Modeling, Control and Optimization of Dynamic Systems Driven by Vector-fields

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Resumo

Nesta tese estamos interessados em desenvolver uma estrutura matemática para a modelização, controlo e optimização de sistemas dinâmicos cuja variável de estado evolui através da interação tanto de equações diferenciais ordinárias como soluções particulares de equações diferenciais às derivadas parciais, cujas dinâmicas podem ser descritas apenas por equações diferenciais ordinárias. Esta estrutura deverá fornecer uma base sólida para o design e controlo de novos sistemas avançados de engenharia que surgem em muitas classes importantes de aplicações, algumas das quais englobam planadores subaquáticos e peixes mecânicos, que pretende ultrapassar as dificuldades inerentes aos resultados disponíveis até ao momento, muitos dos quais baseados em heurísticas e abordagens que combinam um "misto" de resultados de sistemas de controlo para equações diferenciais ordinárias e técnicas numéricas.

A nossa abordagem consiste na obtenção de uma família de problemas robustos de controlo óptimo convencionais, cujas dinâmicas convergem para as do sistema "híbrido" original (sistemas onde a dinâmica envolve equações diferenciais às derivadas parciais e equações diferenciais ordinárias), e na caracterização da solução como um certo limite das condições para os problemas aproximantes.

Os objetivos deste trabalho dizem respeito ao controlo óptimo de sistemas dinâmicos de controlo que evoluem num determinado campo vetorial, especificamente dado por escoamentos de Couette, escoamentos de Poiseuille e gerado por vórtices pontuais, que são soluções particulares das equações de Navier-Stokes e de Euler. O esforço da pesquisa foi no sentido de se obter informação através da aplicação de condições necessárias de optimalidade na forma do Princípio do Máximo de Pontryagin.

Aqui, apresentamos três problemas de controlo óptimo. O primeiro é um problema de tempo mínimo para movimentar uma partícula, de um dado ponto inicial até um ponto final, sujeita a escoamentos de Couette e Poiseuille. No segundo problema minimiza-se o gasto de energia para mover uma partícula, entre dois pontos dados, cuja dinâmica do fluído é dada por um vórtice. E o terceiro é um problema de multiprocessos para o movimento de uma partícula passiva movendo-se num fluido bidimensional cuja dinâmica é dada por um campo vetorial definido, em qualquer intervalo de tempo, por dois vórtices pontuais cujas circulações decaem exponencialmente no tempo, com uma determinada taxa pré-definida.

Palavras-chave: Controlo Óptimo, Princípio do Máximo de Pontryagin, Controlo Óptimo de Multiprocessos, Vórtices Pontuais, Sistemas Dinâmicos, Equações Diferenciais Ordinárias.

Abstract

In this thesis we are interested in developing a mathematical framework for the modeling, control and optimization of dynamic systems whose state variable is driven by interacting both ordinary differential equations and particular solutions of partial differential equations, whose dynamics can only be described by ordinary differential equations. This framework should provide a sound basis for the design and control of new advanced engineering systems arising in many important classes of applications, some of which encompass underwater gliders and mechanical fishes, that overcomes the shortcomings of the currently available, often heuristic--based "mixed" approaches combining ordinary differential equations control systems results and numerical techniques.

The general approach consists in designing a family of well posed, robust, conventional optimal control problems with dynamics converging to the ones of the original "hybrid" system (systems where the dynamics involve partial and ordinary differential equations), and by obtaining the characterization of its solution as a certain type of limit of the conventional conditions for the approximating problems.

The objectives of this work concerns the optimal control of dynamic control systems evolving in a vector-field, specifically given by Couette flows, Poiseuille flows and generated by point vortices, that are particular solutions of the Navier-Stokes and Euler equations. The research effort was deriving necessary conditions of optimality in the form of a Maximum Principle of Pontryagin.

Here, we present three optimal control problems. The first one is the minimum time control problem to move a particle, from one initial point to an end point, advected in a Couette and Poiseuille flows. In the second problem we minimize the energy spent to move a particle, between two given points, driven by a flow generated by one vortex. And the third is an optimal multiprocesses problem of the motion of a passive particle moving in a two dimensional fluid whose dynamics are given by a vector-field defined, in any time interval, by two point vortices whose circulations decay exponentially in time, with a given rate predetermined.

Keywords: Optimal Control, Maximum Principle of Pontryagin, Optimal Multiprocesses, Point Vortices, Dynamical Systems, Ordinary Differential Equations.

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Chapter 1

Introduction

Summary

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1.1 Objectives

The objectives of this thesis concern the optimal control of general nonlinear dynamic control systems evolving in a vector-field which might be affected or not by the control action. More specifically, the main goal consists in deriving necessary conditions of optimality in the form of a Maximum Principle of Pontryagin characterizing optimal control processes for problems with dynamics specified by interacting controlled ordinary and partial differential equations satisfying control, and also, possibly, some constraints. In its full generality, this problem is a tremendous challenge. Thus, we organized this overarching goal into two partial goals of increasing complexity. The following cases have been considered:

- Couette and Poiseuille flows;
- Vector-field generated by a set of vortices.

For each one of them, the following sub-objectives were targeted:

- Mathematical frameworks were defined to obtain the targeted results;
- Formulation of specific optimal control problems amenable to the application of the Maximum Principle;
- Derivation of necessary conditions of optimality in the form of a Maximum Principle;
- Computation of solutions by using analytical procedures, or numerical indirect methods, or a combination of these.

In spite of the extremely relevant motivation underlying the goals set out in this thesis - which is outlined in the next subsection - there are very few results strictly in this area.

The closest works can be found, for example, in Casas et al. (2000), Raymond (1997), Alibert and Raymond (1997), as well as, many other references cited therein. This large body of work - which relies in the seminal work of the mathematicians Jacques-Louis Lions and Pierre-Louis Lions, among others, that laid the foundations in the context of functional analysis, and novel solution concepts not only to solve partial differential equations but also to solve optimal control problems with dynamics governed by partial differential equations, see Vanninathan (1996), Varadhan (1994), Ciarlet (2001), Lax et al. (2001), Mawhin (2001) - is extremely sophisticated as it concerns several general classes of controlled partial differential equations. Moreover, this mathematical framework provides the foundations required to investigate conditions under which the existence of solution to the considered classes of control problems is guaranteed, as well as, to derive necessary conditions of optimality for associated optimal control problems.

However, the class of systems considered in the literature does not encompass the ultimate paradigm to be envisaged with the research strategy of which this thesis is just a first step: Existence and necessary conditions of optimality for optimal control problems whose dynamics are given by mixed ordinary and partial controlled differential, for which the motion defined by controlled ordinary differential is affected by the vector-field and, at the same time the control affects both types of dynamics. As a first step, in this thesis we just study the cases where the dynamics of the flow is given by ordinary differential equations, given by particular solutions of the Euler and Navier-Stokes equations.

Although the control action may, in the more general problem formulation, affect the vector-field behavior, the state of the system encompasses both the state of the vehicle and that of the vector-field in a neighborhood of the vehicle, and the ultimate goal of the control synthesis is not only to obtain a certain evolution of the vector-field, but rather geared towards improving the overall performance the vehicle motion.

1.2 General motivation

This can be organized along two very distinct directions.

- The first one concerns the extending the body of control and optimization theories by developing novel results on the mathematical framework for problem formulation, and deriving results on the existence and optimality conditions for a novel class of systems that was described in the previous section.

- Impact of the results in the previous item in improving the design of advanced engineering systems. There is a huge number of engineering applications of the control systems targeted in this thesis since controlling dynamic systems in vector-fields appear in a very large number of areas like aeronautics, space, mechanical systems (in which lubrication is of interest), maritime transportation, car design, underwater and aerial robotics, renewable power systems, notably from harvesting from the wind, waves, and underwater currents, chemical industries, among many others.

- Natural Sciences, Environmental studies and monitoring that surely requires intensive data sampling from the natural environment vector-fields.

To delimit this section, we focus only on the later item which also intersects the second one.

Most of the sensor platforms being used to gather data from the underwater milieu are powered by rotative motors - mostly fed by electrical batteries - whose propellers and surfaces steer the vehicles along desired paths and trajectories defined by the selected sampling strategies. Other fuels have been considered - like fossil, nuclear, and fuel cells - but, albeit these exhibit several significant advantages, the fact is the solutions based on electrical batteries have, so far, been considered the most competitive ones and hold, by far, the largest market share. So we will not dwell any longer with these other solutions.

Still, one of the most taxing constraints in the design of systems to address the multiple applications based on marine robotic vehicles, is precisely endurance, i.e., the capability of the vehicles accomplishing long duration missions without human intervention or the need of pre-deployed marine basis in which the vehicles, among other operations, can recharge batteries or replace their discharged batteries by newly charged ones. Harvesting energy from the environment has been of course a natural idea that gave rise to a number solutions.

Solar panels have been used to charge batteries of underwater vehicles but this imposes long stays at the surface with the additional drawback of the charging procedure being very slow, sensitive to sea state, and strongly dependent on the weather conditions. Of course, this solution is significantly more competitive for surface vehicles, particularly those that are endowed with one or more sails. These vehicles can combine in a beneficial fashion both solar and wind power harvesting to directly move the vehicle and, at the same time charge electrical batteries for motion - thus improving controllability - and for other functions - communication, computation, etc.

Underwater gliders and wave gliders constitute classes of underwater vehicles that often combine the extraction of energy from solar cells with wave energy. The latter is achieved by controlling the buoyancy of the vehicle (Mahmoudian et al. (2007), Mahmoudian and Woolsey (2010)). The success of underwater gliders and wave gliders has been very significant and stems from their ability to execute missions with highly operational character and extremely long endurance, as, for example, being able to cross the Atlantic Ocean. Unfortunately, they exhibit important behaviour limitations, from which we single out, the up and down motion patterns (saw-teeth like) along the path of interest, and their very poor controllability. Of course, when they surface, GPS data can be obtained and used to redirect the vehicle motion. However, there are still long time periods of motion underwater during which it is not possible to have ways of guiding the glider motion.

However, there are other ways of propelling vehicles in the marine environment inspired in nature: fish undulation. See for example, Liu and Hu (2010), and CORDIS (2013). Most of the work done so far concerns modelling and simulation, but little effort has been devoted to a rigorous mathematical formulation as a control problem enabling the optimization of motion strategies. This is precisely the ultimate goal towards which this thesis constitutes a first but decisive step.

1.3 Contributions

One general contribution consists in providing a first step towards a general framework for general optimal control problems whose dynamics are given by interacting controlled ordinary and particular solutions of the Euler and Navier-Stokes equations. The general approach consists in starting with simple classes of control problems of dynamic systems in vector fields in such a way that the effect of the latter flow fields can be described by ordinary differential equations, and, then, in applying the conventional Maximum Principle of Pontryagin, after ensuring the existence of solution. In many instances this step becomes trivial by the appropriate flow field.

The remaining contributions follow from the deployment of the above research strategy.

The first considered step consists in formulating the minimum time control problem of a particle advected in Couette and Poiseuille flows and in solving it by using the Pontryagin Maximum Principle. Here, not only the dynamics of the control system are defined by a set of ordinary differential equations, but also the conditions resulting from the application of the necessary conditions of optimality can be easily solved in an explicit way.

The second contribution consists in formulating the minimum fuel problem for a passive particle moving in a vector-field generated by a single vortex in the whole space without any boundary constraints, between two given points. Conditions of existence of solution are given and the Pontryagin's Maximum Principle is applied in order to obtain an explicit characterization of the solution.

The third contribution consists in the formulation and resolution of a minimum fuel problem for a particle moving in a two-dimensional fluid whose vector-fields are defined, in any time interval, by two point vortices whose circulations decay exponentially in time, with a given constant rate. The control action is exercised by generating one vortex - specified by its location and respective circulation - at a chosen time, and by varying the exposure of the particle to each one of the vortices in continuum time. A control multiprocess framework is chosen in order to derive necessary conditions of optimality in the form of a Maximum Principle of Pontryagin. These conditions provide relations that suffice to fully determine the optimal control process.

Finally, the last contribution consists in extending the previous problem to the case in which the particle is endowing the usual simple kinematic dynamics (unicycle) considered for Autonomous Underwater Vehicles. In this case, the AUV motion is due to the effects of the water column controlled flow fields, notably, the point vortices which are generated in order to achieve the desired motion features.

1.4 Organization

This thesis is organized in the following way. In the next chapter, 2, details of the state-of-the-art concerning the engineering applications outlined above in 1.2 are presented. Some issues pertinent to the mathematical developments underlying the contributions of the thesis are also raised in this chapter. In chapter 3, some key results in optimal control and their application are discussed. This presentation is focused in the issues that are relevant for the obtained results. In chapter 4, concepts and results of fluid dynamics are discussed. An overview of the general issues will be given and a strong emphasis is placed in the specific flow fields considered in the developments of the thesis. In chapter 5, the first results - notably necessary conditions of optimality in the form of a Maximum Principle of Pontryagin, concerning the optimal control of dynamic control systems in a Couette Poiseuille flow field are presented, proved and illustrated with applications. In chapter 6, the simple case of controlling a dynamic system in a vector field generated by a vortex is investigated. Necessary conditions of optimality are derived and applied to a simple example that serves as an illustration. The optimal control problem of the previous chapter is extended in chapter 7 for the case of two vortices. Here, the problem of minimum time subject to fixed trajectory endpoints is cast in the multiprocesses framework for which the Maximum Principle of Pontryagin is applied. This yields a complex two boundary value problem whose methodology to obtain the solution is outlines. Finally, in the last chapter, some conclusions and prospective future work are addressed.

Chapter 2

State of-the-art

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2.1 Introduction

The development a mathematical framework for the modelling, control and optimization of dynamic control systems whose state variable is driven by interacting ordinary and partial differential equations is still a significant challenge. In Grilo et al. (2012), Grilo et al. (2013), Grilo et al. (2015), Pereira et al. (2017a), Pereira et al. (2017b) and in Grilo et al. (2018), presented is some work aiming at the development of a theory of optimal control of this dynamic systems, whose state evolves due to the interaction of ordinary differential equations with partial differential equations, in which the latter part is replaced by some known particular solution.

Underwater gliders and robotic fishes are two examples of the class of applications whose currently available models we intend to improve. An underwater glider is a winged autonomous underwater vehicle (AUV) that moves by modulating its buoyancy and attitude in the velocity vector fields of its environment. This vehicles are used for long-term, large-scale oceanographic monitoring, undersea surveillance and other applications. The kinetic and dynamic equations that described the vehicle motion can be found in Mahmoudian et al. (2007), Mahmoudian and Woolsey (2010). In Liu and Hu (2010), the motion of the robotic fish is approximated by a model featuring several components. The key advantage of this model is the fact that, instead of being considered a rigid body, the structure of the fish is composed of three parts: head, body and tail.

While the optimal control of systems with dynamics given by ordinary differential equations only has been making great strides in the 20^{th} and 21^{st} centuries (see, among others, Arutyunov et al. (2011), Clarke (1983), Pontryagin et al. (1962)), such a theory for hybrid - in the sense that the controlled dynamics involve ordinary and partial differential equations - systems is still at its infancy.

Given the proposed objectives, the state-of-the-art is extremely vast and encompasses the following components:

(i) The application domain providing the requirements for the novel mathematical framework to be developed, Mahmoudian et al. (2007), Mahmoudian and Woolsey (2010), Liu and Hu (2010), Hou et al. (2007).

(ii) Control of partial differential equations and supporting mathematical methods and tools, Lions (1971), Protas (2008), Doering and Gibbon (1995).

(iii) Dynamic optimization of controlled ordinary differential equations and supporting mathematical methods and tools, Clarke (1980), Clarke (1983), Clarke et al. (1998), Sontag (1998), Arutyunov (2000), Mordukhovich (2006a), Mordukhovich (2006b), Rockafellar and Wets (1998), Vinter (2000), Clarke and Vinter (1989a), Clarke and Vinter (1989b).

Items (ii) and (iii) are well structured and properly documented in a well established literature. Part of this results, essentials to the present work, are presented in chapter 3, devoted to optimal control issues, and chapter 4, dedicated to fluid mechanics.

2.2 Application Domain

This thesis addresses two large classes of control systems: underwater gliders and robotic fishes.

In the literature mentioned in 2.1, the authors study these subjects through numerical algorithms. So far, the studies available rely strongly on numerical methods (simulation) and concern approximating trajectories.

2.2.1 Underwater Gliders

Underwater gliders (see figure 2.1) are highly efficient, winged autonomous underwater vehicles (AUV's) which locomote by modulating their buoyancy and their attitude. The applications include long-term, basin-scale oceanographic sampling and littoral surveillance. The exceptional endurance of underwater gliders is due to their reliance on gravity, weight and buoyancy, for propulsion and attitude control. Most of the work done so far in the direction of optimal control concerns the optimization of power consumption efficiency in their motion.

The glider is modelled as a rigid body (m_{rb}) with two moving mass actuators (m_{p_x} and m_{p_y}) and a variable ballast actuator (m_b). The total vehicle mass is

$$m_v = m_{rb} + m_{p_x} + m_{p_y} + m_b,$$

where m_b can be modulated by control. The variable mass is represented by a mass particle m_b located at the origin of a body-fixed reference frame, and the vehicle's attitude is given by a proper rotation matrix $\mathbf{R}_{IB}$, which maps free vectors from the body-fixed reference frame to a reference frame fixed in inertial space.

Using the notation of the article Mahmoudian et al. (2007) and considering $\mathbf{X}$ the position of the body frame origin with respect to the inertial frame, the vehicle kinematic equations are

$$\left\{ \begin{array}{lcl} \dot{\mathbf{X}} & = & \mathbf{R}_{IB}\upsilon \\ \dot{\mathbf{R}}_{IB} & = & \mathbf{R}_{IB}\hat{\omega} \end{array} \right. \tag{2.1}$$

Figure 2.1: Underwater glider, Slocum.

and the dynamic equations, that relate external forces and moments to the rate of change of linear and angular momentum, are

$$\begin{cases} \dot{\mathbf{p}} & = & \mathbf{p} \times \omega + \tilde{m}g(\mathbf{R}_{IB}^T \mathbf{i}_3) + \mathbf{F}_{visc} \\ \dot{\mathbf{h}} & = & \mathbf{h} \times \omega + \mathbf{p} \times v + (m_p g \mathbf{r}_p + m_{rb} g \mathbf{r}_{rb}) \times (\mathbf{R}_{IB}^T \mathbf{i}_3) + \mathbf{T}_{visc} \end{cases} . \tag{2.2}$$

The first step in the development of underwater gliders was in the efficiency of motion. For this, they started by studying the glider manoeuvrability, developing an approximative analytical expression for steady turning motions for a realistic glider model by applying regular perturbation theory. Because the turning motion results are only approximate, one must incorporate feedback to ensure precise path following. The nature of the steady turn approximations suggests a method for nearly energy-optimal path planning.

2.2.2 Robotic Fishes

The second class of applications is typified by the robotic fish (see figure 2.2). In Liu and Hu (2010) we have an approach to modelling carangiform fish-like motion for multi-joint robotic fish, so we can obtain fish-like behaviours and mimic the body motion of carangiform fish.

The majority of research work has been focused on fish-like propulsion mechanisms, fin materials, remote operation, multi-agent cooperation and mechanical structures. However, the motion of robotic fish has not been investigated extensively, in particular, control schemes along with modelling of carangiform fish-like swimming.

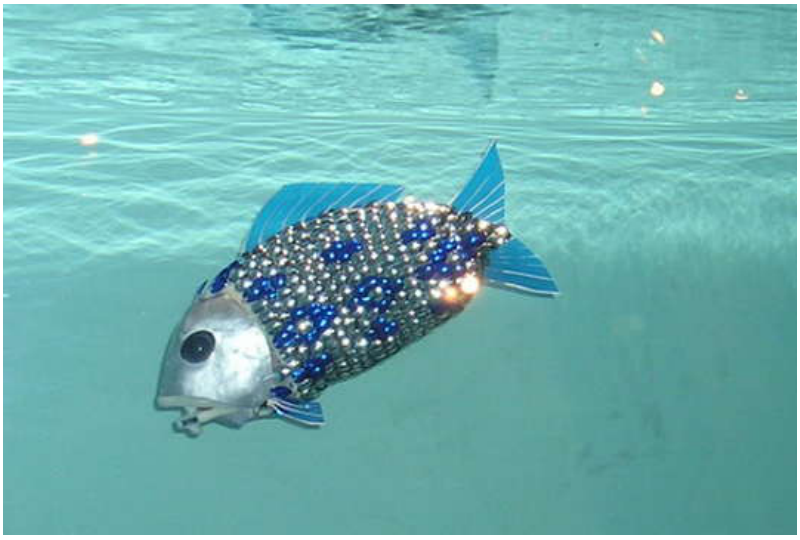

Figure 2.2: Robotic fish.

A given body motion function of fish swimming is firstly converted to a tail motion function which describes the tail motion relative to the head. Then, the tail motion function is discretized into a series of tail postures over time. A digital approximation method calculates the turning angles of joints in the tail to approximate each tail posture, and finally, these angles are grouped into a look-up table, or regressed to a time-dependent function, for practically controlling the tail motors in a multi-joint robotic fish.

The first coordinate system is a world coordinate system $\mathbf{R^w}$, where the origin is fixed at the connection point B between the fish head and tail, and its x-axis is aligned along the swimming direction of fish. The movement of a whole fish body in $\mathbf{R^w}$ is defined as body motion described as $f_B(x,t)$. Another coordinate system is a head-fixed system $\mathbf{R^h}$, where the origin is the same point but its x-axis is aligned along the fish head rather than the swimming direction. The movement of the fish tail in $\mathbf{R^h}$ is called the tail motion denoted as $f_T(x,t)$.

In Liu and Hu (2010) the motion of a fish during cruise straight is described by a travelling wave

$$y = f_B(x,t) = (c_1x + c_2x^2)\sin(kx + wt). \tag{2.3}$$

So the tail motion function corresponding to (2.3) is

$$f_T(x,t) = (c_1x + c_2x^2)\sin(kx + wt) - c_1x\sin(wt). \tag{2.4}$$

2.3 Relevance for the Application Domain

The problems studied in this thesis are just the first step to create better and more realistic models for the motion of the AUV's. The problems studied are simple, corresponding to linear and parabolic velocity profiles (studied in chapter 5), Couette and Poiseuille flows, respectively.

Next, we studied problems whose dynamics is driven by point vortices. First, we considered a fluid with a velocity field created by one vortex (chapter 6), and then we consider two vortices actuating on the flow (chapter 7).

The idea was to create similar conditions like in real life. When we observe a real fish in its environment, it creates its motion using the tail and undulating the body, and this movement produces oscillations/eddies/circulation points in the fluid. The fish uses these eddies as supporting points to move around.

We think that these circulation points created by the fish can be considered like point vortices, thus, we tried to formulate an optimal control problem taking into consideration all these facts: the velocity field of the fluid, the points that the fish creates while moving, and the energy spent to create them. This seems a more realistic approach to get a configuration for the trajectories of the robotic fishes and give the possibility to minimize the energy along its trajectory.

Chapter 3

Preliminary Results on Optimal Control

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3.1 Introduction

In this chapter we present some preliminary results on Optimal Control (OC) Theory, with emphasis on the Maximum Principle of Pontryagin given in Pontryagin et al. (1962), Vinter (2000). We will start by one of simplest, and yet one of the most general, statements of the Optimal Control Problem, and will proceed by listing the key results that are pertinent to the development of the thesis. Of course, there is a vast number of issues - such as, nondegeneracy, well-posedness, sensitivity, higher order conditions, infinity horizon solution concepts, finite approximations to solutions, etc. - that would make sense to be addressed in the quest of solving concrete optimal control problems. However, given the space limitations, we will focus on the key references where these subjects are discussed in detail.

Given the fact that we used the multiprocesses framework in an essential way in chapter 7, we also include a section devoted to this special formulation of optimal control problems that motivate and develop the associated Maximum Principle in two seminal articles, Clarke and Vinter (1989a), and Clarke and Vinter (1989b). The essential idea of this formulation consists in the joint optimization of several dynamic control systems linked by joint constraints and by the cost functional.

3.2 Optimal Control Problem Formulation

An optimal control problem is an optimization problem where the choice variables are functions of time, t, and some of the derivatives of this functions are involved on the dynamics. The main elements of an optimal control problem are:

- the cost functional, that consists of a quantitative criterion for the efficiency of the system;
- the mathematical model which relates the state x to the control u by a system with differential equations, which determines the time evolution of the state variable;
- the constraints on the state variable. The satisfaction of these constraints affects the evolution of the system, and restricts the admissible controls;
- the control constraints.

Thus, a control system is a dynamical system, which evolves over time, where the state variable, x, characterizes the evolution of the system over time (physically, the attribute "dynamic" is due to the existence of energy storages which affect the behaviour of the system), so called trajectory, and the control variable, u, represents the possibility of intervening in order to change the behaviour of the system, so that its performance is optimized.

Optimal control is useful for optimization problems with inter-temporal constraints. Thus, some of the areas of its application are the management of renewable and non-renewable resources, investment strategies, management of financial resources, resources allocation, planning and control of productive systems (manufacturing, chemical, cells, species), definition of therapy protocols, motion planning and control in autonomous mobile robotics, aerospace navigation, synthesis in decision support systems, among others (CORDIS (2013), Fossen (1994), Hou et al. (2007), Liu and Hu (2010), Mahmoudian et al. (2007), Mahmoudian and Woolsey (2010), McGillivary et al. (2012), Melli (September 2008), Triantafyllou et al. (2002)).

A general formulation for an optimal control problem is given by

$$\left\{\begin{array}{ll} \text{Minimize} & g(x(1)) \\ \text{by choosing} & (x,u):[0,1]\to\mathbb{R}^n\times\mathbb{R}^m \\ \text{subject to} & \dot{x}(t)=f(t,x(t),u(t)) \qquad , t\in[0,1]\,\mathcal{L}\text{-a.e.}, \\ & x(0)\;=\;x_0 \\ & u(t)\in\Omega(t)\subset\mathbb{R}^m \end{array}\right. \qquad (P)$$

where x_0 is the initial point of the trajectory.

Let us give the general definitions to understand the problem (P) in its totality:

- dynamic system: $\dot{x}(t)=f(t,x(t),u(t))$, is the system whose state variable conveys its past history. Its future evolution depends not only on the future ("inputs") but also on the current value of the state variable;
- trajectory: $x:[0,1]\to\mathbb{R}^n$, such that x is absolutely continuous, and is the solution, for a given control function, of the differential equation that gives the dynamics, satisfying the boundary conditions and the state constraints (when applicable), $t\in[0,1]\,\mathcal{L}$-a.e.;
- control function: $u\in\mathcal{U}$ with

$$\mathcal{U}=\{u:[0,1]\to\mathbb{R}^m\;:\;u\text{ is measurable, and }u(t)\in\Omega(t),t\in[0,1]\,\mathcal{L}\text{-a.e.}\}\,;$$

- process: (x,u) with $u\in\mathcal{U}$ and x is the corresponding trajectory;
- admissible control process: (x,u) satisfying all the constraints of (P);

- attainable set: $\mathcal{A}(1;(x_0,0))$ is the set of state space points that can be reached from x_0 with admissible control strategies,

$$\mathcal{A}(1;(x_0,0)) = \{x(1) : \forall (x,u) \text{ admissible control process with } x(0)=x_0\} ;$$

- boundary process: control process whose trajectory (or a given function of it) remains in the boundary of the attainable set (or a given function of it);

- strong local minimizer: $(\bar{x},\bar{u})$ admissible control process, such that

$$\exists\, \epsilon > 0,\, \forall (x,u) \text{ admissible control process, } \|x-\bar{x}\|_\infty < \epsilon \Rightarrow g(\bar{x}(1)) \leq g(x(1)).$$

In the formulation of the previous problem we presented a cost function that corresponds to one of the types of optimal control problems. Next we define the several types of the optimal control problems that we can have in our studies.

- Mayer - $g(x(1))$
- Lagrange - $\displaystyle\int_0^1 L(s,x(s),u(s))ds.$
- Bolza - $g(x(1)) + \displaystyle\int_0^1 L(s,x(s),u(s))ds$

If we consider a new variable $z : \dot{z}(t) = L(t,x(t),u(t))$, with $z(0)=0$, then a Lagrange problem becomes a Mayer problem with cost $z(1)$ and a Bolza problem becomes a Mayer problem with cost $g(x(1)) + z(1)$.

Also for the constraints, we can have different types, they are:

- final state: $x(1) = x_1$
- set endpoint: $(x(0),x(1)) \in C_0 \times C_1$
- state: $h(t,x(t)) \leq 0$, $\forall t \in [0,1]$
- mixed state-control: $h(t,x(t),u(t)) \leq 0$, $\forall t \in [0,1]$
- isoperimetric: $\displaystyle\int_0^1 h(s,x(s),u(s))ds = a.$

We now state a general formulation for nonlinear optimal control problem.

$$\left\{\begin{array}{ll} \text{Minimize} & g(x(1)) \\ \text{subject to} & \dot{x}(t) = f(t,x(t),u(t)) \\ & x(0) \;=\; x_0 \\ & u(t) \in \Omega(t) \end{array}\right. , \; t \in [0,1]\, \mathcal{L}\text{-a.e.}, \qquad (P')$$

where x_0 is the initial point of the trajectory.

To guarantee that the problem (P') is well posed, we need some additional hypothesis,

H1: There exists solution to (P) and let us denote by (x^*,u^*) the optimal control process.

H2: There exists K_g and K_f such that, $\forall x,\, y \in \mathbb{R}^n$,

$$\|g(x)-g(y)\| \leq K_g\|x-y\|, \text{ and } \|f(t,x,u)-f(t,y,u)\| \leq K_f\|x-y\|, \; \forall (t,u) \in [0,1]\times\Omega(t).$$

H3: f and g are C^1 in x, being $\nabla_x g$ the gradient of g, and $D_x f$ the Jacobian of f.

H4: $f(\cdot, x, \cdot)$ is Lebesgue measurable in t and Borel measurable in u.

H5: $\Omega(t)$ is compact, $\forall t \in [0,1]$.

H6: $\exists K > 0$ such that $\sup_{u\in\Omega(t)} \Big\{\|f(t,x,u)\|\Big\} \leq K$, $\forall (t,x) \in [0,1] \times \mathbb{R}^n$.

Under these hypotheses, we give the necessary optimality conditions for optimal control problem (P'). Let x^* be an optimal trajectory for (P'). Then, there exists an absolutely continuous function $p : [0,1] \to \mathbb{R}^n$, satisfying

$$-\dot{p}^T(t) = p^T(t) D_x f(t, x^*(t), u^*(t)), \qquad [0,1]\mathcal{L}\text{-a.e. }, \tag{3.1}$$
$$-p^T(1) = \nabla_x g(x^*(1)). \tag{3.2}$$

where $u^* : [0,1] \to \mathbb{R}^m$ is a control strategy, such that $u^*(t)$ maximizes

$$v \to p^T(t) f(t, x^*(t), v) \quad \text{in} \quad \Omega(t), \qquad t \in [0,1]\mathcal{L}\text{-a.e. }. \tag{3.3}$$

The condition 3.3 eliminates the control as it defines implicitly

$$u^*(t) = \bar{u}(x^*(t), p(t)).$$

Then, solving (P') amounts to solve

$$-\dot{p}^T(t) = p^T(t) D_x f(t, x^*(t), \bar{u}(x^*(t), p(t))), \qquad p(1) = -\nabla_x g(x^*(1)),$$
$$\dot{x}^*(t) = f(t, x^*(t), \bar{u}(x^*(t), p(t))), \qquad x(0) = x_0.$$

Therefore, the existence of the solution for the problem requires the following hypotheses:

H'1: g is lower semi-continuous.

H'2: $\Omega(t)$ is compact, $\forall t \in [0,1]$ and $t \to \Omega(t)$ is Borel measurable.

H'3: f is continuous in all of its arguments.

H'4: there exists K_f such that, $\|f(t,x,u) - f(t,y,u)\| \leq K_f\|x-y\|$, $\forall (t,u) \in [0,1] \times \Omega(t)$.

H'5: $\exists K > 0 \,:\, |x \cdot f(t,x,u)| \leq K(1 + \|x\|^2)$ for all values of the arguments of f.

H'6: $f(t, x, \Omega(t))$ is convex $\forall x \in \mathbb{R}^n$ and $\forall t \in [0,1]$.

Until now, we didn't have the state constraints active, but we can have its, like the set inclusion of the state values at the time endpoints, and equality or inequality of the value of some nonlinear function of the graph of the state variable. So, the nonlinear optimal control problem with state constraints is

$$\begin{cases} \text{Minimize} & g(x(1)) \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t)) \\ & (x(0), x(1)) \in C \qquad , \ t \in [0,1]\, \mathcal{L}\text{-a.e.}, \\ & h(t, x(t)) \leq 0 \\ & u(t) \in \Omega(t) \subset \mathbb{R}^m \end{cases} \qquad (P'')$$

with initial point $x(0)$ and terminal point $x(1)$.

The hypotheses **H1-H6** must be valid for (P'') to be well posed. Furthermore, we need two additional hypothesis,

H7: h is continuous in t, differentiable in x and there exists K_h such that,

$$\forall x,\, y \in \mathbb{R}^n,\ \|h(t,x) - h(t,y)\| \leq K_h \|x - y\|,\ \forall t \in [0,1].$$

H8: The set C is compact.

3.3 Pontryagin's Maximum Principle

By the hypotheses, presented on the previous section, and the necessary optimality conditions, we can formulate the Pontryagin's Maximum Principle for problem (P'), as follows:

Theorem 3.1. ***(Maximum Principle of Pontryagin)*** *Let x^* be an optimal trajectory for (P').*

Then, there exists an absolutely continuous function $p : [0,1] \to \mathbb{R}^n$, satisfying

$$-\dot{p}^T(t) = p^T(t) D_x f(t, x^*(t), u^*(t)), \quad [0,1]\mathcal{L}\text{-}a.e. \ , \tag{3.4}$$

$$-p^T(1) = \nabla_x g(x^*(1)), \tag{3.5}$$

where $u^* : [0,1] \to \mathbb{R}^m$ is a control strategy such that $u^*(t)$ maximizes, for almost all $t \in [0,1]$, the pseudo-Hamiltonian or Pontryagin function

$$v \to p^T(t) f(t, x^*(t), v) \quad in \quad \Omega(t). \tag{3.6}$$

Sketch of the Proof:

Let $\varepsilon > 0$ sufficiently small, $\tau \in [0,1]$ to be a Lebesgue point of x^* and consider

$$u_{\varepsilon,\tau} = \begin{cases} \bar{u} & \text{if } t \in (\tau - \varepsilon, \tau] \\ u(t) & \text{if } t \in [0,1] \setminus (\tau - \varepsilon, \tau] \end{cases}$$

and $x_{\varepsilon,\tau}$ is the corresponding trajectory.

We have

$$\begin{aligned} 0 \ \leq \ g(x_{\varepsilon,\tau}(1)) - g(x^*(1)) \ &= \ \nabla_x g(x^*(1))[x_{\varepsilon,\tau}(1) - x^*(1)] + O(\varepsilon) \\ &= \ \nabla_x g(x^*(1)) \Phi(1,\tau)[x_{\varepsilon,\tau}(\tau) - x^*(\tau)] + O(\varepsilon), \end{aligned}$$

where Φ is the state transition matrix of the system $\dot{\xi} = D_x f(t, x^*, u^*)\xi$.

From the Lipschitz continuity of f in x, τ being a Lebesgue point and by defining $\xi(\tau) := f(\tau, x_{\varepsilon,\tau}(\tau), \bar{u}) - f(\tau, x^*(\tau), u^*(\tau))$, we have that

$$\|[x_{\varepsilon,\tau}(\tau) - x^*(\tau)] - \varepsilon\xi(\tau)\| = O(\varepsilon).$$

Defining $p^T(1) := -\nabla_x g(x^*(1))$ and $p^T(t) := -\nabla_x g(x^*(1))\Phi(1,t)$, we have that $p : [0,1] \to \mathbb{R}^m$ satisfy $-\dot{p}^T(t) = p^T(t)\mathrm{D}_x f(t, x^*(t), u^*(t))$.

So,

$$0 \leq \lim_{\varepsilon\to 0} \frac{g(x_{\varepsilon,\tau}(1)) - g(x^*(1))}{\varepsilon} = p^T(\tau)[f(\tau, x^*(\tau), u^*(\tau)) - f(\tau, x^*(\tau), \bar{u})]. \qquad \blacksquare$$

The formulation of the Maximum Principle of Pontryagin for problem (P''), with state constraints, is given by:

Theorem 3.2. ***(Maximum Principle of Pontryagin)*** *Let (x^*, u^*) be an optimal control process for (P'').*

Then, there exist an absolutely continuous function $p : [0,1] \to \mathbb{R}^n$, a non negative number λ, and a non negative measure ν supported on $\{t \in [0,1] : h(t, x^*(t)) = 0\}$, satisfying:

$$\begin{aligned}
&\|p\| + \|\nu\| + \lambda > 0,\\
&-\dot{p}^T(t) = \Big[p^T(t) + \int_{[0,t)} D_x h(s, x^*(s))\nu(ds)\Big] D_x f(t, x^*(t), u^*(t)), \quad [0,1]\mathcal{L}\text{-}a.e.\ ,\\
&\Big(p^T(0), -p^T(1) - \int_{[0,1]} D_x h(t, x^*(t))\nu(dt)\Big) \in \Big(0, \nabla_x g(x^*(1))\Big) + N_C(x^*(0), x^*(1)),
\end{aligned}$$

and $u^* : [0,1] \to \mathbb{R}^m$ is a control strategy such that $u^*(t)$ maximizes the mapping

$$v \to \Big[p^T(t) + \int_{[0,t)} D_x h(s, x^*(s))\nu(ds)\Big] f(t, x^*(t), v) \ \text{in}\ \Omega(t)\ [0,1]\mathcal{L}\text{-}a.e.\ .$$

3.4 Optimal Multiprocesses

In Clarke and Vinter (1989a) we find the theory for the optimal multiprocess problem, and in Clarke and Vinter (1989b) we have some applications of this theory.

A controlled multiprocess system, (x^i, u^i), $i = 1, \ldots N$,, with x^i absolutely continuous function ($x^i(t) \in X^i$) and $u^i \in L^\infty$, consists in a finite number of dynamic control systems which are active in, possibly different, free endpoint time intervals, subject to their own state variable and control constraints, while sharing joint time and state endpoint constraints, that is,

$$\begin{cases} \dot{x}^i = f^i(t, x^i, u^i) \\ h^i(t, x^i) \in C^i & [t_0^i, t_1^i]\,\mathcal{L} - \text{a.e.} \\ u^i \in \mathcal{U}^i \end{cases} \tag{3.7}$$

$$\bar{h}(\{(x^i(t_0^i), x^i(t_1^i)) : i = 1, \ldots, N\}) \in \bar{C}. \tag{3.8}$$

Here, for $i = 1, \ldots, N$, we assumed that the following hypotheses are satisfied:

HM1: $\mathcal{U}^i = \{u \in L^\infty([t_0^i, t_1^i]; \mathbb{R}^m) : u^i(t) \in \Omega^i\}$ is a Borel measurable set, and $\Omega^i \subset \mathbb{R}^m$.

HM2: $C^i \in \mathbb{R}^{k_i}$, and $\bar{C} \in \mathbb{R}^{\bar{k}}$ are closed sets.

HM3: $f^i : [t_0^i, t_1^i] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lebesgue measurable in t and Lipschitz continuous in x^i for any feasible value of u^i for each (t, x^i), $h^i : [t_0^i, t_1^i] \times \mathbb{R}^n \to \mathbb{R}^{k_i}$ is continuous in t and Lipschitz continuous in x^i, and $\bar{h} : \mathbb{R}^{2nN} \to \mathbb{R}^{\bar{k}}$ is Lipschitz continuous in all its arguments.

The goal is to to choose the set of N triples $(x^i(t_0^i), x^i(t_1^i), u^i)$ that optimize a given global performance function $J(x^i(t_0^i), x^i(t_1^i), u^i)$ while the associated controlled multiprocess system satisfies the constraints (3.7) and (3.8).

The optimal multiprocess problem is given by

$$\begin{array}{ll} \text{Minimize} & g(\{t_0^i, t_1^i, x^i(t_0^i), x^i(t_1^i)\}) \\ \text{that satisfy} & \dot{x}^i = f^i(t, x^i, u^i) \\ & h^i(t, x^i) \in C^i \\ & u^i \in \mathcal{U}^i \\ & \{t_0^i, t_1^i, x^i(t_0^i), x^i(t_1^i)\} \subset \Lambda \end{array} \tag{3.9}$$

with g a locally Lipschitz continuous function and Λ a closed set.

Going against the theory developed in Clarke and Vinter (1989a), consider C a given closed set in

$$\prod_i \{(t_0^i, t_1^i, a_0^i) : \, a_0^i \in \mathbb{R}^n, \, t_0^i, t_1^i \in \mathbb{R}, \, t_0^i \leq t_1^i\}$$

and let $\psi : \mathbb{R}^{nN} \to \mathbb{R}^d$ be a given Lipschitz continuous function. We define the reachable set (with respect to C and ψ) to be

$$\mathcal{R}_{\psi,C} = \{\psi(\{y^i(t_1^i)\}) : \, \{t_0^i, t_1^i, y^i(\cdot), u^i(\cdot)\} \text{ is a multiprocess such that} \{t_0^i, t_1^i, y^i(t_0^i)\} \in C\}.$$

We say that a multiprocess $\{t_0^i, t_1^i, y^i(\cdot), u^i\}$ is a boundary multiprocess relative to ψ and C if

$$\{t_0^i, t_1^i, y^i(t_0^i)\} \in C \text{ and } \psi(\{y^i(t_1^i)\}) \in \partial \mathcal{R}_{\psi,C}$$

(∂ denotes boundary).

The following theorem is a necessary condition that a multiprocess be associated with a boundary point of the reachable set.

Theorem 3.3. *Let $\{t_0^i, \, t_1^i, \, x^i(\cdot), \, u^i(\cdot)\}$ be a boundary multiprocess (with respect to ψ and C). Assume that*

$$\text{graph}\{x^i(\cdot)\} \subset \text{interior}\{X^i\}$$

*for $i = 1, \cdots, N$ and that hypotheses **HM1-HM3** are satisfied. Then there exist a vector v of unit length, numbers h_0^i, h_1^i and absolutely continuous functions $p^i(\cdot) : [t_0^i, t_1^i] \to \mathbb{R}^n$ for $i = 1, \cdots, N$ and a number c (whose magnitude is governed by the Lipschitz constant in hypothesis **HM3** together with the Lipschitz rank of ψ restricted to some neighbourhood of $\{x^i(t_1^i)\}$), with the following properties:*

$$-\dot{p}^i(t) \in \partial_x \mathcal{H}^i(t, x^i(t), u^i(t), p^i(t)) \quad \text{a.e.}\, t \in [t_0^i, t_1^i], \tag{3.10}$$

$$\mathcal{H}^i(t, x^i(t), u^i(t), p^i(t)) = \max_{w \in \mathcal{U}^i} \mathcal{H}^i(t, x^i(t), w, p^i(t)) \quad \text{a.e.}\, t \in [t_0^i, t_1^i], \tag{3.11}$$

$$h_0^i \in \text{coess}_{t \to t_0^i} [\sup_{w \in \mathcal{U}^i} \mathcal{H}^i(t, x^i(t_0^i), w, p^i(t_0^i))], \tag{3.12}$$

$$h_1^i \in \text{coess}_{t \to t_1^i} [\sup_{w \in \mathcal{U}^i} \mathcal{H}^i(t, x^i(t_1^i), w, p^i(t_1^i))] \tag{3.13}$$

for $i = 1, \cdots, N$, $\{p^i(t_1^i) \in \partial\psi^*(\{x^i(t_1^i)\})v\}$ *and*
$\{-h_0^i, h_1^i, p^i(t_0^i), p^i(t_1^i)\} \in N_\Lambda(\cdot) + c\nabla_{\Pi_{i=1}^N \{x^i(t_0^i), x^i(t_1^i)\}} g(\{t_0^i, t_1^i, x^(t_0^i), x^i(t_1^i)\})$.

From the theorem 3.3, we have the following maximum principle for solutions to the optimal multiprocess problem.

Theorem 3.4. *Let* $\{t_0^i,\, t_1^i,\, x^i(\cdot),\, u^i(\cdot)\}$ *be a solution to 3.9. Assume that*

$$\text{graph}\{x^i(\cdot)\} \subset \text{interior}\{X^i\}$$

for $i = 1, \cdots, N$ *and that hypotheses* ***HM1-HM3*** *are satisfied. Then there exist a real number* $\lambda \geq 0$, *real numbers* h_0^i, h_1^i, *and absolutely continuous functions* $p^i(\cdot) : [t_0^i, t_1^i] \to \mathbb{R}^n$ *for* $i = 1, \cdots, n$ *and a constant c, such that* $\lambda + \sum_i |p^i(t_1^i)| = 1$ *and we have*

$$-\dot{p}^i(t) \in \partial_x \mathcal{H}^i(t, x^i(t), u^i(t), p^i(t)) \quad \text{a.e.}\, t \in [t_0^i, t_1^i], \tag{3.14}$$

$$\mathcal{H}^i(t, x^i(t), u^i(t), p^i(t)) = \max_{w \in \mathcal{U}^i} \mathcal{H}^i(t, x^i(t), w, p^i(t)) \quad \text{a.e.}\, t \in [t_0^i, t_1^i], \tag{3.15}$$

$$h_0^i \in \text{coess}_{t \to t_0^i} [\sup_{w \in \mathcal{U}^i} \mathcal{H}^i(t, x^i(t_0^i), w, p^i(t_0^i))], \tag{3.16}$$

$$h_1^i \in \text{coess}_{t \to t_1^i} [\sup_{w \in \mathcal{U}^i} \mathcal{H}^i(t, x^i(t_1^i), w, p^i(t_1^i))] \tag{3.17}$$

for $i = 1, \cdots, n$, *and*

$$\{-h_0^i, h_1^i, p^i(t_0^i), -p^i(t_1^i)\} \in c\partial d_\Lambda + \lambda \partial g \tag{3.18}$$

where the generalizes gradients ∂d_Λ *and* ∂g *are evaluated at* $\{t_0^i, t_1^i, x^i(t_0^i), x^i(t_1^i)\}$.

On chapter 7 we present an optimal multiprocesses problem, but in our case we don't need a co-essential of the supreme and the generalized gradients, we use these results taking the limits for h_0^i and h_1^i and computing the gradients of the functions.

Chapter 4

Preliminary Results on Fluid Mechanics

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4.1 Introduction

Fluid mechanics is the area of physics that studies the impact of forces on fluids (Batchelor (1967), Pope (2000)) that respect the conservation of energy, mass, entropy, linear and angular momenta, the latter being expressed by Navier-Stokes equations (Doering and Gibbon (1995)). In particular, we are interested in those whose dynamics is given by vortex systems (Saffman (1992), Chorin (1994), Newton (2001)), Couette and Poiseuille flows (Chossat and Iooss (1994)).

Optimal control in systems governed by partial differential equations is well documented in Lions (1971) for the linear case. The review Protas (2008) addressed is the case of truncated Navier-Stokes solutions, as well as the control of vortex dynamics, singular solutions of the two-dimensional Euler equations.

In this chapter, we present a brief review of some concepts and results of fluids useful in this work.

4.2 Euler and Navier-Stokes Equations

The equations describing the motion of a fluid are the Navier-Stokes equations

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} + \mathbf{f}(\mathbf{x}, t)\,, \tag{4.1}$$

where Δ is the Laplacian operator, $\mathbf{v}$ is the fluid velocity, p is the pressure, ν is the kinematic viscosity of the fluid, and $\mathbf{f}$ is the forcing term. For the case where the fluid is incompressible, we have the additional condition on the velocity field

$$\nabla \cdot \mathbf{v} = 0\,. \tag{4.2}$$

This equation must be supplemented by suitable initial and boundary conditions. Due to the complexity of the equations (4.1)-(4.2), exact solutions are only known for particular cases. Taking $\nu = 0$ in (4.1), we obtain the so-called Euler equations

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{f}(\mathbf{x}, t)\,, \tag{4.3}$$

with the incompressible condition (4.2).

The vorticity field, ω, is the curl of the velocity field, $\mathbf{v}$, i.e.

$$\omega = \nabla \times \mathbf{v}. \tag{4.4}$$

On the plane we have $\omega = \partial_1 v_2 - \partial_2 v_1$. Taking the curl on (4.3) we have

$$\partial_t \omega + (\mathbf{v} \cdot \nabla)\omega = \nabla \times \mathbf{f}(\mathbf{x}, t) \tag{4.5}$$

with the incompressible condition $\nabla \cdot \omega = 0$.

In two-dimensions, $\mathbf{x} = (x, y)$, the Euler equations, thanks to the incompressible condition, can be rewritten in terms of a stream-function, ψ,

$$\frac{\partial \Delta\psi}{\partial t} + J(\Delta\psi, \psi) = 0, \tag{4.6}$$

where the connection between the vorticity and the stream-function is given by the Poisson equation $\Delta\psi = -\omega$. On the plane, the Poisson equation has the Green function

$$G(x, y) = -\frac{1}{2\pi} \log\left(\sqrt{x^2 + y^2}\right). \tag{4.7}$$

So, the solution for Poisson equation can be written as

$$\psi = -\frac{1}{2\pi} \int \log\left(\sqrt{(x - x')^2 + (y - y')^2}\right) \omega(x', y')\, dx'dy'. \tag{4.8}$$

Since $\mathbf{v} = (\partial_2, -\partial_1)\psi$, we have

$$\mathbf{v} = \mathbf{K} * \omega = -\frac{1}{2\pi} \int (\partial_2, -\partial_1) \log\left(\sqrt{(x - x')^2 + (y - y')^2}\right) \omega(x', y')\, dx'dy'\,, \tag{4.9}$$

where $\mathbf{K}(t, x, y) = -\frac{1}{2\pi}(\partial_2, -\partial_1) \log\left(\sqrt{x^2(t) + y^2(t)}\right)$ is the kernel.

4.3 Point Vortices

Vortices are two-dimensional points, each one with its own circulation, that induce a velocity field in the plane. Formally, such velocity field is a singular solution of the two-dimensional incompressible Euler equations on the whole plane, corresponding to the vorticity field $\omega(t,\mathbf{x}) = \sum_{j=1}^{N} \omega_j \delta(\mathbf{x}(t) - \mathbf{x}_j(t))$. Consider N point vortices located in (x_j, y_j), each one with the corresponding circulation ω_j $(j = 1, \cdots, N)$. The evolution of this point vortices, in $\mathbb{R}^2$, is given by the ODE system

$$\left\{\begin{array}{lll} \dot{x}_j & = & -\frac{1}{2\pi}\sum_{l\neq j}^{N} \frac{\omega_l (y_j - y_l)}{(x_j-x_l)^2+(y_j-y_l)^2} \\ \dot{y}_j & = & \frac{1}{2\pi}\sum_{l\neq j}^{N} \frac{\omega_l (x_j - x_l)}{(x_j-x_l)^2+(y_j-y_l)^2} \end{array}\right. , \; j = 1, 2, \cdots, N, \tag{4.10}$$

or in equivalent complex form,

$$\dot{z}_j^* = \frac{1}{2\pi i} \sum_{l\neq j}^{N} \frac{\omega_l}{z_j - z_l}, \tag{4.11}$$

where $z_j = x_j + i y_j$, for $j = 1, 2, \cdots, N$. The system composed by these N-vortices has the Hamiltonian structure defined by

$$\left\{\begin{array}{lll} \omega_j \dot{x}_j & = & \frac{\partial \mathcal{H}}{\partial y_j} \\ \omega_j \dot{y}_j & = & -\frac{\partial \mathcal{H}}{\partial x_j} \end{array}\right. , \; j = 1, 2, \cdots, N,$$

where the Hamiltonian is

$$\mathcal{H} = -\frac{1}{4\pi} \sum_{l\neq j}^{N} \omega_j \omega_l \log\left(|z_j - z_l|\right) . \tag{4.12}$$

The total vorticity of the system is $\omega = \sum_{j=1}^{N} \omega_j$. A point vortex system possesses the invariant quantities:

1. the Hamiltonian $\mathcal{H}$, that is the interaction energy for the vortex system;
2. the moment of vorticity $Q + iP = \sum_{j=1}^{N} \omega_j Z_j$; (when divided by ω we get the center of vorticity of the system)
3. the angular impulse $I = \sum_{j=1}^{N} \omega_j |z_j|^2$.

In Newton (2001) the reader can find a detailed study of the dynamics of N vortices in the plane.

Theorem 4.1. *For $N \leq 3$, the N-vortex problem is integrable for all values of ω_j. If $\omega = 0$, the 4-vortex problem is also integrable.*

The next theorems state the necessary and sufficient conditions for the equilibria or collapse of a system with three vortices (Newton (2001)).

Theorem 4.2. ***(Equilibria)***

1. ***(Fixed Equilibria,*** $N = 3$***)*** *Necessary and sufficient conditions for fixed equilibria in the plane are:*
   (a) *The vortices are collinear with* $(\mathbf{x}_2 - \mathbf{x}_1) = \frac{\omega_2}{\omega_3}(\mathbf{x}_1 - \mathbf{x}_3)$*.*
   (b) $(\sum \omega_j)^2 - \sum \omega_j^2 = 0$*.*
2. ***(Relative Equilibria,*** $N = 3$***)*** *The only relative equilibria are collinear states or equilateral triangles.*
   (a) *All equilateral triangles form relative equilibria that rotate rigidly about their center of vorticity. When* $\omega = 0$*, the center of vorticity is at infinity and the vortices translate in parallel.*
   (b) *Collinear states can form relative equilibria iff the triangle area is* 0*.*

Theorem 4.3. *(*$N = 3$ ***Collapse)*** *Necessary and sufficient conditions for the self-similar collapse of three vortices are:*

1. *The three strengths do not have the same sign, hence we take* $\omega_3 < 0$*.*
2. $\sum_{j=1,\, j\neq l}^{3} \omega_l \omega_j I_{lj}^2 = 0,\ l = 1, 2, 3.$
3. *The initial configuration is not an equilibrium.*
4. *The vortex circuit be positive.*

Vortex models have been employed in a vast range of applications in science and engineering. One of them is the modelling generation of thrust in fish-like locomotion (Protas (2008)). On chapter 7 we can see the idea for this application on robotic fishes, on Pereira et al. (2017a), Pereira et al. (2017b) and Grilo et al. (2018) on AUV's.

On chapters 6 and 7 we study optimal control problems for a motion of a particle that lives in a fluid whose dynamics is given by one vortex or two vortices. The calculations for the case of one vortex advecting one passive particle (vortex with circulation equal 0), and for the case of two vortices advecting one particle are presented on chapter 6.

4.4 Motion of Two Vortices

The dynamic equations for two vortices are

$$\begin{cases} \dot{z}_1^* &= \frac{1}{2\pi i}\frac{\omega_2}{z_1 - z_2} \\ \dot{z}_2^* &= \frac{1}{2\pi i}\frac{\omega_1}{z_2 - z_1} \end{cases} \tag{4.13}$$

with initial conditions $z_1(0) = z_{1,0}$ and $z_2(0) = z_{2,0}$, respectively. Multiplying and dividing (4.13) by $i(z_1 - z_2)^*$ we obtain

$$\begin{cases} \dot{z}_1^* &= \frac{\omega_2 i}{2\pi h^2}(z_1 - z_2) \\ \dot{z}_2^* &= -\frac{\omega_1 i}{2\pi h^2}(z_1 - z_2) \end{cases} \tag{4.14}$$

where $h^2 = |z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Since

$$\frac{dh^2}{dt} = 2((x_1 - x_2)(\dot{x_1} - \dot{x_2}) + (y_1 - y_2)(\dot{y_1} - \dot{y_2})), \tag{4.15}$$

and

$$\dot{z}_1 = \frac{\omega_2 i}{2\pi h^2}(z_1 - z_2) \Leftrightarrow \begin{cases} \dot{x_1} = -\frac{\omega_2(y_1 - y_2)}{2\pi h^2} \\ \dot{y_1} = \frac{\omega_2(x_1 - x_2)}{2\pi h^2} \end{cases} \tag{4.16}$$

$$\dot{z}_2 = -\frac{\omega_1 i}{2\pi h^2}(z_1 - z_2) \Leftrightarrow \begin{cases} \dot{x_2} = \frac{\omega_1(y_1 - y_2)}{2\pi h^2} \\ \dot{y_2} = -\frac{\omega_1(x_1 - x_2)}{2\pi h^2} \end{cases} \tag{4.17}$$

we obtain $\frac{dh^2}{dt} = 0$. This way, we conclude that the distance between the two vortices is constant in time. So, $h = |z_1(0) - z_2(0)|$. In this type of configuration of flow field, we have two cases to be considered: (i) $\omega_1 + \omega_2 = 0$, and (ii) $\omega_1 + \omega_2 \neq 0$.

4.4.1 Case $\omega_1 + \omega_2 = 0$

In this case, we get symmetrical circulations for each one of the vortices. So,

$$\begin{cases} \dot{z}_1 &= \frac{\omega_1 i}{2\pi h^2}(z_2 - z_1) \\ \dot{z}_2 &= \frac{\omega_1 i}{2\pi h^2}(z_2 - z_1) \end{cases}. \tag{4.18}$$

Since $\frac{d}{dt}(z_2 - z_1) = 0$, we have $z_2(t) - z_1(t) = z_2(0) - z_1(0)$, and the motion equations are

$$\begin{cases} z_1(t) &= z_1(0) + \frac{\omega_1 i}{2\pi h^2}(z_2(0) - z_1(0))t \\ z_2(t) &= z_2(0) + \frac{\omega_1 i}{2\pi h^2}(z_2(0) - z_1(0))t \end{cases}. \tag{4.19}$$

Therefore, the vortices are moving in parallel lines, like in figure 4.1.

4.4.2 Case $\omega_1 + \omega_2 \neq 0$

In this case, multiplying the equations (4.16) by $\frac{\omega_1}{\omega_1+\omega_2}$ and (4.17) by $\frac{\omega_2}{\omega_1+\omega_2}$, we get

$$\begin{cases} \frac{\omega_1}{\omega_1+\omega_2}\dot{z}_1 &= \frac{\omega_1\omega_2 i}{2\pi h^2(\omega_1+\omega_2)}(z_1 - z_2) \\ \frac{\omega_2}{\omega_1+\omega_2}\dot{z}_2 &= -\frac{\omega_1\omega_2 i}{2\pi h^2(\omega_1+\omega_2)}(z_1 - z_2) \end{cases}.$$

Thus, $\frac{\omega_1\dot{z}_1+\omega_2\dot{z}_2}{\omega_1+\omega_2} = 0$, allowing us to conclude that

$$\frac{\omega_1 z_1(t) + \omega_2 z_2(t)}{\omega_1 + \omega_2} = \frac{\omega_1 z_1(0) + \omega_2 z_2(0)}{\omega_1 + \omega_2}. \tag{4.20}$$

Rewriting the equations (4.16)-(4.17), and using (4.20) we obtain

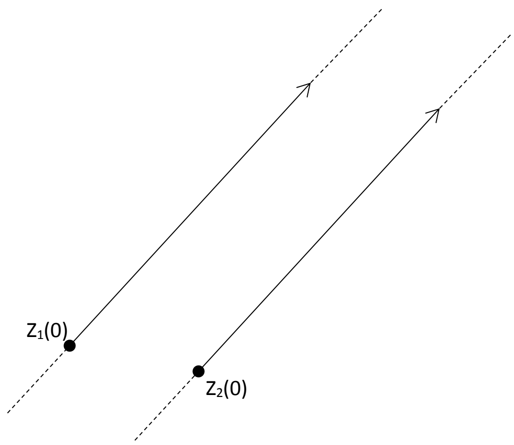

Figure 4.1: Motion of two vortices with circulations $\omega_1 + \omega_2 = 0$.

$$\begin{cases} \dot{z}_1 & = & \frac{(\omega_1+\omega_2)i}{2\pi h^2}\left(z_1 - \frac{\omega_1 z_1(0)+\omega_2 z_2(0)}{\omega_1+\omega_2}\right) \\ \dot{z}_2 & = & \frac{(\omega_1+\omega_2)i}{2\pi h^2}\left(z_2 - \frac{\omega_1 z_1(0)+\omega_2 z_2(0)}{\omega_1+\omega_2}\right) \end{cases} .$$

These are first order linear differential equations, so we get the following trajectories

$$\begin{cases} z_1(t) & = & \frac{1}{\omega_1+\omega_2}\left(\omega_1 z_1(0) + \omega_2 z_2(0) - (z_2(0) - z_1(0)\omega_2 e^{i\Omega t}\right) \\ z_2(t) & = & \frac{1}{\omega_1+\omega_2}\left(\omega_1 z_1(0) + \omega_2 z_2(0) + (z_2(0) - z_1(0)\omega_2 e^{i\Omega t}\right) \end{cases} , \tag{4.21}$$

with $\Omega = \frac{\omega_1+\omega_2}{2\pi h^2}$.

Therefore, the vortices are moving in circle (in concentric circumferences), rotating around their center of circulation, $C = \frac{\omega_1 z_1(0)+\omega_2 z_2(0)}{\omega_1+\omega_2}$, (see figure 4.2).

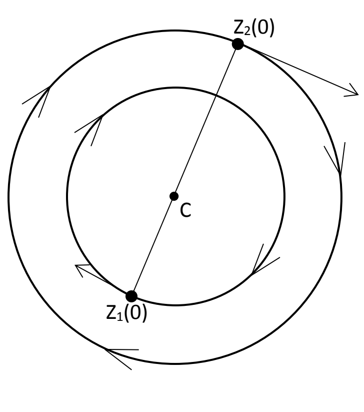

Figure 4.2: Motion of two vortices with circulations $\omega_1 + \omega_2 \neq 0$.

4.5 Couette and Poiseuille Flows

The Couette flow is the flow of a viscous fluid whose dynamics runs between two surfaces one of which moves with respect to the other. The most common configuration of this type of flow takes the form of two parallel plates or the space between two concentric cylinders.

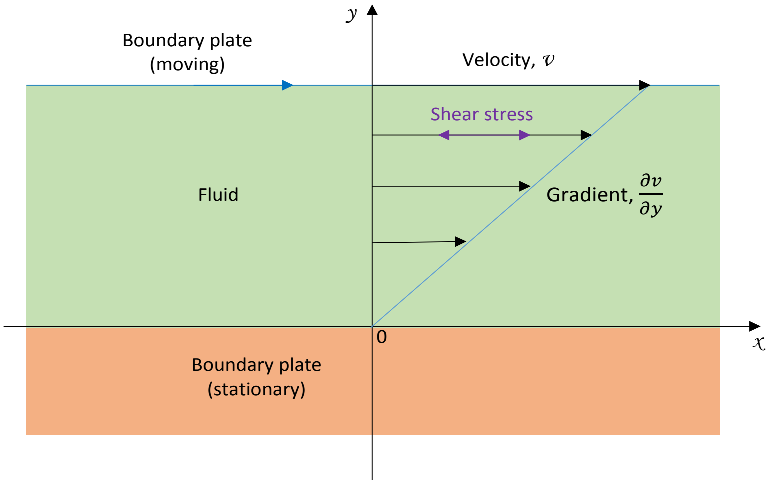

Figure 4.3: Simple Couette configuration using two infinite flat plates.

The Poiseuille flow is the laminar flow through a pipe of uniform and circular cross-section.

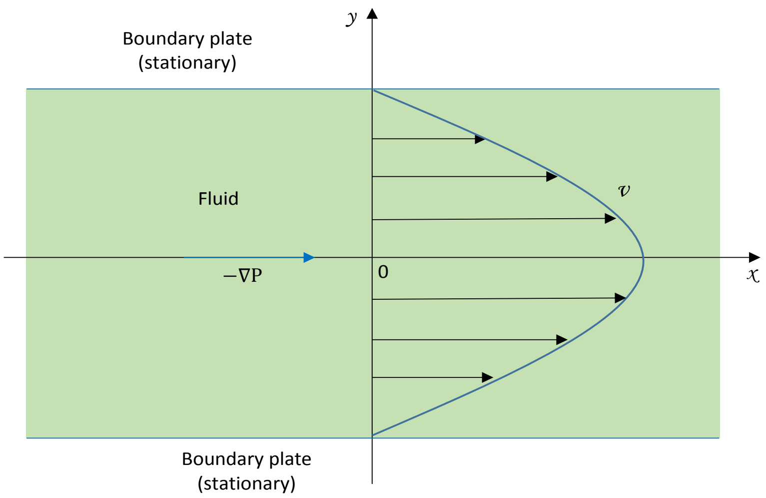

Figure 4.4: Simple Poiseuille configuration using two infinite flat plates.

Both of these flows correspond to particular solutions of the Navier-Stokes equations and are used throughout this work.

Chapter 5

Optimal Control Applied to Couette and Poiseuille Flows

Summary

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5.2 ARTICLE COPY "Optimal control of particle advection in Couette and Poiseuille flows", T. Grilo, F. L. Pereira, S. Gama – Journal Article *Conference Papers in Science*, **Volume 2013, Article ID 783510, Hindawi Publishing Corporation, 2013** **29**

5.3 ARTICLE COPY "On the Optimal Control of Flow Driven Dynamic Systems", T. Grilo, S. Gama, F. L. Pereira – Chapter Book *Mathematics of Energy and Climate Change*, **pages 183–189, Springer, 2015** **35**

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5.1 Introduction

The Couette and Poiseuille steady flows are two particular solutions of the incompressible two-dimensional Navier-Stokes equations (Chossat and Iooss (1994), Temam (1977)), corresponding to two types of fluid velocity profiles (linear and quadratic, respectively), which were exposed in chapter 4.

Here, we present two optimal control problems for a motion of a free particle advected by Couette and Poiseuille flows. We want to steer it from a initial given point to a final point in a minimum time, using the Pontryagin Maximum Principle (Pontryagin et al. (1962)).

In this chapter, we present two published articles (Grilo et al. (2013), Grilo et al. (2015)) about the study of these optimal control problems. In Grilo et al. (2013), we study the case where the particle moves from the initial point $(0, b)$ to the end point (x_f, b), with $0 < b < L$, where L is the width or half of the width of the channel, for linear and parabolic velocity profiles, respectively. This problem was previously studied by us in the article Grilo et al. (2012), but in Grilo et al. (2013) the calculations are explained in more detail. In Grilo et al. (2015), we study a general case, i.e. when the motion of the particle is performed from the initial point $(0, b)$ to the end point (x_f, c), with $0 < b < L$ and $0 < c < L$.

5.2 ARTICLE COPY "Optimal control of particle advection in Couette and Poiseuille flows", T. Grilo, F. L. Pereira, S. Gama – Journal Article *Conference Papers in Science*, Volume 2013, Article ID 783510, Hindawi Publishing Corporation, 2013

Optimal Control of Particle Advection in Couette and Poiseuille Flows

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Conference Paper

Optimal Control of Particle Advection in Couette and Poiseuille Flows

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We present the problem of minimum time control of a particle advected in Couette and Poiseuille flows and solve it by using the Pontryagin maximum principle. This study is a first step of an effort aiming at the development of a mathematical framework for the control and optimization of dynamic control systems whose state variable is driven by interacting ODEs and PDEs which can be applied in the control of underwater gliders and mechanical fishes.

1. Introduction

This paper represents a first step for the optimal control of dynamic systems whose state evolves through the interaction of ordinary differential equations and the partial differential equations, [1, 2], which will provide a sound basis for the design and control of new advanced engineering systems. In Figure 1, two representative examples of the class of applications are considered: (i) underwater gliders, that is, winged autonomous underwater vehicles (AUVs) which locomote by modulating their buoyancy and their attitude in its environment, and (ii) robotic fishes. Motion modeling of these two types of systems can be found in [3, 4] and [5], respectively.

In spite of the key roots of the Optimal Control Theory having been established in the sixties for control systems with dynamics given by ordinary differential equations, [6], its sophistication in multiple directions has been progressing unabated (see, among others, [7, 8]). However, there still remains a large gap in what concerns dynamic control systems driven by partial differential equations, [2], and it is largely inexistent for hybrid systems in the sense that the controlled dynamics involve both partial and ordinary differential equations.

In this paper, we formulate and solve two optimal control problems. Each one of these problems corresponds to a particular solution of the incompressible Navier-Stokes equation in two spatial dimensions. These particular solutions are, respectively, the steady Couette and Poiseuille flows.

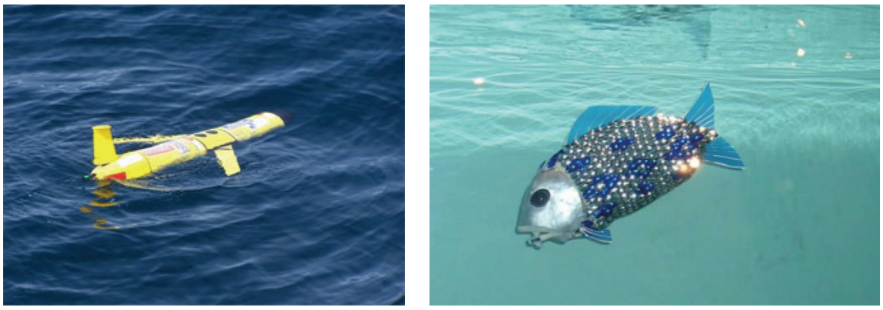
The Couette flow is the steady laminar unidirectional and two-dimensional flow due to the relative motion of two infinite horizontal and parallel rigid plates [9]. The liquid between these two plates is driven by the viscous drag force originated by the uniform motion of the upper plate which moves in the x-direction with velocity v_0 (the lower plate is at rest). In this case, the velocity of such a flow has a linear profile and is given by

$$v(x,y) = (my, 0), \quad x \in \mathbb{R}, \; y \in [0, L] \tag{1}$$

with $m = v_0/L$, the plates being L distance units apart (Figure 2(a)).

The Poiseuille flow is the steady flow due to the presence of a pressure gradient between two fixed (i.e., with zero relative velocity) rigid plates [9]. In this case, a parabolic velocity profile is of the form

$$v(x,y) = \left(a - \frac{a}{L^2} y^2, 0\right), \quad x \in \mathbb{R}, \; y \in [-L, L], \tag{2}$$



(a) (b)

Figure 1: Underwater glider (a), robotic fish (b).

where, now, L is half the distance between the upper and lower plates (Figure 2(b)).

2. Minimum Time Control Problem

Consider a flow in a channel of the width L with a given velocity field and a particle placed in this flow. Let $(0, b)$ be the initial position of the particle, with $0 \leq b \leq L$.

The objective of this problem is to determine the control function $u(\cdot) = (u_x(\cdot), u_y(\cdot))$ to be applied to the particle so that it will move in the channel from the initial position to the end point (x_f, b) in minimum time while subject to the flow field. Let $X(t) = (x(t), y(t))$ be the position of the particle at time t, the control problem can be formulated as follows:

$$\begin{aligned} &\text{Minimize} && T \\ &\text{subject to} && \dot{X}(t) = F(X(t), u(t)), \\ &&& X(0) = (0, b), \\ &&& X(T) = (x_f, b), \qquad \forall t \in [0, T]. \\ &&& y(t) \in [0, L], \\ &&& \|u(t)\|_\infty \leq 1, \end{aligned} \tag{3}$$

In the following two particular cases, for simplicity of notation, we will not indicate the time t as an independent variable of the other variables, although this is the case we are considering.

2.1. Couette Flow: Linear Velocity Profile. In the case of linear flow (with slope $m = v_0/L > 0$), the velocity field of the flow is given by $v(x, y) = (y/m, 0)$. So, the dynamics of this control system are given by $F(X, u) = (y/m + u_x, u_y)$.

The Pontryagin maximum principle, [6], allows us to determine the optimal control $u^* = (u_x^*, u_y^*)$ by using the maximization of Pontryagin's function $H(X, P, u)$ (here, $P = (p_x, p_y)$ is the adjoint variable satisfying $-\dot{P} = \nabla_X H(X, P, u)$, ∇_X being the gradient of H with respect to X) almost everywhere with respect to the Lebesgue measure (from here onwards, functions are specified in this sense), together with the satisfaction of the appropriate boundary conditions. So, being

$$H(X, P, u) = p_x\left(\frac{y}{m} + u_x\right) + \left(p_y + \gamma\right) u_y, \tag{4}$$

where γ is a certain function which reflects the activity of the state constraints of the variable y, it follows that

$$\begin{aligned} -\dot{p}_x &= 0, \\ -\dot{p}_y &= \frac{p_x}{m}, \end{aligned} \tag{5}$$

and, thus,

$$\begin{aligned} p_x &= K_x, \\ p_y &= K_y - \frac{K_x}{m} t, \end{aligned} \tag{6}$$

for some constants $K_x, K_y > 0$. By taking into account that the position of the particle at time t is given by

$$\begin{aligned} x(t) &= \frac{b}{m} t + \int_0^t u_x(\tau)\, d\tau + \frac{1}{m} \int_0^t (t - \tau) u_y(\tau)\, d\tau, \\ y(t) &= b + \int_0^t u_y(\tau)\, d\tau, \end{aligned} \tag{7}$$

we conclude by the maximization of Pontryagin's function that $u_x^*(t) = 1$ (Figure 2) and u_y^* is given by the following.

(i) Consider

$$u_y^*(t) = \begin{cases} 1, & t \in [0, t^*/2[, \\ -1, & t \in]t^*/2, t^*], \end{cases} \tag{8}$$

for the case of $x_f \leq 2(L - b)$. By substituting in the equations of the particle's position, we conclude that the optimal time of (3) is given by

$$t^* = 2\left(\sqrt{(b + m)^2 + m x_f} - (b + m)\right). \tag{9}$$

(ii) Consider

$$u_y^*(t) = \begin{cases} 1, & t \in [0, t_1[, \\ 0, & t \in]t_1, t^* - t_1[, \\ -1, & t \in]t^* - t_1, t^*], \end{cases} \tag{10}$$

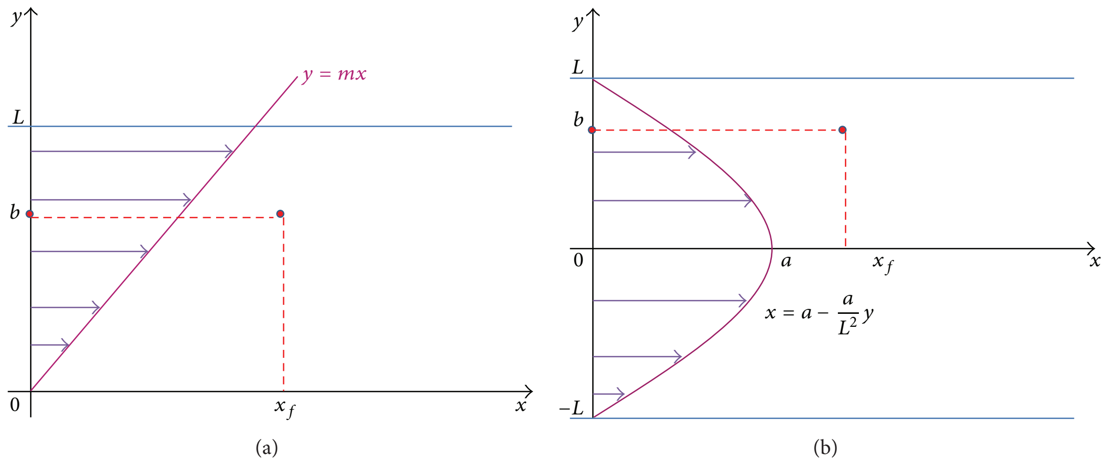


FIGURE 2: Linear (a) and quadratic velocity field (b).

for the case of $x_f > 2(L-b)$, where

$$t_1 = \sqrt{(b+m)^2 + 2m(L-b)} - (b+m), \tag{11}$$

being now

$$t^* = \Big(2(b+m)^2 - 2(b+m)\sqrt{(b+m)^2 + 2m(L-b)} + 2m(L-b) + mx_f\Big) \times \Big(\sqrt{(b+m)^2 + 2m(L-b)}\Big)^{-1}. \tag{12}$$

The two cases in the definition of u_y^* correspond to the situations in which the constraint is inactive and active, respectively.

2.2. Poiseuille Flow: Parabolic Velocity Profile. Similar arguments are applied to the case of flow with parabolic velocity flow (with vertex in $(a, 0)$). Now, the dynamics of the control system is given by

$$F(X,u) = \left(a - \frac{a}{L^2}y^2 + u_x, u_y\right), \tag{13}$$

and Pontryagin's function is given by

$$H(X,P,u) = p_x\left(a - \frac{a}{L^2}y^2 + u_x\right) + \left(p_y + \gamma\right)u_y. \tag{14}$$

It is easy to observe that the state constraint will be inactive along the optimal trajectory and that there is symmetry about the axis $y = 0$. By using these observations in the application of the Pontryagin maximum principle, as well as the fact that the position of the particle is given by

$$x(t) = at - \frac{a}{L^2}\int_0^t y^2(\tau)\,d\tau + \int_0^t u_x(\tau)\,d\tau, \qquad y(t) = b + \int_0^t u_y(\tau)\,d\tau, \tag{15}$$

we conclude that $u_x^*(t) = 1$ (Figure 2) and that u_y^* is defined by the following:

(i)

$$u_y^*(t) = \begin{cases} -1, & t \in [0, t^*/2[, \\ 1, & t \in]t^*/2, t^*], \end{cases} \tag{16}$$

if $x_f \le 2b$, in this case, the minimum time t^* being a root of the polynomial as follows:

$$t^{*3} - 6bt^{*2} + \left(12b^2 - 12L^2 - \frac{12L^2}{a}\right)t^* + \frac{12L^2}{a}x_f = 0, \tag{17}$$

(ii)

$$u_y^*(t) = \begin{cases} -1, & t \in [0, t_1[, \\ 0, & t \in]t_1, t^* - t_1[, \\ 1, & t \in]t^* - t_1, t^*], \end{cases} \tag{18}$$

if $x_f > 2b$, being the optimal time as follows:

$$t^* = \frac{x_f - \left(2a/L^2\right)\left((2/3)t_1^3 - bt_1^2\right)}{a + 1 - (a/L^2)(b - t_1)^2}, \tag{19}$$

where t_1 is half of the value of the t^* obtained in (17) with $x_f = 2b$.

3. Conclusions and Future Work

The case studies discussed here are very simple and their difference concerns only the profile of the velocity of the fluid. Not only the dynamics of the control system are defined by a set of ODEs, but also the conditions resulting from the application of the Pontryagin maximum principle can be easily solved in an explicit way. The next step consists in deriving optimality conditions in the form of a maximum principle leading to the computation of the solution to optimal control problems for which the previous simplifications cannot be exploited. This study suggests that the optimality conditions to be developed will require an adjoint variable satisfying a mixed system with ODEs and PDEs, so that the optimal control can be obtained by maximizing an appropriated Pontryagin function, coupled with appropriate boundary conditions.

Acknowledgments

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5.3 ARTICLE COPY "On the Optimal Control of Flow Driven Dynamic Systems", T. Grilo, S. Gama, F. L. Pereira – Chapter Book *Mathematics of Energy and Climate Change*, pages 183–189, Springer, 2015

On the Optimal Control of Flow Driven Dynamic Systems

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On the Optimal Control of Flow Driven Dynamic Systems

Teresa Grilo, Sílvio Gama and Fernando Lobo Pereira

Abstract The objective of this work is to develop a mathematical framework for the modeling, control and optimization of dynamic control systems whose state variable is driven by interacting ODE's (ordinary differential equations) and solutions of PDE's (partial differential equations). The ultimate goal is to provide a sound basis for the design and control of new advanced engineering systems arising in many important classes of applications, some of which may encompass, for example, underwater gliders and mechanical fishes. For now, the research effort has been focused in gaining insight by applying necessary conditions of optimality for shear flow driven dynamic control systems which can be easily reduced to problems with ODE dynamics. In this article we present the problem of minimum time control of a particle advected in a Couette and Poiseuille flows, and solve it by using the maximum principle.

Key words: Optimal control; Maximum principle; Ordinary differential equations; Dynamical systems.

1 Introduction

The development a mathematical framework for the modeling, control and optimization of dynamic control systems whose state variable is driven by interacting ODE's and PDE's is still a significant challenge. In [5], it is presented some earlier work aiming at the development of a theory of optimal

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control of dynamic systems, [3, 6], whose state evolves due to the interaction of ordinary differential equations with partial differential equations in which the later part is replaced by some known particular solution. Underwater gliders and robotic fishes, figure 1, are two examples of the class of applications whose currently available models we intend to improve.

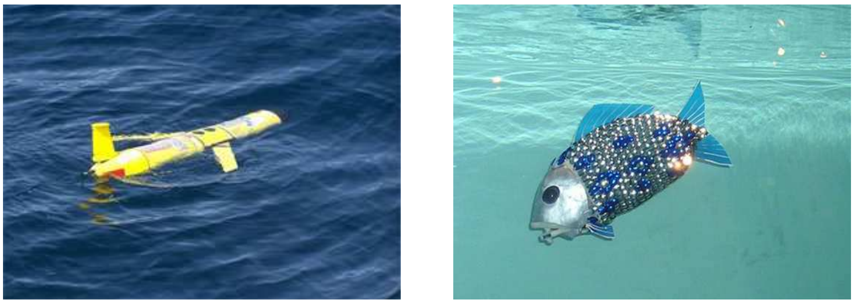

Fig. 1: Underwater glider (left), robotic fish (right).

An underwater glider is a winged autonomous underwater vehicle (AUV) that moves by modulating its buoyancy and attitude in the velocity vector fields of its environment. This vehicles are used for long-term, large-scale oceanographic monitoring, undersea surveillance and other applications. The kinetic and dynamic equations that described the vehicle motion can be found in [8, 9]. In [7], the motion of the robotic fish is approximated by a model featuring several components. The key advantage of this model is the fact that, instead of being considered a rigid body, the structure of the fish is composed of three parts: head, body and tail.

While the optimal control of systems with dynamics given by ordinary differential equations only has been making great strides in the 20^{th} and 21^{st} centuries (see, among others, [1, 4, 10]), such a theory for hybrid - in the sense that the controlled dynamics involve ordinary and partial differential equations - systems is still at its infancy.

Here, we formulate and solve two optimal control problems, where linear and parabolic velocity profiles are considered. Each one of these velocity profiles corresponds to a particular solution of the incompressible two-dimensional Navier-Stokes equation, respectively, the Couette and Poiseuille steady flows.

The Couette flow is the steady laminar unidirectional and two dimensional flow due to the relative motion of two infinite horizontal and parallel rigid plates [2]. The liquid between these two plates is driven by the viscous drag force originated by the uniform motion of the upper plate which moves in the x-direction with velocity v_0 (the lower plate is at rest). In this case, the velocity of such a flow has a linear profile and is given by

$$v(x,y) = (my, 0), \quad x \in \mathbb{R}, \quad y \in [0, L]$$

with $m = v_0/L$, the plates being L distance units apart (figure 2 (left)).

The Poiseuille flow is the steady flow due to the presence of a pressure gradient between two fixed (i.e., with zero relative velocity) rigid plates, [2]. In this case, a parabolic velocity profile is of the form

$$v(x,y) = \left(a - \frac{a}{L^2} y^2, 0\right), \quad x \in \mathbb{R}, \quad y \in [-L, L],$$

where, now, L is half the distance between the upper and lower plates (figure 2 (right)).

2 Minimum time control problem

Let us consider a particle placed in a flow, contained in a channel of width L, with a given velocity field. $(0, b)$ is the initial position of the particle, with $0 \le b \le L$.

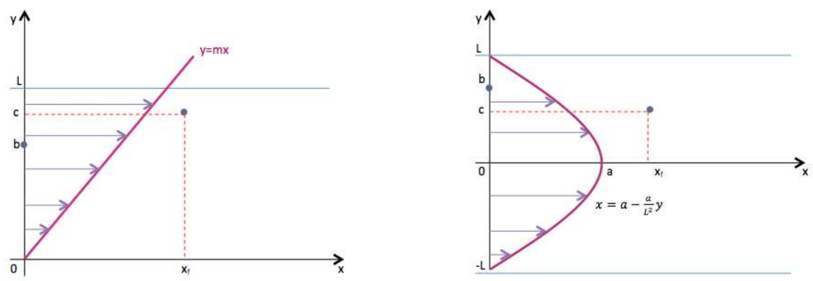

Fig. 2: Linear (left) and quadratic velocity field (right).

Our problem consists in moving the particle in minimum time along a path in the channel connecting a given initial position to a given end point (x_f, c), with $0 \le c \le L$. Since the particle is subject to the flow field, we must determine the value of the control function $u(\cdot) = (u_x(\cdot), u_y(\cdot))$ to be applied so that the conditions of the proposed control problem are satisfied. Let $X(t) = (x(t), y(t))$ be the position of the particle at time t, the control problem can be formulated as follows:

$$\begin{cases} \text{Minimize} & T \\ \text{subject to} & \begin{rcases} \dot{X}(t) = F(X(t), u(t)) \\ X(0) = (0, b) \\ X(T) = (x_f, c) \\ y(t) \in [0, L] \\ \|u(t)\|_\infty \leq 1 \end{rcases} , \forall t \in [0, T]. \end{cases} \tag{1}$$

Remark: From now on, for simplicity of notation, we will not indicate de time t as an independent variable of the others variables, although this is the case we are considering.

The maximum principle, [10], allows us to determine the optimal control $u^* = (u_x^*, u_y^*)$ by using the maximization of the Pontryagin's function $H(X, P, u)$, where $P = (p_x, p_y)$ is the adjoint variable satisfying $-\dot{P} = \nabla_X H(X, P, u)$, being ∇_X the gradient of H with respect to X, almost everywhere with respect to the Lebesgue measure (from here onwards, functions are specified in this sense), together with the satisfaction of the appropriate boundary conditions.

2.1 Couette flow

Consider the case of linear flow, with slope $m = v_0/L > 0$, whose velocity field is given by $v(x, y) = (\frac{y}{m}, 0)$. So, the dynamics of this control system is

$$F(X, u) = (\frac{y}{m} + u_x, u_y)$$

and the position of the particle at time t is given by

$$\begin{cases} x(t) = \frac{b}{m} t + \int_0^t u_x(\tau)\, d\tau + \frac{1}{m} \int_0^t (t - \tau) u_y(\tau)\, d\tau \\ y(t) = b + \int_0^t u_y(\tau)\, d\tau \, . \end{cases}$$

So, the Pontryagin's function is given by

$$H(X, P, u) = p_x(\frac{y}{m} + u_x) + (p_y + \gamma) u_y \, ,$$

where γ is a certain function which reflects the activity of the state constraints of the variable y, it follows from the maximum principle that

$$\begin{cases} -\dot{p}_x = 0 \\ -\dot{p}_y = \frac{p_x}{m} \end{cases} \Leftrightarrow \begin{cases} p_x = K_x \\ p_y = K_y - \frac{K_x}{m} t \end{cases}$$

for some constants $K_x, K_y > 0$.

By taking into account the position of the particle at each instant t, we conclude by the maximization of the Pontryagin's function that $u_x^*(t) = 1$ (figure 2) and u_y^* depends the final position of the particle, $X(T)$.

If $x_f \le 2L - b - c$ the state constraint of the variable y remains inactive and

$$u_y^*(t) = \begin{cases} 1, & t \in [0, \frac{c-b+t^*}{2}[\\ -1, & t \in]\frac{c-b+t^*}{2}, t^*] \end{cases}.$$

By substituting in the equations of the particle's position, we conclude that the optimum time for (1) is given by

$$t^* = \sqrt{(c+b+2m)^2 + (c-b)^2 + 4mx_f} - (c+b+2m).$$

For the case of $x_f > 2L - b - c$ the state constraint of variable y is active and

$$u_y^*(t) = \begin{cases} 1, & t \in [0, t_1[\\ 0, & t \in]t_1, t^* - t_2[\\ -1, & t \in]t^* - t_2, t^*] \end{cases},$$

where $t_1 = \sqrt{(b+m)^2 + 2m(L-b)} - (b+m)$ is the time when the particle is on the boundary of the channel, and $t_2 = \sqrt{(c+m)^2 + 2m(L-c)} - (c+m)$ is the time when the particle leaves the boundary. Now the minimum time is

$$t^* = \sqrt{(b+m)^2 + 2m(L-b)} - (b+m) + \frac{(c+m)^2 + m(b-c) + mx_f}{\sqrt{(b+m)^2 + 2m(L-b)}} - (c+m)A,$$

where $A = \sqrt{\frac{(c+m)^2+2m(L-c)}{(b+m)^2+2m(L-b)}}$.

2.2 Poiseuille flow

Let us consider a flow with a parabolic velocity vector field, with vertex at $(a,0)$. In this case the velocity field is given by $v(x,y) = (a - \frac{a}{L^2}y^2, 0)$. Then, the dynamics of the control system is

$$F(X,u) = (a - \frac{a}{L^2}y^2 + u_x, u_y)$$

and the Pontryagin's function is given by

$$H(X,P,u) = p_x(a - \frac{a}{L^2}y^2 + u_x) + (p_y + \gamma)u_y.$$

It follows from the maximum principle that

$$\begin{cases} -\dot{p}_x = 0 \\ -\dot{p}_y = -\frac{2a}{L^2} p_x y \end{cases} \Leftrightarrow \begin{cases} p_x = K_x \\ p_y = K_y + \frac{2aK_x}{L^2} \displaystyle\int_0^t y(\tau)\, d\tau \end{cases}$$

We remark that there is symmetry with respect to the axis $y = 0$ and, that the state constraint will be inactive along the optimal trajectory. By using these observations in the application of the maximum principle, as well as the fact that the position of the particle is given by

$$\begin{cases} x(t) = at - \frac{a}{L^2} \displaystyle\int_0^t y^2(\tau)\, d\tau + \int_0^t u_x(\tau)\, d\tau \\ y(t) = b + \displaystyle\int_0^t u_y(\tau)\, d\tau, \end{cases}$$

we conclude that $u_x^*(t) = 1$ (figure 2), and that u_y^* is defined by

$$u_y^*(t) = \begin{cases} -1, & t \in [0, \frac{b-c+t^*}{2}[\\ \;\;1, & t \in]\frac{b-c+t^*}{2}, t^*] \end{cases},$$

if $x_f \leq b+c$, being, in this case, the minimum time t^* a root of the polynomial

$$t^{*3} - 3(b+c)t^{*2} + 3\left((c+b)^2 - 4L^2(1+\frac{1}{a})\right)t^* + 3A = 0\,, \tag{2}$$

with $A = b^3 - b^2c - bc^2 + c^3 + \dfrac{4L^2}{a} x_f$.

In the case of $x_f > b+c$ we have

$$u_y^*(t) = \begin{cases} -1, & t \in [0, t_1[\\ \;\;0, & t \in]t_1, t^* - t_2[\\ \;\;1, & t \in]t^* - t_2, t^*] \end{cases},$$

being the optimal time given by

$$t^* = \frac{a(2t_1^3 - 3b(t_1^2 + t_2^2) + 3t_1t_2^2 - t_2^3) - 3L^2 x_f}{3a(b - t_1)^2 - 3aL^2 - 3L^2},$$

where t_1 and t_2 is a half of the value of the t^* obtained in (2) with $x_f = 2b$ and $x_f = 2c$, respectively.

3 Conclusions and Future work

The case studies discussed here are very simple and differ only in the profile of the fluid velocity field. Not only the dynamics of the control system are defined by a set of ODE's, but also the conditions resulting from the application of the maximum principle can be easily solved in an explicit way. The next step consists in deriving optimality conditions in the form of a maximum principle leading to the computation of the solution to optimal control problems for which the above simplifications can not be exploited. This study suggests that the optimality conditions to be developed will require an adjoint variable satisfying a mixed system with ODE's and PDE's, so that the optimal control can be obtained by maximizing an appropriated Pontryagin's function, coupled with appropriate boundary conditions.

Acknowledgements: The first and the third authors gratefully acknowledge the financial support of the FCT funded R&D projects PEST-OE/EEI/UI0147/2014 and PTDC/EEA-CRO/104901/2008. The work of the second author was supported, in part, by FCT through the CMUP – *Centro de Matemática da Universidade do Porto*.

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5.4 Attachment to Chapter 5

The results presented in the papers of this chapter are not given in sufficient details due to publication polices. Here, in this attachment, we show, in detail, the calculations done to solve the minimum time control problems whose dynamics are given by the Couette and Poiseuille steady flows.

5.4.1 Minimum Time Control Problem

As previously discussed in the articles presented in this chapter, our problem consists in moving the particle in minimum time along a path in the channel connecting a given initial position, $(0, b)$ with $0 < b < L$, to a given end point (x_f, c), with $0 \leq c \leq L$ and $x_f > 0$. Since the particle is subject to the flow field, the control problem can be formulated as follows:

$$\begin{cases} \text{Minimize} & T \\ \text{subject to} & \dot{X}(t) = F(X(t), u(t)) \\ & X(0) = (0, b) \\ & X(T) = (x_f, c) \\ & y(t) \in [0, L] \\ & \|u(t)\|_\infty \leq 1 \end{cases} , \forall t \in [0, T], \tag{5.1}$$

where $X(t) = (x(t), y(t))$ is the position of the particle in each instant t and $u(t) = (u_x(t), u_y(t))$ is the control function.

Also here, we omit the presence of time t, for simplicity of notation.

5.4.1.1 Couette Flow: Calculations

In the Couette steady flow we have the linear velocity profile, as in figure 5.1, thus the dynamics for the state variable is $F(X(t), u(t)) = \left(\frac{y}{m} + u_x, u_y\right)$.

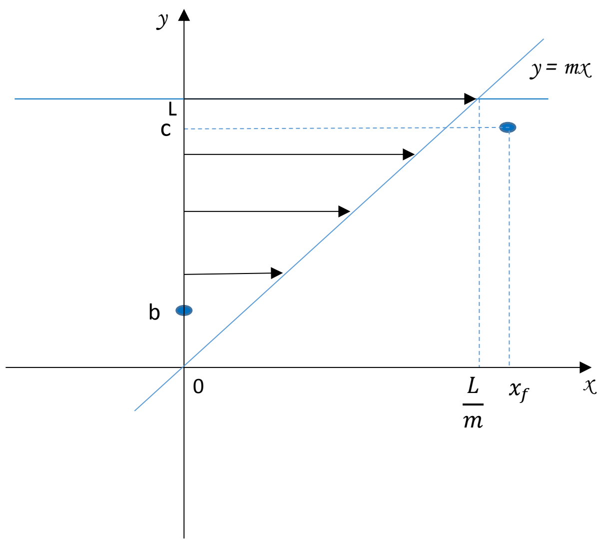

Figure 5.1: Linear velocity field.

We can rewrite the dynamics in the matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + I \begin{bmatrix} u_x \\ u_y \end{bmatrix},$$

that is a first order linear differential equation with exponential matrix given by $e^{At} = \begin{bmatrix} 1 & \frac{t}{m} \\ 0 & 1 \end{bmatrix}$, where A is the matrix of the coefficients of the state variable.

Solving the linear differential equation, using the integral factor method, we obtain

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & \frac{t}{m} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} + \int_0^t \begin{bmatrix} 1 & \frac{t-\tau}{m} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_x(\tau) \\ u_y(\tau) \end{bmatrix} d\tau \,.$$

Therefore,

$$\left\{ \begin{array}{lcl} x(t) & = & \frac{b}{m}t + \int_0^t u_x(\tau)\,d\tau + \frac{1}{m}\int_0^t (t-\tau)u_y(\tau)\,d\tau \\ y(t) & = & b + \int_0^t u_y(\tau)\,d\tau \end{array} \right. . \tag{5.2}$$

On the final point, (x_f, c), the particle's motion equations, (5.2), are satisfied for optimal minimum time t_f^*,

$$\left\{ \begin{array}{lcl} x_f & = & \frac{b}{m}t_f^* + \int_0^{t_f^*} u_x(\tau)\,d\tau + \frac{1}{m}\int_0^{t_f^*} (t_f^*-\tau)u_y(\tau)\,d\tau \\ c-b & = & \int_0^{t_f^*} u_y(\tau)\,d\tau \end{array} \right. . \tag{5.3}$$

Thus, the Pontryagin's function is given by

$$H(X,P,u) = p_x(\frac{y}{m} + u_x) + (p_y + \gamma)u_y \,, \tag{5.4}$$

where γ is a certain function which reflects the activity of the state constraints of the variable y, and from the maximum principle it follows that

$$\left\{ \begin{array}{lcl} -\dot{p}_x & = & 0 \\ -\dot{p}_y & = & \frac{p_x}{m} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{lcl} p_x & = & K_x \\ p_y & = & K_y - \frac{K_x}{m}t \end{array} \right.$$

for some constants $K_x, K_y > 0$, because this is a minimum time control problem with final point (x_f, c) fixed $(x_f > 0)$ and the adjoint variable p satisfy the transversality conditions at time T, so $p_x(t) > 0$.

By maximization of the Pontryagin's function and by the position of the particle in each instant t, we conclude that $u_x^*(t) = 1$, and u_y^* depends the final position of the particle. It is important to emphasize that the fluid is faster near the barrier $y = L$, because of the shape of the velocity field; so, in this problem, to spend the minimum time to move the particle is expectable that it goes near this barrier and gets to the places where the fluid is faster. Next, we will discuss all possible cases.

Case 1. $x_f \leq 2L - b - c$. In this case, we do not need to activate the state constraint of variable y. So,

$$u_y^*(t) = \left\{ \begin{array}{ll} 1, & t \in [0, t_1[\\ -1, & t \in]t_1, t_f^*] \end{array} \right. .$$

There are two subcases.

- $c = b$.

By equations (5.3), we have $\int_0^{t_f^*} u_y(\tau)\,d\tau = 0$, so $t_1 = \frac{t_f^*}{2}$. Substituting u^* on the first equation, we obtain

$$\begin{aligned}
&x_f = \tfrac{b}{m} t_f^* + \int_0^{t_f^*} 1\,d\tau + \frac{1}{m}\int_0^{\frac{t_f^*}{2}} (t_f^* - \tau)\,d\tau - \frac{1}{m}\int_{\frac{t_f^*}{2}}^{t_f^*} (t_f^* - \tau)\,d\tau \Leftrightarrow \\
&(t_f^*)^2 + (4b + 4m)t_f^* - 4mx_f = 0 \Leftrightarrow \\
&t_f^* = 2\left(\sqrt{(b+m)^2 + mx_f} - (b+m)\right) .
\end{aligned}$$

Therefore, $u_y^*(t) = \begin{cases} 1, & t \in [0, \frac{t_f^*}{2}[\\ -1, & t \in]\frac{t_f^*}{2}, t_f^*] \end{cases}$ and the solution of minimum time is $T = 2\left(\sqrt{(b+m)^2 + mx_f} - (b+m)\right)$.

• $c \neq b$.

In this situation, using u^* in equations (5.3), we get

$$c - b = \int_0^{t_1} 1\,d\tau - \int_{t_1}^{t_f^*} 1\,d\tau \;\Leftrightarrow\; t_1 = \frac{c - b + t_f^*}{2}$$

and

$$\begin{aligned}
&x_f = \tfrac{b}{m} t_f^* + \int_0^{t_f^*} 1\,d\tau + \frac{1}{m}\int_0^{t_1} (t_f^* - \tau)\,d\tau - \frac{1}{m}\int_{t_1}^{t_f^*} (t_f^* - \tau)\,d\tau \Leftrightarrow \\
&(t_f^*)^2 + (2c + 2b + 4m)t_f^* - ((c-b)^2 + 4mx_f) = 0 \Leftrightarrow \\
&t_f^* = \sqrt{(c+b+2m)^2 + (c-b)^2 + 4mx_f} - (c+b+2m) .
\end{aligned}$$

Thus, $u_y^*(t) = \begin{cases} 1, & t \in [0, \frac{c-b+t_f^*}{2}[\\ -1, & t \in]\frac{c-b+t_f^*}{2}, t_f^*] \end{cases}$ and the solution of minimum time is $T = \sqrt{(c+b+2m)^2 + (c-b)^2 + 4mx_f} - (c+b+2m)$.

Case 2: $x_f > 2L - b - c$. Here, the state constraint of variable y is active and

$$u_y^*(t) = \begin{cases} 1, & t \in [0, t_1[\\ 0, & t \in [t_1, t_f^* - t_2[\\ -1, & t \in]t_f^* - t_2, t_f^*] \end{cases} .$$

The times t_1 and $t_f^* - t_2$ are the instants when the particle arrives and leaves the boundary of the channel, respectively.

• $c = b$:

By the linearity of the velocity field and the previous case ($c = b$), we can conclude that, if the minimum time to go the end point (xf, b) is $2\left(\sqrt{(b+m)^2 + mx_f} - (b+m)\right)$, the minimum time to make half of the path, with distance $L - b$, is $t_1 = \sqrt{(b+m)^2 + 2m(L-b)} - (b+m)$. In this situation $t_2 = t_1$ because the final and initial points have the same distance from the boundary of the channel.

Applying u^* in (5.3) we obtain

$$
\begin{aligned}
x_f &= \tfrac{b}{m} t_f^* + \int_0^{t_f^*} 1 \, d\tau + \frac{1}{m}\int_0^{t_1} (t_f^* - \tau)\, d\tau - \frac{1}{m}\int_{t_f^*-t_1}^{t_f^*} (t_f^* - \tau)\, d\tau \Leftrightarrow \\
x_f &= \left(\tfrac{b}{m} + \tfrac{1}{m} t_1 + 1\right) t_f^* - \tfrac{1}{m} t_1^2 \Leftrightarrow \\
t_f^* &= \sqrt{(b+m)^2 + 2m(L-b)} - 2(b+m) + \tfrac{(b+m)^2 + m x_f}{\sqrt{(b+m)^2+2m(L-b)}} \, .
\end{aligned}
$$

So, $u_y^*(t) = \begin{cases} 1, & t \in [0, t_1[\\ 0, & t \in [t_1, t_f^* - t_1[\\ -1, & t \in]t_f^* - t_1, t_f^*] \end{cases}$ and the solution of minimum time is $T = \sqrt{(b+m)^2 + 2m(L-b)} - 2(b+m) + \frac{(b+m)^2 + m x_f}{\sqrt{(b+m)^2+2m(L-b)}}$.

• *$c \neq b$:*

By the same arguments given previously, we get $t_1 = \sqrt{(b+m)^2 + 2m(L-b)} - (b+m)$ and $t_2 = \sqrt{(c+m)^2 + 2m(L-c)} - (c+m)$. Substituting in (5.3) we have

$$
\begin{aligned}
x_f &= \tfrac{b}{m} t_f^* + \int_0^{t_f^*} 1 \, d\tau + \frac{1}{m}\int_0^{t_1} (t_f^* - \tau)\, d\tau - \frac{1}{m}\int_{t_f^*-t_2}^{t_f^*} (t_f^* - \tau)\, d\tau \Leftrightarrow \\
x_f &= -\tfrac{1}{2m}(t_f^*)^2 \left(\tfrac{b}{m} + \tfrac{1}{m}(t_1 + t_2) + 1\right) t_f^* - \tfrac{1}{2m}(t_1^2 + t_2^2) \Leftrightarrow \\
t_f^* &= \sqrt{(b+m)^2 + 2m(L-b)} - (b+m) + \tfrac{(c+m)^2 + m(b-c) + m x_f}{\sqrt{(b+m)^2+2m(L-b)}} - (c+m)A \, ,
\end{aligned}
$$

where $A = \sqrt{\frac{(c+m)^2+2m(L-c)}{(b+m)^2+2m(L-b)}}$.

So, $u_y^*(t) = \begin{cases} 1, & t \in [0, t_1[\\ 0, & t \in [t_1, t_f^* - t_2[\\ -1, & t \in]t_f^* - t_2, t_f^*] \end{cases}$ and the solution of minimum time is $T = \sqrt{(b+m)^2 + 2m(L-b)} - (b+m) + \frac{(c+m)^2 + m(b-c) + m x_f}{\sqrt{(b+m)^2+2m(L-b)}} - (c+m)A$.

5.4.1.2 Poiseuille Flow: Calculations

In Poiseuille steady flow, we have the parabolic velocity profile, as in figure 5.2; thus the dynamics for the state variable is $F(X(t), u(t)) = \left(a - \frac{a}{L^2} y^2 + u_x, u_y\right)$.

The motion equations for the particle advected by a Poiseuille flow are

$$
\begin{cases} x(t) &= at - \frac{a}{L^2}\int_0^t y^2(\tau)\, d\tau + \int_0^t u_x(\tau)\, d\tau \\ y(t) &= b + \int_0^t u_y(\tau)\, d\tau \end{cases} . \tag{5.5}
$$

They are satisfied on final point, (x_f, c), for optimal minimum time t_f^*,

$$
\begin{cases} x_f &= a t_f^* - \frac{a}{L^2}\int_0^{t_f^*} y^2(\tau)\, d\tau + \int_0^{t_f^*} u_x(\tau)\, d\tau \\ c - b &= \int_0^{t_f^*} u_y(\tau)\, d\tau \end{cases} , \tag{5.6}
$$

and the Pontryagin's function is given by

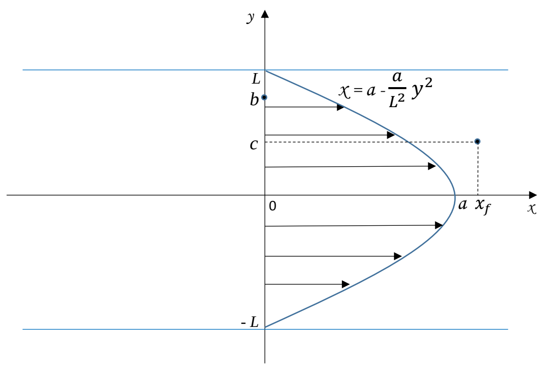

Figure 5.2: Quadratic velocity field.

$$H(X,P,u) = p_x(a - \frac{a}{L^2}y^2 + u_x) + (p_y + \gamma)u_y\,. \tag{5.7}$$

Is important to notice that in this case the state constraint is always inactive, because the flow is stronger in the middle of the channel So, the particle must be controlled to take these stronger velocities to obtain the optimal time, which makes that it never goes to the boundaries of the channel.

From the maximum principle it follows that

$$\begin{cases} -\dot{p}_x &= 0 \\ -\dot{p}_y &= -\frac{2a}{L^2}p_x y \end{cases} \Leftrightarrow \begin{cases} p_x &= K_x \\ p_y &= K_y + \frac{2aK_x}{L^2}\displaystyle\int_0^t y(\tau)\,d\tau \end{cases}$$

for some constants $K_x, K_y > 0$, by the transversality conditions of p.

In figure 5.2, we can see that there is symmetry with respect to the x-axis. Also, we can conclude that $u_x^*(t) = 1$ and u_y^* depends on the final position of the particle. Next, we will discuss the cases that are possible.

Case 1: $x_f \leq b + c$. We have

$$u_y^*(t) = \begin{cases} -1, & t \in [0, t_1[\\ 1, & t \in]t_1, t_f^*] \end{cases}.$$

- $c = b$:

By equations (5.6), we have $t_1 = \frac{t_f^*}{2}$ and

$$x_f = at_f^* + \int_0^{t_f^*} 1\,d\tau - \frac{a}{L^2}\int_0^{\frac{t_f^*}{2}} (b-\tau)^2\,d\tau - \frac{a}{L^2}\int_{\frac{t_f^*}{2}}^{t_f^*} (b - t_f^* + \tau)^2\,d\tau \Leftrightarrow$$
$$x_f = (a+1)t_f^* - \frac{2ab^3}{3L^2} + \frac{2a}{3L^2}(b - \frac{t_f^*}{2})^3 \Leftrightarrow$$
$$(t_f^*)^3 - 6b(t_f^*)^2 + 12\left(b^2 - L^2 - \frac{L^2}{a}\right)t_f^* + \frac{12L^2x_f}{a} = 0\,.$$

Thus, $u_y^*(t) = \begin{cases} -1, & t \in [0, \frac{t_f^*}{2}[\\ 1, & t \in]\frac{t_f^*}{2}, t_f^*] \end{cases}$ and the solution of minimum time is a root of the previous polynomial.

• $c \neq b$.

Inserting $u^*(\cdot)$ in equations (5.6), we have

$$c - b = -\int_0^{t_1} 1\, d\tau + \int_{t_1}^{t_f^*} 1\, d\tau \Leftrightarrow t_1 = \frac{b - c + t_f^*}{2}$$

and

$$x_f = at_f^* + \int_0^{t_f^*} 1\, d\tau - \frac{a}{L^2}\int_0^{\frac{b-c+t_f^*}{2}} (b-\tau)^2\, d\tau - \frac{a}{L^2}\int_{\frac{b-c+t_f^*}{2}}^{t_f^*} (c - t_f^* + \tau)^2\, d\tau \Leftrightarrow$$
$$x_f = (a+1)t_f^* - \tfrac{a}{3L^2}(b^3 + c^3) + \tfrac{2a}{3L^2}(\tfrac{b+c-t_f^*}{2})^3 \Leftrightarrow$$
$$(t_f^*)^3 - 3(b+c)(t_f^*)^2 + 3\left((c+b)^2 - 4L^2(1+\tfrac{1}{a})\right)t_f^* + 3A = 0\,,$$

with $A = b^3 - b^2c - bc^2 + c^3 + \dfrac{4L^2}{a}x_f$. Therefore, $u_y^*(t) = \begin{cases} -1, & t \in [0, \frac{b-c+t_f^*}{2}[\\ 1, & t \in]\frac{b-c+t_f^*}{2}, t_f^*] \end{cases}$ and the solution of the problem is a root of the polynomial expressed above.

Case 2: $x_f > b + c$, here, the u_y^* is given by

$$u_y^*(t) = \begin{cases} -1, & t \in [0, t_1[\\ 0, & t \in [t_1, t_f^* - t_2[\\ 1, & t \in]t_f^* - t_2, t_f^*] \end{cases}.$$

The times t_1 and $t_f^* - t_2$ are the instants when the particle arrives and leave the x-axis, respectively.

• $c = b$:

As in the Couette flow, $t_1 = t_2$ is one half of the value of the t_f^* found in the previous case ($c = b$), with $x_f = 2b$.

By (5.6), we get

$$x_f = at_f^* + \int_0^{t_f^*} 1\, d\tau - \frac{a}{L^2}\int_0^{t_1} (b-\tau)^2\, d\tau - \frac{a}{L^2}\int_{t_1}^{t_f^*-t_1} (b-t_1)^2\, d\tau - \frac{a}{L^2}\int_{t_f^*-t_1}^{t_f^*} (b - t_f^* + \tau)^2\, d\tau \Leftrightarrow$$
$$x_f = (a + 1 - \tfrac{ab^2}{L^2} + \tfrac{2abt_1}{L^2} - \tfrac{at_1^2}{L^2})t_f^* - \tfrac{2abt_1^2}{L^2} + \tfrac{4at_1^3}{3L^2} \Leftrightarrow$$
$$t_f^* = \frac{2abt_1^2 + L^2x_f - \frac{4at_1^3}{3}}{aL^2 + L^2 - ab^2 + 2abt_1 - at_1^2}\,.$$

So, $u_y^*(t) = \begin{cases} -1, & t \in [0, t_1[\\ 0, & t \in [t_1, t_f^* - t_1[\\ 1, & t \in]t_f^* - t_1, t_f^*] \end{cases}$ and the solution of minimum time is $T = \frac{2abt_1^2 + L^2x_f - \frac{4at_1^3}{3}}{aL^2 + L^2 - ab^2 + 2abt_1 - at_1^2}$.

• $c \neq b$:

Here, t_1 and t_2 are 1/2 of t_f^* found in the case 1 ($c = b$), with $x_f = 2b$ and $x_f = 2c$, respectively. Substituting in (5.6), we obtain:

$$x_f = (a+1)t_f^* - \tfrac{a}{L^2}\left[\int_0^{t_1}(b-\tau)^2\,d\tau + \int_{t_1}^{t_f^*-t_2}(b-t_1)^2\,d\tau + \int_{t_f^*-t_2}^{t_f^*}(b-t_f^*-t_1+t_2+\tau)^2\,d\tau\right] \Leftrightarrow$$
$$x_f = (a+1-\tfrac{a}{L^2}(b-t_1)^2)t_f^* + \tfrac{2a}{3L^2}(b-t_1)^3 - \tfrac{ab^3}{3L^2} + \tfrac{a}{L^2}(b-t_1)^2(t_1+t_2) - \tfrac{a}{3L^2}(b-t_1+t_2)^3 \Leftrightarrow$$
$$t_f^* = \tfrac{a(2t_1^3-3b(t_1^2+t_2^2)+3t_1t_2^2-t_2^3)-3L^2x_f}{3a(b-t_1)^2-3aL^2-3L^2}\,.$$

Thus, $u_y^*(t) = \begin{cases} -1, & t \in [0, t_1[\\ 0, & t \in [t_1, t_f^* - t_2[\\ 1, & t \in]t_f^* - t_2, t_f^*] \end{cases}$

and the solution of minimum time is

$$T = \frac{a(2t_1^3 - 3b(t_1^2 + t_2^2) + 3t_1t_2^2 - t_2^3) - 3L^2x_f}{3a(b-t_1)^2 - 3aL^2 - 3L^2}.$$

Chapter 6

Vector-fields Driven by One and Two Point Vortices

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6.1 Introduction

In this chapter, we present two cases of a free particle that lives in a flow whose dynamic system is driven by one and two point vortices, respectively (Batchelor (1967), Newton (2001)).

Here, the control problem consists in moving one particle between two given points, (x_0, y_0) and (x_f, y_f). For this, we apply the necessary conditions of optimality of the Maximum Principle of Pontryagin (Pontryagin et al. (1962)), minimizing the energy spent in this process.

6.2 Vector-field Driven by One Vortex

Consider a flow with a point vortex at the origin $(0, 0)$ with circulation k and let $z(t) = x(t)+y(t)i$ be the position of the particle, placed in this flow, in each instant t. In complex form, the dynamic equations of the particle positioned in z, subject to velocity field given by the vortex, is

$$\dot{z}^*(t) = \frac{k}{2\pi i}\frac{1}{z(t)} \ . \tag{6.1}$$

In the polar coordinates, the position of the particle is given by

$$z(t) = \rho(t)e^{i\theta(t)}$$

and the derivative with respect to t is

$$\dot{z}(t) = \dot{\rho}(t)e^{i\theta(t)} + i\rho(t)\dot{\theta}(t)e^{i\theta(t)} \ .$$

Therefore, the conjugate of the derivative is

$$\dot{z}^*(t) = \dot{\rho}(t)e^{-i\theta(t)} - i\rho(t)\dot{\theta}(t)e^{-i\theta(t)}\,, \tag{6.2}$$

and, from (6.1) and (6.2), we obtain

$$\dot{\rho}(t) - i\rho(t)\dot{\theta}(t) = -i\frac{k}{2\pi}\frac{1}{\rho(t)}\ .$$

Analysing the real and imaginary parts of both sides of this equation, we have the following dynamic equations of the particle, in polar form,

$$\begin{cases} \dot{\rho}(t) = 0 \\ \dot{\theta}(t) = \frac{k}{2\pi}\frac{1}{\rho^2(t)} \end{cases} \tag{6.3}$$

So, the position of the particle, in each instant t, is

$$z(t) = \rho(0)e^{i\left(\frac{kt}{2\pi\rho^2(0)} + \theta(0)\right)}\ . \tag{6.4}$$

In figure 6.1, shown is the trajectory of the particle, with the initial position in (x_0, y_0).

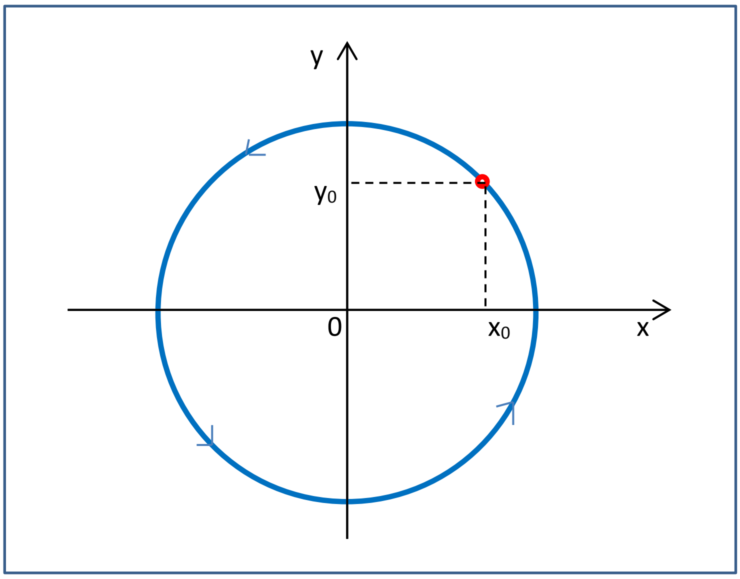

Figure 6.1: Trajectory of the particle.

6.2.1 Control Problem

Consider a flow like the one presented in section 6.2 and a particle placed in the flow, with initial position (x_0, y_0).

The objective of this problem is to determine the control function $u(\cdot)$ to be applied to the particle so that it will move from its initial position to the end point (x_f, y_f), minimizing the cost function $g(X(T))$. Let $X(t) = (x(t), y(t))$ be the position of the particle at time t. The control problem can be formulated as follows:

$$\begin{cases} \text{Minimize} \quad g(X(T)) \\ \text{subject to} \\ \dot{X}(t) = F(X(t), u(t)) \\ X(0) = (x_0, y_0) \\ X(T) = (x_f, y_f) \\ \|u(t)\|_\infty \leq 1 \end{cases} , \forall t \in [0, T]. \tag{6.5}$$

The Maximum Principle (Pontryagin et al. (1962)) allows us to determine the optimal control u^* by using the maximization of the Pontryagin's function $H(X, P, u)$ (here, P is the adjoint variable satisfying $-\dot{P} = \nabla_X H(X, P, u)$, ∇_X being the gradient of H with respect to X) almost everywhere with respect to the Lebesgue measure (from here onwards, functions are specified in this sense), together with the satisfaction of the appropriate boundary conditions.

6.2.1.1 Minimum Energy Problem

The cost function we want to minimize is the energy defined by

$$w(T) = \int_0^T u^2 \, dt \; .$$

necessary to move the particle from its initial point to the final point, in time T.

To have a Mayer's problem, we consider a new variable w that satisfies $\dot{w} = u^2$. Thus, our control problem is formulated as follows:

$$\begin{cases} \text{Minimize} \quad w(T) \\ \text{subject to} \\ \dot{\rho} = u \\ \dot{\theta} = \frac{k}{2\pi} \frac{1}{\rho^2} \\ \dot{w} = u^2 \\ \rho(0) = \sqrt{x_0^2 + y_0^2} \\ \rho(T) = \sqrt{x_f^2 + y_f^2} \\ \tan(\theta(0)) = \frac{y_0}{x_0} \, , \, x_0 \neq 0 \\ \tan(\theta(T)) = \frac{y_f}{x_f} \, , \, x_f \neq 0 \\ w(0) = 0 \\ \|u(t)\|_\infty \leq 1 \end{cases} , \forall t \in [0, T]. \tag{6.6}$$

Remark: From now on, consider $T : \|(x_f, y_f)\|_2^2 \leq \frac{k}{4\pi^2} \leq \|(x_0, y_0)\|_2^2$ and the initial and end points in the first quadrant.

By the dynamic equations, we obtain the equations of motion, according to the control function, given by

$$\left\{\begin{array}{lcl} \rho(t) & = & \sqrt{x_0^2+y_0^2}+\int_0^t u(\tau)\,d\tau \\ \theta(t) & = & \arctan(\frac{y_0}{x_0})+\frac{k}{2\pi}\int_0^t \rho^{-2}(\tau)\,d\tau \\ w(t) & = & \int_0^t u^2(\tau)\,d\tau \end{array}\right. . \tag{6.7}$$

For this problem the Pontryagin's function is

$$H(t,\rho,\theta,w,p_\rho,p_\theta,p_w) = p_\rho u + p_\theta \frac{k}{2\pi\rho^2} + p_w u^2 \tag{6.8}$$

and, by the optimality conditions, the dynamic equations for the adjoint variable are

$$\left\{\begin{array}{lcl} -\dot{p}_\rho & = & -p_\theta\frac{k}{\pi\rho^3} \\ -\dot{p}_\theta & = & 0 \\ -\dot{p}_w & = & 0 \end{array}\right. , \tag{6.9}$$

and, the transversality conditions are

$$\left\{\begin{array}{lcl} -p_\rho(T) & = & -C_\rho \\ -p_\theta(T) & = & -C_\theta \\ -p_w(T) & = & 1 \end{array}\right. , \tag{6.10}$$

thus, by 6.9 and 6.10, we get

$$\left\{\begin{array}{lcl} p_\rho & = & C_\rho - C_\theta\frac{k}{\pi}\int_t^T \rho^{-3}(\tau)\,d\tau \\ p_\theta & = & C_\theta \\ p_w & = & -1 \end{array}\right. .$$

In this control problem we have a fixed time T to arrive on point (x_f, y_f), therefore, at a certain point of the trajectory the particle is free in its motion, which means that the control is not active during a certain time.

In the interval of time , $[0,T]$, applying the Maximum Principle, the control u^* satisfies

$$u^*(t) = \left\{\begin{array}{ll} -1\,, & t\in[0,t') \\ 0\,, & t\in[t',t_1') \\ -1\,, & t\in[t_1',T] \end{array}\right. , \tag{6.11}$$

where t' and t_1' are the instants satisfying

$$\left\{\begin{array}{lcl} \rho(T) & = & \sqrt{x_f^2+y_f^2} \\ \theta(t_1') & = & \theta(t')+2\pi \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl} \sqrt{x_0^2+y_0^2}+\int_0^T u(\tau)\,d\tau & = & \sqrt{x_f^2+y_f^2} \\ \frac{k}{2\pi}\int_0^{t_1'} \rho^{-2}(\tau)\,d\tau & = & \frac{k}{2\pi}\int_0^{t'} \rho^{-2}(\tau)\,d\tau + 2\pi \end{array}\right. \Leftrightarrow$$

$$\left\{\begin{array}{lcl} t_1' & = & T - T_0 + t' \\ t' & = & \sqrt{x_0^2+y_0^2} - \frac{\sqrt{k(T-t_0)}}{2\pi} \end{array}\right. ,$$

where $T_0 = \|(x_0, y_0)\|_2 - \|(x_f, y_f)\|_2$. By this, and using (6.7), the motion equations for the particle, for $t \in [0, T]$ are

$$\rho(t) = \begin{cases} \rho(0) - t\,, & t \in [0, t') \\ \rho(0) - t'\,, & t \in [t', t' + T - T_0) \\ \rho(0) + T - T_0 - t\,, & t \in [t' + T - T_0, T] \end{cases} \tag{6.12}$$

$$\theta(t) = \begin{cases} \theta(0) + \frac{kt}{2\pi\rho(0)(\rho(0)-t)}\,, & t \in [0, t') \\ \theta(0) + \frac{k(\rho(0)t - t'^2)}{2\pi\rho(0)(\rho(0)-t')^2}\,, & t \in [t', t' + T - T_0) \\ \theta(0) + \frac{k(T-T_0)}{2\pi(\rho_0 - t')^2} + \frac{k(t-T+T_0)}{2\pi\rho(0)(\rho(0)+T-T_0-t)}\,, & t \in [t' + T - T_0, T] \end{cases} \tag{6.13}$$

and in figure 6.2 we see the representation of this motion. So, $w(T) = T_0$.

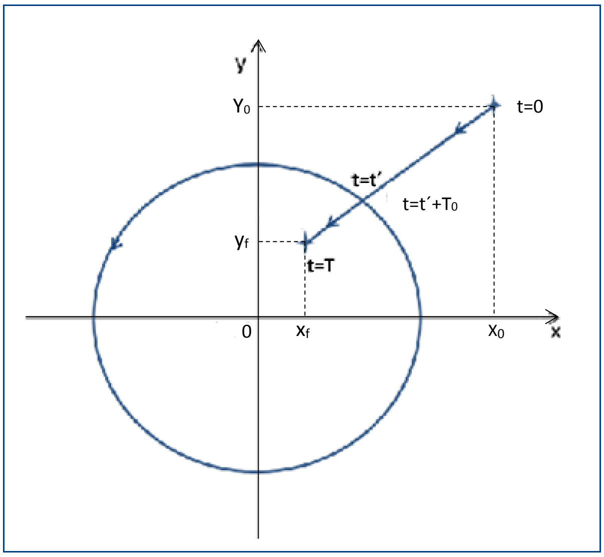

Figure 6.2: Trajectory of the particle subject to the optimal control.

So that, this problem is well posed because we guarantee that the radius, ρ, is always positive, for all $t \in [0, T]$.

6.3 Vector-field Driven by Two Vortices

The issue we are going to address now will be useful for the optimal multiprocesses problem presented in next chapter.

Consider a flow with two point vortices, z_1 and z_2, with circulations k_1 and k_2 (with $k_1 = -k_2 = k$), respectively. Suppose that the initial position of the vortices are $z_1(0) = \frac{h}{2}i$ and $z_2(0) = -\frac{h}{2}i$.

The dynamic equation of a passive particle, positioned at $z(t) = x(t) + iy(t)$, is given by

$$\dot{z}^* = \frac{k}{2\pi i}\left(\frac{1}{z - z_1(t)} - \frac{1}{z - z_2(t)}\right) \tag{6.14}$$

with initial condition $z(0) = z_0$.

Using (4.19) in (6.14) we obtain

$$\dot{z}^* = \frac{k}{2\pi i}\left(\frac{1}{z - \frac{h}{2}i - \frac{k}{2\pi h}t} - \frac{1}{z + \frac{h}{2}i - \frac{k}{2\pi h}t}\right).$$

Making a change of variable, $w(t) = z(t) - \frac{k}{2\pi h}t$, and replacing in the last equation we have

$$\dot{w}^* = \frac{k}{2\pi i}\left(\frac{1}{w - \frac{h}{2}i} - \frac{1}{w + \frac{h}{2}i}\right) - \frac{k}{2\pi h} \tag{6.15}$$

with initial condition $w(0) = z_0 \in \mathbb{C}\backslash\left\{\pm\frac{h}{2}i\right\}$.

The stationary points are very important for the study of this system, so $\dot{w}^* = 0$ occurs when $w = \pm\frac{\sqrt{3}}{2}h$, or on the original variable $z = \pm\frac{\sqrt{3}}{2}h + \frac{kt}{2\pi h}$, which are both stationary point of the system.

Let be now, $w(t) = x(t) + 0i$. Replacing in (6.15) comes

$$\begin{aligned}\dot{x}(t) &= \frac{k}{2\pi i}\left(\frac{1}{x-\frac{h}{2}i} - \frac{1}{x+\frac{h}{2}i}\right) - \frac{k}{2\pi h}\\ &= \frac{k}{2\pi i}\frac{ih}{x^2+\frac{h^2}{4}} - \frac{k}{2\pi h}\\ &= \frac{kh}{2\pi\left(x^2+\frac{h^2}{4}\right)} - \frac{k}{2\pi h}\\ &= \frac{kh}{2\pi}\left(\frac{1}{x^2+\frac{h^2}{4}} - \frac{1}{h^2}\right)\end{aligned} \tag{6.16}$$

with initial condition $w(0) = x_0$.

Solving (6.16), which is an equation of separable variables, we get

$$\begin{aligned}&\dot{x} = \frac{kh}{2\pi}\left(\frac{1}{x^2+\frac{h^2}{4}} - \frac{1}{h^2}\right) \Leftrightarrow\\ &\frac{4x^2+h^2}{3h^2-4x^2}dx = \frac{k}{2\pi h}dt \Rightarrow\\ &-x(t) + \frac{h}{\sqrt{3}}\ln\left|\frac{x+\frac{\sqrt{3}}{2}h}{x-\frac{\sqrt{3}}{2}h}\right| = \frac{kt}{2\pi h} + C\ ,\ C \in \mathbb{R}\,.\end{aligned}$$

Using the initial condition, the solution in an implicit way is

$$\begin{cases} x(t) = \frac{\sqrt{3}}{2}h \cdot \dfrac{-1-\frac{x_0+\frac{\sqrt{3}}{2}h}{x_0-\frac{\sqrt{3}}{2}h}e^{\frac{\sqrt{3}}{2}h\left(\frac{kt}{2\pi h}+x(t)-x_0\right)}}{1-\frac{x_0+\frac{\sqrt{3}}{2}h}{x_0-\frac{\sqrt{3}}{2}h}e^{\frac{\sqrt{3}}{2}h\left(\frac{kt}{2\pi h}+x(t)-x_0\right)}} \\ y(t) = 0 \end{cases}. \tag{6.17}$$

The solution $w(t)$ is always real and $w = \pm\frac{\sqrt{3}}{2}h$ are stationary points, because of this any solution crosses these points. So, we have:

- if $x_0 < -\frac{\sqrt{3}}{2}h$, then $x(t) < -\frac{\sqrt{3}}{2}h$;

- if $-\frac{\sqrt{3}}{2}h < x_0 < \frac{\sqrt{3}}{2}h$, then $-\frac{\sqrt{3}}{2}h < x(t) < \frac{\sqrt{3}}{2}h$;
- if $x_0 > \frac{\sqrt{3}}{2}h$, then $x(t) > \frac{\sqrt{3}}{2}h$.

Thus, if $k > 0$, then the point $\frac{\sqrt{3}}{2}h$ is an attractor and $-\frac{\sqrt{3}}{2}h$ is a repelling point (like shown in figure 6.3), which means that the particle is left behind (in comparison with the motion of the vortices), if $x_0 < -\frac{\sqrt{3}}{2}h$, and goes near $\frac{\sqrt{3}}{2}h$, if $x_0 > -\frac{\sqrt{3}}{2}h$. For the case when $k < 0$, the attractor point is $-\frac{\sqrt{3}}{2}h$ and the repelling is $\frac{\sqrt{3}}{2}h$.

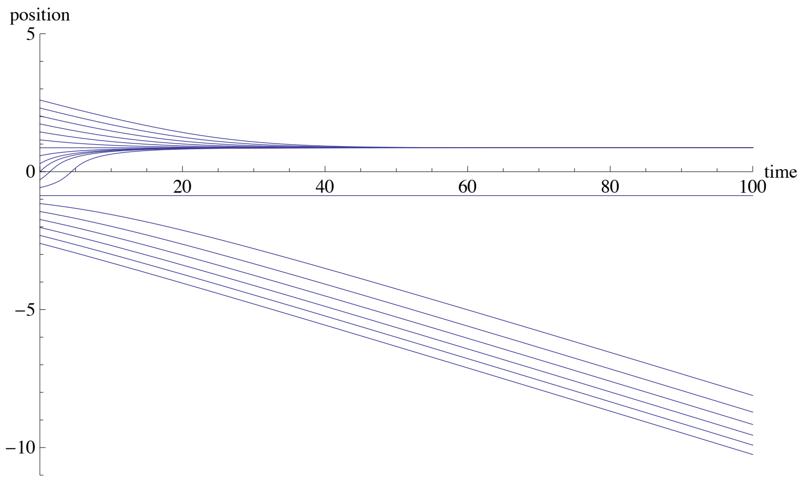

Figure 6.3: Trajectories of the particle subject to the vortices z_1 and z_2 (with $h = 1$), positioned in the x-axis.

Consider now that the passive particle lives in a two-dimensional flow. Suppose that the velocity field is given by the dynamics generated by two point vortices. Here, we are considering that the particle has the ability to create one point vortex such as to optimize the trajectory to reach the final state.

Knowing the position of $z_1(0)$ and the initial position of the particle z_0, through the equation

$$z(t) = \frac{z_2(0) + z_1(0) \pm \sqrt{A}}{2} + \frac{ki}{2\pi h^2}\left(z_2(0) - z_1(0)\right)t\,, \tag{6.18}$$

where

$$A = \frac{z_2(0) - z_1(0)}{z_2^*(0) - z_1^*(0)}\left(|z_2(0)|^2 + |z_1(0)|^2 - z_2(0)z_1^*(0) - z_1(0)z_2^*(0) - 4h^2\right)\,,$$

we obtain the position of $z_2(0)$.

Chapter 7

A Multiprocess Framework for the Optimal AUV Motion Driven by Vortices

Summary

7.1 Introduction **59**

7.2 ARTICLE COPY "Optimal Multi-process Control of a Two Vortex Driven Particle in the Plane", F. L. Pereira, T. Grilo, S. Gama – Journal Article *IFAC - PapersOnline*, **volume 50, Issue 1, pages 2193–2198, ISSN 2405-8963, 2017** **61**

7.3 ARTICLE COPY "Optimal Control Framework for AUV's Motion Planning in Planar Vortices Vector Field", T. Grilo, S. Gama, F. L. Pereira – Proceedings Article *Proceedings-AUV*, **2018** **69**

7.1 Introduction

Frank Clarke and Richard Vinter present in Clarke and Vinter (1989a) the theory of necessary conditions for optimal multiprocesses problems, and in Clarke and Vinter (1989b) some applications for optimization problems in robotics, optics, investment planning, impulse control and renewable resources.

The idea is to apply this theory to get robust conditions to apply in the motion of the robotic fishes or autonomous underwater vehicles (AUV's), considered as a particle. In their motion they are creating circulation points, that we consider, in our study, like point vortices (Newton (2001), Batchelor (1967)), and they are using the impact of this points on their motion.

In our problem we just have two vortices active in each instant, created by the robot, with a decaying circulation. The robot decides the time, the location and the intensity of the vortices that it creates. When one new vortex is created the oldest one, that has been created before, can be considered without impact on the robot, to have again just two vortices active, like illustrated in figure 7.1.

In this chapter, we initiate the formulation for the optimal multiprocesses problem, for the motion of a particle that lies in a fluid whose dynamics is driven by a vector-field given by

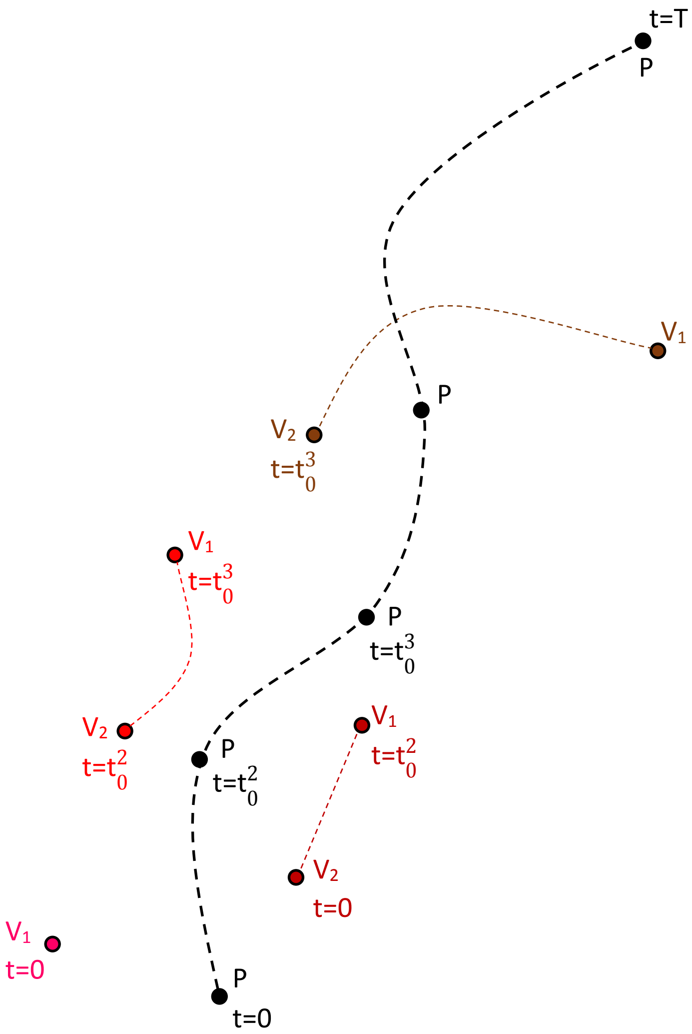

Figure 7.1: Motion of a particle in a two-vortex vector-field.

two point vortices, and the conditions following the application of the Pontryagin Maximum Principle.

In Pereira et al. (2017a), Pereira et al. (2017b) and Grilo et al. (2018) we show the beginning of the results studied for this problem. Here, we only present the first and the last articles mentioned before, because the second one is a compilation of the problem formulation presented in the Pereira et al. (2017a).

7.2 ARTICLE COPY "Optimal Multi-process Control of a Two Vortex Driven Particle in the Plane", F. L. Pereira, T. Grilo, S. Gama – Journal Article *IFAC - PapersOnline*, volume 50, Issue 1, pages 2193–2198, ISSN 2405-8963, 2017

Optimal Multi-process Control of a Two Vortex Driven Particle in the Plane

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Optimal Multi-process Control of a Two Vortex Driven Particle in the Plane ⋆

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Abstract: This article concerns the optimal motion control of a particle moving in a two dimensional fluid whose dynamics are given by a vector field defined, in any time interval, by two point vortices whose circulation decay exponentially in time with a given constant rate. The control action is exercised by generating one vortex - specified by its location and respective circulation - at a chosen time, and by varying the exposure of the particle to each one of the vortices in continuum time. A control multi-process framework is chosen in order to derive necessary conditions of optimality in the form of a Maximum Principle of Pontryagin. These conditions provide relations that suffice to fully determine the optimal control process.

Keywords: Optimal Control, Vortex, Multi-process, Maximum Principle of Pontryagin.

1. INTRODUCTION

In this article we investigate the optimal control problem of driving a particle between two given points with minimal effort in a flow field defined by two point vortices at any point in time, Protas (2008); Batchelor (1992); Newton (2001). The position and initial circulation of each one of the point vortices as well as the fraction of exposure of the particle to each one of the vortices are the controls available to steer it. Moreover, the flow field is such that each point vortex has an exponentially decaying circulation with a given constant decay rate. The objective of the article is to derive necessary conditions of optimality in the form of a Maximum Principle of Pontryagin, (see e.g. Arutyunov et al. (2011); Pontryagin et al. (1962); Vinter (2000)), for this optimal control problem in the multi-process control framework considered in Clarke and Vinter (1989a,b).

A controlled multi-process system,

$$\{(x^i, u^i) \in AC \times L^\infty\}_{i=1}^N,$$

consists in a finite number of dynamic control systems which are active in, possibly different, free endpoint time intervals, subject to their own state variable and control constraints, while sharing joint time and state endpoint constraints, that is,

$$\begin{cases} \dot{x}^i = f^i(t, x^i, u^i) \\ h^i(t, x^i) \in C^i \qquad\qquad [t_0^i, t_1^i]\, \mathcal{L} - \text{a.e.}, \\ u^i \in \mathcal{U}^i \end{cases} \tag{1}$$

$$\bar{h}(\{(x^i(t_0^i), x^i(t_1^i)) : i = 1, \ldots, N\}) \in \bar{C}. \tag{2}$$

Here, for $i = 1, \ldots, N$, where $N \in \mathbb{N}$ might be either given, or a choice variable, $\mathcal{U}^i = \{u \in L^\infty([t_0^i, t_1^i]; \mathbb{R}^m) : u^i(t) \in \Omega^i\}$, $\Omega^i \subset \mathbb{R}^m$, $C^i \in \mathbb{R}^{k_i}$, and $\bar{C} \in \mathbb{R}^{\bar{k}}$ are closed sets, $f^i : [t_0^i, t_1^i] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $h^i : [t_0^i, t_1^i] \times \mathbb{R}^n \to \mathbb{R}^{k_i}$, and $\bar{h} : \mathbb{R}^{2nN} \to \mathbb{R}^{\bar{k}}$ are given functions satisfying mild, possibly nonsmooth, assumptions. To simplify the presentation, we consider N given. Typical assumptions are: the function $\bar{h}$ is Lipschitz continuous in all its arguments, and, for each i, f^i is Lebesgue measurable in t and Lipschitz continuous x^i for any feasible value of u^i and Borel measurable in u^i for each (t, x^i), and h^i is continuous in t and Lipschitz continuous in x^i.

The goal is to optimize a given performance functional $J(\{x^i(t_0^i), x^i(t_1^i), u^i\}_{i=1}^N)$ by choosing a set of N triples $(x^i(t_0^i), x^i(t_1^i), u^i)$, while the controlled multi-process system satisfies the constraints (1) and (2). It is not difficult to conclude that the notions of admissible control multi-process, solution to the optimal control problem and that of Pontryagin type of minimum migrate directly from the corresponding ones already available in the literature, Pontryagin et al. (1962); Arutyunov (2000). For details on this framework, you may check Clarke and Vinter (1989a,b).

To the best of our knowledge, this the first time that a Maximum Principle is derived for a control problem whose dynamics are defined by controlled vortices, and,

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moreover, that a multi-processes framework is considered. It is a fact that the motion of a point mass in vector fields defined by vortices given a priori has already been considered before. In Liu and Hu (2010); Hou et al. (2007), the application of vortices to define the motion of robotic fishes modulated by trigonometric functions is addressed but these vortices created by the fish are periodic and predefined along the entire time interval. However, it is clear that this is a very restrictive context when considering a real world scenario. A much more desirable control system is the one in which the fish uses its tail to generate vortices in order to move from its current position to a selected point while minimizing the spent energy or according to some other criterion.

This is precisely what the problem introduced in this article achieves. Under some simplifying assumptions that will be discussed later, our problem formulation considers a system - environment and fish - that take into account vortices, some of which may be generated by the fish itself, and their impact on its motion. This represents a significant improvement with respect to the simple mimetic approach to the robotic fish locomotion as they enable the computation of certain control strategies of interest. For example, the ones that optimize the fish energy consumption.

In section 2, we present the dynamic equations for a passive particle and for point vortices with constant circulation moving in a two dimensional flow field generated by a set of vortices and, then, particularize to the specific case of interest in this article, in which we have the vector field defined by two point vortices with a given exponentially decaying circulation.

In the ensuing section, we present the general optimal control problem central to this article and explain in detail the rationale behind the chosen controlled dynamics, as well as some of the key assumptions that enable an elegant formulation in the multi-process context which is amenable to the application of the Maximum Principle. Still in this section, we also present a particular instance for which the necessary conditions of optimality will take on a particularly simple form.

In section 4, necessary conditions of optimality in the form of a Maximum Principle of Pontryagin are applied to the problems stated in the previous sections. Some comments on the usage of these conditions in order to carry out the analysis leading to the determination of the optimal solution will be provided. Finally, some conclusions are outlined in the last section.

2. FORMULATION OF THE VECTOR FIELDS DRIVEN BY VORTICES

A vortex is a point with circulation that generates a rotational flow field. Let us consider a two dimensional flow with n_v point vortices located in z_l formulated in the complex plane with circulation k_l, Protas (2008); Batchelor (1992); Newton (2001). Let the superscript $*$ denote the conjugate of the complex number. Then, the evolution of a passive particle $z(\cdot)$ in the vector field is given in the $\mathbb{C}$ by solving the ordinary differential equation

$$\dot{z}^* = \frac{1}{2\pi i}\sum_{l=1}^{n_v}\frac{k_l}{z - z_l}, \quad z(0) = z_0, \tag{3}$$

where vortex l of this flow satisfies, for $l = 1, \ldots, n_v$, the dynamic equation

$$\dot{z}_l^* = \frac{1}{2\pi i}\sum_{j=1, j\neq l}^{n_v}\frac{k_j}{z_l - z_j}, \quad z_l(0) = z_{l,0}. \tag{4}$$

Since we are interested in the state space representation of the systems dynamics, it is important to consider (3) in a real valued state space. Thus, by considering $z = x_a + ix_b$, and $\|x\|$ to be the Euclidean norm of x, we have

$$(\dot{x}_a, \dot{x}_b) = \frac{1}{2\pi}\sum_{l=1}^{n_v} k_l \frac{(-(x_b - x_{b,l}), (x_a - x_{a,l}))}{\|(x_a, x_b) - (x_{a,l}, x_{b,l})\|^2} \tag{5}$$

with $(x_a, x_b)(0) = (x_a, x_b)_0$, and where, in line with (4), we have, by letting $(x_{a,l}, x_{b,l})(0) = (x_{a,l}, x_{b,l})_0$,

$$(\dot{x}_{a,l}, \dot{x}_{b,l}) = \frac{1}{2\pi}\sum_{j=1, j\neq l}^{n_v} k_j \frac{(-(x_{b,l} - x_{b,j}), (x_{a,l} - x_{a,j}))}{\|(x_{a,j}, x_{b,j}) - (x_{a,l}, x_{b,l})\|^2}. \tag{6}$$

As stated above, we consider the case $n_v = 2$ at any time interval, and the circulation decays in time with a prescribed rate, that is, for $l = 1, 2$, we consider the circulation given by $e^{-\delta(t-t_0)}K_l$ where the K_l's are constants. The option $n_v = 2$ is justified by the fact that it is the minimal number of vortices ensuring motion controllability, being the minimality important due to the fact that creating a vortex requires a significant amount of energy from the fish. The fact that the dissipation of energy in the water column is relatively fast makes it reasonable to consider that its circulation decays exponentially in time with a pre-specified rate, and that, after some time, the effect of an "old" vortex can be neglected. These considerations provide the rationale to the vector field dynamics component of the control system model in the problem investigated in this article. The fish has to ensure that it can control its motion and, so, it requires two relevant vortices. On the other hand, since generating a vortex is expensive from the power consumption perspective, the quest for optimality leads to the natural decision to create a new vortex, when the effect of at least one of the currently active ones becomes negligible. This rationale justifies the consideration of an optimal control problem with two "dominant vortices". As we will see in the formulation of the problem in the next section, there is no loss of generality in consider the "oldest" vortex with smaller (independently of the sign) circulation.

Let us now introduce a notation facilitating the representation of the dynamics of the overall system, and, point out to a key property specific of the case $n_v = 2$. Denote by $x_V = col(x_1, x_2, x_3, x_4)$, the position coordinates of the joint vortices vector fields, being $x_{V_1} = col(x_1, x_3)$, and $x_{V_2} = col(x_2, x_4)$, respectively, the first and second vortices. Notice that we have $x_{V_1} = M_1 x_V$ and $x_{V_2} = M_2 x_V$ where $M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, and $M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Now, by using (6) in the case $n_v = 2$ with the notation just considered, the dynamics for the vortices are given by

$$\dot{x}_V = d(x_{V,0})A(t, t_0, K_1, K_2)x_V \tag{7}$$

where $d(x_{V,0}) := \left((x_1(t_0) - x_2(t_0))^2 + (x_3(t_0) - x_4(t_0))^2\right)^{-1}$, and, by denoting by δ the prescribed rate of decay,

$$A(t,t_0,K_1,K_2) := \frac{e^{-\delta(t-t_0)}}{2\pi}\begin{bmatrix} 0 & 0 & -K_2 & K_2 \\ 0 & 0 & K_1 & -K_1 \\ K_2 & -K_2 & 0 & 0 \\ -K_1 & K_1 & 0 & 0 \end{bmatrix}.$$

An important observation with significant implications on the developments of the next two sections is that $\frac{d}{dt}d(x_{V,t}) = 0$. This conclusion can be reached with a few calculations.

3. FORMULATION OF THE MULTI-PROCESS OPTIMAL CONTROL PROBLEM

The formulation of the optimal control problem of the motion of the fish in a given time interval $[0,T]$ involves the specification of (i) the control functional to be optimized, (ii) state and control to be satisfied pointwise in time, (iii) terminal endpoint state and time constraint to be met, and obviously, (iv) controlled dynamic equations that will dictate how the trajectory of the system evolves while satisfying all the constraints.

We started already with the later by explaining how vortices evolve over time. In order to control its motion, the fish has to generate an additional vortex at a certain point in time and then define the extent of its body exposure to the last two vortices during a given time interval initiated at the vortex creation time. The scheme assumes that the vortex with lower circulation has residual impact in the motion, or that the fish has ways to have null exposure to that vortex. Another important issue concerns the definition of the value of the control specifying the extent to the exposure of the fish to a given vortex. Instead of considering the field at the position of the fish, we consider the field at the position of the vortex. This greatly simplifies not only the formulation of the problem but also the complexity of the necessary conditions of optimality. Moreover, from the practical point of view this assumption is reasonable since it is always possible to estimate the value of vortex field at its position from its value at the position of the fish.

From of the above, it is clear that the considered framework is particularly amenable to a multi-process formulation, since the overall time interval $[0,T]$ can be regarded as the union of a set of distinct contiguous time intervals in which the vortex dynamics change as a result of the discrete component of the control action. Let us consider $N \in \mathbb{N}$ adjacent subintervals, i.e., there exists a partition $\{t_i : i = 0,\ldots,N, t_0 = 0, t_{i-1} < t_i, i = 1,\ldots,N, t_N = T\}$, satisfying $[0,T] = \cup_{i=1}^{N} I_i$ where $I_i = [t_{i-1}, t_i]$. Since for each interval, we have to mention its final and initial points, independently of the neighboring intervals, we will consider $I_i = [\sigma_0^i, \sigma_1^i]$, being obvious that $\sigma_0^1 = 0$, $\sigma_1^N = T$, and $\sigma_1^i = \sigma_0^{i+1}$, for $i = 1,\ldots,N-1$.

Let us now specify the state and control variables and its components for each subinterval by attaching an upper index i, $x^i = col(x_0^i, x_V^i, x_p^i, x_K^i)$ at $[\sigma_0^i, \sigma_1^i]$, for $i = 1,\ldots,N$, being

- $x_V^i = col(x_1^i, x_2^i, x_3^i, x_4^i)$ with $x_{V_1}^i = M_1 x_V^i$ and $x_{V_2}^i = M_2 x_V^i$ as explained in the previous section,
- $x_p^i = col(x_5^i, x_6^i)$ is the position of the fish satisfying $\dot{x}_p^i = u_1^i \dot{x}_{V_1}^i + u_2^i \dot{x}_{V_2}^i$, where u_j^i is the control value expressing the extent of exposure of the fish to the vortex j, $j = 1, 2$,
- $x_K^i = col(x_7^i, x_8^i)$ where $x_7^i = \frac{K_1^i}{2\pi}$ and $x_8^i = \frac{K_2^i}{2\pi}$, and
- $x_0^i = d(x_{V,0}^i)$.

First, we remark that we retain the label of matrix A, now given for $t \in [\sigma_0^i, \sigma_1^i]$,

$$A(t,\sigma_0^i,x_7^i,x_8^i) = \mathrm{e}^{-\delta(t-\sigma_0^i)}\begin{bmatrix} 0 & 0 & -x_8^i & x_8^i \\ 0 & 0 & x_7^i & -x_7^i \\ x_8^i & -x_8^i & 0 & 0 \\ -x_7^i & x_7^i & 0 & 0 \end{bmatrix}.$$

Second, it is obvious that we have $\dot{x}_K^i = 0$ and that $\dot{x}_0^i = 0$. Given the trivial dynamics, one may ask the reason why these have been chosen as state variables. The reason is simple: the choice of their initial value affect the evolution the system, or, in the case of x_0^i, if they fully depend explicitly on other variables, their choice facilitates the application of the necessary conditions of optimality.

The decision variables of this problem comprise the feasible initial values of the state variable at the initial time of each interval, i.e., $(x_V^i(\sigma_0^i), x_p^i(\sigma_0^i), x_K^i(\sigma_0^i))$ - notice that, here, $x_0(\sigma_0^i) = d(x_{V,0}^i(\sigma_0^i))$ - and the control functions $u^i = col(u_1^i, u_2^i)$ which are in $L^\infty([\sigma_0^i,\sigma_1^i];\mathbb{R}^2)$ and take values in $\Omega^i = [-w,w]\times[-w,w]$, for a given $w > 0$. In order to facilitate the representation of the dynamics of the system, consider the matrix $U^i = \begin{bmatrix} u_1^i & u_2^i & 0 & 0 \\ 0 & 0 & u_1^i & u_2^i \end{bmatrix}$.

Thus, the dynamics of the system on the interval $[\sigma_0^i, \sigma_1^i]$ are governed by

$$\begin{cases} \dot{x}_0^i &= 0 \\ \dot{x}_V^i &= x_0^i A(t,\sigma_0^i,x_K^i)x_V^i \\ \dot{x}_p^i &= x_0^i U^i A(t,\sigma_0^i,x_K^i)x_V^i \\ \dot{x}_K^i &= 0. \end{cases} \tag{8}$$

Now, we turn to the time and state endpoint constraints.

We start with the time variable. First, note that this formulation applies to either $T > 0$ being pre-defined or also a decision variable of choice. The multi-process time constraints are given by

$$\bar{\sigma} \in \Lambda_\sigma$$

where $\bar{\sigma} = col\left((\sigma_0^i,\sigma_1^i) : i = 1,\ldots,N\right)$, and

$$\Lambda_\sigma := \{\bar{\sigma} \in \mathbb{R}^{2N} : \sigma_0^1 = 0,\ \sigma_1^N = T,\ \sigma_0^i = \sigma_1^{i-1},\ i = 2,\ldots N\}.$$

Later, it will be convenient to use also the notation $\bar{\sigma}_j = col(\bar{\sigma}_j^i : i = 1,\ldots,N)$, for $j = 0, 1$.

Notice that since $x_0^i(\sigma_j^i) = d(x_V^i(\sigma_j^i))$, for $i = 1,\ldots,N$, for $j = 0, 1$, we have

$$\bar{x}_0 = col(x_0^i(\sigma_0^i), x_0^i(\sigma_1^i)) : i = 1,\ldots,N) \in d(\Lambda_V)$$

being the later defined next.

From the above definitions, we have that

$$\bar{x}_V = col\left(col(x_V^i(\sigma_0^i), x_V^i(\sigma_1^i)) : i = 1,\ldots,N\right) \in \Lambda_V$$

where

$$\Lambda_V{:=}\{\bar{x}_V \in \mathbb{R}^{8N}{:}\, M_1x_V^1(0) = x_{V_1,0}^1,\, M_2x_V^1(0) \in \mathbb{R}^2,\; x_V^1(\sigma_1^1) \in \mathbb{R}^4,\, x_V^i(\sigma_1^i) \in \mathbb{R}^4,\; M_1x_V^i(\sigma_0^i) = M_2x_V^{i-1}(\sigma_1^{i-1}),\; M_2x_V^i(\sigma_0^i) \in \mathbb{R}^2,\, i = 2,\ldots,N\}.$$

Since the particle evolves continuously in time, we naturally have that

$$\bar{x}_p = col\left(col(x_p^i(\sigma_0^i), x_p^i(\sigma_1^i)) : i = 1,\ldots,N\right) \in \Lambda_p$$

where

$$\Lambda_p{:=}\{\bar{x}_p{\in}\mathbb{R}^{4N}{:}\, x_5^1(0) = x_{5,0},\, x_6^1(0) = x_{6,0},\, x_5^N(T) = x_{5,T},\; x_6^N(T) = x_{6,T},\, x_5^{i-1}(\sigma_1^{i-1}) = x_5^i(\sigma_0^i),\; x_6^{i-1}(\sigma_1^{i-1}) = x_6^i(\sigma_0^i),\, i = 2,\ldots,N\}.$$

Finally, it remains to characterize the constraints on the circulation. Remark that, for given numbers $\mathbf{K}_2 > \mathbf{K}_1 > 0$, we have $x_7^1(\sigma_0^1) = N_1x_{K,0}^1 \in [-\mathbf{K}_1, \mathbf{K}_1]$ and $x_8^1(\sigma_0^1) = N_2x_{K,0}^1 \in \tilde{V}^1(N_1x_{K,0}^1) := (\bar{V}^1 \cup (-\bar{V}^1))(N_1x_{K,0}^1)$, where $\bar{V}^1(N_1x_{K,0}^1)$ is given by

- $[N_1x_{K,0}^1, \mathbf{K}_2]$ if $N_1x_{K,0}^1 > 0$, and
- $[-\mathbf{K}_2, N_1x_{K,0}^1]$ otherwise,

being $x_{K,0}^1 = x_K^1(\sigma_0^1)$, and N_1 and N_2, respectively, the matrices $[1\;\;0]$ and $[0\;\;1]$.

Moreover, for $i = 2,\ldots,N$, and by denoting $x_{K,0}^i = x_K^i(\sigma_0^i)$ and $g_{i-1,i} = e^{-\delta(\sigma_0^i - \sigma_0^{i-1})}$, we have

- $N_1x_{K,0}^i := g_{i-1,i}N_2x_{K,0}^{i-1}$
- $N_2x_{K,0}^i \in (\bar{V}^i \cup (-\bar{V}^i))(g_{i-1,i}N_2x_{K,0}^{i-1})$, where, for some $1 < \bar{\alpha} < |N_2x_{K,0}^{i-1}|^{-1}g_{i-1,i}^{-1}\bar{\mathbf{K}}$, with $\bar{\mathbf{K}}$ satisfying
$$\bar{\mathbf{K}} \geq \bar{\alpha} \max_{i=2,\ldots,N}\{|g_{i-1,i}N_2x_{K,0}^{i-1}|\}$$
$\bar{V}^i = [\bar{\alpha}g_{i-1,i}N_2x_{K,0}^{i-1}, \bar{\mathbf{K}}]$ or $[-\bar{\mathbf{K}}, \bar{\alpha}g_{i-1,i}N_2x_{K,0}^{i-1}]$ if $N_2x_{K,0}^{i-1}$ is, respectively, positive or negative. Notice that an example of such a $\bar{\mathbf{K}}$ is $\bar{\mathbf{K}} = \mathbf{K}_2$ and, again, let $\tilde{V}^i(N_1x_{K,0}^i) := (\bar{V}^i \cup (-\bar{V}^i))(g_{i-1,i}N_2x_{K,0}^{i-1})$.

The choice of the numbers $\bar{\alpha}$ and $\bar{\mathbf{K}}$ relies on the trade-off between the realism of the two vortices model and the optimality of the solution to the optimal control problem formulated with such a model. Now, we may express the constraints for the set of boundary values of the vortices circulation by

$$\bar{x}_K \in \Lambda_K,$$

where $\bar{x}_K = col\left(col(x_{K,0}^i, x_{K,1}^i) : i = 1,\ldots,N\right)$ (again here, $x_{K,1}^i = x_K^i(\sigma_1^i)$), and

$$\Lambda_K{:=}\{\bar{x}_K \in \mathbb{R}^{4N}{:}\, x_{K,1}^i \in \mathbb{R}^2,\;\text{ for } i = 1,\ldots,N,\; x_{K,0}^1 \in [-\mathbf{K}_1, \mathbf{K}_1] \times \tilde{V}^1(N_1x_{K,0}^1),\; x_{K,0}^i \in \hat{V}^i, \text{ for } i = 2,\ldots,N\},$$

with $\hat{V}^i = \{g_{i-1,i}N_2x_{K,0}^{i-1}\} \times \tilde{V}^i(g_{i-1,i}N_2x_{K,0}^{i-1})$. The constraints to be satisfied by the circulation appear to be somewhat intricate. However, its “raison d'être” ensures that, without, any loss of generality, the “second” or “younger” vortex is always the one with higher circulation. This condition facilitates the formulation of the problem since simpler functions suffice to express the constraints.

Now, we pursue with the specification of the cost functional. We choose it as a weighted sum of the various items contributing to power consumption, notably, the continuum time steering control function, and the energy associated with the creation of the vortices. This last term involves two components: level of circulation and the distance at which the new vortex is generated. Thus, we should have

$$J(\bar{u}, \bar{x}(\bar{\sigma}_0), \bar{\sigma}){:=}\frac{1}{2}\sum_{i=1}^{N}[\alpha_1\int_{\sigma_0^i}^{\sigma_1^i}\|u^i(t)\|^2dt + \alpha_2\|N_2x_K^i(\sigma_0^i)\|^2 + \alpha_3\|x_p^i(\sigma_0^i) - M_2x_V^i(\sigma_0^i)\|^2],$$

where α_1, α_2, and α_3 are the non-negative weighting coefficients.

Then, the multi-process control problem is as follows:

$$\text{(P) Minimize } J(\bar{u}, \bar{x}(\sigma_0), \bar{\sigma})$$
$$\text{subject to } \begin{cases} \text{differential equations (8)} \\ (u_1^i, u_2^i){\in}[-w, w]{\times}[-w, w] \end{cases} [\sigma_0^i, \sigma_1^i]\text{-a.e.}$$
$$\text{for } i = 1,\ldots,N, \text{ and}$$
$$\bar{\sigma}{\in}\Lambda_\sigma, \bar{x}_0{\in}d(\Lambda_V), \bar{x}_V{\in}\Lambda_V, \bar{x}_p{\in}\Lambda_p, \bar{x}_K{\in}\Lambda_K.$$

For future convenience, we express the cost functional $J(\bar{u}, \bar{x}(\bar{\sigma}_0), \bar{\sigma})$ as the sum

$$f_0\left(\{(x(\sigma_0), x(\sigma_1))^i\}_{i=1}^N\right) + \frac{1}{2}\alpha_1\sum_{i=1}^{N}\int_{\sigma_0^i}^{\sigma_1^i}\|u^i(t)\|^2dt,$$

where $f_0\left(\{(x(\sigma_0), x(\sigma_1))^i\}_{i=1}^N\right)$ is given by

$$\frac{1}{2}\sum_{i=1}^{N}\left[\alpha_2\|N_2x_K^i(\sigma_0^i)\|^2 + \alpha_3\|x_p^i(\sigma_0^i) - M_2x_V^i(\sigma_0^i)\|^2\right].$$

4. MAXIMUM PRINCIPLE FOR THE MULTI-PROCESS OPTIMAL CONTROL PROBLEM

Now, we apply the Maximum Principle of Pontryagin for the multi-process optimal control problem (P) described in the previous section. Of course, these conditions are expressed having as a reference the optimal control process which is usually singled out by the superscript “*”. However, in order to mitigate the already heavy burden of the notation, we omit it here.

4.1 The adjoint differential equations

By migrating the notation adopted to the state variable to the adjoint variable p, we have $p^i{=}lin(p_0^i, p_V^i, p_p^i, p_K^i)$ and may express the Pontryagin function as follows

$$H(t, \bar{x}, \bar{p}, \bar{u}, \bar{\sigma}) = \sum_{i=1}^{N}H^i(t, x^i, p^i, u^i, \sigma_0^i)\chi_{[\sigma_0^i, \sigma_1^i]}(t)$$

where $\chi_{[\sigma_0^i, \sigma_1^i]}$ is the indicator function of the interval $[\sigma_0^i, \sigma_1^i]$, and, by denoting $A^i = A(t, \sigma_0^i, x_{K,0}^i)$,

$$H^i(t, x^i, p^i, u^i, \sigma_0^i) = x_0^i(p_V^i + p_p^iU^i)A^ix_V^i - \frac{1}{2}\alpha_1\|u^i\|^2.$$

Now, by denoting the Jacobian of the vector valued map F w.r.t. the variable y by D_yF, we readily conclude that the adjoint system is, for $i = 1,\ldots,N$, given by

$$\begin{cases} -\dot{p}_0^i &= (p_V^i + p_p^i U^i) A^i x_V^i \\ -\dot{p}_V^i &= x_0^i (p_V^i + p_p^i U^i) A^i \\ -\dot{p}_p^i &= 0 \\ -\dot{p}_K^i &= x_0^i (p_V^i + p_p^i U^i) D_{x_K^i}\big(A^i x_V^i\big)\,. \end{cases} \tag{9}$$

The tranversality conditions are given by

$$\big\{(-h_0^i, h_1^i), (p_{0,0}^i, -p_{0,1}^i), (p_{V,0}^i, -p_{V,1}^i), \\ (p_{p,0}^i, -p_{p,1}^i), (p_{K,0}^i, -p_{K,1}^i)\big\}_{i=1}^N \in N_\Lambda(\chi) + \lambda \nabla_\chi \bar{f}_0(\chi).$$

Here, $\lambda \geq 0$, and

- $\chi = \big\{col(\sigma_0^i, \sigma_1^i), col(x_{0,0}^i, x_{0,1}^i), col(x_{V,0}^i, x_{V,1}^i), col(x_{p,0}^i, x_{p,1}^i), col(x_{K,0}^i, x_{K,1}^i)\big\}_{i=1}^N$,
- $z_{\alpha,j}^i = z_\alpha^i(\sigma_j^i)$, for $j = 0,\, 1$,
- $\bar{f}_0(\chi) = f_0(\{x_{V,0}^i, x_{p,0}^i, x_{K,0}^i\}_{i=1}^N)$,
- $\Lambda = \Pi_{i=1}^N \Lambda_i$ where $\Lambda_i = \Lambda_\sigma^i \times d(\Lambda_V^i) \times \Lambda_V^i \times \Lambda_p^i \times \Lambda_K^i$, with Λ_α^i being the i^{th} component of Λ_α, for $\alpha \in \{\sigma, x_0, x_V, x_p, x_K\}$,
- $h_0^i = \sup\limits_{u^i \in [-w,w]^2} \Big\{H^i(t, x^i, p^i, u^i, \sigma_0^i)|_{t=\sigma_0^i}\Big\}$, and
- $h_1^i = \sup\limits_{u^i \in [-w,w]^2} \Big\{H^i(t, x^i, p^i, u^i, \sigma_0^i)|_{t=\sigma_1^i}\Big\}$.

Let us decode two of the above compact adjoint differential equations. First, we note that, by considering $p_V^{iT} = [p_1^i, p_2^i, p_3^i, p_4^i]$, $p_p^{iT} = [p_5^i, p_6^i]$, $p_K^{iT} = [p_7^i, p_8^i]$ and by replacing x_V^i, x_p^i, x_K^i, and u^i by its components, and by considering

$$\Sigma^i = \langle (x_3^i - x_4^i, x_1^i - x_2^i), col(-p_1^i x_8^i + p_2^i x_7^i, p_3^i x_8^i - p_4^i x_7^i)\rangle$$
$$\tilde{\Sigma}^i = \langle (x_3^i - x_4^i, x_1^i - x_2^i), col(-p_5^i, p_6^i)\rangle$$

to simplify the notation, we conclude that

$$H^i(t, x^i, p^i, (u_1^i, u_2^i), \sigma_0^i) = -\frac{1}{2}\alpha_1\left((u_1^i)^2 + (u_2^i)^2\right) \\ + e^{-\delta(t-\sigma_0^i)} x_0^i \left[\Sigma^i + \tilde{\Sigma}^i (u_1^i x_8^i - u_2^i x_7^i)\right].$$

Straightforward computations lead us to the fact that the compact form (9) can be expressed in a detailed form as follows

$$\begin{cases} -\dot{p}_0^i &= e^i(t)\left[\Sigma^i + \tilde{\Sigma}^i(u_1^i x_8^i - u_2^i x_7^i)\right] \\ -\dot{p}_V^i &= -[\dot{p}_1^i, \dot{p}_2^i, \dot{p}_3^i, \dot{p}_4^i] \\ &= e^i(t) x_0^i [x_8^i p_3^i - x_7^i p_4^i + p_6^i(x_8^i u_1^i - x_7^i u_2^i), \\ &\quad -(x_8^i p_3^i - x_7^i p_4^i + p_6^i(x_8^i u_1^i - x_7^i u_2^i)), \\ &\quad -x_8^i p_1^i + x_7^i p_2^i - p_5^i(x_8^i u_1^i - x_7^i u_2^i), \\ &\quad -(-x_8^i p_1^i + x_7^i p_2^i - p_5^i(x_8^i u_1^i - x_7^i u_2^i))] \\ -\dot{p}_p^i &= 0 \\ -\dot{p}_K^i &= -[\dot{p}_7^i, \dot{p}_8^i] \\ &= e^i(t) x_0^i [\langle (x_3^i - x_4^i, x_1^i - x_2^i), (p_2^i, -p_4^i)\rangle - \tilde{\Sigma}^i u_2^i \\ &\quad \langle (x_3^i - x_4^i, x_1^i - x_2^i), (-p_1^i, p_3^i)\rangle + \tilde{\Sigma}^i u_1^i], \end{cases} \tag{10}$$

where $e^i(t) = e^{-\delta(t-\sigma_0^i)}$. It is clear from the above that $\dot{p}_1^i = -\dot{p}_2^i$ and $\dot{p}_3^i = -\dot{p}_4^i$.

4.2 The transversality conditions

In order to express the transversality conditions in detail, we still need to compute both the gradient of the cost functional with respect to the state variable endpoints and the normal cone to the endpoint constraint sets.

The cost functional term that depends on the state variable at its end points only involves the state variable at the initial time. Thus, we may write:

$$(\nabla_{x_K} f_0)^i|_{\sigma_0^i} = \alpha_2 N_2 x_K^i(\sigma_0^i) N_2 = \left[0, \alpha_2 x_8^i(\sigma_0^i)\right] \tag{11}$$
$$(\nabla_{x_p} f_0)^i|_{\sigma_0^i} = \alpha_3 (x_p^i(\sigma_0^i) - M_2 x_V^i(\sigma_0^i))^T \\ = \alpha_3\left[x_5^i(\sigma_0^i) - x_2^i(\sigma_0^i),\ x_6^i(\sigma_0^i) - x_4^i(\sigma_0^i)\right] \tag{12}$$
$$(\nabla_{x_V} f_0)^i|_{\sigma_0^i} = -\alpha_3 (x_p^i(\sigma_0^i) - M_2 x_V^i(\sigma_0^i))^T M_2 \\ = -\alpha_3\left[0, x_5^i(\sigma_0^i) - x_2^i(\sigma_0^i), 0, x_6^i(\sigma_0^i) - x_4^i(\sigma_0^i)\right]. \tag{13}$$

Obviously, $(\nabla_{x_0} f_0)^i|_{\sigma_1^i} = 0$, for all i.

Now, it is the turn to compute the limiting normal cones to the sets Λ_{x_0}, Λ_σ, Λ_V, Λ_p, and Λ_K. We consider the normal cone introduced by Mordukhovich, also known by limiting normal cone, - $N_C^L(c)$ denotes the limiting normal cone of the set C at the point $c \in C$ - which has the advantage of being smaller and, thus, enabling more precise conditions. For a reference on a definition, check Mordukhovich (2006). The derivation of the normal cones to Λ_σ, Λ_V, and Λ_p is straightforward, and it yields:

$$N_{\Lambda_\sigma}^L(\bar{\sigma}) = \{\hat{p}_\sigma \in \mathbb{R}^{2N}: \hat{p}_{\sigma,0}^1 \in \mathbb{R},\ \hat{p}_{\sigma,1}^{i-1} = -\hat{p}_{\sigma,0}^i, i = 2, \ldots, N, \\ \hat{p}_{\sigma,1}^N \in \mathbb{R}\},$$
$$N_{\Lambda_V}^L(\bar{x}_V) = \{\hat{p}_V \in \mathbb{R}^{8N}:\ \hat{p}_{V,0}^1 M_1^T \in \mathbb{R}^2,\ \hat{p}_{V,1}^i = 0,\ \hat{p}_{V,0}^i M_2^T = 0 \\ i = 1, \ldots, N,\ \hat{p}_{V,0}^i M_1^T = -\hat{p}_{V,1}^{i-1} M_2^T, \\ i = 2, \ldots, N\},$$
$$N_{\Lambda_p}^L(\bar{x}_p) = \{\hat{p}_p \in \mathbb{R}^{4N}:\ \hat{p}_{p,0}^1 \in \mathbb{R}^2, \hat{p}_{p,0}^i = -\hat{p}_{p,1}^{i-1}, i = 2, \ldots, N, \\ \hat{p}_{p,1}^N \in \mathbb{R}^2\}.$$

Since $\bar{x}_0 = d(\bar{x}_V)$, the normal cone to Λ_{x_0} can be easily related to $N_{\Lambda_V}^L$ via the gradient of the function d. Since $\nabla d(\bar{x}_V) = 2x_0^2[x_1 - x_2, x_2 - x_1, x_3 - x_4, x_4 - x_3, 0, 0, 0, 0]$, it is not difficult to conclude that

$$N_{x_0}^L = \{\hat{p}_0\colon 2\hat{p}_0 x_0^2 [x_1 - x_2, x_2 - x_1, x_3 - x_4, x_4 - x_3, \\ 0, 0, 0, 0] \in N_{\Lambda_V}^L\}.$$

Due to the specific interdependence of the various time subintervals in the definition of Λ_K, the computation of $N_{\Lambda_K}^L(\bar{x}_K)$ requires some more attention. Let us apply the definition of normal cone applied to the reference point $\hat{x}_{K,0} = col(\hat{x}_{K,0}^i : i = 1, \ldots, N)$ in Λ_K. Without any loss of generality, let us consider $N \geq 3$. Then, $\hat{p}_{K,0} = lin(\hat{p}_{K,0}^i : i = 1, \ldots, N) \in N_{\Lambda_K}^L(\hat{x}_{K,0})$ if, $\forall\, x_{K,0} \in \Lambda_K$,

$$0 \geq \langle \hat{p}_{K,0}, x_{K,0} - \hat{x}_{K,0}\rangle \\ = \langle \hat{p}_{K,0}^1 N_1^T, N_1(x_{K,0}^1 - \hat{x}_{K,0}^1)\rangle + \langle \hat{p}_{K,0}^1 N_2^T, N_2(x_{K,0}^1 - \hat{x}_{K,0}^1)\rangle \\ + \sum_{i=2}^N \left[\langle \hat{p}_{K,0}^i N_1^T, N_1(x_{K,0}^i - \hat{x}_{K,0}^i)\rangle \right. \\ \left. + \langle \hat{p}_{K,0}^i N_2^T, N_2(x_{K,0}^i - \hat{x}_{K,0}^i)\rangle\right].$$

By recalling the definition of the constraint set Λ_K and by regrouping the terms, we readily obtain, $\forall x_K \in \Lambda_K$, the inequality

$$0 \geq \langle \hat{p}^1_{K,0} N_1^T, N_1(x^1_{K,0} - \hat{x}^1_{K,0})\rangle + \sum_{i=1}^{N-1} \langle \hat{p}^i_{K,0} N_2^T + g_{i,i+1}\hat{p}^{i+1}_{K,0} N_1^T, N_2(x^i_{K,0} - \hat{x}^i_{K,0})\rangle + \langle \hat{p}^N_{K,0} N_2^T, N_2(x^N_{K,0} - \hat{x}^N_{K,0})\rangle.$$

By considering all the feasible variations of both components of $x_{K,0}$, we obtain $N^L_K(\hat{x}_{K,0})$ as the set

$$\Big\{ [\hat{p}^i_{K,0}, 0] \in \mathbb{R}^4 \colon \hat{p}^1_{K,0} + [0, g_{1,2}\hat{p}^2_{K,0} N_1^T] \in N^L_{\bar{V}^1(N_1 \hat{x}^1_{K,0})}(\hat{x}^1_{K,0})$$
$$\hat{p}^i_{K,0} N_2^T + g_{i,i+1}\hat{p}^{i+1}_{K,0} N_1^T \in N^L_{\bar{V}^i(g_{i-1,i} N_2 \hat{x}^{i-1}_{K,0})}(\hat{x}^i_{K,0}),$$
$$i = 2, \ldots, N-1,\ \hat{p}^N_{K,0} N_2^T \in N^L_{\bar{V}^N(g_{N-1,N} N_2 \hat{x}^{N-1}_{K,0})}(\hat{x}^N_{K,0}) \Big\}.$$

4.3 The maximum condition

Now, let us compute the optimal candidate control strategy which has to maximizing the Pontryagin function along the optimal trajectory and associated adjoint variable. Due to the quadratic structure of the Hamiltonian and the fact that the control constraints are decoupled, we readily conclude that, $\mathcal{L}$-a.e. in $[\sigma^i_0, \sigma^i_1]$,

$$\hat{u}^i_1(t) = Sat_w\left(\frac{x^i_8(t)x^i_0}{\alpha_1} e^{-\delta(t-\sigma^i_0)}\tilde{\Sigma}^i(t)\right)$$
$$\hat{u}^i_2(t) = Sat_w\left(-\frac{x^i_7(t)x^i_0}{\alpha_1} e^{-\delta(t-\sigma^i_0)}\tilde{\Sigma}^i(t)\right).$$

Here, $Sat_a(z)$ is the saturation function which is defined by taking the values $-a$, z, and a, if, respectively, $z < -a$, $-a \leq z \leq a$, and $z > a$, and the functions $\tilde{\Sigma}^i$, x^i_8 and x^i_7 are evaluated along the optimal control process. This simple structure allows us to obtain the explicit form for the optimal control function given above. To facilitate the analysis to find a control multi-process satisfying the Maximum Principle, the nine situations that might arise are compiled in the Table 1.

Table 1. Optimal Control Function

$\hat{u}^i_1$	$\hat{u}^i_2$	$\bar{H}^i(\hat{u}^i_1, \hat{u}^i_2) - \Sigma^i$
$-w$	$-w$	$-w\left(\tilde{\mathcal{Z}}^i_8 + \tilde{\mathcal{Z}}^i_7 + \alpha_1 w\right)$
$-w$	$-\frac{\tilde{\mathcal{Z}}^i_7}{\alpha_1}$	$-w\left(\tilde{\mathcal{Z}}^i_8 + \frac{\alpha_1 w}{2}\right) + \frac{(\tilde{\mathcal{Z}}^i_7)^2}{2\alpha_1}$
$-w$	w	$-w\left(\tilde{\mathcal{Z}}^i_8 - \tilde{\mathcal{Z}}^i_7 + \alpha_1 w\right)$
$\frac{\tilde{\mathcal{Z}}^i_8}{\alpha_1}$	$-w$	$w\left(-\tilde{\mathcal{Z}}^i_7 - \frac{\alpha_1 w}{2}\right) + \frac{(\tilde{\mathcal{Z}}^i_8)^2}{2\alpha_1}$
$\frac{\tilde{\mathcal{Z}}^i_8}{\alpha_1}$	$\frac{\tilde{\mathcal{Z}}^i_7}{\alpha_1}$	$\frac{1}{2\alpha_1}\left((\tilde{\mathcal{Z}}^i_8)^2 + (\tilde{\mathcal{Z}}^i_7)^2\right)$
$\frac{\tilde{\mathcal{Z}}^i_8}{\alpha_1}$	w	$-w\left(-\tilde{\mathcal{Z}}^i_7 + \frac{\alpha_1 w}{2}\right) + \frac{(\tilde{\mathcal{Z}}^i_8)^2}{2\alpha_1}$
w	$-w$	$w\left(\tilde{\mathcal{Z}}^i_8 - \tilde{\mathcal{Z}}^i_7 - \alpha_1 w\right)$
w	$-\frac{\tilde{\mathcal{Z}}^i_7}{\alpha_1}$	$w\left(\tilde{\mathcal{Z}}^i_8 - \frac{\alpha_1 w}{2}\right) + \frac{(\tilde{\mathcal{Z}}^i_8)^2}{2\alpha_1}$
w	w	$w\left(\tilde{\mathcal{Z}}^i_8 + \tilde{\mathcal{Z}}^i_7 - \alpha_1 w\right)$

Here, for convenience, we omit the time variable and define $\bar{H}^i(\hat{u}^i_1, \hat{u}^i_2) = H^i(t, x^i, p^i, (\hat{u}^i_1, \hat{u}^i_2), \sigma^i_0)$, $\tilde{\mathcal{Z}}^i_7 = -\tilde{\Sigma}^i e^{-\delta(t-\sigma^i_0)} x^i_0 x^i_7$, and $\tilde{\mathcal{Z}}^i_8 = \tilde{\Sigma}^i e^{-\delta(t-\sigma^i_0)} x^i_0 x^i_8$.

5. CONCLUSIONS

This article concerns the derivation of a Maximum Principle for the optimal control of a multi-process control system with a two vortex driven particle in the plane. The obtained conditions supply enough information to select control processes which are candidates for optimality. The problem was presented in full generality and some additional constraints on the initial values of the state variable will simplify the conditions which are amenable for analysis.

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7.3 ARTICLE COPY "Optimal Control Framework for AUV's Motion Planning in Planar Vortices Vector Field", T. Grilo, S. Gama, F. L. Pereira – Proceedings Article *Proceedings-AUV*, 2018 (Submitted)

Optimal Control Framework for AUV's Motion Planning in Planar Vortices Vector Field

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Optimal Control Framework for AUV's Motion Planning in Planar Vortices Vector Field

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***Abstract*—This article concerns the investigation of an optimal control framework based on the Maximum Principle of Pontryagin for the motion planning of Autonomous Underwater Vehicles (AUVs) that take into account not only its own direct propulsion through its actuators, but also the motion effects of the water column flow fields, notably, the point vortices which might be naturally present or intentionally generated in order to achieve the desired motion features. We do not enter into issues related to the AUV mechanical design, but assume that this can such that the shape of the (movable or not) vehicle surfaces, are such that the water column flow field energy can be used with advantage to the required motion. In this context, we simply assume that the vehicle is a point mass unicycle such that the effect of the water column vortices can be controlled to contribute to the desired AUV's longitudinal and rotational velocities. The control action is exercised by generating or using an existing vortex - specified by its location at circulation - at a time as well as by varying the "exposure" of the AUV to each one of the vortices. A control multiprocesses framework is chosen in order to derive necessary conditions of optimality in the form of a Maximum Principle of Pontryagin. These conditions provide enough relations in order to fully specify the optimal motion.**

***Index Terms*—Maximum Principle of Pontryagin, multiprocess system, point vortex**

I. Introduction

The relevance of AUV's as marine platforms carrying sensors to better understand the geophysical and biological phenomena in the ocean bottom and water column as well as in the ocean-atmosphere interface has been widely recognized [1], [2], [3]. The enormous R&D investment, by the research establishment, governance institution, and the multiple types of stakeholders in designing systems in which AUV's, often articulated with other type of vehicles, and human operators play a critical role, has been huge and all the current perspectives point to further growth [4]. This is not surprising - as the marine environment is increasingly regarded has holding important keys for a sustainable permanence of humans on the Earth, the more urgent is to understand the complex, intertwined, myriad of its processes, there are not that many tools for the efficient gathering of the badly required data, and AUV's have revealed to be one of the most important one - and it is attested by the increasing number and sophistication of laboratory facilities (e.g., https://www.eumarinerobots.eu/) and the huge body scientific and technical literature.

Obviously, the challenges are huge. The nature and depth of issues span a wide range: from systems centered, [1], to vehicle or group of vehicle's centered, [6], the ultimate goal is to design systems enabling the extraction of data, and also enable the intervention, in an opaque, often hostile and highly variable milieu with maximal efficiency. This means that it is extremely important to control the all the activities on the AUV or group of AUV's - such as, motion (navigation, guidance, and vehicle actuators control), payload data gathering and intervention, communication, computation, data transmission, and power and safety management - system in order to achieve the planned goals while optimizing the on-board resources. In abstract, this requires solving extremely complex optimization based feedback control problems with minimal power consumption while satisfying the "on-line" requirements. The above issues raise the question of how to take advantage of the huge energy available in the flow fields, abounding in most of the marine environments, in order to mitigate the extent of the challenges. It is not an exaggeration to state that the efforts along this direction have been timid, e.g., [3], when comparing with the effort on vehicle's systems and technologies, from which the above and [5], and references therein are very far from being representative. On the other hand a significant research effort has been spent in designing understanding and developing motion control systems inspired by the remarkable capabilities of propulsion and motion control demonstrated by fish in the underwater environment. See, for example, [3], [12]–[14] and references therein. This is not surprising since this investigation will serve to assess ideas to inspire novel efficient underwater locomotion systems that take advantage of the ocean dynamics.

Here, we provide a step in this direction by investigating the optimal control problem formulated in order to drive a AUV between two given points with minimal effort in a flow field defined by two point vortices at any point in time, [8], [9], [14]. We consider that the position and initial circulation of

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each one of the point vortices, and the fraction of exposure of the AUV - regarded as a unicycle particle - to each one of the vortices are the controls available for propulsion and steering. Moreover, the flow field is such that each point vortex has a exponentially decaying circulation with a given decay rate. Thus, the objective is to formulate the general minimum effort motion control problem outlined above and present and discuss necessary conditions of optimality, in the form of a maximum Principle of Pontryagin, [15], associated its solution. Given the structure of the chosen propulsion ad steering scheme, we cast the optimal control problem of interest in the control multiprocesses framework, first considered in [10], [11].

In section II, we present the dynamic equations for point vortices with constant circulation moving in a plane and, then, we particularize to the specific case of interest in this article, in which we have the vector field defined by two point vortices with a given exponentially decaying circulation. In section III, we present the general optimal control problem central and explain the rationale behind the chosen controlled dynamics, as well as some of the key assumptions that enable an elegant formulation in the multiprocess context which is amenable to the application of the Maximum Principle. In section IV, necessary conditions of optimality in the form of a Maximum Principle of Pontryagin are applied to the problems here stated. Some conclusions are outlined in the last section.

II. Problem formulation

Vortices are two-dimensional points, each one with its own circulation, that induce a velocity field in the plane (formally, such velocity field is a singular solution of the two-dimensional incompressible Euler equations on the whole plane). Consider n point vortices located in (x_i, y_i) with circulation K_i $(i = 1, \ldots, n)$. The evolution of the vector field in $\mathbf{R}^2$ is given the ODE system ([7]–[9]):

$$(\dot{x}_i, \dot{y}_i) = \frac{1}{2\pi} \sum_{j=1, j\neq i}^{n} K_j \, \frac{(-(y_i - y_j), (x_i - x_j))}{\|(x_j, y_j) - (x_i, y_i)\|^2}, \qquad (1)$$

where $(x_i(0), y_i(0)) = (x_0, y_0)$, and $\|\cdot\|$ is the Euclidean distance.

As stated above, we are interested in the case $n = 2$ and the circulation decays in time, that is, we consider the circulation given by $e^{-\delta(t-t_0)} K_i$, where the K_i's are constants $(i = 1, 2)$. The option $n = 2$ is justified by the fact that it is the minimal number of vortices ensuring motion controllability, being the minimality important due to the fact that creating a vortex requires a significant amount of energy from the AUV. The fact that the dissipation of energy in the water column is relatively fast makes it reasonable to consider that its circulation decays exponentially in time with a pre-specified rate, and that, after some time, the effect of an "old" vortex can be neglected. These considerations provide the rationale to the vector field dynamics component of the control system model in the problem investigated in this article. The AUV has to ensure that it can control its motion and, so, it requires two relevant vortices. On the other hand, since generating a vortex is expensive from the power consumption perspective, the quest for optimality leads to the natural decision to create a new vortex, when the effect of at least one of currently active ones becomes negligible. This rationale justifies the consideration of an optimal control problem with two "dominant vortices".

We will introduce a notation that will facilitate the representation of the dynamics of the overall system. Let $x_V = col(x_1, x_2, x_3, x_4)$, $x_{V_1} = col(x_1, x_3)$, and $x_{V_2} = col(x_2, x_4)$, the position coordinates of the, respectively, joint vortices, first vortex, V_1, and second vortex, V_2. We have $x_{V_1} = M_1 x_V$ and $x_{V_2} = M_2 x_V$, where $M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, and $M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Now, by using (1) with $n = 2$, we obtain the following dynamics for the vortices

$$\dot{x}_V = d(x_{V,0}) A(t, t_0, K_1, K_2) x_V, \qquad (2)$$

where $d(x_{V,0}) = \left((x_1(t_0) - x_2(t_0))^2 + (x_3(t_0) - x_4(t_0))^2\right)^{-1}$, and, by denoting by δ the prescribed rate of decay,

$$A(t, t_0, K_1, K_2) := \frac{e^{-\delta(t-t_0)}}{2\pi} \begin{bmatrix} 0 & 0 & -K_2 & K_2 \\ 0 & 0 & K_1 & -K_1 \\ K_2 & -K_2 & 0 & 0 \\ -K_1 & K_1 & 0 & 0 \end{bmatrix}.$$

An important observation with significant implications on the developments of the next two sections is that $\dot{d}(x_{V,t}) = 0$. This conclusion can be reached with a few calculations.

III. Multiprocess optimal control problem

A controlled multiprocess system,

$$\{(x^i, u^i) \in AC \times L^\infty : i = 1, \ldots, N\},$$

consists in a finite number of dynamic control systems which are active in, possibly different, free endpoint time intervals, subject to their own state variable and control constraints, while sharing joint time and state endpoint constraints, that is,

$$\begin{cases} \dot{x} = u_l \cos(\theta) + u_x^i v_x^i \\ \dot{y} = u_l \sin(\theta) + u_y^i v_y^i \\ \dot{\theta} = w + \frac{1+\cos(2\theta)}{2} \\ h^i(t, x, y) \in C^i & [t_0^i, t_1^i]\ \mathcal{L}-\text{a.e.} \\ \bar{u}_l \in [u_m, u_M] \\ \bar{w} \in [-w_M, w_M] \\ \text{dynamics of } (v_x^i, v_y^i) \end{cases} \qquad (3)$$

$$\bar{h}(\{((x, y)(t_0^i), (x, y)(t_1^i)) : i = 1, \ldots, N\}) \in \bar{C}, \qquad (4)$$

where $\bar{u}_l$ and $\bar{w}$ are the longitudinal and angular velocities, and (v_x^i, v_y^i) represents the planar flow field at the vehicle location.

In order to cast the problem in the multiprocesses framework, let us represent the state at the time segment i as follows: the control variables $u_j^i = (u_{x,j}^i, u_{y,j}^i) \in [-\zeta, \zeta]^2$, $j = 1, 2$, define the extent of exposure of the particle to each vortex at each time, and the overall state variable x^i, given by $col(x_0^i, x_V^i, x_p^i, x_K^i, x_\theta^i) \in \mathbf{R}^{10}$, where x_0^i is the distance between both vortices positions, $x_V^i = (x_{V_1}^i, x_{V_2}^i)$, with $x_{V_1}^i = col(x_1^i, x_3^i)$ and $x_{V_2}^i = col(x_2^i, x_4^i)$ are the position of both vortices, $x_p^i = col(x_5^i, x_6^i)$ is the position of the particle, $x_K^i = col(x_7^i, x_8^i)$ the circulation of both vortices and $x_\theta^i = x_9^i$ is the angular component of the particle related with the vortices, satisfies the dynamics in the interval $[\sigma_0^i, \sigma_1^i]$ (being σ_0^i, and σ_1^i decision parameters):

$$\begin{cases} \dot{x}_0^i &= 0 \\ \dot{x}_V^i &= x_0^i A(t, \sigma_0^i, x_K^i) x_V^i \\ \dot{x}_p^i &= u_l \begin{bmatrix} \cos(x_\theta^i) \\ \sin(x_\theta^i) \end{bmatrix} + x_0^i U^i A(t, \sigma_0^i, x_K^i) x_V^i \\ \dot{x}_K^i &= 0 \\ \dot{x}_\theta^i &= w + \frac{1+\cos(2x_\theta^i)}{2} \end{cases} \quad (5)$$

$$A(t, \sigma_0^i, x_K^i) = e^{-\delta(t-\sigma_0^i)} \begin{bmatrix} 0 & 0 & -x_8^i & x_8^i \\ 0 & 0 & x_7^i & -x_7^i \\ x_8^i & -x_8^i & 0 & 0 \\ -x_7^i & x_7^i & 0 & 0 \end{bmatrix}$$

and $U^i = \begin{bmatrix} u_1^i & u_3^i & 0 & 0 \\ 0 & 0 & u_2^i & u_4^i \end{bmatrix}$, with $u^i = col(u_1^i, u_2^i, u_3^i, u_4^i)$, where $u_{V_1}^i = col(u_1^i, u_2^i)$ and $u_{V_2}^i = col(u_3^i, u_4^i)$ the control components associated to the first and the second vortices, respectively, as well as the boundary conditions $(\sigma_0^i, x^i(\sigma_0^i), \sigma_1^i, x^i(\sigma_1^i)) \in \Lambda^i$.

The cost functional to be minimized consists in the overall power consumption required in the overall effort to steer the AUV. The goal is to choose the set of N triples $(x^i(t_0^i), x^i(t_1^i), u^i)$ that optimize a given global performance function $J(x^i(t_0^i), x^i(t_1^i), u^i)$, while the associated controlled multiprocess system satisfies the constraints (3) and (4).

Thus, the application the maximum principle, [10], [11], asserts that, for a given optimal control multiprocess, there exists a multiprocess adjoint variable $\{p^i\}$, $p^i : [\sigma_0^i, \sigma_1^i] \to \mathbf{R}^{10}$, that satisfies the adjoint system, i.e., $-\dot{p}^i = \nabla_{x^i} H^i(t, x^i, p^i, u^i)$, where H^i is the Pontryagin (also known as the un-maximized Hamiltonian) given by $H^i(t, x^i, p^i, u^i) = \langle p^i, f^i(t, x^i, u^i) \rangle$, on $[\sigma_0^i, \sigma_1^i]$, and the transversality condition $(-h_0^i, p^i(\sigma_0^i), h_1^i, -p^i(\sigma_1^i)) \in N_{\Lambda^i}^L$. Moreover, the satisfaction of all endpoint constraints the free end-times and values of state variable at each segment, the optimal control u^i maximizes a.e. in $[\sigma_0^i, \sigma_1^i]$ the map $v \to H^i(t, x^i, p^i, v)$. The specific form of these relations and the fact that they are complete make it amenable to the computation of the optimal control process.

A. Formulation of the optimal control problem

The formulation of the optimal control problem of the motion of the AUV in a given time interval $[0, T]$ involves the specification of (i) the control functional to be optimized, (ii) state and control to be satisfied pointwise in time, (iii) terminal endpoint state and time constraint to be met, and obviously, (iv) controlled dynamic equations that will dictate how the trajectory of the system evolves while satisfying all the constraints.

We started already with the later by explaining how vortices evolve over time. In order to control its motion, the AUV has to generate an additional vortex at a certain point in time and then define the extent of its body exposure to the last two vortices during a given time interval initiated at the vortex creation time. As mentioned earlier, the scheme assumes that the vortex with lower circulation can be neglected, i.e., residual impact in the motion, or that the AUV has ways to have null exposure to that vortex. Another important issue issue concerns the definition of the value of the control specifying the extent to the exposure of the AUV to a given vortex. Instead of considering the field at the position of the AUV, we consider the field at the position of the vortex. This greatly simplifies not only the formulation of the problem but also the complexity of the necessary conditions of optimality.

It is clear that the considered framework is particularly amenable to a multiprocess formulation, since the overall time interval $[0, T]$ can be considered as the union of a set of distinct time intervals in which the vortex dynamics change as a result of the discrete component of the control action. Let us consider $N \in \mathbb{N}$ adjacent subintervals, i.e., there exists a partition $\{t_i : i = 0, \ldots, N, \text{ s.t. } t_0 = 0, t_{i-1} < t_i, i = 1, \ldots, N, t_N = T\}$, satisfying $[0, T] = \bigcup_{i=1}^{N} I_i$, where $I_i = [t_{i-1}, t_i]$. Since for each interval we have to mention its final and initial points, independently of the neighboring intervals, we will consider $I_i = [\sigma_0^i, \sigma_1^i]$, being obvious that $\sigma_0^1 = 0$, $\sigma_1^N = T$, and $\sigma_1^i = \sigma_0^{i+1}$, for $i = 1, \ldots, N-1$. Let us now specify the state and control variables and its components for each subinterval by attaching an upper index i, $x^i = col(x_0^i, x_V^i, x_p^i, x_K^i, x_\theta^i)$ at $[\sigma_0^i, \sigma_1^i]$, being:

- $x_V^i = col(x_1^i, x_2^i, x_3^i, x_4^i)$ with $x_{V_1}^i = M_1 x_V^i$ and $x_{V_2}^i = M_2 x_V^i$ as explained in the previous section,
- $x_p^i = col(x_5^i, x_6^i)$ is the position of the AUV,
- $x_K^i = col(x_7^i, x_8^i)$ where $x_7^i = \dfrac{K_1^i}{2\pi}$ and $x_8^i = \dfrac{K_2^i}{2\pi}$, and
- $x_0^i = d(x_{V,0}^i)$,
- $x_\theta^i = x_9^i$.

For $t \in [\sigma_0^i, \sigma_1^i]$, let

$$A(t, \sigma_0^i, x_7^i, x_8^i) = e^{-\delta(t-\sigma_0^i)} \begin{bmatrix} 0 & 0 & -x_8^i & x_8^i \\ 0 & 0 & x_7^i & -x_7^i \\ x_8^i & -x_8^i & 0 & 0 \\ -x_7^i & x_7^i & 0 & 0 \end{bmatrix}.$$

It is obvious that we have $\dot{x}_K^i = 0$ and $\dot{x}_0^i = 0$. Given the trivial dynamics, one may ask the reason why these have been chosen as state variables. The reason is simple: the choice of their initial value affect the evolution the system, or, in the case of x_0^i, if they fully depend explicitly on other variables, their choice facilitates the application of the necessary conditions of optimality.

In general, the decision variables of this problem comprise, not only, the feasible initial values of the state variable at the initial time of each interval, i.e., $(x_V^i(\sigma_0^i), x_p^i(\sigma_0^i), x_K^i(\sigma_0^i), x_\theta^i(\sigma_0^i))$ - notice that, here,

$x_0(\sigma_0^i) = d(x_{V,0}^i(\sigma_0^i))$ - but also the control functions u^i which are in $L^\infty([\sigma_0^i,\sigma_1^i];\mathbf{R}^4)$ and take values in $\Omega^i = [-\zeta,\zeta]^4$, for some $\zeta > 0$.

Thus, the dynamics of the system on the interval $[\sigma_0^i,\sigma_1^i]$ are governed by (5). Now, we turn to the time and state endpoint constraints. We start with the time variable. First, note that this formulation applies to either $T > 0$ being pre-defined or also a decision variable of choice. The multiprocess time constraints are given by $\bar\sigma \in \Lambda_\sigma$, where $\bar\sigma = col\left((\sigma_0^i,\sigma_1^i) : i = 1,\ldots,N\right)$, and $\Lambda_\sigma := \{\bar\sigma \in \mathbf{R}^{2N} : \sigma_0^1 = 0, \sigma_1^N = T, \sigma_0^i = \sigma_1^{i-1}, i = 2,\ldots N\}$. Later, it will be convenient to use also the notation $\bar\sigma_j = col(\bar\sigma_j^i : i = 1,\ldots,N)$, for $j = 0, 1$. Notice that since $x_0(\sigma_j^i) = d(x_V(\sigma_j^i))$, for $i = 1,\ldots,N$, for $j = 0, 1$, we have

$$\bar x_0 = col(x_0(\sigma_0^i), x_0(\sigma_1^i)) : i = 1,\ldots,N) \in d(\Lambda_V),$$

being the later defined next. From the definitions, we have:

$$\bar x_V = \left((x_V^i(\sigma_0^i), x_V^i(\sigma_1^i)) : i = 1,\ldots,N\right) \in \Lambda_V,$$

where

$$\begin{aligned}\Lambda_V := \{\bar x_V \in \mathbf{R}^{8N} : \; & M_1 x_V^1(0) = x_{V_1,0}^1, \, M_2 x_V^1(0) \in \mathbf{R}^2, \\ & x_V^1(\sigma_1^1) \in \mathbf{R}^4, \, x_V^i(\sigma_1^i) \in \mathbf{R}^4, \\ & M_1 x_V^i(\sigma_0^i) = M_2 x_V^{i-1}(\sigma_1^{i-1}), \\ & M_2 x_V^i(\sigma_0^i) \in \mathbf{R}^2, \, i = 2,\ldots,N\}.\end{aligned}$$

Since the particle evolves continuously in time, we have:

$$\bar x_p = \left((x_p^i(\sigma_0^i), x_p^i(\sigma_1^i)) : i = 1,\ldots,N\right) \in \Lambda_p,$$

where

$$\begin{aligned}\Lambda_p := \{\bar x_p \in \mathbf{R}^{4N} : & x_5^1(0) = x_{5,0}, \, x_6^1(0) = x_{6,0}, \\ & x_5^1(T) = x_{5,T}, x_6^1(T) = x_{6,T}, \, x_5^{i-1}(\sigma_1^{i-1}) = x_5^i(\sigma_0^i), \\ & x_6^{i-1}(\sigma_1^{i-1}) = x_6^i(\sigma_0^i), \, i = 2,\ldots,N\}.\end{aligned}$$

The AUV have an angular component, so we have

$$\bar x_\theta = \left((x_\theta^i(\sigma_0^i), x_\theta^i(\sigma_1^i)) : i = 1,\ldots,N\right) \in \Lambda_\theta,$$

where

$$\begin{aligned}\Lambda_\theta := \{\bar x_\theta \in \mathbf{R}^{2N} : & \, x_\theta^i(\sigma_1^i) \in \mathbf{R}, \, i = 1,\ldots,N, \, x_\theta^1(0) = \theta_0, \\ & x_\theta^{i-1}(\sigma_1^{i-1}) = x_\theta^i(\sigma_0^i), \, i = 2,\ldots,N\}.\end{aligned}$$

Finally, it remains to characterize the constraints on the circulation. Remark that, for given numbers $\mathbf{K}_2 > \mathbf{K}_1 > 0$, we have $x_7^1(\sigma_0^1) = N_1 x_{K,0}^1 \in [-\mathbf{K}_1, \mathbf{K}_1]$ and $x_8^1(\sigma_0^1) = N_2 x_{K,0}^1 \in \tilde V^1(N_1 x_{K,0}^1) := (\bar V^1 \cup (-\bar V^1))(N_1 x_{K,0}^1)$, where $\bar V^1(N_1 x_{K,0}^1)$ is given by

- $[N_1 x_{K,0}^1, \mathbf{K}_2]$, if $N_1 x_{K,0}^1 > 0$, and
- $[-\mathbf{K}_2, N_1 x_{K,0}^1]$, otherwise,

being $x_{K,0}^1 = x_K^1(\sigma_0^1)$, and N_1 and N_2, respectively, the matrices $[1 \;\; 0]$ and $[0 \;\; 1]$. Moreover, for $i = 2,\ldots,N$, and by denoting $x_{K,0}^i = x_K^i(\sigma_0^i)$ and $g_{i-1,i} = e^{-\delta(\sigma_0^i - \sigma_0^{i-1})}$, we have

- $N_1 x_{K,0}^i := g_{i-1,i} N_2 x_{K,0}^{i-1}$,
- $N_2 x_{K,0}^i \in (\bar V^i \cup (-\bar V^i))(g_{i-1,i} N_2 x_{K,0}^{i-1})$, where, for some $1 < \bar\alpha < |N_2 x_{K,0}^{i-1}|^{-1} g_{i-1,i}^{-1} \bar{\mathbf{K}}$, with $\bar{\mathbf{K}}$ satisfying
$$\bar{\mathbf{K}} \ge \bar\alpha \max_{i=2,\ldots,N} \{|g_{i-1,i} N_2 x_{K,0}^{i-1}|\},$$
$\bar V^i = [\bar\alpha g_{i-1,i} N_2 x_{K,0}^{i-1}, \bar{\mathbf{K}}]$ or $[-\bar{\mathbf{K}}, \bar\alpha g_{i-1,i} N_2 x_{K,0}^{i-1}]$ if $N_2 x_{K,0}^{i-1}$ is, respectively, positive or negative. Notice that an example of such a $\bar{\mathbf{K}}$ is $\bar{\mathbf{K}} = \mathbf{K}_2$ and, again, let $\tilde V^i := (\bar V^i \cup (-\bar V^i))(g_{i-1,i} N_2 x_{K,0}^{i-1})$.

The choice of the numbers $\bar\alpha$ and $\bar{\mathbf{K}}$ relies on the trade-off between the realism of the two vortices model and the optimality of such a solution. Now, we may express the constraints for the set of boundary values of the vortices circulation by

$$\bar x_K \in \Lambda_K,$$

where $\bar x_K = col\left((x_{K,0}^i, x_{K,1}^i) : i = 1,\ldots,N\right)$ (again here, $x_{K,1}^i = x_K^i(\sigma_1^i)$), and

$$\begin{aligned}\Lambda_K := \{\bar x_K \in \mathbf{R}^{4N} : \; & x_{K,1}^i \in \mathbf{R}^2, \text{ for } i = 1,\ldots,N, \\ & x_{K,0}^1 \in [-\mathbf{K}_1, \mathbf{K}_1] \times \tilde V^1(N_1 x_{K,0}^1), \\ & x_{K,0}^i \in \hat V^i, \text{ for } i = 2,\ldots,N\},\end{aligned}$$

with $\hat V^i = \{g_{i-1,i} N_2 x_{K,0}^{i-1}\} \times \tilde V^i(g_{i-1,i} N_2 x_{K,0}^{i-1})$. The constraints to be satisfied by the circulation appear to be somewhat intricate. However, its “raison d’łtre” ensures that, without, any loss of generality, the “second” or “younger” vortex is always the one with higher circulation. This condition facilitates the formulation of the problem since simpler functions suffice to express the constraints.

Now, we pursue with the specification of the cost functional. We choose it as a weighted sum of the various items contributing to power consumption, notably, the continuum time steering control function, and the energy associated with the creation of the vortices. This last term involves two components: level of circulation and the distance at which the new vortex is generated. Thus, we should have

$$J(\bar u, \bar x(\bar\sigma_0), \bar\sigma) := \frac{1}{2} \sum_{i=1}^{N} \Big[\alpha_1 \int_{\sigma_0^i}^{\sigma_1^i} \|u^i(t)\|^2 dt + \alpha_2 \|\bar N x_K^i(\sigma_0^i)\|^2 + \alpha_3 \|x_p^i(\sigma_0^i) - M_2 x_V^i(\sigma_0^i)\|^2\Big],$$

where α_1, α_2, and α_3 are the non-negative weighting coefficients, and $\bar N = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

Then, the multiprocesses control process can be defined as follows

(P) Minimize $\quad J(\bar u, \bar x(\sigma_0), \bar\sigma)$

subject to $\begin{cases} \text{differential equations } (5) \\ u^i \in [-\zeta,\zeta]^4 \end{cases}$ $[\sigma_0^i, \sigma_1^i]$-a.e. for $i = 1,\ldots,N$, and

$\bar\sigma \in \Lambda_\sigma, \; \bar x_0 \in d(\Lambda_V), \; \bar x_V \in \Lambda_V, \; \bar x_p \in \Lambda_p, \; \bar x_K \in \Lambda_K, \; \bar x_\theta \in \Lambda_\theta.$

For future convenience, we express the cost functional $J(\bar u, \bar x(\bar\sigma_0), \bar\sigma)$ as the sum

$$f_0\left(\{(x(\sigma_0), x(\sigma_1))^i\}_{i=1}^N\right) + \frac{\alpha_1}{2} \sum_{i=1}^{N} \int_{\sigma_0^i}^{\sigma_1^i} \|u^i(t)\|^2 dt,$$

where $f_0\left(\{(x(\sigma_0),x(\sigma_1))^i\}_{i=1}^N\right)$ is given by

$$\frac{1}{2}\sum_{i=1}^{N}\left[\alpha_2\|\bar{N}x_K^i(\sigma_0^i)\|^2+\alpha_3\|x_p^i(\sigma_0^i)-M_2x_V^i(\sigma_0^i)\|^2\right].$$

IV. Maximum Principle for the multiprocesses optimal control problem

Now, we apply the Maximum Principle of Pontryagin for the multiprocess optimal control problem (P) described in the previous section. Of course, these conditions are expressed having as a reference the optimal control process which is usually singled out by the upperscript "*". However, in order to mitigate the already heavy burden of the notation, we omit it here.

By importing the notation adopted to the primal variable to the adjoint variable p, we have $p^i = lin(p_0^i, p_V^i, p_p^i, p_K^i, p_\theta^i)$ and may express the Pontryagin function as follows

$$H(t,\bar{x},\bar{p},\bar{u},\bar{\sigma})=\sum_{i=1}^{N}H^i(x^i,p^i,u^i,\sigma_0^i)\chi_{[\sigma_0^i,\sigma_1^i]}(t),$$

where $\chi_{[\sigma_0^i,\sigma_1^i]}$ is the indicator function $[\sigma_0^i,\sigma_1^i]$, and

$$\begin{aligned}H^i(\cdot)= \;& x_0^i(p_V^i+p_p^iU^i)A^ix_V^i+u_lp_p^i\begin{bmatrix}\cos(x_\theta^i)\\ \sin(x_\theta^i)\end{bmatrix}\\ &+p_\theta^i(w+\tfrac{1+\cos(2x_\theta^i)}{2})-\tfrac{1}{2}\alpha_1\|u^i\|^2,\end{aligned}$$

with $A^i = A(t,\sigma_0^i,x_{K,0}^i)$. Now, we readily conclude that the adjoint system is, for $i=1,\ldots,N$, given by

$$\left\{\begin{array}{lcl}-\dot{p}_0^i &=& (p_V^i+p_p^iU^i)A^ix_V^i\\ -\dot{p}_V^i &=& x_0^i(p_V^i+p_p^iU^i)A^i\\ -\dot{p}_p^i &=& 0\\ -\dot{p}_K^i &=& x_0^i(p_V^i+p_p^iU^i)D_{x_K^i}\left(A^ix_V^i\right)\\ -\dot{p}_\theta^i &=& u_lp_p^i\begin{bmatrix}-\sin(x_\theta^i)\\ \cos(x_\theta^i)\end{bmatrix}-p_\theta^i\sin(2x_\theta^i)\end{array}\right. \quad (6)$$

where D_yF denotes the Jacobian of the vector valued map F w.r.t. the variable y. The transversality conditions are given by

$$\begin{aligned}&\big\{(-h_0^i,h_1^i),(p_{0,0}^i,-p_{0,1}^i),(p_{V,0}^i,-p_{V,1}^i),(p_{p,0}^i,-p_{p,1}^i),\\ &\qquad (p_{K,0}^i,-p_{K,1}^i),(p_{\theta,0}^i,-p_{\theta,1}^i)\big\}_{i=1}^N\\ &\qquad\in N_\Lambda(\bullet)+\lambda\nabla_{\Pi_{i=1}^N\bullet}\bar{f}_0(\bullet).\end{aligned}$$

Here,

- $\lambda\geq 0$,
- $\bullet=\big\{(\sigma_0^i,\sigma_1^i),(x_{0,0}^i,x_{0,1}^i),(x_{V,0}^i,x_{V,1}^i),(x_{p,0}^i,x_{p,1}^i),(x_{K,0}^i,x_{K,1}^i),(x_{\theta,0}^i,x_{\theta,1}^i)\big\}_{i=1}^N$,
- $z_{\bullet,j}^i=z_\bullet^i(\sigma_j^i)$, for $j=1,2$,
- $\bar{f}_0(\bullet)=f_0(\{x_{V,0}^i,x_{p,0}^i,x_{K,0}^i\}_{i=1}^N)$,
- $\Lambda=\Lambda_\sigma\times d(\Lambda_V)\times\Lambda_V\times\Lambda_p\times\Lambda_K\times\Lambda_\theta$,
- $h_0^i=\sup\limits_{u^i\in[-\zeta,\zeta]^4}\left\{H^i(\bullet,u^i)|_{t=\sigma_0^i}\right\}$, and
- $h_1^i=\sup\limits_{u^i\in[-\zeta,\zeta]^4}\left\{H^i(\bullet,u^i)|_{t=\sigma_1^i}\right\}$.

Let us decode two of the above compact adjoint differential equations. First, we note that, by considering $p_V^i = [p_1^i,p_2^i,p_3^i,p_4^i]$, $p_p^i=[p_5^i,p_6^i]$, $p_K^i=[p_7^i,p_8^i]$, and by replacing x_V^i, x_p^i, x_K^i, and u^i by its components, and by considering

$$\Sigma^i=\langle(-x_3^i+x_4^i,x_1^i-x_2^i),(p_1^ix_8^i-p_2^ix_7^i,p_3^ix_8^i-p_4^ix_7^i)\rangle,$$

to simplify the notation, we conclude that

$$\begin{aligned}H^i(\bullet,u^i) \;=\;& e^{-\delta(t-\sigma_0^i)}x_0^i\left[\Sigma^i+p_5^i(-x_3^i+x_4^i)(u_1^ix_8^i-u_3^ix_7^i)\right.\\ &\left.+p_6^i(x_1^i-x_2^i)(u_2^ix_8^i-u_4^ix_7^i)\right]+p_9^i\left(w+\frac{1+\cos(2x_9^i)}{2}\right)\\ &+u_l(p_5^i\cos(x_9^i)+p_6^i\sin(x_9^i))-\frac{1}{2}\alpha_1\|u^i\|^2.\end{aligned}$$

Straightforward computations lead us to:

$$\left\{\begin{array}{lcl}-\dot{p}_0^i &=& e^{-\delta(t-\sigma_0^i)}\left[\Sigma^i\right.\\ && +p_5^i(-x_3^i+x_4^i)(u_1^ix_8^i-u_3^ix_7^i)\\ && \left.+p_6^i(x_1^i-x_2^i)(u_2^ix_8^i-u_4^ix_7^i)\right]\\ -\dot{p}_V^i &=& -[\dot{p}_1,\,\dot{p}_2,\,\dot{p}_3,\,\dot{p}_4]\\ & & =e^{-\delta(t-\sigma_0^i)}x_0^i[x_8^ip_3^i-x_7^ip_4^i+p_6^i(x_8^iu_2^i-x_7^iu_4^i),\\ && -(x_8^ip_3^i-x_7^ip_4^i+p_6^i(x_8^iu_2^i-x_7^iu_4^i)),\\ && -x_8^ip_1^i+x_7^ip_2^i-p_5^i(x_8^iu_1^i-x_7^iu_3^i),\\ && -(-x_8^ip_1^i+x_7^ip_2^i-p_5^i(x_8^iu_1^i-x_7^iu_3^i))]\\ -\dot{p}_p^i &=& 0\\ -\dot{p}_K^i &=& -[\dot{p}_7,\dot{p}_8]\\ & & =e^{-\delta(t-\sigma_0^i)}x_0^i\cdot\\ && [\langle(x_3^i-x_4^i,x_1^i-x_2^i),(p_2^i+p_5^iu_3^i,-p_4^i-p_6^iu_4^i)\rangle,\\ && \langle(x_3^i-x_4^i,x_1^i-x_2^i),(-p_1^i-p_5^iu_1^i,p_3^i+p_6^iu_2^i)\rangle]\\ -\dot{p}_\theta^i &=& -p_9^i\sin(2x_9^i)\\ && +u_l(-p_5^i\sin(x_9^i)+p_6^i\cos(x_9^i))\end{array}\right. \quad (7)$$

It is clear that $\dot{p}_1^i = -\dot{p}_2^i$ and $\dot{p}_3^i = -\dot{p}_4^i$. In order to express the transversality conditions in detail, we still need to compute both the gradient of the cost functional w.r.t. to the sate variable endpoints and the normal cone to the endpoint constraint sets. The cost functional term, that depends on the state variable at its end points, only involves the state variable at the initial time. Thus, we may write:

$$(\nabla_{x_K}f_0)^i|_{\sigma_0^i} = \alpha_2\bar{N}x_K^i(\sigma_0^i)\bar{N}=[0,\alpha_2x_8^i(\sigma_0^i)] \quad (8)$$

$$\begin{aligned}(\nabla_{x_p}f_0)^i|_{\sigma_0^i} &= \alpha_3(x_p^i(\sigma_0^i)-M_2x_V^i(\sigma_0^i))^T\\ &=\alpha_3[x_5^i(\sigma_0^i)-x_2^i(\sigma_0^i),x_6^i(\sigma_0^i)-x_4^i(\sigma_0^i)]\end{aligned} \quad (9)$$

$$\begin{aligned}(\nabla_{x_V}f_0)^i|_{\sigma_0^i} &= -\alpha_3(x_p^i(\sigma_0^i)-M_2x_V^i(\sigma_0^i))^TM_2\\ &=-\alpha_3[0,x_5^i(\sigma_0^i)-x_2^i(\sigma_0^i),0,x_6^i(\sigma_0^i)-x_4^i(\sigma_0^i)]\end{aligned} \quad (10)$$

Obviously, $(\nabla_xf_0)^i|_{\sigma_1^i}=0,\ \forall i$. The limiting normal cones to the sets Λ_{x_0}, Λ_σ, Λ_V, Λ_p, Λ_θ and Λ_K are – $N_C^L(c)$ denotes the limiting normal cone of the set C at the point $c\in C$ – $(i=2,\cdots,N\,,j=1,\cdots,N)$:

$$N_{\Lambda_\sigma}^L(\bullet)=\{\hat{p}_\sigma\in\mathbf{R}^{2N}:\hat{p}_{\sigma,0}^1\in\mathbf{R},\,\hat{p}_{\sigma,1}^{i-1}=-\hat{p}_{\sigma,0}^i,\hat{p}_{\sigma,1}^N\in\mathbf{R}\},$$

$$N_{\Lambda_V}^L(\bullet)=\{\hat{p}_V\in\mathbf{R}^{8N}:M_1\hat{p}_{V,0}^1\in\mathbf{R}^2,\,\hat{p}_{V,1}^i\in\mathbf{R}^4,\ M_2\hat{p}_{V,0}^i\in\mathbf{R}^2\ M_1\hat{p}_{V,0}^i=-M_2\hat{p}_{V,1}^{i-1}\},$$

$$N_{\Lambda_p}^L(\bullet)=\{\hat{p}_p\in\mathbf{R}^{4N}:\hat{p}_{p,0}^1\in\mathbf{R}^2,\,\hat{p}_{p,0}^i=-\hat{p}_{p,1}^{i-1},\hat{p}_{p,1}^N\in\mathbf{R}^2\},$$

$$N_{\Lambda_\theta}^L(\bullet)=\{\hat{p}_\theta\in\mathbf{R}^{2N}:\hat{p}_{\theta,1}^1\in\mathbf{R},\,\hat{p}_{\theta,1}^{i-1}=-\hat{p}_{\theta,0}^i,\hat{p}_{\theta,0}^1\in\mathbf{R}\}.$$

Since $\bar{x}_0 = d(\bar{x}_V)$, the normal cone to Λ_{x_0} can be easily related to $N^L_{\Lambda_V}$ via the gradient of the function d. Since $\nabla d(\bar{x}_V) = 2x_0^2[x_1 - x_2,\, x_2 - x_1,\, x_3 - x_4,\, x_4 - x_3,\, 0,\, 0,\, 0,\, 0]$, it is not difficult to conclude that

$$N^L_{x_0} = \{\hat{p}_0 : 2\hat{p}_0 x_0^2 [x_1 - x_2,\, x_2 - x_1,\, x_3 - x_4,\, x_4 - x_3,\, 0,\, 0,\, 0,\, 0] \in N^L_{\Lambda_V}\}.$$

Due to the specific interdependence of the various time subintervals in the definition of Λ_K, the computation of $N^L_{\Lambda_K}(\bullet)$ requires some more attention. Let us apply the definition of normal cone applied to the reference point $\hat{x}_{K,0} = col(\hat{x}^i_{K,0} : i = 1, \ldots, N)$ in Λ_K. Without any loss of generality, let us consider $N \geq 3$. Then, $\hat{p}_{K,0} = col(\hat{p}^i_{K,0} : i = 1, \ldots, N) \in N^L_{\Lambda_K}(\hat{x}_{K,0})$, if $\forall\, x_{K,0} \in \Lambda_K$:

$$\begin{aligned} 0 &\geq \langle \hat{p}_{K,0}, x_{K,0} - \hat{x}_{K,0} \rangle \\ &= \langle N_1 \hat{p}^1_{K,0}, N_1(x^1_{K,0} - \hat{x}^1_{K,0})\rangle + \langle N_2 \hat{p}^1_{K,0}, N_2(x^1_{K,0} - \hat{x}^1_{K,0})\rangle \\ &\quad + \sum_{i=2}^{N} \left[\langle N_1 \hat{p}^i_{K,0}, N_1(x^i_{K,0} - \hat{x}^i_{K,0})\rangle \right. \\ &\quad\quad \left. + \langle N_2 \hat{p}^N_{K,0}, N_2(x^N_{K,0} - \hat{x}^N_{K,0})\rangle \right]. \end{aligned}$$

By recalling the definition of the constraint set Λ_K and by regrouping the terms, we readily obtain, $\forall\, x_K \in \Lambda_K$, the inequality

$$\begin{aligned} 0 \geq\ & \langle N_1 \hat{p}^1_{K,0}, N_1(x^1_{K,0} - \hat{x}^1_{K,0})\rangle \\ & + \sum_{i=1}^{N-1} \langle N_2 \hat{p}^i_{K,0} + g_{i,i+1} N_1 \hat{p}^{i+1}_{K,0}, N_2(x^i_{K,0} - \hat{x}^i_{K,0})\rangle \\ & + \langle N_2 \hat{p}^N_{K,0}, N_2(x^N_{K,0} - \hat{x}^N_{K,0})\rangle. \end{aligned}$$

By considering all the feasible variations of both components of $x_{K,0}$, we obtain $N^L_K(\hat{x}_{K,0})$ as the set $(i = 2, \cdots, N)$:

$$\Big\{ (\hat{p}^i_{K,0}, 0) \in \mathbf{R}^4 \colon \hat{p}^1_{K,0} + [0, g_{1,2} N_1 \hat{p}^2_{K,0}] \in N^L_{\tilde{V}^1(N_1 \hat{x}^1_{K,0})}(\hat{x}^1_{K,0}), \\ N_2 \hat{p}^i_{K,0} + g_{i,i+1} N_1 \hat{p}^{i+1}_{K,0} \in N^L_{\tilde{V}^i(g_{i-1,i} N_2 \hat{x}^{i-1}_{K,0})}(\hat{x}^i_{K,0}), \\ N_2 \hat{p}^N_{K,0} \in N^L_{\tilde{V}^N(g_{N-1,N} N_2 \hat{x}^{N-1}_{K,0})}(\hat{x}^N_{K,0}) \Big\}.$$

Now, let us compute the optimal candidate control strategy which has to maximizing the Pontryagin function along the optimal trajectory and associated adjoint variable. Due to the quadratic structure of the Hamiltonian and the fact that the control constraints are decoupled, we readily conclude that, $\mathcal{L}$-a.e. in $[\sigma^i_0, \sigma^i_1]$,

$$\hat{u}^i_1(t) = Sat_\zeta \left(\frac{x^i_0 x^i_8(t) p^i_5}{\alpha_1} e^{-\delta(t - \sigma^i_0)} (-x^i_3(t) + x^i_4(t)) \right),$$

$$\hat{u}^i_2(t) = Sat_\zeta \left(\frac{x^i_0 x^i_8(t) p^i_6}{\alpha_1} e^{-\delta(t - \sigma^i_0)} (x^i_1(t) - x^i_2(t)) \right),$$

$$\hat{u}^i_3(t) = Sat_\zeta \left(-\frac{x^i_0 x^i_7(t) p^i_5}{\alpha_1} e^{-\delta(t - \sigma^i_0)} (-x^i_3(t) + x^i_4(t)) \right),$$

$$\hat{u}^i_4(t) = Sat_\zeta \left(-\frac{x^i_0 x^i_7(t) p^i_6}{\alpha_1} e^{-\delta(t - \sigma^i_0)} (x^i_1(t) - x^i_2(t)) \right).$$

Here, $Sat_a(z)$ is the saturation function which is defined by taking the values $-a$, z, and a, if, respectively, $z < -a$, $-a \leq z \leq a$, and $z > a$, and the functions x^i_8 and x^i_7 are evaluated along the optimal control process. The optimal control multiprocess satisfying the Maximum Principle have 81 possibilities to occur. These results will be presented and discussed in a forthcoming paper.

V. Conclusions

To the best of our knowledge, this the first time that a Maximum Principle is derived for a control problem whose dynamics are defined by controlled vortices, and, moreover, that a multiprocesses framework is considered. Under some reasonable simplifying assumptions, our problem formulation considers a system - environment and AUV - that optimizes the impact of vortices in the AUV motion.

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Sequel of the above article. In the sequel of the above article, we integrated the differential conditions provided by the Maximum Principle and, by using the same notation of IFAC's paper, Pereira et al. (2017a), we continued the investigation of its optimal multiprocesses problem. We give the expressions for the state and adjoint variables in the integral form and the transversality conditions on each time intervals.

In Pereira et al. (2017a) there are some typos whose corrections are:

- Page 5, first column: $p_V^i = [p_1^i,\, p_2^i,\, p_3^i,\, p_4^i]$, $p_p^i = [p_5^i,\, p_6^i]$, $p_K^i = [p_7^i,\, p_8^i]$;
- page 5, second column: $(\nabla_x f_0)^i|_{\sigma_1^i} = 0,\ \forall i$;
- page 5, second column: $N^L_{\Lambda_V}(\bar{x}_V) = \{\hat{p}_V \in \mathbb{R}^{8N} : \hat{p}^1_{V,0}M_1^T \in \mathbb{R}^2,\ \hat{p}^i_{V,1} \in \mathbb{R}^4,$ $\hat{p}^i_{V,0}M_2^T \in \mathbb{R}^2,\ i = 1, \ldots, N, \hat{p}^i_{V,0}M_1^T = -\hat{p}^{i-1}_{V,1}M_2^T,$ $i = 2, \ldots, N\}$.

In this multiprocess problem we have several variables that are constants in time, in each of the subintervals $I_i = \left[\sigma_0^i, \sigma_1^i\right]$. They are, x_0^i, x_K^i, p_p^i, $(p_1+p_2)^i$ and $(p_3+p_4)^i$. So, for $t \in [\sigma_0^i,\, \sigma_1^i]$ we have

$$\begin{cases} x_0^i(t) & = & \bar{x}_0^i \\ x_K^i(t) & = & \bar{x}_K^i \\ p_p^i(t) & = & \bar{p}_P^i \\ (p_1+p_2)^i(t) & = & \bar{p}_{1,2}^i \\ (p_3+p_4)^i(t) & = & \bar{p}_{3,4}^i \end{cases} \Leftrightarrow \begin{cases} x_0^i(t) & = & \frac{1}{(x_{1,0}^i - x_{2,0}^i)^2 + (x_{3,0}^i - x_{4,0}^i)^2} \\ x_K^i(t) & = & [\bar{x}_7^i, \bar{x}_8^i] \\ p_p^i(t) & = & [\bar{p}_5^i, \bar{p}_6^i] \\ p_2^i(t) & = & \bar{p}_{1,2}^i - p_1^i(t) \\ p_4^i(t) & = & \bar{p}_{3,4}^i - p_3^i(t) \end{cases} . \tag{7.1}$$

Remark: Here, we denote the constants, that represent the values taken by the variables on that time interval, with a bar on top of the variables.

For the adjoint variables p_1^i and p_3^i taking the respective dynamic equations and applying 7.1 we have

$$\begin{cases} -\dot{p}_1^i & = & e^i(t)\bar{x}_0^i\left(\bar{x}_8^i p_3^i - \bar{x}_7^i p_4^i + \bar{p}_6^i(u_1^i\bar{x}_8^i - u_2^i\bar{x}_7^i)\right) \\ -\dot{p}_3^i & = & e^i(t)\bar{x}_0^i\left(-\bar{x}_8^i p_1^i + \bar{x}_7^i p_2^i - \bar{p}_5^i(u_1^i\bar{x}_8^i - u_2^i\bar{x}_7^i)\right) \end{cases} \Leftrightarrow$$

$$\begin{cases} -\dot{p}_1^i & = & e^i(t)\bar{x}_0^i\left((\bar{x}_8^i + \bar{x}_7^i)p_3^i - \bar{x}_7^i\bar{p}_{3,4}^i + \bar{p}_6^i(u_1^i\bar{x}_8^i - u_2^i\bar{x}_7^i)\right) \\ -\dot{p}_3^i & = & e^i(t)\bar{x}_0^i\left(-(\bar{x}_8^i + \bar{x}_7^i)p_1^i + \bar{x}_7^i\bar{p}_{1,2}^i - \bar{p}_5^i(u_1^i\bar{x}_8^i - u_2^i\bar{x}_7^i)\right) \end{cases} \Leftrightarrow$$

$$\begin{bmatrix} \dot{p}_1^i \\ \dot{p}_3^i \end{bmatrix} = e^i(t)\bar{x}_0^i\left(\begin{bmatrix} 0 & -(\bar{x}_8^i + \bar{x}_7^i) \\ \bar{x}_8^i + \bar{x}_7^i & 0 \end{bmatrix}\begin{bmatrix} p_1^i \\ p_3^i \end{bmatrix} + \bar{x}_7^i\begin{bmatrix} \bar{p}_{3,4}^i \\ \bar{p}_{1,2}^i \end{bmatrix} + (\bar{x}_8^i u_1^i - \bar{x}_7^i u_2^i)\begin{bmatrix} -\bar{p}_6^i \\ \bar{p}_5^i \end{bmatrix}\right).$$

This is a first order linear differential equation, so, by the integral factor method, we get the next solution

$$\begin{aligned} \begin{bmatrix} p_1^i \\ p_3^i \end{bmatrix} &= (C^i(\omega(t)))^T \int_0^t e^i(\tau)\bar{x}_0^i C^i(\omega(\tau))(\bar{x}_8^i u_1^i(\tau) - \bar{x}_7^i u_2^i(\tau))\begin{bmatrix} -\bar{p}_6^i \\ \bar{p}_5^i \end{bmatrix} d\tau \\ &+ \tfrac{\bar{x}_7^i}{\bar{x}_7^i + \bar{x}_8^i}\left(I - (C^i(\omega(t)))^T\right)\begin{bmatrix} -\bar{p}_{1,2}^i \\ \bar{p}_{3,4}^i \end{bmatrix} + (C^i(\omega(t)))^T\begin{bmatrix} p_{1,0}^i \\ p_{3,0}^i \end{bmatrix}, \end{aligned} \tag{7.2}$$

with $\omega(t)=\frac{1}{\delta}e^i(t)(-1+e^{\delta t})(\bar{x}_8^i+\bar{x}_7^i)\bar{x}_0^i$, and $C^i(\omega(\tau))=\begin{bmatrix}\cos(\omega(t)) & \sin(\omega(t))\\ -\sin(\omega(t)) & \cos(\omega(t))\end{bmatrix}$.

By (7.1) and (7.2) we obtain

$$\begin{bmatrix} p_2^i \\ p_4^i \end{bmatrix} = \begin{bmatrix} \bar{p}_{1,2}^i \\ \bar{p}_{3,4}^i \end{bmatrix} - (C^i(\omega(t)))^T \int_0^t e^i(\tau)\bar{x}_0^i C^i(\omega(\tau))(\bar{x}_8^i u_1^i(\tau) - \bar{x}_7^i u_2^i(\tau)) \begin{bmatrix} -\bar{p}_6^i \\ \bar{p}_5^i \end{bmatrix} d\tau$$
$$-\tfrac{\bar{x}_7^i}{\bar{x}_7^i+\bar{x}_8^i}\left(I-(C^i(\omega(t)))^T\right)\begin{bmatrix} -\bar{p}_{1,2}^i \\ \bar{p}_{3,4}^i \end{bmatrix} - (C^i(\omega(t)))^T \begin{bmatrix} p_{1,0}^i \\ p_{3,0}^i \end{bmatrix}.$$

Now, we will study the evolution of the dynamics for the vortices, so the equations can be written in detail by the following way

$$\left\{\begin{array}{lcl} \dot{x}_1^i &=& -\bar{x}_0^i e^{-\delta(t-\sigma_0^i)}\bar{x}_8^i(x_3^i-x_4^i) \\ \dot{x}_2^i &=& \bar{x}_0^i e^{-\delta(t-\sigma_0^i)}\bar{x}_7^i(x_3^i-x_4^i) \\ \dot{x}_3^i &=& \bar{x}_0^i e^{-\delta(t-\sigma_0^i)}\bar{x}_8^i(x_1^i-x_2^i) \\ \dot{x}_4^i &=& -\bar{x}_0^i e^{-\delta(t-\sigma_0^i)}\bar{x}_7^i(x_1^i-x_2^i) \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl} \bar{x}_7^i\dot{x}_1^i+\bar{x}_8^i\dot{x}_2^i &=& 0 \\ \bar{x}_7^i\dot{x}_3^i+\bar{x}_8\dot{x}_4^i &=& 0 \end{array}\right., \tag{7.3}$$

since $\bar{x}_7^i$ and $\bar{x}_8^i$ are constants for $t\in[\sigma_0^i,\sigma_1^i]$, then

$$\left\{\begin{array}{lcl} \frac{d}{dt}(\bar{x}_7^i x_1^i+\bar{x}_8^i x_2^i) &=& 0 \\ \frac{d}{dt}(\bar{x}_7^i x_3^i+\bar{x}_8^i x_4^i) &=& 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl} \bar{x}_8^i x_2^i &=& \bar{x}_{1,2}^i-\bar{x}_7^i x_1^i \\ \bar{x}_8^i x_4^i &=& \bar{x}_{3,4}^i-\bar{x}_7^i x_3^i \end{array}\right., \tag{7.4}$$

with $\bar{x}_{1,2}^i, \bar{x}_{3,4}^i\in\mathbb{R}$. Thus, substituting (7.4) in (7.3) and using the fact that it is a first order linear differential equation, we obtain

$$\begin{array}{l}
\left\{\begin{array}{lcl} \dot{x}_1^i &=& -e^{-\delta(t-\sigma_0^i)}\bar{x}_0^i\bar{x}_8^i\left(x_3^i-x_4^i\right) \\ \dot{x}_3^i &=& e^{-\delta(t-\sigma_0^i)}\bar{x}_0^i\bar{x}_8^i\left(x_1^i-x_2^i\right) \end{array}\right. \Leftrightarrow \\
\left\{\begin{array}{lcl} \dot{x}_1^i &=& e^{-\delta(t-\sigma_0^i)}\bar{x}_0^i\left(-(\bar{x}_7^i+\bar{x}_8^i)x_3^i+\bar{x}_{3,4}^i\right) \\ \dot{x}_3^i &=& e^{-\delta(t-\sigma_0^i)}\bar{x}_0^i\left((\bar{x}_7^i+\bar{x}_8^i)x_1^i-\bar{x}_{1,2}^i\right) \end{array}\right. \Leftrightarrow \\
\begin{bmatrix} \dot{x}_1^i \\ \dot{x}_3^i \end{bmatrix} = e^{-\delta(t-\sigma_0^i)}\bar{x}_0^i\left(\begin{bmatrix} 0 & -(\bar{x}_7^i+\bar{x}_8^i) \\ \bar{x}_7^i+\bar{x}_8^i & 0 \end{bmatrix}\begin{bmatrix} x_1^i \\ x_3^i \end{bmatrix}+\begin{bmatrix} \bar{x}_{3,4}^i \\ -\bar{x}_{1,2}^i \end{bmatrix}\right) \Leftrightarrow \\
\begin{bmatrix} x_1^i \\ x_3^i \end{bmatrix} = \frac{1}{\bar{x}_7^i+\bar{x}_8^i}\left(I-(C^i(\omega(t)))^T\right)\begin{bmatrix} \bar{x}_{1,2}^i \\ \bar{x}_{3,4}^i \end{bmatrix} + (C^i(\omega(t)))^T\begin{bmatrix} x_{1,0}^i \\ x_{3,0}^i \end{bmatrix}.
\end{array} \tag{7.5}$$

Using this result in (7.4) we have

$$\begin{bmatrix} x_2^i \\ x_4^i \end{bmatrix} = \frac{1}{\bar{x}_8^i}\left(\begin{bmatrix} \bar{x}_{1,2}^i \\ \bar{x}_{3,4}^i \end{bmatrix} - \bar{x}_7^i(C^i(\omega(t)))^T\begin{bmatrix} x_{1,0}^i \\ x_{3,0}^i \end{bmatrix}\right) - \frac{\bar{x}_7^i}{x_8^i(\bar{x}_7^i+\bar{x}_8^i)}\left(I-(C^i(\omega(t)))^T\right)\begin{bmatrix} \bar{x}_{1,2}^i \\ \bar{x}_{3,4}^i \end{bmatrix}.$$

We can see that in a considerable number of equations there appears the terms $x_1^i - x_2^i$ and $x_3^i - x_4^i$, so we will present these calculations below

$$\begin{cases} x_1^i - x_2^i &= x_1^i - \frac{1}{\bar{x}_8^i}\bar{x}_{1,2}^i + \frac{\bar{x}_7^i}{\bar{x}_8^i}x_1^i \\ x_3^i - x_4^i &= x_3^i - \frac{1}{\bar{x}_8^i}\bar{x}_{3,4}^i + \frac{\bar{x}_7^i}{\bar{x}_8^i}x_3^i \end{cases} \Leftrightarrow$$

$$\begin{cases} x_1^i - x_2^i &= \frac{1}{\bar{x}_8^i}((\bar{x}_7^i + \bar{x}_8^i)x_1^i - \bar{x}_{1,2}^i) \\ x_3^i - x_4^i &= \frac{1}{\bar{x}_8^i}((\bar{x}_7^i + \bar{x}_8^i)x_3^i - \bar{x}_{3,4}^i) \end{cases} \Leftrightarrow \tag{7.6}$$

$$\begin{bmatrix} x_1^i - x_2^i \\ x_3^i - x_4^i \end{bmatrix} = \frac{1}{\bar{x}_8^i}(C^i(\omega(t)))^T \left(- \begin{bmatrix} \bar{x}_{1,2}^i \\ \bar{x}_{3,4}^i \end{bmatrix} + (\bar{x}_7^i + \bar{x}_8^i) \begin{bmatrix} x_{1,0}^i \\ x_{3,0}^i \end{bmatrix} \right).$$

By the dynamics of the fish, we can get the trajectory for it given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_5^i \\ \dot{x}_6^i \end{bmatrix} &= e^i(t)\bar{x}_0^i \begin{bmatrix} -(x_3^i - x_4^i)(\bar{x}_8^i u_1^i - \bar{x}_7^i u_2^i) \\ (x_1^i - x_2^i)(\bar{x}_8^i u_1^i - \bar{x}_7^i u_2^i)) \end{bmatrix} \Leftrightarrow \\ \begin{bmatrix} x_5^i \\ x_6^i \end{bmatrix} &= \begin{bmatrix} x_{5,0}^i \\ x_{6,0}^i \end{bmatrix} + \int_0^t e^i(\tau)\bar{x}_0^i \begin{bmatrix} -(x_3^i - x_4^i)(\bar{x}_8^i u_1^i(\tau) - \bar{x}_7^i u_2^i(\tau)) \\ (x_1^i - x_2^i)(\bar{x}_8^i u_1^i(\tau) - \bar{x}_7^i u_2^i(\tau)) \end{bmatrix} d\tau \Leftrightarrow \\ \begin{bmatrix} x_5^i \\ x_6^i \end{bmatrix} &= \begin{bmatrix} x_{5,0}^i \\ x_{6,0}^i \end{bmatrix} + \int_0^t e^i(\tau)\bar{x}_0^i(\bar{x}_8^i u_1^i(\tau) - \bar{x}_7^i u_2^i(\tau)) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (x_{V_1}^i(\tau) - x_{V_2}^i(\tau))d\tau\,. \end{aligned}$$

Therefore, the Pontryagin's function, for $t \in [\sigma_0^i, \sigma_1^i]$, is

$$\begin{aligned} H^i(t, x^i, p^i, u^i, \sigma_0^i) = e^i(t)\bar{x}_0^i \big(&-(x_3^i(t) - x_4^i(t))(\bar{x}_8^i p_1^i(t) - \bar{x}_7^i p_2^i(t) + \bar{p}_5^i(\bar{x}_8^i u_1^i(t) - \bar{x}_7^i u_2^i(t)) \\ &+(x_1^i(t) - x_2^i(t))(\bar{x}_8^i p_3^i(t) - \bar{x}_7^i p_4^i(t) + \bar{p}_6^i(\bar{x}_8^i u_1^i(t) - \bar{x}_7^i u_2^i(t)))\big) \\ &-\tfrac{1}{2}\alpha_1((u_1^i)^2 + (u_2^i)^2). \end{aligned} \tag{7.7}$$

The transversality conditions are, in an explicit way, for each one of the time intervals, $I_i = [\sigma_0^i, \sigma_1^i]$,

For $i = 1$:

$$\begin{cases} -\hat{p}_\sigma^1(0) &= \bar{p}_{\sigma,0}^1 \\ \hat{p}_0^1(0) &= \bar{p}_{0,0}^1 \\ \hat{p}_1^1(0) &= \bar{p}_{1,0}^1 \\ \hat{p}_2^1(0) &= \bar{p}_{2,0}^1 - \alpha_3(x_5^1(0) - x_2^1(0)) \\ \hat{p}_3^1(0) &= \bar{p}_{3,0}^1 \\ \hat{p}_4^1(0) &= \bar{p}_{4,0}^1 - \alpha_3(x_6^1(0) - x_4^1(0)) \\ \hat{p}_5^1(0) &= \bar{p}_{5,0}^1 + \alpha_3(x_5^1(0) - x_2^1(0)) \\ \hat{p}_6^1(0) &= \bar{p}_{6,0}^1 + \alpha_3(x_6^1(0) - x_4^1(0)) \\ \hat{p}_7^1(0) &= \bar{p}_{7,0}^1 \\ \hat{p}_8^1(0) &= \bar{p}_{8,0}^1 \end{cases} \quad \text{and} \quad \begin{cases} \hat{p}_\sigma^1(\sigma_1^1) &= -\hat{p}_\sigma^2(\sigma_0^2) \\ -\hat{p}_0^1(\sigma_1^1) &= \hat{p}_0^2(\sigma_0^2) \\ -\hat{p}_1^1(\sigma_1^1) &= \bar{p}_{1,1}^1 \\ -\hat{p}_2^1(\sigma_1^1) &= \hat{p}_1^2(\sigma_0^2) \\ -\hat{p}_3^1(\sigma_1^1) &= \bar{p}_{3,1}^1 \\ -\hat{p}_4^1(\sigma_1^1) &= \hat{p}_3^2(\sigma_0^2) \\ -\hat{p}_5^1(\sigma_1^1) &= \hat{p}_5^2(\sigma_0^2) \\ -\hat{p}_6^1(\sigma_1^1) &= \hat{p}_6^2(\sigma_0^2) \\ -\hat{p}_7^1(\sigma_1^1) &= 0 \\ -\hat{p}_8^1(\sigma_1^1) &= 0 \end{cases}.$$

For $i = 2, \cdots, N-1$*:*

$$\left\{\begin{array}{lcl} -\hat{p}^i_\sigma(\sigma^i_0) & = & \hat{p}^{i-1}_\sigma(\sigma^{i-1}_1) \\ \hat{p}^i_0(\sigma^i_0) & = & -\hat{p}^{i-1}_0(\sigma^{i-1}_1) \\ \hat{p}^i_1(\sigma^i_0) & = & -\hat{p}^{i-1}_2(\sigma^{i-1}_1) \\ \hat{p}^i_2(\sigma^i_0) & = & \bar{p}^i_{2,0} - \alpha_3(x^i_5(\sigma^i_0) - x^i_2(\sigma^i_0)) \\ \hat{p}^i_3(\sigma^i_0) & = & -\hat{p}^{i-1}_4(\sigma^{i-1}_1) \\ \hat{p}^i_4(\sigma^i_0) & = & \bar{p}^i_{4,0} - \alpha_3(x^i_6(\sigma^i_0) - x^i_4(\sigma^i_0)) \\ \hat{p}^i_5(\sigma^i_0) & = & -\hat{p}^{i-1}_5(\sigma^{i-1}_1) + \alpha_3(x^i_5(\sigma^i_0) - x^i_2(\sigma^i_0)) \\ \hat{p}^i_6(\sigma^i_0) & = & -\hat{p}^{i-1}_6(\sigma^{i-1}_0) + \alpha_3(x^i_6(\sigma^i_0) - x^i_4(\sigma^i_0)) \\ \hat{p}^i_7(\sigma^i_0) & = & \bar{p}^i_{7,0} \\ \hat{p}^i_8(\sigma^i_0) & = & \bar{p}^i_{8,0} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} \hat{p}^i_\sigma(\sigma^i_1) & = & -\hat{p}^{i+1}_\sigma(\sigma^{i+1}_0) \\ -\hat{p}^i_0(\sigma^i_1) & = & \hat{p}^{i+1}_0(\sigma^{i+1}_0) \\ -\hat{p}^i_1(\sigma^i_1) & = & \bar{p}^i_{1,1} \\ -\hat{p}^i_2(\sigma^i_1) & = & \hat{p}^{i+1}_1(\sigma^{i+1}_0) \\ -\hat{p}^i_3(\sigma^i_1) & = & \bar{p}^i_{3,1} \\ -\hat{p}^i_4(\sigma^i_1) & = & \hat{p}^{i+1}_3(\sigma^{i+1}_0) \\ -\hat{p}^i_5(\sigma^i_1) & = & \hat{p}^{i+1}_5(\sigma^{i+1}_0) \\ -\hat{p}^i_6(\sigma^i_1) & = & \hat{p}^{i+1}_6(\sigma^{i+1}_0) \\ -\hat{p}^i_7(\sigma^i_1) & = & 0 \\ -\hat{p}^i_8(\sigma^i_1) & = & 0 \end{array}\right. .$$

For $i = N$*:*

$$\left\{\begin{array}{lcl} -\hat{p}^N_\sigma \sigma^N_0) & = & \hat{p}^{N-1}_\sigma(\sigma^{N-1}_1) \\ \hat{p}^N_0(\sigma^N_0) & = & -\hat{p}^{N-1}_0(\sigma^{N-1}_1) \\ \hat{p}^N_1(\sigma^N_0) & = & -\hat{p}^{N-1}_2(\sigma^{N-1}_1) \\ \hat{p}^N_2(\sigma^N_0) & = & \bar{p}^N_{2,0} - \alpha_3(x^N_5(\sigma^N_0) - x^N_2(\sigma^N_0)) \\ \hat{p}^N_3(\sigma^N_0) & = & -\hat{p}^{N-1}_4(\sigma^{N-1}_1) \\ \hat{p}^N_4(\sigma^N_0) & = & \bar{p}^N_{4,0} - \alpha_3(x^N_6(\sigma^N_0) - x^N_4(\sigma^N_0)) \\ \hat{p}^N_5(\sigma^N_0) & = & -\hat{p}^{N-1}_5(\sigma^{N-1}_1) + \alpha_3(x^N_5(\sigma^N_0) - x^N_2(\sigma^N_0)) \\ \hat{p}^N_6(\sigma^N_0) & = & -\hat{p}^{N-1}_6(\sigma^{N-1}_0) + \alpha_3(x^N_6(\sigma^N_0) - x^N_4(\sigma^N_0)) \\ \hat{p}^N_7(\sigma^N_0) & = & \bar{p}^N_{7,0} \\ \hat{p}^N_8(\sigma^N_0) & = & \bar{p}^N_{8,0} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} \hat{p}^N_\sigma(T) & = & \bar{p}^N_{\sigma,1} \\ -\hat{p}^N_0(T) & = & \bar{p}^N_{0,1} \\ -\hat{p}^N_1(T) & = & \bar{p}^N_{1,1} \\ -\hat{p}^N_2(T) & = & \bar{p}^N_{2,1} \\ -\hat{p}^N_3(T) & = & \bar{p}^N_{3,1} \\ -\hat{p}^N_4(T) & = & \bar{p}^N_{4,1} \\ -\hat{p}^N_5(T) & = & \bar{p}^N_{5,1} \\ -\hat{p}^N_6(T) & = & \bar{p}^N_{6,1} \\ -\hat{p}^N_7(T) & = & 0 \\ -\hat{p}^N_8(T) & = & 0 \end{array}\right. .$$

Chapter 8

Conclusions and Future Work

The bulk of this Ph.D. thesis is consecrated to the derivation of necessary conditions of optimality in a form of a Maximum Principle of Pontryagin to solve the optimal control problems presented in chapters 5, 6, and 7.

In chapter 5, the minimum time control problems were solved for the motion of a passive particle, between two given points, advected by a Couette or Poiseuille flow. Using the Maximum Principle of Pontryagin we found the solutions. These simple problems constituted a first approach to the more ambitious optimal control problem, namely a particle driven by a flow/solution governed directly by the Euler or Navier-Stokes equation. Here, we have simply considered known solutions of these equations.

The minimal energy problem to move a particle from one initial point to a final destination, driven by a flow generated by one vortex, was solved in chapter 6, as well as the corresponding necessary conditions of optimality in a form of a Maximum Principle of Pontryagin. With this problem, we have started the study of optimal control problems to move particles in a fluid. This was the first effort to gain some insight and the main idea to the optimal multiprocesses problem presented in chapter 7.

An open question, the most important in this thesis, is to obtain solutions for the optimal multiprocesses problems presented in chapter 7, where the ideas and formulations are explained in Pereira et al. (2017a), Pereira et al. (2017b) and Grilo et al. (2018). In the future, some approaches to solve them may consist in the relaxation of the conditions imposed to this problems, to gain "sensibility" to extrapolate the results to more general and robust cases.

From our point of view, among all these approaches, it can be considered that, at first, the two vortices have constant circulations, in each of the time intervals $[\sigma_0^i, \sigma_1^i]$, which makes that the Pontryagin function, $H^i(t, x^i, p^i, u^i)$, be constant, for $i = 1, \cdots, N$. We can try to solve this problem for $N = 2$ and $N = 3$, and then extend (if possible) to arbitrary values for N, in order to derive the optimality conditions in the form of a Maximum Principle of Pontryagin. In a second step, and with the insight of the previous case, we consider the original problem (with decaying circulations), and solve it for $N = 2$ and $N = 3$, and then to try to derive the optimality conditions N, in the form of the Maximum Principle. Of course, the numerical methods are always available to obtain the numerical approximation of the solution for these problems.

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