WELL-POSEDNESS FOR DEGENERATE THIRD ORDER EQUATIONS WITH DELAY AND APPLICATIONS TO INVERSE PROBLEMS

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Abstract. In this paper, we study well-posedness for the following third-order in time equation with delay

(0.1) $$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad t \in [0, 2\pi]$$

where α, β, γ are real numbers, A and B are linear operators defined on a Banach space X with domains $D(A)$ and $D(B)$ such that $D(A) \cap D(B) \subset D(M) \cap D(N)$; $u(t)$ is the state function taking values in X and $u_t : (-\infty, 0] \to X$ defined as $u_t(\theta) = u(t+\theta)$ for $\theta < 0$ belongs to an appropriate phase space where F and G are bounded linear operators. Using operator-valued Fourier multipliers techniques we provide optimal conditions for well-posedness of equation (0.1) in periodic Lebesgue-Bochner spaces $L^p(\mathbb{T}, X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and periodic Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$. A novel application to an inverse problem is given.

1. Introduction

Well-posedness for abstract degenerate (also called Sobolev type) evolution equations have been studied in the literature since a long time ago. Anufrieva [5] studied well-posedness of the second order degenerate equation

$$Mu''(t) = Au(t) + Bu'(t), \quad t > 0,$$

where A, B are closed linear operators defined on a Banach space X and M is bounded with $KerM \neq 0$. She used the technique of integrated semigroups and she obtained a criterion for the well-posedness of the above problem in terms of the behavior of the operator $M(\lambda^2 M - \lambda B - A)^{-1}$. Barbu and Favini [9] studied the inhomogeneous equation

(1.2) $$(Mu)'(t) = Au(t) + f(t), \quad t \in [0, 2\pi],$$

where M and A are closed linear operators on X with $D(A)$ continuously embedded in $D(M)$ and A has a bounded inverse. They associate to (1.2) the periodicity condition $Mu(0) = Mu(2\pi)$ and consider the operational method by Grisvard [29] to treat the equation. One of the main hypothesis considered by Barbu and Favini is that the operator $M(\lambda M - A)^{-1}$ must satisfy the estimate

$$\|M(\lambda M - A)^{-1}\| \leq C(1 + |\lambda|)^{-\beta}, \quad \lambda \in \Sigma_\alpha,$$

where $\Sigma_\alpha := \{z \in \mathbb{C} : \Re(z) \geq -c(1 + |\mathfrak{Im}\,(z)|)^\alpha\}$, c is a positive constant and $0 < \beta \leq \alpha \leq 1$. More recently, Favini and Marinoschi [25] studied a concrete degenerate PDE problem that arises in fast diffusion. For a general treatment of degenerate differential equations in Banach spaces we refer the reader to the classical textbook [23] by Favini and Yagi, where the method

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of multivalued operators is developed and several interesting examples are given, and the more recent monograph [46] by Sviridyuk and Fedorov. It is important to observe that there is a strong connection between well posedness and inverse problems [3] and consequently some optimal control problems. To this regard, we quote references [2], [9], [22] and [26]. See also our last section in this paper.

On the other hand, since the pioneer work by Amann [4], Arendt and Bu [6] and Weis [44] methods based on operator-valued Fourier multipliers theorems have been considered by many authors in the study of well-posedness of abstract evolution equations in Banach spaces. See [6, 7, 8, 19, 31, 32, 33, 34, 40, 41] and references therein.

In [37], the authors used operator-valued Fourier multiplier methods to provide conditions on the symbol $(\lambda M - A)^{-1}$ to characterize well-posedness in Lebesgue-Bochner, Besov and Triebel-Lizorkin vector-valued spaces for the degenerate first order Cauchy problem (1.2). In [39] the authors also investigated well-posedness of equation (1.2) adding a delay term. The case $M = Id$ was investigated by Arendt, Bu and Kim in [6, 7, 16].

In [12] S. Bu investigates, using the same method just described, the second-order degenerate equation

$$(Mu')'(t) = Au(t) + f(t), \quad t \in [0, 2\pi]$$

with conditions $u(0) = u(2\pi)$ and $Mu'(t) = Mu'(2\pi)$, where A and M are closed linear operators defined on a UMD Banach space X that satisfy $D(A) \subset D(M)$. See also the work by Bu and Cai [17, 14] for second order degenerate differential equations with delay, and the paper by Chill and Srivastava [18] where L^p-well-posedness for second-order differential equations in case $M = Id$ is studied.

In [13], Cai and Bu studied the following third order equation:

$$\begin{cases} \alpha(Mu)'''(t) + (Mu)''(t) = \beta Au(t) + \gamma Bu'(t) + f(t), \quad t \in [0, 2\pi] \\ Mu(0) = Mu(2\pi), \quad (Mu)'(0) = (Mu)'(2\pi), \quad (Mu)''(0) = (Mu)''(2\pi). \end{cases} \tag{1.3}$$

This is an important model in acoustics for wave propagation in viscous thermally relaxing fluids. See [28, 32, 40] and references therein for details. They obtain a characterization of the well-posedness of (1.3) in vector valued Lebesgue-Bochner, Besov and Triebel-Lizorkin spaces using operator-valued Fourier multiplier methods. This characterization involves Rademacher boundedness conditions of the operator-valued symbol $(\alpha\lambda^3 M + \lambda^2 M - \beta A - \gamma\lambda B)^{-1}$ defined on a UMD-space X. These results extend the ones provided in [43] where the case $M = Id$ was considered. In both cases, they use sophisticate representations of the above symbol in order to prove Mikhlin type multiplier conditions [13, Proposition 2.2 and Proposition 3.1], [43, Lemma 4.3 and Lemma 5.3].

On the other hand, delay equations appear in computational and applied contexts that have been the subject of research by many authors in the last decades, see for instance [11, 30, 20]. The study of well-posedness for delay differential equations has been also studied by many authors. In [36], Lizama considered the following first order finite delay equation:

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in [0, 2\pi]. \tag{1.4}$$

Here $u_t : (-\infty, 0] \to X$ defined as $u_t(\theta) = u(t + \theta)$ for $\theta < 0$ belongs to an abstract phase space $\mathcal{S}$ and F is a bounded linear operator from $\mathcal{S}$ to X. He characterized the existence and uniqueness of periodic solutions of the inhomogeneous abstract equation (1.4) and provided sufficient and necessary conditions for L^p-well-posedness for such a problem in terms of the symbol $(\lambda - A - F_\lambda)^{-1}$ where

$$F_\lambda x = F(e_\lambda x), \quad e_\lambda(t) = e^{i\lambda t}, \quad \lambda \in \mathbb{R}, \quad t \leq 0, \quad x \in X.$$

On the other hand, Hölder well-posedness of (1.4) was considered in [38] and Besov and Triebel-Lizorkin well-posedness of (1.4) was investigated by Bu and Fang in [15].

In [27] Fu and Li characterized the well-posedness in vector-valued Lebesgue spaces (resp. Besov and Triebel-Lizorkin) using operator-valued Fourier multipliers for the second order differential equation with delay given by

$$u''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t), t \in [0, 2\pi], \tag{1.5}$$

where A and B are linear operators defined on a Banach space X. In the proof of their main results, these authors assume that the pair (A, B) is coercive and a combination of conditions in terms of uniform boundedness and R-boundedness of the symbol $(\lambda^2 + \lambda B - \lambda G_\lambda - F_\lambda + A)^{-1}$. We also observe that an intricate representation of the above symbol is necessary in order to prove certain Mikhlin type bounds [27, Lemma 4.2].

The aim of this paper is to study the following degenerate equation with delay

$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad t \in [0, 2\pi], \tag{1.6}$$

where α, β, γ are real numbers, A and B are linear operators defined on a Banach space X with domains $D(A)$ and $D(B)$ such that $D(A) \cap D(B) \subset D(M) \cap D(N)$; $u(t)$ is the state function taking values in X and $u_t : (-\infty, 0] \to X$ defined as $u_t(\theta) = u(t + \theta)$ for $\theta < 0$ belongs to an appropriate phase space, being F and G bounded linear operators.

Compared with the non degenerate case which was studied in [43] and without delay, the nature of equation (1.6) leads to a different treatment that requires new tools.

As remarked in [35], there are three important notions that are needed in the study of maximal regularity of abstract equations by operator-valued Fourier multiplier theorems, namely $n-$regularity of scalar sequences, M-boundedness and $MR-$boundedness of order n for operator sequences. The concept of $n-$regular sequences was introduced in [33] as a discrete version of the notion of $k-$regularity used in [45]. Define the differences $\Delta^k M_n$ by $\Delta^0 M_n = M_n$, $\Delta^1 M_n = \Delta M_n = M_{n+1} - M_n$, and $\Delta^{k+1} M_n = \Delta(\Delta^k M_n)$, for $k \geq 1$. If $\{M_n\}$ is the operator family under consideration, $M-$ boundedness (resp. $MR-$boundedness) of order m ($m \in \mathbb{N} \cup \{0\}$) means that the sequences $\{n^j \Delta^j M_n\}$ are bounded (resp. $R-$bounded) for $0 \leq j \leq m$.

One of the main problems when dealing with the above mentioned concepts, is the absence of practical criteria for MR-boundedness (resp. M-boundedness) for the manipulation of structurally complicated operator-valued symbols. The first original contribution of this paper is to show a new computable condition for M-boundedness and MR-boundedness of order 1 and 2. More precisely, we prove:

Theorem 1.1. *Let $T : D(T) \subset X \to X$ be a closed linear operator defined in a Banach space X. For each $k \in \mathbb{Z}$, let $H_k : X \to D(T)$ be a sequence of bounded and linear operators such that $0 \in \rho(H_k)$ for all $k \in \mathbb{Z}$. Suppose that $(c_k)_{k \in \mathbb{Z}} \subset \mathbb{C}$ is a 1-regular sequence and denote*

$$M_k := c_k T H_k, \tag{1.7}$$

and

$$L_k := (H_k^{-1} - H_{k+1}^{-1}) H_k. \tag{1.8}$$

If $\{M_k : k \in \mathbb{Z}\}$ and $\{kL_k : k \in \mathbb{Z}\}$ are R-bounded sets, then $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded of order 1. *If, in addition, $(c_k)_{k\in\mathbb{Z}}$ is* 2*-regular and the set $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ is R-bounded, then $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded of order* 2.

For instance in the simplest case $c_k = 1, T = M$ and $H_k = (ikM - A)^{-1}$ for the equation (1.2), we easily obtain $M_k = M(ikM - A)^{-1}$ and $L_k = -iM(ikM - A)^{-1}$. In particular, we note that R-boundedness of the set $\{kL_k\}_{k\in\mathbb{Z}}$ characterizes L^p-well-posedness of (1.2) in UMD Banach spaces [37, Theorem 3.3].

Our second main contribution in this paper is a characterization of L^p- well posedness for degenerate third-order differential equations with infinite delay which has not been considered in the literature yet.

More specifically, we succeed in proving the following:

Theorem 1.2. *Let $1 < p < \infty$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume A, B and M, N are closed linear operators defined on a UMD space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$. Assume that $\{kG_k : k \in \mathbb{Z}\}$ is R-bounded. The following assertions are equivalent:*

(i) *For each $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p-solution of* (1.6).

(ii) *For each $k \in Z$ the operator*
$$N_k = [i\alpha k^3 M + k^2 N + \beta A + \gamma(ik)B + (ik)G_k + F_k]^{-1},$$
exists as a bounded linear operator in X and the sets $\{i\alpha k^3 MN_k\}_{k\in\mathbb{Z}}$, $\{k^2 NN_k\}_{k\in\mathbb{Z}}$, $\{\gamma kBN_k\}_{k\in\mathbb{Z}}$, $\{kN_k\}_{k\in\mathbb{Z}}$ are R-bounded.

Compared with earlier results by Fu and Li [27, Theorem 3.4] in case $\alpha = 0$ and $N = I$, we note that they assumed that the set $\{k^2N_k\}_{k\in\mathbb{Z}}$ must be R-bounded as well as uniform boundedness of the set $\{kBN_k\}_{k\in\mathbb{Z}}$. However, they assumed a stronger hypothesis of coercivity on the pair (A, B) that in our case is not necessary. Compared with a recent result by Bu and Cai [14] we observe that our result generalizes their main theorem ([14, Theorem 2.2]) for $B = I$ and $\alpha = 0$. In such a case our condition on R-boundedness of the set $\{kG_k\}_{k\in\mathbb{Z}}$ implies that the set $\{k(G_{k+1} - G_k\}$ must be R-bounded, which is assumed by Bu and Cai.

In the case of Besov and Triebel-Lizorkin scales of periodic vector-valued spaces, R-boundedness can be replaced by uniform boundedness and the spectral condition $0 \in \rho(M)$. We remark that this condition is only required when the delays are present in the equation. In such a case, our new characterization is stated as follows:

Theorem 1.3. *Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume A, B and M, N are closed linear operators defined on a Banach space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$ and $0 \in \rho(M)$. Assume that $\{kG_k : k \in \mathbb{Z}\}$ is uniformly bounded. The following assertions are equivalent:*

(i) *The equation*
$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad t \in [0, 2\pi]$$
is $B^s_{p,q}$-well posed;

(ii) *$\mathbb{Z} \subset \rho_{M,N}(A, B)$ and the sets $\{i\alpha k^3 MN_k\}_{\in\mathbb{Z}}$, $\{k^2 NN_k\}_{\in\mathbb{Z}}$, $\{\gamma kBN_k\}_{k\in\mathbb{Z}}$ are uniformly bounded.*

Observe that no additional restriction on the Banach space X is necessary. Other interesting fact of our findings that contrasts with the case of Lebesgue-Bochner vector-valued spaces is that we only need three conditions of boundedness instead of four. In this way, our result complements one of the main theorems by Cai and Bu [13, Theorem 3.3]. Note that in the case $M = N = I, \alpha = 0$ and $B = I$ condition (ii) reduces to inquire that the set $\{k^2 N_k\}_{k\in\mathbb{Z}}$ must be uniformly bounded, which recovers exactly [27, Theorem 4.3]. In addition, Theorem 1.3 extends the main result of [27, Section 4] to the case $B \neq I$. In this way, the obtained results furnish extra information on optimal conditions. Finally, we consider the following inverse problem:

$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + f(t)z, \quad t \in [0, 2\pi], \tag{1.9}$$

with the additional information

$$\Phi[Mu'(t)] = g(t) \quad \Phi[Nu'(t)] = h(t), \tag{1.10}$$

where $z \in X, \Phi \in X^*$ and the unknown (u, f) is to be determined. The main novelty, and third main contribution of this paper, is the treatment of the above problem combining our new results on well posedness and an original method recently introduced by Al Horani and Favini [3] which consists in reducing the inverse problem to a direct evolution equation, where perturbations $A + A_1$ and $B + B_1$ substitutes the operators A and B, respectively. As observed in [3], the described inverse problem can be also faced in the form of optimal control problems.

2. Preliminaries

Let $\mathbb{T} = \mathbb{R}\backslash 2\pi\mathbb{Z}$. Given $1 \leq p < \infty$, let $L^p(\mathbb{T}, X)$ be the space of all Bochner measurable vector-valued, p-integrable functions on $\mathbb{T}$. The k-th Fourier coefficient of a function $f \in L^1(\mathbb{T}, X)$ is defined as

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t)dt$$

for all $k \in \mathbb{Z}$. We denote by $f \in L^1(\mathbb{T}, X)$, a function that can be periodically extended to the left onto the interval $(-\infty, 0]$. Then, the k-th Fourier coeficient in t of $f_t(\theta) := f(t + \theta)$, $t \in \mathbb{T}$, $\theta \leq 0$ is $\hat{f}_t(k) = e^{ik\theta}\hat{f}(k)$.

In what follows, we will introduce the notation $\mathcal{B}(X, Y)$ for the space of bounded linear operators from X into Y endowed with the uniform operator topology. We abbreviate $\mathcal{B}(X)$ whenever $X \equiv Y$.

Definition 2.1. *Let X and Y be Banach spaces and $1 \leq p < \infty$. We say that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an L^p-Fourier multiplier if, for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.*

Let X be a Banach space. We define the means

$$\|(x_1, ..., x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \Big\| \sum_{j=1}^n \epsilon_j x_j \Big\|,$$

for $x_1, ..., x_n \in X$.

Definition 2.2. *Let X and Y be Banach spaces. A set $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded if there is a constant $c \geq 0$ such that*

$$\|(T_1x_1, ..., T_nx_n)\|_R \leq c\|(x_1, ..., x_n)\|_R, \tag{2.1}$$

for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$, $n \in \mathbb{N}$. The least c such that (2.1) is satisfied is called the R-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.

Denote by r_j the j-th Rademacher function, that is $r_k(t) := sign(\sin(2^k\pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector-valued function $t \to r_j(t)x$. An equivalent definition using the Rademacher functions replaces (2.1) by

$$\Big\|\sum_{k=1}^{n} r_k \otimes T_k x_k\Big\|_{L^2((0,1);Y)} \leq c\Big\|\sum_{k=1}^{n} r_k \otimes x_k\Big\|_{L^2((0,1);X)}. \tag{2.2}$$

In the next proposition we summarize some importante properties concerning R-bounded sets. For a proof and related information, we refer the reader to the monograph [19] by Denk, Hieber and Prüss.

Proposition 2.3. *The following properties hold:*

- *Given a R-bounded set $\mathcal{T} \subset \mathcal{B}(X,Y)$, it follows that*
$$\sup\{||T|| : T \in \mathcal{T}\} \leq R(\mathcal{T}).$$
- *If X and Y are Hilbert spaces, R- boundedness is equivalent to uniform boundedness.*
- *Given X, Y Banach spaces, then if $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X,Y)$ are R-bounded, then*
$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$
is also R-bounded and $R(\mathcal{T} + \mathcal{S}) \leq R(\mathcal{T}) + R(\mathcal{S})$.
- *Given X, Y, Z Banach spaces, then if $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ are R-bounded, then*
$$\mathcal{T}\mathcal{S} = \{TS : T \in \mathcal{T}, S \in \mathcal{S}\}$$
is also R-bounded and $R(\mathcal{T}\mathcal{S}) \leq R(\mathcal{T})R(\mathcal{S})$.
- *A set $\mathcal{T} \in \mathcal{B}(X)$ defined by $\mathcal{T} = \{\lambda I : \lambda \in \Omega\}$ with Ω a bounded set is R-bounded.*

Let A, B, M, N be closed linear operators defined on a Banach space X. We now introduce the notion of M-resolvent of A and B. Under the assumption that $D(A) \cap D(B) \subset D(M) \cap D(N)$, the M-resolvent of A and B is defined as

$$\rho_{M,N}(A,B) := \{s \in \mathbb{R} : \alpha is^3 M + s^2 N + \beta A + \gamma isB + isG_s + F_s : [D(A) \cap D(B)] \to X$$

$$\text{is invertible and } [\alpha is^3 M + s^2 N + \beta A + \gamma isB + isG_s + F_s]^{-1} \in \mathcal{B}(X)\}. \tag{2.3}$$

Here $[D(A) \cap D(B)]$ denotes a Banach space under the norm $\|x\|_{[D(A)\cap D(B)]} := \|x\| + \|Ax\| + \|Bx\|$.

Let $\{T_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ be a sequence of operators. We set

$$\Delta^0 T_k = T_k, \quad \Delta T_k = \Delta^1 T_k = T_{k+1} - T_k$$

and for $n \in \mathbb{N}$ with $n \geq 2$ we have

$$\Delta^n T_k = \Delta(\Delta^{n-1} T_k).$$

Definition 2.4. [33] *We say that a sequence $\{T_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is M-bounded of order n ($n \in \mathbb{N} \cup \{0\}$), if*

$$\sup_{0\leq l\leq n}\sup_{k\in\mathbb{Z}} ||k^l \Delta^l T_k|| < \infty. \tag{2.4}$$

To be more explicit when $n=0$, M-boundedness of order n for $\{T_k\}$ simply means that $\{T_k\}$ is bounded. For $n=1$ this is equivalent to

$$\sup_{k\in\mathbb{Z}} ||T_k|| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} ||k\,(T_{k+1}-T_k)|| < \infty. \tag{2.5}$$

When $n=2$ we require in addition to (2.5) that

$$\sup_{k\in\mathbb{Z}} ||k^2\,(T_{k+2}-2T_{k+1}+T_k)|| < \infty. \tag{2.6}$$

Remark 2.5. *(i) Analogously, we define M- boundedness of order n in case of sequences $\{a_k\}_{k\in\mathbb{Z}}$ of real or complex numbers (this amounts to taking $M_k = a_k I$ in $\mathcal{B}(X)$).*

(ii) Note that if $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are M-bounded of order n then $\{M_k \pm N_k\}_{k\in\mathbb{Z}}$ is M-bounded of order n. In fact, the set of $n-$bounded sequences is a vector space. This is obvious from the definition.

(iii) If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are sequences in $\mathcal{B}(Y,Z)$ and $\mathcal{B}(X,Y)$ that are M-bounded of order n, then $\{M_k\ N_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Z)$ is also M-bounded of the same order.

Remark 2.6. *A simple but very useful rule for the Δ operator acting on the product of two sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{T_k\}_{k\in\mathbb{Z}}$ is the following:*

$$\Delta(a_kT_k) = \Delta(a_k)T_k + a_{k+1}\Delta T_k, \quad k\in\mathbb{Z}. \tag{2.7}$$

In particular, applying this rule to the identity $\frac{1}{a_k}a_k = 1$ we get

$$\Delta(\frac{1}{a_k}) = -\frac{\Delta(a_k)}{a_ka_{k+1}}. \tag{2.8}$$

The following definition was first considered in [33].

Definition 2.7. *A sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ is called n-regular if the sequences $\{k^p\frac{\Delta^p c_k}{c_k}\}_{k\in\mathbb{Z}}$ are bounded for all $p=1,\dots,n$.*

We prove the following lemma that will be needed later.

Lemma 2.8. *Let $(c_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ be a 2-regular sequence. If we denote by $d_k = \frac{\Delta c_k}{c_k}$, then the sequence $(k^2\Delta d_k)_{k\in\mathbb{Z}}$ is bounded.*

Proof. A simple computation using (2.7) and (2.8) shows that

$$\Delta d_k = \Delta\left(\frac{1}{c_k}\right)\Delta c_k + \frac{1}{c_{k+1}}\Delta^2 c_k = -\frac{\Delta c_k}{c_k}\frac{\Delta c_k}{c_{k+1}} + \frac{\Delta^2 c_k}{c_{k+1}},$$

then $k^2\Delta d_k = -\frac{k\Delta c_k}{c_k}\frac{k\Delta c_k}{c_{k+1}} + \frac{k^2\Delta^2 c_k}{c_k}$ is bounded from the 2-regularity of $(c_k)_{k\in\mathbb{Z}}$. □

We recall the following definition.

Definition 2.9. *We say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is MR-bounded of order n, if for each $0\le l\le n$ the set*

$$\{k^l\Delta^l M_k : k\in\mathbb{Z}\}, \tag{2.9}$$

is R-bounded.

Remark 2.10. *A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is MR-bounded of order 1 if the sets*

$$\{M_k : k \in \mathbb{Z}\} \text{ and } \{k\,(M_{k+1} - M_k) : k \in \mathbb{Z}\} \tag{2.10}$$

are R-bounded.

If in addition we have that the set

$$\{k^2\,(M_{k+1} - 2M_k + M_{k-1}) : k \in \mathbb{Z}\} \tag{2.11}$$

is R-bounded then $\{M_k\}_{k\in\mathbb{Z}}$ is MR-bounded of order 2.

Remark 2.11. *In Hilbert spaces MR-bounded and M-bounded are identical concepts. In general, MR-bounded implies R-bounded which in turn implies boundedness. Moreover, note that MR-boundedness implies M-boundedness.*

3. Well-posedness in L^p-spaces

Let $1 \le p < \infty$ and X be a Banach space. In this section, we want to give optimal conditions that can describe the well-posedness of the problem

$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{3.12}$$

in 2π-periodic vector valued L^p-spaces. In other words, we want to obtain a complete characterization on the existence, uniqueness and well posedness of the problem only in terms of the data of the problem. Here A, B, N and M are closed linear operators such that $D(A) \cap D(B) \subset D(N) \cap D(M)$ and $F, G \in \mathcal{B}(L^p(-2\pi, 0); X), X)$. We first recall some definitions and results that will be needed to prove the main theorem of this section. We define the vector-valued function spaces:

$$W^{n,p}_{per}(\mathbb{T}, X) := \{u \in L^p(\mathbb{T}, X) : \text{there exists } v \in L^p(\mathbb{T}, X),\ \hat{v}(k) = (ik)^n \hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

We define the following space of maximal regularity.

$$\begin{aligned} &S_p(A, B, M, N) := \\ &\{u \in W^{1,p}_{per}(\mathbb{T}, [D(A) \cap D(B)]) \cap L^p(\mathbb{T}, [D(A) \cap D(B)]) : \\ &u' \in L^p(\mathbb{T}, [D(A) \cap D(B)]), Mu' \in W^{2,p}_{per}(\mathbb{T}, X), Nu' \in W^{1,p}_{per}(\mathbb{T}, X)\}. \end{aligned}$$

The space $S_p(A, B, M, N)$ is a Banach space with the norm

$$\begin{aligned} ||u||_{S_p(A,B,M,N)} :=& ||u||_p + \|u'\|_p + \|Bu'\|_p + ||Au||_p + ||(Nu')'||_p \\ &+ ||(Mu')''||_p + ||Nu'||_p + \|Mu'\|_p. \end{aligned}$$

Denote $e_\lambda(\theta) = e^{i\lambda\theta}$ for all $\lambda \in \mathbb{R}$ and $\theta \le 0$, and define $(e_\lambda \otimes x)(t) := e_\lambda(t)x,\ x \in X,\ t \in \mathbb{R}$. Clearly $e_\lambda \otimes x \in L^p(-2\pi, 0), X)$ for each $x \in X$ and $\lambda \in \mathbb{R}$. We define the one-parameter families of operators $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{G_\lambda\}_{\lambda\in\mathbb{R}}$ by

$$F_\lambda(x) = F(e_\lambda \otimes x) \quad \text{and} \quad G_\lambda(x) = G(e_\lambda \otimes x), \quad \lambda \in \mathbb{R}, \quad x \in X. \tag{3.13}$$

Remark 3.12. *From (3.13) we have that $F_k x := F(e_k \otimes x)$, $G_k x := G(e_k \otimes x)$, $x \in X$ are bounded and linear operators for each $k \in \mathbb{Z}$. Moreover, when $u \in L^p(\mathbb{T}, X)$ we have*

$$\widehat{Fu.}(k) = F_k \hat{u}(k), \quad \widehat{Gu.}(k) = G_k \hat{u}(k) \tag{3.14}$$

This implies that $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers (cf. Definition 2.1), and

$$\|Fu_t\| \le \|F\|\|u.\|_p = \|F\|\|u\|_p,$$

and thus $Fu., Gu. \in L^p(\mathbb{T}, X)$. This justifies why we do not consider this property in the definition of the space of maximal regularity.

We include the following useful and stronger result that is contained in the proof of Lemma 3.2 in [27].

Lemma 3.13. *The sets $\{F_k : k \in \mathbb{Z}\}$ and $\{G_k : k \in \mathbb{Z}\}$ are R-bounded.*

The following two theorems establish the equivalence between R-boundedness and the fact of being an L^p-multiplier. They can be found in [6].

Theorem 3.14. *Let X, Y be UMD spaces. If a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is MR-bounded of order 1, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier whenever $1 < p < \infty$.*

Theorem 3.15. *Let X, Y be Banach spaces, $1 \leq p < \infty$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ be an L^p-Fourier multiplier. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

We next introduce the following definition.

Definition 3.16. *Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{T}, X)$. We call $u \in S_p(A, B, M, N)$ a strong L^p-solution of equation (3.12) if it satisfies (3.12) for all $t \in \mathbb{T}$. We say that equation (3.12) is L^p-well-posed if for each $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p-solution of equation (3.12).*

As a consequence of the Closed Graph Theorem we get the following estimate:

Remark 3.17. *If equation (3.12) is L^p-well-posed, then there exists a constant $C > 0$ such that for each $f \in L^p(\mathbb{T}, X)$, we have*

$$||u||_{S_p(A,B,M,N)} \leq C||f||_{L^p}.$$

The following is the first main result in this paper. It provides an important criterion for MR-boundedness in the context of maximal regularity for abstract evolution equations.

Theorem 3.18. *Let $T : D(T) \subset X \to X$ be a closed linear operator defined in a Banach space X. For each $k \in \mathbb{Z}$ let $H_k : X \to D(T)$ be a sequence of bounded and linear operators such that $0 \in \rho(H_k)$ for all $k \in \mathbb{Z}$. Suppose that $(c_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is a 1-regular sequence and denote*

$$M_k := c_k T H_k, \tag{3.15}$$

and

$$L_k := (H_k^{-1} - H_{k+1}^{-1})H_k. \tag{3.16}$$

If $\{M_k : k \in \mathbb{Z}\}$ and $\{kL_k : k \in \mathbb{Z}\}$ are R-bounded sets, then $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded of order 1. If, in addition, $(c_k)_{k\in\mathbb{Z}}$ is 2-regular and the set $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ is R-bounded, then $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded of order 2.

Proof. A computation shows that

$$\begin{aligned} \Delta M_k &= (\Delta c_k)TH_k + c_{k+1}T\Delta H_k = \frac{\Delta c_k}{c_k} M_k + c_{k+1}T(H_{k+1} - H_k) \\ &= \frac{\Delta c_k}{c_k} M_k + c_{k+1}TH_{k+1}(I - H_{k+1}^{-1}H_k) \\ &= \frac{\Delta c_k}{c_k} M_k + M_{k+1}L_k. \end{aligned} \tag{3.17}$$

Then $k\Delta M_k = \frac{k\Delta c_k}{c_k} M_k + M_{k+1} kL_k$. Since $(c_k)_{k\in\mathbb{Z}}$ is 1-regular and kL_k is R-bounded, it follows that $\{k\Delta M_k : k \in \mathbb{Z}\}$ is R-bounded. It shows that $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded of order 1.

Let denote $d_k = \frac{\Delta c_k}{c_k}$, then from (3.17) we obtain

$$\Delta^2 M_k = (\Delta d_k) M_k + \frac{\Delta c_{k+1}}{c_{k+1}} (\Delta M_k) + (\Delta M_{k+1})(L_k) + M_{k+2}(\Delta L_k).$$

Therefore, we have

$$k^2\Delta^2 M_k = (k^2\Delta d_k) M_k + \frac{k\Delta c_{k+1}}{c_{k+1}} (k\Delta M_k) + (k\Delta M_{k+1})(kL_k) + M_{k+2}(k^2\Delta L_k).$$

Since $(c_k)_{k\in\mathbb{Z}}$ is 2-regular, $(k^2\Delta d_k)_{k\in\mathbb{Z}}$ is bounded from Lemma 2.8 and $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ is R-bounded by hypothesis. It follows by Proposition 2.3 that $\{k^2\Delta^2 M_k : k \in \mathbb{Z}\}$ is R-bounded. Therefore, $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded of order 2.

□

Let A, B and M, N be closed linear operators such that $D(A) \cap D(B) \subset D(M) \cap D(N)$. Assume $\mathbb{Z} \subset \rho_{M,N}(A, B)$. We denote

$$N_k := -[b_k M + a_k N + \beta A + ik\gamma B + ikG_k + F_k]^{-1}, \quad a_k = k^2, \quad b_k = i\alpha k^3, \quad k \in \mathbb{Z}, \tag{3.18}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are fixed constants.

The following proposition is the key technical tool for the proof of our second main result.

Proposition 3.19. *Let A, B and M be closed linear operators defined on a UMD space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$ and $\alpha, \beta, \gamma \in \mathbb{R}$. If $\mathbb{Z} \subset \rho_{M,N}(A, B)$ and $\{ik^3 MN_k : k \in \mathbb{Z}\}$, $\{k^2 NN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$ and $\{kG_k : k \in \mathbb{Z}\}$ are R-bounded sets, then the sets $\{k(N_k^{-1} - N_{k+1}^{-1})N_k\}_{k\in\mathbb{Z}}$, $\{k\Delta(k^3 MN_k)\}_{k\in\mathbb{Z}}$, $\{k\Delta(k^2 NN_k)\}_{k\in\mathbb{Z}}$,$\{k\Delta(kBN_k)\}_{k\in\mathbb{Z}}$ and $\{k\Delta(kN_k)\}_{k\in\mathbb{Z}}$ are R-bounded sets.*

Proof. Let $M_k = ik^3 MN_k$. In order to show that M_k is an L^p-multiplier it is sufficient to show that $\{k\Delta M_k : k \in \mathbb{Z}\}$ is R-bounded. We apply Theorem 3.18 with $c_k = ik^3$, which is clearly a 1-regular sequence, $H_k = N_k$ and $T = M$. By hypothesis $\{M_k : k \in \mathbb{Z}\}$ is R-bounded, then it only remains to show that $\{kL_k : k \in \mathbb{Z}\}$ is R-bounded. Indeed,

$$\begin{aligned} kL_k &= k({N_k}^{-1} - N_{k+1}^{-1})N_k \\ &= k[\Delta b_k M + \Delta a_k N + i\gamma B + i(k+1)G_{k+1} - ikG_k + (F_{k+1} - F_k)]N_k \\ &= k[\Delta b_k M + (2k+1)N + i\gamma B + i(k+1)G_{k+1} - ikG_k + (F_k - F_{k+1})]N_k \\ &= \alpha \frac{k\Delta b_k}{b_k} M_k + \frac{2k+1}{k}(k^2 NN_k) + i\gamma(kBN_k) + i(k+1)G_{k+1}(kN_k) - i(kG_k)(kN_k) \\ &+ (F_k - F_{k+1})(kN_k). \end{aligned}$$

Note that the set $\{F_k\}_{k\in\mathbb{Z}}$ is bounded by Lemma 3.13. By hypothesis and Proposition 2.3 it follows that $\{kL_k : k \in \mathbb{Z}\}$ is R-bounded. The R-boundedness of $\{k\Delta(kBN_k) : k \in \mathbb{Z}\}$, $\{k\Delta(kN_k) : k \in \mathbb{Z}\}$ and $\{k\Delta(k^2 NN_k) : k \in \mathbb{Z}\}$ follows similarly applying Theorem 3.18 with $c_k = k$, $T = B$ and $H_k = N_k$ in the first case, $c_k = k$, $T = I$ and $H_k = N_k$ in the second case and $c_k = k^2$, $T = N$ and $H_k = N_k$ in the third case. □

Remark 3.20. *From the proof, we observe that theorem 3.18 and proposition 3.19 also holds if "R-bounded" is replaced by "uniformly bounded".*

As a consequence, we obtain immediately the following corollary.

Corollary 3.21. *Let A, B and M be closed linear operators defined on a UMD space X such that $D(A)\cap D(B)\subset D(M)\cap D(N)$ and $\alpha,\beta,\gamma\in\mathbb{R}$. If $\mathbb{Z}\subset\rho_{M,N}(A,B)$ and $\{ik^3MN_k : k\in\mathbb{Z}\}$, $\{kBN_k : k\in\mathbb{Z}\}$, $\{k^2NN_k : k\in\mathbb{Z}\}$, $\{kN_k : k\in\mathbb{Z}\}$ and $\{kG_k : k\in\mathbb{Z}\}$ are R-bounded sets, then $(ik^3MN_k)_{k\in\mathbb{Z}}$, $(k^2NN_k)_{k\in\mathbb{Z}}$, $(kBN_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.*

We now show the second main result of this section that provides a computable criterion to characterize the well-posedness of equation (3.12).

Theorem 3.22. *Let $1<p<\infty$ and $\alpha,\beta,\gamma\in\mathbb{R}$. Assume A, B and M, N are closed linear operators defined on a UMD space X such that $D(A)\cap D(B)\subset D(M)\cap D(N)$. Assume that $\{kG_k : k\in\mathbb{Z}\}$ is R-bounded. The following assertions are equivalent:*

- (i) *Equation (3.12) is L^p-well posed;*
- (ii) *$\mathbb{Z}\subset\rho_{M,N}(A,B)$ and the sets $\{ik^3\alpha MN_k : k\in\mathbb{Z}\}$, $\{k^2NN_k : k\in\mathbb{Z}\}$, $\{k\gamma BN_k : k\in\mathbb{Z}\}$ $\{kN_k : k\in\mathbb{Z}\}$ are R-bounded.*

Proof. We first prove $(i)\implies(ii)$. Assume that equation (3.12) is L^p-well posed. Let $k\in\mathbb{Z}$ and $y\in X$ be given. Then we define $f\in L^p(\mathbb{T},X)$ as $f(t)=e^{ikt}y$. It is clear that $\hat{f}(k)=y$ and $\hat{f}(n)=0$ for $n\neq k$. By hypothesis, there exists a unique $u\in S_p(A,B,M,N)$ that satisfies:

$$\alpha(Mu)'''(t)+(Nu)''(t)=\beta Au(t)+\gamma Bu'(t)+Gu'_t+Fu_t+f(t),\quad t\in[0,2\pi]. \tag{3.19}$$

Observe that by the linearity of F and G we get that

$$\widehat{Gu'_t}(k)=G(e^{ik\theta}\hat{u}'(k))=G(e^{ik\theta}ik\hat{u}(k))=ikG_k\hat{u}_k$$

and

$$\widehat{Fu_t}(k)=F(e^{ik\theta}\hat{u}(k))=F_k\hat{u}_k\quad k\in\mathbb{Z}.$$

Taking Fourier transform in both sides of (3.19) we get:

$$-[b_kM+a_kN+\beta A+ik\gamma B+ikG_k+F_k]\hat{u}(k)=y \tag{3.20}$$

and

$$-[b_nM+a_nN+\beta A+\gamma(in)B+(in)G_n+F_n]\hat{u}(n)=0,\quad n\neq k. \tag{3.21}$$

This shows that $-[b_kM+a_kN+\beta A+\gamma(ik)B+(ik)G_k+F_k]$ is surjective. In order to show injectivity, let $x\in D(A)\cap D(B)$ be such that

$$-[b_kM+a_kN+\beta A+\gamma(ik)B+(ik)G_k+F_k]x=0.$$

Let $u(t)=e^{ikt}x$ for $t\in\mathbb{T}$. It is clear that $u\in S_p(A,B,M,N)$ and u solves (3.12) for $f=0$. Then by uniqueness, $x=0$ and injectivity follows directly. Therefore, $-[b_kM+a_kN+\beta A+\gamma(ik)B+(ik)G_k+F_k]$ is bijective from $D(A)\cap D(B)$ onto X. Moreover, $-[b_kM+a_kN+\beta A+\gamma(ik)B+(ik)G_k+F_k]^{-1}\in\mathcal{B}(X)$. Indeed, given $y\in X$ and $k\in\mathbb{Z}$ let $f(t)=e^{ikt}y$ and let u be the corresponding solution of (3.12) for f. Then

$$\hat{u}(n)=\begin{cases} 0, & \text{if}\quad n\neq k;\\ -[b_kM+a_kN+\beta A+\gamma(ik)B+(ik)G_k+F_k]^{-1}y, & \text{if}\quad n=k.\end{cases}$$

This implies $u(t)=-e^{-ikt}[b_kM+a_kN+\beta A+\gamma(ik)B+(ik)G_k+F_k]^{-1}y$ by uniqueness. By remark 3.17 there exists $C>0$ independent of y and k such that

$$\|u\|_{S_p(A,B,M,N)}\le C||f||_{L^p}.$$

As a consequence,

$$||[b_kM + a_kN + \beta A + \gamma(ik)B + (ik)G_k + F_k]^{-1}|| \leq C,$$

for all $k \in \mathbb{Z}$. This proves the claim. We have shown that $\mathbb{Z} \subset \rho_{M,N}(A,B)$. Let $M_k = ik^3MN_k$, $S_k = k^2NN_k$, $H_k = kBN_k$ and $R_k = kN_k$ with $k \in \mathbb{Z}$, where N_k is defined in (3.18). Let show that $(M_k)_{k\in\mathbb{Z}}, (S_k)_{k\in\mathbb{Z}}, (H_k)_{k\in\mathbb{Z}}$ and $(R_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. Given $f \in L^p(\mathbb{T}, X)$, there exists $u \in S_p(A,B,M,N)$ which is a solution of equation (3.19) by assumption. Taking Fourier transforms on both sides of (3.19), we get that $\hat{u}(k) \in D(A) \cap D(B)$ and

$$-[b_kM + a_kN + \beta A + \gamma(ik)B + (ik)G_k + F_k]\hat{u}(k) = \hat{f}(k), \quad k \in \mathbb{Z}.$$

Since $-[b_kM+a_kN+\beta A+\gamma(ik)B+(ik)G_k+F_k]$ is invertible it follows that $\hat{u}(k) = -N_k\hat{f}(k), k \in \mathbb{Z}$. As $u \in S_p(A,B,M,N)$ we get that

$$[\widehat{(Mu)'''}](k) = -ik^3M\hat{u}(k) = ik^3MN_k\hat{f}(k) = M_k\hat{f}(k) \tag{3.22}$$

$$[\widehat{(Nu)''}](k) = -k^2N\hat{u}(k) = k^2NN_k\hat{f}(k) = S_k\hat{f}(k) \tag{3.23}$$

$$\widehat{u'}(k) = ik\hat{u}(k) = ikN_k\hat{f}(k) = -iR_k\hat{f}(k) \quad \text{and} \tag{3.24}$$

$$\widehat{Bu'}(k) = ikB\hat{u}(k) = ikBN_k\hat{f}(k) = -iH_k\hat{f}(k). \tag{3.25}$$

Finally, since $(Mu)''', (Nu)'', u'$ and $Bu' \in L^p(\mathbb{T}, X)$ we get that $(M_k)_{k\in\mathbb{Z}}, (S_k)_{k\in\mathbb{Z}}, (H_k)_{k\in\mathbb{Z}}$ and $(R_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers, proving the claim. Then, by Theorem 3.15, we conclude that the sets $\{M_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$, $\{H_k : k \in \mathbb{Z}\}$ and $\{R_k : k \in \mathbb{Z}\}$ are R-bounded, proving (ii).

Let now show $(ii) \implies (i)$. We assume that $\mathbb{Z} \subset \rho_{M,N}(A,B)$ and the sets $\{ik^3MN_k : k \in \mathbb{Z}\}$, $\{k^2NN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded. Let $M_k = ik^3MN_k$, $S_k = k^2NN_k$, $H_k = kBN_k$ and $R_k = kN_k$ with $k \in \mathbb{Z}$. It follows from Corollary 3.21 that $(M_k)_{k\in\mathbb{Z}}, (S_k)_{k\in\mathbb{Z}}, (H_k)_{k\in\mathbb{Z}}$ and $(R_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.

Note that:

$$-I_X = \alpha M_k + S_k + \beta AN_k + \gamma iH_k + ikG_kN_k + F_kN_k. \tag{3.26}$$

Next, observe that the R-boundedness of the set $\{kN_k\}_{k\in\mathbb{Z}}$ implies that the set $\{k(N_{k+1}-N_k)\}$ is also R-bounded. It follows from Theorem 3.14 that $\{N_k\}_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier. In particular, $N_k \in \mathcal{B}(X, [D(A) \cap D(B)])$.

From (3.26) we deduce that $\{AN_k\}_{k\in\mathbb{Z}}$ is also an L^p-Fourier multiplier since the sum and product of L^p-Fourier multipliers is still an L^p-Fourier multiplier.

Then for all $f \in L^p(\mathbb{T}, X)$ there exist $w, u_2 \in L^p(\mathbb{T}, [D(A) \cap D(B)])$ and $u_1 \in L^p(\mathbb{T}, X)$ satisfying

$$\hat{w}(k) = N_k\hat{f}(k), \quad \hat{u}_1(k) = AN_k\hat{f}(k), \quad \hat{u}_2(k) = ikN_k\hat{f}(k). \tag{3.27}$$

Consequently, $\hat{u}_2(k) = ik\hat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w \in W^{1,p}_{per}(\mathbb{T}; [D(A) \cap D(B)])$ [6, Lemma 2.1] and $w'(t) = u_2(t)$ a.e. [6, Lemma 3.1]. In particular, $w' \in L^p(\mathbb{T}, [D(A) \cap D(B)])$.

Since $\{ikBN_k\}_{k\in Z}$ is an L^p-Fourier multiplier, we have that there exists $v \in L^p(\mathbb{T}, X)$ such that

$$\hat{v}(k) = B(ikN_k)\hat{f}(k) = B\hat{u}_2(k) = B\widehat{w'}(k), \quad k \in \mathbb{Z}. \tag{3.28}$$

It follows from [6, Lemma 3.1] that $w'(t) \in D(B)$ and $Bw'(t) = v(t)$ a.e. In addition, $Bw' \in L^p(\mathbb{T}, X)$.

Since the set $\{k^2NN_k\}_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier by hypothesis, we deduce that there exists $u \in L^p(\mathbb{T}, X)$ such that

$$\hat{u}(k) = -k^2NN_k\hat{f}(k) = ikN\hat{u}_2(k) = ikN\widehat{w'}(k) = ik\widehat{Nw'}(k), \tag{3.29}$$

where we have used that N is closed. It implies that $Nw' \in W^{1,p}_{per}(\mathbb{T}; X)$ and $u(t) = (Nw')'(t)$ a.e. In particular. $(Nw')' \in L^p(\mathbb{T}, X)$.

Analogously, since the set $\{k^3MN_k\}_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier, there exists $r \in L^p(\mathbb{T}, X)$ such that

$$\hat{r}(k) = -ik^3MN_k\hat{f}(k) = -k^2M\hat{u}_2(k) = (ik)^2\widehat{Mw'}(k), \tag{3.30}$$

because M is closed. We deduce that $Mw' \in W^{2,p}_{per}(\mathbb{T}, X)$ and $(Mw')''(t) = r(t)$ a.e. Moreover, $(Mw')'' \in L^p(\mathbb{T}, X)$.

We note that the sets $\{G_k\}_{k\in\mathbb{Z}}$ and $\{F_k\}_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers by Remark 3.12. Thus $\{ikG_kN_k\}_{k\in\mathbb{Z}}$ and $\{F_kN_k\}_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as the product of L^p-Fourier multipliers is still an L^p-Fourier multiplier. Therefore, exists $s_1 \in L^p(\mathbb{T}, X)$ such that

$$\hat{s}_1(k) = ikG_kN_k\hat{f}(k) = G_k\hat{u}_2(k) = G_k\widehat{w'}(k) = \widehat{Gw'_{\cdot}}(k), \quad k \in \mathbb{Z}.$$

We conclude that $Gw'_{\cdot} \in L^p(\mathbb{T}, X)$. Analogously, we have that there exists $s_2 \in L^p(\mathbb{T}, X)$ such that

$$\hat{s}_2(k) = F_kN_k\hat{f}(k) = F_k\hat{w}(k) = \widehat{Fw_{\cdot}}(k),$$

and hence $Fw_{\cdot} \in L^p(\mathbb{T}, X)$. We have shown that $w \in S_p(A, B, M, N)$. Finally, from the identity (3.26) we obtain

$$\begin{aligned}
\alpha\widehat{(Mw')''}(k) + \widehat{(Nw')'}(k) &= \alpha\hat{r}(k) + \hat{u}(k) = (-\alpha ik^3MN_k - k^2NN_k)\hat{f}(k) = -(\alpha M_k + S_k)\hat{f}(k) \\
&= (I_X + \beta AN_k + i\gamma H_k + ikG_kN_k + F_kN_k)\hat{f}(k) \\
&= \hat{f}(k) + \beta A\hat{w}(k) + \gamma B\widehat{w'}(k) + \widehat{Gw'_{\cdot}}(k) + \widehat{Fw_{\cdot}}(k).
\end{aligned}$$

This implies that

$$\alpha(Mw')''(t) + (Nw')'(t) = f(t) + \beta Aw(t) + \gamma Bw'(t) + Gw'_t + Fw_t,$$

by the uniqueness theorem (see [6, Pag. 314]).

It only remains to prove uniqueness. Indeed, let $w \in S_p(A, B, M, N)$ satisfying

$$\alpha(Mw')''(t) + (Nw')'(t) = \beta Aw(t) + \gamma Bw'(t) + Gw'_t + Fw_t, t \in \mathbb{T}. \tag{3.31}$$

Taking Fourier transform in (3.31), we get that $-[b_kM + a_kN + \beta A + \gamma(ik)B + (ik)G_k + F_k]\hat{w}(k) = 0$ for all $k \in \mathbb{Z}$. Hence $w = 0$ as $\mathbb{Z} \subset \rho_{M,N}(A, B)$ by hypothesis. Thus, (3.12) is L^p-well-posed. This completes the proof.

□

Remark 3.23. *We point out the following*

(i) *Our result avoids the coercivity condition on the pair (A, B) assumed in* [27, Section 3] *where the case $M = 0, N = I$ is studied. In contrast, we have to assume extra conditions on R-boundedness.*

(ii) *The hypothesis of R-boundedness of the set $\{kG_k : k \in \mathbb{Z}\}$ is natural as it is required in Theorem 4.7 of* [27] *when showing maximal regularity in Triebel-Lizorkin spaces.*

(iii) *L^p-well posedness does not depend on the parameter p, that is, if equation* (3.12) *is L^p-well posed for some $1 < p < \infty$ then it is L^p-well posed for all $1 < p < \infty$.*

4. Well-posedness in Besov and Triebel-Lizorkin spaces

In this section, we show the well-posedness of problem (3.12) in periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and periodic Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$. We first introduce the notion of vector-valued periodic Besov spaces (see [7]).

Let $\mathbb{S}(\mathbb{R})$ be the space of all rapidly decreasing smooth functions on $\mathbb{R}$ and $\mathcal{D}(\mathbb{T})$ the space of infinitely differentiable functions defined on $\mathbb{T}$ endowed with the locally convex topology defined by the seminorms

$$||f||_n = \sup_{x\in\mathbb{T}} |f^{(n)}(x)| \quad \text{for} \quad n \in \mathbb{N}.$$

Let $\mathcal{D}'(\mathbb{T}, X) := \mathcal{B}(\mathcal{D}(\mathbb{T}), X)$, we consider the dyadic-like sets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \le 2\}, \quad I_k = \{t \in \mathbb{R} : 2^{k-1} \le |t| \le 2^{k+1}\}, \quad k \in \mathbb{N}.$$

Let $\phi(\mathbb{R})$ be the set of systems $\phi = (\phi_k)_{k\in\mathbb{N}} \subset \mathbb{S}(\mathbb{R})$ such that $\sup(\phi_k) \subset \overline{I_k}$ for each $k \in \mathbb{N}$, $\sum_{k\in\mathbb{N}} \phi_k(x) = 1$ for $x \in \mathbb{R}$ and for each $\alpha \in \mathbb{N}$, $\sup_{x\in\mathbb{R},k\in\mathbb{N}} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty$. Let $\phi = (\phi_k)_{k\in\mathbb{N}} \subset \phi(\mathbb{R})$ be fixed, for all $1 \le p, q \le \infty$, $s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T}, X) = \left\{ f \in \mathcal{D}'(\mathbb{T}, X) : ||f||_{B^s_{p,q}} := \left(\sum_{j\ge0} 2^{sjq} \left|\left| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \right|\right|_p^q \right)^{1/q} < \infty \right\}.$$

The space $B^s_{p,q}(\mathbb{T}, X)$ endowed with the norm $||.||_{B^s_{p,q}}$ is a Banach space.

Remark 4.24. *We note the following*

(i) *The space $B^s_{p,q}(\mathbb{T}, X)$ is independent of the choice of ϕ.*
(ii) *If $s_1 \le s_2$ then $B^{s_1}_{p,q}(\mathbb{T}, X) \subset B^{s_2}_{p,q}(\mathbb{T}, X)$ and the embedding is continuous.*
(iii) *If $s > 0$, then $B^s_{p,q}(\mathbb{T}, X) \subset L^p(\mathbb{T}, X)$, and $f \in B^{s+1}_{p,q}(\mathbb{T}, X)$ if and only if f is differentiable a.e. on $\mathbb{T}$ and $f' \in B^s_{p,q}(\mathbb{T}, X)$.*

In particular, (iii) implies that if $u \in B^s_{p,q}(\mathbb{T}, X)$ is such that there exists $v \in B^s_{p,q}(\mathbb{T}, X)$ satisfying $\hat{v}(k) = ik\hat{u}(k)$ when $k \in Z$, then $u \in B^{s+1}_{p,q}(\mathbb{T}, X)$ and $u' = v$ [6, Lemma 2.1].

In this section we consider the equation

$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{4.32}$$

in 2π-periodic vector valued $B^s_{p,q}$-spaces. Once again A, B, N and M are closed linear operators such that $D(A) \cap D(B) \subset D(N) \cap D(M)$ and $F, G \in \mathcal{B}(B^s_{p,q}(-2\pi, 0); X), X)$.

Remark 4.25. *As observed in* [14, Section 3]*, $F_k, G_k \in \mathcal{B}(X)$ and there exists a constant $C > 0$ such that*

$$\|F_k\| \le C\|F\|, \quad \|G_k\| \le C\|G\|, \quad k \in \mathbb{Z}.$$

Moreover, when $u \in B^s_{p,q}(\mathbb{T}, X)$ then

$$\widehat{Fu.}(k) = F_k\hat{u}(k), \quad \widehat{Gu.}(k) = G_k\hat{u}(k), \quad k \in \mathbb{Z}.$$

However, in contrast with vector-valued Lebesgue spaces, the functions $Fu.$ and $Gu'_.$ are not necessarily in $B^s_{p,q}(\mathbb{T}, X)$ for $1 \le p, q \le \infty$, $s > 0$. This particularity, will require the use of new tools in the proof of the main result in this section.

Let $1 \le p, q \le \infty$, $s > 0$. We define the maximal regularity space that describe the $B^s_{p,q}$-well posedness of the equation (4.32) by

$$S_{p,q,s}(A,B,M,N) := \{u \in B^{s+1}_{p,q}(\mathbb{T},[D(A)\cap D(B)]) \cap B^{s}_{p,q}(\mathbb{T},[D(A)\cap D(B)]) : \\ Mu' \in B^{s+2}_{p,q}(\mathbb{T},X), Nu' \in B^{s+1}_{p,q}(\mathbb{T},X), Au, Bu', Fu., Gu'_. \in B^{s}_{p,q}(\mathbb{T},X)\}.$$

The vectorial space $S_{p,q,s}(A,B,M,N)$ is a Banach space with the norm

$$||u||_{S_{p,q,s}(A,B,M,N)} := ||u||_{B^s_{p,q}} + ||u'||_{B^s_{p,q}} + ||Au||_{B^s_{p,q}} + ||Nu'||_{B^s_{p,q}} + ||(Nu')'||_{B^s_{p,q}} + ||(Mu')''||_{B^s_{p,q}} \\ + ||Bu'||_{B^s_{p,q}} + ||Mu'||_{B^s_{p,q}} + ||Fu.||_{B^s_{p,q}} + ||Gu'_.||_{B^s_{p,q}}.$$

We now provide the formal definition of $B^s_{p,q}$- well posedness of equation (4.32).

Definition 4.26. *Let $1 \le p, q < \infty$, $s > 0$ and $f \in B^s_{p,q}(\mathbb{T},X)$. We call $u \in S_{p,q,s}(A,B,M,N)$ a strong $B_{p,q}$-solution of (4.32) if it satisfies (4.32) for all $t \in \mathbb{T}$. We say that (4.32) is $B^s_{p,q}$-well-posed if for each $f \in B^s_{p,q}(\mathbb{T},X)$, there exists a unique strong $B^s_{p,q}$-solution of (4.32).*

Remark 4.27. *If (4.32) is $B^s_{p,q}$-well-posed, by the Closed Graph Theorem, there exists a constant $C > 0$ such that for each $f \in B^s_{p,q}(\mathbb{T},X)$, we have*

$$||u||_{S_{p,q,s}(A,B,M,N)} \le C||f||_{B^s_{p,q}}.$$

We now introduce the notion of $B^s_{p,q}$- Fourier multiplier introduced in that will be needed for our characterization of well-posedness of equation (4.32) in Besov spaces (see [6]).

Definition 4.28. *Let X, Y be Banach spaces, $1 \le p, q < \infty$, $s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$, we say that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier if, for each $f \in B^s_{p,q}(\mathbb{T},X)$ there exists $u \in B^s_{p,q}(\mathbb{T},Y)$ such that*

$$\hat{u}(k) = M_k \hat{f}(k)$$

for all $k \in \mathbb{Z}$.

The following important result, which was proved in [7], provides a sufficient condition for an operator valued symbol to be a $B^s_{p,q}$- Fourier multiplier.

Theorem 4.29. *Let X, Y be Banach spaces. If $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is M-bounded of order 2, then for $1 \le p, q \le \infty$, $s \in \mathbb{R}$ the set $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.*

Some important properties of $B^s_{p,q}$- Fourier multipliers can be found in [7]. Some of them are the following:

Remark 4.30. *We point out that:*

(i) *The sum and product of $B^s_{p,q}$-Fourier multipliers is a $B^s_{p,q}$- Fourier multiplier too.*
(ii) *If $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier then it is uniformly bounded.*

The following result shows necessary conditions for certain sets to be $B^s_{p,q}$-Fourier multipliers and will be needed in the proof of the main theorem of this section.

Proposition 4.31. *Let A, B and M be closed linear operators defined on a UMD space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$ and $\alpha, \beta, \gamma \in \mathbb{R}$. If $\mathbb{Z} \subset \rho_{M,N}(A,B)$, $0 \in \rho(M)$ and $\{ik^3 MN_k : k \in \mathbb{Z}\}$, $\{k^2 NN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$ and $\{kG_k : k \in \mathbb{Z}\}$ are uniformly bounded, then $(ik^3 MN_k)_{k\in\mathbb{Z}}$, $(k^2 NN_k)_{k\in\mathbb{Z}}$, $(kBN_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers.*

Proof. Let $M_k = i\alpha k^3 M N_k$. In order to show that M_k is a $B^s_{p,q}$-Fourier multiplier and according Theorem 4.29 we need to prove that $\sup_{k\in\mathbb{Z}}(||M_k||+||k\Delta M_k||) < \infty$ and $\sup_{k\in\mathbb{Z}} ||k^2\Delta^2 M_k|| < \infty$. The first assertion is a consequence of the hypothesis and Proposition 3.19 (cf. Remark 3.17). In order to show the second one we apply Theorem 3.18 with $c_k = i\alpha k^3$, which is clearly a 2-regular sequence, $H_k = N_k$ and $T = M$. By hypothesis $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. Moreover, by Proposition 3.19 it follows that $\sup_{k\in\mathbb{Z}} \|kL_k\| < \infty$, then it only remains to show that $\sup_{k\in\mathbb{Z}} \|k^2\Delta L_k\| < \infty$. Indeed, we have

$$L_k = ({N_k}^{-1} - N_{k+1}^{-1})N_k = [-\Delta b_k M - \Delta a_k N - i\gamma B + ikG_k - i(k+1)G_{k+1} + (F_k - F_{k+1})]N_k.$$

Then,

$$\begin{aligned}
k^2\Delta L_k &= k^2[(b_{k+1} - b_{k+2})MN_{k+1} - (b_k - b_{k+1})MN_k] \\
&+ k^2[(a_{k+1} - a_{k+2})NN_{k+1} - (a_k - a_{k+1})NN_k] \\
&- i\gamma k^2 B\Delta N_k \\
&+ [i(k+1)G_{k+1}k^2N_{k+1} - i(k+2)G_{k+2}k^2N_{k+2} - ikG_k k^2 N_k + i(k+1)G_{k+1}k^2N_k] \\
&+ [(F_{k+1} - F_{k+2})k^2N_{k+1} - (F_k - F_{k+1})k^2N_k],
\end{aligned}$$

where $a_k = k^2$ and $b_k = i\alpha k^3$. It remains to prove that each summand in the right hand side is bounded. In fact, a calculation shows the identity

$$(b_{k+1} - b_{k+2})N_{k+1} - (b_k - b_{k+1})N_k = -(\Delta^2 b_k)N_{k+1} + \frac{\Delta b_k}{b_k}\big[(b_kN_k - b_{k+1}N_{k+1}) + N_{k+1}(\Delta b_k)\big].$$

Therefore

$$\begin{aligned}
k^2[(b_{k+1} - b_{k+2})MN_{k+1} - (b_k - b_{k+1})MN_k] &= -k^2\frac{(\Delta^2 b_k)}{b_k}\frac{b_k}{b_{k+1}}(b_{k+1}MN_{k+1}) \\
&+ k\frac{\Delta b_k}{b_k}[k(b_kMN_k - b_{k+1}MN_{k+1}) \\
&+ b_{k+1}MN_{k+1}\frac{b_k}{b_{k+1}}\left\{\frac{k(\Delta b_k)}{b_k}\right\}^2].
\end{aligned}$$

Since the sequence b_k is 2-regular, $M_k = b_kMN_k$ and $k\Delta M_k$ are bounded, the above identity shows that

$$\sup_{k\in\mathbb{Z}} \|k^2[(b_{k+1} - b_{k+2})MN_{k+1} - (b_k - b_{k+1})MN_k]\| < \infty.$$

Analogously and following the same procedure since a_k is also 2-regular, $S_k = a_kNN_k$ and $k\Delta S_k$ are bounded, we obtain that

$$\sup_{k\in\mathbb{Z}} \|k^2[(a_{k+1} - a_{k+2})NN_{k+1} - (a_k - a_{k+1})NN_k]\| < \infty.$$

Next, note that in Proposition 3.19 it was shown that $\sup_{k\in\mathbb{Z}} \|k\Delta(kBN_k)\| < \infty$. Since

$$k\Delta(kBN_k) = k[BN_k + (k+1)B\Delta N_k],$$

and $\sup_{k\in\mathbb{Z}} \|kBN_k\| < \infty$ by hypothesis, we deduce that $\sup_{k\in\mathbb{Z}} \|k^2B\Delta N_k\| < \infty$.

For the following two terms, observe that $0 \in \rho(M)$ and the identity $k^2N_k = \frac{1}{k}M^{-1}M_k$ for $k \neq 0$, implies

$$\sup_{k\in\mathbb{Z}} \|k^2N_k\| < \infty.$$

This fact together with the hypothesis on the boundedness of the sets $\{kG_k\}_{k\in\mathbb{Z}}$ and $\{F_k\}_{k\in\mathbb{Z}}$ proves the claim. Consequently, $(ik^3MN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.

Let now $M_k = k^2NN_k$. It is clear by Proposition 3.19 that $\sup_{k\in\mathbb{Z}}(||M_k|| + ||k\Delta M_k||) < \infty$. To prove that $\sup_{k\in\mathbb{Z}} ||k^2\Delta^2 M_k|| < \infty$ we apply the second part of Theorem 3.18 with $c_k = k^2, H_k = N_k$ and $T = N$. By hypothesis and Proposition 3.19 it follows that $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$ and $\sup_{k\in\mathbb{Z}} \|kL_k\| < \infty$, respectively. It remains to show that $\sup_{k\in\mathbb{Z}} \|k^2\Delta L_k\| < \infty$, where L_k is exactly the same that in the above computation. Therefore, $(k^2NN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.

Let now $M_k = kBN_k$. It is clear by Proposition 3.19 that $\sup_{k\in\mathbb{Z}}(||M_k|| + ||k\Delta M_k||) < \infty$. To prove that $\sup_{k\in\mathbb{Z}} ||k^2\Delta^2 M_k|| < \infty$ we apply the second part of Theorem 3.18 with $c_k = k, H_k = N_k$ and $T = B$. By hypothesis and Proposition 3.19 it follows that $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$ and $\sup_{k\in\mathbb{Z}} \|kL_k\| < \infty$, respectively. It remains to show that $\sup_{k\in\mathbb{Z}} \|k^2\Delta L_k\| < \infty$, where L_k is exactly the same that in the above computation. Therefore, $(kBN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.

Finally, we set $M_k = kN_k$. Again Proposition 3.19 shows that $\sup_{k\in\mathbb{Z}}(||M_k|| + ||k\Delta M_k||) < \infty$. We apply Theorem 3.18 with $c_k = k, H_k = N_k$ and $T = I$. We have $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$ and $\sup_{k\in\mathbb{Z}} \|kL_k\| < \infty$ by hypothesis and Proposition 3.19, respectively. Then the fact that $\sup_{k\in\mathbb{Z}} \|k^2\Delta L_k\| < \infty$ was just proved. Consequently, $(kN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. □

We shall enunciate the main result of this section. The proof follows essentially the same steps than the proof of Theorem 3.22. However, we include here the proof in order to make clear to the reader the essential changes that have to be done in order to treat with the delay terms and how to assume less hypothesis than in Theorem 3.22.

Theorem 4.32. *Let $1 \le p, q \le \infty$, $s > 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume A, B and M, N are closed linear operators defined on a Banach space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$ and $0 \in \rho(M)$. Assume that $\{kG_k : k \in \mathbb{Z}\}$ is uniformly bounded. The following assertions are equivalent:*

(i) *The equation*

$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad t \in [0, 2\pi]$$

is $B^s_{p,q}$-well posed;

(ii) *$\mathbb{Z} \subset \rho_M(A, B)$ and the sets $\{i\alpha k^3MN_k : k \in \mathbb{Z}\}$, $\{k^2NN_k : k \in \mathbb{Z}\}$ and $\{\gamma kBN_k : k \in \mathbb{Z}\}$ are uniformly bounded.*

Proof. $(i) \implies (ii)$ follows the same lines of Theorem 3.22 and therefore is omitted. We prove $(ii) \implies (i)$. We assume that $\mathbb{Z} \subset \rho_{M,N}(A, B)$ and the sets $\{ik^3MN_k : k \in \mathbb{Z}\}$, $\{k^2NN_k : k \in \mathbb{Z}\}$ and $\{kBN_k : k \in \mathbb{Z}\}$ are uniformly bounded.

Since $0 \in \rho(M)$, the identities $kN_k = \frac{1}{k^2}M^{-1}(k^3MN_k)$ and $k^2N_k = \frac{1}{k}M^{-1}(k^3MN_k)$ show that the sets $\{kN_k : k \in \mathbb{Z}\}$ and $\{k^2N_k : k \in \mathbb{Z}\}$ are also uniformly bounded. Therefore the sets $\{k(N_{k+1} - N_k)\}_{k\in\mathbb{Z}}$ and $\{k^2(N_{k+2} - 2N_{k+1} + N_k)\}_{k\in\mathbb{Z}}$ are uniformly bounded and hence by Theorem 4.29 the set $\{N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. Let $M_k = ik^3MN_k$, $S_k = k^2NN_k$, $H_k = kBN_k$ and $R_k = kN_k$ with $k \in \mathbb{Z}$. It follows from Proposition 4.31 that $(M_k)_{k\in\mathbb{Z}}, (S_k)_{k\in\mathbb{Z}}, (H_k)_{k\in\mathbb{Z}}$ and $(R_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers.

Let $f \in B^s_{p,q}(\mathbb{T}, X)$ be given. Since $\{N_k\}_{k\in\mathbb{Z}}$ and $\{R_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers, there exists $w, u_2 \in B^s_{p,q}(\mathbb{T}, [D(A) \cap D(B)])$ such that

$$\hat{w}(k) = N_k\hat{f}(k), \quad \hat{u}_2(k) = ikN_k\hat{f}(k). \tag{4.33}$$

Consequently, $\hat{u}_2(k) = ik\hat{w}(k)$ for $k \in \mathbb{Z}$ and we obtain $w \in B^{s+1}_{p,q}(\mathbb{T}; [D(A) \cap D(B)])$ and $w'(t) = u_2(t)$ a.e. (cf. Remark 4.24).

Moreover, there exist $r, u \in B^s_{p,q}(\mathbb{T}, X)$ satisfying that

$$\hat{r}(k) = -M_k\hat{f}(k) = -ik^3MN_k\hat{f}(k) = (ik)^2M\hat{u}_2(k) = (ik)^2\widehat{Mw'}(k), \tag{4.34}$$

and

$$\hat{u}(k) = -S_k\hat{f}(k) = -k^2NN_k\hat{f}(k) = ikN\hat{u}_2(k) = ikN\widehat{w'}(k) = ik\widehat{Nw'}(k), \tag{4.35}$$

where we have used that M and N are closed linear operators. We deduce that $(Mw')'' = r$, $Mw' \in B^{s+2}_{p,q}(\mathbb{T}, X)$ and $(Nw')' = u$, $Nw' \in B^{s+1}_{p,q}(\mathbb{T}, X)$.

Now, we claim that $(F_kN_k)_{k\in\mathbb{Z}}$ and $(ikG_kN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers. Indeed, we first show that the sets $\{k\Delta(F_kN_k)\}_{k\in\mathbb{Z}}$ and $\{k\Delta(kG_kN_k)\}_{k\in\mathbb{Z}}$ are uniformly bounded. In fact, from the identity $N_k = \frac{1}{k}(kN_k)$ we deduce that $\{N_k\}_{k\in\mathbb{Z}}$ is uniformly bounded. It follows from Remark 4.25 that $\{F_kN_k : k \in \mathbb{Z}\}$ and $\{ikG_kN_k : k \in \mathbb{Z}\}$ are uniformly bounded. On the other hand, the identities

$$k(F_{k+1}N_{k+1} - F_kN_k) = F_{k+1}(\frac{k}{k+1}R_{k+1}) - F_k(kN_k)$$

and

$$k[i(k+1)G_{k+1}N_{k+1} - ikG_kN_k] = i\frac{k^2}{(k+1)^2}(k+1)G_{k+1}R_{k+1} - ikG_k(R_k) + i\frac{k}{k+1}G_{k+1}R_{k+1},$$

show that the sets $\{k\Delta(F_kN_k)\}_{k\in\mathbb{Z}}$ and $\{k\Delta(kG_kN_k)\}_{k\in\mathbb{Z}}$ are uniformly bounded. This proves the claim.

It only remains to show that the sets $\{k^2\Delta(F_kN_k)\}_{k\in\mathbb{Z}}$ and $\{k^2\Delta(kG_kN_k)\}_{k\in\mathbb{Z}}$ are uniformly bounded. In fact, since $0 \in \rho(M)$ and the identity $k^2N_k = \frac{1}{k}M^{-1}M_k$ for $k \neq 0$, we obtain that the set $\{k^2N_k\}_{k\in\mathbb{Z}}$ is uniformly bounded. Therefore, the identities

$$k^2(F_{k+1}N_{k+1} - F_kN_k) = F_{k+1}(\frac{k^2}{(k+1)^2}(k+1)^2N_{k+1}) - F_k(k^2N_k)$$

and

$$k^2[i(k+1)G_{k+1}N_{k+1} - ikG_kN_k] = i\frac{k^2}{(k+1)^2}(k+1)G_{k+1}(k+1)^2N_{k+1} - ikG_k(k^2N_k),$$

show that the sets $\{k^2\Delta(F_kN_k)\}_{k\in\mathbb{Z}}$ and $\{k^2\Delta(kG_kN_k)\}_{k\in\mathbb{Z}}$ are uniformly bounded since they are sum and product of uniformly bounded sets. Then, by Theorem 4.29, our claim follows i.e. $(F_kN_k)_{k\in\mathbb{Z}}$ and $(ikG_kN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers. From this and (4.33) it follows that there exist $s_1, s_2 \in B^s_{p,q}(\mathbb{T}, X)$ such that

$$\hat{s}_1(k) = ikG_kN_k\hat{f}(k) = G_k\hat{u}_2(k) = G_k\widehat{w'}(k) = \widehat{Gw'_{\cdot}}(k), \tag{4.36}$$

and

$$\hat{s}_2(k) = F_kN_k\hat{f}(k) = F_k\hat{w}(k) = \widehat{Fw_{\cdot}}(k). \tag{4.37}$$

Therefore, $Gw'_{\cdot} = s_1 \in B^s_{p,q}(\mathbb{T}, X)$ and $Fw_{\cdot} = s_2 \in B^s_{p,q}(\mathbb{T}, X)$.

From (3.26) we deduce that $(AN_k)_{k\in\mathbb{Z}}$ is also an $B^s_{p,q}$-Fourier multiplier since it can be expressed as the sum and product of $B^s_{p,q}$-Fourier multipliers. Therefore, there exists $u_1 \in B^s_{p,q}(\mathbb{T},X)$, such that

$$\hat{u}_1(k) = AN_k\hat{f}(k) = A\hat{w}(k), \tag{4.38}$$

where we have used (4.33) in the last equality. By [6, Lemma 3.1] we obtain $w(t) \in D(A)$ and $Aw = u_1 \in B^s_{p,q}(\mathbb{T},X)$. Finally, since $\{H_k\}_{k\in\mathbb{Z}}$ is $B^s_{p,q}$-Fourier multiplier, we obtain that there exists $v \in B^s_{p,q}(\mathbb{T},X)$ such that

$$\hat{v}(k) = iH_k\hat{f}(k) = B(ikN_k)\hat{f}(k) = B\hat{u}_2(k) = B\widehat{w'}(k). \tag{4.39}$$

It follows from [6, Lemma 3.1] that $w'(t) \in D(B)$ and $Bw' = v \in B^s_{p,q}(\mathbb{T},X)$. From (3.26) note that:

$$\hat{f}(k) = -\alpha M_k\hat{f}(k) - S_k\hat{f}(k) - \beta AN_k\hat{f}(k) - \gamma iH_k\hat{f}(k) - ikG_kN_k\hat{f}(k) - F_kN_k\hat{f}(k). \tag{4.40}$$

Replacing (4.34) - (4.39) in (4.40) we obtain by the uniqueness of the Fourier coefficients that

$$\alpha(Mw')''(t) + (Nw')'(t) = \beta Aw(t) + \gamma Bw'(t) + Gw'_t + Fw_t + f(t), \quad t \in \mathbb{T}.$$

This shows the existence. It only remains to prove uniqueness. Indeed, let $w \in S_{p,q,s}(A,B,M,N)$ satisfying

$$\alpha(Mw')''(t) + (Nw')'(t) = \beta Aw(t) + \gamma Bw'(t) + Gw'_t + Fw_t, \quad t \in \mathbb{T}. \tag{4.41}$$

Taking Fourier transform in (4.41), we get that $-[b_kM + a_kN + \beta A + \gamma(ik)B + (ik)G_k + F_k]\hat{w}(k) = 0$, for all $k \in \mathbb{Z}$. Hence $w = 0$ as $\mathbb{Z} \subset \rho_{M,N}(A,B)$ by hypothesis. Thus, equation (4.32) is $B^s_{p,q}$-well-posed. □

Since the second statement of Theorem 4.32 does not depend on the parameters p,q and s, the following result follows immediately.

Corollary 4.33. *Let X be a Banach space and let A, B and M be closed linear operators defined on a Banach space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$ and $\alpha,\beta,\gamma \in \mathbb{R}$. Then equation (4.32) is $B^s_{p,q}$-well-posed for some $1 \le p,q \le \infty$, $s > 0$ if and only if it is $B^s_{p,q}$-well-posed for all $1 \le p,q \le \infty$, $s > 0$.*

To finish this section, we consider well-posedness in periodic Triebel-Lizorkin spaces $F^s_{p,q}$ with $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. We omit the definition and properties of these spaces but we refer the reader to [16] for the details.

We consider the problem:

$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad t \in \mathbb{T} := [0,2\pi], \tag{4.42}$$

in 2π-periodic vector valued $F^s_{p,q}$-spaces. Here A, B, N and M are closed linear operators such that $D(A) \cap D(B) \subset D(N) \cap D(M)$ and $F, G \in \mathcal{B}(F^s_{p,q}(-2\pi,0);X),X)$.

We define the solution space of the $F^s_{p,q}$- well-posedness of (4.42) by

$$\begin{aligned}&F_{p,q,s}(A,B,M,N) := \{u \in F^{s+1}_{p,q}(\mathbb{T},[D(A)\cap D(B)]) \cap F^s_{p,q}(\mathbb{T},[D(A)\cap D(B)]) : \\ &Mu' \in F^{s+2}_{p,q}(\mathbb{T},X), Nu' \in F^{s+1}_{p,q}(\mathbb{T},X), Au, Bu', Fu., Gu'_. \in F^s_{p,q}(\mathbb{T},X)\}.\end{aligned}$$

The definition of $F^s_{p,q}$- well posedness of equation (4.42) is as follows.

Definition 4.34. *Let $1 \le p < \infty, 1 \le q \le \infty$, $s > 0$ and $f \in F^s_{p,q}(\mathbb{T}, X)$. We call $u \in F_{p,q,s}(A, B, M)$ a strong $F_{p,q}$-solution of (4.32) if it satisfies (4.32) for all $t \in \mathbb{T}$. We say that (4.32) is $F^s_{p,q}$-well-posed if for each $f \in F^s_{p,q}(\mathbb{T}, X)$, there exists a unique strong $F^s_{p,q}$-solution of (4.32).*

Using a similar argument as the one in the proof of Theorem 4.32, we obtain the following characterization of the $F^s_{p,q}$- well posedness of equation (4.32). In order to prove this result we use the operator-valued Fourier multiplier theorem proved in [16]. We omit the details.

Theorem 4.35. *Let $1 \le p < \infty, 1 \le q \le \infty$, $s > 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume A, B and M are closed linear operators defined on a Banach space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$ and $0 \in \rho(M)$. Assume that $\{kG_k : k \in \mathbb{Z}\}$ is uniformly bounded. The following assertions are equivalent:*

(i) *The equation*

$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + Gu'_t + Fu_t + f(t), \quad t \in [0, 2\pi]$$

is $F^s_{p,q}$-well posed;

(ii) *$\mathbb{Z} \subset \rho_M(A, B)$ and the sets $\{i\alpha k^3 MN_k : k \in \mathbb{Z}\}$, $\{k^2 NN_k : k \in \mathbb{Z}\}$, $\{\gamma kBN_k : k \in \mathbb{Z}\}$ are uniformly bounded.*

Similarly to Corollary 4.33 we obtain the following result.

Corollary 4.36. *Let X be a Banach space and let A, B and M, N be closed linear operators defined on a Banach space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Then equation (4.32) is $F^s_{p,q}$-well-posed for some $1 \le p < \infty, 1 \le q \le \infty$, $s > 0$ if and only if it is $F^s_{p,q}$-well-posed for all $1 \le p < \infty, 1 \le q \le \infty$, $s > 0$.*

5. Applications

We first consider the following inverse problem:

$$\alpha(Mu')''(t) + (Nu')'(t) = \beta Au(t) + \gamma Bu'(t) + f(t)z, \quad t \in [0, 2\pi], \tag{5.43}$$

with the additional information

$$\Phi[Mu'(t)] = g(t) \quad \Phi[Nu'(t)] = h(t), \tag{5.44}$$

where $z \in X, \Phi \in X^*$ and the unknown (u, f) is to be determined.

This kind of inverse problems was recently studied by Al Horani and Favini [3] when A is the generator of a semigroup in X in case $B = I, M = N = 0$ [3, Theorem 2.1] and under the assumption that B is dissipative defined on a Hilbert space in case $M = 0, N = I$ [3, Theorem 4.1]. In this section, we consider existence and uniqueness of solutions for the general case (5.43)-(5.44) under new hypotheses, as the ones given in our main results in the previous sections. Observe that we can considerably relax the hypotheses on A, B, M and N thanks to the remarkable fact that in our main theorems we do not require any assumptions on generation of semigroups or even cosine operator functions.

Our identification result in case of vector-valued Lebesgue spaces read as follows.

Theorem 5.37. *Suppose that A, B, M and N are closed linear operators defined on a UMD space X such that $D(A) \cap D(B) \subset D(M) \cap D(N)$. Suppose that $z \in X, \Phi \in X^*, \Phi[z] \neq 0, g \in W^{2,p}_{per}(\mathbb{T}, \mathbb{C}), h \in W^{1,p}_{per}(\mathbb{T}, \mathbb{C})$. We define the following operators*

$$A_1 u = -\frac{\Phi[Au]}{\Phi[z]} z, \quad u \in D(A_1) = D(A),$$

and

$$B_1 v = -\frac{\Phi[Bv]}{\Phi[z]} z, \quad v \in D(B_1) = D(B),$$

and assume that for each $k \in \mathbb{Z}$ the operator

$$N_k := [i\alpha k^3 M + k^2 N + \beta(A + A_1) + ik\gamma(B + B_1)]^{-1}$$

exists as a bounded linear operator in X and the sets $\{i\alpha k^3 MN_k\}_{k\in\mathbb{Z}}$,$\{k^2 MN_k\}_{k\in\mathbb{Z}}$, $\{\gamma kBN_k\}_{k\in\mathbb{Z}}$ and $\{kN_k\}_{k\in\mathbb{Z}}$ are R-bounded. Then, (5.43)-(5.44) *admits a unique strong solution*

$$(u, f) \in W^{1,p}_{per}(\mathbb{T}, [D(A) \cap D(B)]) \cap L^p_{per}(\mathbb{T}, [D(A) \cap D(B)]) \times L^p(\mathbb{T}, \mathbb{C}).$$

Proof. Applying Φ to (5.43) and taking into account (5.44), we obtain

$$f(t) = \frac{1}{\Phi[z]}\Big[\alpha g''(t) + h'(t) - \beta\Phi[Au(t)] - \gamma\Phi[Bu'(t)]\Big]. \tag{5.45}$$

Therefore, the given inverse problem (5.43)-(5.44) translates into the following direct problem:

$$\begin{aligned}\alpha(Mu')''(t) + (Nu')'(t) &= \beta Au(t) + \gamma Bu'(t) + [\alpha g''(t) + h'(t)]\frac{z}{\Phi[z]} - \beta\frac{\Phi[Au(t)]}{\Phi[z]}z - \gamma\frac{\Phi[Bu'(t)]}{\Phi[z]}z \\ &= \beta(A + A_1)u(t) + \gamma(B + B_1)u'(t) + [\alpha g''(t) + h'(t)]\frac{z}{\Phi[z]}. \end{aligned} \tag{5.46}$$

Since $\alpha g'' + h' \in L^p(\mathbb{T})$, it follows from the hypothesis and Theorem 3.22 that the inverse problem (5.43)-(5.44) admits a unique strong solution $u \in S_{p,q,s}(A, B, M, N)$. Hence, the pair (u, f), where f is given by (5.45), solves the identification problem with regularity $u \in W^{1,p}_{per}(\mathbb{T}, [D(A) \cap D(B)]) \cap L^p_{per}(\mathbb{T}, [D(A) \cap D(B)])$ with $u' \in L^p(\mathbb{T}, [D(A) \cap D(B)])$ and $f \in L^p(\mathbb{T}, \mathbb{C})$. ☐

We remark that analogous results can be established using theorems 4.32 and 4.35. Finally, we provide the following simple example to illustrate the treatment of equations with delay.

Example 5.38. *We consider the following integro-differential equation with delay*

$$\alpha\frac{\partial^3(q(x)u(t,x))}{\partial t^3} + \frac{\partial^2(q(x)u(t,x))}{\partial t^2} = \beta\frac{\partial^2 u}{\partial x^2}(t,x) + u(t-\tau,x) + f(t,x), \quad t, x \in \mathbb{T} := [0, 2\pi], \tag{5.47}$$

where $\alpha, \beta \in \mathbb{R}$, $\tau \in (0, 2\pi)$, f is a given function on $\mathbb{T} \times \mathbb{T}$ and q is a measurable function such that the multiplication operator

$$(Mf)(t) = q(t)f(t)$$

with domain $D(M) := \{f \in L^2(\mathbb{T}) \,|\, q \cdot m \in L^2(\mathbb{T})\}$ has a bounded inverse. This last property holds if and only if $0 \notin q_{ess}(\mathbb{T})$, the essential range of q [21, Chapter I, Proposition 4.10]*; In such case, we know that $\|M^{-1}\| = \|q^{-1}\|_\infty := \sup\{|\lambda| \,:\, \lambda \in q^{-1}_{ess}(\mathbb{T})\}$* [21, Proposition 4.10].

We define the operator $(A, D(A))$ on $L^2(\mathbb{T})$ as

$$Af = f'', \quad D(A) = \{f \in L^2(\mathbb{T}) | \quad f', f'' \in L^2(\mathbb{T}), \quad f(0) = f(2\pi) = 0\}.$$

It is known that $\sigma(A) = \sigma_p(A) = \{-n^2 : n \in \mathbb{N}\}$. *We define the operator* $F : L^p((-2\pi, 0), L^2(\mathbb{T})) \to L^2(\mathbb{T})$ *by* $F(\varphi) = \varphi(-\tau)$. *It is clear that* F *is linear and* $\|F(\varphi)\| \leq \|\varphi\|_p$. *Then equation* (5.47) *labels into the scheme of* (3.12) *with* $A = \Delta = \frac{\partial^2}{\partial x^2}$ *the one-dimensional Dirichlet Laplacian on* $L^2(\mathbb{T})$, $M = N$ *the multiplication operator by* q *on* $L^2(\mathbb{T})$, $\gamma = 0$ *and* $G = 0$.

By [24] *(see also the introduction) it is known that there exists a constant* $c > 0$ *such that*

$$\|M(zM - \Delta)^{-1}\| \leq \frac{c}{1 + |z|} \tag{5.48}$$

for all z *such that* $\Re(z) \geq -c(1 + |\mathfrak{Im}\, z|)$. *We will show that for all* α, β *such that*

$$|\alpha| \geq \frac{1}{c} \quad \text{and} \quad \beta > c\|q^{-1}\|_\infty \tag{5.49}$$

equation (5.47) *is* $L^p(\mathbb{T}, L^2(\mathbb{T}))$*-well posed.*

Indeed, let $z_k = -\frac{k^2}{\beta} - i\frac{\alpha k^3}{\beta}$. *From* (5.49) *it follows that* $\Re(z_k) \geq -c(1 + |\mathfrak{Im}\, z_k|)$ *for all* $k \in \mathbb{Z}$. *Then we get from estimate* (5.48) *that*

$$\|M((k^2 + i\alpha k^3)M + \beta\Delta)^{-1}\| \leq \frac{c}{\beta + |k^2 + i\alpha k^3|}.$$

On the other hand, since the multiplication operator M *is invertible on* $L^2(\mathbb{T})$ *we have*

$$\begin{aligned}\|((k^2 + i\alpha k^3)M + \beta\Delta)^{-1}F_k\| &\leq \|M^{-1}\|\|M((k^2 + i\alpha k^3)M + \beta\Delta)^{-1}\|\|F\| \\ &\leq \|M^{-1}\|\frac{c}{\beta + |k^2 + i\alpha k^3|} \leq \frac{c}{\beta}\|q^{-1}\|_\infty < 1.\end{aligned}$$

We also get using the Neumann's series that

$$\|(I + F_k((k^2 + i\alpha k^3)M + \beta\Delta)^{-1})^{-1}\| < \frac{1}{1 - \frac{c}{\beta}\|M^{-1}\|}.$$

We conclude that $N_k = [(k^2 + i\alpha k^3)M + \beta\Delta]^{-1}[I + F_k((k^2 + i\alpha k^3)M + \beta\Delta)^{-1}]^{-1}$ *is well defined,* $\mathbb{Z} \subset \rho_M(\Delta)$ *and*

$$\|kN_k\| \leq \|M^{-1}\|\frac{c|k|}{\beta + |k^2 + i\alpha k^3|}\frac{1}{1 - \frac{c}{\beta}\|M^{-1}\|}.$$

It follows that $\sup_k ||kN_k|| < \infty$ *and analogously it can be checked that* $\sup_k \|k^3MN_k\| < \infty$. *From part* (ii) *of Theorems 3.22 it follows that equation* (5.47) *is* L^p*-well posed for all* $1 < p < \infty$. *Moreover, from theorems 4.32 and 4.35 it is also* $B^s_{p,q}$ *and* $F^s_{p,q}$*-well posed for all* $1 \leq p < \infty, 1 \leq q \leq \infty$, $s > 0$.

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References

[1] L. Abadías, C. Lizama and M. Murillo-Arcila. *Hölder regularity for the Moore-Gibson-Thomson equation with infinite delay.* Commun. Pure Appl. Anal. 17 (1) (2018), 243-265.

[2] K. Abbaoui and Y. Cherruault. *New ideas for solving identification and optimal control problems related to biomedical systems.* Int. J. Biomed. Comput. 36 (3) (1994), 181–186.

[3] M. Al Horani and A. Favini. *Perturbation method for first- and complete second-order differential equations.* J. Optim. Theory Appl. 166 (3) (2015), 949–967.

[4] H. Amann. *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications.* Math. Nachr. 186 (1997), 5–56.
[5] U.A. Anufrieva. *A degenerate Cauchy problem for a second-order equation. A well-posedness criterion.* (Russian) Differ. Uravn. 34 (8) (1998), 1131–1133. Translation in: Differential Equations 34 (8) (1999), 1135–1137.
[6] W. Arendt and S. Bu. *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity.* Math. Z. 240 (2) (2002), 311–343.
[7] W. Arendt and S. Bu. *Operator-valued Fourier multipliers on periodic Besov spaces and applications.* Proc. Edinburgh Math. Soc. 47 (2004), 15–33.
[8] W. Arendt, C. Batty and S. Bu. *Fourier multipliers for Holder continuous functions and maximal regularity.* Studia Math. 160 (2004), 23–51.
[9] V. Barbu and A. Favini. *Periodic problems for degenerate differential equations.* Dedicated to the memory of Pierre Grisvard. Rend. Inst. Mat. Univ. Trieste 28 suppl., (1997) 29–57.
[10] V. Barbu and N. H. Pavel. *Determining the acoustic impedance in the 1-D wave equation via an optimal control problem.* Siam J. Control Optim. 35 (5) (1997), 1544–1556.
[11] A. Bátkai and S. Piazzera. *Semigroups for Delay Equations.* Research Notes in Mathematics, 10, Wellesley, MA, 2005.
[12] S. Bu. *Well-posedness of second order degenerate differential equations in vector-valued function spaces.* Studia Math. 214 (2013), 1–16.
[13] S. Bu and G. Cai. *Periodic solutions of third-order degenerate differential equations in vector-valued functional spaces.* Israel J. Math. 212 (2016), 163–188.
[14] S. Bu and G. Cai. *Well-posedness of second-order degenerate differential equations with finite delay in vector-valued function spaces.* Pacific J. Math. 288 (1) (2017), 27–46.
[15] S. Bu and Y. Fang. *Periodic solutions of delay equations in Besov spaces and Triebel-Lizorkin spaces.* Taiwanese J. Math. 13 (2009), 1063–1076.
[16] S. Bu and J. Kim. *Operator-valued Fourier multipliers on periodic Triebel spaces.* Acta Math. Sinica (English Series) 21 (2005), 1049–1056.
[17] G. Cai and S. Bu. *Well-posedness of second order degenerate integro-differential equations with infinite delay in vector-valued function spaces.* Math. Nachr. 289 (4) (2016), 436–451.
[18] R. Chill and S. Srivastava. *L_p-maximal regularity for second order Cauchy problems.* Math. Z. 251 (2005), 751–781.
[19] R. Denk, M. Hieber and J. Prüss. *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type.* Mem. Amer. Math. Soc. 166 (2003), no. 788.
[20] O. Diekmann, S. van Giles, S. Verduyn Lunel and H. Walter. Delay Equations. Springer, New York, 1995.
[21] K. J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Graduate Texts in Math. 194, Springer, New York, 2000.
[22] M. Fabrizio, A. Favini and G. Marinoschi. *An optimal control problem for a singular system of solid liquid phase-transition.* Numer. Funct. Anal. Optim. 31 (2010), 989–1022.
[23] A. Favini and A. Yagi. *Degenerate differential equations in Banach spaces.* Monographs and Textbooks in Pure and Applied Mathematics, 215. Marcel Dekker, Inc., New York, 1999.
[24] A. Favini and A. Yagi. *Degenerate Differential Equations in Banach Spaces, Monographs and Textbooks in Pure and Applied Mathematics.* Vol. 215, Dekker, New York, 1999.
[25] A. Favini and G. Marinoschi. *Periodic behavior for a degenerate fast diffusion equation.* J. Math. Anal. Appl. 351 (2) (2009), 509–521.
[26] A. Favini and G. Marinoschi. *Identification of the time derivative coefficients in a fast diffusion degenerate equation.* J. Optim. Theory Appl. 145 (2010), 249–269.
[27] X.L. Fu and M. Li. *Maximal regularity of second-order evolution equations with infinite delay in Banach spaces.* Studia Math. 224 (3) (2014), 199–219.
[28] G. C. Gorain. *Boundary stabilization of nonlinear vibrations of a flexible structure in a bounded domain in R^n.* J. Math. Anal. Appl. 319 (2006), 635–650.
[29] P. Grisvard. *Équations différentielles abstraites.* Ann. Sci. École Norm. Sup. 2 (1969), 311-395.
[30] J.K. Hale and W. Huang. *Global geometry of the stable regions for two delay differential equations.* J. Math. Anal. Appl. 178 (1993), 344–362.

[31] Y. Hino, s. Murakami and T. Naito. *Functional Differential equations with Infinite Delay.* Lecture Notes in Math., 1473, Springer, Berlin, 1991.

[32] B. Kaltenbacher, I. Lasiecka and M. Pospieszalska. *Well-posedness and exponential decay of the energy in the nonlinear Moore-Gibson-Thomson equation arising in high intensity ultrasound.* Math. Models Meth. Appl. Sci. 22 (2012), 1250035, 34.

[33] V. Keyantuo and C. Lizama. *Fourier multipliers and integro-differential equations in Banach spaces.* J. London Math. Soc. 69 (2004), 737–750.

[34] V. Keyantuo and C. Lizama. *Maximal regularity for a class of integro-differential equations with infinite delay in Banach spaces.* Studia Math. 168 (2005), 25–50.

[35] V.Keyantuo, C. Lizama and V. Poblete. *Periodic solutions of integro-differential euations in vector-valued function spaces.* J. Differential Equations, 246(3) (2009), 1007–1037.

[36] C. Lizama. *Fourier multipliers and periodic solutions of delay equations in Banach spaces.* J. Math. Anal. Appl. 324 (2006), 921–933.

[37] C. Lizama and R. Ponce. *Periodic solutions of degenerate differential equations in vector valued function spaces.* Studia Math. 202 (2011), 49–63.

[38] C. Lizama and V. Poblete. *Maximal regularity of delay equations in Banach spaces.* Studia Math. 175 (2006), 91–102.

[39] C. Lizama and R. Ponce. *Maximal regularity for degenerate differential equations with infinite delay in periodic vector-valued function spaces.* Proc. Edinburgh Math. Soc. 56 (2013), 853–871.

[40] R. Marchand, T. Mcdevitt and R. Triggiani. *An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in highintensity ultrasound: structural decomposition, spectral analysis, exponential stability.* Math. Meth. Appl. Sci. 35 (2012), 1896–1929.

[41] V. Poblete. *Solutions of second-order integro-differential equations on periodic Besov spaces.* Proc. Edinburgh Math. Soc. 50 (2007), 477–492.

[42] V. Poblete. *Maximal regularity of second-order equations with delay.* J. Differential Equations 246 (2009), 261–276.

[43] V. Poblete and J. C. Pozo. *Periodic solutions of an abstract third-order differential equation.* Studia Math. 215 (2013), 195–219.

[44] L.Weis. *Operator-valued Fourier multiplier theorems and maximal Lp-regularity.* Math. Ann. 319 (2001), 735–758.

[45] J. Prüss. Evolutionary Integral Equations and Applications. Monographs in Math. 87, Birkhäuser, Heidelberg, 1993.

[46] G. A. Sviridyuk and V. E. Fedorov. Linear Sobolev type Equations and Degenerate Semigroups of Operators. De Gruyter, Utrecht, 2003.

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