf{a}_5\mathsf{a}_1)_{\mathsf{a}_1\to 1} = -8\mathsf{a}_2\mathsf{a}_6\mathsf{a}_5, \tag{4.54}$$

which is the expected result.

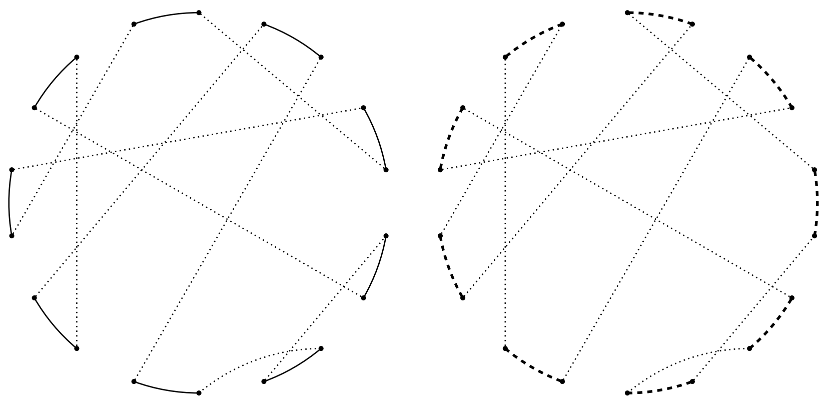


Figure 4.4: Two bicoloured components of the diagram associated to the trace in example 4.3.12.

Example 4.3.12. *We can compute a larger example, like the contraction of an 18 letter word, to demonstrate the efficiency of this contraction formalism. Let*

$$\begin{aligned}\operatorname{tr}(\mathsf{w}) &= \operatorname{tr}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_1\mathsf{a}_6\mathsf{a}_2\mathsf{a}_8\mathsf{a}_9\mathsf{a}_{10}\mathsf{a}_9\mathsf{a}_3\mathsf{a}_{10}\mathsf{a}_{14}\mathsf{a}_4\mathsf{a}_8\mathsf{a}_6\mathsf{a}_{14}) \\ &= (-2)^3 \operatorname{tr}(\mathsf{a}_4\mathsf{a}_3\mathsf{a}_2\mathsf{a}_6\mathsf{a}_2\mathsf{a}_8\mathsf{a}_{10}\mathsf{a}_3\mathsf{a}_{10}\mathsf{a}_6\mathsf{a}_8\mathsf{a}_4) \\ &= (-2)^5 \operatorname{tr}(\mathsf{a}_4\mathsf{a}_3\mathsf{a}_6\mathsf{a}_8\mathsf{a}_3\mathsf{a}_6\mathsf{a}_8\mathsf{a}_4) \\ &= (-2)^7 \operatorname{tr}(\mathsf{a}_4\mathsf{a}_6\mathsf{a}_6\mathsf{a}_4) \\ &= (-2)^{13} \end{aligned} \tag{4.55}$$

where we already combined multiple contractions in the same line and chose an efficient order of contractions. For our formalism we simply count the number chords ($k = 9$), and cycles ($c_3(D(\mathsf{w})) = 0$, $c_2(D(\mathsf{w})) = 3$, two depicted in fig. 4.4 on the left and one on the right). Hence we have indeed $\operatorname{tr}(\mathsf{w}) = -(-2)^{9+3+0}2 = (-2)^{13}$.

4.3.3 Multiple traces

Above we considered contraction of single traces but theorem 4.3.9 can be generalised to arbitrary products of traces – including contraction of letters occurring in different traces – without much effort.

Consider first two words $\mathsf{w}_1, \mathsf{w}_2 \in \mathsf{D}$ without any shared letters and $D_i = D(\mathsf{w}_i) \in \mathcal{D}_{k_i}^{n_i}$ for $i = 1, 2$ their respective chord diagrams. Multiplying their traces gives

$$\operatorname{tr}(\mathsf{w}_1)\operatorname{tr}(\mathsf{w}_2) = (-2)^{k_1+k_2+c_2(D_1)+c_2(D_2)+c_3(D_1)+c_3(D_2)}\bar{\mathsf{w}}(D_1)\bar{\mathsf{w}}(D_2) \tag{4.56}$$

where

$$\begin{aligned}\bar{\mathsf{w}}(D_1)\bar{\mathsf{w}}(D_2) = &\prod_{i=1}^{\ell_1'} \operatorname{sym}(\mathsf{u}_{1,i}) \prod_{j=1}^{\ell_2'} \operatorname{sym}(\mathsf{u}_{2,j}) + \prod_{i=1}^{\ell_1'} \operatorname{sym}(\mathsf{u}_{1,i}^*) \prod_{j=1}^{\ell_2'} \operatorname{sym}(\mathsf{u}_{2,j}^*) \\ &+ \prod_{i=1}^{\ell_1'} \operatorname{sym}(\mathsf{u}_{1,i}) \prod_{j=1}^{\ell_2'} \operatorname{sym}(\mathsf{u}_{2,j}^*) + \prod_{i=1}^{\ell_1'} \operatorname{sym}(\mathsf{u}_{1,i}^*) \prod_{j=1}^{\ell_2'} \operatorname{sym}(\mathsf{u}_{2,j}). \end{aligned} \tag{4.57}$$

Consider the disjoint union of the two chord diagrams $D_{12} = D_1 \sqcup D_2$. The terms in eq. (4.57) can be interpreted as two different cycle words associated to D_{12}, corresponding to two different colourings of the base cycles of D_1 and D_2.

Assuming that all diagrams use the same colour for their chords, there are 2^ℓ possible colourings of the ℓ base cycles with the other two colours - visible as four terms in eq. (4.57). Combining the terms pairwise (the two in the upper line and the two in the lower line) one has a sum over the $2^{\ell-1}$ *relative colourings* of the base cycles. Earlier we defined the map D from Dirac words D to chord diagrams. Clearly it can be extended to word tuples $(\mathsf{w}_1, \dots, \mathsf{w}_\ell) \in \mathsf{D}^\ell$, mapping them to chord diagrams with ℓ base cycles as long as the concatenation $\mathsf{w}_1 \cdots \mathsf{w}_\ell \in \mathsf{D}$, i.e. as long as no letter appears more than twice in the tuple. Clearly the set of tricoloured cycles and therefore the projection of such a chord diagram then also depends on the choice of colouring c. Similarly the cycle numbers $c_2(D,c)$ and $c_3(D,c)$ depend now on the choice of colouring. Using this we can simply extend the cycle word definition 4.3.7 to a generalised chord diagram together with a particular colouring as

$$\bar{\mathsf{w}}(D,c) := \prod_{i=1}^{\ell'} \mathrm{sym}(\mathsf{u}_i) + \prod_{i=1}^{\ell'} \mathrm{sym}(\mathsf{u}_i^*). \tag{4.58}$$

where of course ℓ' and all the u_i are now also colour dependent via π_0^c. For each fixed colouring it is of the same form as the cycle word for a single chord diagram, so theorem 4.3.9 can be applied term by term in a sum over all possible colourings and extends fully to generalised chord diagrams. In particular the addition of a chord between different base cycles (corresponding to contraction of letters in different traces) can be treated as part of case 2 (new chord between vertices in different tricoloured cycles) in the proof. Hence, we can give the following corollary to theorem 4.3.9.

Corollary 4.3.13. *Let* $(\mathsf{w}_1, \dots, \mathsf{w}_\ell) \in \mathsf{D}^\ell$ *be a tuple of Dirac words such that* $D \equiv D(\mathsf{w}_1, \dots, \mathsf{w}_\ell) \in \mathcal{D}_k^{(n_1,\dots,n_\ell)}$*. Then*

$$\mathrm{tr}(\mathsf{w}_1) \cdots \mathrm{tr}(\mathsf{w}_\ell) = (-1)^\ell (-2)^k \sum_c (-2)^{c_2(D,c)+c_3(D,c)} \bar{\mathsf{w}}(D,c). \tag{4.59}$$

Proof. For $k = 0$ one simply has an ℓ-fold product of the single trace case,

$$\begin{aligned} \mathrm{tr}(\mathsf{w}_1) \cdots \mathrm{tr}(\mathsf{w}_\ell) &= \prod_{j=1}^{\ell} 2\,\mathrm{sym}(\mathsf{w}_j + \mathsf{w}_j^*) \\ &= 2^\ell \Big(\mathrm{sym}(\mathsf{w}_1) \cdots \mathrm{sym}(\mathsf{w}_\ell) + \mathrm{sym}(\mathsf{w}_1) \cdots \mathrm{sym}(\mathsf{w}_{\ell-1})\,\mathrm{sym}(\mathsf{w}_\ell^*) \\ &\qquad + \cdots + \mathrm{sym}(\mathsf{w}_1^*) \cdots \mathrm{sym}(\mathsf{w}_\ell^*) \Big) \\ &= 2^\ell \sum_c \bar{\mathsf{w}}(D,c) \\ &= (-1)^\ell (-2)^0 \sum_c (-2)^{0+\ell} \bar{\mathsf{w}}(D,c), \end{aligned}$$

where $c_3(D,c) = \ell$ is simply the number of base cycles, independent of the colouring, since there are no chords between the different bases yet. Then we proceed for each colouring as in theorem 4.3.9 to finish the proof. □

With the physics application of this identity in mind it is particularly interesting that we get a factor of $(-1)^\ell$. It will precisely be absorbed by the factors of -1 that we associate to each closed fermion loop, i.e. each trace appearing in the integrand.

Example 4.3.14. *Consider the chord diagram depicted with its two different relative colourings in fig. 4.5. Labelling counter-clockwise and starting with the uppermost vertex of the left base cycle it corresponds to*

$$\begin{aligned} \mathsf{g}_{1,5}\mathsf{g}_{6,12}\mathsf{g}_{8,10}\mathsf{g}_{9,11}\mathsf{g}_{14,15}\, &\mathrm{tr}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6\mathsf{a}_7\mathsf{a}_8)\,\mathrm{tr}(\mathsf{a}_9\mathsf{a}_{10}\mathsf{a}_{11}\mathsf{a}_{12}\mathsf{a}_{13}\mathsf{a}_{14}\mathsf{a}_{15}\mathsf{a}_{16}) \\ &= (-2)^4\,\mathrm{tr}(\mathsf{a}_4\mathsf{a}_3\mathsf{a}_2\mathsf{a}_6\mathsf{a}_7\mathsf{a}_8)\,\mathrm{tr}(\mathsf{a}_8\mathsf{a}_6\mathsf{a}_{13}\mathsf{a}_{16}) \\ &= -(-2)^6\Big(\mathrm{tr}(\mathsf{a}_4\mathsf{a}_3\mathsf{a}_2\mathsf{a}_7\mathsf{a}_{13}\mathsf{a}_{16}) + \mathrm{tr}(\mathsf{a}_4\mathsf{a}_3\mathsf{a}_2\mathsf{a}_{13}\mathsf{a}_{16}\mathsf{a}_7)\Big). \end{aligned} \tag{4.60}$$

We have a total of five chords in two base cycles, so $k = 5$, $\ell = 2$. The two colourings each have only one 3-cycle, with corresponding words

$$\mathsf{u}_1 = \mathsf{a}_3\mathsf{a}_4\mathsf{a}_{16}\mathsf{a}_{13}\mathsf{a}_7\mathsf{a}_2 \quad \textit{and} \quad \mathsf{u}_2 = \mathsf{a}_3\mathsf{a}_4\mathsf{a}_7\mathsf{a}_{16}\mathsf{a}_{13}\mathsf{a}_2 \tag{4.61}$$

respectively, which are up to cyclic shifts and reversal the two words in the traces in eq. (4.60). Both have three 2-cycles, the two bases and one between vertices 14 *and* 15*. Therefore we compute*

$$\begin{aligned} &\frac{(-2)^5}{4}\bigg((-2)^{3+1}\Big(\mathrm{sym}(\mathsf{u}_1) + \mathrm{sym}(\mathsf{u}_1^*)\Big) + (-2)^{3+1}\Big(\mathrm{sym}(\mathsf{u}_2) + \mathrm{sym}(\mathsf{u}_2^*)\Big)\bigg) \\ &\quad = -(-2)^6\Big(\mathrm{tr}(\mathsf{u}_1) + \mathrm{tr}(\mathsf{u}_2)\Big). \end{aligned} \tag{4.62}$$

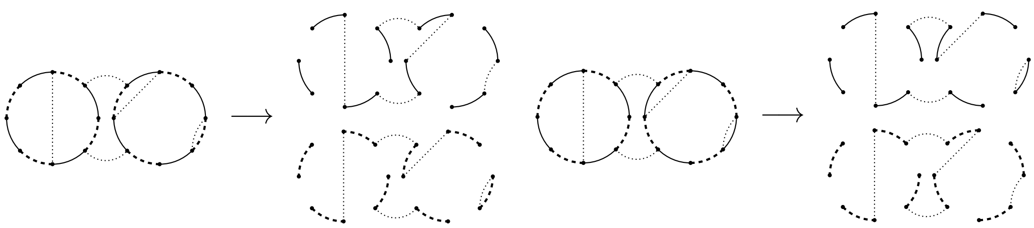


Figure 4.5: The bicoloured subgraphs of the same chord diagram for two different relative colourings. One can see the different structures that result in different cycle words and numbers.

4.4 Contraction of the tensor structure

Now that we have worked out a concise formalism to compute the contraction of Dirac matrices we apply it to the integrand. For concreteness and simplicity we work in the quenched case ($\ell = 1$) with Feynman gauge ($\varepsilon = 0$). As discussed before (e.g. eq. (3.12) for the gauge or corollary 4.3.13 for multiple traces) the generalisations are mostly a matter of adding a summation sign or a factor, but do not introduce significant conceptual complexity.

4.4.1 From Feynman graphs to chord diagrams

Consider first a graph Γ with $\operatorname{res}(\Gamma) = \sim\!\sim\!\sim$. We can use the notion of chord diagram introduced in this chapter to reinterpret the sum over pairings in eq. (2.110). Clearly, a pairing as defined in section 2.3.4 is nothing but a set of chords, in a chord diagram whose base cycle is given by the matrices in the trace, and with vertices labelled by fermion edges and vertices of a Feynman graph, or just fermion edges in the Feynman gauge case of eq. (2.111). In these diagrams the two external vertices $v_\mu, v_\nu \in V_\Gamma$ label two 2-valent vertices without a chord incident to it. Additionally, a diagram corresponding to a term in $I_\Gamma^{(k,0)}$ has $2k$ more such free vertices. These, however, are contracted with the momenta in each $\mathrm{X}_\Gamma^{e,\mu_e} =: q^{\mu_e} x_\Gamma^e$. Hence, with $\not{q}\not{q} = q^2$ all possible cycle words (see def. 4.3.7) for $k > 0$ reduce to three cases corresponding to the three labels used in section 3.1.2:

- If $\operatorname{sgn}(v_\mu, v_\nu) = +1$, then

$$\begin{aligned}
\bar{\mathsf{w}}(D) &= \operatorname{sym}(\mathsf{u}\mathsf{a}_\mu\mathsf{v}\mathsf{a}_\nu)\prod_i \operatorname{sym}(\mathsf{w}_i) + \operatorname{sym}(\mathsf{a}_\nu\mathsf{u}\mathsf{a}_\mu\mathsf{v})\prod_i \operatorname{sym}(\mathsf{w}_i^*), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |\mathsf{u}|, |\mathsf{v}|, |\mathsf{w}_i| \text{ even} \\
&\sim (q^2)^{|\mathsf{u}|+|\mathsf{v}|+\sum|\mathsf{w}_i|}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \\
&= 2g^{\mu\nu}(q^2)^k. \qquad (4.63)
\end{aligned}$$

- If $\operatorname{sgn}(v_\mu, v_\nu) = 0$, then

$$\begin{aligned}
\bar{\mathsf{w}}(D) &= \operatorname{sym}(\mathsf{u}\mathsf{a}_\mu\mathsf{v}\mathsf{a}_\nu)\prod_i \operatorname{sym}(\mathsf{w}_i) + \operatorname{sym}(\mathsf{a}_\nu\mathsf{u}\mathsf{a}_\mu\mathsf{v})\prod_i \operatorname{sym}(\mathsf{w}_i^*), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad |\mathsf{u}|, |\mathsf{v}| \text{ odd}, \ |\mathsf{w}_i| \text{ even} \\
&\sim (q^2)^{k-1}(\not{q}\gamma^\mu\not{q}\gamma^\nu + \gamma^\nu\not{q}\gamma^\mu\not{q}) \\
&= 2(2q^\mu q^\nu - q^2 g^{\mu\nu})(q^2)^{k-1}. \qquad (4.64)
\end{aligned}$$

- If $\operatorname{sgn}(v_\mu, v_\nu) = -1$, then

$$\begin{aligned}
\bar{\mathsf{w}}(D) &= \operatorname{sym}(\mathsf{u}\mathsf{a}_\mu)\operatorname{sym}(\mathsf{v}\mathsf{a}_\nu)\prod_i \operatorname{sym}(\mathsf{w}_i) + \operatorname{sym}(\mathsf{a}_\nu\mathsf{u})\operatorname{sym}(\mathsf{a}_\mu\mathsf{v})\prod_i \operatorname{sym}(\mathsf{w}_i^*), \\
&\qquad\qquad\qquad\qquad |\mathsf{u}|, |\mathsf{v}| \text{ odd}, \ |\mathsf{w}_i| \text{ even} \\
&\sim (q^2)^{k-1}\frac{1}{2}(\not{q}\gamma^\mu + \gamma^\mu\not{q})(\not{q}\gamma^\nu + \gamma^\nu\not{q}) \\
&= 2q^\mu q^\nu (q^2)^{k-1}.
\end{aligned} \tag{4.65}$$

Consider now again the superficially renormalised integral for the photon, from eq. (3.32), in Feynman gauge:

$$\phi_\Gamma^R := (q^2 g^{\mu\nu} - q^\mu q^\nu) L \int_{\sigma_\Gamma} \Omega_\Gamma \frac{\varphi_\Gamma J_\Gamma^{(0,0)} + J_{\Gamma,0}^{(1,0)} - J_{\Gamma,+}^{(1,0)}}{\Psi_\Gamma^{h_1+3}}. \tag{4.66}$$

We want to express each $J_\Gamma^{(k,0)}$ explicitly as a sum over chord diagrams, each summand with the correct power of (-2) from the contraction. To do so, define D_Γ to be the chord diagram associated Γ, i.e. each base cycle corresponds to a closed fermion cycle and is labelled with fermion edges and vertices of Γ. We want D_Γ to have $h_1 = |E_\Gamma^{(p)}| + 1$ fixed chords, one of them between the external vertices[3], the rest corresponding to the photon edges. In general gauge (cf. eq. (3.12)) the coefficient of ε^n, would contain a sum over multiple such diagrams, corresponding to all possible choices of $|E_\Gamma^{(p)}| - n$ photon edges to fix chords. Since these fixed chords are always the same we can project them out and define $D_\Gamma^0 = \pi_0(D_\Gamma)$. For examples see fig. 4.6.

Furthermore, for any chord diagram $D_0 \in \mathcal{D}_{N-m}^n$ with $n \in \mathbb{N}^\ell$ and $\sum n_i = N$ define $\bar{\mathcal{D}}^k(D_0) \subseteq \mathcal{D}_{N-m+k}^n$ for all $0 \le k \le m$ to be the set

$$\bar{\mathcal{D}}^k(D_0) := \{D \in \mathcal{D}_{N-m+k}^n \mid D_0 \subset D\}. \tag{4.67}$$

That is, $\bar{\mathcal{D}}^k(D_0)$ is the set of chord diagrams that results from adding k chords to D_0 in all possible ways. For a Feynman graph Γ, with the chord diagram D_Γ^0 we define then

$$\mathcal{D}_\Gamma^k := \bar{\mathcal{D}}^{h_1-k}(D_\Gamma^0). \tag{4.68}$$

Moreover, looking forward to the treatment subdivergences in the next section, we also define the subset $\mathcal{D}_{\Gamma,\gamma}^{k,l} \subset \mathcal{D}_\Gamma^{k+l}$ as those diagrams in $\mathcal{D}_\Gamma^{k+l}$ that satisfy the restrictions

[3] We want to include this chord because below we want to have complete chord diagrams without free vertices for $k = 0$. Its effect on the factors will be discussed shortly.

- $u, v \in E^{(f)}_{\Gamma/\gamma}$ or $u, v \in E^{(f)}_{\gamma}$ for all chords $(u,v) \in E^0_D$,
- $V^{(2)}_D \cap E^{(f)}_{\Gamma/\gamma} = 2k$ and $V^{(2)}_D \cap E^{(f)}_{\gamma} = 2l$.

In other words, only chords between vertices labelled by edges of either subgraph or cograph are allowed, and there are k and l chords missing in each of these two components.

With this notation $I^{(k,0)}_\Gamma$, which is the k-th summand in eq. (2.111) with Kirchhoff polynomials factored out, can be written

$$g^{\mu\nu} I^{(k,0)}_\Gamma = \bar{\gamma}_\Gamma \Big(g^{\mu\nu} \prod_{e \in E^{(p)}_\Gamma} g^{\mu_{\partial_+(e)}\mu_{\partial_-(e)}} \Big) \sum_{A \in \binom{E^{(f)}_\Gamma}{2(m-k)}} \Big(\prod_{l \notin A} \mathrm{X}^{l,\mu_l}_\Gamma \Big) \sum_{P \in \mathcal{P}_2(A)} \prod_{(i,j) \in P} \frac{g^{\mu_i \mu_j} \chi^{(i|j)}_\Gamma}{2}$$
$$= 2^{k-h_1} \bar{\gamma}_\Gamma \Big(\prod_{(u,v) \in E^0_{D_\Gamma}} g^{\mu_u \mu_v} \Big) \sum_{D \in \mathcal{D}^k_\Gamma} \Big(\prod_{(u,v) \in E^0_D} g^{\mu_u \mu_v} \chi^{(u|v)}_\Gamma \Big) \prod_{w \in V^{(2)}_D} \mathrm{X}^{w,\mu_w}_\Gamma . \tag{4.69}$$

Now theorem 4.3.9 is applied to compute the contractions. This is simplified by the observation that $c_2(D_\Gamma) = 0$, since the fixed chords are attached to every other vertex. Hence, the projection to D^0_Γ does not change this number. While $c_3(D_\Gamma)$ differs from graph to graph (see fig. 4.6), this does not influence the result, since D_Γ with some additional chords and D^0_Γ with those same chords will have the same tricoloured cycles in the end. However, we need to include a $(-2)^{h_1}$ for the removed chords. All overall factors combine to

$$2^{k-h_1} \cdot \underbrace{(-2)^{h_1}}_{\text{fixed chords}} \cdot \underbrace{(-2)^{h_1-k}}_{\text{other chords}} \cdot \underbrace{2(q^2)^k}_{\text{cycle word}} = 2^{h_1+1}(-q^2)^k. \tag{4.70}$$

As discussed above, the cycle word $\bar{\mathsf{w}}(D)$ reduces to powers of q^2 with a factor of 2 if each letter a_i is replaced with $\gamma_{\mu_i} \mathrm{X}^{i,\mu_i}_\Gamma = \not{x}^i_\Gamma$. These factors also include $(-1)^2 = 1$ from the overall minus sign in theorem 4.3.9 which is cancelled by the sign in $\bar{\gamma}_\Gamma = -\operatorname{tr}(\cdots)$. We find

$$g^{\mu\nu} I^{(k,0)}_\Gamma = 2^{h_1+1}(-q^2)^k \underbrace{\sum_{D \in \mathcal{D}^k_\Gamma} (-2)^{\tilde{c}(D)} \Big(\prod_{(u,v) \in E^0_D} \chi^{(u|v)}_\Gamma \Big) \prod_{w \in V^{(2)}_D} x^w_\Gamma}_{=:2Z^k_\Gamma} . \tag{4.71}$$

The sum of bicoloured and tricoloured cycles is abbreviated $\tilde{c}(D) = c_2(D) + c_3(D)$. We also used foreknowledge of chapter 5 to introduce the notation Z^k_Γ. For now it is merely an abbreviation, but in the next chapter we will find some more meaning in it, and also explain the choice of the factor 2.

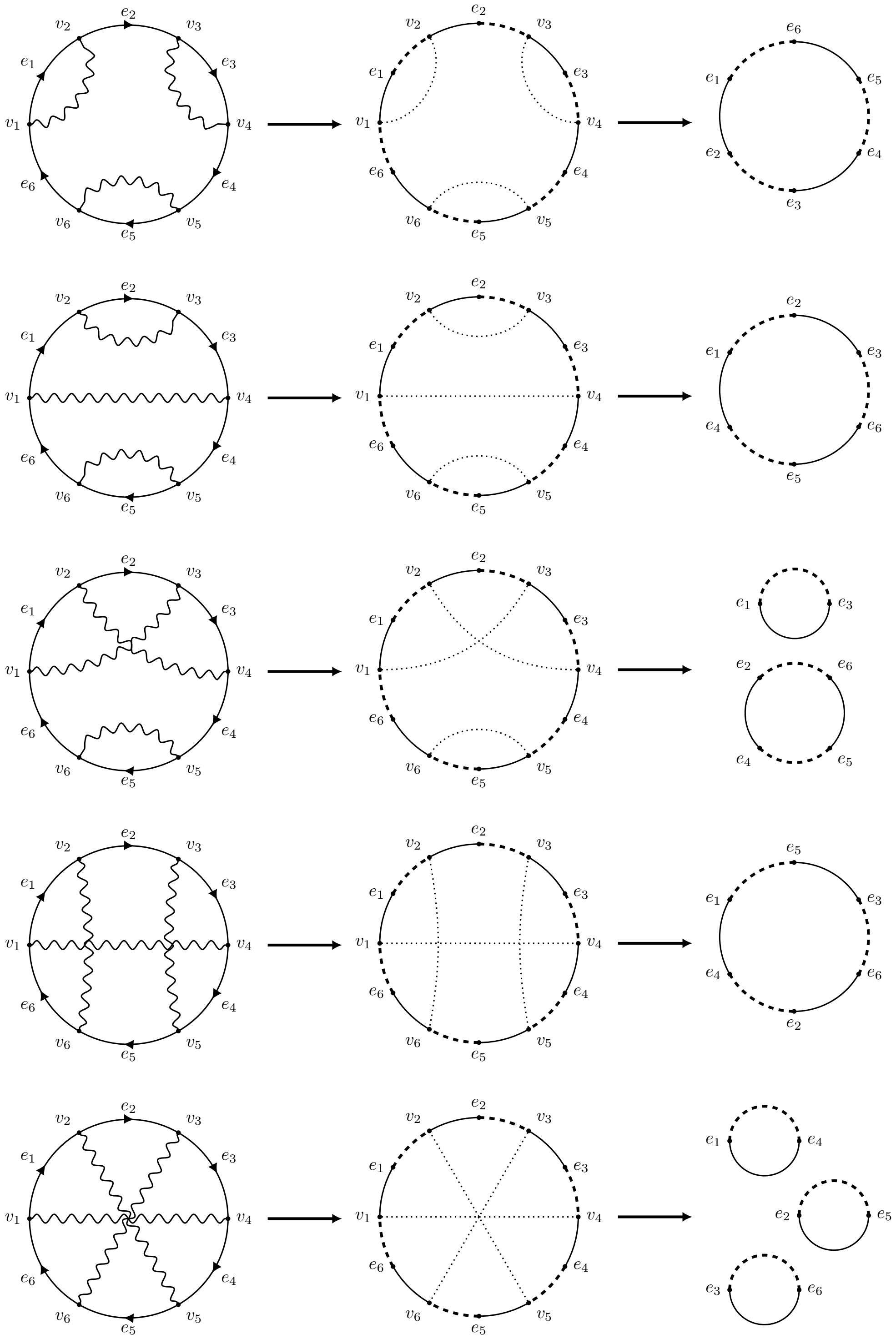


Figure 4.6: Completions, associated chord diagrams D_Γ and their projections D^0_Γ for all 3-loop photon propagator graphs in Feynman gauge.

From this we can now reconstruct $J_\Gamma^{(k,0)}$. For $k=0$ one simply has $g^{\mu\nu} I_\Gamma^{(0,0)} = 4J_\Gamma^{(0,0)}$, such that $J_\Gamma^{(0,0)} = 2^{h_1} Z_\Gamma^0$. For $k > 0$ we have the three different cases. Let $\mathcal{D}^k_{\Gamma,\bullet} \subseteq \mathcal{D}^k_\Gamma$ be the subset of diagrams belonging to one such case and $Z^k_{\Gamma,\bullet}$ the corresponding sum. These diagrams are classified by demanding that after removal of the chord between the two external vertices the signum $\operatorname{sgn}(u,v)$ of these vertices is $\pm$ or 0. In chapter 6 and especially section 6.2 we give examples for this.

Contracting the tensor structures found in eqs. (4.63), (4.64), and (4.65) one then finds factors of 4 ,−2, and 1. Hence,

$$J_{\Gamma,a}^{(k,0)} = 2^{h_1}(-2)^{f(a)}(-1)^k Z^k_{\Gamma,a} \tag{4.72}$$

with

$$f(a) = \begin{cases} 0 & \text{if } a = +, \\ 1 & \text{if } a = 0, \\ 2 & \text{if } a = -. \end{cases} \tag{4.73}$$

Inserting into eq. (4.66) then yields

$$\phi_\Gamma^R := (q^2 g^{\mu\nu} - q^\mu q^\nu) 2^{h_1} L \int_{\sigma_\Gamma} \Omega_\Gamma \frac{\varphi_\Gamma Z_\Gamma^0 + 2Z^1_{\Gamma,0} + Z^1_{\Gamma,+}}{\Psi_\Gamma^{h_1+3}}. \tag{4.74}$$

4.4.2 Open fermion lines

Next we need to look into fermion and vertex graphs again. In both cases the idea is to close the fermion line, apply the theorem for contraction, then correct factors to get the desired integrand.

The fermion propagator

In this case all free vertices are contracted with momenta q^{μ_e}, so we do not need to differentiate between the three cases that appear in the photon and vertex. We close the external fermion by considering

$$\begin{aligned} \operatorname{tr}(\not{q} I_\Gamma^{(k,0)}) &= \operatorname{tr}(\not{q}\bar{\gamma}_\Gamma) \Bigg(\prod_{e \in E_\Gamma^{(p)}} g^{\mu_{\partial_+(e)}\mu_{\partial_-(e)}} \Bigg) \sum_{A \in \binom{E_\Gamma^{(f)}}{2(m-k)}} \Bigg(\prod_{l \notin A} \mathrm{X}_\Gamma^{l,\mu_l} \Bigg) \sum_{P \in \mathcal{P}_2(A)} \prod_{(i,j) \in P} \frac{g^{\mu_i \mu_j} \chi_\Gamma^{(i|j)}}{2} \\ &= 2^{k-h_1+1} \operatorname{tr}(\gamma_{\mu_0}\bar{\gamma}_\Gamma) \Bigg(\prod_{(u,v) \in E^0_{D_\Gamma}} g^{\mu_u\mu_v} \Bigg) \sum_{D \in \mathcal{D}_\Gamma^{k+1}} \Bigg(\prod_{(u,v) \in E^0_D} g^{\mu_u\mu_v} \chi_\Gamma^{(u|v)} \Bigg) \prod_{w \in V_D^{(2)}} \mathrm{X}_\Gamma^{w,\mu_w}. \end{aligned} \tag{4.75}$$

By convention we define $X_\Gamma^{0,\mu_0} = q^{\mu_0}$, and therefore $x_\Gamma^0 = 1$, for the added dummy vertex. The contraction then yields

$$\mathrm{tr}(\not{q} I_\Gamma^{(k,0)}) = -2^{h_1+1}(-q^2)^{k+1} \underbrace{\sum_{D\in\mathcal{D}_\Gamma^{k+1}} (-2)^{\tilde{c}(D)} \Big(\prod_{(u,v)\in E_D^0} \chi_\Gamma^{(u|v)} \Big) \prod_{w\in V_D^{(2)}} x_\Gamma^w}_{=:2Z_\Gamma^{k+1}} \tag{4.76}$$

and we immediately find

$$J_\Gamma^{(k,0)} = \frac{1}{4(q^2)^{k+1}} \mathrm{tr}(\not{q} I_\Gamma^{(k,0)}) = 2^{h_1}(-1)^k Z_\Gamma^{k+1}. \tag{4.77}$$

The superficially renormalised integral is found by inserting this into eq. (3.27):

$$\phi_\Gamma^R = -2^{h_1} L \not{q} \int_{\sigma_\Gamma} \frac{Z_\Gamma^1}{\Psi_\Gamma^{h_1+2}} \Omega_\Gamma \tag{4.78}$$

The vertex

The fermion line is closed by adding a vertex. We also include the external photon line as a fixed chord in the corresponding D_Γ. In other words, we consider

$$\begin{aligned}
\mathrm{tr}(\gamma_\mu I_\Gamma^{(k,0)}) &= \mathrm{tr}(\gamma_\nu \bar{\gamma}_\Gamma) \Big(g^{\nu\mu} \prod_{e\in E_\Gamma^{(p)}} g^{\mu_{\partial_+(e)}\mu_{\partial_-(e)}} \Big) \\
&\quad \times \sum_{A\in\binom{E_\Gamma^{(f)}}{2(m-k)}} \Big(\prod_{l\notin A} X_\Gamma^{l,\mu_l} \Big) \sum_{P\in\mathcal{P}_2(A)} \prod_{(i,j)\in P} \frac{g^{\mu_i\mu_j}\chi_\Gamma^{(i|j)}}{2} \\
&= 2^{k-h_1} \mathrm{tr}(\gamma_{\mu_0}\bar{\gamma}_\Gamma) \Big(\prod_{(u,v)\in E_{D_\Gamma}^0} g^{\mu_u\mu_v} \Big) \\
&\quad \times \sum_{D\in\mathcal{D}_\Gamma^k} \Big(\prod_{(u,v)\in E_D^0} g^{\mu_u\mu_v} \chi_\Gamma^{(u|v)} \Big) \prod_{w\in V_D^{(2)}} X_\Gamma^{w,\mu_w} \\
&= 2^{h_1+3}(-q^2)^k Z_\Gamma^k.
\end{aligned} \tag{4.79}$$

The factor is again computed as in eq. (4.70), except that we have $h_1 + 1$ fixed chords instead of h_1 and an overall (-1) since there was no closed fermion loop in the original Γ. Studying the three cases yields the same results as in the photon case, except that we lose another factor of $4 = \mathrm{tr}(\mathbb{1})$ from the trace. Inserting results into eq. (3.35) then gives the integral

$$\phi_\Gamma^R := -\int_{\sigma_\Gamma} \Omega_\Gamma \frac{2^{h_1-1}}{\Psi_\Gamma^{h_1+2}} \Big(\gamma^\mu L Z_\Gamma^0 + 4\frac{\not{q} q^\mu}{q^2} \sum_{k=1}^{h_1} \frac{Z_{\Gamma,-}^k - Z_{\Gamma,0}^k}{(-\varphi_\Gamma)^k} \Big). \tag{4.80}$$

4.4.3 Subdivergences

The contraction of Dirac matrices for a term in the forest formula gets the exact same overall factors. For example, first contracting a product of matrices corresponding to a fermion subgraph within a trace of an overall photon propagator graph, followed by contraction of the remaining trace clearly gives the same result as just contracting the whole trace. The only thing that happens for such terms is that the sum over chord diagrams is restricted to the subsets $\mathcal{D}^{k,l}_{\Gamma,\gamma}$. With these subsets we can define the corresponding Z-polynomials as sums

$$Z^{k,l}_{\Gamma,\gamma} := \frac{1}{2} \sum_{D\in\mathcal{D}^{k,l}_{\Gamma,\gamma}} (-2)^{\tilde{c}(D)} \Big(\prod_{(u,v)\in E^0_{D,\Gamma/\gamma}} \chi^{(u|v)}_{\Gamma/\gamma} \Big) \Big(\prod_{(u,v)\in E^0_{D,\gamma}} \chi^{(u|v)}_{\gamma} \Big) \Big(\prod_{w\in V^{(2)}_{D,\Gamma/\gamma}} x^w_{\Gamma/\gamma} \Big) \Big(\prod_{w\in V^{(2)}_{D,\gamma}} x^w_{\gamma} \Big) \tag{4.81}$$

where the products over chords and vertices are simply split into products over the respective subsets labelled by cograph or subgraph edges. Up to integer factors this expression corresponds to a product $J^{(k)}_{\Gamma/\gamma} J^{(l)}_{\gamma}$ as seen in chapter 3. Further indices at the graphs, e.g. $Z^{k,l}_{\Gamma_+,\gamma_0}$, indicate the analogously defined expression with a further restricted set of chord diagrams such that they correspond to one of the three possible cases, $\pm$ or 0, discussed for the J polynomials.

Applying all this to a term in the forest formula we find the following. For a vertex subdivergence in a photon graph as in eq. (3.42) (again modified for Feynman gauge) this gives

$$\begin{aligned} M^{\Gamma}_{\gamma} :=& 2^{h_1} \frac{\Omega_\Gamma}{\Psi_{\Gamma/\gamma}^{h_1^{\Gamma/\gamma}+3} \Psi_\gamma^{h_1^\gamma+2}} \Big(\Big(\varphi_{\Gamma/\gamma} Z^{0,0}_{\Gamma,\gamma} + 2 Z^{1,0}_{\Gamma_0,\gamma} + Z^{1,0}_{\Gamma_+,\gamma} \Big) \log \Delta^\Gamma_\gamma \\ &+ \sum_{k=1}^{h_1^{\Gamma/\gamma}} \sum_{l=1}^{h_1^\gamma} \tilde{Z}^{k,l}_{\Gamma,\gamma} \Big(-s\Psi_\gamma \Big)^{k-1} \Big(-\Psi_{\Gamma/\gamma} \Big)^{l} \Big(\Upsilon^\Gamma_\gamma(s)^{-(k+l-1)} - \Upsilon^\Gamma_\gamma(1)^{-(k+l-1)} \Big) \Big). \end{aligned} \tag{4.82}$$

Here we used yet another abbreviation,

$$\tilde{Z}^{k,l}_{\Gamma,\gamma} = 2 \Big(Z^{k,l}_{\Gamma_0,\gamma_0} - Z^{k,l}_{\Gamma_0,\gamma_+} \Big) + \Big(Z^{k,l}_{\Gamma_+,\gamma_0} - Z^{k,l}_{\Gamma_+,\gamma_+} \Big), \tag{4.83}$$

derived from the abbreviation in eq. (3.40), but including the factors from the last section.

For a fermion subdivergence one finds the analogous result from eq. (3.47), except for minor modifications.

$$
\begin{aligned}
M_\gamma^\Gamma :=& 2^{h_1} \frac{\Omega_\Gamma}{\Psi_{\Gamma/\gamma}^{h_1^{\Gamma/\gamma}+3} \Psi_\gamma^{h_1^\gamma+2}} \Big(- \Big(\varphi_{\Gamma/\gamma} Z_{\Gamma,\gamma}^{0,1} + 2 Z_{\Gamma_0,\gamma}^{1,1} + Z_{\Gamma_+,\gamma}^{1,1} \Big) \log \Delta_\gamma^\Gamma \\
&+ \sum_{k=1}^{h_1^{\Gamma/\gamma}} \sum_{l=2}^{h_1^\gamma} \tilde{Z}_{\Gamma,\gamma}^{k,l} \Big(- s\Psi_\gamma \Big)^{k-1} \Big(- \Psi_{\Gamma/\gamma} \Big)^l \Big(\Upsilon_\gamma^\Gamma(s)^{-(k+l-1)} - \Upsilon_\gamma^\Gamma(1)^{-(k+l-1)} \Big) \Big).
\end{aligned}
\tag{4.84}
$$

While we do now have decent control over the integrand, it is still rather ugly. The chord diagram sums in each Z_Γ^k grow factorially, the splitting into three different sums for $k \geq 1$ is very inconvenient, and for subdivergences it only gets worse. Hence, the next step is trying to understand these Z-polynomials a bit better.

5 Structure of the integrand II: Polynomial identities

"If I had a world of my own, everything would be nonsense. Nothing would be what it is, because everything would be what it isn't. And contrary wise, what is, it wouldn't be. And what it wouldn't be, it would. You see?"

Lewis Carroll[1] , *Alice's Adventures in Wonderland & Through the Looking-Glass*, 1871

In the previous chapter we achieved reduction of the QED integral to a scalar integral, but the integrand still contains a sum over a factorially growing set of chord diagrams (e.g. $(2h_1 - 1)!!$ for Z_Γ^0 in Feynman gauge). Computations suggest that these sums can be written in a different form that leads to cancellations with the Kirchhoff polynomials in the denominator, reducing the overall size of the integrand considerably. These cancellations are most notable in the case of chord diagrams with full sets of chords (as in Z_Γ^0) for which we find the main result of this chapter, theorem 5.2.4. Sums over chord diagrams with free vertices are much more complicated and will need further study in future work, but we will discuss some observations that might lead to similar simplifications for them.

The observation that motivates the methods of this chapter is the fact that the cycle polynomials can be interpreted as a special case of *Dodgson polynomials*, which we will introduce in section 5.1. This means in particular that they satisfy the Dodgson identity eq. (5.6), which is the main tool needed to prove the desired summation identity in section 5.2.

[1]Real name: Charles Lutwidge Dodgson (1832-1898)

5.1 Dodgson polynomials

The Dodgson polynomials were introduced by Francis Brown in [28, sec. 2]. They are motivated by the definition of the Kirchhoff polynomial as determinant of the graph matrix $M(G)$ and are given by minors of $M(G)$.

5.1.1 Definition and properties

Let G be a connected graph, $M(G)$ its graph matrix as defined in eq. (2.13) and

$$\Psi_G = \det(M(G))$$

as in eq. (2.15) its Kirchhoff polynomial. From now on we use subscript M_I (superscript M^I) to denote the matrix that results from deleting all rows (all columns) in the index set I. We also use square brackets $M[I, J]$ for the matrix in which all rows except those in I and all columns except those in J are deleted. It is well known that

$$\Psi_{G\backslash e} = \det\left(M(G)^{\{e\}}_{\{e\}}\right) \qquad \text{and} \qquad \Psi_{G/\!/e} = \det\left(M(G)|_{\alpha_e=0}\right), \tag{5.1}$$

which suggests

Definition 5.1.1. (Dodgson polynomials)
Let $I, J, K \subseteq E_G$ with $|I| = |J|$. Then we define the Dodgson polynomial of G to be

$$\Psi^{I,J}_{G,K} := \det\left(M(G)^J_I\right)_{\alpha_e=0\ \forall e\in K}. \tag{5.2}$$

Different choices of $M(G)$, i.e. different orderings of edges and vertices corresponding to permutations of rows and columns, may change the sign of the Dodgson polynomials, an ambiguity that is discussed in more detail below.

Other notable properties are:

- **Passing to a minor:** For all $A, B \subset E_G$

$$\Psi^{I,J}_{G\backslash A/\!/B,K} = \Psi^{I\cup A,J\cup A}_{G,K\cup B}. \tag{5.3}$$

  Thus we can and will usually assume $I \cap J = \emptyset = K$ and omit the subscript K from the notation.

- **Plücker identities:**

$$\sum_{k=1}^{n} \Psi_G^{\{i_1,\dots,i_{n-1},j_k\},\{i_n,j_1,\dots,\widehat{j_k}\dots,j_n\}} = 0 \tag{5.4}$$

- **Determinant identities:** Based on the Desnanot-Jacobi identity [54]

$$\det(\mathrm{adj}(M)[I,J]) = \det(M)^{|I|-1}\det(M_J^I) \tag{5.5}$$

for determinants one finds identities of the type

$$\Psi_G^{\{i_1\},\{i_3\}}\Psi_G^{\{i_2\},\{i_4\}} - \Psi_G^{\{i_1\},\{i_4\}}\Psi_G^{\{i_2\},\{i_3\}} = \Psi_G\Psi_G^{\{i_1,i_2\},\{i_3,i_4\}}. \tag{5.6}$$

This case ($|I| = 2 = |J|$) is also called *Dodgson identity*[1] and its generalisations are the crucial tool that we will work with below.

- **Combinatoric interpretation:** In the case of $I \cap J = \emptyset = K$ the combinatoric interpretation for Dodgson polynomials given by Brown in [28, prop. 23] simplifies to

$$\Psi_G^{I,J} = \sum_{T \subset E_G \setminus (I \cup J)} \pm \prod_{e \notin T} \alpha_e \tag{5.7}$$

where the sum is over edge subsets T that are simultaneously spanning trees of $(G \setminus I) /\!\!/ J$ and $(G \setminus J) /\!\!/ I$.

5.1.2 Dodgson cycle polynomials

The relation

$$\chi_G^{(e|e)} = \Psi_{G\setminus e} = \Psi_G^{\{e\},\{e\}} \tag{5.8}$$

suggests a possible connection between cycle and Dodgson polynomials, and indeed we find

Proposition 5.1.2.

$$\chi_G^{(i|j)} = \pm\Psi_G^{\{i\},\{j\}} \tag{5.9}$$

for all $i, j \in E_G$.

[1]Somewhat confusingly, it is also occasionally called *Lewis Carroll identity* and both names are sometimes used to refer to the determinant identity eq. (5.5) [23]. Here we follow the conventions of [28].

Proof. For $i = j$ the proof is done and for $i \neq j$ we use eq. (5.7),

$$\Psi_G^{\{i\},\{j\}} = \sum_{T \subset E_G \setminus \{i,j\}} \pm \prod_{e \notin T} \alpha_e. \tag{5.10}$$

Translating the sum over spanning trees into a sum over cycles works analogously to the proofs in section 2.3.3. Decompose the sum over spanning trees into a double sum over paths $P \subset G \setminus i$ and spanning trees of the corresponding graph $(G \setminus i) /\!\!/ P$ where all paths are between endpoints $\partial_+(i)$ and $\partial_-(i)$ and contain the edge j. Then adding i to each path completes it into a simple cycle $C_P = P \cup \{i\} \in \mathcal{C}_G^{[1]}$ that contains both i and j, and the corresponding monomials of $\chi_G^{(i|j)}$ and $\Psi_G^{\{i\},\{j\}}$ indeed agree, at least up to sign.

The signs $o_{C_1}(i,j)$, $o_{C_2}(i,j)$ of two partial polynomials $\Psi_{G/\!\!/C_1}$ and $\Psi_{G/\!\!/C_2}$ in $\chi_G^{(i|j)}$ differ if and only if $C_1 \cup C_2$ is – up to contraction of longer paths to single edges – isomorphic to K_4:

Comparing this with the discussion of signs in Dodgson polynomials in section 2 of [32], one finds that the endpoints of i are precisely the transposed vertices given in [32, corollary 17] as a criterion for opposite signs. Therefore all partial polynomials have the correct relative signs and only the overall sign ambiguity of Dodgson polynomials remains, concluding the proof. □

It should be noted that the sign ambiguity of the Dodgson polynomials is of course not entirely absent from the cycle polynomials – the choice one has to make is simply moved from the order of rows and columns in a matrix to the orientations of edges in G. Since we always considered our graphs together with some such fixed choice from the very beginning it does not appear in the combinatorial definition of the cycle polynomials and from now on we fix the choice of the graph matrix such that the signs of $\chi_G^{(i|j)}$ and $\Psi_G^{\{i\},\{j\}}$ agree.

The equality suggests the definition of a higher order cycle polynomial via the Dodgson identity eq. (5.6).

Definition 5.1.3. (Dodgson cycle polynomials)
Let G be a connected graph and $\chi_G^{(i|j)}$ for all $i, j \in E_G$ the cycle polynomial as in definition 2.3.2. Then define an alphabet $\mathsf{A} = \{\mathsf{a}_i \mid i \in E_G\}$ in which each letter is associated to an edge of G and consider two words u, v over this alphabet with

no duplicate or shared letters and $|\mathsf{u}| = k = |\mathsf{v}|$. *The Dodgson cycle polynomial is then defined as* $\chi_G^{(\mathsf{a}_i|\mathsf{a}_j)} := \chi_G^{(i|j)}$ *if* $k = 1$ *and*

$$\chi_G^{(\mathsf{u}|\mathsf{v})} := \Psi_G^{1-k} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} \chi_G^{(\mathsf{u}_i|\sigma_i(\mathsf{v}))} \tag{5.11}$$

for $2 \leq k \leq h_1(G)$.

In this we simply recursively define $\chi_G^{(\mathsf{u}|\mathsf{v})}$ for words of length k by repeatedly using the Dodgson identity eq. (5.6) or its generalisations derived from eq. (5.5). For $k = 2, 3$ one has

$$\chi_G^{(\mathsf{a}_1\mathsf{a}_2|\mathsf{a}_3\mathsf{a}_4)} = \Psi_G^{-1}\Big(\chi_G^{(\mathsf{a}_1|\mathsf{a}_3)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_4)} - \chi_G^{(\mathsf{a}_1|\mathsf{a}_4)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_3)}\Big), \tag{5.12}$$

$$\begin{aligned}\chi_G^{(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3|\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6)} = \Psi_G^{-2}\Big(&\chi_G^{(\mathsf{a}_1|\mathsf{a}_4)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_5)}\chi_G^{(\mathsf{a}_3|\mathsf{a}_6)} - \chi_G^{(\mathsf{a}_1|\mathsf{a}_4)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_6)}\chi_G^{(\mathsf{a}_3|\mathsf{a}_5)}\\ &+ \chi_G^{(\mathsf{a}_1|\mathsf{a}_5)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_4)}\chi_G^{(\mathsf{a}_3|\mathsf{a}_6)} - \chi_G^{(\mathsf{a}_1|\mathsf{a}_5)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_6)}\chi_G^{(\mathsf{a}_3|\mathsf{a}_4)}\\ &+ \chi_G^{(\mathsf{a}_1|\mathsf{a}_6)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_4)}\chi_G^{(\mathsf{a}_3|\mathsf{a}_5)} - \chi_G^{(\mathsf{a}_1|\mathsf{a}_6)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_5)}\chi_G^{(\mathsf{a}_3|\mathsf{a}_4)}\Big).\end{aligned} \tag{5.13}$$

Note that this also permits an expansion (essentially the cofactor expansion of the determinant) that yields, e.g. for $k = 3$,

$$\chi_G^{(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3|\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6)} = \Psi_G^{-1}\Bigg(\chi_G^{(\mathsf{a}_1|\mathsf{a}_4)}\chi_G^{(\mathsf{a}_2\mathsf{a}_3|\mathsf{a}_5\mathsf{a}_6)} - \chi_G^{(\mathsf{a}_1|\mathsf{a}_5)}\chi_G^{(\mathsf{a}_2\mathsf{a}_3|\mathsf{a}_4\mathsf{a}_6)} + \chi_G^{(\mathsf{a}_1|\mathsf{a}_6)}\chi_G^{(\mathsf{a}_2\mathsf{a}_3|\mathsf{a}_4\mathsf{a}_5)}\Bigg). \tag{5.14}$$

which will be very useful later on.

Defining the polynomials like this, imposes an ordering on the indices instead of using unordered sets I, J. This yields a symmetry $\chi_G^{(\mathsf{u}|\mathsf{v})} = \operatorname{sgn}(\sigma)\chi_G^{(\mathsf{u}|\sigma(\mathsf{v}))}$ for all permutations of letters in the words, which we will be able to exploit for our purposes below.

5.2 Summation of chord diagrams and Dodgson polynomials

5.2.1 Chord diagram summation without polynomials

Before we start summing Dodgson polynomials it is quite instructive to consider sums over chord diagrams whose summands merely contain the integer factors that resulted from the traces, i.e. sums of the form $\sum_D (-2)^{\tilde{c}(D)}$. First, consider the sum over all diagrams resulting from addition of a single chord in all possible ways.

Theorem 5.2.1. *Let* $D_0 \in \mathcal{D}^n_{k-1}$ *with* $n = (n_1, \ldots, n_\ell)$ *and* $1 \leq k \leq \sum n_i$. *Furthermore, denote with* $m = n - k + 1$ *the number of missing chords in* D_0. *Then*

$$\sum_{D \in \bar{\mathcal{D}}(D_0)} (-2)^{\tilde{c}(D)} = -m(m+1) \cdot (-2)^{\tilde{c}(D_0)}. \tag{5.15}$$

Proof. Let (u, v) denote the newly added chord in a given diagram D. Since we only care about the change in total number of cycles the five cases of proposition 4.3.5 can be collected into only three cases here. Cases *1.a*, *1.b* and *1.d* become one, characterised by $\operatorname{sgn}(u, v) = 1$, while cases *1.c* and *2.* are characterised by $\operatorname{sgn}(u, v) = 0$ and $\operatorname{sgn}(u, v) = -1$ respectively. With this notation we can conveniently denote the change in number of cycles from D_0 to D as

$$\tilde{c}(D) = \tilde{c}(D_0) + \operatorname{sgn}(u, v). \tag{5.16}$$

We first compute explicitly the cases $k = n$ and $k = n - 1$ to illustrate the idea of the proof and then prove for general k.

For $k = n$ ($m = 1$) there are only two free vertices, which are the endpoints of the two paths of a single tricoloured cycle and there is only one possibility of adding a chord, with $\operatorname{sgn}(u, v) = 1$. Hence,

$$\sum_{D \in \bar{\mathcal{D}}(D_0)} (-2)^{\tilde{c}(D)} = (-2)^{\tilde{c}(D)} = (-2)^{\tilde{c}(D_0)+1} = -2 \cdot (-2)^{\tilde{c}(D_0)}. \tag{5.17}$$

For $k = n - 1$ ($m = 2$) there are 6 ways to add a chord and the four free vertices can be arranged either in a single tricoloured cycle with four paths or two tricoloured cycles with two paths each.

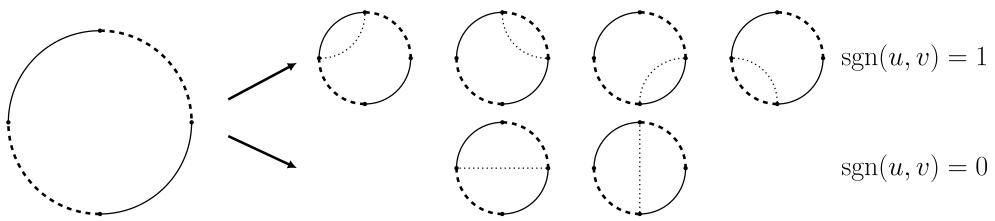


Figure 5.1: Visualisation of the six cases with $m = 2$ and one tricoloured cycle.

If it is a single tricoloured cycle, then four of the six ways to add a chord yield $\operatorname{sgn}(u, v) = 1$, with $d_-(u, v) = 1$ and $d_+(u, v) = 3$. The other possibility corresponds to $\operatorname{sgn}(u, v) = 0$, with $d_\pm = 2$, such that overall

$$\sum_{D \in \bar{\mathcal{D}}(D_0)} (-2)^{\tilde{c}(D)} = 4(-2)^{\tilde{c}(D_0)+1} + 2(-2)^{\tilde{c}(D_0)} = -6 \cdot (-2)^{\tilde{c}(D_0)}. \tag{5.18}$$

If there are two tricoloured cycles, then four possible chord additions are between those two cycles, i.e. $\mathrm{sgn}(u, v) = -1$, and the other two have $\mathrm{sgn}(u, v) = 1$, analogous to the $m = 1$ case. One finds the same result,

$$\sum_{D \in \bar{\mathcal{D}}(D_0)} (-2)^{\tilde{c}(D)} = 4(-2)^{\tilde{c}(D_0)-1} + 2(-2)^{\tilde{c}(D_0)+1} = -6 \cdot (-2)^{\tilde{c}(D_0)}. \tag{5.19}$$

In general there are $\binom{2m}{2}$ possibilities of adding a chord to D_0 and the $2m$ free vertices can be partitioned into up to m tricoloured cycles (base cycles of $\pi_0(D_0)$) as follows:

$$\begin{aligned} &(2m) \\ &(2m-2, 2), (2m-4, 4), \ldots, (\lceil m \rceil, \lfloor m \rfloor) \\ &\vdots \\ &(2, 2, \ldots, 2) \end{aligned}$$

Two observations allow us to collect all terms in each of these cases. First, consider a single tricoloured cycle on $2m$ vertices. There are $2m$ possible vertices for u. Now one simply counts the number of possible choices for the other vertex that yield a chord additions with odd or even signum, and divides by 2 because of symmetry under exchange of u and v. One finds that this gives m^2 instances of $\mathrm{sgn}(u, v) = 1$ (odd length segments) and $m(m-1)$ of case $\mathrm{sgn}(u, v) = 0$ (even length segments).

Secondly, consider a set of ℓ tricoloured cycles on $2m_i$ vertices respectively and count only the number of possibilities to add a chord between any two of them. There are $\binom{\ell}{2}$ choices of two cycles, each of which contribute $2m_i \cdot 2m_j$ possibilities to add a chord such that we can express the total number of possibilities as $E_2(2m_1, \ldots, 2m_\ell) = 4E_2(m_1, \ldots, m_\ell)$, the evaluation of the elementary symmetric polynomial of degree 2.

Combining these two results we find that, for a set of ℓ tricoloured cycles on $2m_i$ vertices with $\sum m_i = m$ one has the following number of chord additions corresponding to each case:

$$\mathrm{sgn}(u, v) = \begin{cases} +1 & \longrightarrow \quad \sum m_i^2 \\ \ \ 0 & \longrightarrow \quad \sum m_i(m_i - 1) \\ -1 & \longrightarrow \quad 4E_2(m_1, \ldots, m_\ell) \end{cases}$$

Now it remains to be shown that the sum $\sum_{D \in \bar{\mathcal{D}}(D_0)} (-2)^{\tilde{c}(D)}$ yields the same result, regardless of the particular tricoloured cycle partition present in D_0. Assume D_0 contains ℓ tricoloured cycles with m_i free vertices such that $(m_1, \ldots, m_\ell)$

with $\sum m_i = m$ is the corresponding integer partition of $m \geq 1$. Then

$$\begin{aligned}
\sum_{D \in \bar{\mathcal{D}}(D_0)} (-2)^{\tilde{c}(D)} &= (-2)^{\tilde{c}(D_0)} \left(-2 \sum_{i=1}^{\ell} m_i^2 + \sum_{i=1}^{\ell} m_i(m_i - 1) - \frac{1}{2} 4E_2(m_1, \ldots, m_\ell) \right) \\
&= (-2)^{\tilde{c}(D_0)} \left(-\sum_{i=1}^{\ell} m_i - \underbrace{\left(\sum_{i=1}^{\ell} m_i^2 + 2E_2(m_1, \ldots, m_\ell) \right)}_{= \left(\sum_{i=1}^{\ell} m_i \right)^2 = m^2} \right) \\
&= -m(m+1) \cdot (-2)^{\tilde{c}(D_0)}.
\end{aligned} \tag{5.20}$$

□

Inspired by the above theorem we can now consider iterations $\bar{\mathcal{D}}^k(D_0)$, i.e. sums over sets of chord diagrams, which result from adding multiple chords in all possible ways.

Corollary 5.2.2. *Let $D_0 \in \mathcal{D}^n_{N-m}$ with $1 \leq m \leq N = \sum n_i$ and $\bar{\mathcal{D}}(D_0)$ as in theorem 5.2.1 above. Then, for $1 \leq k \leq m$*

$$\sum_{D \in \bar{\mathcal{D}}^k(D_0)} (-2)^{\tilde{c}(D)} = (-1)^k k! \binom{m}{k} \binom{m+1}{k} (-2)^{\tilde{c}(D_0)}. \tag{5.21}$$

In particular, one finds the sum over all completions of D_0 for $k = m$

$$\sum_{D \in \bar{\mathcal{D}}^m(D_0)} (-2)^{\tilde{c}(D)} = (-1)^m (m+1)! (-2)^{\tilde{c}(D_0)} \tag{5.22}$$

and the sum over all diagrams with a given base cycle structure n and a full set of chords for $k = m = N$

$$\sum_{D \in \mathcal{D}^n_N} (-2)^{\tilde{c}(D)} = 4(-1)^N (N+1)! \tag{5.23}$$

Proof. For $k = 1$ the statement is just theorem 5.2.1 since $\binom{m}{1}\binom{m+1}{1} = m(m+1)$. For $k > 1$ we make use of theorem 5.2.1 iteratively, collect the factors and divide by $k!$ to account for the different permutations of chord additions that result in the same diagrams:

$$
\begin{aligned}
\sum_{D\in\bar{\mathcal{D}}^k(D_0)}(-2)^{\tilde{c}(D)} &= \frac{1}{k!}\sum_{D_1\in\bar{\mathcal{D}}(D_0)}\cdots\sum_{D_k\in\bar{\mathcal{D}}(D_{k-1})}(-2)^{\tilde{c}(D_k)}\\
&= \frac{(-1)}{k!}(m-k+1)(m-k+2)\sum_{D_1\in\bar{\mathcal{D}}(D_0)}\cdots\sum_{D_{k-1}\in\bar{\mathcal{D}}(D_{k-2})}(-2)^{\tilde{c}(D_{k-1})}\\
&\ \vdots\\
&= \frac{(-1)^k}{k!}\left(\prod_{i=1}^{k}(m-k+i)(m-k+i+1)\right)(-2)^{\tilde{c}(D_0)}\\
&= \frac{(-1)^k}{k!}\frac{m!}{(m-k)!}\frac{(m+1)!}{(m-k+1)!}(-2)^{\tilde{c}(D_0)}\\
&= (-1)^k k!\binom{m}{k}\binom{m+1}{k}(-2)^{\tilde{c}(D_0)} \qquad (5.24)
\end{aligned}
$$

□

This result already captures a lot of the information of these sums. In fact, the r.h.s. of eq. (5.22), which contains exactly the same type of sum as in e.g. Z^0_Γ, is exactly the factor that we will also find with polynomials.

The next step is now to introduce a very particular new polynomial, a combination of Dodgson cycle polynomials with words of length ≥ 1 (instead of just single letters) that accompanies these integer factors.

5.2.2 Partition polynomials

Let $D \in \mathcal{D}^n_0$ with $n \in \mathbb{N}^\ell$ and label all its vertices with letters from an alphabet $\mathsf{A} = \{\mathsf{a}_i \mid i \in V_D\}$. Consider all pairs of monomial words (u, v) of length $|\mathsf{u}| = N = |\mathsf{v}|$ over this alphabet such that uv contains each letter exactly once. Then the symmetries

$$\chi_\Gamma^{(\mathsf{u}|\mathsf{v})} = \chi_\Gamma^{(\mathsf{v}|\mathsf{u})} \quad \text{and} \quad \chi_\Gamma^{(\mathsf{u}|\mathsf{v})} = \operatorname{sgn}(\sigma)\chi_\Gamma^{(\mathsf{u}|\sigma(\mathsf{v}))} \quad \forall \sigma \in S_N, \qquad (5.25)$$

induce an equivalence relation[2] on these words via

$$(\mathsf{u}, \mathsf{v}) \sim (\mathsf{u}', \mathsf{v}') \iff \chi_\Gamma^{(\mathsf{u}|\mathsf{v})} = \pm\chi_\Gamma^{(\mathsf{u}'|\mathsf{v}')}, \qquad (5.26)$$

[2]This may seem somewhat redundant: We just end up with the unordered sets of the original Dodgson polynomials again! But rest assured, the ordering given by the words does actually play an important role below.

or equivalently

$$(\mathsf{u},\mathsf{v}) \sim (\mathsf{u}',\mathsf{v}') \iff \exists\, \sigma,\sigma' \in S_N \text{ s.t. } \mathsf{u}' = \sigma(\mathsf{u}),\ \mathsf{v}' = \sigma'(\mathsf{v}). \tag{5.27}$$

Let P denote the corresponding set of equivalence classes of pairs (u,v) that satisfy the above mentioned properties. For the two coloured subsets of base edges E_D^1 and E_D^2 define the corresponding subsets $\mathsf{P}_i \subset \mathsf{P}$ by imposing an additional constraint: For all edges $(u,v) \in E_D^i$ we demand that the two corresponding letters do not appear in the same word, i.e. $\mathsf{a}_u \in \mathsf{u}$ and $\mathsf{a}_v \in \mathsf{v}$ or vice versa. The full set of equivalence classes is then the union $\mathsf{P} = \mathsf{P}_1 \cup \mathsf{P}_2$. Moreover, the P_i intersect only in exactly one element, which, assuming the vertices of D are labelled consecutively within each base cycle, is the class of pairs that contain all letters labelled with odd numbers in one word and those labelled with even numbers in the other. Finally, we need to fix one distinguished representative of each class with respect to which we consider permutations. Assuming some arbitrary ordering of i-coloured base edges $(u_1,v_1),\ldots,(u_N,v_N) \in E_D^i$ each equivalence class contains exactly one element that we notate $(\mathsf{u}_{\mathrm{id}},\mathsf{v}_{\mathrm{id}})$ such that a_{u_j} and a_{v_j} are the j-th letters of u_{id} and v_{id}, or vice versa. For any other ordering of base edges the designated element would be related to $(\mathsf{u}_{\mathrm{id}},\mathsf{v}_{\mathrm{id}})$ by the same permutation in both words, such that the choice of ordering on E_D^i does not matter.

For all $(\mathsf{u},\mathsf{v}) \in \mathsf{P}$ and partitions of i-coloured base edges $\mathcal{E} = (E_1,\ldots,E_{|\mathcal{E}|}) \in \mathcal{P}(E_D^i)$ define a map $\lambda_{\mathcal{E}}$ as follows. Let

$$V_j := \bigcup_{(u,v)\in E_j} \{u,v\} \subseteq V_D, \tag{5.28}$$

be the set of vertices in the part E_j and consider the restriction

$$(\mathsf{u}_j,\mathsf{v}_j) = (\mathsf{u},\mathsf{v})|_{\mathsf{a}_k=1\ \forall k\in V_D\setminus V_j} \tag{5.29}$$

of (u,v) to the alphabet corresponding to these vertices. In each $(\mathsf{u}_j,\mathsf{v}_j)$ all letters not associated to this part of the partition are removed but, critically, the order of the remaining letters is preserved. Then

$$\lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v}) := \begin{cases} \{(\mathsf{u}_1,\mathsf{v}_1),\ldots,(\mathsf{u}_{|\mathcal{E}|},\mathsf{v}_{|\mathcal{E}|})\} & \text{if } |\mathsf{u}_j| = |\mathsf{v}_j| \text{ for all } 1 \le j \le |\mathcal{E}|, \\ \emptyset & \text{else.} \end{cases} \tag{5.30}$$

The concatenations $\mathsf{u}_1\cdots\mathsf{u}_{|\mathcal{E}|}$ and $\mathsf{v}_1\cdots\mathsf{v}_{|\mathcal{E}|}$ are then permutations of u and v and we define

$$\mathrm{sgn}_{\mathcal{E}}(\mathsf{u},\mathsf{v}) := \begin{cases} 0 & \text{if } \lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v}) = \emptyset, \\ \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma') & \text{else,} \end{cases} \tag{5.31}$$

where $\sigma,\sigma' \in S_N$ are the permutations with $\sigma(\mathsf{u}_{\mathrm{id}}) = \mathsf{u}_1\cdots\mathsf{u}_{|\mathcal{E}|}$ and $\sigma'(\mathsf{v}_{\mathrm{id}}) = \mathsf{v}_1\cdots\mathsf{v}_{|\mathcal{E}|}$. With this we are now ready to insert these types of words into certain combinations of Dodgson polynomials, which we will call partition polynomials.

Definition 5.2.3. (Partition polynomials)
Let Γ *be a QED Feynman graph with the associated chord diagram* D_Γ *such that* $\pi_0(D_\Gamma) = D^0_\Gamma \in \mathcal{D}^n_0$ *and* $\mathcal{D}^0_\Gamma \simeq \mathcal{D}^n_N$ *with* $n \in \mathbb{N}^\ell$, $N = \sum_i n_i = h_1(\Gamma)$. *Then we define the partition polynomial of* Γ *to be*

$$Z^0_\Gamma(\alpha) := \sum_{\mathcal{E}\in\mathcal{P}(E^1_D)} (-\Psi_\Gamma)^{N-|\mathcal{E}|}(|\mathcal{E}|+1)! \sum_{(\mathsf{u},\mathsf{v})\in\mathsf{P}_2} \operatorname{sgn}_{\mathcal{E}}(\mathsf{u},\mathsf{v}) \prod_{(\mathsf{u}',\mathsf{v}')\in\lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v})} \chi_\Gamma^{(\mathsf{u}'|\mathsf{v}')}. \tag{5.32}$$

Moreover, for $1 \leq l \leq N$ *let*

$$Z^0_\Gamma\Big|_l := \sum_{\substack{\mathcal{E}\in\mathcal{P}(E^1_D)\\ |\mathcal{E}|=l}} \sum_{(\mathsf{u},\mathsf{v})\in\mathsf{P}_2} \operatorname{sgn}_{\mathcal{E}}(\mathsf{u},\mathsf{v}) \prod_{(\mathsf{u}',\mathsf{v}')\in\lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v})} \chi_\Gamma^{(\mathsf{u}'|\mathsf{v}')} \tag{5.33}$$

such that

$$Z^0_\Gamma = \sum_{l=1}^{N} (-\Psi_\Gamma)^{N-l}(l+1)!\; Z^0_\Gamma\Big|_l. \tag{5.34}$$

Note that using partitions $\mathcal{P}(E^2_D)$ in the first and words $(\mathsf{u},\mathsf{v}) \in \mathsf{P}_1$ in the second sum yields the exact same polynomial. This symmetry is not quite obvious from this definition but will become so in the proof of theorem 5.2.4 below.

5.2.3 The summation theorem

The statement of the main theorem is now that the polynomial Z^0_Γ, defined like this, is in fact equal to the expression that we abbreviated with this notation in section 4.4.

Theorem 5.2.4.

$$Z^0_\Gamma = \frac{1}{2} \sum_{D\in\mathcal{D}^0_\Gamma} (-2)^{\tilde{c}(D)} \prod_{(u,v)\in E^0_D} \chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)}. \tag{5.35}$$

In order to prove this we first need some auxiliary results. First we attempt to study the summation by essentially working backwards and looking at sums $\sum \chi_\Gamma^{(\mathsf{u}_{\mathrm{id}}|\mathsf{v}_{\mathrm{id}})}$ for $(\mathsf{u}_{\mathrm{id}}, \mathsf{v}_{\mathrm{id}}) \in \mathsf{P}_2$, which appear in the partition polynomial for the single part partition $\mathcal{E} = \{E^1_D\}$.

Lemma 5.2.5. *Let* P_j *be as above and* $c^j_2(D)$ *the number of two-coloured cycles consisting of chords and* j*-coloured base edges (such that* $c_2(D) = c^1_2(D) + c^2_2(D)$*). Then*

$$\sum_{(\mathsf{u}_{\mathrm{id}},\mathsf{v}_{\mathrm{id}})\in\mathsf{P}_j} \chi_\Gamma^{(\mathsf{u}_{\mathrm{id}}|\mathsf{v}_{\mathrm{id}})} = (-\Psi_\Gamma)^{1-N} \sum_{D\in\mathcal{D}^0_\Gamma} (-2)^{c^j_2(D)-1} \prod_{(u,v)\in E^0_D} \chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)}, \tag{5.36}$$

Proof. Quick computations show that the claim holds for all n with $N = \sum n_i = 1, 2$, and even $N = 3$ is only mildly tedious, as shown below in example 5.2.6. We now reduce the l.h.s. of 5.36 to a sum over expressions corresponding $N-1$, in order to prove by induction.

Consider a word pair $(\mathsf{x}_{11}\cdots\mathsf{x}_{1N}, \mathsf{x}_{21}\cdots\mathsf{x}_{2N})$ with all $\mathsf{x}_{ij} \in \mathsf{A}$ different letters. Assuming this word is a representative $(\mathsf{u}_{\mathrm{id}}, \mathsf{v}_{\mathrm{id}}) \in \mathsf{P}_j$, each pair $(\mathsf{x}_{1k}, \mathsf{x}_{2k})$ of k-th letters corresponds to a base edge of $E^j_{D_0}$, for a chord diagram $D_0 \in \mathcal{D}^n_0$. With eq. (5.11) its Dodgson polynomial can be written as

$$\begin{aligned}\Psi_\Gamma \chi_\Gamma^{(\mathsf{x}_{11}\cdots\mathsf{x}_{1N}|\mathsf{x}_{21}\cdots\mathsf{x}_{2N})} &\\ &= \sum_{k=1}^{N}(-1)^{1+k}\chi_\Gamma^{(\mathsf{x}_{11}|\mathsf{x}_{2k})}\chi_\Gamma^{(\mathsf{x}_{12}\cdots\mathsf{x}_{1N}|\mathsf{x}_{21}\cdots\hat{\mathsf{x}}_{2k}\cdots\mathsf{x}_{2N})}\\ &= \chi_\Gamma^{(\mathsf{x}_{11}|\mathsf{x}_{21})}\chi_\Gamma^{(\mathsf{x}_{12}\cdots\mathsf{x}_{1N}|\mathsf{x}_{22}\cdots\mathsf{x}_{2N})} - \sum_{k=2}^{N}\chi_\Gamma^{(\mathsf{x}_{11}|\mathsf{x}_{2k})}\chi_\Gamma^{(\mathsf{x}_{1k}\mathsf{x}_{12}\cdots\hat{\mathsf{x}}_{1k}\cdots\mathsf{x}_{1N}|\mathsf{x}_{21}\cdots\hat{\mathsf{x}}_{2k}\cdots\mathsf{x}_{2N})}.\end{aligned} \tag{5.37}$$

Moving the letter x_{1k} in the second line guarantees that the k-th letter pairs are still paired up in the expansion. In fact, the word pairs

$$(\mathsf{x}_{1k}\mathsf{x}_{12}\cdots\hat{\mathsf{x}}_{1k}\cdots\mathsf{x}_{1N}, \mathsf{x}_{21}\cdots\hat{\mathsf{x}}_{2k}\cdots\mathsf{x}_{2N}) \tag{5.38}$$

are the representatives $(\mathsf{u}'_{\mathrm{id}}, \mathsf{v}'_{\mathrm{id}})$ of an equivalence class of word pairs associated to the diagram $\pi_0(D)$, where D is D_0 together with the chord corresponding to the letter pair $(\mathsf{x}_{11}, \mathsf{x}_{2k})$. The sum over all equivalence classes in P_j can be realised by summing word pairs of the form

$$(\mathsf{x}_{(1+t_1)1}\cdots\mathsf{x}_{(1+t_N)N},\ \mathsf{x}_{(2-t_1)1}\cdots\mathsf{x}_{(2-t_N)N}) \tag{5.39}$$

over all N-tuples in $\mathcal{T} = \{t \in \{0,1\}^N \mid t_1 = 0\}$. One finds

$$\begin{aligned}\Psi_\Gamma \sum_{t\in\mathcal{T}} \chi_\Gamma^{(\mathsf{x}_{(1+t_1)1}\cdots\mathsf{x}_{(1+t_N)N}|\mathsf{x}_{(2-t_1)1}\cdots\mathsf{x}_{(2-t_N)N})} &\\ = \chi_\Gamma^{(\mathsf{x}_{11}|\mathsf{x}_{21})}\sum_{t\in\mathcal{T}}\chi_\Gamma^{(\mathsf{x}_{(1+t_2)2}\cdots\mathsf{x}_{(1+t_N)N}|\mathsf{x}_{(2-t_2)2}\cdots\mathsf{x}_{(2-t_N)N})} &\\ - \sum_{k=2}^{N}\sum_{t\in\mathcal{T}}\chi_\Gamma^{(\mathsf{x}_{11}|\mathsf{x}_{(2-t_k)k})}\chi_\Gamma^{(\mathsf{x}_{(1+t_k)k}\mathsf{x}_{(1+t_2)2}\cdots\hat{\mathsf{x}}_{(1+t_k)k}\cdots\mathsf{x}_{(1+t_N)N}|\mathsf{x}_{21}\cdots\hat{\mathsf{x}}_{(2-t_k)k}\cdots\mathsf{x}_{(2-t_N)N})}&.\end{aligned} \tag{5.40}$$

Now we want to translate this back to vertices of a chord diagram. Let $u, v \in V_{D_0}$

such that $\mathsf{x}_{11} = \mathsf{a}_u$, $\mathsf{x}_{21} = \mathsf{a}_v$ and $(u, v) \in E^j_{D_0}$. Then eq. (5.40) becomes

$$\Psi_\Gamma \sum_{(\mathsf{u}_{\mathrm{id}},\mathsf{v}_{\mathrm{id}})\in\mathsf{P}_j} \chi_\Gamma^{(\mathsf{u}_{\mathrm{id}}|\mathsf{v}_{\mathrm{id}})} = 2\chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)} \sum_{(\mathsf{u}'_{\mathrm{id}},\mathsf{v}'_{\mathrm{id}})\in\mathsf{P}_j^{u,v}} \chi_\Gamma^{(\mathsf{u}'_{\mathrm{id}}|\mathsf{v}'_{\mathrm{id}})} - \sum_{\substack{w\in V_{D_0}\\ w\neq u,v}} \chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_w)} \sum_{(\mathsf{u}'_{\mathrm{id}},\mathsf{v}'_{\mathrm{id}})\in\mathsf{P}_j^{u,w}} \chi_\Gamma^{(\mathsf{u}'_{\mathrm{id}}|\mathsf{v}'_{\mathrm{id}})}, \tag{5.41}$$

where $\mathsf{P}_j^{u,v}$ and $\mathsf{P}_j^{u,w}$ are the classes of word pairs after addition of the chords (u, v) or (u, w) respectively. Replacing these sums with the corresponding r.h.s. of eq. (5.36) finishes the proof, where the factor of $-(-2)$ in the first term corresponds to the addition of the cycle that consists of the j-coloured base edge (u, v) and the chord between those same vertices. All other chords (u, w) added to D_0 do not add two-coloured cycles but only split, twist or merge base cycles when projected out with π_0. □

Example 5.2.6. *Consider as an example $N = 3$ with a single base cycle. Label vertices consecutively from 1 to 6 and choose j to be the colour of $(1, 2)$. Then the sum over word pairs in P_j on the l.h.s. of eq. (5.36) is*

$$\chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_3\mathsf{a}_5|\mathsf{a}_2\mathsf{a}_4\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_4\mathsf{a}_5|\mathsf{a}_2\mathsf{a}_3\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_3\mathsf{a}_6|\mathsf{a}_2\mathsf{a}_4\mathsf{a}_5)} + \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_4\mathsf{a}_6|\mathsf{a}_2\mathsf{a}_3\mathsf{a}_5)}. \tag{5.42}$$

Expanding each term as defined in eq. (5.11) yields Ψ_Γ^{-2} times 24 terms, 15 of which are distinct, such that one finds

$$\begin{aligned}
&4\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_2)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_4)}\chi_\Gamma^{(\mathsf{a}_5|\mathsf{a}_6)} - 2\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_2)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_5)}\chi_\Gamma^{(\mathsf{a}_4|\mathsf{a}_6)} - 2\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_2)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_6)}\chi_\Gamma^{(\mathsf{a}_4|\mathsf{a}_5)}\\
&-2\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_3)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_4)}\chi_\Gamma^{(\mathsf{a}_5|\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_3)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_5)}\chi_\Gamma^{(\mathsf{a}_4|\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_3)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_6)}\chi_\Gamma^{(\mathsf{a}_4|\mathsf{a}_5)}\\
&-2\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_4)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_3)}\chi_\Gamma^{(\mathsf{a}_5|\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_4)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_5)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_4)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_6)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_5)}\\
&+\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_5)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_3)}\chi_\Gamma^{(\mathsf{a}_4|\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_5)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_4)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_6)} - 2\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_5)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_6)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_4)}\\
&+\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_6)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_3)}\chi_\Gamma^{(\mathsf{a}_4|\mathsf{a}_5)} + \chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_6)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_4)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_5)} - 2\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_6)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_5)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_4)}.
\end{aligned}$$

Now one can simply check each summand by counting the cycles of the corresponding chord diagram, while keeping in mind that only the bicoloured cycles with chords and j-coloured base edges are counted. For example, in the first term each factor corresponds to a chord $(1, 2)$, $(3, 4)$, $(5, 6)$, each spanning exactly one of the j-coloured base edges. Hence, there are three such cycles and $(-2)^{c_2^j(D)-1} = 4$.

The obvious next questions is now: Can we find such an identity for all partitions? Indeed, we can.

Lemma 5.2.7. *Let $\mathcal{E} \in \mathcal{P}(E_D^1)$ be any partition of* 1*-coloured base edges. Then*

$$\sum_{(\mathsf{u},\mathsf{v})\in\mathsf{P}_2} \operatorname{sgn}_{\mathcal{E}}(\mathsf{u},\mathsf{v}) \prod_{(\mathsf{u}',\mathsf{v}')\in\lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v})} \chi_\Gamma^{(\mathsf{u}'|\mathsf{v}')} = (-1)^{1-|\mathcal{E}|}(-\Psi_\Gamma)^{|\mathcal{E}|-N} \sum_{D\in\mathcal{D}|_{\mathcal{E}}} (-2)^{c_2^2(D)-1} \prod_{(u,v)\in E_D^0} \chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)}, \tag{5.43}$$

where $\mathcal{D}|_{\mathcal{E}} \subset \mathcal{D}_N^n \simeq \mathcal{D}_\Gamma^0$ is the subset of complete chord diagrams with base cycles given by n (and vertices labelled by edges of Γ) that is restricted by demanding that all chords of a diagram can only connect vertices that lie within the same part of $\mathcal{E}$.

Proof. Begin with partitions with exactly two parts $\mathcal{E} = \{E_1, E_2\}$ in which one part consists of only one edge, say $E_1 = \{(u, v)\}$. Then for all $(\mathsf{u},\mathsf{v}) \in \mathsf{P}_2$ the pair $(\mathsf{a}_u, \mathsf{a}_v)$ is contained in $\lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v})$ such that $\chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)}$ simply factors out. Proposition 5.2.5 can be applied to the remaining terms to finish the proof for partitions of this type, and in particular one notes that the definition of $\lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v})$ precisely yields the restriction of chord diagrams to this partition. Moreover, repetition of this argument proves the claim for all partitions with at most one part that contains more than one edge.

Assume now again a partition with two parts $\mathcal{E} = \{E_1, E_2\}$, but also with $|E_1| = 2$ and $|E_2| \geq 2$. The Dodgson polynomial $\chi_\Gamma^{(\mathsf{u}_1'|\mathsf{v}_1')}$, where $(\mathsf{u}_1', \mathsf{v}_1') \in \lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v})$ is the word pair corresponding to part E_1 of the partition, can be expanded analogous to eq. (5.36), with the caveat that the restriction to E_1 removes all terms from the sum that correspond to vertices not in that part (including the first term, which contributes the factor 2, if the fixed vertex u and the vertex v that it shares a 2-coloured base edge with are not in the same part of the 1-coloured base edge partition). Then the coefficient of each $\chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)}$ corresponds to a partition with part sizes $\{|E_1| - 1 = 1, |E_2|\}$, which was discussed above, such that term by term application of the identity proves it for this type of partition. Further repetition of this argument finishes the proof for all partitions, independent of number or size of parts. $\square$

The final ingredient for the proof of this chapter's main theorem is an identity allowing summation of Stirling numbers of the second kind[3]. To prove it we need a certain identity relating Stirling numbers and the classical polylogarithm. However, while the literature contains a number of well known identities that do so, they are all either similar but not obviously equivalent to the one we need, or appear without proof. Moreover, the commonly cited references (e.g. [2, 91, 122],

[3]James Stirling (1692-1770), *Methodus Differentialis*, 1730. See also [130]. The Stirling numbers of the second kind $S(k, l)$ count the ways to partition a set of k elements into l non-empty sets.

among many others) all appear to cite each other or unavailable older literature, so it may actually be somewhat elucidating to derive everything we need ourselves.

Proposition 5.2.8. *Let*

$$\mathrm{Li}_s(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^s} \qquad |z| < 1, \; s \in \mathbb{Z} \tag{5.44}$$

be the classical polylogarithm and $S(k,l)$ *be the Stirling number of the second kind. Then*

$$\mathrm{Li}_{-k+1}(z) = (-1)^k \sum_{l=1}^{k} S(k,l) \frac{(l-1)!}{(z-1)^l} \tag{5.45}$$

for integers $k \geq 2$.

Proof. For $k = 2$ the r.h.s. is

$$\frac{1}{z-1} + \frac{1}{(z-1)^2} = \frac{z}{(1-z)^2} = z\partial_z \frac{1}{1-z} = z\partial_z \sum_{l=0}^{\infty} z^l = \sum_{l=1}^{\infty} l z^l = \mathrm{Li}_{-1}(z). \tag{5.46}$$

Now proceed by induction

$$\begin{aligned} \mathrm{Li}_{-k+1}(z) = z\partial_z \, \mathrm{Li}_{-k+2}(z) &= (-1)^{k-1} \sum_{l=1}^{k-1} S(k-1,l) z\partial_z \frac{1}{(z-1)^l}(l-1)! \\ &= (-1)^k \sum_{l=1}^{k-1} S(k-1,l) \frac{z}{(z-1)^{l+1}} l!, \end{aligned} \tag{5.47}$$

and use partial fraction decomposition to find

$$S(k-1,l)\frac{z}{(z-1)^{l+1}} l! = l S(k-1,l) \frac{(l-1)!}{(z-1)^l} + S(k-1,l) \frac{l!}{(z-1)^{l+1}} \tag{5.48}$$

Using the recurrence relation $S(k,l) = S(k-1,l-1) + l(S(k-1,l)$ the first term is further rewritten as

$$l S(k-1,l) \frac{(l-1)!}{(z-1)^l} = S(k,l) \frac{(l-1)!}{(z-1)^l} - S(k-1,l-1) \frac{(l-1)!}{(z-1)^l} \tag{5.49}$$

In the sum one now has a telescopic cancellation involving the second terms of eqs. (5.48) and (5.49). The only remaining terms are

$$\frac{S(k-1,0)}{z-1} = 0 \qquad \text{and} \qquad S(k-1,k-1)\frac{(k-1)!}{(z-1)^k} = S(k,k)\frac{(k-1)!}{(z-1)^k},$$

as well as the first part of the r.h.s. of eq. (5.49) summed up to $l = k-1$, such that overall

$$\mathrm{Li}_{-k+1}(z) = (-1)^k \sum_{l=1}^{k} S(k,l) \frac{(l-1)!}{(z-1)^l}.$$

□

Lemma 5.2.9. *Let $S(k,l)$ be the Stirling number of the second kind. Then*

$$\sum_{l=1}^{k} S(k,l)(-1)^l (l+1)! = (-2)^k \qquad \forall k \geq 1.$$

Proof. For $k=1$ the claim is checked directly. For $k \geq 2$ we use the identity derived for the polylogarithm in proposition 5.2.8 and note that a change of the argument allows us to write

$$(-1)^k \,\mathrm{Li}_{-k+1}\left(1+\frac{1}{z}\right) = \sum_{l=1}^{k} S(k,l) z^l (l-1)! \tag{5.50}$$

with $z < -1$. Now let

$$\begin{aligned} L(z) := \sum_{l=1}^{k} S(k,l) z^l (l+1)! &= z\partial_z^2 z \sum_{l=1}^{k} S(k,l) z^l (l-1)! \\ &= (-1)^k z\partial_z^2 z \,\mathrm{Li}_{-k+1}\left(1+\frac{1}{z}\right). \end{aligned} \tag{5.51}$$

Computing the derivative one finds

$$L(z) = \frac{(-1)^k}{(z+1)^2}\left(\mathrm{Li}_{-k-1}\left(1+\frac{1}{z}\right) - \mathrm{Li}_{-k}\left(1+\frac{1}{z}\right)\right). \tag{5.52}$$

Both polylogarithms start with terms linear in $(z+1)/z$, yielding divergences when evaluating at $z=-1$, but upon closer inspection we see that they precisely cancel each other. With $z < -1$ one has $|1+1/z| < 1$ such that we are able to employ the classical sum representation of the polylogarithm, of which only the first two terms are of interest to us:

$$\begin{aligned} L(z) &= \frac{(-1)^k}{(z+1)^2}\left(\sum_{t=1}^{\infty} t^{k+1}\left(\frac{z+1}{z}\right)^t - \sum_{t=1}^{\infty} t^k \left(\frac{z+1}{z}\right)^t\right) \\ &= \frac{(-1)^k}{(z+1)^2}\left(\frac{z+1}{z} + 2^{k+1}\left(\frac{z+1}{z}\right)^2 - \frac{z+1}{z} - 2^k\left(\frac{z+1}{z}\right)^2 + \mathcal{O}\left(\left(\frac{z+1}{z}\right)^3\right)\right) \\ &= (-2)^k\left(\frac{1}{z^2} + \frac{1}{(z+1)^2}\mathcal{O}\left(\left(\frac{z+1}{z}\right)^3\right)\right). \end{aligned} \tag{5.53}$$

Now we can safely take the limit $z \to -1$ to find

$$\sum_{l=1}^{k} S(k,l)(-1)^l(l+1)! = L(-1) = (-2)^k. \tag{5.54}$$

□

Proof of theorem 5.2.4. First, use lemma 5.2.7 to rewrite the partition polynomial as

$$\begin{aligned} Z_\Gamma^0 &= \sum_{\mathcal{E}\in\mathcal{P}(E_D^1)} (-\Psi_\Gamma)^{N-|\mathcal{E}|}(|\mathcal{E}|+1)! \sum_{(\mathsf{u},\mathsf{v})\in\mathsf{P}_2} \operatorname{sgn}_{\mathcal{E}}(\mathsf{u},\mathsf{v}) \prod_{(\mathsf{u}',\mathsf{v}')\in\lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v})} \chi_\Gamma^{(\mathsf{u}'|\mathsf{v}')} \\ &= \sum_{\mathcal{E}\in\mathcal{P}(E_D^1)} (-1)^{|\mathcal{E}|+1}(|\mathcal{E}|+1)! \sum_{D\in\mathcal{D}|_{\mathcal{E}}} (-2)^{c_2^2(D)-1} \prod_{(u,v)\in E_D^0} \chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)}. \end{aligned}$$

The sum already contains $c_2^2(D)$, the number of 2-coloured cycles. Regarding cycles of the other colour we can make the following observation: In each diagram with $c_2^1(D) \leq N$ the 1-coloured cycles can themselves be interpreted as a partition of E_D^1 in which each part is given by the base edges connected to each other by chords. The diagrams in $\mathcal{D}|_{\mathcal{E}}$ can only have chords connecting base edges within the same part of $\mathcal{E}$, so each part in the partition given by the 1-coloured cycles has to be a subset of a part of $\mathcal{E}$. Counting the number of ways of partitioning the $c_2^1(D)$ cycles of a given diagram into partitions with $|\mathcal{E}|$ parts (i.e. counting the number of partitions $\mathcal{E}$ with a certain number of parts such that $\mathcal{D}|_{\mathcal{E}}$ contains the given diagram D) one finds precisely the Stirling numbers of the second kind $S(c_2^1(D), |\mathcal{E}|)$. Using this, we can exchange summation over diagrams and partitions and find

$$\begin{aligned} Z_\Gamma^0 &= \sum_{\mathcal{E}\in\mathcal{P}(E_D^1)} (-1)^{|\mathcal{E}|+1}(|\mathcal{E}|+1)! \sum_{D\in\mathcal{D}|_{\mathcal{E}}} (-2)^{c_2^2(D)-1} \prod_{(u,v)\in E_D^0} \chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)} \\ &= \frac{1}{2} \sum_{D\in\mathcal{D}_N^n} (-2)^{c_2^2(D)} \left(\prod_{(u,v)\in E_D^0} \chi_\Gamma^{(\mathsf{a}_u|\mathsf{a}_v)} \right) \sum_{l=1}^{c_2^1(D)} S(c_2^1(D), l)(-1)^l(l+1)!. \end{aligned}$$

Now lemma 5.2.9 is applied to evaluate the sum to $(-2)^{c_2^1(D)}$, which finishes the proof. □

Application to the integrand

Let us return to eq. (4.74). When inserting our definition for Z^0_Γ from eq. (5.34) it becomes

$$\begin{aligned}\phi^R_\Gamma &= (q^2 g^{\mu\nu} - q^\mu q^\nu) 2^{h_1} L \int_{\sigma_\Gamma} \Omega_\Gamma \frac{\varphi_\Gamma Z^0_\Gamma + 2Z^1_{\Gamma,0} + Z^1_{\Gamma,+}}{\Psi_\Gamma^{h_1+3}} \\ &= (q^2 g^{\mu\nu} - q^\mu q^\nu) 2^{h_1} L \int_{\sigma_\Gamma} \frac{\Omega_\Gamma}{\Psi_\Gamma^{h_1+3}} \Big(2Z^1_{\Gamma,0} + Z^1_{\Gamma,+} + \varphi_\Gamma \sum_{l=1}^{h_1} (-\Psi_\Gamma)^{h_1-l} (l+1)! Z^0_\Gamma\Big|_l \Big) \\ &= (q^2 g^{\mu\nu} - q^\mu q^\nu) 2^{h_1} L \int_{\sigma_\Gamma} \Omega_\Gamma \Big(\frac{2Z^1_{\Gamma,0} + Z^1_{\Gamma,+}}{\Psi_\Gamma^{h_1+3}} + \varphi_\Gamma \sum_{l=1}^{h_1} (-1)^{h_1-l} (l+1)! \frac{Z^0_\Gamma\big|_l}{\Psi_\Gamma^{3+l}} \Big).\end{aligned} \tag{5.55}$$

For the vertex integral from eq. (4.80) the simplification is similar, although there it is confined to the coefficient of γ^μ.

5.3 Polynomials for incomplete chord diagrams

Now that Z^0_Γ is well understood we can proceed to the much more complicated Z^k_Γ, for $k > 0$. Contrary to Z^0_Γ they contain not only the cycle polynomials, but also the physical momentum evaluation of

$$\mathrm{X}^{e,\mu}_\Gamma = \frac{1}{2\alpha_e} \frac{\partial}{\partial \xi_{e,\mu}} \beta_\Gamma = \alpha_e^{-1} \sum_{e' \in E_\Gamma} \xi^\mu_{e'} \beta_\Gamma^{(e|e')},$$

which we had defined in eq. (2.104). In section 2.2, eq. (2.47), we had an expression for the second Symanzik polynomial in terms of matrices:

$$Q'^T \tilde{L}'^{-1} Q' = \frac{\alpha_{E_\Gamma}}{\Psi_\Gamma} \sum_{v_1, v_2 \in V'_\Gamma} q_{v_1} \cdot q_{v_2} (-1)^{v_1+v_2} \det(\tilde{L}'^{\{v_1\}}_{\{v_2\}})$$

Replacing Q' with the vector $\vartheta^T = (\sum_e I_{ev} \xi_e)_{v \in V'_\Gamma}$ yields the bond polynomial $\beta_\Gamma(\alpha, \xi) = \Phi_\Gamma(\alpha, \xi)$ instead. This we can now again interpret in terms of the graph matrix, using the block matrix identity from eq. (2.14):

$$\det(M(G)^{\{v_1\}}_{\{v_2\}}) = \alpha_{E_\Gamma} \det(\tilde{L}'^{\{v_1\}}_{\{v_2\}}) \tag{5.56}$$

Note that, while the determinant $\det(M(G)^{\{v_1\}}_{\{v_2\}})$ again has an ambiguous sign depending on ordering of rows and columns of the matrix, the factor $(-1)^{v_1+v_2}$ from the cofactor inversion ensures that the l.h.s. $Q'^T \tilde{L}'^{-1} Q'$ is independent of that choice.

5.3.1 Vertex-indexed Dodgson polynomials

This suggests a generalisation of Dodgson polynomials to include indices from the vertex set of Γ:

$$\Psi_{G,\emptyset}^{\{v_1\},\{v_2\}} := \det(M(G)_{\{v_2\}}^{\{v_1\}}) \tag{5.57}$$

In particular, the determinant identity eq. (5.5) is completely independent of this particular choice of matrix and works for any choice of columns or rows to delete. Hence, the Dodgson identity still works with vertices as indices. For the case of singleton index sets as above, this is up to the sign also the spanning forest polynomial $\Phi_G^{\{v_0\},\{v_1,v_2\}}$ from [32, Def. 9], where $v_0 \in V_\Gamma \setminus V'_\Gamma$ is the deleted vertex (as we see it is closely related to the second Symanzik polynomial, hence the letter Φ is used here again). In general it is defined as

$$\Phi_G^P = \sum_{F=T_1\sqcup\cdots\sqcup T_k\in\mathcal{T}^{[k]}} \alpha_{E_G\setminus F}, \tag{5.58}$$

where $P = P_1,\ldots,P_k$ is a partition of vertices of G and F is a spanning k-forest such that the vertices of P_i are contained in T_i.

In this sense, $\Phi_G^{\{v_0\},\{v_1,v_2\}}$ is the fixed-sign version of a vertex-indexed Dodgson polynomial $\Psi_{G,\emptyset}^{\{v_1\},\{v_2\}}$, just like the cycle polynomials $\chi_\Gamma^{(e_1|e_2)}$ were fixed-sign versions of $\Psi_{G,\emptyset}^{\{e_1\},\{e_2\}}$. Hence, we reuse our previous notation to define

$$\chi_G^{(\mathsf{a}_{v_1}|\mathsf{a}_{v_2})} := \Phi_G^{\{v_0\},\{v_1,v_2\}}, \tag{5.59}$$

now with words indexing it, as in the previous case of cycle Dodgson polynomials. In this notation the Dodgson identity again takes the form

$$\chi_G^{(\mathsf{a}_1|\mathsf{a}_2)}\chi_G^{(\mathsf{a}_3|\mathsf{a}_4)} - \chi_G^{(\mathsf{a}_1|\mathsf{a}_3)}\chi_G^{(\mathsf{a}_2|\mathsf{a}_4)} = \Psi_\Gamma\chi_G^{(\mathsf{a}_1\mathsf{a}_4|\mathsf{a}_2\mathsf{a}_3)}, \tag{5.60}$$

and generalisations are analogous to eq. (5.11). Note that, where edge indices lower the degree of the polynomial, such that $\deg(\chi_\Gamma^{(\mathsf{w}_1|\mathsf{w}_2)}) = h_1 - |\mathsf{w}_i|$, if the letters of both w_i correspond to edges, the vertex indices do the opposite! For single letters, $\deg(\chi_\Gamma^{(\mathsf{a}_i|\mathsf{a}_j)}) = \deg(\Phi_\Gamma^{\{v_0\},\{i,j\}}) = h_1 + 1$, such that the polynomial with two-letter words on the r.h.s. has to have degree $h_1 + 2$. Another property we get by courtesy of the spanning forest polynomial is that

$$\chi_G^{(\mathsf{a}_v|\mathsf{a}_v)} = \Psi_{G|_{v=v_0}}. \tag{5.61}$$

In other words, equal indices correspond to identification of that vertex with v_0 in the graph, analogous to the edge-indexed case

$$\chi_G^{(\mathsf{a}_e|\mathsf{a}_e)} = \Psi_{G\setminus e}, \tag{5.62}$$

which indicated deletion of an edge.

In $\Phi_G^{\{v_0\},\{v_1,v_2\}}$ the vertex v_0 that is initially deleted from the graph matrix is explicit. We will see below that it is actually useful to consider Dodgson polynomials coming from different such choices. Hence, we use from now on the subscript $\chi_{G,v_0}^{(\mathsf{a}_1|\mathsf{a}_2)}$ to indicate it. Note that this is different from the subscript K in the usual Dodgson polynomial $\Psi_{G,K}^{I,J}$, which indicates contracted edges and is always empty for us.

5.3.2 Dodgson polynomials in the bond polynomial

It is surely worthwhile to study these types of polynomials and their interconnections more in the future, but for now we want to concentrate on the problem at hand. Can we express the $\beta_\Gamma^{(e|e')}$ (which are indexed by edges) in terms of Dodgson polynomials with vertex indices? And, more importantly, will this help simplifying products like $\mathrm{X}_\Gamma^{e,\mu}\mathrm{X}_\Gamma^{f,\nu}$?

We have

$$\begin{aligned}\beta_G &= \sum_{u,v\in V_G'} \vartheta_u\vartheta_v(-1)^{u+v}\Psi_G^{\{u\},\{v\}}\\ &= \sum_{u,v\in V_G'}\Big(\sum_{e,f\in E_G} I_{eu}I_{fv}\xi_e\xi_f\Big)\Phi_G^{\{v_0\},\{u,v\}}\\ &= \sum_{e,f\in E_G}\xi_e\xi_f\underbrace{\sum_{u,v\in V_G'} I_{eu}I_{fv}\Phi_G^{\{v_0\},\{u,v\}}}_{=\beta_G^{(e|f)}}.\end{aligned} \tag{5.63}$$

Inserting into eq. (2.104), the definition of $\mathrm{X}_G^{e,\mu}$, one finds

$$\mathrm{X}_G^{e,\mu} = \alpha_e^{-1}\sum_{u,v\in V_G'} I_{eu}\vartheta_v^\mu\Phi_G^{\{v_0\},\{u,v\}}. \tag{5.64}$$

Now, if we move to the physical case, i.e. a Feynman graph Γ in which we evaluate the formal parameters ξ_e to physical momenta, then we see that

$$\vartheta_v \to \begin{cases} 0 & \text{if } v\in V_\Gamma^{int},\\ -q_v & \text{if } v\in V_\Gamma^{ext},\end{cases} \tag{5.65}$$

and since there are exactly two vertices u_1, u_2 with $\partial(e) = (u_1, u_2)$ for a given edge such that $I_{eu_1} = -I_{eu_2} \neq 0$ for these two,

$$\mathrm{X}_\Gamma^{e,\mu} = -\alpha_e^{-1}\sum_{v\in V_\Gamma^{ext}} q_v^\mu\Big(\Phi_G^{\{v_0\},\{u_2,v\}} - \Phi_G^{\{v_0\},\{u_1,v\}}\Big). \tag{5.66}$$

Note that $\Phi_G^{\{v_0\},\{u_i,v\}} = 0$ if either of the two endpoints of e is v_0. Moreover, accounting for cancellations between spanning forests that appear in both polynomials, the difference can be written as

$$\Phi_G^{\{v_0\},\{u_2,v\}} - \Phi_G^{\{v_0\},\{u_1,v\}} = \Phi_G^{\{v_0,u_1\},\{u_2,v\}} - \Phi_G^{\{v_0,u_2\},\{u_1,v\}}. \tag{5.67}$$

If we now specialise to the case of only two external vertices v_1, v_2 (or at least only two with non-vanishing momenta), then this reduces further to

$$\mathrm{X}_\Gamma^{e,\mu} = q^\mu \alpha_e^{-1}\Big(\Phi_\Gamma^{\{v_0\},\{u_1,v_1\}} + \Phi_\Gamma^{\{v_0\},\{u_2,v_2\}} - \Phi_\Gamma^{\{v_0\},\{u_2,v_1\}} - \Phi_\Gamma^{\{v_0\},\{u_1,v_2\}}\Big) \tag{5.68}$$

In order to get the correct overall sign we emphasise again that e is directed from $\partial_-(e) = u_1$ to $\partial_+(e) = u_2$, and $q_{v_1} = q = -q_{v_2}$.

By the same principle as eq. (5.67) we can explicitly remove terms that would cancel between these four summands:

$$\begin{aligned}
&\Phi_\Gamma^{\{v_0\},\{u_1,v_1\}} - \Phi_\Gamma^{\{v_0\},\{u_2,v_1\}} + \Phi_\Gamma^{\{v_0\},\{u_2,v_2\}} - \Phi_\Gamma^{\{v_0\},\{u_1,v_2\}} \\
&\quad = \Phi_\Gamma^{\{v_0,u_2\},\{u_1,v_1\}} - \Phi_\Gamma^{\{v_0,u_1\},\{u_2,v_1\}} + \Phi_\Gamma^{\{v_0,u_1\},\{u_2,v_2\}} - \Phi_\Gamma^{\{v_0,u_2\},\{u_1,v_2\}} \\
&\quad = \Phi_\Gamma^{\{v_0,u_2,v_2\},\{u_1,v_1\}} + \Phi_\Gamma^{\{v_0,u_1,v_1\},\{u_2,v_2\}} - \Phi_\Gamma^{\{v_0,u_2,v_1\},\{u_1,v_2\}} - \Phi_\Gamma^{\{v_0,u_1,v_2\},\{u_2,v_1\}} \\
&\quad = \Phi_\Gamma^{\{u_1,v_1\},\{u_2,v_2\}} - \Phi_\Gamma^{\{u_1,v_2\},\{u_2,v_1\}}
\end{aligned} \tag{5.69}$$

Much to our delight, this is now explicitly independent of the arbitrarily chosen vertex v_0. We can re-expand eq. (5.69) by including terms cancelled between the two to get

$$\Phi_\Gamma^{\{u_1,v_1\},\{u_2,v_2\}} - \Phi_\Gamma^{\{u_1,v_2\},\{u_2,v_1\}} = \Phi_\Gamma^{\{v_1\},\{u_2,v_2\}} - \Phi_\Gamma^{\{v_1\},\{u_1,v_2\}}. \tag{5.70}$$

Here we need to be very careful with our notation and be mindful of what we are doing. Previously, we explicitly had $v_0 \neq v_1, v_2$, since the sums in eq. (5.63) and eq. (5.64) are over vertices in V'_Γ. However, this does not stop us from simply considering Dodgson polynomials w.r.t. a different choice, which is what we do for $\Phi_\Gamma^{\{v_1\},\{u_i,v_2\}}$. With the different choice indicated by a subscript as discussed above, we now have

$$\mathrm{X}_\Gamma^{e,\mu} = q^\mu \alpha_e^{-1}\Big(\chi_{G,v_1}^{(\mathsf{a}_{u_2}|\mathsf{a}_{v_2})} - \chi_{G,v_1}^{(\mathsf{a}_{u_1}|\mathsf{a}_{v_2})}\Big). \tag{5.71}$$

Having found a significantly nicer form for $\mathrm{X}_\Gamma^{e,\mu} = q^\mu x_\Gamma^e$, we can consider products like $x_\Gamma^e x_\Gamma^f$ that appear in the integrands. Let u, v be the two external vertices, and choose u to be the vertex whose row and column gets deleted in the graph matrix. Then let a, b, c, d be the not necessarily distinct endpoints of edges e and f, with directions $\partial(e) = (a, b)$ and $\partial(f) = (c, d)$ and use letters $\mathsf{a} \equiv \mathsf{a}_a$, $\mathsf{b} \equiv \mathsf{a}_b$, etc. With this preparation the product is now

$$\begin{aligned}
\alpha_e \alpha_f x^e_\Gamma x^f_\Gamma &= \left(\chi^{(\mathsf{b}|\mathsf{v})}_{\Gamma,u} - \chi^{(\mathsf{a}|\mathsf{v})}_{\Gamma,u}\right)\left(\chi^{(\mathsf{d}|\mathsf{v})}_{\Gamma,u} - \chi^{(\mathsf{c}|\mathsf{v})}_{\Gamma,u}\right) \\
&= \chi^{(\mathsf{b}|\mathsf{v})}_{\Gamma,u}\chi^{(\mathsf{d}|\mathsf{v})}_{\Gamma,u} - \chi^{(\mathsf{a}|\mathsf{v})}_{\Gamma,u}\chi^{(\mathsf{d}|\mathsf{v})}_{\Gamma} - \chi^{(\mathsf{b}|\mathsf{v})}_{\Gamma,u}\chi^{(\mathsf{c}|\mathsf{v})}_{\Gamma} + \chi^{(\mathsf{a}|\mathsf{v})}_{\Gamma,u}\chi^{(\mathsf{c}|\mathsf{v})}_{\Gamma,u} \\
&= \chi^{(\mathsf{v}|\mathsf{v})}_{\Gamma,u}\left(\chi^{(\mathsf{b}|\mathsf{d})}_{\Gamma,u} - \chi^{(\mathsf{a}|\mathsf{d})}_{\Gamma,u} - \chi^{(\mathsf{b}|\mathsf{c})}_{\Gamma,u} + \chi^{(\mathsf{a}|\mathsf{c})}_{\Gamma,u}\right) \\
&\quad - \Psi_\Gamma\left(\chi^{(\mathsf{bv}|\mathsf{dv})}_{\Gamma,u} - \chi^{(\mathsf{av}|\mathsf{dv})}_{\Gamma,u} - \chi^{(\mathsf{bv}|\mathsf{cv})}_{\Gamma,u} + \chi^{(\mathsf{av}|\mathsf{cv})}_{\Gamma,u}\right).
\end{aligned} \tag{5.72}$$

The coefficient of $\chi^{(\mathsf{v}|\mathsf{v})}_{\Gamma,u}$ in the first summand is exactly the sum from eq. (5.69) with different labels, such that

$$\begin{aligned}
\chi^{(\mathsf{b}|\mathsf{d})}_{\Gamma,u} - \chi^{(\mathsf{a}|\mathsf{d})}_{\Gamma,u} - \chi^{(\mathsf{b}|\mathsf{c})}_{\Gamma,u} + \chi^{(\mathsf{a}|\mathsf{c})}_{\Gamma,u} &= \Phi^{\{u\},\{b,d\}}_\Gamma - \Phi^{\{u\},\{a,d\}}_\Gamma - \Phi^{\{u\},\{b,c\}}_\Gamma + \Phi^{\{u\},\{a,c\}}_\Gamma \\
&= \Phi^{\{a,c\},\{b,d\}}_\Gamma - \Phi^{\{b,c\},\{a,d\}}_\Gamma .
\end{aligned} \tag{5.73}$$

$\chi^{(\mathsf{v}|\mathsf{v})}_{\Gamma,u}$ itself is the Kirchhoff polynomial of $\Gamma|_{u=v}$, which is sometimes denoted with $\Gamma^\bullet$, if u, v are the only two external vertices of a Feynman graph. Moreover, in that case this Kirchhoff polynomial is also $\Psi_{\Gamma^\bullet} = \varphi_\Gamma$. The coefficients of Ψ_Γ can be interpreted as

$$\chi^{(\mathsf{av}|\mathsf{dv})}_{\Gamma,u} = \chi^{(\mathsf{a}|\mathsf{d})}_{\Gamma^\bullet,u}, \tag{5.74}$$

such that they add up to

$$\Phi^{\{a,c\},\{b,d\}}_{\Gamma^\bullet} - \Phi^{\{b,c\},\{a,d\}}_{\Gamma^\bullet}, \tag{5.75}$$

just like eq. (5.73). After putting all of this together we have proved the statement of the following theorem.

Theorem 5.3.1. *Let Γ be a QED Feynman graph with only two non-zero external momenta $q_u = q = -q_v$ at vertices $u, v \in V_\Gamma$, and $\Gamma^\bullet = \Gamma|_{u=v}$. Let furthermore $e, f \in E_\Gamma$ be any two edges of Γ (which includes $e = f$ and e, f adjacent to each other). Then*

$$\begin{aligned}
\alpha_e \alpha_f x^e_\Gamma x^f_\Gamma &= \Psi_{\Gamma^\bullet}\left(\Phi^{\{\partial_-(e),\partial_-(f)\},\{\partial_+(e),\partial_+(f)\}}_\Gamma - \Phi^{\{\partial_+(e),\partial_-(f)\},\{\partial_-(e),\partial_+(f)\}}_\Gamma\right) \\
&\quad - \Psi_\Gamma\left(\Phi^{\{\partial_-(e),\partial_-(f)\},\{\partial_+(e),\partial_+(f)\}}_{\Gamma^\bullet} - \Phi^{\{\partial_+(e),\partial_-(f)\},\{\partial_-(e),\partial_+(f)\}}_{\Gamma^\bullet}\right) \\
&= \Psi_{\Gamma^\bullet}\beta^{(e|f)}_\Gamma - \Psi_\Gamma \beta^{(e|f)}_{\Gamma^\bullet}.
\end{aligned} \tag{5.76}$$

The fact that the particular combinations of spanning forest polynomials we have here are in fact the bond polynomials $\beta_\Gamma^{(e|f)}$ and $\beta_{\Gamma^\bullet}^{(e|f)}$ may have gotten somewhat obscured, but it becomes clear when retracing our last steps from eq. (5.63) onward.

If $e \neq f$, which is always the case in the integrands, then we can further simplify this. Remembering lemma 2.3.8 from chapter 2,

$$\beta_\Gamma^{(e|f)} = -\alpha_e \alpha_f \chi_\Gamma^{(e|f)} \qquad \text{for all } e \neq f,$$

we can write

$$x_\Gamma^e x_\Gamma^f = -\Psi_{\Gamma^\bullet} \chi_\Gamma^{(e|f)} + \Psi_\Gamma \chi_{\Gamma^\bullet}^{(e|f)}. \tag{5.77}$$

Remark 5.3.2. *The r.h.s. of eq. (5.77) can also be written in a different and rather curious manner. Let* $\tilde{\Gamma} = (E_\Gamma \cup \{e_0\}, V_\Gamma)$ *be the graph* Γ *together with the "external edge"* $e_0 = (v, u)$ *between the two external vertices. Then* $\Gamma^\bullet = \tilde{\Gamma} /\!\!/ e_0$ *and* $\Gamma = \tilde{\Gamma} \setminus e_0$ *and with the contraction-deletion relations we can write*

$$x_\Gamma^e x_\Gamma^f = -\det \begin{pmatrix} \Psi_{\tilde{\Gamma}}|_{\alpha_0=0} & \partial_0 \Psi_{\tilde{\Gamma}} \\ \chi_{\tilde{\Gamma}}^{(e|f)}|_{\alpha_0=0} & \partial_0 \chi_{\tilde{\Gamma}}^{(e|f)} \end{pmatrix}.$$

This is very reminiscent of the five-invariant [28] (see also [13, eq. (8.13)], where it was first observed, but not yet named),

$${}^5\Psi_\Gamma(i,j,k,l,m) = \pm \det \begin{pmatrix} \Psi_\Gamma^{\{i,j\},\{k,l\}}|_{\alpha_m=0} & \partial_m \Psi_\Gamma^{\{i,j\},\{k,l\}} \\ \Psi_\Gamma^{\{i,k\},\{j,l\}}|_{\alpha_m=0} & \partial_m \Psi_\Gamma^{\{i,k\},\{j,l\}} \end{pmatrix}. \tag{5.78}$$

The five-invariant is the first obstruction to linear reducibility that one may encounter in the parametric integration algorithm after the first five integrations of the variables $\alpha_i, \ldots, \alpha_m$ *of a period integral* $\int_{\sigma_G} \frac{\Omega_G}{\Psi_G^2}$.

6 Applications

Die Praxis sollte das Ergebnis des Nachdenkens sein.

Hermann Hesse, *Freunde*, 1907

We work out a number of examples in great detail. All integrations are computed in Maple[1] using Erik Panzer's `hyperInt` [108]. Aside from the integration procedures we only need a handful of other procedures provided by `hyperInt` – e.g. `forestPolynomial` for the spanning forest polynomial – and a few dozen lines of miscellaneous code for things like partition polynomials or counting cycles in chord diagrams. However, here our focus lies on exposition, so even the small examples will take up some space when computed explicitly by hand.

6.1 1-loop graphs

6.1.1 Photon

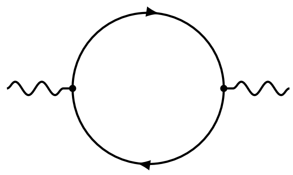


Figure 6.1: The 1-loop photon propagator.

[1] Maple™ is a trademark of Waterloo Maple Inc. [1].

Let us construct the integral in this simple case. There are no subdivergences, $\Psi_\Gamma = \alpha_1 + \alpha_2$ and $\varphi_\Gamma = \alpha_1\alpha_2$. The chord diagram D^0_Γ has just two vertices in one base cycle. The only possible cycle polynomial is $\chi^{(1|2)}_\Gamma = 1$, and consequently,

$$Z^0_\Gamma = (-\Psi_\Gamma)^{h_1-1}(1+1)!Z^0_\Gamma\Big|_1 = 2. \tag{6.1}$$

Since there is only a single possible chord to be added to D^0_Γ, there is also only a single diagram with one chord missing, which has one tricoloured cycle. It belongs to case 0 (as discussed in section 4.4), so $Z^1_{\Gamma,+} = 0$ and

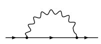
$$Z^1_{\Gamma,0} = \frac{1}{2}(-2)^1 x^1_\Gamma x^2_\Gamma = \alpha_1\alpha_2. \tag{6.2}$$

Altogether

$$\begin{aligned}\phi^R_\Gamma &= (q^2 g^{\mu\nu} - q^\mu q^\nu)2^{h_1} L \int_{\sigma_\Gamma} \Omega_\Gamma \frac{\varphi_\Gamma Z^0_\Gamma + 2Z^1_{\Gamma,0} + Z^1_{\Gamma,+}}{\Psi_\Gamma^{h_1+3}} \\ &= 8L(g^{\mu\nu}q^2 - q^\mu q^\nu)\int_{\sigma_\Gamma} \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^4}\Omega_\Gamma \\ &= \frac{4}{3}L(g^{\mu\nu}q^2 - q^\mu q^\nu). \end{aligned} \tag{6.3}$$

The factor $\frac{4}{3}$ is exactly the 1-loop coefficient of the QED beta function in the conventions of [26, 76, 114].

6.1.2 Fermion

Now $\Gamma =$  . The fermion is the least complicated type of graph, so we use the opportunity to illustrate the difference between Feynman gauge and general gauge. Here, too, we have $\Psi_\Gamma = \alpha_1 + \alpha_2$ and $\varphi_\Gamma = \alpha_1\alpha_2$, with e_2 being the photon edge. In general gauge,

$$\begin{aligned}\bar\chi^{(1|2)}_\Gamma &= \chi^{(1|2)}_\Gamma = 1, \\ \bar\chi^{(2|2)}_\Gamma &= \chi^{(2|2)}_\Gamma + \frac{2\Psi_\Gamma}{\varepsilon\alpha_2} = 1 + 2\frac{\alpha_1+\alpha_2}{\varepsilon\alpha_2}, \\ x^1_\Gamma &= \alpha_2, \qquad x^2_\Gamma = -\alpha_1,\end{aligned}$$

where we remember the map $\bar e(v_i) = e_2$ from eq. (2.107). Completing the graph by connecting the external edge, labelled e_0, one has an associated chord diagram on four vertices in a single base cycle $v_1 - e_0 - v_2 - e_1$ and we defined $x^0_\Gamma = 1$ and $\chi^{(e|e_0)}_\Gamma = 0$. Chords (v_i, e_1) yield a single bicoloured cycle while (v_1, v_2) does not (they have $\operatorname{sgn}(v_i, e_1) = 1$ and $\operatorname{sgn}(v_1, v_2) = 0$). Moreover, since there are still free vertices we must not forget to count the tricoloured cycle that is still present. Hence,

$$\begin{aligned}
2Z^1_\Gamma &= (-2)^1\Big(\bar{\chi}^{(2|2)}_\Gamma x^0_\Gamma x^1_\Gamma + \bar{\chi}^{(0|1)}_\Gamma x^2_\Gamma x^2_\Gamma\Big) + (-2)^2\Big(2\bar{\chi}^{(0|2)}_\Gamma x^1_\Gamma x^2_\Gamma + 2\bar{\chi}^{(1|2)}_\Gamma x^0_\Gamma x^2_\Gamma\Big) \\
&= \Big(-2\alpha_2\Big(1 + 2\frac{\alpha_1+\alpha_2}{\varepsilon\alpha_2}\Big) - 8\alpha_1\Big) \\
&= -\frac{2}{\varepsilon\alpha_2}\Big(2\alpha_2(\alpha_1+\alpha_2) + \varepsilon\alpha_2(4\alpha_1+\alpha_2)\Big).
\end{aligned}$$

We had defined Z^k_Γ only in Feynman gauge, but the computation goes through in general gauge when simply using $\bar{\chi}^{(i|j)}_\Gamma$ instead of $\chi^{(i|j)}_\Gamma$ and carrying the factor $\varepsilon\alpha_e$ for each photon edge.

Finally, since we have general gauge the factors differ slightly from eq. (4.78). D_Γ does not have a fixed chord corresponding to the photon, so instead of $2^{h_1} = 2$ we have $2^0 = 1$, and the power of the denominator is $2h_1 + 2 = 4$ instead of $h_1 + 2 = 3$. The integral is therefore

$$\begin{aligned}
\phi^R_\Gamma &= \not{q}L \int_{\sigma_\Gamma}\Big(\frac{2\alpha_2}{(\alpha_1+\alpha_2)^3} + \varepsilon\frac{\alpha_2(4\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2)^4}\Big)\Omega_\Gamma \\
&= (1+\varepsilon)\not{q}L. \qquad (6.4)
\end{aligned}$$

Here we recover the well-known result that the 1-loop electron self-energy vanishes in Landau gauge $\varepsilon \to -1$.

6.1.3 Vertex

The vertex is already complicated, so we return to Feynman gauge. Now $\Psi_\Gamma = \alpha_1 + \alpha_2 + \alpha_3$ and $\varphi_\Gamma = (\alpha_1 + \alpha_2)\alpha_3$, with e_3 being the photon edge and no momentum entering at the external photon. Again, the only cycle polynomial is $\chi^{(1|2)}_\Gamma$ and $Z^0_\Gamma = (-\Psi_\Gamma)^{h_1-1}(1+1)!Z^0_\Gamma\Big|_1 = 2$. For the higher order contributions there is also only one diagram (similar to the 1-loop photon), such that $Z^1_{\Gamma,-} = 0$ and

$$Z^1_{\Gamma,0} = \frac{1}{2}(-2)x^1_\Gamma x^2_\Gamma = \alpha_3^2, \qquad (6.5)$$

which the reader is encouraged to double-check with eq. (5.77). Inserting this into eq. (4.80) yields

$$\begin{aligned}\phi_\Gamma^R &= -\int_{\sigma_\Gamma} \Omega_\Gamma \Big(\gamma^\mu L \frac{2}{(\alpha_1+\alpha_2+\alpha_3)^3} + 4 \frac{\not{q} q^\mu}{q^2} \frac{\alpha_3}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)^3} \Big) \\ &= -\gamma^\mu L - 2\frac{\not{q} q^\mu}{q^2}.\end{aligned}$$

6.2 2-loop photon propagator

At two loops we have three graphs with two different topologies.

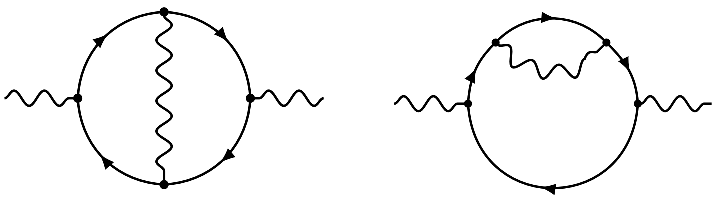
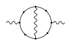


Figure 6.2: The 2-loop photon propagators.

6.2.1 Vertex subdivergences

For $\Gamma =$  one has

$$\Psi_\Gamma = (\alpha_1+\alpha_4)(\alpha_2+\alpha_3) + \alpha_5(\alpha_1+\alpha_2+\alpha_3+\alpha_4) \tag{6.6}$$

$$\varphi_\Gamma = \alpha_1\alpha_4(\alpha_2+\alpha_3+\alpha_5) + \alpha_2\alpha_3(\alpha_1+\alpha_4+\alpha_5) + \alpha_5(\alpha_1\alpha_3+\alpha_2\alpha_4) \tag{6.7}$$

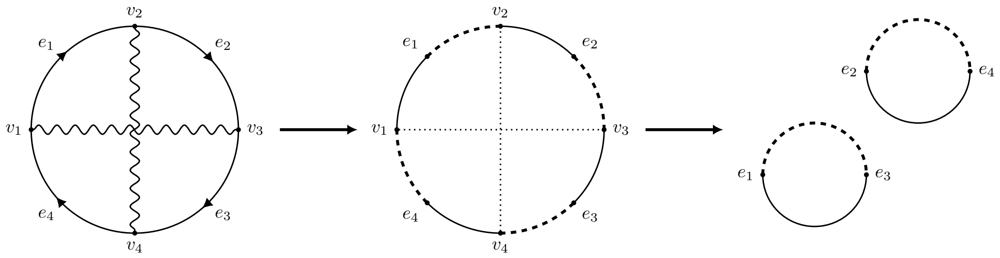
where the edge labelling is e_1-e_4 clockwise starting from the left external vertex for fermions and e_5 for the photon. The cycle polynomials involving fermion edges are

$$\chi_\Gamma^{(1|2)} = \chi_\Gamma^{(1|3)} = \chi_\Gamma^{(2|4)} = \chi_\Gamma^{(3|4)} = \alpha_5 \tag{6.8}$$

$$\chi_\Gamma^{(1|4)} = \alpha_2+\alpha_3+\alpha_5 \tag{6.9}$$

$$\chi_\Gamma^{(2|3)} = \alpha_1+\alpha_4+\alpha_5 \tag{6.10}$$

and the completion, associated chord diagram D_Γ, and projection D_Γ^0 thereof are:



The chord diagram sum version of Z^0_Γ is

$$2Z^0_\Gamma = (-2)^2\chi_\Gamma^{(1|2)}\chi_\Gamma^{(3|4)} + (-2)^4\chi_\Gamma^{(1|3)}\chi_\Gamma^{(2|4)} + (-2)^2\chi_\Gamma^{(1|4)}\chi_\Gamma^{(2|3)}$$
$$= 24\alpha_5^2 + 4\underbrace{\big(\alpha_5(\alpha_1+\alpha_2+\alpha_3+\alpha_4) + (\alpha_2+\alpha_3)(\alpha_1+\alpha_4)\big)}_{=\Psi_\Gamma}, \tag{6.11}$$

where the appearance of the Kirchhoff polynomial that we get in the partition polynomial happens to be obvious. Let us nonetheless work it out for the sake of didactics.

Partition polynomial contribution Z^0_Γ

The partitions of 1-coloured edges of D^0_Γ are $\mathcal{E}_1 = \{\{(e_1,e_3),(e_2,e_4)\}\}$ consisting of one part and $\mathcal{E}_2 = \{\{(e_1,e_3)\},\{(e_2,e_4)\}\}$ with two parts. P_2 contains two equivalence classes of word pairs represented by, for example, $(\mathsf{a}_1\mathsf{a}_2,\mathsf{a}_3\mathsf{a}_4)$ and $(\mathsf{a}_1\mathsf{a}_4,\mathsf{a}_3\mathsf{a}_2)$. Then $\lambda_{\mathcal{E}_1}$ of these two pairs is just the pair again, since $|\mathcal{E}_1| = 1$, and therefore $\mathrm{sgn}_{\mathcal{E}_1}$ is also positive for both, such that

$$\begin{aligned} Z^0_\Gamma\Big|_1 &= \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_2|\mathsf{a}_3\mathsf{a}_4)} + \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_4|\mathsf{a}_3\mathsf{a}_2)} \\ &= \frac{1}{\Psi_\Gamma}\Big(2\chi_\Gamma^{(1|3)}\chi_\Gamma^{(2|4)} - \chi_\Gamma^{(1|2)}\chi_\Gamma^{(3|4)} - \chi_\Gamma^{(1|4)}\chi_\Gamma^{(2|3)}\Big) \\ &= -1. \end{aligned} \tag{6.12}$$

For the other partition we have

$$\lambda_{\mathcal{E}_2}(\mathsf{a}_1\mathsf{a}_2,\mathsf{a}_3\mathsf{a}_4) = \{(\mathsf{a}_1,\mathsf{a}_3),(\mathsf{a}_2,\mathsf{a}_4)\}, \tag{6.13}$$

$$\lambda_{\mathcal{E}_2}(\mathsf{a}_1\mathsf{a}_4,\mathsf{a}_3\mathsf{a}_2) = \{(\mathsf{a}_1,\mathsf{a}_3),(\mathsf{a}_4,\mathsf{a}_2)\}, \tag{6.14}$$

where we see that the parts are again such that the signs are all positive. Thus,

$$Z^0_\Gamma\Big|_2 = 2\chi_\Gamma^{(1|3)}\chi_\Gamma^{(2|4)} = 2\alpha_5^2 \tag{6.15}$$

and overall we find that

$$\begin{aligned}2Z^0_\Gamma &= 2(-\Psi_\Gamma)^{h_1-1}(1+1)!Z^0_\Gamma\Big|_1 + 2(-\Psi_\Gamma)^{h_1-2}(2+1)!Z^0_\Gamma\Big|_2)\\ &= 4\Psi_\Gamma Z^0_\Gamma\Big|_1 + 12Z^0_\Gamma\Big|_2\\ &= -8\chi_\Gamma^{(1|3)}\chi_\Gamma^{(2|4)} + 4\chi_\Gamma^{(1|2)}\chi_\Gamma^{(3|4)} + 4\chi_\Gamma^{(1|4)}\chi_\Gamma^{(2|3)} + 24\chi_\Gamma^{(1|3)}\chi_\Gamma^{(2|4)}\\ &= 4\chi_\Gamma^{(1|2)}\chi_\Gamma^{(3|4)} + 16\chi_\Gamma^{(1|3)}\chi_\Gamma^{(2|4)} + 4\chi_\Gamma^{(1|4)}\chi_\Gamma^{(2|3)} \qquad (6.16)\end{aligned}$$

is indeed equal to the sum from eq. (6.11). Already at two loops we can see the effect of this cancellation – only two terms remain! The size of this part of the integrand is now entirely dominated by the second Symanzik polynomial:

$$2^2 L\varphi_\Gamma\Big((-1)^1 2!\frac{-1}{\Psi_\Gamma^4} + (-1)^2 3!\frac{2\alpha_5^2}{\Psi_\Gamma^5}\Big)\Omega_\Gamma = 2^3 L\varphi_\Gamma\Big(\frac{1}{\Psi_\Gamma^4} + \frac{6\alpha_5^2}{\Psi_\Gamma^5}\Big)\Omega_\Gamma \qquad (6.17)$$

Note that here we included the remaining factor $2^{h_1}L$ from eq. (5.55).

Other contributions $Z^1_{\Gamma,\bullet}$

For $Z^1_{\Gamma,\bullet}$ there are 6 possible diagrams, corresponding to the choices of any two of the four fermion edges. There are two of each type, visualised in fig. 6.3. Since we only need $Z^1_{\Gamma,+}$ and $Z^1_{\Gamma,0}$ we can ignore the diagrams with chords (e_1, e_2) and (e_3, e_4). For the others we need the product formula eq. (5.77). First, for $Z^1_{\Gamma,+}$ we have

$$x^1_\Gamma x^3_\Gamma = -\alpha_5\Psi_{\Gamma^\bullet} - \Psi_\Gamma\alpha_2\alpha_4, \qquad x^2_\Gamma x^4_\Gamma = -\alpha_5\Psi_{\Gamma^\bullet} - \Psi_\Gamma\alpha_1\alpha_3. \qquad (6.18)$$

such that

$$Z^1_{\Gamma,+} = \frac{1}{2}\Big((-2)^3\chi_\Gamma^{(1|3)}x^2_\Gamma x^4_\Gamma + (-2)^3\chi_\Gamma^{(2|4)}x^1_\Gamma x^3_\Gamma\Big) = 8\alpha_5^2\Psi_{\Gamma^\bullet} + 4\Psi_\Gamma\alpha_5(\alpha_1\alpha_3 + \alpha_2\alpha_4). \qquad (6.19)$$

Now we repeat this for $x^1_\Gamma x^4_\Gamma$ and $x^2_\Gamma x^3_\Gamma$, which appear in $Z^1_{\Gamma,0}$. Here,

$$x^1_\Gamma x^4_\Gamma = -\,\Psi_{\Gamma^\bullet}(\alpha_2 + \alpha_3 + \alpha_5) + \Psi_\Gamma\alpha_2\alpha_3, \qquad (6.20)$$

$$x^2_\Gamma x^3_\Gamma = -\,\Psi_{\Gamma^\bullet}(\alpha_1 + \alpha_4 + \alpha_5) + \Psi_\Gamma\alpha_1\alpha_4, \qquad (6.21)$$

which means that

$$\begin{aligned}Z^1_{\Gamma,0} =& \frac{1}{2}\Big((-2)^1\chi_\Gamma^{(1|4)}x^2_\Gamma x^3_\Gamma + (-2)^1\chi_\Gamma^{(2|3)}x^1_\Gamma x^4_\Gamma\Big)\\ =& 2\Psi_{\Gamma^\bullet}(\alpha_1 + \alpha_4 + \alpha_5)(\alpha_2 + \alpha_3 + \alpha_5)\\ &- \Psi_\Gamma(\alpha_1\alpha_4(\alpha_2 + \alpha_3 + \alpha_5) + \alpha_2\alpha_3(\alpha_1 + \alpha_4 + \alpha_5)). \qquad (6.22)\end{aligned}$$

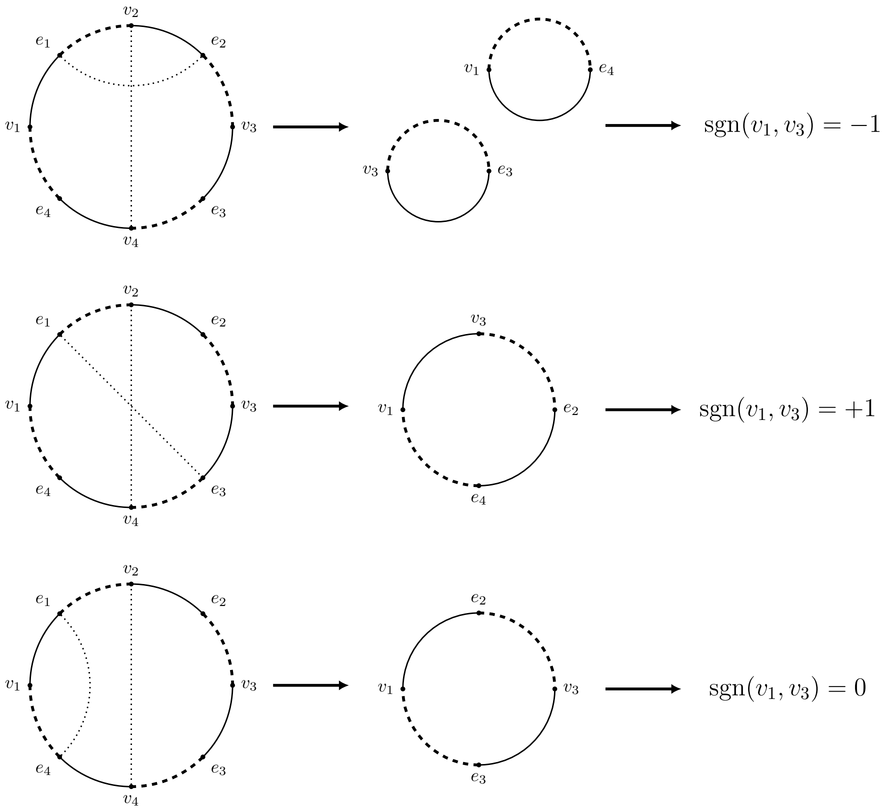


Figure 6.3: Examples for all three types of diagrams that occur in Z^1_Γ

In the integrand we have the combination

$$
\begin{aligned}
&2Z^1_{\Gamma,0} + Z^1_{\Gamma,+} \\
&\quad = 4\Psi_{\Gamma\bullet}\Big(\underbrace{(\alpha_1+\alpha_4+\alpha_5)(\alpha_2+\alpha_3+\alpha_5)+2\alpha_5^2}_{=\Psi_\Gamma+3\alpha_5^2}\Big) \\
&\qquad - 2\Psi_\Gamma\Big(\underbrace{\alpha_1\alpha_4(\alpha_2+\alpha_3+\alpha_5)+\alpha_2\alpha_3(\alpha_1+\alpha_4+\alpha_5)-2\alpha_5(\alpha_1\alpha_3+\alpha_2\alpha_4)}_{=\varphi_\Gamma-3\alpha_5(\alpha_1\alpha_3+\alpha_2\alpha_4)}\Big). \\
&\quad = 2\varphi_\Gamma\Psi_\Gamma + 12\varphi_\Gamma\alpha_5^2 + 6\Psi_\Gamma\alpha_5(\alpha_1\alpha_3+\alpha_2\alpha_4).
\end{aligned}
\tag{6.23}
$$

Referring back to eq. (5.55) again, these terms contribute

$$
2^3L\Big(\frac{\varphi_\Gamma}{\Psi_\Gamma^4}+\frac{6\varphi_\Gamma\alpha_5^2}{\Psi_\Gamma^5}+\frac{3\alpha_5(\alpha_1\alpha_3+\alpha_2\alpha_4)}{\Psi_\Gamma^4}\Big) \tag{6.24}
$$

to the integrand. This is exactly the same as the r.h.s. of eq. (6.17) except for a remainder

$$R_\Gamma = 2^3 3L \frac{\alpha_5(\alpha_1\alpha_3 + \alpha_2\alpha_4)}{\Psi_\Gamma^4}. \tag{6.25}$$

This is an observation that can also be made at 3-loops (and at 1 loop, where it was rather trivial with $R_\Gamma = 0$), which leads us to believe that it is indeed a general phenomenon. Consequently, the first term in the forest formula is now nothing more than

$$M_\emptyset^\Gamma = 2^4 L\Big(\frac{\varphi_\Gamma}{\Psi_\Gamma^4} + \frac{6\varphi_\Gamma\alpha_5^2}{\Psi_\Gamma^5} + \frac{3}{2}\frac{\alpha_5(\alpha_1\alpha_3+\alpha_2\alpha_4)}{\Psi_\Gamma^4}\Big)\Omega_\Gamma. \tag{6.26}$$

Subdivergences

Now we still have to subtract the subdivergences. We consider the vertex subdivergence on the right. The other works identically with edge variables relabelled. We can recycle our results from section 6.1,

$$\Psi_{\Gamma/\gamma} = \alpha_1 + \alpha_4 \qquad \Psi_\gamma = \alpha_2 + \alpha_3 + \alpha_5,$$

$$\varphi_{\Gamma/\gamma} = \alpha_1\alpha_4 \qquad \varphi_\gamma = \alpha_2(\alpha_3 + \alpha_5),$$

and assemble them into

$$\Upsilon_\gamma^\Gamma(s) = s\alpha_1\alpha_4(\alpha_2+\alpha_3+\alpha_5) + \alpha_2(\alpha_1+\alpha_4)(\alpha_3+\alpha_5). \tag{6.27}$$

Additionally, we need various $Z_{\Gamma,\gamma}^{k,l}$ (cf. eq. (4.81)). The only cycle polynomials that can occur are $\chi_{\Gamma/\gamma}^{(1|4)}$ and $\chi_\gamma^{(2|3)}$, both of which are 1. Hence, $Z_{\Gamma,\gamma}^{0,0} = \frac{1}{2}(-2)^2 = 2$. Removing the chord in the cograph leaves us with $Z_{\Gamma_+,\gamma}^{1,0} = 0$ and

$$Z_{\Gamma_0,\gamma}^{1,0} = \frac{1}{2}(-2)^1 x_{\Gamma/\gamma}^1 x_{\Gamma/\gamma}^4 \chi_\gamma^{(2|3)} = \alpha_1\alpha_4 = \varphi_{\Gamma/\gamma}. \tag{6.28}$$

Finally, we also need $\tilde{Z}_{\Gamma,\gamma}^{1,1}$ from eq. (4.83). There is only one diagram, which is (looking back at the discussion of 1-loop graphs in section 6.1) of type 0 in both co- and subgraph. Hence,

$$\begin{aligned}\tilde{Z}_{\Gamma,\gamma}^{1,1} &= 2\Big(Z_{\Gamma_0,\gamma_0}^{1,1} - Z_{\Gamma_0,\gamma_+}^{1,1}\Big) + \Big(Z_{\Gamma_+,\gamma_0}^{1,1} - Z_{\Gamma_+,\gamma_+}^{1,1}\Big)\\ &= 2Z_{\Gamma_0,\gamma_0}^{1,1}\\ &= 2\frac{1}{2}(-2)^2 x_{\Gamma/\gamma}^1 x_{\Gamma/\gamma}^4 x_\gamma^2 x_\gamma^3\\ &= 4\varphi_{\Gamma/\gamma}\alpha_5^2. \end{aligned} \tag{6.29}$$

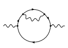
So, with eq. (4.82), the subdivergence contribution is

$$M_{\gamma}^{\Gamma} = \frac{2^4 \varphi_{\Gamma/\gamma}}{\Psi_{\Gamma/\gamma}^4 \Psi_{\gamma}^3} \left(\log \frac{\Upsilon_{\gamma}^{\Gamma}(s)}{\Upsilon_{\gamma}^{\Gamma}(1)} - \alpha_5^2 \Psi_{\Gamma/\gamma} \left(\frac{1}{\Upsilon_{\gamma}^{\Gamma}(s)} - \frac{1}{\Upsilon_{\gamma}^{\Gamma}(1)} \right) \right) \Omega_{\Gamma}$$

After subtraction of both subdivergences one finds

$$\begin{aligned} \phi_{\Gamma}^{R} &= (g^{\mu\nu} q^2 - q^{\mu} q^{\nu}) \int_{\sigma_\Gamma} \left(M_{\emptyset}^{\Gamma} - M_{\gamma_1}^{\Gamma} - M_{\gamma_2}^{\Gamma} \right) \\ &= (g^{\mu\nu} q^2 - q^{\mu} q^{\nu}) \left(\frac{56}{9} L - \frac{4}{3} L^2 \right). \end{aligned} \tag{6.30}$$

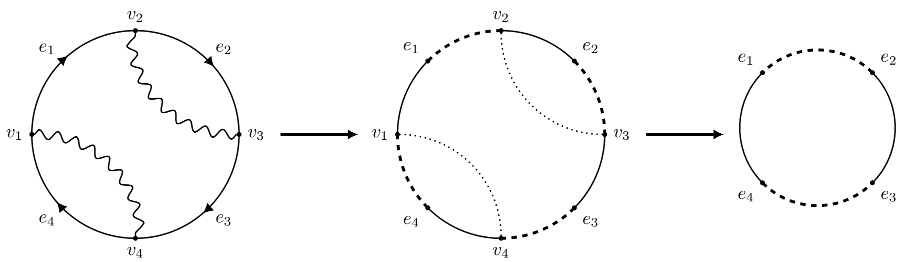
6.2.2 Fermion subdivergence

For $\Gamma =$  we have to include the squashing as discussed in section 3.3.2. Here it is easier to begin with the subdivergence. Choosing to squash e_3, such that $\bar{\Gamma} = \Gamma /\!\!/ e_3$, we have

$$\Psi_{\Gamma/\gamma} = \alpha_1 + \alpha_4 \qquad \Psi_{\gamma} = \alpha_2 + \alpha_5, \tag{6.31}$$

$$\varphi_{\Gamma/\gamma} = \alpha_1 \alpha_4 \qquad \varphi_{\gamma} = \alpha_2 \alpha_5, \tag{6.32}$$

where fermion edges are again labelled clockwise, starting from the left external vertex (see also the chord diagram below, where v_1, v_4 are the external vertices).



Next we compute

$$Z_{\Gamma,\gamma}^{0,1} = \frac{(-2)^2}{2} \chi_{\Gamma/\gamma}^{(1|4)} x_{\gamma}^2 = 2\alpha_5, \tag{6.33}$$

$$Z_{\Gamma_0,\gamma}^{1,1} = \frac{(-2)^1}{2} x_{\Gamma/\gamma}^1 x_{\Gamma/\gamma}^4 x_{\gamma}^2 = \alpha_1 \alpha_4 \alpha_5, \tag{6.34}$$

$$Z_{\Gamma_+,\gamma}^{1,1} = 0. \tag{6.35}$$

Hence, $\varphi_{\Gamma/\gamma} Z^{0,1}_{\Gamma,\gamma} + 2Z^{1,1}_{\Gamma_0,\gamma} + Z^{1,1}_{\Gamma_+,\gamma} = 4\varphi_{\Gamma/\gamma}\alpha_5$ and we immediately find

$$M^\Gamma_\gamma = -\frac{2^4\alpha_1\alpha_4\alpha_5}{(\alpha_1+\alpha_4)^4(\alpha_2+\alpha_5)^3}\log\frac{s\alpha_1\alpha_4(\alpha_2+\alpha_5)+\alpha_2\alpha_5(\alpha_1+\alpha_4)}{\alpha_1\alpha_4(\alpha_2+\alpha_5)+\alpha_2\alpha_5(\alpha_1+\alpha_4)}\Omega_{\bar{\Gamma}} \tag{6.36}$$

from eq. (4.84).

Squashing

Now following the prescription of section 3.3.2 we build $M^\Gamma_\emptyset$. Reducing eq. (3.50) to Feynman gauge the upper limit of the sum reduces to $h^\gamma_1 - 1 = 0$, so there is only one summand. The only thing left to do is taking M^Γ_γ without the logarithm and substituting

$$\alpha_1 \to \alpha_1 + \frac{\varphi_\gamma}{\Psi_\gamma} = \alpha_1 + \frac{\alpha_2\alpha_5}{\alpha_2+\alpha_5}, \tag{6.37}$$

where e_1 is the connector of γ that was not squashed. One finds

$$\frac{\left(\alpha_1 + \frac{\alpha_2\alpha_5}{\alpha_2+\alpha_5}\right)\alpha_4}{(\alpha_1 + \frac{\alpha_2\alpha_5}{\alpha_2+\alpha_5} + \alpha_4)^4(\alpha_2+\alpha_5)^3} = \frac{\left(\alpha_1\alpha_2+\alpha_1\alpha_5+\alpha_2\alpha_5\right)\alpha_4}{((\alpha_1+\alpha_4)(\alpha_2+\alpha_5)+\alpha_2\alpha_5)^4} = \frac{\varphi_{\bar{\Gamma}}}{\Psi^4_{\bar{\Gamma}}}. \tag{6.38}$$

and therefore

$$M^\Gamma_\emptyset = -2^4 L\alpha_5\frac{\varphi_{\bar{\Gamma}}}{\Psi^4_{\bar{\Gamma}}}\Omega_{\bar{\Gamma}}. \tag{6.39}$$

The integration result is

$$\begin{aligned}\phi^R_\Gamma &= (g^{\mu\nu}q^2 - q^\mu q^\nu)\int_{\sigma_{\bar{\Gamma}}}\left(M^\Gamma_\emptyset - M^\Gamma_\gamma\right)\\ &= (g^{\mu\nu}q^2 - q^\mu q^\nu)\left(-\frac{10}{9}L + \frac{2}{3}L^2\right),\end{aligned} \tag{6.40}$$

such that we recover the well-known

$$\left(\frac{56}{9}L - \frac{4}{3}L^2\right) + 2\left(-\frac{10}{9}L + \frac{2}{3}L^2\right) = 4L \tag{6.41}$$

as the full 2-loop result after adding both integrals together with the right multiplicity.

6.3 3-loop photon propagators

At 3 loops the integrals start to become much more involved. For example, Z_Γ^0 is now already a polynomial of degree $h_1(h_1-1)=6$, compared to just 2 at two loops, and the number of chord diagrams rises to 15 (in Feynman gauge) such that the reduction to $h_1=3$ small summands in the partition polynomial now becomes significant.

Hence, we will not treat all 8 topologies in as much detail as the graphs above. Instead, we focus on one, the graph with crossed photon lines in fig. 6.4h. It is conceptually simple, since it has only two subdivergences, consisting of primitive vertex graphs, such that everything works completely analogous to the 2-loop graph with vertex subdivergences. While it is conceptually easy, it is also the graph that takes by far the most computing time, so it benefits the most from the simplification provided by the partition polynomial.

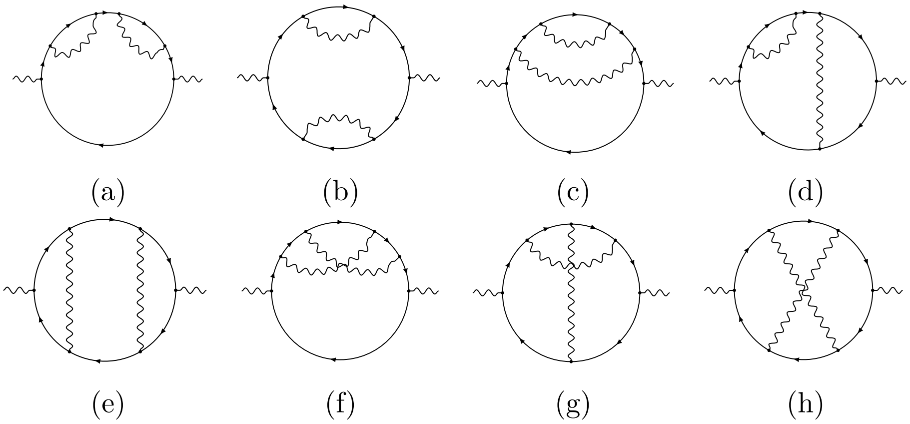
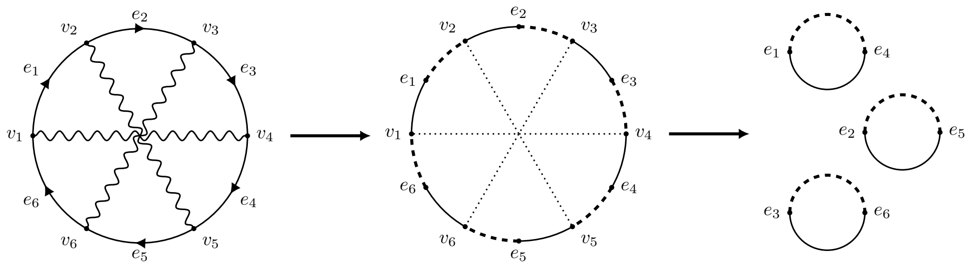


Figure 6.4: The 3-loop topologies with one fermion cycle.

Label edges and vertices as in



with v_1, v_4 being the external vertices and $e_7=(v_2,v_5)$ and $e_8=(v_3,v_6)$ the two photon edges. The Kirchhoff and second Symanzik polynomial consist of 36 and

45 monomials, so we refrain from writing them out in full here. An example for a cycle polynomial is:

$$\chi_\Gamma^{(1|6)} = \alpha_2(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_8) + (\alpha_3 + \alpha_4)(\alpha_5 + \alpha_7 + \alpha_8) + \alpha_7(\alpha_5 + \alpha_8) \quad (6.42)$$

This one has so many terms since e_1 and e_6 share all their cycles, because they are incident to the same external (i.e. 2-valent) vertex of Γ. Others are simpler:

$$\chi_\Gamma^{(1|3)} = \chi_\Gamma^{(1|4)} = \chi_\Gamma^{(3|6)} = \chi_\Gamma^{(4|6)} = -\alpha_2\alpha_5 + \alpha_7\alpha_8 \quad (6.43)$$

Here we have an example of monomials with different signs, which is due to the fact that the two corresponding cycles are twisted relative to each other (as discussed in the proof of proposition 5.1.2). One cycle is the fermion cycle, the other crosses via both photon edges. The subdivergences are analogous to the 2-loop case. The only notable difference is that, since now $h_1^\gamma = 2$ the sum in eq. (4.82) has one more term. Hence, we focus on the structure of $M_\emptyset^\Gamma$, beginning with the partition polynomial.

The partition polynomial

The word pairs we get from D_Γ^0 are

$$(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_5\mathsf{a}_6), \quad (\mathsf{a}_1\mathsf{a}_5\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_2\mathsf{a}_6), \quad (\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_5\mathsf{a}_3), \quad (\mathsf{a}_1\mathsf{a}_5\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_2\mathsf{a}_3).$$

For $Z_\Gamma^0\big|_1$ we have the single partition $\mathcal{E} = \{\{(e_1, e_4), (e_2, e_5), (e_3, e_6)\}\}$, such that

$$\begin{aligned} Z_\Gamma^0\Big|_1 &= \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3|\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6|\mathsf{a}_4\mathsf{a}_5\mathsf{a}_3)} + \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_5\mathsf{a}_3|\mathsf{a}_4\mathsf{a}_2\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_5\mathsf{a}_6|\mathsf{a}_4\mathsf{a}_2\mathsf{a}_3)} \\ &= 1 + 0 + 0 + 1 = 2. \end{aligned} \quad (6.44)$$

For $Z_\Gamma^0\big|_2$ we have three partitions with two parts, consisting of one and two edges respectively. For $\mathcal{E}_1 = \{\{(e_1, e_4)\}, \{(e_2, e_5), (e_3, e_6)\}\}$ one has

$$\begin{aligned} \lambda_{\mathcal{E}_1}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_5\mathsf{a}_6) &= \{(\mathsf{a}_1, \mathsf{a}_4), (\mathsf{a}_2\mathsf{a}_3, \mathsf{a}_5\mathsf{a}_6)\}, \\ \lambda_{\mathcal{E}_1}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_5\mathsf{a}_3) &= \{(\mathsf{a}_1, \mathsf{a}_4), (\mathsf{a}_2\mathsf{a}_6, \mathsf{a}_5\mathsf{a}_3)\}, \\ \lambda_{\mathcal{E}_1}(\mathsf{a}_1\mathsf{a}_5\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_2\mathsf{a}_6) &= \{(\mathsf{a}_1, \mathsf{a}_4), (\mathsf{a}_5\mathsf{a}_3, \mathsf{a}_2\mathsf{a}_6)\}, \\ \lambda_{\mathcal{E}_1}(\mathsf{a}_1\mathsf{a}_5\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_2\mathsf{a}_3) &= \{(\mathsf{a}_1, \mathsf{a}_4), (\mathsf{a}_5\mathsf{a}_6, \mathsf{a}_2\mathsf{a}_3)\}. \end{aligned}$$

All permutations give positive signs and the corresponding polynomial is

$$2\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_4)}\Big(\chi_\Gamma^{(\mathsf{a}_2\mathsf{a}_3|\mathsf{a}_5\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_2\mathsf{a}_6|\mathsf{a}_5\mathsf{a}_3)}\Big) = -2(-\alpha_2\alpha_5 + \alpha_7\alpha_8)(\alpha_7 + \alpha_8), \quad (6.45)$$

which is also the polynomial one finds analogously for $\mathcal{E}_3 = \{\{(e_1, e_4), (e_2, e_5)\}, \{(e_3, e_6)\}\}$. For the third, $\mathcal{E}_2 = \{\{(e_1, e_4), (e_3, e_6)\}, \{(e_2, e_5)\}\}$, the words are

$$\lambda_{\mathcal{E}_2}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_5\mathsf{a}_6) = \{(\mathsf{a}_2, \mathsf{a}_5), (\mathsf{a}_1\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_6)\},$$
$$\lambda_{\mathcal{E}_2}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_5\mathsf{a}_3) = \{(\mathsf{a}_2, \mathsf{a}_5), (\mathsf{a}_1\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_3)\},$$
$$\lambda_{\mathcal{E}_2}(\mathsf{a}_1\mathsf{a}_5\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_2\mathsf{a}_6) = \{(\mathsf{a}_5, \mathsf{a}_2), (\mathsf{a}_1\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_6)\},$$
$$\lambda_{\mathcal{E}_2}(\mathsf{a}_1\mathsf{a}_5\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_2\mathsf{a}_3) = \{(\mathsf{a}_5, \mathsf{a}_2), (\mathsf{a}_1\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_3)\}.$$

Again, the total sign is always positive but this time with $\operatorname{sgn}(\sigma) = -1 = \operatorname{sgn}(\sigma')$, where e.g. $\mathsf{a}_2\mathsf{a}_1\mathsf{a}_3 = \sigma(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3)$ and $\mathsf{a}_5\mathsf{a}_4\mathsf{a}_6 = \sigma'(\mathsf{a}_4\mathsf{a}_5\mathsf{a}_6)$. The polynomial is

$$\begin{aligned} &2\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_5)}\left(\chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_3|\mathsf{a}_4\mathsf{a}_6)} + \chi_\Gamma^{(\mathsf{a}_1\mathsf{a}_6|\mathsf{a}_4\mathsf{a}_3)}\right) \\ &\qquad = 2(-(\alpha_1+\alpha_6)(\alpha_3+\alpha_4)+\alpha_7\alpha_8)(\alpha_2+\alpha_5+\alpha_7+\alpha_8). \end{aligned} \tag{6.46}$$

The last polynomial is always of the same form. The only partition $\mathcal{E} = \{\{(e_1, e_4)\}, \{(e_2, e_5)\}, \{(e_3, e_6)\}\}$ has each base edge in a separate part such that

$$\lambda_{\mathcal{E}}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_5\mathsf{a}_6) = \{(\mathsf{a}_1, \mathsf{a}_4), (\mathsf{a}_2, \mathsf{a}_5), (\mathsf{a}_3, \mathsf{a}_6)\},$$
$$\lambda_{\mathcal{E}}(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_5\mathsf{a}_3) = \{(\mathsf{a}_1, \mathsf{a}_4), (\mathsf{a}_2, \mathsf{a}_5), (\mathsf{a}_6, \mathsf{a}_3)\},$$
$$\lambda_{\mathcal{E}}(\mathsf{a}_1\mathsf{a}_5\mathsf{a}_3, \mathsf{a}_4\mathsf{a}_2\mathsf{a}_6) = \{(\mathsf{a}_1, \mathsf{a}_4), (\mathsf{a}_5, \mathsf{a}_2), (\mathsf{a}_3, \mathsf{a}_6)\},$$
$$\lambda_{\mathcal{E}}(\mathsf{a}_1\mathsf{a}_5\mathsf{a}_6, \mathsf{a}_4\mathsf{a}_2\mathsf{a}_3) = \{(\mathsf{a}_1, \mathsf{a}_4), (\mathsf{a}_5, \mathsf{a}_2), (\mathsf{a}_6, \mathsf{a}_3)\}.$$

and

$$\begin{aligned} Z_\Gamma^0\Big|_3 &= 4\chi_\Gamma^{(\mathsf{a}_1|\mathsf{a}_4)}\chi_\Gamma^{(\mathsf{a}_2|\mathsf{a}_5)}\chi_\Gamma^{(\mathsf{a}_3|\mathsf{a}_6)} \\ &= (-\alpha_2\alpha_5+\alpha_7\alpha_8)^2(-(\alpha_1+\alpha_6)(\alpha_3+\alpha_4)+\alpha_7\alpha_8). \end{aligned} \tag{6.47}$$

The number of terms in the three polynomials is 1, 22 and 15 respectively. For comparison, the full chord diagram sum consists of 437 monomials. Here we especially also see how hidden the factorisation of the Kirchhoff polynomials can be. The expression $a\Psi_\Gamma^2 Z_\Gamma^0\Big|_1 + b\Psi_\Gamma Z_\Gamma^0\Big|_2 + cZ_\Gamma^0\Big|_3$ should have $36^2 \cdot 1 + 36 \cdot 22 + 15 = 2103$ terms. But the same monomials may of course occur in different parts and add up or cancel to yield the 437 that are left in the sum, obscuring the pattern. See also table 6.1 for the reduction observed for other graphs.

The remainder

For $2Z_{\Gamma,0}^1 + Z_{\Gamma,+}^1$ one can again make similar observations as in the 2-loop case, but it takes much more effort. Naively computing it, $2Z_{\Gamma,0}^1 + Z_{\Gamma,+}^1$ has 4751 terms.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
$\#Z^0_\Gamma$	9	9	44	84	348	231	448	437
$\#Z^0_\Gamma/\Psi^6_\Gamma$	9	9	44	16	16	73	53	38

Table 6.1: Number of terms in Z^0_Γ compared to Z^0_Γ/Ψ^6_Γ after cancellations for all graphs from fig. 6.4a to 6.4h.

By working through the

$$\binom{2h_1}{2}(2h_1 - 3) = h_1(2h_1 - 1)!! = 45 \tag{6.48}$$

chord diagrams with one chord missing and manually applying the Dodgson identity one can indeed whittle down $(2Z^1_{\Gamma,0} + Z^1_{\Gamma,+})/\Psi^6_\Gamma$ to a few dozen terms that do in particular include Z^0_Γ as expected. However, doing this by hand is probably nigh impossible beyond 3 loops, so finding a summation theorem that applies to these subsets of chord diagrams should be the highest priority for future work.

6.4 Future work

There are a number of open questions that one could tackle next. In what is certainly no coincidence, most of them appear to be a variation of the same problem: Summation of specific subsets of chord diagrams.

In the immediate future one should of course first attempt to finish the simplification of the integrand by finding a summation identity for $2Z^1_{\Gamma,0} + Z^1_{\Gamma,+}$. When that is done, generalisation to subgraph/cograph terms as in eq. (4.82) is next. In that case, chords may always only connect vertices of a chord diagram that both correspond to the cograph or subgraph. For simple subgraphs one can therefore see quite easily that $\tilde{c}(D_\Gamma) = \tilde{c}(D_{\Gamma/\gamma}) + \tilde{c}(D_\gamma) + c$, with a small subgraph-dependent integer c, simply by cutting out a segment of the base corresponding to the subgraph and gluing the respective cut base edges back together to get $D_{\Gamma/\gamma}$ and D_γ. It remains to be investigated how well this works in general (for nested, disjoint subdivergenves), if it reflects the Hopf algebraic structure of the graphs on the level of the chord diagrams, and, if so, how it relates to the standard Hopf algebraic structure of chord diagrams (see e.g. [40, chapter 4]). This would lead to a factorisation, such that the summation identities can be applied to each element in a forest of subdivergences.

6.4.1 Generalisations

In this thesis we dealt with a rather special case of a quantum field theory and then specialised more to be able to concisely discuss new ideas. Having understood it, for the most part, one should look to other cases.

Gauge and masses

As was mentioned before, a general gauge treatment rather than Feynman gauge mostly just adds tedious summations, and larger and more chord diagrams. Initially, the same holds true for the massive case, but at the renormalisation stage masses considerably complicate the situation. Looking at the scalar case with masses [29, eq. 74], one would suspect that combining this with the already involved renormalisation procedure we went through in chapter 3 will blow up the notation to the point of it being unpractical, but in principle it should be no problem to work it out.

Other theories

Already in the introduction we discussed that the study of quantum electrodynamics is ultimately a stepping stone to other gauge theories and, eventually, the full standard model. Following the corolla polynomial formalism discussed in section 2.2.3, many of our results should be transferable to other theories. The peculiarities of non-abelian theories that did not appear in this thesis – like 4-valent vertices – are entirely worked out in [96]. Ultimately, the problem should always reduce to sorting through cycle polynomials resulting from derivatives of the bond polynomial (second Symanzik polynomial with auxiliary momenta).

While there is currently no corolla polynomial that applies to gravity, some of our results can still be useful for that setting. The Feynman rules for an n-point graviton vertex contain $2n$ space-time indices and computation of contractions between such vertices are needed when studying certain generalised Slavnov-Taylor identities for gravity. It should certainly be possible to adapt our diagrammatic approach from chapter 4 to that setting. In a best case scenario this might then lead to a combinatorial proof of the renormalisability of gravity. The diagrammatic contraction also appears to be applicable to certain open problems in quantum field theories arising from non-commutative geometry[2], and bears some similarity to otherwise unrelated methods in the theory of tensor models [19, 80].

Finally, it might also be interesting to consider scalar QED in this setting. In a certain sense, it is dual to the Feynman gauge case we mostly concentrated on –

[2] "Non-commutative geometry and quantum field theory", talk given by Walter van Suijlekom at the *Summer school on structures in local quantum field theory*, Les Houches (June 15, 2018)

instead of removing the complications due to the complicated photon propagator, scalar QED replaces the fermion with a scalar particle. Since scalar QED also contains 4-valent vertices it seems like an advisable exercise before tackling QCD or other more complicated theories.

6.4.2 Higher order computations

Given a simpler integrand one should try to compute new results with it. In the last section we focused on small integrals that can sensibly be written on a page, but with a completely automatised integrand generation there is nothing stopping us from going to 4, 5, or possibly 6 loops. While parametric integrals are generally not the most efficient in terms of pure computing time, this might be useful for some integrals that were not accessible before. However, depending on what precisely one wants to do, it might be wise to put this off until the general structure of the integrand is even more well studied.

6.4.3 Cancellations

The original motivation for this thesis was the observed cancellation of $\zeta(3)$ in the QED beta function [26, 76, 114]. Now that the integrand of a single graph is reasonably well understood, we can start comparing different graphs in order to possibly find reasons for these cancellations directly on the level of the integrand. The sum over graphs can itself be interpreted as a sum over chord diagrams, with QED graphs only differing in the distribution of the photon edges. The problem, however, would be the appearance of different graph polynomials. In order to continue the combinatorial studies of this thesis to that level, it appears that we might actually need to find relations between graph polynomials of different graphs, or rather, view the graph polynomials of all graphs at a given loop number as special cases of a bigger amplitude-level graph polynomial.

6.4.4 More general polynomials

We have seen a multitude of relations between different graph polynomials. Kirchhoff and Symanzik, Cycle and Bond, Dodgson, Corolla, etc. They all are interrelated in one way or other, which suggests a search for a polynomial that contains them all as special cases. The Kirchhoff and Corolla are already known to relate to the multivariate Tutte polynomial, so maybe we can use this to shed some light on underlying structures.

6.4.5 Gauge sets

Cvitanović observed that when collecting QED vertex diagrams in “minimal gauge invariant subsets”, or just gauge sets, their sums add up to curious values [50, 52]. Specifically, while every separate contribution is of the order 10^1 to 10^2, their sums are small multiples of $1/2$, up to small corrections of order 10^{-2}. A gauge set is the set (k, m, m') of all diagrams that have k photons crossing from one fermion leg of a vertex graph to the other and m/m' photons beginning and ending on the same leg (see fig. 6.5).

The sign of the sum is correctly (empirically) predicted by $(-1)^{m+m'}$, but little else is known about them. However, these observations lead to the estimate

$$\frac{1}{2}(g-2) \approx \cdots + \frac{n}{2}\left(\frac{\alpha}{\pi}\right)^n + \cdots \tag{6.49}$$

for the n-th order contribution of $g - 2$, which is summable to all orders, in contradiction of Dyson's famous argument [56] that estimated

$$\frac{1}{2}(g-2) \approx \cdots + n^n\left(\frac{\alpha}{\pi}\right)^n + \cdots . \tag{6.50}$$

Since a gauge set can clearly be interpreted as a particular subset of the chord diagrams this seems like a problem related to the work of this thesis. Moreover, it is not unreasonable to believe that the cancellations observed here are manifestations of the same structure that leads to cancellations in the beta function.

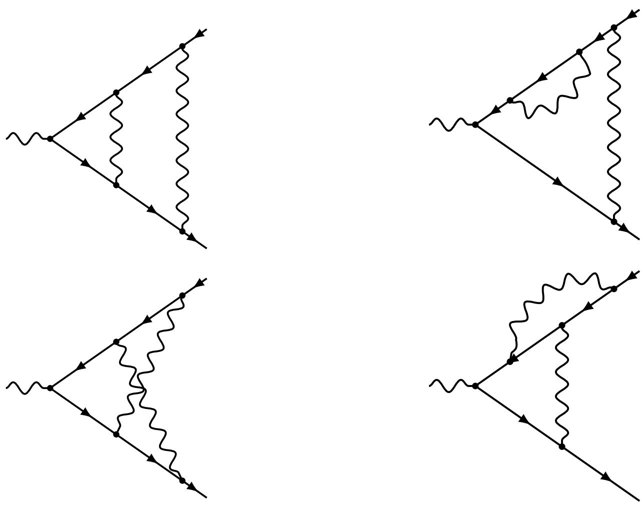


Figure 6.5: The two graphs belonging to the gauge set $(2, 0, 0)$ on the left and $(1, 1, 0)$ on the right. $(1, 0, 1)$ would be the time reversal of the latter.

6.4.6 Finiteness of the 4-photon vertex

Consider the integrand of the 4-photon vertex, following our considerations for the other cases in section 3.1.2. It has $|E_\Gamma^{(f)}| = 2h_1 + 2$ and $|E_\Gamma^{(p)}| = h_1 - 1$ such that indeed $\omega_4(\Gamma) = 4h_1 - 2h_1 - 2 - 2(h_1 - 1) = 0$ for all such graphs. The unrenormalised integrand is

$$I_\Gamma = \frac{e^{-\frac{\Phi_\Gamma}{\Psi_\Gamma}}}{\Psi_\Gamma^{2h_1+2}} \sum_{k=0}^{2h_1} \frac{I_\Gamma^{(k)}}{\Psi_\Gamma^k}, \tag{6.51}$$

where only $I_\Gamma^{(0)}$ is logarithmically divergent and the other summands are convergent. The tensor structure of $I_\Gamma^{(0)}$ contains four uncontracted vertices. Since all others are contracted with metric tensors, not momenta (since this is the term corresponding to pairings of all fermion edges and internal vertices), that leaves only three possible coefficients such that

$$g^{\mu\nu} g^{\sigma\rho} I_{\Gamma,a}^{(0)} + g^{\mu\sigma} g^{\nu\rho} I_{\Gamma,b}^{(0)} + g^{\mu\rho} g^{\sigma\nu} I_{\Gamma,c}^{(0)}. \tag{6.52}$$

In the 1-loop case one easily checks that up to permutation $-2I_{\Gamma,a}^{(0)} = I_{\Gamma,b}^{(0)} = I_{\Gamma,c}^{(0)}$ such that this reduces to

$$-2g^{\mu\nu} g^{\sigma\rho} + g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\sigma\nu}. \tag{6.53}$$

The 6 graphs needed to complete the full 1-loop amplitude simply correspond to the 6 permutations of three of the four indices, such that the sum indeed vanishes.

The task is now to prove a relation between the integrands $I_{\Gamma,\bullet}$ for any loop number, which is essentially again a problem involving identities between sums over certain subsets of chord diagrams.

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Symbols and abbreviations

	a scalar propagator	eq. (2.29),	p. 19
	a fermion propagator	eq. (2.30),	p. 19
	a photon propagator	eq. (2.31),	p. 19
	a QED vertex	eq. (2.32),	p. 19
A	an alphabet,	sec. 4.2.1,	p. 67
$\mathsf{a}, \mathsf{b}, \ldots$	letters or words	sec. 4.2.1,	p. 67
α_i	a Schwinger parameter,	sec. 1.2.4,	p. 6
α	a tuple of Schwinger parameters	sec. 2.1.1,	p. 12
α_S	a product of Schwinger parameters	eq. (2.4),	p. 13
A	a diagonal matrix of Schwinger parameters	sec. 2.1.1,	p. 14
B, $\mathcal{B}_G$	a bond subgraph, set of bonds of G	sec. 2.1.1,	p. 10
β_G, $\beta_G^{(e_i\|e_j)}$	bond polynomials of G	def. 2.3.3,	p. 29
C, $\mathcal{C}_G^{[1]}$, $\mathcal{C}_G^{[i]}$	a cycle subgraph, set of simple cycles of G, set of all edge-disjoint unions of i simple cycles	sec. 2.1.1,	p. 10
χ_G, $\chi_G^{(e_i\|e_j)}$	cycle polynomials of G	def. 2.3.2,	p. 29
$\bar{\chi}_\Gamma^{(e_i\|e_j)}$	an auxiliary cycle polynomial	eq. (2.103),	p. 39
$\chi_G^{(\mathsf{u}\|\mathsf{v})}$	cycle Dodgson polynomials of G	def. 5.1.3,	p. 98
$\mathrm{X}_\Gamma^{e,\mu}$	a derivative of the bond polynomial	eq. (2.104),	p. 39
x_Γ^e	momentum independent version of $\mathrm{X}_\Gamma^{e,\mu}$	sec. 4.4,	p. 86
$c_2, c_3, \tilde{c}$	number of bi-/tricoloured cycles in a chord diag.	eq. (4.38),	p. 74
∂	a map from edges of a graph to pairs of vertices	sec. 2.1.1,	p. 9

$\partial_\pm$	start and target vertex of a graph	sec. 2.1.1,	p. 9
∂_Γ	derivatives w.r.t. ξ_e in I_Γ for QED	eq. (2.56),	p. 26
D ($\bar{\mathsf{D}}$)	the set of (fully contracted) Dirac words	def. 4.2.1,	p. 67
D	a chord diagram	sec. 4.3	p. 73
D_Γ, D_Γ^0	the chord diagram associated to Γ, its projection	sec. 4.4.1	p. 87
$\mathcal{D}_k^n$	a set of chord diagrams, k chords, base with $2\sum n_i$ vertices, $n = (n_1, \dots, n_\ell)$	def. 4.3.1	p. 73
$\bar{\mathcal{D}}^k$	set of k-fold chord additions to a chord diagram	sec. 4.67	p. 87
$\mathcal{D}_\Gamma^k$	set of $(h_1 - k)$-fold chord additions to D_Γ^0	sec. 4.68	p. 87
E_G, e	the set of edges of a graph, an edge	sec. 2.1.1,	p. 9
$E_\Gamma^{(f)}$, $E_\Gamma^{(p)}$	the sets of fermion and photon edges of a QED Feynman graph	sec. 2.1.2,	p. 15
$\mathcal{F}_\Gamma$	the forests of subdivergences	sec. 3.3.4,	p. 60
G	a graph	sec. 2.1.1,	p. 9
Γ	a Feynman graph	sec. 2.1.2,	p. 15
g, γ	subgraphs of (Feynman) graphs		
$G \setminus e$, $G /\!\!/ e$	deletion and contraction of an edge e from G	sec. 2.1.1,	p. 11
Γ/γ, $\Gamma /\!\!/ \gamma$	algebraic and edge contraction of a subgraph	sec. 3.1.1,	p. 46
γ^μ	a Dirac matrix	sec. 4.1,	p. 63
$\bar{\gamma}_\Gamma$	the Dirac matrix structure of Γ	sec. 2.1.3,	p. 20
h_1	the first Betti number of a graph	sec. 2.1.1,	p. 10
I, (I')	the (reduced) incidence matrix of a graph	eq. (2.10),	p. 14
$\tilde{I}_\Gamma$	the momentum space integrand of ϕ_Γ		
	scalar, unrenormalised (also $\tilde{S}_\Gamma$)	sec. 2.1.3,	p. 18
	QED, unrenormalised, massless	eq. (2.51),	p. 25
I_Γ	the parametric integrand of ϕ_Γ		
	scalar, unrenormalised (also S_Γ)	eq. (2.50),	p. 24
	fermion, unrenormalised	eq. (3.9),	p. 49
	photon, unrenormalised	eq. (3.10),	p. 50
	QED vertex, unrenormalised	eq. (3.11),	p. 50

$I_\Gamma^{(k)}$	the k-th summand ofI_Γ	sec. 3.1.2,	p. 49
$J_\Gamma^{(k)}$	the momentum independent part of $I_\Gamma^{(k)}$	sec. 3.1.2,	p. 49
$J_\Gamma^{(k,l)}$	the coefficient of ε^l in $J_\Gamma^{(k)}$	eq. (3.12),	p. 50
k_e	internal momentum ($\neq$ loop momentum) associated to an edge e	eq. (2.28),	p. 18
ℓ	number of base cycles in a chord diagram	sec. 4.3,	p. 73
L, (L')	the (reduced) Laplacian matrix of a graph	eq. (2.11),	p. 14
$\tilde{L}'$	the weighted reduced Laplacian matrix of a graph	sec. 2.2.1,	p. 23
$\lambda_{\mathcal{E}}(\mathsf{u},\mathsf{v})$	a partitioning map for word pairs	sec. 5.2.2	p. 104
M	the graph matrix of G	eq. (2.13),	p. 15
M_f^Γ	a term in the forest formula (only for res$(\Gamma) = \sim\!\sim$)		
	no subdivergence	eq. (3.43),	p. 57
	vertex subdivergence	eq. (3.42),	p. 57
	fermion subdivergence	eq. (3.47),	p. 58
	with squashing	eq. (3.50),	p. 59
ω_Γ^D	the superficial degree of divergence	eq. (3.1),	p. 45
Ω_Γ	projective volume form	sec. 3.1.3,	p. 52
Φ_Γ	the second Symanzik polynomial of Γ	eq. (2.20),	p. 17
φ_Γ	momentum independent second Symanzik,	eq. (2.22),	p. 17
ϕ_Γ	the Feynman integral of Γ		
	in momentum space	eq. (2.28),	p. 18
	parametric	eq. (3.6),	p. 49
ϕ_Γ^R	the renormalised (parametric) Feynman integral		
	fermion, superficially	eq. (3.27),	p. 53
	photon, superficially	eq. (3.32),	p. 54
	QED vertex, superficially	eq. (3.35),	p. 55
	photon, vertex subdivergence	eq. (3.44),	p. 57
	photon, fermion subdivergence	eq. (3.51),	p. 60
π_0	projection map for chord diagrams	eq. (4.37),	p. 74
Ψ_G	the Kirchhoff polynomial of G	eq. (2.3),	p. 13
$\Psi_{G,K}^{I,J}$	Dodgson polynomials	def. 5.1.1,	p. 96
q, q_v	external momentum, entering at vertex v	sec. 2.1.3,	p. 18

res(Γ)	the residue of a Feynman graph	sec. 2.1.2,	p. 16
σ_Γ	a certain subset of projective space	eq. (3.20),	p. 52
$\mathrm{sgn}(u, v)$	signum of two vertices in a chord diagram	sec. 4.3.1,	p. 76
$\mathrm{sgn}_{\mathcal{E}}(\mathsf{u}, \mathsf{v})$	the sign of a partitioning of a word pair	sec. 5.2.2	p. 104
$\mathrm{sym}(\mathsf{w})$,	symmetrisation of a word	eq. (4.16),	p. 69
T, $\mathcal{T}_G^{[k]}$	a tree, set of spanning k-forests of G	sec. 2.1.1,	p. 10
V_G, v	the set of vertices of a graph, a vertex	sec. 2.1.1,	p. 9
V_Γ^{ext}, V_Γ^{int}	the sets of external and internal vertices of a QED Feynman graph	sec. 2.1.2,	p. 16
$V_D^{(2)}$	subset of 2-valent vertices of a chord diagram	sec. 4.3,	p. 73
$\bar{\mathsf{w}}(D)$	cycle word of a chord diagram	def. 4.3.7,	p. 78
ξ_e	auxiliary momentum associated to e	sec. 2.2.2,	p. 25
x_γ^Γ , $\mathrm{x}_\gamma^{\Gamma,0}$, Υ_γ^Γ	combinations of graph polynomials	sec. 3.1.3,	p. 51
Z_Γ^k	abbreviation for a sum of chord diagrams	eq. (4.71),	p. 88
Z_Γ^0, $Z_\Gamma^0\Big\vert_l$	the partition polynomials	def. 5.2.3,	p. 105
BPHZ	Bogoliubov-Parasiuk-Hepp-Zimmermann	sec. 3.1.1,	p. 45
QFT	quantum field theory	sec. 1.2.1,	p. 3
QED	quantum electrodynamics	sec. 1.2.5,	p. 6